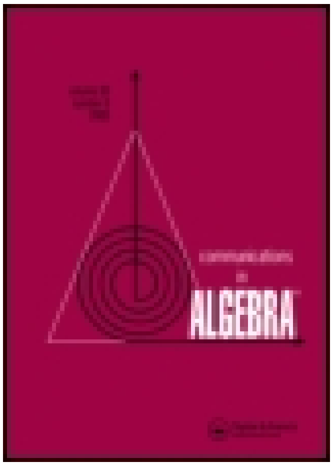

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ON MULTIPLICATIVE TRANSFER

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Introduction

In representation theory and cohomology theory of finite groups there are two kinds of transfer maps: additive and multiplicative. The purpose of this paper is to study properties of multiplicative transfer in an abstract setting.

Green and Dress introduced Mackey functors to give a unified treatment of various additive transfers ([8], [4]). Fix a finite group G. A Mackey functor on the category of G-sets is a function S which assigns to each G-set X an abelian group $S(X)$ and to each G-map $f\colon X \to Y$ homomorphisms $f^*\colon S(Y) \to S(X)$, $f_*\colon S(X) \to S(Y)$ satisfying certain axioms.

We here consider a Mackey functor S with extra structure as follows. Each $S(X)$ is a commutative ring, and for each G-map $f\colon X \to Y$ a multiplicative map $f_\star\colon S(X) \to S(Y)$ is defined. We call f^*, f_*, $f_\star$ a restriction, trace, norm map respectively, and call such S a TNR-functor.

An example is the representation ring functor A. Fix a field k. For a subgroup H of G, A takes the G-set G/H to the representation ring of $k[H]$, that is, the Grothendieck ring of $k[H]$ with respect to direct sums. For a natural G-map $f\colon G/K \to G/H$ with $K \leq H \leq G$, the maps f^*, f_*, $f_\star$ are induced by the restriction, ordinary induction, tensor induction of representations, respectively.

Let us outline the contents of the paper. We give the definition of TNR-functors in Section 2, and some natural examples of them in Section 3.

Fulton and MacPherson developed a formal calculus of multiplicative transfer ([7]). In Section 4 we discuss their addition formula for a norm map. In Section 5, using the mixed transfers introduced by them, we show that the restriction $f^*: S(Y) \to S(X)$ is an integral ring map .

In a number of important cases, each $S(X)$ is the Grothendieck ring of a certain category $\mathcal{C}(X)$ with $\oplus$ and $\otimes$, and $f_\star: S(X) \to S(Y)$ is induced by a certain functor $\mathcal{C}(f): \mathcal{C}(X) \to \mathcal{C}(Y)$ preserving $\otimes$. But, since $\mathcal{C}(f)$ does not preserve $\oplus$, it is not obvious how to define $f_\star$ on differences of objects of $\mathcal{C}(X)$. Theorem 6.1 provides a method of extending a norm map between semi-rings to one between the associated rings. This is based on our reformulation of the addition formula.

Let $\mathcal{U}$ be the category of TNR-functors. In Sections 7 and 8 we construct a category U so that $\mathcal{U}$ is isomorphic to the category of functors $U \to \{\text{sets}\}$ preserving finite products. This may be compared with Lindner's construction for Mackey functors ([9]).

Notation and Conventions. A finite group G is fixed throughout. $\mathcal{S}$ denotes the category of sets. $\mathcal{S}_f^G$ denotes the category of finite left G-sets. We simply say G-sets for finite left G-sets. A semi-ring is a set together with binary operations $+, \cdot$ and elements $0, 1$ which satisfy the axioms of a ring except the existence of the inversion for $(+, 0)$. Semi-rings and rings are assumed to be commutative. Homomorphisms of semi-rings and rings preserve 0 and 1.

1. Direct images and exponential diagrams of G-sets

Let X, Y be finite G-sets and $f: X \to Y$ a G-map. Denote by $\mathcal{S}_f^G/X$ the category of G-sets over X. The pullback functor

$$\begin{array}{ccc} \mathcal{S}_f^G/Y & \to & \mathcal{S}_f^G/X \\ (B \to Y) & \mapsto & (X \times_Y B \to X) \end{array}$$

has a right adjoint

$$\begin{array}{ccc} \mathcal{S}_f^G/X & \to & \mathcal{S}_f^G/Y \\ (A \xrightarrow{p} X) & \mapsto & (\Pi_f A \xrightarrow{q} Y), \end{array}$$

where we make q from p as follows. For each $y \in Y$, the fibre $q^{-1}(y)$ is the set of maps $s\colon f^{-1}(y) \to A$ such that $p(s(x)) = x$ for all $x \in f^{-1}(y)$. If $\sigma \in G$ and $s \in q^{-1}(y)$, the map ${}^{\sigma}s\colon f^{-1}(\sigma \cdot y) \to A$ taking x to $\sigma \cdot s(\sigma^{-1} \cdot x)$ belongs to $q^{-1}(\sigma \cdot y)$. The operation $(\sigma, s) \mapsto {}^{\sigma}s$ makes $\Pi_f A$ a G-set and q a G-map.

We have a commutative diagram

$$\begin{array}{ccccc}
X & \xleftarrow{p} & A & \xleftarrow{e} & X \times_Y \Pi_f A \\
{\scriptstyle f}\downarrow & & & & \downarrow{\scriptstyle f'} \\
Y & & \xleftarrow[q]{\qquad\qquad} & & \Pi_f A,
\end{array}$$

where f' is the projection and e is the evaluation map $(x, s) \mapsto s(x)$. A diagram in $\mathcal{S}_f^G$ which is isomorphic to this is called an exponential diagram.

The following properties are easily verified.

(1.1) If

$$\begin{array}{ccccc}
\cdot & \xleftarrow{p} & \cdot & \xleftarrow{l} & \cdot \\
{\scriptstyle f}\downarrow & & & & \downarrow{\scriptstyle f'} \\
\cdot & & \xleftarrow[q]{\qquad\qquad} & & \cdot
\end{array}$$

is an exponential diagram and

$$\begin{array}{ccccccc}
\cdot & \xleftarrow{f_1} & \cdot & \xleftarrow{p_1} & \cdot & \xleftarrow{l_1} & \cdot \\
{\scriptstyle y}\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle x'} \\
\cdot & \xleftarrow[f]{} & \cdot & \xleftarrow[p]{} & \cdot & \xleftarrow[l]{} & \cdot
\end{array}
\qquad
\begin{array}{ccccc}
\cdot & \xleftarrow{q_1} & \cdot & \xleftarrow{f_1'} & \cdot \\
{\scriptstyle y}\downarrow & & \downarrow & & \downarrow{\scriptstyle x'} \\
\cdot & \xleftarrow[q]{} & \cdot & \xleftarrow[f']{} & \cdot
\end{array}$$

are pullback diagrams, then

$$\begin{array}{ccccc}
\cdot & \xleftarrow{p_1} & \cdot & \xleftarrow{l_1} & \cdot \\
{\scriptstyle f_1}\downarrow & & & & \downarrow{\scriptstyle f_1'} \\
\cdot & & \xleftarrow[q_1]{\qquad\qquad} & & \cdot
\end{array}$$

is an exponential diagram.

(1.2) If

$$\begin{array}{ccccc} \cdot & \xleftarrow{p} & \cdot & \xleftarrow{l} & \cdot \\ f\downarrow & & & & \downarrow f' \\ \cdot & & \xleftarrow[q]{} & & \cdot \end{array} \qquad \begin{array}{ccccc} \cdot & \xleftarrow{s'} & \cdot & \xleftarrow{m} & \cdot \\ f'\downarrow & & & & \downarrow f'' \\ \cdot & & \xleftarrow[t]{} & & \cdot \end{array}$$

are exponential diagrams and

$$\begin{array}{ccc} \cdot & \xleftarrow{l'} & \cdot \\ s\downarrow & & \downarrow s' \\ \cdot & \xleftarrow[l]{} & \cdot \end{array}$$

is a pullback diagram, then

$$\begin{array}{ccccc} \cdot & \xleftarrow{ps} & \cdot & \xleftarrow{l'm} & \cdot \\ f\downarrow & & & & \downarrow f'' \\ \cdot & & \xleftarrow[qt]{} & & \cdot \end{array}$$

is an exponential diagram.

