HIGHER K-THEORY VIA UNIVERSAL INVARIANTS

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Abstract

Using the formalism of Grothendieck's derivators, we construct the universal localizing invariant *of differential graded (dg) categories. By this we mean a morphism $\mathscr{U}_l$ from the pointed derivator* $\mathsf{HO}(\mathsf{dgcat})$ *associated with the Morita homotopy theory of dg categories to a triangulated strong derivator $\mathscr{M}_{\mathrm{dg}}^{\mathrm{loc}}$ such that $\mathscr{U}_l$ commutes with filtered homotopy colimits, preserves the point, sends each exact sequence of dg categories to a triangle, and is universal for these properties.*

Similarly, we construct the universal additive invariant *of dg categories, that is, the universal morphism of derivators $\mathscr{U}_a$ from* $\mathsf{HO}(\mathsf{dgcat})$ *to a strong triangulated derivator $\mathscr{M}_{\mathrm{dg}}^{\mathrm{add}}$ that satisfies the first two properties but the third one only for split exact sequences. We prove that Waldhausen's K-theory becomes corepresentable in the target of the universal additive invariant. This is the first conceptual characterization of Quillen and Waldhausen's K-theory (see [34], [43]) since its definition in the early 1970s. As an application, we obtain for free the higher Chern characters from K-theory to cyclic homology.*

Contents

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1. Introduction

Differential graded (dg) categories enhance our understanding of triangulated categories appearing in algebra and geometry (see [21]). They are considered noncommutative schemes by Drinfeld [9], [10] and Kontsevich [23], [24] in their program of noncommutative algebraic geometry (i.e., the study of dg categories and their homological invariants).

In this article, using the formalism of Grothendieck's derivators, we construct the *universal localizing invariant* of dg categories (cf. [22]). By this, we mean a morphism $\mathscr{U}_l$ from the pointed derivator $\mathsf{HO}(\mathsf{dgcat})$ associated with the Morita homotopy theory of dg categories (see [37], [38]) to a triangulated strong derivator $\mathscr{M}^{\text{loc}}_{\text{dg}}$ such that $\mathscr{U}_l$ commutes with filtered homotopy colimits, preserves the point, sends each exact sequence of dg categories to a triangle, and is universal for these properties. Because of its universality property reminiscent of motives (see Kontsevich's preprint [25, Section 4.1]), we call $\mathscr{M}^{\text{loc}}_{\text{dg}}$ the (stable) *localizing motivator* of dg categories.

Similarly, we construct the *universal additive invariant* of dg categories (i.e., the universal morphism of derivators $\mathscr{U}_a$ from $\mathsf{HO}(\mathsf{dgcat})$ to a strong triangulated derivator $\mathscr{M}^{\text{add}}_{\text{dg}}$ that satisfies the first two properties but the third one only for split exact sequences. We call $\mathscr{M}^{\text{add}}_{\text{dg}}$ the *additive motivator* of dg categories.

We prove that Waldhausen's K-theory spectrum appears as a spectrum of morphisms in the base category $\mathscr{M}^{\text{add}}_{\text{dg}}(e)$ of the additive motivator. This shows us that Waldhausen's K-theory is completely characterized by its additive property. Intuitively, Waldhausen's K-theory is the universal construction with values in a stable context which satisfies additivity.

To the best of the author's knowledge, this is the first conceptual characterization of Quillen and Waldhausen's K-theory (see [34], [43]) since its definition in the early 1970s. This result gives us a completely new way of thinking about algebraic K-theory and furnishes us for free with the higher Chern characters from K-theory to cyclic homology (see [26]).

The corepresentation of K-theory as a spectrum of morphisms extends our results in [37], where we corepresented K_0 using functors with values in *additive categories* rather than morphisms of derivators with values in strong *triangulated derivators*.

For example, the mixed complex construction (see [20]), from which all variants of cyclic homology can be deduced, and the nonconnective algebraic K-theory (see [35]) are localizing invariants and factor through $\mathscr{U}_l$ and through $\mathscr{U}_a$. The connective algebraic K-theory (see [43]) is an example of an additive invariant that is not a localizing one. We prove that it becomes corepresentable in $\mathscr{M}^{\mathrm{add}}_{\mathrm{dg}}$ (see Theorem 15.10).

Our construction is similar in spirit to those of Meyer and Nest [31], Cortiñas and Thom [8], and Garkusha [12]. It splits into several general steps and also offers some insight into the relationship between the theory of derivators (see [7], [13], [14], [19], [29]) and the classical theory of Quillen model categories (see [33]). Derivators allow us to state and prove precise universal properties and to dispense with many of the technical problems one faces in using model categories.

In Section 2, we recall the notion of the Grothendieck derivator and point out its connection with that of small homotopy theory in the sense of Heller [14]. In Section 3, we recall Cisinski's theory of derived Kan extensions (see [4]), and in Section 4, we develop Cisinski's ideas on the Bousfield localization of derivators (see [6]). In particular, we characterize the derivator associated with a left Bousfield localization of a Quillen model category by a universal property (see Theorem 4.4). This is based on a constructive description of the local weak equivalences.

In Section 5, starting from a Quillen model category $\mathscr{M}$ satisfying some compactness conditions, we construct a morphism of prederivators

$$\mathsf{HO}(\mathscr{M}) \xrightarrow{\mathbb{R}h} \mathsf{L}_\Sigma \mathsf{Hot}_{\mathscr{M}_f}$$

which commutes with filtered homotopy colimits, has a derivator as a target, and is universal for these properties. In Section 6, we study morphisms of pointed derivators, and in Section 7, we prove a general result that guarantees that small weak generators are preserved under left Bousfield localizations. In Section 8, we recall Heller's stabilization construction (see [14]) and we prove that this construction takes finitely generated unstable theories to compactly generated stable theories. We establish the connection between Heller's stabilization and Hovey and Schwede's stabilization (see [17], [36]) by proving that if we start with a pointed Quillen model category that satisfies some mild generation hypotheses, then the two stabilization procedures yield equivalent results. This allows us to characterize Hovey and Schwede's construction by a universal property and, in particular, to give a very simple characterization of the classical category of spectra in the sense of Bousfield and Friedlander [2]. In Section 9, by applying the general arguments from Sections 3 – 8 to the Morita homotopy theory of dg categories (see [37], [38]), we construct the universal morphism of derivators

$$\mathscr{U}_t : \mathsf{HO}(\mathsf{dgcat}) \longrightarrow \mathsf{St}(\mathsf{L}_{\Sigma,P}\mathsf{Hot}_{\mathsf{dgcat}_f})$$

which commutes with filtered homotopy colimits, preserves the point, and has a strong triangulated derivator as a target. For every inclusion $\mathscr{A} \overset{K}{\hookrightarrow} \mathscr{B}$ of a full dg subcategory, we have an induced morphism

$$S_K : \mathsf{cone}\big(\mathscr{U}_t(\mathscr{A} \overset{K}{\hookrightarrow} \mathscr{B})\big) \to \mathscr{U}_t(\mathscr{B}/\mathscr{A}),$$

where $\mathscr{B}/\mathscr{A}$ denotes Drinfeld's dg quotient. By applying the localization techniques of Section 4 and using the fact that the derivator $\mathsf{St}(\mathsf{L}_{\Sigma,P}\mathsf{Hot}_{\mathsf{dgcat}_f})$ admits a stable Quillen model, we invert the morphisms S_K and finally obtain the universal localizing invariant of dg categories

$$\mathscr{U}_l : \mathsf{HO}(\mathsf{dgcat}) \longrightarrow \mathscr{M}^{\mathrm{loc}}_{\mathrm{dg}}.$$

We establish a connection between the triangulated category $\mathscr{M}^{\mathrm{loc}}_{\mathrm{dg}}(e)$ and Waldhausen's K-theory by showing that Waldhausen's $S_\bullet$-construction corresponds to the suspension functor in $\mathscr{M}^{\mathrm{loc}}_{\mathrm{dg}}(e)$. In Section 11, we prove that the derivator $\mathscr{M}^{\mathrm{loc}}_{\mathrm{dg}}$ admits a stable Quillen model given by a left Bousfield localization of a category of presheaves of spectra. In Section 12, we introduce the concept of an upper triangular dg category and construct a Quillen model structure on this class of dg categories, which satisfies strong compactness conditions. In Section 13, we establish the connection between upper triangular dg categories and split short exact sequences, and we use the Quillen model structure of Section 12 to prove an approximation result (see Proposition 13.2). In Section 14, by applying the techniques of Section 4, we construct the universal morphism of derivators

$$\mathscr{U}_u : \mathsf{HO}(\mathsf{dgcat}) \longrightarrow \mathscr{M}^{\mathrm{unst}}_{\mathrm{dg}}$$

which commutes with filtered homotopy colimits, preserves the point, and sends each split short exact sequence to a homotopy cofiber sequence. We prove that Waldhausen's K-theory space construction appears as a fibrant object in $\mathscr{M}^{\mathrm{unst}}_{\mathrm{dg}}$. This allows us to obtain the weak equivalence of simplicial sets

$$\mathsf{Map}\big(\mathscr{U}_u(k), S^1 \wedge \mathscr{U}_u(\mathscr{A})\big) \overset{\sim}{\longrightarrow} |N.wS_\bullet\mathscr{A}_f|$$

and the isomorphisms

$$\pi_{i+1}\mathsf{Map}\big(\mathscr{U}_u(k), S^1 \wedge \mathscr{U}_u(\mathscr{A})\big) \overset{\sim}{\longrightarrow} K_i(\mathscr{A}), \quad \forall i \geq 0$$

(see Proposition 14.12).

In Section 15, we stabilize the derivator $\mathscr{M}^{\mathrm{unst}}_{\mathrm{dg}}$ using the fact that it admits a Quillen model and finally obtain the universal additive invariant of dg categories

$$\mathscr{U}_a : \mathsf{HO}(\mathsf{dgcat}) \longrightarrow \mathscr{M}^{\mathrm{add}}_{\mathrm{dg}}.$$

Connective algebraic K-theory is additive and so factors through $\mathscr{U}_a$. We prove that for a small dg category $\mathscr{A}$, its connective algebraic K-theory corresponds to a fibrant resolution of $\mathscr{U}_a(\mathscr{A})[1]$ (see Theorem 15.9). Using the fact that the Quillen model for $\mathscr{M}^{\mathrm{add}}_{\mathrm{dg}}$ is enriched over spectra, we prove our main corepresentability theorem.

Let $\mathscr{A}$ and $\mathscr{B}$ be small dg categories with $\mathscr{A} \in \mathsf{dgcat}_f$.

THEOREM 1.1 (see Theorem 15.10)
We have the following weak equivalence of spectra:

$$\mathsf{Hom}^{\mathsf{Sp}^{\mathbb{N}}}\big(\mathscr{U}_a(\mathscr{A}), \mathscr{U}_a(\mathscr{B})[1]\big) \stackrel{\sim}{\longrightarrow} K^c\big(\mathsf{rep}_{\mathrm{mor}}(\mathscr{A}, \mathscr{B})\big),$$

where $K^c(\mathsf{rep}_{\mathrm{mor}}(\mathscr{A}, \mathscr{B}))$ *denotes Waldhausen's connective* K*-theory spectrum of* $\mathsf{rep}_{\mathrm{mor}}(\mathscr{A}, \mathscr{B})$.

In the triangulated base category $\mathscr{M}^{\mathrm{add}}_{\mathrm{dg}}(e)$ of the additive motivator, we have the following.

PROPOSITION 1.2 (see Proposition 16.1)
We have the following isomorphisms of abelian groups:

$$\mathsf{Hom}_{\mathscr{M}^{\mathrm{add}}_{\mathrm{dg}}(e)}\big(\mathscr{U}_a(\mathscr{A}), \mathscr{U}_a(\mathscr{B})[-n]\big) \stackrel{\sim}{\longrightarrow} K_n\big(\mathsf{rep}_{\mathrm{mor}}(\mathscr{A}, \mathscr{B})\big), \quad \forall n \geq 0.$$

Remark 1.3
Note that if, in Theorem 1.1 (resp., Proposition 1.2), we consider that $\mathscr{A} = k$, we have

$$\mathsf{Hom}^{\mathsf{Sp}^{\mathbb{N}}}\big(\mathscr{U}_a(k), \mathscr{U}_a(\mathscr{B})[1]\big) \stackrel{\sim}{\longrightarrow} K^c(\mathscr{B}),$$

respectively,

$$\mathsf{Hom}_{\mathscr{M}^{\mathrm{add}}_{\mathrm{dg}}(e)}\big(\mathscr{U}_a(k), \mathscr{U}_a(\mathscr{B})[-n]\big) \stackrel{\sim}{\longrightarrow} K_n(\mathscr{B}), \quad \forall n \geq 0.$$

This shows that Waldhausen's connective K-theory spectrum (resp., groups) becomes corepresentable in $\mathscr{M}^{\mathrm{add}}_{\mathrm{dg}}$ (resp., in $\mathscr{M}^{\mathrm{add}}_{\mathrm{dg}}(e)$).

In Section 16, we show that our corepresentability theorem furnishes us for free with the higher Chern characters from K-theory to cyclic homology.

THEOREM 1.4 (see Theorem 16.3)
The corepresentability theorem (Theorem 15.10) furnishes us with the higher Chern characters

$$\mathrm{ch}_{n,r} : K_n(-) \longrightarrow HC_{n+2r}(-), \quad n, r \geq 0.$$

In Section 17, we point out some questions that deserve further investigation.

2. Preliminaries

In this section, following [7, Definition 1.11], we recall the notion of the Grothendieck derivator.

Notation 2.1

We denote by CAT (resp., Cat) the 2-category of categories (resp., small categories). The empty category is written $\emptyset$, and the 1-point category (i.e., the category with one object and one identity morphism) is written e. If X is a small category, X^{op} is the opposite category associated to X. If $u : X \to Y$ is a functor, and if y is an object of Y, one defines the category X/y as follows: the objects are the couples (x, f), where x is an object of X, and f is a map in Y from $u(x)$ to y; a map from (x, f) to (x', f') in X/y is a map $\xi : x \to x'$ in X such that $f'u(\xi) = f$. The composition law in X/y is induced by the composition law in X. Dually, one defines $y\backslash X$ by the formula $y\backslash X = (X^{\mathrm{op}}/y)^{\mathrm{op}}$. We then have the canonical functors

$$X/y \to X \qquad \text{and} \qquad y\backslash X \to X$$

defined by projection $(x, f) \mapsto x$. One can easily check that one gets the following pullback squares of categories:

$$\begin{array}{ccc} X/y & \longrightarrow & X \\ {\scriptstyle u/y}\downarrow & & \downarrow{\scriptstyle u} \\ Y/y & \longrightarrow & Y \end{array} \qquad\qquad \begin{array}{ccc} y\backslash X & \longrightarrow & X \\ {\scriptstyle y\backslash u}\downarrow & & \downarrow{\scriptstyle u} \\ y\backslash Y & \longrightarrow & Y \end{array}$$

If X is any category, we let $p_X : X \to e$ be the canonical projection functor. Given any object x of X, we write $x : e \to X$ for the unique functor that sends the object of e to x. The objects of X (or, equivalently, the functors $e \to X$) are called the *points* of X.

If X and Y are two categories, we denote by $\mathsf{Fun}(X, Y)$ the category of functors from X to Y. If $\mathscr{C}$ is a 2-category, one writes $\mathscr{C}^{\mathrm{op}}$ for its dual 2-category: $\mathscr{C}^{\mathrm{op}}$ has the same objects as $\mathscr{C}$, and for any two objects X and Y, the category $\mathsf{Fun}_{\mathscr{C}^{\mathrm{op}}}(X, Y)$ of 1-arrows from X to Y in $\mathscr{C}^{\mathrm{op}}$ is $\mathsf{Fun}_{\mathscr{C}}(Y, X)^{\mathrm{op}}$.

Definition 2.2

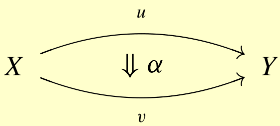
A *prederivator* is a strict 2-functor from $\mathrm{Cat}^{\mathrm{op}}$ to the 2-category of categories

$$\mathbb{D} : \mathrm{Cat}^{\mathrm{op}} \longrightarrow \mathrm{CAT}.$$

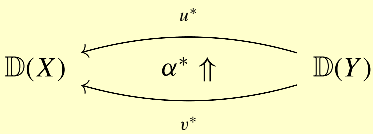
More explicitly, for any small category $X \in \mathrm{Cat}$, one has a category $\mathbb{D}(X)$. For any functor $u : X \to Y$ in Cat, one gets a functor

$$u^* = \mathbb{D}(u) : \mathbb{D}(Y) \longrightarrow \mathbb{D}(X).$$

For any morphism of functors



one has a morphism of functors

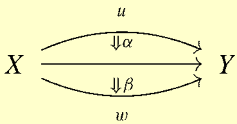


Of course, all these data have to verify some coherence conditions, namely, the following.

(a) For any composable maps in Cat, $X \xrightarrow{u} Y \xrightarrow{v} Z$,

$$(vu)^* = u^* v^* \qquad \text{and} \qquad 1_X^* = 1_{\mathbb{D}(X)}.$$

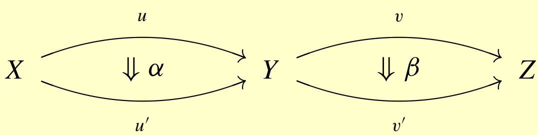
(b) For any composable 2-cells in Cat,



we have

$$(\beta\alpha)^* = \alpha^* \beta^* \qquad \text{and} \qquad 1_u^* = 1_{u^*}.$$

(c) For any 2-diagram in Cat,



we have $(\beta\alpha)^* = \alpha^* \beta^*$.

Example 2.3

Let M be a small category. The prederivator $\underline{M}$ naturally associated with M is defined as

$$X \mapsto \underline{M}(X),$$

where $\underline{M}(X) = \mathsf{Fun}(X^{\mathrm{op}}, M)$ is the category of presheaves over X with values in M.

Example 2.4

Let $\mathscr{M}$ be a category endowed with a class of maps called *weak equivalences* (e.g., $\mathscr{M}$ can be the category of bounded complexes in a given abelian category, and the weak equivalences can be the quasi isomorphisms). For any small category X, we define the weak equivalences in $\mathscr{M}(X)$ to be the morphisms of presheaves which are termwise weak equivalences in $\mathscr{M}$. We can then define $\mathbb{D}_{\mathscr{M}}(X)$ as the localization of $\mathscr{M}(X)$ by the weak equivalences. It is clear that, for any functor $u : X \to Y$, the inverse image functor

$$\mathscr{M}(Y) \to \mathscr{M}(X),$$
$$F \mapsto u^*(F) = F \circ u$$

respects weak equivalences, so that it induces a well-defined functor

$$u^* : \mathbb{D}_{\mathscr{M}}(X) \longrightarrow \mathbb{D}_{\mathscr{M}}(X).$$

The 2-functoriality of localization implies that we have a prederivator $\mathbb{D}_{\mathscr{M}}$.

Let X be a small category, and let x be an object of X. Given an object $F \in \mathbb{D}(X)$, we write $F_x = x^*(F)$. The object F_x is called the *fiber of F at the point x*.

For a prederivator $\mathbb{D}$, define its *opposite* to be the prederivator $\mathbb{D}^{\mathrm{op}}$ given by the formula $\mathbb{D}^{\mathrm{op}}(X) = \mathbb{D}(X^{\mathrm{op}})^{\mathrm{op}}$ for all small categories X.

Definition 2.5

Let $\mathbb{D}$ be a prederivator. A map $u : X \to Y$ in Cat has a *cohomological direct image functor* (resp., a *homological direct image functor*) in $\mathbb{D}$ if the inverse image functor

$$u^* : \mathbb{D}(Y) \longrightarrow \mathbb{D}(X)$$

has a right adjoint (resp., a left adjoint)

$$u_* : \mathbb{D}(X) \longrightarrow \mathbb{D}(Y) \qquad (\text{resp., } u_! : \mathbb{D}(X) \to \mathbb{D}(Y)),$$

called the *cohomological direct image functor* (resp., *homological direct image functor*) associated to u.

Notation 2.6

Let X be a small category, and let $p = p_X : X \to e$. If p has a cohomological direct image functor in $\mathbb{D}$, one defines, for any object F of $\mathbb{D}(X)$, the object of global sections of F as

$$\Gamma_*(X, F) = p_*(F).$$

Dually, if p has a homological direct image in $\mathbb{D}$, then for any object F of $\mathbb{D}(X)$, one sets

$$\Gamma_!(X, F) = p_!(F).$$

Notation 2.7

Let $\mathbb{D}$ be a prederivator, and let

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ {\scriptstyle u'}\downarrow & \Downarrow\alpha & \downarrow{\scriptstyle u} \\ Y' & \xrightarrow[w]{} & Y \end{array}$$

be a 2-diagram in Cat. By 2-functoriality, one obtains the 2-diagram

$$\begin{array}{ccc} \mathbb{D}(X') & \xleftarrow{v^*} & \mathbb{D}(X) \\ {\scriptstyle u'^*}\uparrow & \Uparrow\alpha^* & \uparrow{\scriptstyle u^*} \\ \mathbb{D}(Y') & \xleftarrow[w^*]{} & \mathbb{D}(Y) \end{array}$$

If we assume that the functors u and u' both have cohomological direct images in $\mathbb{D}$, then one can define the *base change morphism* induced by α,

$$\beta : w^*u_* \to u'_*v^*,$$

$$\begin{array}{ccc} \mathbb{D}(X') & \xleftarrow{v^*} & \mathbb{D}(X) \\ {\scriptstyle u'_*}\downarrow & \Uparrow\beta & \downarrow{\scriptstyle u_*} \\ \mathbb{D}(Y') & \xleftarrow[w^*]{} & \mathbb{D}(Y) \end{array}$$

as follows. The counit $u^*u_* \to 1_{\mathbb{D}(X)}$ induces a morphism $v^*u^*u_* \to v^*$ and, by composition with α^*u_*, a morphism $u'^*w^*u_* \to v^*$. This gives β by adjunction.

This construction is used in the following situation: let $u : X \to Y$ be a map in Cat, and let y be a point of Y. According to Notation 2.1, we have a functor $j : X/y \to X$, defined by the formula $j(x, f) = x$, where $f : u(x) \to y$ is a morphism in Y. If $p : X/y \to e$ is the canonical map, then one obtains the 2-diagram below, where α denotes the 2-cell defined by the formula $\alpha_{(x,f)} = f$:

$$\begin{array}{ccc} X/y & \xrightarrow{\;j\;} & X \\ {\scriptstyle p}\Big\downarrow & \Downarrow\alpha & \Big\downarrow{\scriptstyle u} \\ e & \xrightarrow[\;y\;]{} & Y \end{array}$$

Using Notation 2.6, the associated base change morphism gives rise to a canonical morphism

$$u_*(F)_y \to \Gamma_*(X/y, F/y)$$

for any object $F \in \mathbb{D}(X)$, where $F/y = j^*(F)$. Dually, one has canonical morphisms

$$\Gamma_!(y\backslash X, y\backslash F) \to u_!(F)_y,$$

where $y\backslash F = k^*(F)$ and k denotes the canonical functor from $y\backslash X$ to X.

Notation 2.8

Let X and Y be two small categories. Using the 2-functoriality of $\mathbb{D}$, one defines a functor

$$d_{X,Y} : \mathbb{D}(X \times Y) \to \mathsf{Fun}\big(X^{\mathrm{op}}, \mathbb{D}(Y)\big)$$

as follows. Setting $X' = X \times Y$, we have a canonical functor

$$\mathsf{Fun}(Y, X')^{\mathrm{op}} \longrightarrow \mathsf{Fun}\big(\mathbb{D}(X'), \mathbb{D}(Y)\big)$$

which defines a functor

$$\mathsf{Fun}(Y, X')^{\mathrm{op}} \times \mathbb{D}(X') \longrightarrow \mathbb{D}(Y)$$

and then a functor

$$\mathbb{D}(X') \longrightarrow \mathsf{Fun}\big(\mathsf{Fun}(Y, X')^{\mathrm{op}}, \mathbb{D}(Y)\big).$$

Using the canonical functor

$$X \to \mathsf{Fun}(Y, X \times Y), \qquad x \mapsto \big(y \mapsto (x, y)\big),$$

this gives the desired functor.

In particular, for any small category X, one gets a functor

$$d_X = d_{X,e} : \mathbb{D}(X) \to \mathsf{Fun}\big(X^{\mathrm{op}}, \mathbb{D}(e)\big).$$

If F is an object of $\mathbb{D}(X)$, then $d_X(F)$ is the presheaf on X with values in $\mathbb{D}(e)$ defined by

$$x \mapsto F_x.$$

Definition 2.9

A *derivator* is a prederivator $\mathbb{D}$ with the following properties.

(Der1) (Nontriviality axiom) For any finite set I and any family $\{X_i, i \in I\}$ of small categories, the canonical functor

$$\mathbb{D}\Big(\coprod_{i \in I} X_i\Big) \longrightarrow \prod_{i \in I} \mathbb{D}(X_i)$$

is an equivalence of categories.

(Der2) (Conservativity axiom) For any small category X, the family of functors

$$x^* : \mathbb{D}(X) \to \mathbb{D}(e),$$
$$F \mapsto x^*(F) = F_x$$

corresponding to the points x of X is conservative. In other words, if $\varphi : F \to G$ is a morphism in $\mathbb{D}(X)$, so that for any point x of X the map $\varphi_x : F_x \to G_x$ is an isomorphism in $\mathbb{D}(e)$, then φ is an isomorphism in $\mathbb{D}(X)$.