(1.3) If

$$\begin{array}{ccccc} \cdot & \xleftarrow{p} & \cdot & \xleftarrow{l} & \cdot \\ f\downarrow & & & & \downarrow f' \\ \cdot & & \xleftarrow[q]{} & & \cdot \end{array}$$

$$\begin{array}{ccccc} \cdot & \xleftarrow{q} & \cdot & \xleftarrow{m} & \cdot \\ g\downarrow & & & & \downarrow g'' \\ \cdot & & \xleftarrow[r]{} & & \cdot \end{array}$$

are exponential diagrams and

$$\begin{array}{ccc} \cdot & \xleftarrow{m'} & \cdot \\ f'\downarrow & & \downarrow f'' \\ \cdot & \xleftarrow[m]{} & \cdot \end{array}$$

is a pullback diagram, then

$$\begin{array}{ccccc} \cdot & \xleftarrow{p} & \cdot & \xleftarrow{lm'} & \cdot \\ {\scriptstyle gf}\downarrow & & & & \downarrow{\scriptstyle g''f''} \\ \cdot & & \xleftarrow[r]{} & & \cdot \end{array}$$

is an exponential diagram.

2. TNR-functors

A semi-TNR-functor S is a function which assigns to each finite G-set X a semi-ring $S(X)$ and to each G-map $f\colon X \to Y$ three maps $f^*\colon S(Y) \to S(X)$, $f_*\colon S(X) \to S(Y)$, $f_\star\colon S(X) \to S(Y)$ in such a way that the following conditions are satisfied.

(2.1)(i) If

$$X_1 \xrightarrow{i_1} X \xleftarrow{i_2} X_2$$

is a sum diagram of G-sets, then

$$S(X_1) \xleftarrow{i_1^*} S(X) \xrightarrow{i_2^*} S(X_2)$$

is a product diagram of sets. $S(\emptyset)$ consists of a single element.

(ii) f^*, f_*, and $f_\star$ are homomorphisms of semi-rings, additive monoids and multiplicative monoids, respectively.

(iii) If $f\colon X \to Y$, $g\colon Y \to Z$ are G-maps, then

$$(gf)^* = f^*g^*, \quad (gf)_* = g_*f_*, \quad (gf)_\star = g_\star f_\star$$

and

$$(1_X)^* = (1_X)_* = (1_X)_\star = 1_{S(X)}.$$

(iv) If

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow[f]{} & Y \end{array}$$

is a pullback diagram, then

$$q^* f_* = f'_* p^*, \quad q^* f_\star = f'_\star p^*.$$

(v) If

$$\begin{array}{ccccc} X & \xleftarrow{g} & Z & \xleftarrow{e} & X' \\ {\scriptstyle f}\downarrow & & & & \downarrow{\scriptstyle f'} \\ Y & & \xleftarrow[h]{\hspace{3em}} & & Y' \end{array}$$

is an exponential diagram, then

$$f_\star g_* = h_* f'_\star e^*.$$

If all $S(X)$ are rings, S is called a TNR-functor. A morphism $\varphi\colon S \to T$ of semi-TNR-functors consists of semi-ring maps $\varphi(X)\colon S(X) \to T(X)$ for all G-sets X which commute with f^*, f_*, $f_\star$ for all G-maps f. We denote by $\mathcal{U}_+$ the category of semi-TNR-functors and by $\mathcal{U}$ the full subcategory of $\mathcal{U}_+$ consisting of TNR-functors.

Remark 2.2. The formulas of (iv) are often called Mackey double coset formulas. (v) is a generalization of the distributive law. See [7, Proposition 8.6] and the proof of [4, Lemma 8.1(b)].

We notice some easy consequences of (2.1).

(2.3) Let S be a semi-TNR-functor.

(i) Let $\iota\colon \emptyset \to X$ be the unique map. Then

$$\iota_*(0) = 0, \quad \iota_\star(0) = 1.$$

(ii) Let $l\colon X \to X + Y$, $r\colon Y \to X + Y$ be the canonical injections into the disjoint sum. Then the inverse of the bijection $(l^*, r^*)\colon S(X+Y) \to S(X) \times S(Y)$ is given by

$$(a, b) \mapsto l_*(a) + l_*(b)$$

and also by

$$(a, b) \mapsto l_\star(a) r_\star(b).$$

(iii) Let $\nabla\colon X+X \to X$ be the canonical folding map. Then the following diagrams are commutative.

$$\begin{array}{ccc} S(X+X) & \xrightarrow{(l^*,r^*)} & S(X)\times S(X) \\ \nabla_*\downarrow & \swarrow \text{addition} & \\ S(X) & & \end{array}$$

$$\begin{array}{ccc} S(X+X) & \xrightarrow{(l^*,r^*)} & S(X)\times S(X) \\ \nabla_\star\downarrow & \swarrow \text{multiplication} & \\ S(X). & & \end{array}$$

(iv) (Projection formula) If $f\colon X \to Y$ is a G-map and $a \in S(X)$, $b \in S(Y)$, then

$$f_*(af^*(b)) = f_*(a)b.$$

PROOF. (i) $S(\emptyset) = \{0\} = \{1\}$, and ι_*, $\iota_\star$ preserve 0, 1, respectively.

(ii) We shall prove that the second map is a section of (l^*, r^*). By the pullback diagram

$$\begin{array}{ccccc} X & \xrightarrow{1} & X & \longleftarrow & \emptyset \\ 1\downarrow & & l\downarrow & & \downarrow \\ X & \xrightarrow[l]{} & X+Y & \xleftarrow[r]{} & Y \end{array}$$

and (i), we have $l^*l_\star(a) = a$, $l^*r_\star(b) = 1$ for $a \in S(X)$, $b \in S(Y)$. Similarly $r^*l_\star(a) = 1$, $r^*r_\star(b) = b$. Hence $l^*(l_\star(a)r_\star(b)) = a$, $r^*(l_\star(a)r_\star(b)) = b$.

(iii) For $a, a' \in S(X)$, we have $\nabla_\star(l_\star(a)r_\star(a')) = (\nabla l)_\star(a)(\nabla r)_\star(a') = aa'$. By this and (ii) the second diagram commutes.

(iv) We have an exponential diagram

$$\begin{array}{ccccc} Y+Y & \xleftarrow{f+1} & X+Y & \xleftarrow{1+f} & X+X \\ \nabla\downarrow & & & & \downarrow\nabla \\ Y & & \xleftarrow[f]{\qquad\qquad} & & X. \end{array}$$

Hence $\nabla_\star(f+1)_*(c) = f_*\nabla_\star(1+f)^*(c)$ for $c \in S(X+Y)$. By (ii) this is equivalent to the asserted formula.

A semi-Mackey ring functor S is a function which assigns to each G-set X a semi-ring $S(X)$ and to each G-map $f\colon X \to Y$ two maps $f^*\colon S(Y) \to S(X)$, $f_*\colon S(X) \to S(Y)$ in such a way that they satisfy (2.1)(i)–(iv) (for f^*, f_*) and (2.3)(iv). If all $S(X)$ are rings, S is called a Mackey ring functor.

Thus a semi-TNR-functor is a semi-Mackey ring functor, when $f_\star$ are forgotten.

3. Examples of TNR-functors

(3.1) *Invariant ring functors.* Let R be a G-ring, that is, a ring with G-action. For a G-set X, we put

$$\tilde{R}(X) = \mathrm{Map}_G(X, R) = \{G\text{-maps } X \to R\}.$$

This is a ring by pointwise addition and multiplication. If H is a subgroup of G, then $\tilde{R}(G/H)$ is isomorphic to the invariant ring R^H. For a G-map $f\colon X \to Y$, we define

$$f^*\colon \tilde{R}(Y) \to \tilde{R}(X)$$
$$f_*\colon \tilde{R}(X) \to \tilde{R}(Y)$$
$$f_\star\colon \tilde{R}(X) \to \tilde{R}(Y)$$

by the formulas

$$f^*(\psi)(x) = \psi(f(x))$$
$$f_*(\varphi)(y) = \sum_{x \in f^{-1}(y)} \varphi(x)$$
$$f_\star(\varphi)(y) = \prod_{x \in f^{-1}(y)} \varphi(x)$$

for $\varphi \in \tilde{R}(X)$, $\psi \in \tilde{R}(Y)$. Then $\tilde{R}(X)$, f^*, f_*, $f_\star$ form a TNR-functor $\tilde{R}$.

(3.2) *The Burnside ring functor.* For Burnside rings we refer to [4], [5], [10], [11]. Let X be a G-set. Let $\Omega_+(X)$ be the set of isomorphism classes $[A \to X]$ of G-sets over X. The categorical sum and product in $\mathcal{S}_f^G/X$ give $\Omega_+(X)$ a semi-ring structure. For a G-map $f\colon X \to Y$, we define

$$f^*\colon \Omega_+(Y) \to \Omega_+(X)$$
$$f_*\colon \Omega_+(X) \to \Omega_+(Y)$$
$$f_\star\colon \Omega_+(X) \to \Omega_+(Y)$$

by the formulas

$$f^*[B \to Y] = [X \times_Y B \to X]$$
$$f_*[A \to X] = [A \to X \to Y]$$
$$f_\star[A \to X] = [\Pi_f A \to Y].$$

Then $\Omega_+(X)$, f^*, f_*, $f_\star$ form a semi-TNR-functor Ω_+. This readily follows from (1.1)–(1.3). One can complete the semi-rings $\Omega_+(X)$ into rings $\Omega(X)$ by adjoining additive inverses. For a subgroup H of G, $\Omega(G/H)$ is the Burnside ring of H. One can define maps $f^*: \Omega(Y) \to \Omega(X)$, $f_*, f_\star: \Omega(X) \to \Omega(Y)$ so that $\Omega(X)$, f^*, f_*, $f_\star$ form a TNR-functor Ω and the canonical maps $\Omega_+(X) \to \Omega(X)$ form a morphism $\Omega_+ \to \Omega$ of semi-TNR-functors. This is well-known, and will be proved in a more general setting in Section 6.

(3.3) *Representation ring functors.* Let k be a field. The functor A in Introduction is a TNR-functor. If we replace $A(G/H)$ by the Grothendieck ring $A_0(G/H)$ of $k[H]$ with respect to exact sequences, we similarly obtain a TNR-functor A_0.

(3.4) *Cohomology ring functors.* Let R be a G-ring. Then we have a TNR-functor $h(R)$ as follows. If H is a subgroup of G, $h(R)(G/H)$ is the cohomology ring $\bigoplus_{n\geq 0} H^{2n}(H, R)$. If $f: G/K \to G/H$ is the natural surjection with $K \leq H \leq G$, then f^*, f_*, $f_\star$ are the restriction map, Eckmann's transfer, Evens' transfer ([6]), respectively.

4. Addition formula

The addition formula of a norm map in cohomology was given by Fulton and MacPherson [7]. In our context it is formulated as follows.

Let $f: X \to Y$ be a G-map. Consider the G-set

$$V = \{(y, C) \mid y \in Y, C \subset f^{-1}(y)\}.$$

Then $X \times_Y V = U + U'$, where

$$U = \{(x, C) \mid x \in X, C \subset f^{-1}f(x), x \in C\}$$
$$U' = \{(x, C) \mid x \in X, C \subset f^{-1}f(x), x \notin C\}.$$

We have commutative diagrams of G-sets

$$\begin{array}{ccc} X & \xleftarrow{\;r\;} & U \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle t} \\ Y & \xleftarrow[\;s\;]{} & V \end{array} \qquad\qquad \begin{array}{ccc} X & \xleftarrow{\;r'\;} & U' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle t'} \\ Y & \xleftarrow[\;s\;]{} & V, \end{array}$$

where s, r, r' are the projections, and t, t' take (x, C) to $(f(x), C)$. By a T-diagram (resp. an F-diagram) we mean a diagram in $\mathcal{S}_f^G$ which is isomorphic to the left (resp. right) one (T stands for true, F false).

PROPOSITION 4.1. *Let S be a semi-TNR-functor. Then*

$$f_\star(a + a') = s_*(t_\star r^*(a) \cdot t'_\star r'^*(a'))$$

for $a, a' \in S(X)$, and

$$f_\star(0) = j_*(1),$$

where $j\colon Y - f(X) \to Y$ is the inclusion map.

PROOF. We have the exponential diagram

$$\begin{array}{ccccc} X & \xleftarrow{\;\nabla\;} & X + X & \xleftarrow{\;r + r'\;} & U + U' \\ {\scriptstyle f}\downarrow & & & & \downarrow{\scriptstyle (t, t')} \\ Y & & \xleftarrow[\;s\;]{} & & V, \end{array}$$

where ∇ is the folding map. Let $a'' \in S(X + X)$ be a unique element which restricts to a on the left X and a' on the right X. Then

$$f_\star \nabla_*(a'') = s_*(t, t')_\star (r + r')^*(a'').$$

By (2.3)(iii) we have

$$\nabla_*(a'') = a + a'$$
$$(t, t')_\star (r + r')^*(a'') = t_\star r^*(a) \cdot t'_\star r'^*(a'),$$

hence the first formula follows.

By the exponential diagram

$$\begin{array}{ccccc} X & \xleftarrow{\;i\;} & \emptyset & \longleftarrow & \emptyset \\ {\scriptstyle f}\downarrow & & & & \downarrow{\scriptstyle k} \\ Y & & \xleftarrow[\;j\;]{\hspace{4em}} & & Y - f(X), \end{array}$$

and (2.3)(i) we have

$$f_\star(0) = f_\star i_*(0) = j_* k_\star 1^*(0) = j_*(1).$$

We give another version of the addition formula. For this we first define a new product on $S(V)$. Let $V^{(2)}$ be the G-set of triples (y, C_1, C_2) of $y \in Y$ and mutually disjoint subsets C_1, C_2 of $f^{-1}(y)$. Let p_1, p_2, m be the G-maps $V^{(2)} \to V$ taking (y, C_1, C_2) to (y, C_1), (y, C_2), $(y, C_1 \cup C_2)$, respectively, and $i\colon Y \to V$ the G-map taking y to $(y, \emptyset)$.