(Der3) (Direct image axiom) Any functor in Cat has a cohomological direct image functor and a homological direct image functor in $\mathbb{D}$ (see Definition 2.5).

(Der4) (Base change axiom) For any functor $u : X \to Y$ in Cat, any point y of Y, and any object F in $\mathbb{D}(X)$, the canonical base change morphisms (see Notation 2.7)

$$u_*(F)_y \to \Gamma_*(X/y, F/y) \qquad \text{and} \qquad \Gamma_!(y\backslash X, y\backslash F) \to u_!(F)_y$$

are isomorphisms in $\mathbb{D}(e)$.

(Der5) (Essential surjectivity axiom) Let I be the category corresponding to the graph

$$0 \leftarrow 1.$$

For any small category X, the functor

$$d_{I,X} : \mathbb{D}(I \times X) \to \mathsf{Fun}\big(I^{\mathrm{op}}, \mathbb{D}(X)\big)$$

(see Notation 2.8) is full and essentially surjective.

Example 2.10
By [3], any Quillen model category $\mathscr{M}$ gives rise to a derivator denoted $\mathsf{HO}(\mathscr{M})$.

We denote by $\mathsf{Ho}(\mathscr{M})$ the homotopy category of $\mathscr{M}$. By definition, it equals $\mathsf{HO}(\mathscr{M})(e)$.

Definition 2.11
A derivator $\mathbb{D}$ is *strong* if, for every finite free category X and every small category Y, the natural functor

$$\mathbb{D}(X \times Y) \longrightarrow \mathsf{Fun}\big(X^{\mathrm{op}}, \mathbb{D}(Y)\big)$$

(see Notation 2.8) is full and essentially surjective.

Note that a strong derivator is the same thing as a small homotopy theory in the sense of Heller [14]. Note also that by [5, proposition 2.15], $\mathsf{HO}(\mathscr{M})$ is a strong derivator.

Definition 2.12
A derivator $\mathbb{D}$ is *regular* if, in $\mathbb{D}$, sequential homotopy colimits commute with finite products and homotopy pullbacks.

Notation 2.13
Let X be a category. Remember that a *sieve* (or a *crible*) in X is a full subcategory U of X such that for any object x of X, if there exists a morphism $x \to u$ with u in U, then x is in U. Dually, a *cosieve* (or a *cocrible*) in X is a full subcategory Z of X such that for any morphism $z \to x$ in X, if z is in Z, then so is x.

A functor $j : U \to X$ is an *open immersion* if it is injective on objects, fully faithful, and if $j(U)$ is a sieve in X. Dually, a functor $i : Z \to X$ is a *closed immersion* if it is injective on objects, fully faithful, and if $i(Z)$ is a cosieve in X. One can easily show that open immersions and closed immersions are stable by composition and pullback.

Definition 2.14
A derivator $\mathbb{D}$ is *pointed* if it satisfies the following property.

(Der6) (Exceptional axiom) For any closed immersion $i : Z \to X$ in Cat, the cohomological direct image functor

$$i_* : \mathbb{D}(Z) \longrightarrow \mathbb{D}(X)$$

has a right adjoint

$$i^! : \mathbb{D}(X) \longrightarrow \mathbb{D}(Z)$$

called the *exceptional inverse image functor* associated to i. Dually, for any open immersion $j : U \to X$, the homological direct image functor

$$j_! : \mathbb{D}(U) \longrightarrow \mathbb{D}(X)$$

has a left adjoint

$$j^? : \mathbb{D}(X) \longrightarrow \mathbb{D}(U)$$

called the *coexceptional inverse image functor* associated to j.

Let $\square$ be the category given by the commutative square

$$\begin{array}{ccc} (0,0) & \longleftarrow & (0,1) \\ \uparrow & & \uparrow \\ (1,0) & \longleftarrow & (1,1) \end{array}$$

We are interested in two of its subcategories. The subcategory $\lrcorner$ is

$$\begin{array}{ccc} & & (0,1) \\ & & \uparrow \\ (1,0) & \longleftarrow & (1,1) \end{array}$$

and $\ulcorner$ is the subcategory

$$\begin{array}{ccc} (0,0) & \longleftarrow & (0,1) \\ \uparrow & & \\ (1,0) & & \end{array}$$

We thus have two inclusion functors

$$\sigma : \lrcorner \to \square \qquad \text{and} \qquad \tau : \ulcorner \to \square$$

(σ is an open immersion, and τ is a closed immersion). A *global commutative square* in $\mathbb{D}$ is an object of $\mathbb{D}(\square)$. A global commutative square C in $\mathbb{D}$ is thus locally of shape

$$\begin{array}{ccc} C_{0,0} & \longrightarrow & C_{0,1} \\ \downarrow & & \downarrow \\ C_{1,0} & \longrightarrow & C_{1,1} \end{array}$$

in $\mathbb{D}(e)$.

A global commutative square C in $\mathbb{D}$ is *Cartesian* (or a *homotopy pullback square*) if, for any global commutative square B in $\mathbb{D}$, the canonical map

$$\mathsf{Hom}_{\mathbb{D}(\square)}(B, C) \longrightarrow \mathsf{Hom}_{\mathbb{D}(\lrcorner)}\big(\sigma^*(B), \sigma^*(C)\big)$$

is bijective. Dually, a global commutative square B in $\mathbb{D}$ is *co-Cartesian* (or a *homotopy pushout square*) if, for any global commutative square C in $\mathbb{D}$, the canonical map

$$\mathsf{Hom}_{\mathbb{D}(\square)}(B, C) \longrightarrow \mathsf{Hom}_{\mathbb{D}(\ulcorner)}\big(\tau^*(B), \tau^*(C)\big)$$

is bijective.

As $\square$ is isomorphic to its opposite $\square^{\mathrm{op}}$, one can see that a global commutative square in $\mathbb{D}$ is Cartesian (resp., co-Cartesian) if and only if it is co-Cartesian (resp., Cartesian) as a global commutative square in $\mathbb{D}^{\mathrm{op}}$.

Definition 2.15
A derivator $\mathbb{D}$ is *triangulated* or *stable* if it is pointed and satisfies the following axiom.
(Der7) (Stability axiom) A global commutative square in $\mathbb{D}$ is Cartesian if and only if it is co-Cartesian.

THEOREM 2.16 (see [30])
For any triangulated derivator $\mathbb{D}$ and small category X, the category $\mathbb{D}(X)$ has a canonical triangulated structure.

Let $\mathbb{D}$ and $\mathbb{D}'$ be derivators. We denote by $\underline{\mathsf{Hom}}(\mathbb{D}, \mathbb{D}')$ the category of all morphisms of derivators; we denote by $\underline{\mathsf{Hom}}_!(\mathbb{D}, \mathbb{D}')$ the category of morphisms of derivators which commute with homotopy colimits [4, Remark 3.25]; and we denote by $\underline{\mathsf{Hom}}_{\mathrm{flt}}(\mathbb{D}, \mathbb{D}')$ the category of morphisms of derivators which commute with filtered homotopy colimits.

3. Derived Kan extensions

Let A be a small category, and let $\mathsf{Fun}(A^{\mathrm{op}}, \mathrm{Sset})$ be the Quillen model category of simplicial presheaves on A endowed with the projective model structure (see [42]). We have at our disposal the functor

$$\begin{aligned} A &\xrightarrow{h} \mathsf{Fun}(A^{\mathrm{op}}, \mathrm{Sset}), \\ X &\longmapsto \mathsf{Hom}_A(?, X), \end{aligned}$$

where $\mathsf{Hom}_A(?, X)$ is considered as a constant simplicial set.

The functor h gives rise to a morphism of prederivators

$$\underline{A} \xrightarrow{h} \mathsf{HO}\big(\mathsf{Fun}(A^{\mathrm{op}}, \mathrm{Sset})\big).$$

Using the notation of [4], we denote by Hot_A the derivator $\mathsf{HO}(\mathsf{Fun}(A^{\mathrm{op}}, \mathrm{Sset}))$. The following results are proved in [4]. Let $\mathbb{D}$ be a derivator.

THEOREM 3.1
The morphism h induces an equivalence of categories

$$\begin{array}{c} \underline{\mathsf{Hom}}(\underline{A}, \mathbb{D}) \\ \varphi \big\downarrow \big\uparrow h^* \\ \underline{\mathsf{Hom}}_!(\mathsf{Hot}_A, \mathbb{D}) \end{array}$$

Proof
This theorem is equivalent to [4, corollaire 3.26] since we have

$$\underline{\mathsf{Hom}}(\underline{A}, \mathbb{D}) \simeq \mathbb{D}(A^{\mathrm{op}}). \qquad \square$$

LEMMA 3.2
We have an adjunction

$$\begin{array}{c} \underline{\mathsf{Hom}}(\mathsf{Hot}_A, \mathbb{D}) \\ \mathrm{inc} \big\uparrow \big\downarrow \Psi \\ \underline{\mathsf{Hom}}_!(\mathsf{Hot}_A, \mathbb{D}) \end{array}$$

where

$$\Psi(F) := \varphi(F \circ h).$$

Proof
We construct a universal 2-morphism of functors

$$\epsilon : \mathrm{inc} \circ \Psi \longrightarrow \mathrm{Id}.$$

Let F be a morphism of derivators belonging to $\underline{\mathsf{Hom}}(\mathsf{Hot}_A, \mathbb{D})$. Let L be a small category, and let X be an object of $\mathsf{Hot}_A(L)$. Recall from [4, section 1] that we have the diagram

$$\begin{array}{ccc} & \nabla \int X & \\ {}^{\pi}\swarrow & & \searrow^{\varpi} \\ L^{\mathrm{op}} & & A \end{array}$$

Now, let p be the functor π^{op}, and let q be the functor ϖ^{op}. By the dual of [4, proposition 1.15], we have the functorial isomorphism

$$p_! q^*(h) \xrightarrow{\sim} X.$$

Finally, let $\epsilon_L(X)$ be the composed morphism

$$\epsilon_L(X) : \Psi(F)(X) = p_! q^* F(h) = p_! F(q^* h) \to F(p_! q^* h) \xrightarrow{\sim} F(X),$$

and note, using Theorem 3.1, that ϵ induces an adjunction. □

4. Localization: Model categories versus derivators

Let $\mathscr{M}$ be a left proper, cellular Quillen model category (see [15]).

We start by fixing a frame on $\mathscr{M}$ (see [15, Definition 16.6.21]). Let D be a small category, and let F be a functor from D to $\mathscr{M}$. We denote by hocolim F the object of $\mathscr{M}$, as in [15, Definition 19.1.2]. Let S be a set of morphisms in $\mathscr{M}$, and denote by $\mathsf{L}_S\mathscr{M}$ the left Bousfield localization of $\mathscr{M}$ by S.

Note that the Quillen adjunction

$$\begin{array}{c} \mathscr{M} \\ {\scriptstyle \mathrm{Id}}\downarrow \uparrow{\scriptstyle \mathrm{Id}} \\ \mathsf{L}_S\mathscr{M} \end{array}$$

induces a morphism of derivators

$$\gamma : \mathsf{HO}(\mathscr{M}) \xrightarrow{\mathbb{L}\,\mathrm{Id}} \mathsf{HO}(\mathsf{L}_S\mathscr{M})$$

which commutes with homotopy colimits.

PROPOSITION 4.1

Let $\mathscr{W}_S$ be the smallest class of morphisms in $\mathscr{M}$ satisfying the following properties.

(a) *Every element in S belongs to $\mathscr{W}_S$.*

(b) *Every weak equivalence of $\mathscr{M}$ belongs to $\mathscr{W}_S$.*

(c) *If, in a commutative triangle, two out of three morphisms belong to $\mathscr{W}_S$, then so does the third one. The class $\mathscr{W}_S$ is stable under retractions.*

(d) *Let D be a small category, and let F and G be functors from D to $\mathscr{M}$. If η is a morphism of functors from F to G such that for every object d in D, $F(d)$ and $G(d)$ are cofibrant objects and the morphism $\eta(d)$ belongs to $\mathscr{W}_S$, then so does the morphism*

$$\operatorname{hocolim} F \longrightarrow \operatorname{hocolim} G.$$

Then the class $\mathscr{W}_S$ equals the class of S-local equivalences in $\mathscr{M}$ (see [15, Definition 3.1.4]).

Proof

The class of S-local equivalences satisfies properties (a) – (d). Properties (a) and (b) are satisfied by definition, [15, Propositions 3.2.3, 3.2.4] imply property (c), and [15, Proposition 3.2.5] implies property (d).

Let us now show that, conversely, each S-local equivalence is in $\mathscr{W}_S$. Let

$$X \overset{g}{\to} Y$$

be an S-local equivalence in $\mathscr{M}$. Without loss of generality, we can suppose that X is cofibrant. Indeed, let $Q(X)$ be a cofibrant resolution of X, and consider the diagram

$$\begin{array}{ccc} Q(X) & & \\ {\scriptstyle \pi}\downarrow {\scriptstyle \sim} & \searrow^{g\circ\pi} & \\ X & \xrightarrow[g]{} & Y \end{array}$$

Note that since π is a weak equivalence, g is an S-local equivalence if and only if $g \circ \pi$ is also.

By [15, Theorem 4.3.6], g is an S-local equivalence if and only if the morphism $\mathsf{L}_S(g)$ appearing in the diagram

$$\begin{array}{ccc} X & \xrightarrow{\;g\;} & Y \\ {\scriptstyle j(X)}\downarrow & & \downarrow{\scriptstyle j(Y)} \\ \mathsf{L}_S X & \xrightarrow[\mathsf{L}_S(g)]{} & \mathsf{L}_S Y \end{array}$$

is a weak equivalence in $\mathscr{M}$. This shows that it is enough to prove that $j(X)$ and $j(Y)$ belong to $\mathscr{W}_S$. Apply the small-object argument to the morphism

$$X \longrightarrow *$$

using the set $\widetilde{\Lambda(S)}$ (see [15, Proposition 4.2.5]). We have the factorization

$$\begin{array}{ccc} X & \longrightarrow & * \\ & {\scriptstyle j(X)}\searrow \quad \nearrow & \\ & \mathsf{L}_S(X) & \end{array}$$

where $j(X)$ is a relative $\widetilde{\Lambda(S)}$-cell complex.

We now prove the two following stability conditions concerning the class $\mathscr{W}_S$.

(S1) Consider the pushout

$$\begin{array}{ccc} W_0 & \longrightarrow & W_2 \\ {\scriptstyle f}\downarrow & \lrcorner & \downarrow{\scriptstyle f_*} \\ W_1 & \longrightarrow & W_3 \end{array}$$

where W_0, W_1, and W_2 are cofibrant objects in $\mathscr{M}$ and f is a cofibration that belongs to $\mathscr{W}_S$. Observe that f_* corresponds to the colimit of the morphism of diagrams

$$\begin{array}{ccccc} W_0 & = & W_0 & \longrightarrow & W_2 \\ {\scriptstyle f}\downarrow & & \| & & \| \\ W_1 & \xleftarrow[f]{} & W_0 & \longrightarrow & W_2 \end{array}$$

Now, both [15, Proposition 19.9.4] and property (d) imply that f_* belongs to $\mathscr{W}_S$.

(S2) Consider the diagram

$$\mathbf{X} : X_0 \overset{f_0}{\rightarrowtail} X_1 \overset{f_1}{\rightarrowtail} X_2 \overset{f_2}{\rightarrowtail} X_3 \rightarrowtail \cdots$$

in $\mathscr{M}$, where the objects are cofibrant and the morphisms are cofibrations that belong to the class $\mathscr{W}_S$. Observe that the transfinite composition of $\mathbf{X}$ corresponds to the colimit of the morphism of diagrams

$$\begin{array}{ccccccc} X_0 & = & X_0 & = & X_0 & = & \cdots \\ \| & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1 \circ f_0} & & \\ X_0 & \underset{f_0}{\rightarrowtail} & X_1 & \underset{f_1}{\rightarrowtail} & X_2 & \rightarrowtail & \cdots \end{array}$$

Now, since $\mathbf{X}$ is a Reedy cofibrant diagram on category with fibrant constants (see [15, Definition 15.10.1]), [15, Theorem 19.9.1] and property (d) imply that the transfinite composition of $\mathbf{X}$ belongs to $\mathscr{W}_S$. Note that the above argument can be immediately generalized to a transfinite composition of a λ-sequence, where λ denotes an ordinal (see [15, Section 10.2]).

Now, the construction of the morphism $j(X)$ and the stability conditions (S1) and (S2) show us that it is enough to prove that the elements of $\widetilde{\Lambda(S)}$ belong to $\mathscr{W}_S$. By [15, Proposition 4.2.5], it is sufficient to show that the set

$$\Lambda(S) = \left\{ \tilde{\mathbf{A}} \otimes \Delta[n] \coprod_{\tilde{\mathbf{A}} \otimes \partial\Delta[n]} \tilde{\mathbf{B}} \otimes \Delta[n] \xrightarrow{\Lambda(g)} \tilde{\mathbf{B}} \otimes \Delta[n] \,\middle|\, (A \xrightarrow{g} B) \in S,\ n \geq 0 \right\}$$

of horns in S is contained in $\mathscr{W}_S$. Recall from [15, Definition 4.2.1] that $\tilde{g} : \tilde{\mathbf{A}} \to \tilde{\mathbf{B}}$ denotes a cosimplicial resolution of $g : A \to B$ and $\tilde{g}$ is a Reedy cofibration. We have the diagram

$$\begin{array}{ccc} \tilde{\mathbf{A}} \otimes \partial\Delta[n] & \xrightarrow{\tilde{g} \otimes 1} & \tilde{\mathbf{B}} \otimes \partial\Delta[n] \\ {\scriptstyle 1 \otimes i}\downarrow & & \downarrow{\scriptstyle 1 \otimes i} \\ \tilde{\mathbf{A}} \otimes \Delta[n] & \underset{\tilde{g} \otimes 1}{\longrightarrow} & \tilde{\mathbf{B}} \otimes \Delta[n] \end{array}$$

Observe that the morphism

$$\tilde{\mathbf{A}} \otimes \Delta[n] \xrightarrow{\tilde{g}\otimes 1} \tilde{\mathbf{B}} \otimes \Delta[n]$$

identifies with

$$\tilde{\mathbf{A}}^n \xrightarrow{\tilde{g}^n} \tilde{\mathbf{B}}^n,$$

and so it belongs to $\mathscr{W}_S$. Now, the morphism

$$\tilde{\mathbf{A}} \otimes \partial\Delta[n] \xrightarrow{\tilde{g}\otimes \mathbf{1}_{\partial\Delta[n]}} \tilde{\mathbf{B}} \otimes \partial\Delta[n]$$

corresponds to the induced map of latching objects

$$\mathsf{L}_n\tilde{\mathbf{A}} \longrightarrow \mathsf{L}_n\tilde{\mathbf{B}},$$

which is a cofibration in $\mathscr{M}$ by [15, Proposition 15.3.11].

Now, by [15, Propositions 15.10.4, 16.3.12; Theorem 19.9.1], we have the commutative diagram

$$\begin{array}{ccc} \underset{\partial(\overrightarrow{\Delta}\downarrow[n])}{\operatorname{hocolim}} \tilde{\mathbf{A}} & \longrightarrow & \underset{\partial(\overrightarrow{\Delta}\downarrow[n])}{\operatorname{hocolim}} \tilde{\mathbf{B}} \\ {\scriptstyle\sim}\downarrow & & \downarrow{\scriptstyle\sim} \\ \mathsf{L}_n\tilde{\mathbf{A}} & \longrightarrow & \mathsf{L}_n\tilde{\mathbf{B}} \end{array}$$

where the vertical arrows are weak equivalences and $\partial(\overrightarrow{\Delta}\downarrow [n])$ denotes the category of strictly increasing maps with target $[n]$. By property (d) of the class $\mathscr{W}_S$, we conclude that $\tilde{g} \otimes \mathbf{1}_{\partial\Delta[n]}$ belongs to $\mathscr{W}_S$.

We have the diagram

$$\begin{array}{ccc} \tilde{\mathbf{A}} \otimes \Delta[n] \underset{\tilde{\mathbf{A}}\otimes\partial\Delta[n]}{\coprod} \tilde{\mathbf{B}} \otimes \Delta[n] & & \\ {\scriptstyle I}\uparrow & \searrow^{\Lambda(g)} & \\ \tilde{\mathbf{A}} \otimes \Delta[n] & \xrightarrow[\tilde{g}\otimes 1]{} & \tilde{\mathbf{B}} \otimes \Delta[n] \end{array}$$

Note that the morphism I belongs to $\mathscr{W}_S$ by the stability condition (S1) applied to the morphism

$$\tilde{\mathbf{A}} \otimes \partial\Delta[n] \xrightarrow{\tilde{g}\otimes 1} \tilde{\mathbf{B}} \otimes \partial\Delta[n],$$

which is a cofibration and belongs to $\mathscr{W}_S$. Since the morphism $\tilde{g} \otimes 1$ belongs to $\mathscr{W}_S$, so does $\Lambda(g)$.

This proves the proposition. $\square$

Let $\mathbb{D}$ be a derivator, and let S be a class of morphisms in $\mathbb{D}(e)$.

Definition 4.2 (*see Cisinski [6]*)
The derivator $\mathbb{D}$ admits a *left Bousfield localization* by S if there exists a morphism of derivators

$$\gamma : \mathbb{D} \to \mathsf{L}_S\mathbb{D}$$

which commutes with homotopy colimits, sends the elements of S to isomorphisms in $\mathsf{L}_S\mathbb{D}(e)$, and satisfies the universal property: for every derivator $\mathbb{D}'$, the morphism γ induces an equivalence of categories

$$\underline{\mathsf{Hom}}_!(\mathsf{L}_S\mathbb{D}, \mathbb{D}') \xrightarrow{\gamma^*} \underline{\mathsf{Hom}}_{!,S}(\mathbb{D}, \mathbb{D}'),$$

where $\underline{\mathsf{Hom}}_{!,S}(\mathbb{D}, \mathbb{D}')$ denotes the category of morphisms of derivators which commute with homotopy colimits and send the elements of S to isomorphisms in $\mathbb{D}'(e)$.

LEMMA 4.3
Suppose that $\mathbb{D}$ is a triangulated derivator, suppose that S is stable under the loop-space functor $\Omega(e) : \mathbb{D}(e) \to \mathbb{D}(e)$ (see [7]), and suppose that $\mathbb{D}$ admits a left Bousfield localization $\mathsf{L}_S\mathbb{D}$ by S.

Then $\mathsf{L}_S\mathbb{D}$ is also a triangulated derivator.

Proof
Recall from [7, Remark 1.19] that since $\mathbb{D}$ is a triangulated derivator, we have the following equivalence:

$$\begin{array}{c} \mathbb{D} \\ \Sigma \uparrow \downarrow \Omega \\ \mathbb{D} \end{array}$$

Note that both morphisms of derivators, Σ and Ω, commute with homotopy colimits. Since S is stable under the functor $\Omega(e) : \mathbb{D}(e) \to \mathbb{D}(e)$ and $\mathbb{D}$ admits a left Bousfield localization $\mathsf{L}_S\mathbb{D}$ by S, we have an induced morphism

$$\Omega : \mathsf{L}_S\mathbb{D} \to \mathsf{L}_S\mathbb{D}.$$

Let s be an element of S. We now show that the image of s by the functor $\gamma \circ \Sigma$ is an isomorphism in $\mathsf{L}_S\mathbb{D}(e)$. For this, consider the category $\ulcorner$ (see Section 2) and the functors

$$(0,0) : e \to \ulcorner \qquad \text{and} \qquad p : \ulcorner \to e.$$

Now, recall from [14, Section 7] that

$$\Omega(e) := p_! \circ (0,0)_*.$$

This description shows us that the image of s under the functor $\gamma \circ \Sigma$ is an isomorphism in $\mathsf{L}_S\mathbb{D}(e)$ because γ commutes with homotopy colimits. In conclusion, we have an induced adjunction

$$\begin{array}{c} \mathsf{L}_S\mathbb{D} \\ \Sigma \uparrow \downarrow \Omega \\ \mathsf{L}_S\mathbb{D} \end{array}$$

which is clearly an equivalence. This proves the lemma. □

THEOREM 4.4 (see Cisinski [6])
The morphism of derivators

$$\gamma : \mathsf{HO}(\mathscr{M}) \xrightarrow{\mathbb{L}\,\mathrm{Id}} \mathsf{HO}(\mathsf{L}_S\mathscr{M})$$

is a left Bousfield localization of $\mathsf{HO}(\mathscr{M})$ *by the image of the set* S *in* $\mathsf{Ho}(\mathscr{M})$.

Proof
Let $\mathbb{D}$ be a derivator.