Let S be a semi-TNR-functor, or more generally, a semi-Mackey ring functor. For $a, b \in S(V)$ we put

$$a \vee b = m_*(p_1^*(a) \cdot p_2^*(b)).$$

LEMMA 4.2. *The additive monoid $S(V)$ is a semi-ring with multiplication $\vee$ and unit element $i_*(1)$.*

PROOF. We shall verify only the associativity of $\vee$. Let $V^{(3)}$ be the G-set of quadruples (y, C_1, C_2, C_3) of $y \in Y$ and mutually disjoint subsets C_1, C_2, C_3 of $f^{-1}(y)$. Define G-maps p_{12}, p_{23}, m_{12}, $m_{23}\colon V^{(3)} \to V^{(2)}$ and p_1, p_2, p_3, $m\colon V^{(3)} \to V$ by

$$\begin{aligned} p_{ij}(y, C_1, C_2, C_3) &= (y, C_i, C_j) \\ m_{12}(y, C_1, C_2, C_3) &= (y, C_1 \cup C_2, C_3) \\ m_{23}(y, C_1, C_2, C_3) &= (y, C_1, C_2 \cup C_3) \\ p_i(y, C_1, C_2, C_3) &= (y, C_i) \\ m(y, C_1, C_2, C_3) &= (y, C_1 \cup C_2 \cup C_3) \end{aligned}$$

for $1 \leq i \leq j \leq 3$. Then we have a pullback diagram

$$\begin{array}{ccccc} V^{(3)} & \xrightarrow{p_{12}} & V^{(2)} & \xleftarrow{p_{23}} & V^{(3)} \\ {\scriptstyle m_{12}}\downarrow & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m_{23}} \\ V^{(2)} & \xrightarrow[p_1]{} & V & \xleftarrow[p_2]{} & V^{(2)}. \end{array}$$

Now let $a, b, c \in S(V)$. By the Mackey formula (2.1)(iv) for the left square and the projection formula (2.3)(iv) for m_{12}, we compute

$$\begin{aligned} (a \vee b) \vee c &= m_*(p_1^* m_*(p_1^*(a) \cdot p_2^*(b)) \cdot p_2^*(c)) \\ &= m_*(m_{12*} p_{12}^*(p_1^*(a) \cdot p_2^*(b)) \cdot p_2(c)) \\ &= m_* m_{12*}(p_{12}^*(p_1^*(a) \cdot p_2^*(b)) \cdot m_{12}^* p_2^*(c)) \\ &= m_*(p_1^*(a) \cdot p_2^*(b) \cdot p_3^*(c)). \end{aligned}$$

Using the right square similarly, we find that $a \vee (b \vee c)$ is the same.

For $n \geq 0$ we put

$$\begin{aligned} V_n &= \{(y, C) \in V \mid \#C = n\} \\ U_n &= \{(x, C) \in U \mid \#C = n\}. \end{aligned}$$

Then U, V are the disjoint unions of U_n, V_n, respectively. We have commutative diagrams

$$\begin{array}{ccc} X & \xleftarrow{r_n} & U_n \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle t_n} \\ Y & \xleftarrow[s_n]{} & V_n \end{array} \tag{4.3}$$

with r_n, s_n, t_n the restrictions of r, s, t.

The new semi-ring $S(V)$ has a grading given by the decomposition $S(V) \cong \bigoplus_n S(V_n)$. If S is a semi-TNR-functor, the map $t_* r^* \colon S(X) \to S(V)$ decomposes into the sum of the maps $t_{n*} r_n^* \colon S(X) \to S(V_n)$. Note also that $t_{0*} r_0^*(a) = 1$.

PROPOSITION 4.4. *Let S be a semi-TNR-functor. We have*

$$t_* r^*(a + a') = t_* r^*(a) \vee t_* r^*(a')$$

for $a, a' \in S(X)$, *and*

$$t_\star r^*(0) = i_*(1).$$

PROOF. Consider the following G-sets and G-maps for $i = 1, 2$:

$$\begin{array}{lll} U_i^{(2)} = \{(x, C_1, C_2) \mid (f(x), C_1, C_2) \in V^{(2)}, x \in C_i\} & & \\ n_i\colon\ U_i^{(2)} \to V^{(2)} & (x, C_1, C_2) \mapsto (f(x), C_1, C_2) \\ q_i\colon\ U_i^{(2)} \to U & (x, C_1, C_2) \mapsto (x, C_i) \\ l_i\colon\ U_i^{(2)} \to U & (x, C_1, C_2) \mapsto (x, C_1 \cup C_2). \end{array}$$

Then

$$\begin{array}{ccc} U & \xleftarrow{l_1} & U_1^{(2)} \\ t\downarrow & & \downarrow n_1 \\ V & \xleftarrow[m]{} & V^{(2)} \end{array} \qquad \begin{array}{ccc} U & \xleftarrow{l_2} & U_2^{(2)} \\ t\downarrow & & \downarrow n_2 \\ V & \xleftarrow[m]{} & V^{(2)} \end{array} \qquad \begin{array}{ccc} U & \xleftarrow{q_i} & U_i^{(2)} \\ t\downarrow & & \downarrow n_i \\ V & \xleftarrow[p_i]{} & V^{(2)} \end{array}$$

are a T-diagram, an F-diagram, a pullback diagram, respectively.

By the addition formula for $t_\star$ we have

$$\begin{aligned} t_\star r^*(a + a') &= m_*(n_{1\star} l_1^* r^*(a) \cdot n_{2\star} l_2^* r^*(a')) \\ &= m_*(n_{1\star} q_1^* r^*(a) \cdot n_{2\star} q_2^* r^*(a')) \\ &= m_*(p_1^* t_\star r^*(a) \cdot p_2^* t_\star r^*(a')) \\ &= t_\star r^*(a) \vee t_\star r^*(a'). \end{aligned}$$

Since $V - t(U) = i(Y)$, we have

$$t_\star r^*(0) = t_\star(0) = i_*(1).$$

This proves the proposition.

As an application of this proposition, we show that $f_\star$ is an algebraic map in the sense of Dress [3]. Let K be an abelian monoid, L an abelian group and $\varphi\colon K \to L$ a map. We say $\deg\varphi = 0$ if φ is a constant map. For $a \in K$ let $D_a\varphi\colon K \to L$ be the map $x \mapsto \varphi(x + a) - \varphi(x)$. Inductively, for $n > 0$ we say $\deg\varphi \le n$ if $\deg(D_a\varphi) \le n - 1$ for all $a \in K$. We say φ is algebraic if $\deg\varphi < \infty$.

LEMMA 4.5. (i) *Let $\varphi, \psi\colon K \to L$ be algebraic maps with K, L abelian groups. If φ and ψ coincide on a submonoid generating K, then $\varphi = \psi$.*

(ii) *Let $\varphi\colon K \to L$ and $\psi\colon L \to M$ be algebraic maps with L, M abelian groups. Then $\psi \circ \varphi$ is an algebraic map.*

PROOF. See [5, Section 5].

LEMMA 4.6. *Let K be an abelian monoid and $R = \bigoplus_{n\geq 0} R_n$ a graded ring. Let φ be a monoid map of K into the multiplicative group $1 + \prod_{n>0} R_n$ with components $\varphi_n : K \to R_n$. Then $\deg \varphi_n \leq n$.*

PROOF. This follows inductively from

$$\varphi_n(x+a) - \varphi_n(x) = \sum_{i=0}^{n-1} \varphi_i(x)\varphi_{n-i}(a).$$

The degree $\deg f$ of a G-map $f\colon X \to Y$ is the function on Y given by $(\deg f)(y) = \# f^{-1}(y)$.

PROPOSITION 4.7. *Let S be a TNR-functor. Then $f_*\colon S(X) \to S(Y)$ is an algebraic map with respect to the additive structures of $S(X)$, $S(Y)$. If $\deg f \leq n$, then $\deg f_* \leq n$.*

PROOF. We may assume Y has only one orbit. Then f has a constant degree, say, n. Applying Lemma 4.6 to the map

$$t_* r^*\colon S(X) \to 1 + \prod_{k>0} S(V_k),$$

we know that $\deg(t_{k*}r_k^*) \leq k$ for all k. Since $t_{n*}r_n^* = f_*$, we have $\deg f_* \leq n$.

5. Mixed transfers $f_*^{(n)}$

In this section we introduce transfers $f_*^{(n)}\colon S(X) \to S(Y)$ for $n \geq 0$ after Fulton and MacPherson [7]. We can imagine $f_*^{(n)}(a)$ as the n^{th} elementary symmetric polynomial of the conjugates of a. In particular, f^* is an integral ring map. We also give a direct proof of a formula of [7, Remark 10.10], which relates $f_*(a^m)$ with $f_*^{(n)}(a)$. The later sections are independent of this section.

We fix a TNR-functor S and a G-map $f\colon X \to Y$. With the natation of (4.3) we put

$$f_*^{(n)} = s_{n*}t_{n\star}r_n^*\colon S(X) \to S(Y).$$

Clearly $f_*^{(0)}(a) = 1$, $f_*^{(1)}(a) = f_*(a)$ for all $a \in S(X)$. $f_*^{(n)}(a) = f_\star(a)$ if f has a constant degree n, and $f_*^{(n)}(a) = 0$ if $\deg f < n$.

PROPOSITION 5.1. *If* $\deg f \leq n$, *then*

$$\sum_{k=0}^{n}(-a)^{n-k}f^*f_*^{(k)}(a) = 0$$

for all $a \in S(X)$. *In particular,* $S(X)$ *is integral over the subring* $f^*(S(Y))$.

LEMMA 5.2. *Let* $f''\colon X+X' \to Y$ *be a* G*-map with components* $f\colon X \to Y$ *and* $f'\colon X' \to Y$. *If* $a'' \in S(X+X')$ *corresponds to* $(a, a') \in S(X) \times S(X')$ *through the natural bijection, then*

$$f_*^{\prime\prime(n)}(a'') = \sum_{i+j=n} f_*^{(i)}(a)f_*^{\prime(j)}(a').$$

PROOF. Let

$$\begin{array}{ccc} X & \xleftarrow{\ r\ } & U \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle t} \\ Y & \xleftarrow[\ s\ ]{} & V \end{array} \qquad\qquad \begin{array}{ccc} X' & \xleftarrow{\ r'\ } & U' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle t'} \\ Y & \xleftarrow[\ s'\ ]{} & V' \end{array}$$

be T-diagrams. Then

$$\begin{array}{ccc} X + X' & \xleftarrow{\ r''\ } & U \times_Y V' + V \times_Y U' \\ {\scriptstyle f''}\downarrow & & \downarrow{\scriptstyle t''} \\ Y & \xleftarrow[\ s''\ ]{} & V \times_Y V' \end{array}$$

is a T-diagram, where

$$s'' = s \times_Y s', \qquad t'' = (t \times_Y 1, 1 \times_Y t'), \qquad r'' = rp + r'p'$$

and p (resp. p') is the projection to the left (resp. right) factor. We have

$$\begin{aligned} t''_* r''^*(a'') &= (t \times_Y 1)_*(rp)^*(a) \cdot (1 \times_Y t')_*(r'p')^*(a') \\ &= p^* t_* r^*(a) \cdot p'^* t'_* r'^*(a'), \end{aligned}$$

where the projections $V \times_Y V' \to V$, $V \times_Y V' \to V'$ are denoted by p, p' again. Define $V'_n, s'_n, \ldots, (V \times_Y V')_n, s''_n, \ldots$ similarly to V_n, $s_n, \ldots$. Then

$$(V \times_Y V')_n = \bigcup_{i+j=n} V_i \times_Y V'_j.$$

Hence

$$\begin{aligned} f''^{(n)}_*(a'') &= s''_{n*} t''_{n*} r''^*_n(a'') \\ &= \sum_{i+j=n} (s_i \times_Y s'_j)_*(p^* t_{i*} r^*_i(a) \cdot p'^* t'_{j*} r'^*_j(a')) \\ &= \sum_{i+j=n} s_{i*} t_{i*} r^*_i(a) \cdot s'_{j*} t'_{j*} r'^*_j(a') \\ &= \sum_{i+j=n} f^{(i)}_*(a) f'^{(j)}_*(a'). \end{aligned}$$

PROOF OF PROPOSITION 5.1. Let X' be the complement of the diagonal in $X \times_Y X$ and $g_1, g_2 \colon X' \to X$ the projections. We have a pullback diagram

$$\begin{array}{ccc} X & \xleftarrow{(1,g_2)} & X + X' \\ f \downarrow & & \downarrow (1,g_1) \\ Y & \xleftarrow[f]{} & X. \end{array}$$

Applying Lemma 5.2 to the map $(1, g_1)$, we have

$$\begin{aligned} f^* f^{(k)}_*(a) &= (1, g_1)^{(k)}_*(1, g_2)^*(a) \\ &= \sum_{i+j=k} 1^{(i)}_*(a) \cdot g^{(j)}_{1*} g^*_2(a) \\ &= g^{(k)}_{1*} g^*_2(a) + a g^{(k-1)}_{1*} g^*_2(a) \end{aligned}$$

for all k. Hence

$$\sum_{k=0}^{n} (-a)^{n-k} f^* f^{(k)}_*(a) = g^{(n)}_{1*} g^*_2(a).$$

If $\deg f \le n$, then $\deg g_1 < n$, so $g^{(n)}_{1*} g^*_2(a) = 0$. This proves the proposition.

We next consider $f_*(a^n)$, an analogue of the n^{th} power sum.

PROPOSITION 5.3. *We have*

$$s_{n*}(t_{n*}(1) \cdot t_{n*}r_n^*(a)) = \sum_{i=1}^{n}(-1)^{i-1} f_*(a^i) f_*^{(n-i)}(a)$$

for $a \in S(X)$, $n \geq 0$.

S is said to be cohomological if $g_*(1) = d$ for any G-map g of constant degree d (Green [8]). For example, the functors of (3.1), (3.4) are cohomological. In this case the above formula takes the following form.

COROLLARY 5.4. *If S is cohomological, then*

$$n f_*^{(n)}(a) = \sum_{i=1}^{n}(-1)^{i-1} f_*(a^i) f_*^{(n-i)}(a).$$

Fulton and MacPherson deduced this formula from a general principle relating symmetric polynomials with transfers [7, Section 10]. We shall give a direct proof.

LEMMA 5.5. *Set* $b_n = r_{n*}t_n^* t_{n*} r_n^*(a)$ *for* $n \geq 0$. *Then*

$$b_n = a(f^* f_*^{(n-1)}(a) - b_{n-1})$$

for $n > 0$.

PROOF. Let W be the complement of the diagonal in $U_n \times_{V_n} U_n$ and $q_1, q_2 \colon W \to U_n$ the projections. By the pullback diagram

$$\begin{array}{ccc} U_n & \xleftarrow{(1,q_2)} & U_n + W \\ t_n \downarrow & & \downarrow (1, q_1) \\ V_n & \xleftarrow[t_n]{} & U_n \end{array}$$

we have $t_n^* t_{n*}(x) = x \cdot q_{1*}q_2^*(x)$ for all $x \in S(U_n)$. Hence

$$\begin{aligned} b_n &= r_{n*}(r_n^*(a) \cdot q_{1*}q_2^* r_n^*(a)) \\ &= a \cdot r_{n*}q_{1*}q_2^* r_n^*(a). \end{aligned}$$

The set W consists of triples (x_1, x_2, C) such that $x_1, x_2 \in X$, $x_1 \neq x_2$, $f(x_1) = f(x_2)$, $C \subset f^{-1}f(x_1)$, $x_1, x_2 \in C$. We have a commutative diagram

$$\begin{array}{ccccc} U_n & \xleftarrow{q_1} & W & \xrightarrow{q_2} & U_n \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle r_n} \\ V_{n-1} & \xleftarrow[t_{n-1}]{} & U_{n-1} & \xrightarrow[r_{n-1}]{} & X \end{array}$$

with the left square cartesian, where l, m are given by

$$l(x, C) = (f(x), C - \{x\})$$
$$m(x_1, x_2, C) = (x_2, C - \{x_1\}).$$

Hence

$$\begin{aligned} b_n &= a \cdot r_{n*} q_{1*} m^* r_{n-1}^*(a) \\ &= a \cdot r_{n*} l^* t_{n-1*} r_{n-1}^*(a). \end{aligned}$$