The morphism γ admits a fully faithful right adjoint

$$\sigma : \mathsf{HO}(\mathsf{L}_S\mathscr{M}) \longrightarrow \mathsf{HO}(\mathscr{M}).$$

Therefore, the induced functor

$$\gamma^* : \underline{\mathsf{Hom}}_!\big(\mathsf{HO}(\mathsf{L}_S\mathscr{M}), \mathbb{D}\big) \longrightarrow \underline{\mathsf{Hom}}_{!,S}\big(\mathsf{HO}(\mathscr{M}), \mathbb{D}\big)$$

admits a left adjoint σ^*, and $\sigma^*\gamma^* = (\gamma\sigma)^*$ is isomorphic to the identity. Therefore, γ^* is fully faithful. We now show that γ^* is essentially surjective. Let F be an object of $\underline{\mathsf{Hom}}_{!,S}(\mathsf{HO}(\mathscr{M}), \mathbb{D})$. Note that since $\mathbb{D}$ satisfies the conservativity axiom, it is sufficient to show that the functor

$$F(e) : \mathsf{Ho}(\mathscr{M}) \to \mathbb{D}(e)$$

sends the images in $\mathsf{Ho}(\mathscr{M})$ of S-local equivalences of $\mathscr{M}$ to isomorphisms in $\mathbb{D}(e)$. The morphism F then becomes naturally a morphism of derivators

$$\overline{F} : \mathsf{HO}(\mathsf{L}_S\mathscr{M}) \to \mathbb{D},$$

so that $\gamma^*(\overline{F}) = F$. Now, since F commutes with homotopy colimits, the functor

$$\mathscr{M} \to \mathsf{Ho}(\mathscr{M}) \to \mathbb{D}(e)$$

sends the elements of $\mathscr{W}_S$ to isomorphisms. This proves the theorem since, by Proposition 4.1, the class $\mathscr{W}_S$ equals the class of S-local equivalences in $\mathscr{M}$. □

5. Filtered homotopy colimits

Let $\mathscr{M}$ be a cellular Quillen model category with I the set of generating cofibrations. Suppose that the domains and codomains of the elements of I are cofibrant, $\aleph_0$-compact, $\aleph_0$-small, and homotopically finitely presented (see [41, Definition 2.1.1]).

Example 5.1

Consider the quasi-equivalent (resp., quasi-equiconic; resp., Morita) Quillen model structure on dgcat constructed in [37] and [38].

Recall that a dg functor $F : \mathscr{C} \to \mathscr{E}$ is a quasi equivalence (resp., quasi-equiconic; resp., a Morita dg functor) if it satisfies one of the following conditions.

(C1) The dg category $\mathscr{C}$ is empty, and all the objects of $\mathscr{E}$ are contractible.

(C2) For every object $c_1, c_2 \in \mathscr{C}$, the morphism of complexes from $\mathsf{Hom}_{\mathscr{C}}(c_1, c_2)$ to $\mathsf{Hom}_{\mathscr{E}}(F(c_1)), F(c_2))$ is a quasi isomorphism, and the functor $\mathsf{H}^0(F)$ (resp., $\mathsf{H}^0(\text{pre-tr}(F))$; resp., $\mathsf{H}^0(\text{pre-tr}(F))^\flat$) is essentially surjective.

Observe that the domains and codomains of the set I of generating cofibrations in dgcat satisfy the conditions above for all the Quillen model structures.

The following proposition is a simplification of [41, Proposition 2.2].

PROPOSITION 5.2

Let $\mathscr{M}$ be a Quillen model category that satisfies conditions (C1) and (C2) in Example 5.1. Then we have the following.

(1) *A filtered colimit of trivial fibrations is a trivial fibration.*

(2) *For any filtered diagram X_i in $\mathscr{M}$, the natural morphism*

$$\operatorname*{hocolim}_{i\in I} X_i \longrightarrow \operatorname*{colim}_{i\in I} X_i$$

is an isomorphism in $\mathsf{Ho}(\mathscr{M})$.

(3) *Any object X in $\mathscr{M}$ is equivalent to a filtered colimit of strict finite I-cell objects.*

(4) *An object X in $\mathscr{M}$ is homotopically finitely presented if and only if it is equivalent to a retract of a strict finite I-cell object.*

Proof

The proof of (1) – (3) is exactly the same as [41, proof of Proposition 2.2]. The proof of (4) is also the same once we observe that the domains and codomains of the elements of the set I are already homotopically finitely presented by hypothesis. □

In everything that follows, we fix

(i) a cosimplicial resolution functor

$$\big(\Gamma(-) : \mathscr{M} \to \mathscr{M}^{\Delta},\, i\big)$$

in the model category $\mathscr{M}$ (see [15, Definition 16.1.8]), which means that for every object X in $\mathscr{M}$, $\Gamma(X)$ is cofibrant in the Reedy model structure on $\mathscr{M}^{\Delta}$ and that

$$i(X) : \Gamma(X) \xrightarrow{\sim} c^*(X)$$

is a weak equivalence on $\mathscr{M}^{\Delta}$, where $c^*(X)$ denotes the constant cosimplicial object associated with X;

(ii) a fibrant resolution functor

$$\big((-)_f : \mathscr{M} \to \mathscr{M},\, \epsilon\big)$$

in the model category $\mathscr{M}$ (see [15, Definition 8.1.2]).

Definition 5.3

Let $\mathscr{M}_f$ be the smallest full subcategory of $\mathscr{M}$ such that

(i) $\mathscr{M}_f$ contains (a representative of the isomorphism class of) each strictly finite I-cell object of $\mathscr{M}$; and

(ii) the category $\mathscr{M}_f$ is stable under the functors $(-)_f$ and $\Gamma(-)^n$, $n \geq 0$.

Remark 5.4

Note that $\mathscr{M}_f$ is a small category, and note that every object in $\mathscr{M}_f$ is weakly equivalent to a strict finite I-cell.

We have the inclusion

$$\mathscr{M}_f \overset{I}{\hookrightarrow} \mathscr{M}.$$

Definition 5.5
Let S be the set of preimages of the weak equivalences in $\mathscr{M}$ under the functor i.

LEMMA 5.6
The induced functor

$$\mathscr{M}_f[S^{-1}] \xrightarrow{\mathsf{Ho}(I)} \mathsf{Ho}(\mathscr{M})$$

is fully faithful, where $\mathscr{M}_f[S^{-1}]$ denotes the localization of $\mathscr{M}$ by the set S.

Proof
Let X, Y be objects of $\mathscr{M}_f$. Notice that $(Y)_f$ is a fibrant resolution of Y in $\mathscr{M}$ which belongs to $\mathscr{M}_f$, and notice that

$$\begin{array}{ccc} \Gamma(X)^0 \coprod \Gamma(X)^0 & \longrightarrow & \Gamma(X)^0 \\ {\scriptstyle d^0 \coprod d^1} \downarrow & \nearrow_{s^0} & \\ \Gamma(X)^1 & & \end{array}$$

is a cylinder object for $\Gamma(X)^0$ (see [15, Proposition 16.1.6]). Since $\mathscr{M}_f$ is also stable under the functors $\Gamma(-)^n$, $n \geq 0$, this cylinder object also belongs to $\mathscr{M}_f$. This implies that if, in the construction of the homotopy category $\mathsf{Ho}(\mathscr{M})$, as in [15, Theorem 8.35], we restrict ourselves to $\mathscr{M}_f$, we recover $\mathscr{M}_f[S^{-1}]$ as a full subcategory of $\mathsf{Ho}(\mathscr{M})$. This implies the lemma. □

We denote by $\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset})$ the Quillen model category of simplicial presheaves on $\mathscr{M}_f$ endowed with the projective model structure (see Section 3). Let Σ be the image in $\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset})$ by the functor h (see Section 3) of the set S in $\mathscr{M}_f$. Since the category $\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset})$ is cellular and left proper, its left Bousfield localization by the set Σ exists (see [15, Definition 3.1.1]). We denote it by $\mathsf{L}_\Sigma\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset})$. We have a composed functor that we still denote by h:

$$h : \mathscr{M}_f \to \mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset}) \stackrel{\mathrm{Id}}{\to} \mathsf{L}_\Sigma\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset}).$$

Now, consider the functor

$$\begin{aligned} \underline{h} : \mathscr{M} &\longrightarrow \mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset}), \\ X &\longmapsto \mathsf{Hom}\big(\Gamma(-), X\big)_{|\mathscr{M}_f}. \end{aligned}$$

We also have a composed functor that we still denote by $\underline{h}$:

$$\underline{h} : \mathscr{M} \to \mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset}) \xrightarrow{\mathrm{Id}} \mathsf{L}_\Sigma\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset}).$$

Now, observe that the natural equivalence

$$i(-) : \Gamma(-) \longrightarrow c^*(-)$$

induces, for every object X in $\mathscr{M}_f$, a morphism $\Psi(X)$ in $\mathsf{L}_\Sigma\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset})$,

$$\Psi(X) : \ h(X) = \mathsf{Hom}\big(c^*(-), X\big) \longrightarrow \mathsf{Hom}\big(\Gamma(-), X\big) =: (\underline{h} \circ I)(X),$$

which is functorial in X.

LEMMA 5.7
The functor $\underline{h}$ preserves weak equivalences between fibrant objects.

Proof
Let X be a fibrant object in $\mathscr{M}$. We have an equivalence

$$\mathsf{Hom}\big(\Gamma(Y), X\big) \stackrel{\sim}{\longrightarrow} \mathsf{Map}_{\mathscr{M}}(Y, X)$$

(see [15, Definition 17.4.1]). This implies the lemma. □

Remark 5.8
Lemma 5.7 implies that the functor $\underline{h}$ admits a right derived functor

$$\begin{aligned} \mathbb{R}\underline{h} : \mathsf{Ho}(\mathscr{M}) &\longrightarrow \mathsf{Ho}\big(\mathsf{L}_\Sigma\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset})\big), \\ X \quad &\longmapsto \quad \mathsf{Hom}\big(\Gamma(-), X_f\big)_{|\mathscr{M}_f}. \end{aligned}$$

Since the functor

$$h : \mathscr{M}_f \to \mathsf{L}_\Sigma\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset})$$

sends, by definition, the elements of S to weak equivalences, we have an induced morphism

$$\mathsf{Ho}(h) : \mathscr{M}_f[S^{-1}] \to \mathsf{Ho}\big(\mathsf{L}_\Sigma\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset})\big).$$

Remark 5.9
Note that [42, Lemma 4.2.2] implies that for every X in $\mathscr{M}_f$, the morphism $\Psi(X)$,

$$\Psi(X) : \ \mathsf{Ho}(h)(X) \longrightarrow \big(\mathbb{R}\underline{h} \circ \mathsf{Ho}(I)\big)(X),$$

is an isomorphism in $\mathsf{Ho}(\mathsf{L}_\Sigma\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset}))$.

This shows that the functors

$$\mathsf{Ho}(h),\ \mathbb{R}\underline{h}\circ\mathsf{Ho}(I):\ \mathcal{M}_f[S^{-1}]\to\mathsf{Ho}\big(\mathsf{L}_\Sigma\mathsf{Fun}(\mathcal{M}_f^{\mathrm{op}},\mathrm{Sset})\big)$$

are canonically isomorphic, and so we have the diagram

$$\begin{array}{ccc}
\mathcal{M}_f[S^{-1}] & \xrightarrow{\ \mathsf{Ho}(I)\ } & \mathsf{Ho}(\mathcal{M}), \\
{\scriptstyle \mathsf{Ho}(h)}\big\downarrow & \swarrow{\scriptstyle \mathbb{R}\underline{h}} & \\
\mathsf{Ho}\big(\mathsf{L}_\Sigma\mathsf{Fun}(\mathcal{M}_f^{\mathrm{op}},\mathrm{Sset})\big) & &
\end{array}$$

which is commutative up to isomorphism.

LEMMA 5.10
The functor $\mathbb{R}\underline{h}$ commutes with filtered homotopy colimits.

Proof
Let $\{Y_i\}_{i\in I}$ be a filtered diagram in $\mathcal{M}$. We can suppose, without loss of generality, that Y_i is fibrant in $\mathcal{M}$. By Proposition 5.2, the natural morphism

$$\underset{i\in I}{\mathrm{hocolim}}\, Y_i \longrightarrow \underset{i\in I}{\mathrm{colim}}\, Y_i$$

is an isomorphism in $\mathsf{Ho}(\mathcal{M})$ and $\mathrm{colim}_{i\in I}\, Y_i$ is also fibrant. Since the functor

$$\mathsf{Ho}\big(\mathsf{Fun}(\mathcal{M}_f^{\mathrm{op}},\mathrm{Sset})\big)\xrightarrow{\ \mathbb{L}\,\mathrm{Id}\ }\mathsf{Ho}\big(\mathsf{L}_\Sigma\mathsf{Fun}(\mathcal{M}_f^{\mathrm{op}},\mathrm{Sset})\big)$$

commutes with homotopy colimits that, in $\mathsf{Ho}(\mathsf{Fun}(\mathcal{M}_f^{\mathrm{op}},\mathrm{Sset}))$, are calculated objectwise, it is sufficient to show that the morphism

$$\underset{i\in I}{\mathrm{hocolim}}\,\mathbb{R}\underline{h}(Y_i)(X)\longrightarrow\mathbb{R}\underline{h}(\underset{i\in I}{\mathrm{colim}}\, Y_i)(X)$$

is an isomorphism in $\mathsf{Ho}(\mathrm{Sset})$ for every object X in $\mathcal{M}_f$. Now, since every object X in $\mathcal{M}_f$ is homotopically finitely presented (see Proposition 5.2), we have the following equivalences:

$$\begin{aligned}
\mathbb{R}\underline{h}(\underset{i\in I}{\mathrm{colim}}\, Y_i)(X) &= \mathsf{Hom}\big(\Gamma(X),\underset{i\in I}{\mathrm{colim}}\, Y_i\big)\\
&\simeq \mathsf{Map}\big(\Gamma(X),\underset{i\in I}{\mathrm{colim}}\, Y_i\big)\\
&\simeq \underset{i\in I}{\mathrm{colim}}\,\mathsf{Map}(X,Y_i)\\
&\simeq \underset{i\in I}{\mathrm{hocolim}}\,\mathbb{R}\underline{h}(Y_i)(X).
\end{aligned}$$

This proves the lemma. □

We now denote by $\mathsf{L}_\Sigma\mathsf{Hot}_{\mathscr{M}_f}$ the derivator associated with $\mathsf{L}_\Sigma\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset})$, and we denote by $\underline{\mathscr{M}_f}[S^{-1}]$ the prederivator $\underline{\mathscr{M}_f}$ localized at the set S (see Examples 2.3, 2.4).

Observe that the morphism of functors

$$\Psi : h \longrightarrow \underline{h} \circ I$$

induces a 2-morphism of derivators

$$\overline{\Psi} : \mathsf{Ho}(h) \longrightarrow \mathbb{R}\underline{h} \circ \mathsf{Ho}(I).$$

LEMMA 5.11
The 2-morphism $\overline{\Psi}$ is an isomorphism.

Proof
For the terminal category e, the 2-morphism $\overline{\Psi}$ coincides with the morphism of functors of Remark 5.9. Since this one is an isomorphism, so too is $\overline{\Psi}$ by conservativity. This proves the lemma. □

As before, we have the diagram

$$\begin{array}{ccc} \underline{\mathscr{M}_f}[S^{-1}] & \xrightarrow{\mathsf{Ho}(I)} & \mathsf{HO}(\mathscr{M}) \\ {\scriptstyle \mathsf{Ho}(h)}\downarrow & \swarrow {\scriptstyle \mathbb{R}\underline{h}} & \\ \mathsf{L}_\Sigma\mathsf{Hot}_{\mathscr{M}_f} & & \end{array}$$

which is commutative up to isomorphism in the 2-category of prederivators. Note that by Lemma 5.10, $\mathbb{R}\underline{h}$ commutes with filtered homotopy colimits.

Let $\mathbb{D}$ be a derivator.

LEMMA 5.12
The morphism of prederivators

$$\underline{\mathscr{M}_f}[S^{-1}] \xrightarrow{\mathsf{Ho}(h)} \mathsf{L}_\Sigma\mathsf{Hot}_{\mathscr{M}_f}$$

induces an equivalence of categories

$$\underline{\mathsf{Hom}}_!(\mathsf{L}_\Sigma\mathsf{Hot}_{\mathscr{M}_f}, \mathbb{D}) \xrightarrow{\mathsf{Ho}(h)^*} \underline{\mathsf{Hom}}(\underline{\mathscr{M}_f}[S^{-1}], \mathbb{D}).$$

Proof

The category $\underline{\mathsf{Hom}}_!(\mathsf{L}_\Sigma\mathsf{Hot}_{\mathscr{M}_f}, \mathbb{D})$ is equivalent, by Theorem 4.4, to the category $\underline{\mathsf{Hom}}_{!,\Sigma}(\mathsf{Hot}_{\mathscr{M}_f}, \mathbb{D})$. This last category identifies, under the equivalence

$$\underline{\mathsf{Hom}}_!(\mathsf{Hot}_{\mathscr{M}_f}, \mathbb{D}) \to \underline{\mathsf{Hom}}(\underline{\mathscr{M}_f}, \mathbb{D})$$

given by Theorem 3.1, with the full subcategory of $\underline{\mathsf{Hom}}(\underline{\mathscr{M}_f}, \mathbb{D})$ consisting of the morphisms of prederivators which send the elements of S to isomorphisms in $\mathbb{D}(e)$. Now, observe that this last category identifies with $\underline{\mathsf{Hom}}(\underline{\mathscr{M}_f}[S^{-1}], \mathbb{D})$ by definition of the localized prederivator $\underline{\mathscr{M}_f}[S^{-1}]$. This proves the lemma. □

Recall from [11, Section 9.5] that the cosimplicial resolution functor $\Gamma(-)$ which we fixed in the beginning of this section allows us to construct a Quillen adjunction:

$$\begin{array}{c} \mathscr{M} \\ \mathrm{Re}\uparrow\;\downarrow \underline{h}=\mathrm{sing} \\ \mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset}). \end{array}$$

Since the functor Re sends the elements of Σ to weak equivalences in $\mathscr{M}$, we have the Quillen adjunction

$$\begin{array}{c} \mathscr{M} \\ \mathrm{Re}\uparrow\;\downarrow \underline{h} \\ \mathsf{L}_\Sigma\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset}) \end{array}$$

and a natural weak equivalence

$$\eta : \mathrm{Re} \circ \underline{h} \overset{\sim}{\longrightarrow} I$$

(see [11, Theorem 2.3]).

This implies that we have the diagram

$$\begin{array}{ccc} \underline{\mathscr{M}_f}[S^{-1}] & \xrightarrow{\mathsf{Ho}(I)} & \mathsf{Ho}(\mathscr{M}) \\ {\scriptstyle \mathsf{Ho}(\underline{h})}\downarrow & \underset{\mathbb{R}\underline{h}}{\overset{\mathbb{L}\mathrm{Re}}{\rightleftarrows}} & \\ \mathsf{Ho}\big(\mathsf{L}_\Sigma\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset})\big) & & \end{array}$$

which is commutative up to isomorphism.

We now claim that $\mathbb{L}\,\mathrm{Re} \circ \mathbb{R}\underline{h}$ is naturally isomorphic to the identity. Indeed, by Proposition 5.2, each object of $\mathscr{M}$ is isomorphic in $\mathsf{Ho}(\mathscr{M})$ to a filtered colimit of strict finite I-cell objects. Since $\mathbb{R}\underline{h}$ and $\mathbb{L}\,\mathrm{Re}$ commute with filtered homotopy colimits and $\mathbb{L}\,\mathrm{Re} \circ \mathsf{Ho}(h) \simeq \mathrm{Id}$, we conclude that $\mathbb{L}\,\mathrm{Re} \circ \mathbb{R}\underline{h}$ is naturally isomorphic to the identity. This implies that the morphism $\mathbb{R}\underline{h}$ is fully faithful.

Now, observe that the natural weak equivalence η induces a 2-isomorphism, so we obtain the diagram

$$\begin{array}{ccc} \underline{\mathscr{M}_f}[S^{-1}] & \xrightarrow{\mathsf{Ho}(I)} & \mathsf{HO}(\mathscr{M}) \\ {\scriptstyle \mathsf{Ho}(h)}\downarrow & \nearrow{\scriptstyle \mathbb{L}\,\mathrm{Re}} \quad \swarrow{\scriptstyle \mathbb{R}\underline{h}} & \\ \mathsf{L}_{\Sigma}\mathsf{Hot}_{\mathscr{M}_f} & & \end{array}$$

which is commutative up to isomorphism in the 2-category of prederivators. Note that $\mathbb{L}\,\mathrm{Re} \circ \mathbb{R}\underline{h}$ is naturally isomorphic to the identity (by conservativity), and so the morphism of derivators $\mathbb{R}\underline{h}$ is fully faithful.

Let $\mathbb{D}$ be a derivator.

THEOREM 5.13

The morphism of derivators

$$\mathsf{HO}(\mathscr{M}) \xrightarrow{\mathbb{R}\underline{h}} \mathsf{L}_{\Sigma}\mathsf{Hot}_{\mathscr{M}_f}$$

induces an equivalence of categories

$$\underline{\mathsf{Hom}}_{!}(\mathsf{L}_{\Sigma}\mathsf{Hot}_{\mathscr{M}_f}, \mathbb{D}) \xrightarrow{\mathbb{R}\underline{h}^*} \underline{\mathsf{Hom}}_{\mathrm{flt}}\big(\mathsf{HO}(\mathscr{M}), \mathbb{D}\big),$$

where $\underline{\mathsf{Hom}}_{\mathrm{flt}}(\mathsf{HO}(\mathscr{M}), \mathbb{D})$ *denotes the category of morphisms of derivators which commute with filtered homotopy colimits.*

Proof

We have the adjunction

$$\begin{array}{c} \underline{\mathsf{Hom}}\big(\mathsf{HO}(\mathscr{M}), \mathbb{D}\big) \\ {\scriptstyle \mathbb{R}\underline{h}^*}\uparrow\ \downarrow{\scriptstyle \mathbb{L}\,\mathrm{Re}^*} \\ \underline{\mathsf{Hom}}(\mathsf{L}_{\Sigma}\mathsf{Hot}_{\mathscr{M}_f}, \mathbb{D}) \end{array}$$

with $\mathbb{R}\underline{h}^*$ a fully faithful functor.

Now, note that the adjunction of Lemma 3.2 naturally induces an adjunction

$$\begin{array}{c}\underline{\mathsf{Hom}}(\mathsf{L}_\Sigma \mathsf{Hot}_{\mathscr{M}_f}, \mathbb{D}) \\ \big\uparrow \big\downarrow \Psi \\ \underline{\mathsf{Hom}}_!(\mathsf{L}_\Sigma \mathsf{Hot}_{\mathscr{M}_f}, \mathbb{D})\end{array}$$

This implies that the composed functor

$$\mathbb{R}\underline{h}^* : \underline{\mathsf{Hom}}_!(\mathsf{L}_\Sigma \mathsf{Hot}_{\mathscr{M}_f}, \mathbb{D}) \longrightarrow \underline{\mathsf{Hom}}_{\mathrm{flt}}\big(\mathsf{HO}(\mathscr{M}), \mathbb{D}\big)$$

is fully faithful.

We now show that this functor is essentially surjective.

Let F be an object of $\underline{\mathsf{Hom}}_{\mathrm{flt}}(\mathsf{HO}(\mathscr{M}), \mathbb{D})$. Consider the morphism

$$\mathbb{L}\,\mathrm{Re}^*(F) := F \circ \mathbb{L}\,\mathrm{Re}.$$

Note that this morphism does not necessarily commute with homotopy colimits. Now, by the above adjunction, we have a universal 2-morphism

$$\varphi : \Psi\big(\mathbb{L}\,\mathrm{Re}^*(F)\big) \longrightarrow \mathbb{L}\,\mathrm{Re}^*(F).$$

Consider the 2-morphism

$$\mathbb{R}\underline{h}^* : \mathbb{R}\underline{h}^*\big((\Psi \circ \mathbb{L}\,\mathrm{Re}^*)(F)\big) \longrightarrow (\mathbb{R}\underline{h}^* \circ \mathbb{L}\,\mathrm{Re}^*)(F) \simeq F.$$

Now, we show that this 2-morphism is a 2-isomorphism. By conservativity, it is sufficient to show this for the case of the terminal category e. For this, observe that $\mathbb{R}\underline{h}^*(\varphi)$ induces an isomorphism

$$\Psi\big(\mathbb{L}\,\mathrm{Re}^*(F)\big) \circ \mathbb{R}\underline{h} \circ \mathsf{Ho}(I) \longrightarrow F \circ \mathsf{Ho}(I).$$

Now, each object of $\mathscr{M}$ is isomorphic, in $\mathsf{Ho}(\mathscr{M})$, to a filtered colimit of strict finite I-cell objects. Since F and $\Psi(\mathbb{L}\,\mathrm{Re}^*(F))$ commute with filtered homotopy colimits, $\mathbb{R}\underline{h}^*(\varphi)$ induces an isomorphism. This shows that the functor $\mathbb{R}\underline{h}^*$ is essentially surjective.

This proves the theorem. □

6. Pointed derivators

Recall from Section 5 that we have constructed a derivator $\mathsf{L}_\Sigma \mathsf{Hot}_{\mathscr{M}_f}$ associated with a Quillen model category $\mathscr{M}$ satisfying suitable compactness assumptions.

Now, suppose that $\mathsf{Ho}(\mathscr{M})$ is pointed; that is, suppose that the morphism

$$\emptyset \longrightarrow *$$

in $\mathscr{M}$, where $\emptyset$ denotes the initial object and $*$ the terminal one, is a weak equivalence. Consider the morphism

$$P : \widetilde{\emptyset} \longrightarrow h(\emptyset),$$

where $\widetilde{\emptyset}$ denotes the initial object in $\mathsf{L}_{\Sigma}\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset})$.

Observe that since $\mathbb{R}\underline{h}$ admits a left adjoint, $h(\emptyset)$ identifies with the terminal object in

$$\mathsf{Ho}\big(\mathsf{L}_{\Sigma}\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset})\big)$$

because

$$h(\emptyset) = \mathsf{Ho}(h)(\emptyset) \xrightarrow{\sim} \mathbb{R}\underline{h} \circ \mathsf{Ho}(I)(\emptyset) \xrightarrow{\sim} \mathbb{R}\underline{h}(*).$$

We denote by

$$\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset})$$

the left Bousfield localization of $\mathsf{L}_{\Sigma}\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset})$ at the morphism P.

Note that the category

$$\mathsf{Ho}\big(\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset})\big)$$

is now a pointed one.

We have the morphisms of derivators

$$\begin{array}{c} \mathsf{Ho}(\mathscr{M}) \\ {\scriptstyle \mathbb{L}\mathrm{Re}}\uparrow\;\downarrow{\scriptstyle \mathbb{R}\underline{h}} \\ \mathsf{L}_{\Sigma}\mathsf{Hot}_{\mathscr{M}_f} \\ {\scriptstyle \Phi}\downarrow \\ \mathsf{L}_{\Sigma,P}\mathsf{Hot}_{\mathscr{M}_f} \end{array}$$

By construction, we have a pointed morphism of derivators

$$\mathsf{HO}(\mathscr{M}) \xrightarrow{\Phi\circ\mathbb{R}\underline{h}} \mathsf{L}_{\Sigma,P}\mathsf{Hot}_{\mathscr{M}_f}$$

which commutes with filtered homotopy colimits and preserves the point.

Let $\mathbb{D}$ be a pointed derivator.

PROPOSITION 6.1
The morphism of derivators $\Phi \circ \mathbb{R}\underline{h}$ induces an equivalence of categories

$$\underline{\mathsf{Hom}}_!(\mathsf{L}_{\Sigma,P}\mathsf{Hot}_{\mathscr{M}_f}, \mathbb{D}) \xrightarrow{(\Phi\circ\mathbb{R}\underline{h})^*} \underline{\mathsf{Hom}}_{\mathrm{flt},p}\big(\mathsf{HO}(\mathscr{M}), \mathbb{D}\big),$$

where $\underline{\mathsf{Hom}}_{\mathrm{flt},p}(\mathsf{HO}(\mathscr{M}), \mathbb{D})$ denotes the category of morphisms of derivators which commute with filtered homotopy colimits and preserve the point.

Proof
By Theorem 4.4, we have an equivalence of categories

$$\underline{\mathsf{Hom}}_!(\mathsf{L}_{\Sigma,P}\mathsf{Hot}_{\mathscr{M}_f}, \mathbb{D}) \xrightarrow{\Phi^*} \underline{\mathsf{Hom}}_{!,P}(\mathsf{L}_{\Sigma}\mathsf{Hot}_{\mathscr{M}_f}, \mathbb{D}).$$

By Theorem 5.13, we have an equivalence of categories

$$\underline{\mathsf{Hom}}_!(\mathsf{L}_{\Sigma}\mathsf{Hot}_{\mathscr{M}_f}, \mathbb{D}) \xrightarrow{\mathbb{R}\underline{h}^*} \underline{\mathsf{Hom}}_{\mathrm{flt}}\big(\mathsf{HO}(\mathscr{M}), \mathbb{D}\big).$$

We now show that under this last equivalence, the category $\underline{\mathsf{Hom}}_{!,P}(\mathsf{L}_{\Sigma}\mathsf{Hot}_{\mathscr{M}_f}, \mathbb{D})$ identifies with $\underline{\mathsf{Hom}}_{\mathrm{flt},p}(\mathsf{HO}(\mathscr{M}), \mathbb{D})$. Let F be an object of $\underline{\mathsf{Hom}}_{!,P}(\mathsf{L}_{\Sigma}\mathsf{Hot}_{\mathscr{M}_f}, \mathbb{D})$. Since F commutes with homotopy colimits, it preserves the initial object. This implies that $F \circ \mathbb{R}\underline{h}$ belongs to $\underline{\mathsf{Hom}}_{\mathrm{flt},p}(\mathsf{HO}(\mathscr{M}), \mathbb{D})$.

Now, let G be an object of $\underline{\mathsf{Hom}}_{\mathrm{flt},p}(\mathsf{HO}(\mathscr{M}), \mathbb{D})$. Consider, as in the proof of Theorem 5.13, the morphism

$$\Psi\big(\mathbb{L}\,\mathrm{Re}^*(G)\big) : \mathsf{L}_{\Sigma}\mathsf{Hot}_{\mathscr{M}_f} \longrightarrow \mathbb{D}.$$

Since $\Psi(\mathbb{L}\,\mathrm{Re}^*(G))$ commutes with homotopy colimits, by construction it sends $\widetilde{\emptyset}$ to the point of $\mathbb{D}$. Observe also that $h(\emptyset)$ is sent to the point of $\mathbb{D}$ because

$$\Psi\big(\mathbb{L}\,\mathrm{Re}^*(G)\big)\big(h(\emptyset)\big) \simeq G(\emptyset).$$

This proves the proposition. □

7. Small weak generators

Let $\mathscr{N}$ be a pointed, left proper, compactly generated Quillen model category, as in [41, Definition 2.1]. Observe that, in particular, this implies that $\mathscr{N}$ is finitely generated, as in [16, Section 7.4]. We denote by $\mathscr{G}$ the set of cofibers of the generating cofibrations I in $\mathscr{N}$. By [16, Corollary 7.4.4], the set $\mathscr{G}$ is a set of small weak generators for $\mathsf{Ho}(\mathscr{N})$ (see [16, Definitions 7.2.1, 7.2.2]). Let S be a set of morphisms in $\mathscr{N}$ between objects that are homotopically finitely presented (see [41, Definition 2.1]), and let $\mathsf{L}_S\mathcal{N}$ be the left Bousfield localization of $\mathcal{N}$ by S. We have an adjunction

$$\begin{array}{c} \mathsf{Ho}(\mathcal{N}) \\ \mathbb{L}\,\mathrm{Id} \downarrow \;\; \uparrow \mathbb{R}\,\mathrm{Id} \\ \mathsf{Ho}(\mathsf{L}_S\mathcal{N}) \end{array}$$

LEMMA 7.1
The image of the set $\mathcal{G}$ under the functor $\mathbb{L}\,\mathsf{Id}$ is a set of small weak generators in $\mathsf{Ho}(\mathsf{L}_S\mathcal{N})$.

Proof
The previous adjunction is equivalent to

$$\begin{array}{c} \mathsf{Ho}(\mathcal{N}) \\ (-)_f \downarrow \;\; \uparrow \\ \mathsf{Ho}(\mathcal{N})_S \end{array}$$

where $\mathsf{Ho}(\mathcal{N})_S$ denotes the full subcategory of $\mathsf{Ho}(\mathcal{M})$ formed by the S-local objects of $\mathcal{N}$ and $(-)_f$ denotes a fibrant resolution functor in $\mathsf{L}_S\mathcal{N}$ (see [15, Definition 4.3.2]). Clearly, this implies that the image of the set $\mathcal{G}$ under the functor $(-)_f$ is a set of weak generators in $\mathsf{Ho}(\mathsf{L}_S\mathcal{N})$.

We now show that the S-local objects in $\mathcal{N}$ are stable under filtered homotopy colimits. Let $\{X_i\}_{i\in I}$ be a filtered diagram of S-local objects. By Proposition 5.2, we have an isomorphism

$$\underset{i\in I}{\mathrm{hocolim}}\, X_i \xrightarrow{\sim} \underset{i\in I}{\mathrm{colim}}\, X_i$$

in $\mathsf{Ho}(\mathcal{N})$. We now show that $\mathrm{colim}_{i\in I}\, X_i$ is an S-local object. Let $g : A \to B$ be an element of S. We have at our disposal the following commutative diagram:

$$\begin{array}{ccc} \mathsf{Map}(B, \underset{i\in I}{\mathrm{colim}}\, X_i) & \xrightarrow{g^*} & \mathsf{Map}(A, \underset{i\in I}{\mathrm{colim}}\, X_i) \\ \sim \uparrow & & \uparrow \sim \\ \underset{i\in I}{\mathrm{colim}}\, \mathsf{Map}(B, X_i) & \xrightarrow[\underset{i\in I}{\mathrm{colim}}\, g_i^*]{} & \underset{i\in I}{\mathrm{colim}}\, \mathsf{Map}(A, X_i) \end{array}$$

Now, observe that since A and B are homotopically finitely presented objects, the vertical arrows in the diagram are isomorphisms in $\mathsf{Ho}(\mathrm{Sset})$. Since each object X_i is S-local, the morphism g_i^* is an isomorphism in $\mathsf{Ho}(\mathrm{Sset})$ and so is $\mathrm{colim}_{i\in I}\, g_i^*$. This implies that $\mathrm{colim}_{i\in I}\, X_i$ is an S-local object. This shows that the inclusion

$$\mathsf{Ho}(\mathscr{N})_S \hookrightarrow \mathsf{Ho}(\mathscr{N})$$

commutes with filtered homotopy colimits, so the image of the set $\mathscr{G}$ under the functor $(-)_f$ consists of small objects in $\mathsf{Ho}(\mathsf{L}_S\mathscr{N})$.

This proves the lemma. □

Recall from Section 6 that we have constructed a pointed derivator $\mathsf{L}_{\Sigma,P}\mathsf{Hot}_{\mathscr{M}_f}$. We now construct a strictly pointed Quillen model category whose associated derivator is equivalent to $\mathsf{L}_{\Sigma,P}\mathsf{Hot}_{\mathscr{M}_f}$. Consider the pointed Quillen model category

$$* \downarrow \mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset}) = \mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet).$$

We have the Quillen adjunction

$$\begin{array}{c} \mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet) \\ {\scriptstyle (-)_+}\uparrow \downarrow {\scriptstyle U} \\ \mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset}) \end{array}$$

where U denotes the forgetful functor.

We denote by $\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)$ the left Bousfield localization of $\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)$ by the image of the set $\Sigma \cup \{P\}$ under the functor $(-)_+$. We denote by $\mathsf{L}_{\Sigma,P}\mathsf{Hot}_{\mathscr{M}_{f\bullet}}$ the derivator associated with $\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)$.

Remark 7.2

Since the derivators associated with $\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset})$ and $\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)$ are characterized by the same universal property, we have a canonical equivalence of pointed derivators

$$\mathsf{L}_{\Sigma,P}\mathsf{Hot}_{\mathscr{M}_f} \xrightarrow{\sim} \mathsf{L}_{\Sigma,P}\mathsf{Hot}_{\mathscr{M}_{f\bullet}}.$$

Note also that the category $\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)$ endowed with the projective model structure is pointed, left proper, compactly generated, and note that the domains and codomains of the elements of the set $(\Sigma \cup \{P\})_+$ are homotopically finitely presented objects. Therefore, by Lemma 7.1, the set

$$\mathscr{G} = \{\mathbf{F}^X_{\Delta[n]_+/\partial\Delta[n]_+} \mid X \in \mathscr{M}_f, n \geq 0\}$$

of cofibers of the generating cofibrations in $\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)$ is a set of small weak generators in $\mathsf{Ho}(\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet))$.

8. Stabilization

Let $\mathbb{D}$ be a regular pointed strong derivator.

Heller [14, Section 8] constructs a universal morphism to a triangulated strong derivator

$$\mathbb{D} \xrightarrow{\text{stab}} \mathsf{St}(\mathbb{D})$$

which commutes with homotopy colimits.

This construction is done in two steps. First, consider the ordered set

$$\mathbf{V} := \left\{(i, j) \,\middle|\, |i - j| \leq 1\right\} \subset \mathbb{Z} \times \mathbb{Z}$$

naturally as a small category. We denote by

$$\dot{\mathbf{V}} := \left\{(i, j) \,\middle|\, |i - j| = 1\right\} \subset \mathbb{V}$$

the full subcategory of points on the boundary.

Now, let $\mathsf{Spec}(\mathbb{D})$ be the full subderivator of $\mathbb{D}_{\mathbf{V}}$ (see [7, Definition 3.4]) formed by the objects X in $\mathbb{D}_{\mathbf{V}}(L)$ whose image under the functor

$$\mathbb{D}_{\mathbf{V}}(L) = \mathbb{D}(\mathbf{V} \times L) \longrightarrow \mathsf{Fun}\big(\dot{\mathbf{V}}^{\mathrm{op}}, \mathbb{D}(L)\big)$$

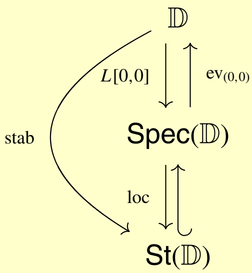
is of the form

$$\begin{array}{ccccc}
 & & * & \longrightarrow & X_{(1,1)} \cdots \\
 & & \uparrow & & \uparrow \\
* & \longrightarrow & X_{(0,0)} & \longrightarrow & * \\
\uparrow & & \uparrow & & \\
\cdots X_{(-1,-1)} & \longrightarrow & * & &
\end{array}$$

(see [14, Section 8]). We have an evaluation functor $\mathrm{ev}_{(0,0)} : \mathsf{Spec}(\mathbb{D}) \to \mathbb{D}$ which admits a left adjoint $L[0, 0]$.

Finally, let $\mathsf{St}(\mathbb{D})$ be the full reflexive subderivator of $\mathsf{Spec}(\mathbb{D})$ consisting of the Ω-spectra, as defined in [14, Section 8].

We have the following adjunctions:



in the 2-category of derivators.

Let $\mathbb{T}$ be a triangulated strong derivator. The following theorem is proved in [14, Corollary 10.1].

THEOREM 8.1
The morphism stab *induces an equivalence of categories*

$$\underline{\mathsf{Hom}}_!\big(\mathsf{St}(\mathbb{D}), \mathbb{T}\big) \xrightarrow{\mathrm{stab}^*} \underline{\mathsf{Hom}}_!(\mathbb{D}, \mathbb{T}).$$

LEMMA 8.2
Let $\mathscr{G}$ be a set of objects in $\mathbb{D}(e)$ which satisfies the following conditions.

(A1) *If, for each g in $\mathscr{G}$, we have*

$$\mathsf{Hom}_{\mathbb{D}(e)}(g, X) = \{*\},$$

then X is isomorphic to $*$, where $*$ denotes the terminal and initial object in $\mathbb{D}(e)$.

(A2) *For every set K and each g in $\mathscr{G}$, the canonical map*

$$\operatorname*{colim}_{\substack{S\subseteq K\\ S \text{ finite}}} \mathsf{Hom}_{\mathbb{D}(e)}\Big(g, \coprod_{\alpha\in S} X_\alpha\Big) \xrightarrow{\sim} \mathsf{Hom}_{\mathbb{D}(e)}\Big(g, \coprod_{\alpha\in S} X_\alpha\Big)$$

is bijective.

Then the set

$$\big\{\Sigma^n \mathrm{stab}(g) \,\big|\, g \in \mathscr{G},\ n \in \mathbb{Z}\big\}$$

of objects in $\mathsf{St}(\mathbb{D})(e)$, where Σ denotes the suspension functor in $\mathsf{St}(\mathbb{D})(e)$, satisfies conditions (A1) and (A2).

Proof
Let $\underline{X}$ be an object of $\mathsf{St}(\mathbb{D})(e)$. Suppose that for each g in $\mathscr{G}$ and n in $\mathbb{Z}$, we have

$$\mathsf{Hom}_{\mathsf{St}(\mathbb{D})(e)}\big(\Sigma^n \mathrm{stab}(g), \underline{X}\big) = \{*\}.$$

Then by the isomorphisms

$$\begin{aligned}\mathsf{Hom}_{\mathsf{St}(\mathbb{D})(e)}\big(\Sigma^n \mathrm{stab}(g), \underline{X}\big) &\simeq \mathsf{Hom}_{\mathsf{St}(\mathbb{D})(e)}\big(\mathrm{stab}(g), \Omega^n \underline{X}\big)\\ &\simeq \mathsf{Hom}_{\mathbb{D}(e)}(g, \mathrm{ev}_{(0,0)}\Omega^n \underline{X})\\ &\simeq \mathsf{Hom}_{\mathbb{D}(e)}(g, \mathrm{ev}_{(n,n)}\underline{X}),\end{aligned}$$

we conclude that for all n in $\mathbb{Z}$, we have

$$\mathrm{ev}_{(n,n)}\underline{X} = *.$$

By the conservativity axiom, $\underline{X}$ is isomorphic to $*$ in $\mathsf{St}(\mathbb{D})(e)$. This shows condition (A1). Now, observe that condition (A2) follows from the isomorphisms

$$\begin{aligned}\mathsf{Hom}_{\mathsf{St}(\mathbb{D})(e)}\Big(\Sigma^n \mathrm{stab}(g), \bigoplus_{\alpha\in K}\underline{X_\alpha}\Big) &\simeq \mathsf{Hom}_{\mathsf{St}(\mathbb{D})(e)}\Big(\mathrm{stab}(g), \Omega^n\bigoplus_{\alpha\in K}\underline{X_\alpha}\Big)\\ &\simeq \mathsf{Hom}_{\mathsf{St}(\mathbb{D})(e)}\Big(\mathrm{stab}(g), \bigoplus_{\alpha\in K}\Omega^n\underline{X_\alpha}\Big)\\ &\simeq \mathsf{Hom}_{\mathbb{D}(e)}\Big(g, \mathrm{ev}_{(0,0)}\coprod_{\alpha\in K}\Omega^n\underline{X_\alpha}\Big)\\ &\simeq \mathsf{Hom}_{\mathbb{D}(e)}\Big(g, \coprod_{\alpha\in K}\mathrm{ev}_{(0,0)}\Omega^n\underline{X_\alpha}\Big)\\ &\simeq \operatorname*{colim}_{\substack{S\subseteq K\\ S \text{ finite}}} \mathsf{Hom}_{\mathbb{D}(e)}\Big(g, \coprod_{\alpha\in S}\mathrm{ev}_{(0,0)}\Omega^n\underline{X_\alpha}\Big)\\ &\simeq \operatorname*{colim}_{\substack{S\subseteq K\\ S \text{ finite}}} \mathsf{Hom}_{\mathbb{D}(e)}\Big(g, \mathrm{ev}_{(0,0)}\coprod_{\alpha\in S}\Omega^n\underline{X_\alpha}\Big)\\ &\simeq \operatorname*{colim}_{\substack{S\subseteq K\\ S \text{ finite}}} \mathsf{Hom}_{\mathsf{St}(\mathbb{D})(e)}\Big(\mathrm{stab}(g), \bigoplus_{\alpha\in S}\Omega^n\underline{X_\alpha}\Big)\\ &\simeq \operatorname*{colim}_{\substack{S\subseteq K\\ S \text{ finite}}} \mathsf{Hom}_{\mathsf{St}(\mathbb{D})(e)}\Big(\Sigma^n\mathrm{stab}(g), \bigoplus_{\alpha\in S}\underline{X_\alpha}\Big).\end{aligned}$$ □

LEMMA 8.3
Let $\mathbb{T}$ be a triangulated derivator, and let $\mathscr{G}$ be a set of objects in $\mathbb{T}(e)$ which satisfies conditions (1) and (2) of Lemma 8.2.

Then for every small category L and every point $x : e \to L$ in L, the set

$$\big\{x_!(g)\,\big|\, g\in\mathscr{G},\ x : e\to L\big\}$$

satisfies conditions (1) and (2) in the category $\mathbb{T}(L)$.

Proof
Suppose that

$$\mathsf{Hom}_{\mathbb{T}(L)}\big(x_!(g), M\big) = \{*\}$$

for every $g \in \mathscr{G}$ and every point x in L. Then by adjunction, x^*M is isomorphic to $*$ in $\mathbb{T}(e)$, and so by the conservativity axiom, M is isomorphic to $*$ in $\mathbb{T}(L)$. This shows condition (A1). Condition (A2) follows from the following isomorphisms:

$$\begin{aligned}\mathsf{Hom}_{\mathbb{T}(L)}\Big(x_!(g), \bigoplus_{\alpha\in K} M_\alpha\Big) &\simeq \mathsf{Hom}_{\mathbb{T}(e)}\Big(g, x^*\bigoplus_{\alpha\in K} M_\alpha\Big)\\ &\simeq \mathsf{Hom}_{\mathbb{T}(e)}\Big(g, \bigoplus_{\alpha\in K} x^*M_\alpha\Big)\\ &\simeq \bigoplus_{\alpha\in K}\mathsf{Hom}_{\mathbb{T}(e)}(g, x^*M_\alpha)\\ &\simeq \bigoplus_{\alpha\in K}\mathsf{Hom}_{\mathbb{T}(L)}\big(x_!(g), M_\alpha\big).\end{aligned}$$

□

Remark 8.4
Note that if $\mathbb{D}$ is a regular pointed strong derivator and we have at our disposal a set $\mathscr{G}$ of objects in $\mathbb{D}(e)$ which satisfies conditions (A1) and (A2), then Lemmas 7.1 and 8.3 imply that $\mathsf{St}(\mathbb{D})(L)$ is a compactly generated triangulated category for every small category L.

Relation with Hovey and Schwede's stabilization
We now relate Heller's construction in [14] with the construction of spectra as it is done by Hovey [17] and Schwede [36].

Let $\mathscr{M}$ be a pointed, simplicial, left proper, cellular, almost finitely generated Quillen model category (see [17, Definition 4.1]), where sequential colimits commute with finite products and homotopy pullbacks. This implies, in particular, that the associated derivator $\mathsf{HO}(\mathscr{M})$ is regular.

Example 8.5
Consider the category $\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)$ defined in Section 7. Note that the category of pointed simplicial presheaves $\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)$ is pointed, simplicial, left proper, cellular, and even finitely generated (see [17, Definition 4.1]). Since limits and colimits in $\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)$ are calculated objectwise, we conclude that sequential colimits commute with finite products. Now, by [15, Theorem 4.1.1], the category $\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)$ is also pointed, simplicial, left proper, and cellular.

Now, observe that the domains and codomains of each morphism in $\Lambda((\Sigma \cup \{P\})_+)$ (see [15, Definition 4.2.1]) are finitely presented since the forgetful functor

$$\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet) \to \mathsf{Fun}(\mathscr{M}^{\mathrm{op}}, \mathrm{Sset})$$

commutes with filtered colimits and homotopy pullbacks. Now, by [15, Proposition 4.2.4], we conclude that a morphism $A \xrightarrow{f} B$ in $\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)$, with B a local object, is a local fibration if and only if it has the right lifting property (RLP) with respect to the set

$$J \cup \Lambda\big((\Sigma \cup \{P\})_+\big),$$

where J denotes the set of generating acyclic cofibrations in $\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)$. This shows that $\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)$ is almost finitely generated.

Recall from [36, Section 1.2] that since $\mathscr{M}$ is a pointed, simplicial model category, we have a Quillen adjunction

$$\begin{array}{c} \mathscr{M} \\ \Sigma(-) \downarrow \uparrow \Omega(-) \\ \mathscr{M} \end{array}$$

where $\Sigma(X)$ denotes the suspension of an object X , that is, the pushout of the diagram

$$\begin{array}{ccc} X \otimes \partial\Delta^1 & \longrightarrow & X \otimes \Delta^1 \\ \downarrow & & \\ * & & \end{array}$$

Recall also that Hovey [17] and Schwede [36] construct a stable Quillen model category $\mathsf{Sp}^{\mathbb{N}}(\mathscr{M})$ of spectra associated with $\mathscr{M}$ and with the left Quillen functor $\Sigma(-)$. We have the Quillen adjunction (see [17])

$$\begin{array}{c} \mathscr{M} \\ \Sigma^\infty \downarrow \uparrow \mathrm{ev}_0 \\ \mathsf{Sp}^{\mathbb{N}}(\mathscr{M}) \end{array}$$

and thus a morphism to a strong triangulated derivator

$$\mathsf{HO}(\mathscr{M}) \xrightarrow{\mathbb{L}\Sigma^\infty} \mathsf{HO}\big(\mathsf{Sp}^{\mathbb{N}}(\mathscr{M})\big)$$

which commutes with homotopy colimits.

By Theorem 8.1, we have at our disposal a diagram

$$\begin{array}{ccc} \mathsf{HO}(\mathscr{M}) & & \\ {\scriptstyle \text{stab}}\downarrow & \overset{\mathbb{L}\Sigma^\infty}{\searrow} & \\ \mathsf{St}\big(\mathsf{HO}(\mathscr{M})\big) & \underset{\varphi}{\longrightarrow} & \mathsf{HO}\big(\mathsf{Sp}^{\mathbb{N}}(\mathscr{M})\big) \end{array}$$

which is commutative up to isomorphism in the 2-category of derivators.

Now, suppose also that we have a set $\mathscr{G}$ of small weak generators in $\mathsf{Ho}(\mathscr{M})$, as in [17, Definitions 7.2.1, 7.2.2]. Suppose also that each object of $\mathscr{G}$ considered in $\mathscr{M}$ is cofibrant, finitely presented, homotopy finitely presented, and has a finitely presented cylinder object.

Example 8.6
Observe that the category $\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)$ is pointed and finitely generated. By [16, Corollary 7.4.4], the set

$$\mathscr{G} = \{\mathbf{F}^X_{\Delta[n]_+/\partial\Delta[n]_+} \mid X \in \mathscr{M}_f, n \geq 0\}$$

is a set of small weak generators in $\mathsf{Ho}(\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet))$. Since the domains and codomains of the set

$$(\Sigma \cup \{P\})_+$$

are homotopically finitely presented objects, Lemma 7.1 implies that $\mathscr{G}$ is a set of small weak generators in $\mathsf{Ho}(\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet))$. Clearly, the elements of $\mathscr{G}$ are cofibrant, finitely presented, and have a finitely presented cylinder object. They are also homotopically finitely presented.

Under the hypotheses above on the category $\mathscr{M}$, we have the following comparison theorem.

THEOREM 8.7
The induced morphism of triangulated derivators

$$\varphi : \mathsf{St}\big(\mathsf{HO}(\mathscr{M})\big) \longrightarrow \mathsf{HO}\big(\mathsf{Sp}^{\mathbb{N}}(\mathscr{M})\big)$$

is an equivalence.