By the pullback diagram

$$\begin{array}{ccc} X & \xleftarrow{(r_{n-1}, r_n)} & U_{n-1} + U_n \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle (t_{n-1}, l)} \\ Y & \xleftarrow[s_{n-1}]{} & V_{n-1} \end{array}$$

we have $f^* s_{n-1*}(x) = r_{n-1*} t_{n-1}^*(x) + r_{n*} l^*(x)$ for all $x \in S(V_{n-1})$. Hence

$$\begin{aligned} b_n &= a(f^* s_{n-1*} t_{n-1*} r_{n-1}^*(a) - r_{n-1*} t_{n-1}^* t_{n-1*} r_{n-1}^*(a)) \\ &= a(f^* f_*^{(n)}(a) - b_{n-1}). \end{aligned}$$

PROOF OF PROPOSITION 5.3. From Lemma 5.5 we obtain

$$b_n = \sum_{i=1}^{n} (-1)^{i-1} a^i \cdot f^* f_*^{(n-i)}(a).$$

Applying f_* to the both sides, we have

$$f_*(b_n) = \sum_{i=1}^{n} (-1)^{i-1} f_*(a^i) f_*^{(n-i)}(a).$$

On the other hand,

$$\begin{aligned} f_*(b_n) &= f_* r_{n*} t_n^* t_{n\star} r_n^*(a) \\ &= s_{n*} t_{n*} t_n^* t_{n\star} r_n^*(a) \\ &= s_{n*}(t_{n*}(1) \cdot t_{n\star} r_n^*(a)). \end{aligned}$$

This proves the proposition.

REMARK 5.6. Proposition 5.1 is trivial if f is normal, that is, if f is a natural surjection $G/K \to G/H$ with $K \leq H \leq G$ and K is normal in H. Indeed, let $r_\sigma \colon G/K \to G/K$ be the right multiplication by $\sigma \in H/K$. Then $f^* f_\star^{(n)}(a)$ is the n^{th} elementary symmetric polynomial of $r_\sigma^*(a)$ for all $\sigma \in H/K$. Moreover, in this case the ring map $f^* \colon S(G/H) \to S(G/K)$ is H/K-normal in the sense of Dress [2].

REMARK 5.7. For the representation ring functor A of (3.3), the integrality of $\mathbb{C} \otimes f^*$ was proved by a different method in [1]. For the other functors of Section 3, the integrality of f^* is well-known.

6. From semi-TNR-functors to TNR-functors

For an abelian monoid M there exists a universal abelian group γM with monoid map $k_M \colon M \to \gamma M$. γM is an abelian group with generators $k_M(m)$ for $m \in M$ and relations $k_M(m + m') = k_M(m) + k_M(m')$ for $m, m' \in M$. If M is a semi-ring, the group γM made from the additive monoid of M has a unique ring structure such that k_M is a semi-ring map.

THEOREM 6.1. *Let S be a semi-TNR-functor. Then the function which assigns the set $\gamma S(X)$ to each G-set X has a unique structure of a TNR-functor such that the maps $k_{S(X)}$ form a morphism of semi-TNR-functors.*

PROOF. We give the above ring structure to each $\gamma S(X)$. Let $f \colon X \to Y$ be a G-map. It is clear that f^*, f_* of S uniquely extend to additive maps $f^* \colon \gamma S(Y) \to \gamma S(X)$, $f_* \colon \gamma S(X) \to \gamma S(Y)$, respectively. We claim that there exists also a unique algebraic map $f_\star \colon \gamma S(X) \to \gamma S(Y)$ extending $f_\star$ of S. The uniqueness is guaranteed by Lemma 4.5(i). In Section 4 we constructed the homomorphism $\chi := t_\star r^*$ from the additive monoid $S(X)$ into the monoid $S(V)$ with multiplication $\vee$. Its image lies in $1 + \prod_{n>0} S(V_n)$, and if $j: Y \to V$ denotes the map $y \mapsto (y, f^{-1}(y))$, then $j^*\chi = f_\star$. Since $1 + \prod_{n>0} \gamma S(V_n)$ is a group, there exists a unique monoid map $\tilde{\chi}: \gamma S(X) \to \gamma S(V)$ extending χ. Then $f_\star := j^*\tilde{\chi}: \gamma S(X) \to \gamma S(Y)$ is an extension of $f_\star: S(X) \to S(Y)$. It is algebraic by the proof of Proposition 4.7.

By Lemma 4.5 we know that the maps f^*, f_*, $f_\star$ for $\gamma S(\ )$ satisfy (2.1). This proves the theorem.

Let us denote by γS the TNR-functor constructed above and by κ_S the morphism $S \to \gamma S$ with components $k_{S(X)}$. The following is clear.

PROPOSITION 6.2. *$\kappa_S: S \to \gamma S$ is a universal morphism from S to a TNR-functor.*

REMARK 6.3. There is a lemma of Dress stating that any algebraic map φ from an abelian monoid K to an abelian group L extends to an algebraic map $\tilde{\varphi}: \gamma K \to L$ ([3], [5, Lemma 5.6.15]). In the above proof we did not use the lemma because the direct construction was available .

7. Category U_+

Lindner observed that Mackey functors are precisely additive functors from the category of *spans* $[Y \leftarrow A \to X]$ of G-maps to the category of abelian groups ([9]). We aim to give a similar interpretation to TNR-functors. In this section we construct a category U_+ such that $\mathcal{U}_+$ is isomorphic to the category of functors $U_+ \to \mathcal{S}$ preserving finite products.

We say two diagrams $Y \leftarrow B \leftarrow A \to X$ and $Y \leftarrow B' \leftarrow A' \to X$ in $\mathcal{S}_f^G$ are isomorphic if there are G-isomorphisms $A \to A'$, $B \to B'$ making the diagram

$$\begin{array}{ccccccc} Y & \longleftarrow & B & \longleftarrow & A & \longrightarrow & X \\ \| & & \downarrow & & \downarrow & & \| \\ Y & \longleftarrow & B' & \longleftarrow & A' & \longrightarrow & X \end{array}$$

commutative. Let $U_+(X,Y)$ be the set of the isomorphism classes $[Y \leftarrow B \leftarrow A \to X]$ of diagrams $Y \leftarrow B \leftarrow A \to X$. We define an operation $\circ: U_+(Y,Z) \times U_+(X,Y) \to U_+(X,Z)$ by

$$[Z \leftarrow D \leftarrow C \to Y] \circ [Y \leftarrow B \leftarrow A \to X] = [Z \leftarrow \tilde{D} \leftarrow A'' \to X],$$

where the maps in the right side are composites of the maps in the diagram

$$\begin{array}{ccccccccc}
 & & & & \tilde{D} & \leftarrow & \tilde{C} & \leftarrow & A'' \\
 & & & \swarrow & & & \downarrow & & \downarrow \\
Z & \leftarrow & D & \leftarrow & C & \leftarrow & B' & \leftarrow & A' \\
 & & & & \downarrow & & \downarrow & & \downarrow \\
 & & & & Y & \leftarrow & B & \leftarrow & A \\
 & & & & & & & & \downarrow \\
 & & & & & & & & X.
\end{array}$$

Here the three squares are pullback diagrams and the pentagon is an exponential diagram.

For a G-set X we put

$$I_X = [X \xleftarrow{1} X \xleftarrow{1} X \xrightarrow{1} X].$$

PROPOSITION 7.1. *The operation* $\circ$ *is associative. The element* I_X *satisfies the unit condition with respect to* $\circ$.