The proof of Theorem 8.7 consists in verifying the conditions of the following general proposition.

PROPOSITION 8.8

Let $F : \mathbb{T}_1 \to \mathbb{T}_2$ be a morphism of strong triangulated derivators. Suppose that the triangulated categories $\mathbb{T}_1(e)$ and $\mathbb{T}_2(e)$ are compactly generated, and suppose that there is a set $\mathscr{G} \subset \mathbb{T}_1(e)$ of compact generators which is stable under suspensions and satisfies the following conditions:

(a) *$F(e)$ induces bijections*

$$\mathsf{Hom}_{\mathbb{T}_1(e)}(g_1, g_2) \to \mathsf{Hom}_{\mathbb{T}_2(e)}(Fg_1, Fg_2), \quad \forall g_1, g_2 \in \mathscr{G};$$

(b) *the set of objects $\{Fg \mid g \in \mathscr{G}\}$ is a set of compact generators in $\mathbb{T}_2(e)$.*

Then the morphism F is an equivalence of derivators.

Proof

Conditions (a) and (b) imply that $F(e)$ is an equivalence of triangulated categories (see [32, Section 4]).

Now, let L be a small category. We show that conditions (a) and (b) are also verified by $F(L)$, $\mathbb{T}_1(L)$, and $\mathbb{T}_2(L)$. By Lemma 8.3, the sets

$$\left\{x_!(g) \,\middle|\, g \in \mathscr{G}, \quad x : e \to L\right\} \qquad \text{and} \qquad \left\{x_!(Fg) \,\middle|\, g \in \mathscr{G}, \quad x : e \to L\right\}$$

consist of compact generators for $\mathbb{T}_1(L)$ (resp., $\mathbb{T}_2(L)$) which are stable under suspensions. Since F commutes with homotopy colimits $F(x_!(g)) = x_!(Fg)$, the isomorphisms

$$\begin{aligned}\mathsf{Hom}_{\mathbb{T}_1(L)}\big(x_!(g_1), x_!(g_2)\big) &\simeq \mathsf{Hom}_{\mathbb{T}_1(e)}\big(g_1, x^*x_!(g_2)\big)\\ &\simeq \mathsf{Hom}_{\mathbb{T}_2(e)}\big(F(g_1), F(x^*x_!(g_2))\big)\\ &\simeq \mathsf{Hom}_{\mathbb{T}_2(e)}\big(Fg_1, x^*F(x_!(g_2))\big)\\ &\simeq \mathsf{Hom}_{\mathbb{T}_2(L)}\big(x_!F(g_1), x_!F(g_2)\big)\end{aligned}$$

imply the proposition. □

Let us now prove Theorem 8.7.

Proof of Theorem 8.7

Let us first prove condition (b) of Proposition 8.8. Since the set $\mathscr{G}$ of small generators in $\mathsf{Ho}(\mathscr{M})$ satisfies the conditions of Lemma 8.2, we have a set

$$\left\{\Sigma^n \mathrm{stab}(g) \,\middle|\, g \in \mathscr{G}, \ n \in \mathbb{Z}\right\}$$

of compact generators in $\mathsf{St}(\mathsf{HO}(\mathscr{M}))(e)$ which is stable under suspensions. We now show that the set

$$\left\{\Sigma^n \mathbb{L}\Sigma^\infty(g) \,\middle|\, g \in \mathscr{G},\, n \in \mathbb{Z}\right\}$$

is a set of compact generators in $\mathsf{Ho}(\mathsf{Sp}^{\mathbb{N}}(\mathscr{M}))$. These objects are compact because the functor $\mathbb{R}\,\mathrm{ev}_0$ in the adjunction

$$\begin{array}{c}\mathsf{Ho}(\mathscr{M})\\ \mathbb{L}\Sigma^\infty \big\downarrow \big\uparrow \mathbb{R}\,\mathrm{ev}_0\\ \mathsf{Ho}\big(\mathsf{Sp}^{\mathbb{N}}(\mathscr{M})\big)\end{array}$$

commutes with filtered homotopy colimits. We now show that they form a set of generators. Let Y be an object in $\mathsf{Ho}(\mathsf{Sp}^{\mathbb{N}}(\mathscr{M}))$ which we can suppose, without loss of generality, to be an Ω-spectrum (see [17, Definition 3.1]). Suppose that

$$\mathsf{Hom}\big(\Sigma^n \mathbb{L}\Sigma^\infty(g_i), Y\big) \simeq \operatorname*{colim}_m \mathsf{Hom}(g_i, \Omega^m Y_{m+p}) = \{*\}, \quad n \geq 0.$$

Since Y is an Ω-spectrum, we have

$$Y_p = *, \quad \forall p \geq 0.$$

This implies that Y is isomorphic to $*$ in $\mathsf{Ho}(\mathsf{Sp}^{\mathbb{N}}(\mathscr{M}))$.

We now show condition (a). Let g_1 and g_2 be objects in $\mathscr{G}$. Observe that we have the following isomorphisms (see [14, Proposition 8.2]):

$$\begin{aligned}
&\mathsf{Hom}_{\mathsf{St}(\mathsf{HO}(\mathscr{M}))(e)}\big(\mathrm{stab}(g_1), \mathrm{stab}(g_2)\big)\\
&\simeq \mathsf{Hom}_{\mathsf{Ho}(\mathscr{M})}\big(g_1, (\mathrm{ev}_{(0,0)} \circ \mathrm{loc} \circ L[0,0])(g_2)\big)\\
&\simeq \mathsf{Hom}_{\mathsf{Ho}(\mathscr{M})}\big(g_1, \mathrm{ev}_{(0,0)}(\mathrm{hocolim}\,(L[0,0](g_2) \to \Omega\sigma L[0,0](g_2) \to \cdots))\big)\\
&\simeq \mathsf{Hom}_{\mathsf{Ho}(\mathscr{M})}\big(g_1, \mathrm{hocolim}\,\mathrm{ev}_{(0,0)}(L[0,0](g_2) \to \Omega\sigma L[0,0](g_2) \to \cdots)\big)\\
&\simeq \operatorname*{colim}_j \mathsf{Hom}_{\mathsf{Ho}(\mathscr{M})}\big(g_1, \Omega^j \Sigma^j(g_2)\big).
\end{aligned}$$

Now, by [17, Corollary 4.13], we have

$$\begin{aligned}
\mathsf{Hom}_{\mathsf{Ho}(\mathsf{Sp}^{\mathbb{N}}(\mathscr{M}))}\big(\mathbb{L}\Sigma^\infty(g_1), \mathbb{L}\Sigma^\infty(g_2)\big) &\simeq \mathsf{Hom}_{\mathsf{Ho}(\mathsf{Sp}^{\mathbb{N}}(\mathscr{M}))}\big(\Sigma^\infty(g_1), (\Sigma^\infty(g_2))_f\big)\\
&\simeq \operatorname*{colim}_m \mathsf{Hom}_{\mathsf{Ho}(\mathscr{M})}\big(g_1, \Omega^m(\Sigma^m(g_2))_f\big),
\end{aligned}$$

where $(\Sigma^\infty(g_2))_f$ denotes a levelwise fibrant resolution of $\Sigma^\infty(g_2)$ in the category $\mathsf{Sp}^{\mathbb{N}}(\mathscr{M})$.

Now, note that since g_2 is cofibrant, so is $\Sigma^m(g_2)$, and so we have the isomorphism

$$\Omega^m\big(\Sigma^m(g_2)\big)_f \xrightarrow{\sim} (\mathbb{R}\Omega)^m \circ (\mathbb{L}\Sigma)^m(g_2)$$

in $\mathsf{Ho}(\mathsf{SP}^{\mathbb{N}}(\mathscr{M}))$. This implies that for $j \geq 0$, we have an isomorphism

$$\Omega^j\Sigma^j(g_2) \xrightarrow{\sim} \Omega^j\big(\Sigma^j(g_2)\big)_f$$

in $\mathsf{Ho}(\mathsf{Sp}^{\mathbb{N}}(\mathscr{M}))$, and so

$$\mathsf{Hom}_{\mathsf{St}(\mathsf{HO}(\mathscr{M}))(e)}\big(\mathrm{stab}(g_1), \mathrm{stab}(g_2)\big) = \mathsf{Hom}_{\mathsf{Ho}(\mathsf{Sp}^{\mathbb{N}}(\mathscr{M}))}\big(\mathbb{L}\Sigma^\infty(g_1), \mathbb{L}\Sigma^\infty(g_2)\big).$$

Now, let p be an integer. Note that

$$\mathsf{Hom}_{\mathsf{St}(\mathsf{HO}(\mathscr{M}))(e)}\big(\mathrm{stab}(g_1), \Sigma^p\mathrm{stab}(g_2)\big) = \operatorname*{colim}_j \mathsf{Hom}_{\mathsf{Ho}(\mathscr{M})}\big(g_1, \Omega^j\Sigma^{j+p}(g_2)\big),$$

and note that

$$\mathsf{Hom}_{\mathsf{Ho}(\mathsf{Sp}^{\mathbb{N}}(\mathscr{M}))}\big(\mathbb{L}\Sigma^\infty(g_1), \Sigma^p\mathbb{L}\Sigma^\infty(g_2)\big) = \operatorname*{colim}_m \mathsf{Hom}_{\mathsf{Ho}(\mathscr{M})}\big(g_1, \Omega^m(\Sigma^{m+p}(g_2))_f\big).$$

This proves condition (a), and so the theorem is proved. □

Remark 8.9
If we consider, for $\mathscr{M}$, the category $\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)$, we have equivalences of derivators

$$\varphi : \mathsf{St}(\mathsf{L}_{\Sigma,P}\mathsf{Hot}_{\mathscr{M}_{f_\bullet}}) \xrightarrow{\sim} \mathsf{HO}\big(\mathsf{Sp}^{\mathbb{N}}(\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet))\big) \xleftarrow{\sim} \mathsf{St}(\mathsf{L}_{\Sigma,P}\mathsf{Hot}_{\mathscr{M}_f}).$$

Let $\mathbb{D}$ be a strong triangulated derivator.

Now, by Theorem 8.1 and Proposition 6.1, we have the following proposition.

PROPOSITION 8.10
We have an equivalence of categories

$$\underline{\mathsf{Hom}}_!\big(\mathsf{St}(\mathsf{L}_{\Sigma,P}\mathsf{Hot}_{\mathsf{dgcat}_f}), \mathbb{D}\big) \xrightarrow{(\mathrm{stab}\circ\Phi\circ\mathbb{R}\underline{h})^*} \underline{\mathsf{Hom}}_{\mathsf{flt},p}\big(\mathsf{HO}(\mathsf{dgcat}), \mathbb{D}\big).$$

Since the category $\mathrm{Sset}_\bullet$ satisfies all the conditions of Theorem 8.7, we have, after Bousfield and Friedlander [2, Definition 2.4], the following characterization of the classical category of spectra, by a universal property.

PROPOSITION 8.11
We have an equivalence of categories

$$\underline{\mathsf{Hom}}_!\big(\mathsf{HO}(\mathsf{Sp}^{\mathbb{N}}(\mathrm{Sset}_\bullet)), \mathbb{D}\big) \xrightarrow{\sim} \mathbb{D}(e).$$

Proof
By Theorems 8.7 and 3.1, we have the following equivalences:

$$\begin{aligned}\underline{\mathsf{Hom}}_!\big(\mathsf{HO}(\mathsf{Sp}^{\mathbb{N}}(\mathrm{Sset}_\bullet)), \mathbb{D}\big) &\simeq \underline{\mathsf{Hom}}_!\big(\mathsf{HO}(\mathrm{Sset}_\bullet), \mathbb{D}\big)\\ &= \underline{\mathsf{Hom}}_!(\mathsf{Hot}_\bullet, \mathbb{D})\\ &\simeq \underline{\mathsf{Hom}}_!(\mathsf{Hot}, \mathbb{D})\\ &\simeq \mathbb{D}(e).\end{aligned}$$

This proves the proposition. □

Remark 8.12
An analogous characterization of the category of spectra, but in the context of stable ∞-categories, is proved in [27, Corollary 17.6].

9. dg quotients

Recall from [37] that we have at our disposal a Morita Quillen model structure on the category of small dg categories dgcat (see Example 5.1). As shown in [37], the homotopy category $\mathsf{Ho}(\mathsf{dgcat})$ is pointed. In the following, we consider this Quillen model structure. We denote by I its set of generating cofibrations.

Notation 9.1
We denote by $\mathscr{E}$ the set of inclusions of full dg subcategories

$$\mathscr{G} \hookrightarrow \mathscr{H},$$

where $\mathscr{H}$ is a strict finite I-cell.

Recall that we have a morphism of derivators

$$\mathscr{U}_t := (\mathrm{stab}) \circ \Phi \circ \mathbb{R}\underline{h} : \mathsf{HO}(\mathsf{dgcat}) \longrightarrow \mathsf{St}(\mathsf{L}_{\Sigma,P}\mathsf{Hot}_{\mathsf{dgcat}_f})$$

which commutes with filtered homotopy colimits and preserves the point.

Let us now make some general arguments.

Let $\mathbb{D}$ be a pointed derivator. We denote by M the category associated to the graph

$$0 \leftarrow 1.$$

Consider the functor $t = 1 : e \to M$. Since the functor t is an open immersion (see Notation 2.8) and the derivator $\mathbb{D}$ is pointed, the functor

$$t_! : \mathbb{D}(e) \to \mathbb{D}(M)$$

has a left adjoint

$$t^{?} : \mathbb{D}(M) \to \mathbb{D}(e)$$

(see [7, Definition 1.13]). We denote it by

$$\mathsf{cone} : \mathbb{D}(M) \to \mathbb{D}(e).$$

Let $F : \mathbb{D} \to \mathbb{D}'$ be a morphism of pointed derivators. Note that we have a natural transformation of functors

$$S : \mathsf{cone} \circ F(M) \to F(e) \circ \mathsf{cone}.$$

PROPOSITION 9.2

Let $\mathscr{A} \overset{R}{\hookrightarrow} \mathscr{B}$ be an inclusion of a full dg subcategory, and let $\ulcorner_R$

$$\begin{array}{ccc} \mathscr{A} & \overset{R}{\hookrightarrow} & \mathscr{B} \\ \downarrow & & \\ 0 & & \end{array}$$

be the associated object in $\mathsf{HO}(\mathsf{dgcat})(\ulcorner)$, where 0 denotes the terminal object in $\mathsf{Ho}(\mathsf{dgcat})$. Then there exist a filtered category J and an object D_R in $\mathsf{HO}(\mathsf{dgcat})(\ulcorner \times J)$ such that

$$p_!(D_R) \overset{\sim}{\longrightarrow} \ulcorner_R,$$

where $p : \ulcorner \times J \to \ulcorner$ denotes the projection functor. Moreover, for every point $j : e \to J$ in J, the object $(1 \times j)^*$ in $\mathsf{HO}(\mathsf{dgcat})(\ulcorner)$ is of the form

$$0 \leftarrow Y_j \overset{L_j}{\to} X_j,$$

where $Y_j \overset{L_j}{\to} X_j$ belongs to the set $\mathscr{E}$.

Proof

Apply the small-object argument to the morphism

$$\emptyset \longrightarrow \mathscr{B}$$

using the set of generating cofibrations I, and obtain the factorization

$$\begin{array}{ccc} \emptyset & \longrightarrow & \mathscr{B} \\ & {\scriptstyle i}\searrow \quad \nearrow{\scriptstyle p}\ {\scriptstyle\sim} & \\ & Q(\mathscr{B}) & \end{array}$$

where i is an I-cell. Now, consider the following fiber product:

$$\begin{array}{ccc} p^{-1}(\mathscr{A}) & \hookrightarrow & Q(\mathscr{B}) \\ {\scriptstyle\sim}\downarrow & \ulcorner & \downarrow{\scriptstyle\sim}\ {\scriptstyle p} \\ \mathscr{A} & \overset{J}{\hookrightarrow} & \mathscr{B} \end{array}$$

Note that $p^{-1}(\mathscr{A})$ is a full dg subcategory of $Q(\mathscr{B})$.

Now, by Proposition 5.2, we have an isomorphism

$$\operatorname*{colim}_{j \in J} X_j \xrightarrow{\sim} Q(\mathscr{B}),$$

where J is the filtered category of inclusions of strict finite sub-I-cells X_j into $Q(\mathscr{B})$.

For each $j \in J$, consider the fiber product

$$\begin{array}{ccc} Y_j & \hookrightarrow & X_j \\ \downarrow & \ulcorner & \downarrow \\ p^{-1}(\mathscr{A}) & \hookrightarrow & Q(\mathscr{B}) \end{array}$$

In this way, we obtain a morphism of diagrams

$$\{Y_j\}_{j \in J} \hookrightarrow \{X_j\}_{j \in J}$$

such that for each j in J, the inclusion

$$Y_j \hookrightarrow X_j$$

belongs to the set $\mathscr{E}$ and J is filtered.

Consider now the diagram D_I,

$$\{0 \leftarrow Y_j \hookrightarrow X_j\}_{j \in J},$$

in the category $\mathsf{Fun}(\ulcorner \times J, \mathsf{dgcat})$. Now, note that we have the isomorphism

$$\operatorname*{colim}_{j\in J}\{0 \leftarrow Y_j \hookrightarrow X_j\} \xrightarrow{\sim} \left\{0 \leftarrow p^{-1}(\mathscr{A}) \hookrightarrow Q(\mathscr{B})\right\}$$

in $\mathsf{Fun}(\ulcorner, \mathsf{dgcat})$ and the weak equivalence

$$\begin{array}{ccccc} \{0 & \longleftarrow & p^{-1}(\mathscr{A}) & \hookrightarrow & Q(\mathscr{B})\} \\ \| & & \downarrow \sim & & \downarrow \sim \\ \{0 & \longleftarrow & \mathscr{A} & \hookrightarrow & \mathscr{B}\} \end{array}$$

in $\mathsf{Fun}(\ulcorner, \mathsf{dgcat})$ when endowed with the projective model structure (see [15, Theorem 11.6.1]). Since $\mathsf{Fun}(\ulcorner, \mathsf{dgcat})$ is clearly also compactly generated, we have an isomorphism

$$\operatorname*{hocolim}_{j\in J}(0 \leftarrow Y_j \rightarrow X_j) \xrightarrow{\sim} \operatorname*{colim}_{j\in J}(0 \leftarrow Y_j \rightarrow X_j).$$

Finally, note that D_R is an object of $\mathsf{HO}(\mathsf{dgcat})(\ulcorner \times J)$, and note that $p_!(D_R)$, where $p : \ulcorner \times J \rightarrow J$ denotes the projection functor, identifies with

$$\operatorname*{hocolim}_{i\in J}(0 \leftarrow Y_j \rightarrow X_j).$$

This proves the proposition. □

Notation 9.3
We denote by $\mathscr{E}_{\mathrm{st}}$ the set of morphisms S_L, where L belongs to the set $\mathscr{E}$.

Let $\mathbb{D}$ be a strong triangulated derivator.

THEOREM 9.4
If

$$G : \mathsf{St}(\mathsf{L}_{\Sigma,P}\mathsf{Hot}_{\mathsf{dgcat}_f}) \rightarrow \mathbb{D}$$

is a morphism of triangulated derivators commuting with arbitrary homotopy colimits and is such that $G(e)(S_L)$ is invertible for each L in $\mathscr{E}$, then $G(e)(S_K)$ is invertible for each inclusion $K : \mathscr{A} \hookrightarrow \mathscr{B}$ of a full dg subcategory.

Proof
Let $\mathscr{A} \overset{K}{\hookrightarrow} \mathscr{B}$ be an inclusion of a full dg subcategory. Consider the morphism

$$\varphi_K := \varphi(\ulcorner_K) : (i_! \circ \mathscr{U}_T)(\ulcorner_K) \longrightarrow (\mathscr{U}_T \circ i_!)(\ulcorner_K)$$

in $\mathsf{St}(\mathsf{L}_{\Sigma,P}\mathsf{Hot}_{\mathsf{dgcat}_f})(\square)$.

Let D_K be the object of $\mathsf{HO}(\mathsf{dgcat})(\ulcorner \times J)$ constructed in Proposition 9.2. In particular, $p'_!(D_K) \xrightarrow{\sim} \ulcorner_K$, where $p' : \ulcorner \times J \to \ulcorner$ denotes the projection functor.

The inclusion $i : \ulcorner \hookrightarrow \square$ induces a commutative square

$$\begin{array}{ccc}
\mathsf{HO}(\mathsf{dgcat})(\square \times J) & \xrightarrow{\mathscr{U}_T(\square \times J)} & \mathsf{St}(\mathsf{L}_{\Sigma,P}\mathsf{Hot}_{\mathsf{dgcat}_f})(\square \times J) \\
\downarrow{\scriptstyle (i\times 1)^*} & & \downarrow{\scriptstyle (i\times 1)^*} \\
\mathsf{HO}(\mathsf{dgcat})(\ulcorner \times J) & \xrightarrow[\mathscr{U}_T(\square \times J)]{} & \mathsf{St}(\mathsf{L}_{\Sigma,P}\mathsf{Hot}_{\mathsf{dgcat}_f})(\ulcorner \times J)
\end{array}$$

and a morphism

$$\Psi : \big((i \times 1)_! \circ \mathscr{U}_T(\ulcorner \times J)\big)(D_K) \longrightarrow \big(\mathscr{U}_T(\square \times J) \circ (i \times 1)_!\big)(D_K).$$

We now show that

$$p_! \Psi \xrightarrow{\sim} \varphi_K,$$

where $p : \square \times J \to \square$ denotes the projection functor.

The fact that we have the commutative square

$$\begin{array}{ccc}
\square & \xleftarrow{p} & \square \times J \\
\uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i\times 1} \\
\ulcorner & \xleftarrow[p']{} & \ulcorner \times J
\end{array}$$

and the fact that the morphism of derivators $\mathscr{U}_T$ commutes with filtered homotopy colimits imply the equivalences

$$\begin{aligned}
&p_! \Psi \\
&\quad = p_! \circ (i \times 1)_! \circ \mathscr{U}_T(\ulcorner \times j)(D_K) \longrightarrow p_! \circ \mathscr{U}_T(\square \times J) \circ (i \times 1)_!(D_k) \\
&\quad \simeq i_! \circ p'_! \circ \mathscr{U}_T(\ulcorner \times J)(D_k) \longrightarrow \mathscr{U}_T(\square \times J) \circ p_! \circ (i \times 1)_!(D_K) \\
&\quad \simeq i_! \circ \mathscr{U}_T(\ulcorner) \circ p'_!(D_K) \longrightarrow \mathscr{U}_T(\square) \circ i_! \circ p'_!(D_K) \\
&\quad \simeq \big(i_! \circ \mathscr{U}_T(\ulcorner)\big)(\ulcorner_K) \longrightarrow \big(\mathscr{U}_T(\square) \circ i_!\big)(\ulcorner_K) \\
&\quad = \varphi_J .
\end{aligned}$$

This shows that

$$p_!(\Psi) \xrightarrow{\sim} \varphi_K .$$

We now show that Ψ is an isomorphism. For this, by conservativity, it is enough to show that for every object $j : e \to J$ in J, the morphism

$$(1 \times j)^*(\Psi)$$

is an isomorphism in $\mathsf{St}(\mathsf{L}_{\Sigma,P}\mathsf{Hot}_{\mathsf{dgcat}_f})(\square)$. Recall from Proposition 9.2 that $(1 \times j)^*(D_K)$ identifies with

$$\{0 \leftarrow Y_j \overset{L_j}{\hookrightarrow} X_j\},$$

where L_j belongs to $\mathscr{E}$. We now show that $(1 \times j)^*(\Psi)$ identifies with φ_{L_j}, which, by hypothesis, is an isomorphism.

Now, the commutative diagram

$$\begin{array}{ccc} \square & \xrightarrow{1\times j} & \square \times J \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i\times i} \\ \ulcorner & \xrightarrow[1\times j]{} & \ulcorner \times J \end{array}$$

and the dual of [4, proposition 2.8] imply that we have the following equivalences:

$$\begin{aligned} &(1\times j)^*\Psi \\ &\quad = \big((1 \times j)^* \circ (i \times 1)_! \circ \mathscr{U}_T(\ulcorner \times J)\big)(D_K) \to \big((1 \times j)^* \circ \mathscr{U}_T(\square \times J) \circ (i \times 1)_!\big)(D_K) \\ &\quad \simeq \big(i_! \circ (1 \times j)^* \circ \mathscr{U}_T(\ulcorner \times J)\big)(D_K) \longrightarrow \big(\mathscr{U}_T(\square \times J) \circ (1 \times j)^* \circ (i \times 1)_!\big)(D_K) \\ &\quad \simeq \big(i_! \circ \mathscr{U}_T(\ulcorner) \circ (1 \times j)^*\big)(D_K) \longrightarrow \big(\mathscr{U}_T(\square) \circ i_! \circ (1 \times j)^*\big)(D_K) \\ &\quad \simeq i_! \circ \mathscr{U}_T(\ulcorner)(\ulcorner_{L_j}) \longrightarrow \mathscr{U}_T(\square) \circ i_!(\ulcorner_{L_j}) \\ &\quad = \varphi_{L_j}. \end{aligned}$$

Since, by hypothesis, φ_{L_j} is an isomorphism and the morphism G commutes with homotopy colimits, the theorem is proved. $\square$

10. The universal localizing invariant

Recall from Theorem 8.7 and Remark 8.9 that if we consider for the category $\mathscr{M}$ the category $\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)$ (see Example 8.6), then we have an equivalence of triangulated derivators

$$\varphi : \mathsf{St}(\mathsf{L}_{\Sigma,P}\mathsf{Hot}_{\mathsf{dgcat}_f}) \xrightarrow{\sim} \mathsf{HO}\big(\mathsf{Sp}^{\mathbb{N}}(\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet))\big).$$

Now, stabilize the set $\mathscr{E}_{\mathrm{st}}$ defined in Section 9 under the functor loop space, and choose for each element of this stabilized set a representative in the category

$\mathsf{Sp}^{\mathbb{N}}(\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet))$. We denote the set of these representatives by $\widetilde{\mathscr{E}_{\mathrm{st}}}$. Since $\mathsf{Sp}^{\mathbb{N}}(\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet))$ is a left proper, cellular Quillen model category (see [17, Definition 12.1.1]), its left Bousfield localization by $\widetilde{\mathscr{E}_{\mathrm{st}}}$ exists. We denote it by $\mathsf{L}_{\widetilde{\mathscr{E}_{\mathrm{st}}}}\mathsf{Sp}^{\mathbb{N}}(\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet))$. By Lemma 4.3, it is a stable Quillen model category.