PROOF. The unit condition is easy to see. The associativity is a consequence of (1.1)–(1.3). We omit the detail.

We define a category U_+ as follows. The objects of U_+ are precisely the finite G-sets. For G-sets X and Y we put $\mathrm{Hom}_{U_+}(X,Y) = U_+(X,Y)$. The composition of morphisms is the operation $\circ$ and the identity morphisms are I_X.

We associate with a G-map $f\colon X \to Y$ three morphisms

$$T_f = [Y \xleftarrow{f} X \xleftarrow{1} X \xrightarrow{1} X] \in U_+(X,Y)$$
$$N_f = [Y \xleftarrow{1} Y \xleftarrow{f} X \xrightarrow{1} X] \in U_+(X,Y)$$
$$R_f = [X \xleftarrow{1} X \xleftarrow{1} X \xrightarrow{f} Y] \in U_+(Y,X).$$

PROPOSITION 7.2. (i) $[Y \xleftarrow{h} B \xleftarrow{g} A \xrightarrow{f} X] = T_h \circ N_g \circ R_f$.

(ii) *If* $f\colon X \to Y$, $g\colon Y \to C$ *are* G-*maps, then*

$$R_{gf} = R_f \circ R_g, \quad R_{1_X} = I_X$$
$$T_{gf} = T_g \circ T_f, \quad T_{1_X} = I_X$$
$$N_{gf} = N_g \circ N_f, \quad N_{1_X} = I_X.$$

(iii) *If*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow[f]{} & Y \end{array}$$

is a pullback diagram, then

$$R_q \circ T_f = T_{f'} \circ R_p$$
$$R_q \circ N_f = N_{f'} \circ R_p.$$

(iv) *If*

$$\begin{array}{ccccc} X & \xleftarrow{g} & C & \xleftarrow{e} & X' \\ {\scriptstyle f}\downarrow & & & & \downarrow{\scriptstyle f'} \\ Y & & \xleftarrow[h]{\qquad\qquad} & & Y' \end{array}$$

is an exponential diagram, then

$$N_f \circ T_g = T_h \circ N_{f'} \circ R_e.$$

PROOF. Easy and omitted.

PROPOSITION 7.3. *The category U_+ has a presentation by generators R_f, T_f, N_f and relations in Proposition 7.2(ii)–(iv).*

PROOF. It is enough to observe that one can reduce any word of R_f, T_f, N_f to a word of the form $T_h \circ N_g \circ R_f$, using the relations in Proposition 7.2(ii)–(iv).

REMARK 7.4. If one defines U_+ by the above presentation, then one will have to prove the uniqueness of the normal form $T_h \circ N_g \circ R_f$.

We next aim to make $U_+(X, Y)$ into a semi-ring.

PROPOSITION 7.5. (i) *If*

$$X_1 \xrightarrow{i_1} X \xleftarrow{i_2} X_2$$

is a sum diagram in S_f^G, then

$$X_1 \xleftarrow{R_{i_1}} X \xrightarrow{R_{i_2}} X_2$$

is a product diagram in U_+. $\emptyset$ is a final object in U_+.

(ii) Let X be a G-set and $\nabla\colon X+X \to X$ the folding map, $\iota\colon \emptyset \to X$ the unique map. Then X has a structure of a semi-ring object of U_+ with addition T_∇, additive unit T_ι, multiplication N_∇, multiplicative unit N_ι.

(iii) If $f\colon X \to Y$ is a G-map, then the morphisms R_f, T_f, N_f of U_+ preserve the above structures of semi-rings, additive monoids, multiplicative monoids on X and Y, respectively.

PROOF. (i) For any G-set Y, the map

$$(R_{i_1} \circ (?), R_{i_2} \circ (?))\colon U_+(Y, X) \to U_+(Y, X_1) \times U_+(Y, X_2)$$

is bijective because

$$R_{i_\nu} \circ [X \leftarrow B \leftarrow A \to Y] = [X_\nu \leftarrow B_\nu \leftarrow A_\nu \to Y],$$

where

$$\begin{array}{ccccc} X_\nu & \longleftarrow & B_\nu & \longleftarrow & A_\nu \\ {\scriptstyle i_\nu}\downarrow & & \downarrow & & \downarrow \\ X & \longleftarrow & B & \longleftarrow & A \end{array}$$

are pullback diagrams for $\nu = 1, 2$.

$U_+(Y, \emptyset)$ consists of the single element $[\emptyset \leftarrow \emptyset \leftarrow \emptyset \to Y]$.

(ii) The associative, commutative, and unit conditions are easily verified. We prove only the distributivity. We have an exponential diagram

$$\begin{array}{ccccc} X+X & \xleftarrow{1+\nabla} & X+X+X & \xleftarrow{(\nabla+1+1)(1+\tau+1)} & X+X+X+X \\ \nabla\downarrow & & & & \downarrow\nabla+\nabla \\ X & & \xleftarrow[\nabla]{\hspace{6em}} & & X+X \end{array}$$

where $\tau\colon X+X \to X+X$ is the twisting map. Hence we have a commutative diagram in U_+

$$\begin{array}{ccccc} X+X & \xleftarrow{T_{1+\nabla}} & X+X+X & \xrightarrow{R_{(\nabla+1+1)(1+\tau+1)}} & X+X+X+X \\ N_\nabla\downarrow & & & & \downarrow N_{\nabla+\nabla} \\ X & & \xleftarrow[N_\nabla]{\hspace{6em}} & & X+X. \end{array}$$

This means the distributive law.

(iii) It is clear that T_f (resp. N_f) preserves the additive (resp. multiplicative) structure. The Mackey formulas for the pullback diagrams

$$\begin{array}{ccc} X+X & \xrightarrow{f+f} & Y+Y \\ \nabla\downarrow & & \downarrow\nabla \\ X & \xrightarrow[f]{} & Y \end{array} \qquad\qquad \begin{array}{ccc} \emptyset & \longrightarrow & \emptyset \\ \downarrow & & \downarrow \\ X & \xrightarrow[f]{} & Y \end{array}$$

show that R_f preserves the both structures. This finishes the proof.

Let X, Y be G-sets. The semi-ring structure of Y as an object of U_+ makes the hom-set $U_+(X,Y)$ a semi-ring. Explicitly:

PROPOSITION 7.6. *The semi-ring structure of* $U_+(X,Y)$ *is given as follows.*

$$\begin{aligned} &[Y \leftarrow B \leftarrow A \rightarrow X] + [Y \leftarrow B' \leftarrow A' \rightarrow X] \\ &\quad = [Y \leftarrow B + B' \leftarrow A + A' \rightarrow X] \\ &0 = [Y \leftarrow \emptyset \leftarrow \emptyset \rightarrow X] \\ &[Y \leftarrow B \leftarrow A \rightarrow X] \cdot [Y \leftarrow B' \leftarrow A' \rightarrow X] \\ &\quad = [Y \leftarrow B \times_Y B' \leftarrow A \times_Y B' + B \times_Y A' \rightarrow X] \\ &1 = [Y \leftarrow Y \leftarrow \emptyset \rightarrow X]. \end{aligned}$$

PROOF. We prove only the last half. We have

$$\begin{aligned} &[Y \leftarrow B \leftarrow A \rightarrow X] \cdot [Y \leftarrow B' \leftarrow A' \rightarrow X] \\ &= N_\nabla \circ [Y + Y \leftarrow B + B' \leftarrow A + A' \rightarrow X] \end{aligned}$$

with $\nabla\colon Y+Y \rightarrow Y$ the folding map. The diagram

$$\begin{array}{ccccccc} & & B \times_Y B' & \leftarrow & B \times_Y B' + B \times_Y B' & \leftarrow & A \times_Y B' + B \times_Y A' \\ & \swarrow & & \text{exp} & \downarrow & \text{pb} & \downarrow \\ Y & \leftarrow & Y+Y & \leftarrow & B+B' & \leftarrow & A+A' \\ & & & & & & \downarrow \\ & & & & & & X \end{array}$$

gives the third formula.