Remark 10.1
Since the localization morphism

$$\gamma : \mathsf{St}(\mathsf{L}_{\Sigma,P}\mathsf{Hot}_{\mathsf{dgcat}_f}) \xrightarrow{\mathbb{L}\,\mathrm{Id}} \mathsf{HO}\big(\mathsf{L}_{\widetilde{\mathscr{E}_{\mathrm{st}}}}\mathsf{Sp}^{\mathbb{N}}(\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet))\big)$$

commutes with homotopy colimits and inverts the set of morphisms $\mathscr{E}_{\mathrm{st}}$, Theorem 9.4 allows us to conclude that it inverts all morphisms S_K for each inclusion $\mathscr{A} \hookrightarrow \mathscr{B}$ of a full dg subcategory.

Definition 10.2

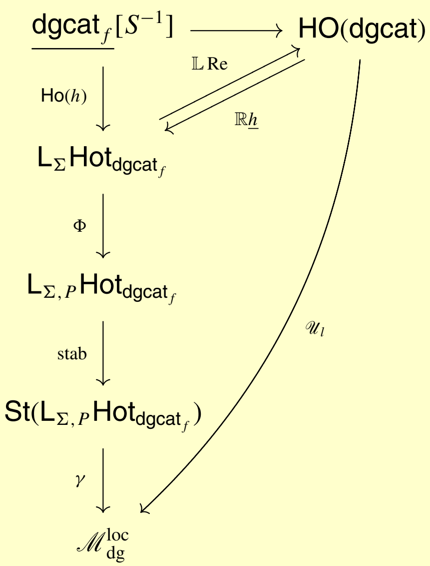
(i) The *localizing motivator* of dg categories $\mathscr{M}_{\mathrm{dg}}^{\mathrm{loc}}$ is the triangulated derivator associated with the stable Quillen model category

$$\mathsf{L}_{\widetilde{\mathscr{E}_{\mathrm{st}}}}\mathsf{Sp}^{\mathbb{N}}\big(\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)\big).$$

(ii) The *universal localizing invariant* of dg categories is the canonical morphism of derivators

$$\mathscr{U}_l : \mathsf{HO}(\mathsf{dgcat}) \to \mathscr{M}_{\mathrm{dg}}^{\mathrm{loc}}.$$

We sum up the construction of $\mathscr{M}_{\mathrm{dg}}^{\mathrm{loc}}$ in the following diagram:



Observe that the morphism of derivators $\mathscr{U}_l$ is pointed, commutes with filtered homotopy colimits, and satisfies the following condition:

(Dr) for every inclusion $\mathscr{A} \overset{K}{\hookrightarrow} \mathscr{B}$ of a full dg subcategory, the canonical morphism

$$S_K : \mathsf{cone}\big(\mathscr{U}_l(\mathscr{A} \overset{K}{\hookrightarrow} \mathscr{B})\big) \to \mathscr{U}_l(\mathscr{B}/\mathscr{A})$$

is invertible in $\mathscr{M}^{\mathrm{loc}}_{\mathrm{dg}}(e)$.

We now give a conceptual characterization of condition (Dr). Let us now denote by I the category associated with the graph $0 \leftarrow 1$.

LEMMA 10.3

The isomorphism classes in $\mathsf{HO}(\mathsf{dgcat})(I)$ *associated with the inclusions* $\mathscr{A} \overset{K}{\hookrightarrow} \mathscr{B}$ *of full dg subcategories coincide with the classes of homotopy monomorphisms in* dgcat *(see [40, Section 2]).*

Proof

Recall from [40, Section 2] that in a model category $\mathscr{M}$, a morphism $X \overset{f}{\to} Y$ is a homotopy monomorphism if, for every object Z in $\mathscr{M}$, the induced morphism of simplicial sets

$$\mathsf{Map}(Z, X) \overset{f_*}{\to} \mathsf{Map}(Z, Y)$$

induces an injection on π_0 and isomorphisms on all π_i for $i > 0$ (for all base points).

Now, by [40, Lemma 2.4], a dg functor $\mathscr{A} \overset{F}{\to} \mathscr{B}$ is a homotopy monomorphism on the quasi-equivalent Quillen model category in dgcat if and only if it is *quasi-fully faithful*; that is, for any two objects X and Y in $\mathscr{A}$, the morphism of complexes $\mathsf{Hom}_{\mathscr{A}}(X, Y) \to \mathsf{Hom}_{\mathscr{B}}(FX, FY)$ is a quasi isomorphism.

Recall that by [37, corollaire 5.10], the mapping space functor $\mathsf{Map}(\mathscr{A}, \mathscr{B})$ in the Morita Quillen model category identifies with the mapping space $\mathsf{Map}(\mathscr{A}, \mathscr{B}_f)$ in the quasi-equivalent Quillen model category, where $\mathscr{B}_f$ denotes a Morita fibrant resolution of $\mathscr{B}$. This implies that a dg functor $\mathscr{A} \overset{F}{\to} \mathscr{B}$ is a homotopy monomorphism if and only if $\mathscr{A}_f \overset{F_f}{\to} \mathscr{B}_f$ is a quasi–fully faithful dg functor.

Now, note that an inclusion $\mathscr{A} \hookrightarrow \mathscr{B}$ of a full dg subcategory is a homotopy monomorphism. Conversely, let $\mathscr{A} \overset{F}{\to} \mathscr{B}$ be a homotopy monomorphism. Consider the diagram

$$\begin{array}{ccc} \widetilde{\mathscr{A}}_f & \hookrightarrow & \mathscr{B}_f \\ {\scriptstyle \pi}\uparrow & & \| \\ \mathscr{A}_f & \xrightarrow{F_f} & \mathscr{B}_f \\ {\scriptstyle \sim}\uparrow & & \uparrow{\scriptstyle \sim} \\ \mathscr{A} & \xrightarrow[F]{} & \mathscr{B} \end{array}$$

where $\widetilde{\mathscr{A}}_f$ denotes the full dg subcategory of $\mathscr{B}_f$ whose objects are those in the image by the dg functor F_f. Since F_f is a quasi–fully faithful dg functor, the dg functor π is a quasi equivalence. This proves the lemma. □

Remark 10.4

Lemma 10.3 shows that condition (Dr) is equivalent to the following:

(Dr′) for every homotopy monomorphism $\mathscr{A} \xrightarrow{F} \mathscr{B}$ in $\mathsf{HO}(\mathsf{dgcat})(I)$, the canonical morphism

$$\mathsf{cone}\big(\mathscr{U}_l(\mathscr{A} \xrightarrow{F} \mathscr{B})\big) \to \mathscr{U}_l\big(\mathsf{cone}(F)\big)$$

is invertible in $\mathscr{M}^{\mathrm{loc}}_{\mathrm{dg}}(e)$.

Let $\mathbb{D}$ be a strong triangulated derivator.

THEOREM 10.5

The morphism $\mathscr{U}_l$ induces an equivalence of categories

$$\underline{\mathsf{Hom}}_!(\mathscr{M}^{\mathrm{loc}}_{\mathrm{dg}}, \mathbb{D}) \xrightarrow{\mathscr{U}_l^*} \underline{\mathsf{Hom}}_{\mathrm{flt,\,Dr,}\,p}\big(\mathsf{HO}(\mathsf{dgcat}), \mathbb{D}\big),$$

where $\underline{\mathsf{Hom}}_{\mathrm{flt,\,Dr,}\,p}(\mathsf{HO}(\mathsf{dgcat}), \mathbb{D})$ denotes the category of morphisms of derivators which commute with filtered homotopy colimits, satisfy condition (Dr), and preserve the point.

Proof

By Theorem 4.4, we have the following equivalence of categories:

$$\underline{\mathsf{Hom}}_!(\mathscr{M}^{\mathrm{loc}}_{\mathrm{dg}}\mathbb{D}) \xrightarrow{\gamma^*} \underline{\mathsf{Hom}}_{!,\widetilde{\mathscr{E}}_{\mathrm{st}}}\big(\mathsf{St}(\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)), \mathbb{D}\big).$$

We now show that we have the following equivalence of categories:

$$\begin{aligned}&\underline{\mathsf{Hom}}_{!,\widetilde{\mathscr{E}_{\mathrm{st}}}}\big(\mathsf{St}(\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}},\mathrm{Sset}_\bullet)),\mathbb{D}\big)\\ &\quad\stackrel{\sim}{\to}\underline{\mathsf{Hom}}_{!,\mathscr{E}_{\mathrm{st}}}\big(\mathsf{St}(\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}},\mathrm{Sset}_\bullet)),\mathbb{D}\big).\end{aligned}$$

Let G be an element of $\underline{\mathsf{Hom}}_{!,\mathscr{E}_{\mathrm{st}}}\big(\mathsf{St}(\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}},\mathrm{Sset}_\bullet)),\mathbb{D}\big)$, and let s be an element of $\mathscr{E}_{\mathrm{st}}$. We show that the image of s under the functor $G(e)\circ\Omega(e)$ is an isomorphism in $\mathbb{D}(e)$. Recall from the proof of Lemma 4.3 that the functor $G(e)$ commutes with $\Sigma(e)$. Since the suspension and loop-space functors in $\mathbb{D}(e)$ are inverse from each other, we conclude that the image of s under the functor $G(e)\circ\Omega(e)$ is an isomorphism in $\mathbb{D}(e)$. Now, simply observe that the category on the right-hand side of the above equivalence identifies with $\underline{\mathsf{Hom}}_{\mathrm{flt},\,\mathrm{Dr},\,p}(\mathsf{HO}(\mathsf{dgcat}),\mathbb{D})$ under the equivalence

$$\underline{\mathsf{Hom}}_{!}\big(\mathsf{St}(\mathsf{L}_{\Sigma,P}\mathsf{Hot}_{\mathsf{dgcat}_f}),\mathbb{D}\big)\xrightarrow{(\mathrm{stab}\circ\Phi\circ\mathbb{R}\underline{h})^*}\underline{\mathsf{Hom}}_{\mathrm{flt},\,p}\big(\mathsf{HO}(\mathsf{dgcat}),\mathbb{D}\big)$$

of Proposition 8.10.

This proves the theorem. □

Notation 10.6

We call an object of the right-hand-side category of Theorem 10.5 a *localizing invariant* of dg categories.

We now present some examples.

Hochschild and cyclic homology

Let $\mathscr{A}$ be a small k-flat k-category. The *Hochschild chain complex* of $\mathscr{A}$ is the complex concentrated in homological degrees $p\geq 0$ whose pth component is the sum of the

$$\mathscr{A}(X_p,X_0)\otimes\mathscr{A}(X_p,X_{p-1})\otimes\mathscr{A}(X_{p-1},X_{p-2})\otimes\cdots\otimes\mathscr{A}(X_0,X_1),$$

where $X_0,\ldots,X_p$ range through the objects of $\mathscr{A}$, endowed with the differential

$$d(f_p\otimes\cdots\otimes f_0)=f_{p-1}\otimes\cdots\otimes f_0f_p+\sum_{i=1}^{p}(-1)^i f_p\otimes\cdots\otimes f_if_{i-1}\otimes\cdots\otimes f_0.$$

Via the cyclic permutations

$$t_p(f_{p-1}\otimes\cdots\otimes f_0)=(-1)^p f_0\otimes f_{p-1}\otimes\cdots\otimes f_1,$$

this complex becomes a precyclic chain complex and thus gives rise to a *mixed complex* $C(\mathscr{A})$, that is, a dg module over the dg algebra $\Lambda=k[B]/(B^2)$, where B is of degree

-1 and $dB = 0$. All variants of cyclic homology only depend on $C(\mathscr{A})$ considered in $\mathscr{D}(\Lambda)$. For example, the cyclic homology of $\mathscr{A}$ is the homology of the complex $C(\mathscr{A}) \overset{\mathbb{L}}{\otimes}_{\Lambda} k$ (see [18, Theorem 1.3]).

If $\mathscr{A}$ is a k-flat differential graded category, its mixed complex is the sum-total complex of the bicomplex obtained as the natural reinterpretation of the above complex. If $\mathscr{A}$ is an arbitrary dg k-category, its Hochschild chain complex is defined as that of a k-flat (e.g., a cofibrant) resolution of $\mathscr{A}$. The following theorem is proved in [20, Theorem 1.5].

THEOREM 10.7
The map $\mathscr{A} \mapsto C(\mathscr{A})$ yields a morphism of derivators

$$\mathsf{HO}(\mathsf{dgcat}) \to \mathsf{HO}(\Lambda - \mathrm{Mod}),$$

which commutes with filtered homotopy colimits, preserves the point, and satisfies condition (Dr).

Remark 10.8
By Theorem 10.5, the morphism of derivators C factors through $\mathscr{U}_l$ and so gives rise to a morphism of derivators

$$C : \mathscr{M}^{\mathrm{loc}}_{\mathrm{dg}} \to \mathsf{HO}(\Lambda - \mathrm{Mod}).$$

Nonconnective K-theory
Let $\mathscr{A}$ be a small dg category. Its nonconnective K-theory spectrum $K(A)$ is defined by applying Schlichting's construction (see [35, Theorem 3]) to the Frobenius pair associated with the category of cofibrant perfect $\mathscr{A}$-modules. (To the empty dg category, we associate zero.) Recall that the conflations in the Frobenius category of cofibrant perfect $\mathscr{A}$-modules are the short exact sequences that split in the category of graded $\mathscr{A}$-modules.

THEOREM 10.9
The map $\mathscr{A} \mapsto K(\mathscr{A})$ yields a morphism of derivators

$$\mathsf{HO}(\mathsf{dgcat}) \to \mathsf{HO}(\mathrm{Spt})$$

to the derivator associated with the category of spectra which commutes with filtered homotopy colimits, preserves the point, and satisfies condition (Dr).

Proof
Note that [35, Proposition 11.15], which is an adaptation of [39, Theorem 1.9.8], implies that we have a well-defined morphism of derivators

$$\mathsf{HO}(\mathsf{dgcat}) \to \mathsf{HO}(\mathrm{Spt}).$$

Also, [35, Lemma 6.3] implies that this morphism commutes with filtered homotopy colimits, and [35, Theorem 11.10] implies that condition (Dr) is satisfied. □

Remark 10.10
By Theorem 10.5, the morphism of derivators K factors through $\mathscr{U}_l$ and so gives rise to a morphism of derivators

$$K : \mathscr{M}^{\mathrm{loc}}_{\mathrm{dg}} \to \mathsf{HO}(\mathrm{Spt}).$$

We now establish a connection between Waldhausen's $S_\bullet$-construction (see [43, Section 1.3]) and the suspension functor in the triangulated category $\mathscr{M}^{\mathrm{loc}}_{\mathrm{dg}}(e)$. Let $\mathscr{A}$ be a Morita fibrant dg category (see [37, remarque 5.4]). Note that $\mathsf{Z}^0(\mathscr{A})$ carries a natural exact category structure obtained by pulling back the graded-split structure on $\mathscr{C}_{\mathrm{dg}}(\mathscr{A})$ along the Yoneda functor

$$\begin{aligned} h : \mathsf{Z}^0(\mathscr{A}) &\longrightarrow \mathscr{C}_{\mathrm{dg}}(\mathscr{A}), \\ A &\mapsto \mathsf{Hom}^\bullet(?, A). \end{aligned}$$

Notation 10.11
Note that the simplicial category $S_\bullet\mathscr{A}$, obtained by applying Waldhausen's $S_\bullet$-construction to $\mathsf{Z}^0(\mathscr{A})$, admits a natural enrichment over the complexes. We denote by $S_\bullet\mathscr{A}$ the simplicial Morita fibrant dg category obtained.

Recall that Δ denotes the simplicial category, and recall that $p : \Delta \to e$ denotes the projection functor.

PROPOSITION 10.12
There is a canonical isomorphism in $\mathscr{M}^{\mathrm{loc}}_{\mathrm{dg}}(e)$,

$$p_!\mathscr{U}_l(S_\bullet\mathscr{A}) \xrightarrow{\sim} \mathscr{U}_l(\mathscr{A})[1].$$

Proof
As in [28, Section 3.3], we consider the sequence in $\mathsf{HO}(\mathsf{dgcat})(\Delta)$,

$$0 \to \mathscr{A}_\bullet \to PS_\bullet\mathscr{A} \to S_\bullet\mathscr{A} \to 0,$$

where $\mathscr{A}_\bullet$ denotes the constant simplicial dg category with value $\mathscr{A}$ and $PS_\bullet\mathscr{A}$ denotes the path object of $S_\bullet\mathscr{A}$. For each point $n : e \to \Delta$, the nth component of the above sequence is the following short exact sequence in $\mathsf{Ho}(\mathsf{dgcat})$:

$$0 \to \mathscr{A} \xrightarrow{I} PS_n\mathscr{A} = S_{n+1}\mathscr{A} \xrightarrow{Q} S_n\mathscr{A} \to 0,$$

where I maps $A \in \mathscr{A}$ to the constant sequence

$$0 \to A \xrightarrow{\mathrm{Id}} A \xrightarrow{\mathrm{Id}} \cdots \xrightarrow{\mathrm{Id}} A$$

and Q maps a sequence

$$0 \to A_0 \to A_1 \to \cdots \to A_n$$

to

$$A_1/A_0 \to \cdots \to A_n/A_0.$$

Since the morphism of derivators $\mathscr{U}_l$ satisfies condition (Dr), the conservativity axiom implies that we obtain a triangle

$$\mathscr{U}_l(\mathscr{A}_\bullet) \to \mathscr{U}_l(PS_\bullet) \to \mathscr{U}_l(S_\bullet) \to \mathscr{U}_l(\mathscr{A}_\bullet)[1]$$

in $\mathscr{M}^{\mathrm{loc}}_{\mathrm{dg}}(\Delta)$. By applying the functor $p_!$, we obtain the following triangle:

$$p_!\mathscr{U}_l(\mathscr{A}_\bullet) \to p_!\mathscr{U}_l(PS_\bullet\mathscr{A}) \to p_!\mathscr{U}_l(S_\bullet\mathscr{A}) \to p_!\mathscr{U}_l(\mathscr{A}_\bullet)[1]$$

in $\mathscr{M}^{\mathrm{loc}}_{\mathrm{dg}}(e)$. We now show that we have natural isomorphisms

$$p_!\mathscr{U}_l(\mathscr{A}_\bullet) \xrightarrow{\sim} \mathscr{U}_l(\mathscr{A})$$

and

$$p_!\mathscr{U}_l(PS_\bullet\mathscr{A}) \xrightarrow{\sim} 0$$

in $\mathscr{M}^{\mathrm{loc}}_{\mathrm{dg}}(e)$, where 0 denotes the zero object in the triangulated category $\mathscr{M}^{\mathrm{loc}}_{\mathrm{dg}}(e)$. This clearly implies the proposition. Since the morphisms of derivators Φ, stab, and γ commute with homotopy colimits, it is enough to show that we have isomorphisms

$$p_!\mathbb{R}\underline{h}(\mathscr{A}_\bullet) \xrightarrow{\sim} \mathbb{R}\underline{h}(\mathscr{A})$$

and

$$p_!\mathbb{R}\underline{h}(PS_\bullet\mathscr{A}) \xrightarrow{\sim} *$$

in $\mathsf{Hot}_{\mathsf{dgcat}_f}(e)$, where $*$ denotes the terminal object in $\mathsf{Hot}_{\mathsf{dgcat}_f}(e)$. Note that since $\mathscr{A}$ and $PS_n\mathscr{A}$, $n \geq 0$, are Morita fibrant dg categories, we have natural isomorphisms

$$\underline{h}(\mathscr{A}_\bullet) \xrightarrow{\sim} \mathbb{R}\underline{h}(\mathscr{A}_\bullet)$$

and

$$\underline{h}(PS_\bullet\mathscr{A}) \xrightarrow{\sim} \mathbb{R}\underline{h}(PS_\bullet\mathscr{A})$$

in $\mathsf{Hot}_{\mathsf{dgcat}_f}(\Delta)$.

Now, since homotopy colimits in $\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset})$ are calculated objectwise and since $h(\mathscr{A}_\bullet)$ is a constant simplicial object in $\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset})$, [15, Corollary 18.7.7] implies that we have an isomorphism

$$p_!\mathbb{R}\underline{h}(\mathscr{A}_\bullet) \xrightarrow{\sim} \mathbb{R}\underline{h}(\mathscr{A})$$

in $\mathscr{M}^{\mathrm{loc}}_{\mathrm{dg}}(e)$.

Note also that since $PS_\bullet\mathscr{A}$ is a contractible simplicial object (see [28, Section 3.3]), so is $\underline{h}(PS_\bullet\mathscr{A})$. Since homotopy colimits in $\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset})$ are calculated objectwise, we have an isomorphism

$$p_!\mathbb{R}\underline{h}(PS_\bullet\mathscr{A}) \xrightarrow{\sim} *$$

in $\mathsf{Hot}_{\mathsf{dgcat}_f}(e)$.

This proves the proposition. □

11. A Quillen model in terms of presheaves of spectra

In this section, we construct another Quillen model category whose associated derivator is the localizing motivator of dg categories $\mathscr{M}^{\mathrm{loc}}_{\mathrm{dg}}$.

Consider the Quillen adjunction

$$\begin{array}{c} \mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet) \\ {\scriptstyle\Sigma^\infty}\downarrow\ \uparrow{\scriptstyle \mathrm{ev}_0} \\ \mathsf{Sp}^{\mathbb{N}}\big(\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)\big). \end{array}$$

Recall from Section 7 that we have a set of morphisms $(\Sigma \cup \{P\})_+$ in the category $\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)$. Now, stabilize the image of this set by the derived functor $\mathbb{L}\Sigma^\infty$ under the functor loop space in $\mathsf{Ho}\big(\mathsf{Sp}^{\mathbb{N}}(\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet))\big)$. For each one of the morphisms thus obtained, choose a representative in the model category $\mathsf{Sp}^{\mathbb{N}}(\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet))$.

Notation 11.1
Let us denote this set by G, and let us denote by $\mathsf{L}_G\mathsf{Sp}^{\mathbb{N}}(\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet))$ the associated left Bousfield localization.

PROPOSITION 11.2
We have an equivalence of triangulated strong derivators

$$\mathsf{HO}\big(\mathsf{Sp}^{\mathbb{N}}(\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet))\big) \xrightarrow{\sim} \mathsf{HO}\big(\mathsf{L}_G\mathsf{Sp}^{\mathbb{N}}(\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet))\big).$$

Proof
Observe that Theorems 4.4 and 8.7 imply that both derivators have the same universal property. This proves the proposition. □

Remark 11.3
Note that the stable Quillen model category

$$\mathsf{Sp}^{\mathbb{N}}\big(\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)\big)$$

identifies with

$$\mathsf{Fun}\big(\mathscr{M}_f^{\mathrm{op}}, \mathsf{Sp}^{\mathbb{N}}(\mathrm{Sset}_\bullet)\big)$$

endowed with the projective model structure.

The above considerations imply the following proposition.

PROPOSITION 11.4
We have an equivalence of derivators

$$\mathsf{HO}\big(\mathsf{L}_{\widetilde{\mathscr{E}}_{\mathrm{st}},G}\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathsf{Sp}^{\mathbb{N}}(\mathrm{Sset}_\bullet))\big) \xrightarrow{\sim} \mathscr{M}_{\mathrm{dg}}^{\mathrm{loc}}.$$

12. Upper triangular dg categories

In this section, we study upper triangular dg categories using the formalism of Quillen's homotopical algebra. In Section 13, we relate this important class of dg categories with split short exact sequences in $\mathsf{Ho}(\mathsf{dgcat})$.

Definition 12.1
An *upper triangular* dg category $\underline{\mathscr{B}}$ is given by an upper triangular matrix

$$\underline{\mathscr{B}} := \begin{pmatrix} \mathscr{A} & X \\ 0 & \mathscr{C} \end{pmatrix},$$

where $\mathscr{A}$ and $\mathscr{C}$ are small dg categories and X is an $\mathscr{A}$-$\mathscr{C}$-bimodule.

A morphism $\underline{F} : \underline{\mathscr{B}} \to \underline{\mathscr{B}'}$ of upper triangular dg categories is given by a triple $\underline{F} := (F_{\mathscr{A}}, F_{\mathscr{C}}, F_X)$, where $F_{\mathscr{A}}$ (resp., $F_{\mathscr{C}}$) is a dg functor from $\mathscr{A}$ to $\mathscr{A}'$ (resp., from $\mathscr{C}$ to $\mathscr{C}'$) and F_X is a morphism of $\mathscr{A}$-$\mathscr{C}$-bimodules from X to X'. (We consider X' endowed with the action induced by $F_{\mathscr{A}}$ and $F_{\mathscr{C}}$.) The composition is the natural one.

Notation 12.2
We denote by $\mathsf{dgcat}^{\mathrm{tr}}$ the category of upper triangular dg categories.

Let $\underline{\mathscr{B}} \in \mathsf{dgcat}^{\mathrm{tr}}$.

Definition 12.3
Let $|\underline{\mathscr{B}}|$ be the *totalization* of $\underline{\mathscr{B}}$, that is, the small dg category whose set of objects is the disjoint union of the set of objects of $\mathscr{A}$ and $\mathscr{C}$ and whose morphisms are given by

$$\mathsf{Hom}_{|\underline{\mathscr{B}}|}(x, x') := \begin{cases} \mathsf{Hom}_{\mathscr{A}}(x, x') & \text{if } x,\ x' \in \mathscr{A}, \\ \mathsf{Hom}_{\mathscr{C}}(x, x') & \text{if } x,\ x' \in \mathscr{C}, \\ X(x, x') & \text{if } x \in \mathscr{A},\ x' \in \mathscr{C}, \\ 0 & \text{if } x \in \mathscr{C},\ x' \in \mathscr{A}. \end{cases}$$

We have the adjunction

$$\begin{array}{c} \mathsf{dgcat}^{\mathrm{tr}} \\ {\scriptstyle |-|}\big\downarrow \; \big\uparrow{\scriptstyle I} \\ \mathsf{dgcat} \end{array}$$

where

$$I(\mathscr{B}') := \begin{pmatrix} \mathscr{B}' & \mathsf{Hom}_{\mathscr{B}'}(-,-) \\ 0 & \mathscr{B}' \end{pmatrix}.$$

LEMMA 12.4
The category $\mathsf{dgcat}^{\mathrm{tr}}$ *is complete and cocomplete.*

Proof
Let $\{\underline{\mathscr{B}_j}\}_{j \in J}$ be a diagram in $\mathsf{dgcat}^{\mathrm{tr}}$. Observe that the upper triangular dg category

$$\begin{pmatrix} \operatorname*{colim}_{j \in J} \mathscr{A}_j & \operatorname*{colim}_{j \in J} |\underline{\mathscr{B}_j}|(-,-) \\ 0 & \operatorname*{colim}_{j \in J} \mathscr{C}_j \end{pmatrix},$$

where $\mathrm{colim}_{j\in J}\,|\underline{\mathscr{B}_j}|(-,-)$ is the $\mathrm{colim}_{j\in J}\,\mathscr{A}_j$-$\mathrm{colim}_{j\in J}\,\mathscr{C}_j$-bimodule naturally associated with the dg category $\mathrm{colim}_{j\in J}\,|\underline{\mathscr{B}_j}|$, corresponds to $\mathrm{colim}_{j\in J}\,\underline{\mathscr{B}_j}$. Observe also that the upper triangular dg category

$$\begin{pmatrix} \lim\limits_{j\in J} \mathscr{A}_j & \lim\limits_{j\in J} X_j \\ 0 & \lim\limits_{j\in J} \mathscr{C}_j \end{pmatrix}$$

corresponds to $\lim_{j\in J}\,\underline{\mathscr{B}_j}$. This proves the lemma. □

Notation 12.5
Let $p_1(\underline{\mathscr{B}}) := \mathscr{A}$, and let $p_2(\underline{\mathscr{B}}) := \mathscr{C}$.

We have at our disposal the adjunction

$$\begin{array}{c} \mathsf{dgcat}^{\mathrm{tr}} \\ E \Big\uparrow \Big\downarrow p_1\times p_2 \\ \mathsf{dgcat}\times\mathsf{dgcat} \end{array}$$

where

$$E(\mathscr{B}',\mathscr{B}'') := \begin{pmatrix} \mathscr{B}' & 0 \\ 0 & \mathscr{B}'' \end{pmatrix}.$$

Recall from [37, théorème 5.3] and [38, théorème 5.3] that dgcat admits a structure of cofibrantly generated Quillen model category whose weak equivalences are the Morita dg functors. This structure clearly induces a componentwise model structure on $\mathsf{dgcat}\times\mathsf{dgcat}$ which is also cofibrantly generated.

PROPOSITION 12.6
The category $\mathsf{dgcat}^{\mathrm{tr}}$ *admits a structure of cofibrantly generated Quillen model category whose weak equivalences (resp., fibrations) are the morphisms* $\underline{F}:\underline{\mathscr{B}}\to\underline{\mathscr{B}'}$ *such that* $(p_1\times p_2)(\underline{F})$ *are quasi equivalences (resp., fibrations) in* $\mathsf{dgcat}\times\mathsf{dgcat}$.

Proof
We show that the previous adjunction $(E, p_1\times p_2)$ verifies conditions (1) and (2) of [15, Theorem 11.3.2].

(1) Since the functor E is also a right adjoint to $p_1\times p_2$, the functor $p_1\times p_2$ commutes with colimits, and so condition (1) is verified.

(2) Let J (resp., $J \times J$) be the set of generating trivial cofibrations in dgcat (resp., in $\mathsf{dgcat} \times \mathsf{dgcat}$). Since the functor $p_1 \times p_2$ commutes with filtered colimits, it is enough to prove the following. Let $\underline{G} : \underline{\mathscr{B}'} \to \underline{\mathscr{B}''}$ be an element of the set $E(J \times J)$, let $\underline{\mathscr{B}}$ be an object in $\mathsf{dgcat}^{\mathrm{tr}}$, and let $\underline{\mathscr{B}'} \to \underline{\mathscr{B}}$ be a morphism in $\mathsf{dgcat}^{\mathrm{tr}}$. Consider the pushout in $\mathsf{dgcat}^{\mathrm{tr}}$:

$$\begin{array}{ccc} \underline{\mathscr{B}'} & \longrightarrow & \underline{\mathscr{B}} \\ {\scriptstyle \underline{G}}\downarrow & \lrcorner & \downarrow{\scriptstyle \underline{G}_*} \\ \underline{\mathscr{B}''} & \longrightarrow & \underline{\mathscr{B}''} \coprod_{\underline{\mathscr{B}'}} \underline{\mathscr{B}} \end{array}$$

We now prove that $(p_1 \times p_2)(\underline{G}_*)$ is a weak equivalence in $\mathsf{dgcat} \times \mathsf{dgcat}$. Observe that the image of the previous pushout under the functors p_1 and p_2 corresponds to the following two pushouts in dgcat:

$$\begin{array}{ccc} \mathscr{A}' & \longrightarrow & \mathscr{A} \\ {\scriptstyle G_{\mathscr{A}'}}\downarrow\wr & \lrcorner & \downarrow{\scriptstyle G_{\mathscr{A}'_*}} \\ \mathscr{A}'' & \longrightarrow & \mathscr{A}'' \coprod_{\mathscr{A}'} \mathscr{A} \end{array} \qquad \begin{array}{ccc} \mathscr{C}' & \longrightarrow & \mathscr{C} \\ {\scriptstyle G_{\mathscr{C}'}}\downarrow\wr & \lrcorner & \downarrow{\scriptstyle G_{\mathscr{C}'_*}} \\ \mathscr{C}'' & \longrightarrow & \mathscr{C}'' \coprod_{\mathscr{C}'} \mathscr{C} \end{array}$$

Since $G_{\mathscr{A}'_*}$ and $G_{\mathscr{C}'_*}$ belong to J, the morphism

$$(p_1 \times p_2)(\underline{G}_*) = (G_{\mathscr{A}'_*}, G_{\mathscr{C}'_*})$$

is a weak equivalence in $\mathsf{dgcat} \times \mathsf{dgcat}$. This proves condition (2).

The proposition is then proved. □

Let $\underline{\mathscr{B}}, \underline{\mathscr{B}'} \in \mathsf{dgcat}^{\mathrm{tr}}$.

Definition 12.7
A morphism $\underline{F} : \underline{\mathscr{B}} \to \underline{\mathscr{B}'}$ is a *total Morita dg functor* if $F_{\mathscr{A}}$ and $F_{\mathscr{C}}$ are Morita dg functors (see [37, section 5], [38, section 5]), and F_X is a quasi isomorphism of $\mathscr{A}$-$\mathscr{C}$-bimodules.

Remark 12.8
Notice that if $\underline{F}$ is a total Morita dg functor, then $|\underline{F}|$ is a Morita dg functor in dgcat, but the converse is not true.

THEOREM 12.9

The category $\mathsf{dgcat}^{\mathrm{tr}}$ *admits a structure of cofibrantly generated Quillen model category whose weak equivalences* $\mathcal{W}$ *are the total Morita dg functors and whose fibrations are the morphisms* $\underline{F} : \underline{\mathcal{B}} \to \underline{\mathcal{B}'}$ *such that* $F_{\mathcal{A}}$ *and* $F_{\mathcal{C}}$ *are Morita fibrations (see [37]) and* F_X *is a componentwise surjective morphism of bimodules.*

Proof

The proof is based on enlarging the set $E(I \times I)$ (resp., $E(J \times J)$) of generating cofibrations (resp., generating trivial cofibrations) of the Quillen model structure of Proposition 12.6.

Let $\tilde{I}$ be the set of morphisms in $\mathsf{dgcat}^{\mathrm{tr}}$,

$$\begin{pmatrix} k & S^{n-1} \\ 0 & k \end{pmatrix} \hookrightarrow \begin{pmatrix} k & D^n \\ 0 & k \end{pmatrix}, \quad n \in \mathbb{Z},$$

where S^{n-1} is the complex $k[n-1]$ and D^n is the mapping cone on the identity of S^{n-1}. The k-k-bimodule S^{n-1} is sent to D^n by the identity on k in degree $n-1$.

Consider also the set $\tilde{J}$ of morphisms in $\mathsf{dgcat}^{\mathrm{tr}}$,

$$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \hookrightarrow \begin{pmatrix} k & D^n \\ 0 & k \end{pmatrix}, \quad n \in \mathbb{Z}.$$

Observe that a morphism $\underline{F} : \underline{\mathcal{B}} \to \underline{\mathcal{B}'}$ in $\mathsf{dgcat}^{\mathrm{tr}}$ has the RLP with respect to the set $\tilde{J}$ (resp., $\tilde{I}$) if and only if F_X is a componentwise surjective morphism (resp., surjective quasi isomorphism) of $\mathcal{A}$-$\mathcal{C}$-bimodules.

Define $I := E(I \times I) \cup \tilde{I}$ as the set of *generating cofibrations* in $\mathsf{dgcat}^{\mathrm{tr}}$, and define $J := E(J \times J) \cup \tilde{J}$ as the set of *generating trivial cofibrations*. We now prove that conditions (1)–(6) of [16, Theorem 2.1.19] are satisfied; this is clearly the case for conditions (1)–(3).

(4) We now prove that J-cell $\subset \mathcal{W}$ (see [15]). Since, by Proposition 12.6, we have $E(J \times J)$-cell $\subset \mathcal{W}'$, where $\mathcal{W}'$ denotes the weak equivalences of Proposition 12.6, it is enough to prove that pushouts with respect to any morphism in $\tilde{J}$ belong to $\mathcal{W}$. Let n be an integer, and let $\underline{\mathcal{B}}$ be an object in $\mathsf{dgcat}^{\mathrm{tr}}$. Consider the following pushout in $\mathsf{dgcat}^{\mathrm{tr}}$:

$$\begin{array}{ccc} \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} & \xrightarrow{\underline{T}} & \underline{\mathcal{B}} \\ \Big\downarrow & \lrcorner & \Big\downarrow \underline{R} \\ \begin{pmatrix} k & D^n \\ 0 & k \end{pmatrix} & \longrightarrow & \underline{\mathcal{B}'} \end{array}$$

Notice that the morphism $\underline{T}$ corresponds to specifying an object A in $\mathscr{A}$ and an object C in $\mathscr{C}$. The upper triangular dg category $\underline{\mathscr{B}'}$ is then obtained from $\underline{\mathscr{B}}$ by gluing a new morphism of degree n from A to C. Observe that $R_{\mathscr{A}}$ and $R_{\mathscr{C}}$ are the identity dg functors, and observe that R_X is a quasi isomorphism of bimodules. This shows that $\underline{R}$ belongs to $\mathscr{W}$, and so condition (4) is proved.

(5), (6) We now show that $\mathrm{RLP}(I) = \mathrm{RLP}(J) \cap \mathscr{W}$. The proof of Proposition 12.6 implies that $\mathrm{RLP}(E(I \times I)) = \mathrm{RLP}(E(J \times J)) \cap \mathscr{W}'$. Let $\underline{F} : \underline{\mathscr{B}} \to \underline{\mathscr{B}'}$ be a morphism in $\mathrm{RLP}(\tilde{I})$. Clearly, $\underline{F}$ belongs to $\mathrm{RLP}(\tilde{J})$, and F_X is a quasi isomorphism of bimodules. This shows that $\mathrm{RLP}(I) \subset \mathrm{RLP}(J) \cap \mathscr{W}$. Now, let $\underline{F} : \underline{\mathscr{B}} \to \underline{\mathscr{B}'}$ be a morphism in $\mathrm{RLP}(\tilde{J}) \cap \mathscr{W}$. Clearly, $\underline{F}$ belongs to $\mathrm{RLP}(\tilde{I})$, and so $\mathrm{RLP}(J) \cap \mathscr{W} \subset \mathrm{RLP}(I)$. This proves conditions (5) and (6).

This proves the theorem. □

Remark 12.10

Notice that the Quillen model structure of Theorem 12.9 is cellular (see [15, Definition 12.1.1]), and notice that the domains and codomains of I (the set of generating cofibrations) are cofibrant, $\aleph_0$-compact, $\aleph_0$-small, and homotopically finitely presented (see [41, Definition 2.1.1]). This implies that we are in the conditions of Proposition 5.2, so any object $\underline{\mathscr{B}}$ in $\mathsf{dgcat}^{\mathrm{tr}}$ is weakly equivalent to a filtered colimit of strict finite I-cell objects.

PROPOSITION 12.11

If $\underline{\mathscr{B}}$ is a strict finite I-cell object in $\mathsf{dgcat}^{\mathrm{tr}}$, then $p_1(\underline{\mathscr{B}})$, $p_2(\underline{\mathscr{B}})$, and $|\underline{\mathscr{B}}|$ are strict finite I-cell objects in dgcat.

Proof

We consider the following inductive argument.

(A) Note that the initial object in $\mathsf{dgcat}^{\mathrm{tr}}$ is

$$\begin{pmatrix} \emptyset & 0 \\ 0 & \emptyset \end{pmatrix},$$

and notice that it is sent to $\emptyset$ (the initial object in dgcat) by the functors p_1, p_2, and $|-|$.

(B) Suppose that $\underline{\mathscr{B}}$ is an upper triangular dg category such that $p_1(\underline{\mathscr{B}})$, $p_2(\underline{\mathscr{B}})$, and $|\underline{\mathscr{B}}|$ are strict finite I-cell objects in dgcat. Let $\underline{G} : \underline{\mathscr{B}'} \to \underline{\mathscr{B}''}$ be an element of the set I in $\mathsf{dgcat}^{\mathrm{tr}}$ (see the proof of Theorem 12.9), and let $\underline{\mathscr{B}'} \to \underline{\mathscr{B}}$ be a morphism.

Consider the following pushout in $\mathsf{dgcat}^{\mathrm{tr}}$:

$$\begin{array}{ccc} \underline{\mathscr{B}'} & \longrightarrow & \underline{\mathscr{B}} \\ {\scriptstyle \underline{G}}\downarrow & \lrcorner & \downarrow{\scriptstyle \underline{G}_*} \\ \underline{\mathscr{B}''} & \longrightarrow & \mathsf{PO} \end{array}$$

We now prove that $p_1(\mathsf{PO})$, $p_2(\mathsf{PO})$, and $|\mathsf{PO}|$ are strict finite I-cell objects in dgcat. We consider the following two cases.

(1) $\underline{G}$ belongs to $E(I \times I)$. Observe that $p_1(\mathsf{PO})$, $p_2(\mathsf{PO})$, and $|\mathsf{PO}|$ correspond exactly to the following pushouts in dgcat:

$$\begin{array}{ccc} \mathscr{A}' & \longrightarrow & \mathscr{A} \\ {\scriptstyle G_{\mathscr{A}'}}\downarrow{\scriptstyle \sim} & \lrcorner & \downarrow \\ \mathscr{A}'' & \longrightarrow & p_1(\mathsf{PO}) \end{array} \qquad \begin{array}{ccc} \mathscr{C}' & \longrightarrow & \mathscr{C} \\ {\scriptstyle G_{\mathscr{C}'}}\downarrow{\scriptstyle \sim} & \lrcorner & \downarrow \\ \mathscr{C}'' & \longrightarrow & p_2(\mathsf{PO}) \end{array}$$

$$\begin{array}{ccc} \mathscr{A}' \coprod \mathscr{C}' & \longrightarrow & |\underline{\mathscr{B}}| \\ {\scriptstyle G_{\mathscr{A}'} \amalg G_{\mathscr{C}'}}\downarrow & \lrcorner & \downarrow \\ \mathscr{A}'' \coprod \mathscr{C}'' & \longrightarrow & |\mathsf{PO}| \end{array}$$

Since $G_{\mathscr{A}'}$ and $G_{\mathscr{C}'}$ belong to I, this case is proved.

(2) $\underline{G}$ belongs to $\tilde{I}$. Observe that $p_1(\mathsf{PO})$ identifies with $\mathscr{A}$, $p_2(\mathsf{PO})$ identifies with $\mathscr{C}$, and $|\mathsf{PO}|$ corresponds to the following pushout in dgcat :

$$\begin{array}{ccc} \mathscr{C}(n) & \longrightarrow & |\underline{\mathscr{B}}| \\ {\scriptstyle S(n)}\downarrow & \lrcorner & \downarrow \\ \mathscr{P}(n) & \longrightarrow & |\mathsf{PO}| \end{array}$$

where $S(n)$ is a generating cofibration in dgcat (see [38, section 2]). This proves this case.

The proposition is proved. □

13. Split short exact sequences

In this section, we establish the connection between split short exact sequences of dg categories and upper triangular dg categories.

Definition 13.1
A split short exact sequence of dg categories is a short exact sequence of dg categories (see [21, Theorem 4.11]) which is Morita equivalent to one of the form

$$0 \longrightarrow \mathscr{A} \underset{i_{\mathscr{A}}}{\overset{R}{\rightleftarrows}} \mathscr{B} \underset{P}{\overset{i_{\mathscr{C}}}{\rightleftarrows}} \mathscr{C} \longrightarrow 0$$

where we have $P \circ i_{\mathscr{A}} = 0$, R is a dg functor right adjoint to $i_{\mathscr{A}}$, $i_{\mathscr{C}}$ is a dg functor right adjoint to P, and we have $P \circ i_{\mathscr{C}} = \mathrm{Id}_{\mathscr{C}}$ and $R \circ i_{\mathscr{A}} = \mathrm{Id}_{\mathscr{A}}$ via the adjunction morphisms.

To a split short exact sequence, we can naturally associate the upper triangular dg category

$$\underline{\mathscr{B}} := \begin{pmatrix} \mathscr{A} & \mathsf{Hom}_{\mathscr{B}}\big(i_{\mathscr{C}}(-), i_{\mathscr{A}}(-)\big) \\ 0 & \mathscr{C} \end{pmatrix}.$$

Conversely, to an upper triangular dg category $\underline{\mathscr{B}}$ such that $\mathscr{C}$ admits a zero object (e.g., if $\mathscr{C}$ is Morita fibrant), we can associate a split short exact sequence

$$0 \longrightarrow \mathscr{A} \underset{i_{\mathscr{A}}}{\overset{R}{\rightleftarrows}} |\underline{\mathscr{B}}| \underset{P}{\overset{i_{\mathscr{C}}}{\rightleftarrows}} \mathscr{C} \longrightarrow 0$$

where P and R are the projection dg functors. Moreover, this construction is functorial in $\underline{\mathscr{B}}$ and sends total Morita equivalences to Morita-equivalent split short exact sequences. Note also that by Lemma 12.4, this functor preserves colimits.

PROPOSITION 13.2
Every split short exact sequence of dg categories is weakly equivalent to a filtered homotopy colimit of split short exact sequences whose components are strict finite I-cell objects in dgcat.

Proof
Let

$$0 \longrightarrow \mathscr{A} \underset{i_{\mathscr{A}}}{\overset{R}{\rightleftarrows}} \mathscr{B} \underset{P}{\overset{i_{\mathscr{C}}}{\rightleftarrows}} \mathscr{C} \longrightarrow 0$$

be a split short exact sequence of dg categories. We can suppose that $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ are Morita-fibrant dg categories (see [37, remarque 5.4]). Consider the upper triangular dg category

$$\underline{\mathscr{B}} := \begin{pmatrix} \mathscr{A} & \mathsf{Hom}_{\mathscr{B}}\big(i_{\mathscr{C}}(-), i_{\mathscr{A}}(-)\big) \\ 0 & \mathscr{C} \end{pmatrix}.$$

Now, by Remark 12.10, $\underline{\mathscr{B}}$ is equivalent to a filtered colimit of strict finite I-cell objects in $\mathsf{dgcat}^{\mathrm{tr}}$. Consider the image of this diagram by the functor, described following Definition 13.1, which sends an upper triangular dg category to a split short exact sequence. By Proposition 12.11, the components of each split short exact sequence of this diagram are strict finite I-cell objects in dgcat. Since the category dgcat satisfies the conditions of Proposition 5.2, filtered homotopy colimits are equivalent to filtered colimits, and so the proposition is proved. □

14. Quasi additivity

Recall from Section 10 that we have at our disposal the Quillen model category $\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset})$ which is *homotopically pointed*; that is, the morphism $\emptyset \to *$, from the initial object $\emptyset$ to the terminal one $*$, is a weak equivalence. We now consider a strictly pointed Quillen model.

PROPOSITION 14.1
We have a Quillen equivalence

$$\begin{array}{c} * \downarrow \mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset}) \\ {\scriptstyle (-)_+}\uparrow \downarrow {\scriptstyle U} \\ \mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset}) \end{array}$$

where U denotes the forgetful functor.

This follows from the fact that the category $\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset})$ is homotopically pointed and from the following general argument.

PROPOSITION 14.2
Let $\mathscr{M}$ be a homotopically pointed Quillen model category. We have a Quillen equivalence

$$\begin{array}{c} * \downarrow \mathscr{M} \\ {\scriptstyle (-)_+}\uparrow \downarrow {\scriptstyle U} \\ \mathscr{M} \end{array}$$

where U denotes the forgetful functor.

Proof
Clearly, the functor U preserves cofibrations, fibrations, and weak equivalences by construction. Let now $N \in \mathscr{M}$, and let $M \in * \downarrow \mathscr{M}$. Consider the following commutative diagram in $\mathscr{M}$:

$$\begin{array}{ccc} N \simeq \emptyset \coprod N & \xrightarrow{\quad f \quad} & U(M) \\ & {\scriptstyle i \amalg \mathbf{1}} \searrow^{\sim} \quad \nearrow {\scriptstyle f^\sharp} & \\ & * \coprod N & \end{array}$$

where $f^\sharp$ is the morphism that corresponds to f, considered as a morphism in $\mathscr{M}$ under the adjunction, and $i : \emptyset \xrightarrow{\sim} *$. Since the morphism $i \amalg \mathbf{1}$ corresponds to the homotopy colimit of i and 1, which are both weak equivalences, Proposition 4.1 implies that $i \amalg \mathbf{1}$ is a weak equivalence. Now, by the two-out-of-three property, the morphism f is a weak equivalence if and only if $f^\sharp$ is also. This proves the proposition. □

Notation 14.3
Let $\mathscr{A}$ and $\mathscr{B}$ be small dg categories. We denote by $\mathsf{rep}_{\mathrm{mor}}(\mathscr{A}, \mathscr{B})$ the full dg subcategory of $\mathscr{C}_{\mathrm{dg}}(\mathscr{A}_c^{\mathrm{op}} \otimes \mathscr{B})$, where $\mathscr{A}_c$ denotes a cofibrant resolution of $\mathscr{A}$ whose objects are the bimodules X such that $X(?, A)$ is a compact object in $\mathscr{D}(\mathscr{B})$ for all $A \in \mathscr{A}_c$ and which are cofibrant as bimodules. We denote by $w\mathscr{A}$ the category of homotopy equivalences of $\mathscr{A}$, and we denote by $N.w\mathscr{A}$ its nerve.

Now, consider the morphism

$$\mathsf{Ho}(\mathsf{dgcat}) \to \mathsf{Ho}\big(* \downarrow \mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset})\big),$$
$$\mathscr{A} \mapsto \begin{cases} \mathsf{Hom}_{\mathsf{dgcat}}\big(\Gamma(?), \mathscr{A}_f\big)_+ \\ \quad \simeq \mathsf{Map}_{\mathsf{dgcat}}(?, \mathscr{A})_+ \\ \quad \simeq N.w\,\mathsf{rep}_{\mathrm{mor}}(?, \mathscr{A})_+ \end{cases}$$

which by Sections 5 and 6 and Proposition 14.1 corresponds to the component $(\Phi \circ \mathbb{R}\underline{h}\,)(e)$ of the morphism of derivators

$$\Phi \circ \mathbb{R}\underline{h} : \mathsf{HO}(\mathsf{dgcat}) \longrightarrow \mathsf{L}_{\Sigma,P}\mathsf{Hot}_{\mathsf{dgcat}_f}$$

(see Proposition 6.1). Observe that the simplicial presheaf $N.w\,\mathsf{rep}_{\mathrm{mor}}(?, \mathscr{A})$ is already canonically pointed.