We have $1 = N_\iota \circ [\emptyset \leftarrow \emptyset \leftarrow \emptyset \rightarrow X]$. The diagram

$$\begin{array}{ccccccc} & & Y & \leftarrow & \emptyset & \leftarrow & \emptyset \\ & \swarrow & \text{exp} & & \downarrow & \text{pb} & \downarrow \\ Y & \leftarrow & \emptyset & \leftarrow & \emptyset & \leftarrow & \emptyset \\ & & & & & & \downarrow \\ & & & & & & X \end{array}$$

gives the last formula.

Let us denote by $[U_+, \mathcal{S}]_0$ the category of functors $U_+ \rightarrow \mathcal{S}$ preserving finite products, where $\mathcal{S}$ is the category of sets.

PROPOSITION 7.7. *We have an isomorphism of categories* $\mathcal{U}_+ \cong [U_+, \mathcal{S}]_0$.

PROOF. Objects $S \in \mathcal{U}_+$ and $F \in [U_+, \mathcal{S}]_0$ correspond to each other if

$$S(X) = F(X), \quad f^* = F(R_f), \quad f_* = F(T_f), \quad f_\star = F(N_f).$$

Indeed, given $S \in \mathcal{U}_+$, these formulas determine a functor $F\colon U_+ \rightarrow \mathcal{S}$ by Proposition 7.3, and F preserves finite products by (2.1)(i) and Proposition 7.5(i).

Conversely, given $F \in [U_+, \mathcal{S}]_0$, the functions $S(\ )$, $(\ )^*$, $(\ )_*$, $(\ )_\star$ determined by the above formulas satisfy (i), (iii), (iv), (v) of (2.1). Since each G-set X is a semi-ring object of U_+ and F preserves finite products, $F(X)$ becomes a semi-ring. By Proposition 7.5(iii), f^*, f_*, $f_\star$ are homomorphisms of semi-rings, additive monoids, multiplicative monoids, respectively. Thus we obtain an object $S \in \mathcal{U}_+$.

8. Category U

In this section we construct a category U such that $\mathcal{U}$ is isomorphic to the category of functors $U \rightarrow \mathcal{S}$ preserving finite products.

Let X, Y be G-sets. With the notation of Section 6, we make from the semi-ring $U_+(X,Y)$ the ring

$$U(X,Y) = \gamma U_+(X,Y)$$

together with the canonical semi-ring map

$$k\colon U_+(X,Y) \rightarrow U(X,Y).$$

PROPOSITION 8.1. *There exists a unique category U satisfying the following conditions:*

(i) $\mathrm{Obj}(U) = \mathrm{Obj}(\mathcal{S}_f^G)$.

(ii) $\mathrm{Hom}_U(X,Y) = U(X,Y)$.

(iii) *The maps* $k\colon U_+(X,Y) \to U(X,Y)$ *and the identity on* $\mathrm{Obj}(\mathcal{S}_f^G)$ *form a functor* $k\colon U_+ \to U$.

(iv) *This functor k preserves finite products.*

PROOF. We identify $\mathcal{U}_+ = [U_+, \mathcal{S}]_0$ by Proposition 7.7. Let X be a G-set. Applying Theorem 6.1 to the hom-functor $U_+(X,?) \in [U_+, \mathcal{S}]_0$, we obtain a unique functor $U(X,?) \in [U_+, \mathcal{S}]_0$ such that it assigns to each G-set Y the set $U(X,Y)$ and such that the maps $k\colon U_+(X,Y) \to U(X,Y)$ form a morphism $\kappa\colon U_+(X,?) \to U(X,?)$ in $[U_+, \mathcal{S}]_0$. This functor $U(X,?)$ takes the semi-ring structure of $Y \in U_+$ to the original ring structure of $U(X,Y)$.

Let $\alpha \in U(X,Y)$. By Yoneda's lemma α corresponds to a morphism $\alpha^\sharp\colon U_+(Y,?) \to U(X,?)$ in $[U_+, \mathcal{S}]_0$. By Proposition 6.2 there exists a unique morphism $\alpha^\natural$ in $[U_+, \mathcal{S}]_0$ making the triangle

$$\begin{array}{ccc} U_+(Y,?) & \xrightarrow{\alpha^\sharp} & U(X,?) \\ \kappa\big\downarrow & \nearrow \alpha^\natural & \\ U(Y,?) & & \end{array}$$

commute.

Now we define the category U. The objects and morphisms of U are given by (i) and (ii). The composition $\circ\colon U(Y,Z) \times U(X,Y) \to U(X,Z)$ is given by $\beta \circ \alpha = \alpha^\natural(\beta)$. The identity morphisms are $k(I_X)$.

One can easily verify that U is a category and the maps $k\colon U_+(X,Y) \to U(X,Y)$ form a functor $k\colon U_+ \to U$. Since $U(X,?)\colon U_+ \to \mathcal{S}$ preserves finite products for each X, so does $k\colon U_+ \to U$.

The uniqueness of U can be seen by reversing the above argument. This finishes the proof.

Since $k\colon U_+ \to U$ preserves finite products, each object X of U has the semi-ring structure $k(T_\nabla)$, $k(T_\iota)$, $k(N_\nabla)$, $k(N_\iota)$. This induces the original ring structure on $U(Y,X)$ for every Y. Hence X is a ring object of U. The additive inversion morphism is $-k(I_X)$.

Let us denote by $[U, \mathcal{S}]_0$ the category of functors $U \to \mathcal{S}$ preserving finite products.

THEOREM 8.2. *We have an isomorphism of categories* $\mathcal{U} \cong [U, \mathcal{S}]_0$.

PROOF. We shall show that we have a commutative diagram

$$\begin{array}{ccc} \mathcal{U}_+ & \underset{\sigma}{\overset{\sim}{\to}} & [U_+, \mathcal{S}]_0 \\ \bigcup & & \uparrow \\ \mathcal{U} & \overset{\sim}{\to} & [U, \mathcal{S}]_0, \end{array}$$

where σ is the isomorphism of Proposition 7.7 and the right arrow is the restriction by $k\colon U_+ \to U$.

Let $F' \in [U, \mathcal{S}]_0$. Each G-set X is a ring object of U, so $F'(X)$ is a ring. Hence $F' \circ k \in \sigma(\mathcal{U})$.

Conversely, let $F \in \sigma(\mathcal{U})$. Write $\mathcal{S}(A, B) = \mathrm{Hom}_{\mathcal{S}}(A, B)$. For each G-set X, the functor $\mathcal{S}(F(X), F(?)) \in [U_+, \mathcal{S}]_0$ belongs to $\sigma(\mathcal{U})$, and F induces a morphism $F_X : U_+(X, ?) \to \mathcal{S}(F(X), F(?))$ in $[U_+, \mathcal{S}]_0$. Then, by Proposition 6.2 there exists a unique morphism F'_X in $[U_+, \mathcal{S}]_0$ such that the diagram

$$\begin{array}{ccc} U_+(X, ?) & \overset{F_X}{\to} & \mathcal{S}(F(X), F(?)) \\ \kappa \downarrow & \nearrow F'_X & \\ U(X, ?) & & \end{array}$$

commutes. The maps $F'_X(Y)\colon U(X, Y) \to \mathcal{S}(F(X), F(Y))$ for all X, Y form a functor $F'\colon U \to \mathcal{S}$ such that $F = F' \circ k$. This proves the theorem.

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