PROPOSITION 14.4
The canonical morphism

$$\Psi : N.w\,\mathsf{rep}_{\mathrm{mor}}(?, \mathscr{A})_+ \to N.w\,\mathsf{rep}_{\mathrm{mor}}(?, \mathscr{A})$$

is a weak equivalence in $* \downarrow \mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset})$.

Proof
Observe that $N.w\,\mathsf{rep}_{\mathrm{mor}}(?, \mathscr{A})$ is a fibrant object in $* \downarrow \mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset})$, and observe that the canonical morphism Ψ corresponds to the counit of the adjunction of Proposition 14.1. Since this adjunction is a Quillen equivalence, the proposition is proved. □

Recall now from Remark 7.2 that we have a canonical equivalence of pointed derivators

$$\mathsf{L}_{\Sigma,P}\mathsf{Hot}_{\mathsf{dgcat}_f} \xrightarrow{\sim} \mathsf{L}_{\Sigma,P}\mathsf{Hot}_{\mathsf{dgcat}_{f_\bullet}},$$

where $\mathsf{L}_{\Sigma,P}\mathsf{Hot}_{\mathsf{dgcat}_{f_\bullet}}$ is the derivator associated with the Quillen model category $\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)$. For the remainder of this article, we consider this Quillen model. We have the morphism of derivators

$$\Phi \circ \mathbb{R}\underline{h} : \mathsf{HO}(\mathsf{dgcat}) \longrightarrow \mathsf{L}_{\Sigma,P}\mathsf{Hot}_{\mathsf{dgcat}_{f_\bullet}},$$

which commutes with filtered homotopy colimits and preserves the point.

Notation 14.5
(i) We denote by $\mathscr{E}^s$ the set of retractions of dg categories

$$\mathscr{G} \underset{i_{\mathscr{G}}}{\overset{R}{\rightleftarrows}} \mathscr{H}$$

where $\mathscr{G}$ and $\mathscr{H}$ are strict finite I-cell objects in dgcat, $i_{\mathscr{G}}$ is a fully faithful dg functor, R is a right adjoint to $i_{\mathscr{G}}$, and $R \circ i_{\mathscr{G}} = \mathrm{Id}_{\mathscr{G}}$.
(ii) We denote by $\mathscr{E}^s_{\mathrm{un}}$ the set of morphisms S_L in $\mathsf{L}_{\Sigma,P}\mathsf{Hot}_{\mathsf{dgcat}_{f_\bullet}}(e)$ (see Section 9), where L belongs to the set $\mathscr{E}^s$.

Now, choose for each element of the set $\mathscr{E}^s_{\mathrm{un}}$ a representative in the category $\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)$. We denote this set of representatives by $\widetilde{\mathscr{E}^s_{\mathrm{un}}}$. Since $\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)$ is a left proper, cellular Quillen model category (see [15, Definition 12.1.1]) its left Bousfield localization by $\widetilde{\mathscr{E}^s_{\mathrm{un}}}$ exists. We denote it by $\mathsf{L}_{\widetilde{\mathscr{E}^s_{\mathrm{un}}}}\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)$, and we denote by $\mathsf{L}_{\widetilde{\mathscr{E}^s_{\mathrm{un}}}}\mathsf{L}_{\Sigma,P}\mathsf{Hot}_{\mathsf{dgcat}_{f_\bullet}}$ the associated derivator. We have the morphism of derivators

$$\Psi : \mathsf{L}_{\Sigma,P}\mathsf{FunHot}_{\mathsf{dgcat}_{f_\bullet}} \to \mathsf{L}_{\widetilde{\mathscr{E}^s_{\mathrm{un}}}}\mathsf{L}_{\Sigma,P}\mathsf{FunHot}_{\mathsf{dgcat}_{f_\bullet}}.$$

Remark 14.6

(i) Notice that by construction, the domains and codomains of the set $\widetilde{\mathscr{E}^s_{\mathrm{un}}}$ are homotopically finitely presented objects. Therefore, by Lemma 7.1, the set

$$\mathscr{G} = \{\mathbf{F}^X_{\Delta[n]_+/\partial\Delta[n]_+} \mid X \in \mathscr{M}_f, n \geq 0\}$$

of cofibers of the generating cofibrations in $\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)$ is a set of small weak generators in $\mathsf{Ho}(\mathsf{L}_{\widetilde{\mathscr{E}^s_{\mathrm{un}}}}\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathscr{M}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet))$.

(ii) Notice also that Proposition 13.2 implies that variants of Proposition 9.2 and Theorem 9.4 are also verified: simply consider the set $\mathscr{E}^s$ instead of $\mathscr{E}$ and a retraction of dg categories instead of an inclusion of a full dg subcategory. The proofs are exactly the same.

Definition 14.7

(i) The *unstable motivator* of dg categories $\mathscr{M}^{\mathrm{unst}}_{\mathrm{dg}}$ is the derivator associated with the Quillen model category

$$\mathsf{L}_{\widetilde{\mathscr{E}^s_{\mathrm{un}}}}\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet).$$

(ii) The *universal unstable invariant* of dg categories is the canonical morphism of derivators

$$\mathscr{U}_u : \mathsf{HO}(\mathsf{dgcat}) \to \mathscr{M}^{\mathrm{unst}}_{\mathrm{dg}}.$$

Let $\mathscr{M}$ be a left proper, cellular model category, let S be a set of maps in $\mathscr{M}$, and let $\mathsf{L}_S\mathscr{M}$ be the left Bousfield localization of $\mathscr{M}$ with respect to S (see [15, Definition 12.1.1]). Recall from [15, Theorem 4.1.1] that an object X in $\mathsf{L}_S\mathscr{M}$ is fibrant if X is fibrant in $\mathscr{M}$ and if, for every element $f : A \to B$ of S, the induced map of homotopy function complexes $f^* : \mathsf{Map}(B, X) \to \mathsf{Map}(A, X)$ is a weak equivalence.

PROPOSITION 14.8

An object $F \in \mathsf{L}_{\widetilde{\mathscr{E}^s_{\mathrm{un}}}}\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)$ *is fibrant if and only if it satisfies the following conditions:*

(1) $F(\mathscr{B}) \in \mathrm{Sset}_\bullet$ *is fibrant for all* $\mathscr{B} \in \mathsf{dgcat}_f$;

(2) $F(\emptyset) \in \mathrm{Sset}_\bullet$ *is contractible;*

(3) *for every Morita equivalence* $\mathscr{B} \xrightarrow{\sim} \mathscr{B}'$ *in* dgcat_f, *the morphism* $F(\mathscr{B}') \xrightarrow{\sim} F(\mathscr{B})$ *is a weak equivalence in* $\mathrm{Sset}_\bullet$ *;*

(4) *every split short exact sequence*

$$0 \longrightarrow \mathscr{B}' \underset{i_{\mathscr{B}'}}{\overset{R}{\rightleftarrows}} \mathscr{B} \underset{P}{\overset{i_{\mathscr{B}''}}{\rightleftarrows}} \mathscr{B}'' \longrightarrow 0$$

in dgcat_f *induces a homotopy fiber sequence*

$$F(\mathscr{B}'') \to F(\mathscr{B}) \to F(\mathscr{B}')$$

in Sset.

Proof
Clearly, condition (1) corresponds to the fact that F is fibrant in $\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)$. Now, observe that $\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)$ is a simplicial Quillen model category with the simplicial action given by

$$\begin{aligned} \mathrm{Sset} \times \mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet) &\to \mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet), \\ (K, F) &\mapsto K_+ \wedge F, \end{aligned}$$

where $K_+ \wedge F$ denotes the componentwise smash product. This simplicial structure and the construction of the localized Quillen model category $\mathsf{L}_{\widetilde{\mathscr{E}^s_{\mathrm{un}}}} \mathsf{L}_{\Sigma, P} \mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)$ (see Section 7) allow us to recover conditions (2) and (3). Condition (4) follows from the construction of the set $\widetilde{\mathscr{E}^s_{\mathrm{un}}}$ and from the fact that the functor

$$\mathsf{Map}(?, F) : \mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)^{\mathrm{op}} \to \mathrm{Sset}$$

transforms homotopy cofiber sequences into homotopy fiber sequences.

This proves the proposition. □

Let $\mathscr{A}$ be a Morita fibrant dg category. Recall from Notation 10.11 that $S_\bullet \mathscr{A}$ denotes the simplicial Morita fibrant dg category obtained by applying Waldhausen's $S_\bullet$-construction to the exact category $\mathsf{Z}^0(\mathscr{A})$ and by remembering the enrichment in complexes.

Notation 14.9
We denote by $K(\mathscr{A}) \in \mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)$ the simplicial presheaf

$$\mathscr{B} \mapsto |N.wS_\bullet \, \mathsf{rep}_{\mathrm{mor}}(\mathscr{B}, \mathscr{A})|,$$

where $| - |$ denotes the fibrant realization functor of bisimplicial sets.

PROPOSITION 14.10
The simplicial presheaf $K(\mathscr{A})$ *is fibrant in* $\mathsf{L}_{\widetilde{\mathscr{E}^s_{\text{un}}}}\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}_f^{\text{op}}, \text{Sset}_\bullet)$.

Proof
Observe that $K(\mathscr{A})$ satisfies conditions (1)–(3) of Proposition 14.8. We now prove that Waldhausen's fibration theorem [43, Theorem 1.6.4] implies condition (4). Apply the contravariant functor $\mathsf{rep}_{\text{mor}}(?, \mathscr{A})$ to the split short exact sequence

$$0 \longrightarrow \mathscr{B}' \underset{i_{\mathscr{B}'}}{\overset{R}{\leftrightarrows}} \mathscr{B} \underset{P}{\overset{i_{\mathscr{B}''}}{\leftrightarrows}} \mathscr{B}'' \longrightarrow 0$$

and obtain a split short exact sequence

$$0 \longrightarrow \mathsf{rep}_{\text{mor}}(\mathscr{B}'', \mathscr{A}) \leftrightarrows \mathsf{rep}_{\text{mor}}(\mathscr{B}, \mathscr{A}) \leftrightarrows \mathsf{rep}_{\text{mor}}(\mathscr{B}', \mathscr{A}) \longrightarrow 0$$

Now, consider the Waldhausen category $v\,\mathsf{rep}_{\text{mor}}(\mathscr{B}, \mathscr{A}) := \mathsf{Z}^0(\mathsf{rep}_{\text{mor}}(\mathscr{B}, \mathscr{A}))$, where the weak equivalences are the morphisms f such that $\mathsf{cone}(f)$ is contractible. Consider also the Waldhausen category $w\,\mathsf{rep}_{\text{mor}}(\mathscr{B}, \mathscr{A})$, which has the same cofibrations as $v\,\mathsf{rep}_{\text{mor}}(\mathscr{B}, \mathscr{A})$ but in which the weak equivalences are the morphisms f such that $\mathsf{cone}(f)$ belongs to $\mathsf{rep}_{\text{mor}}(\mathscr{B}'', \mathscr{A})$. Observe that we have the inclusion $v\,\mathsf{rep}_{\text{mor}}(\mathscr{B}, \mathscr{A}) \subset w\,\mathsf{rep}_{\text{mor}}(\mathscr{B}, \mathscr{A})$ and an equivalence $\mathsf{rep}_{\text{mor}}(\mathscr{B}, \mathscr{A})^w \simeq \mathsf{Z}^0(\mathsf{rep}_{\text{mor}}(\mathscr{B}', \mathscr{A}))$ (see [43, Section 1.6]). The conditions of [43, Theorem 1.6.4] are satisfied, and so we have a homotopy fiber sequence

$$|N.wS_\bullet\,\mathsf{rep}_{\text{mor}}(\mathscr{B}'', \mathscr{A})| \to |N.wS_\bullet\,\mathsf{rep}_{\text{mor}}(\mathscr{B}, \mathscr{A})| \to |N.wS_\bullet\,\mathsf{rep}_{\text{mor}}(\mathscr{B}', \mathscr{A})|$$

in Sset. This proves the proposition. □

Let $p : \Delta \to e$ be the projection functor.

PROPOSITION 14.11
The objects

$$S^1 \wedge N.w\,\mathsf{rep}_{\text{mor}}(?, \mathscr{A}) \qquad \textit{and} \qquad |N.wS_\bullet\,\mathsf{rep}_{\text{mor}}(?, \mathscr{A})| = K(\mathscr{A})$$

are canonically isomorphic in $\mathsf{Ho}(\mathsf{L}_{\widetilde{\mathscr{E}^s_{\text{un}}}}\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}_f^{\text{op}}, \text{Sset}_\bullet))$.

Proof
As in [28, Section 3.3], we consider the sequence in $\mathsf{HO}(\mathsf{dgcat})(\Delta)$,

$$0 \longrightarrow \mathscr{A}_\bullet \stackrel{I}{\longrightarrow} PS_\bullet\mathscr{A} \stackrel{Q}{\longrightarrow} S_\bullet\mathscr{A} \longrightarrow 0,$$

where $\mathscr{A}_\bullet$ denotes the constant simplicial dg category with value $\mathscr{A}$ and $PS_\bullet\mathscr{A}$ denotes the path object of $S_\bullet\mathscr{A}$. By applying the morphism of derivators $\mathscr{U}_u$ to this sequence, we obtain the canonical morphism

$$S_I : \mathrm{cone}\big(\mathscr{U}_u(\mathscr{A}_\bullet \stackrel{I}{\to} PS_\bullet\mathscr{A})\big) \to \mathscr{U}_u(S_\bullet\mathscr{A})$$

in $\mathscr{M}^{\mathrm{unst}}_{\mathrm{dg}}(\Delta)$. We now prove that for each point $n : e \to \Delta$, the nth component of S_I is an isomorphism in $\mathsf{L}_{\widetilde{\mathscr{E}^s_{\mathrm{un}}}}\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)$. For each point $n : e \to \Delta$, we have a split short exact sequence in $\mathsf{Ho}(\mathsf{dgcat})$,

$$0 \longrightarrow \mathscr{A} \underset{I_n}{\overset{R_n}{\rightleftarrows}} PS_n\mathscr{A} = S_{n+1}\mathscr{A} \underset{Q_n}{\overset{S_n}{\rightleftarrows}} S_n\mathscr{A} \longrightarrow 0$$

where I_n maps $A \in \mathscr{A}$ to the constant sequence

$$0 \to A \stackrel{\mathrm{Id}}{\to} A \stackrel{\mathrm{Id}}{\to} \cdots \stackrel{\mathrm{Id}}{\to} A,$$

Q maps a sequence

$$0 \to A_0 \to A_1 \to \cdots \to A_n$$

to

$$A_1/A_0 \to \cdots \to A_n/A_0,$$

S_n maps a sequence

$$0 \to A_0 \to A_1 \to \cdots \to A_{n-1}$$

to

$$0 \to 0 \to A_0 \to \cdots \to A_{n-1},$$

and R_n maps a sequence

$$0 \to A_0 \to A_1 \to \cdots \to A_{n-1}$$

to A_0. Now, by construction of $\mathsf{L}_{\widetilde{\mathscr{E}^s_{\mathrm{un}}}}\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)$, the canonical morphisms

$$S_{I_n} : \mathrm{cone}\big(\mathscr{U}_u(\mathscr{A} \xrightarrow{I_n} PS_n\mathscr{A})\big) \to \mathscr{U}_u(S_n\mathscr{A}), \quad n \in \mathbb{N}$$

are isomorphisms in $\mathscr{M}^{\mathrm{unst}}_{\mathrm{dg}}(e)$. Since homotopy colimits in the model category $\mathsf{L}_{\widetilde{\mathscr{E}^s_{\mathrm{un}}}}\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)$ are calculated objectwise, the nth component of S_I identifies with S_{I_n}, and so by the conservativity axiom, S_I is an isomorphism in $\mathscr{M}^{\mathrm{unst}}_{\mathrm{dg}}(\Delta)$. This implies that we obtain the homotopy co-Cartesian square

$$\begin{array}{ccc} p_!\mathscr{U}_u(\mathscr{A}_\bullet) & \longrightarrow & p_!\big(\mathscr{U}_u(PS_\bullet\mathscr{A})\big) \\ \downarrow & & \downarrow \\ * & \longrightarrow & p_!\big(\mathscr{U}_u(S_\bullet\mathscr{A})\big). \end{array}$$

As in the proof of Proposition 10.12, we show that $p_!\mathscr{U}_u(\mathscr{A}_\bullet)$ identifies with $N.w\,\mathsf{rep}_{\mathrm{mor}}(?, \mathscr{A}) = \mathscr{U}_u(\mathscr{A})$ and that $p_!(\mathscr{U}_u(PS_\bullet\mathscr{A}))$ is contractible. Since we have the equivalence

$$p_!\big(\mathscr{U}_u(S_\bullet\mathscr{A})\big) = p_!\big(N.w\,\mathsf{rep}_{\mathrm{mor}}(?, S_\bullet\mathscr{A})\big) \xrightarrow{\sim} |N.wS_\bullet\,\mathsf{rep}_{\mathrm{mor}}(?, \mathscr{A})|$$

and $N.w\,\mathsf{rep}_{\mathrm{mor}}(?, \mathscr{A})$ is cofibrant in $\mathsf{L}_{\widetilde{\mathscr{E}^s_{\mathrm{un}}}}\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)$, the proposition is proved. □

PROPOSITION 14.12

We have the following weak equivalence of simplicial sets:

$$\mathsf{Map}\big(\mathscr{U}_u(k), S^1 \wedge \mathscr{U}_u(\mathscr{A})\big) \xrightarrow{\sim} |N.wS_\bullet\mathscr{A}_f|$$

in $\mathsf{L}_{\widetilde{\mathscr{E}^s_{\mathrm{un}}}}\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)$. *In particular, we have the isomorphisms*

$$\pi_{i+1}\mathsf{Map}\big(\mathscr{U}_u(k), S^1 \wedge \mathscr{U}_u(\mathscr{A})\big) \xrightarrow{\sim} K_i(\mathscr{A}), \quad \forall i \geq 0.$$

Proof

This follows from Propositions 14.10 and 14.11 and from the weak equivalences

$$\begin{aligned} \mathsf{Map}\big(\mathscr{U}_u(k), S^1 \wedge \mathscr{U}_u(\mathscr{A})\big) &\simeq \mathsf{Map}\big(\mathbb{R}\underline{h}(k), K(\mathscr{A})\big) \\ &\simeq \big(K(\mathscr{A})\big)(k) \\ &\simeq |N.wS_\bullet\mathscr{A}_f|. \end{aligned}$$

□

15. The universal additive invariant

Consider the Quillen model category $\mathsf{L}_{\widetilde{\mathscr{E}^s_{\mathrm{un}}}}\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)$ constructed in Section 14. The definition of the set $\widetilde{\mathscr{E}^s_{\mathrm{un}}}$ and the same arguments as those of Examples 8.5 and 8.6 allow us to conclude that $\mathsf{L}_{\widetilde{\mathscr{E}^s_{\mathrm{un}}}}\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)$ satisfies the conditions of Theorem 8.7. In particular, we have an equivalence of triangulated derivators

$$\mathsf{St}(\mathscr{M}^{\mathrm{unst}}_{\mathrm{dg}}) \xrightarrow{\sim} \mathsf{HO}\big(\mathsf{Sp}^{\mathbb{N}}(\mathsf{L}_{\widetilde{\mathscr{E}^s_{\mathrm{un}}}}\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet))\big).$$

Definition 15.1

(i) The *additive motivator* of dg categories $\mathscr{M}^{\mathrm{add}}_{\mathrm{dg}}$ is the triangulated derivator associated with the stable Quillen model category

$$\mathsf{Sp}^{\mathbb{N}}\big(\mathsf{L}_{\widetilde{\mathscr{E}^s_{\mathrm{un}}}}\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)\big).$$

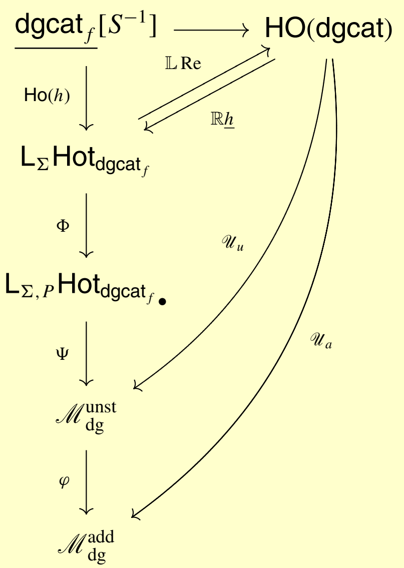
(ii) The *universal additive invariant* of dg categories is the canonical morphism of derivators

$$\mathscr{U}_a : \mathsf{HO}(\mathsf{dgcat}) \to \mathscr{M}^{\mathrm{add}}_{\mathrm{dg}}.$$

Remark 15.2

Observe that Remarks 14.6 and 8.4 imply that $\mathscr{M}^{\mathrm{add}}_{\mathrm{dg}}$ is a compactly generated triangulated derivator.

We sum up the construction of $\mathscr{M}^{\mathrm{add}}_{\mathrm{dg}}$ in the diagram



Observe that the morphism of derivators $\mathcal{U}_a$ is pointed, commutes with filtered homotopy colimits, and satisfies the following condition:

(A) for every split short exact sequence

$$0 \longrightarrow \mathscr{A} \underset{i_{\mathscr{A}}}{\overset{R}{\leftrightarrows}} \mathscr{B} \underset{P}{\overset{i_{\mathscr{C}}}{\leftrightarrows}} \mathscr{C} \longrightarrow 0$$

in $\mathsf{Ho}(\mathsf{dgcat})$, we have a split triangle

$$\mathcal{U}_a(\mathscr{A}) \underset{i_{\mathscr{A}}}{\overset{R}{\leftrightarrows}} \mathcal{U}_a(\mathscr{B}) \underset{P}{\overset{i_{\mathscr{C}}}{\leftrightarrows}} \mathcal{U}_a(\mathscr{C}) \longrightarrow \mathcal{U}_a(\mathscr{A})[1]$$

in $\mathscr{M}^{\mathrm{add}}_{\mathrm{dg}}(e)$. This implies that the dg functors $i_{\mathscr{A}}$ and $i_{\mathscr{C}}$ induce an isomorphism

$$\mathcal{U}_a(\mathscr{A}) \oplus \mathcal{U}_a(\mathscr{C}) \xrightarrow{\sim} \mathcal{U}_a(\mathscr{B})$$

in $\mathscr{M}^{\mathrm{add}}_{\mathrm{dg}}(e)$.

Remark 15.3

Since the dg category $\mathscr{B}$ in the above split short exact sequence is Morita equivalent to the dg category $E(\mathscr{A}, \mathscr{B}, \mathscr{C})$ (see [43, Section 1.1]), condition (A) is equivalent to the additivity property stated by Waldhausen in [43, Proposition 1.3.2].

Let $\mathbb{D}$ be a strong triangulated derivator.

THEOREM 15.4

The morphism $\mathcal{U}_a$ induces an equivalence of categories

$$\underline{\mathsf{Hom}}_!(\mathscr{M}^{\mathrm{add}}_{\mathrm{dg}}, \mathbb{D}) \xrightarrow{\mathcal{U}_a^*} \underline{\mathsf{Hom}}_{\mathrm{flt,\,(A)},\,p}\big(\mathsf{HO}(\mathsf{dgcat}), \mathbb{D}\big),$$

where $\underline{\mathsf{Hom}}_{\mathrm{flt,\,(A)},\,p}(\mathsf{HO}(\mathsf{dgcat}), \mathbb{D})$ denotes the category of morphisms of derivators which commute with filtered homotopy colimits, satisfy condition (A), and preserve the point.

Proof

By Theorem 8.1, we have an equivalence of categories

$$\underline{\mathsf{Hom}}_!(\mathscr{M}^{\mathrm{add}}_{\mathrm{dg}}, \mathbb{D}) \xrightarrow{\sim} \underline{\mathsf{Hom}}_!(\mathscr{M}^{\mathrm{unst}}_{\mathrm{dg}}, \mathbb{D}).$$

By Theorem 4.4, we have an equivalence of categories

$$\underline{\mathsf{Hom}}_!(\mathscr{M}^{\mathrm{unst}}_{\mathrm{dg}}, \mathbb{D}) \xrightarrow{\sim} \underline{\mathsf{Hom}}_{!,\mathscr{E}^s_{\mathrm{un}}}(\mathsf{L}_{\Sigma,P}\mathsf{Hot}_{\mathsf{dgcat}_{f_\bullet}}, \mathbb{D}).$$

Now, we observe that since $\mathbb{D}$ is a strong triangulated derivator, the category $\underline{\mathsf{Hom}}_{!,\mathscr{E}^s_{\mathrm{un}}}(\mathsf{L}_{\Sigma,P}\mathsf{Hot}_{\mathsf{dgcat}_{f_\bullet}}, \mathbb{D})$ identifies $\underline{\mathsf{Hom}}_{\mathrm{flt},\,(\mathrm{A}),\,p}(\mathsf{HO}(\mathsf{dgcat}), \mathbb{D})$. This proves the theorem. □

Notation 15.5
We call an object of the right-hand-side category of Theorem 15.4 an *additive invariant* of dg categories.

Example 15.6
(i) The Hochschild and cyclic homology and the nonconnective K-theory defined in Section 10 are examples of additive invariants.
(ii) Another example is given by the classical Waldhausen's connective K-theory spectrum

$$K^c : \mathsf{HO}(\mathsf{dgcat}) \to \mathsf{HO}(\mathrm{Spt})$$

(see [43]).

Remark 15.7
By Theorem 15.4, the morphism of derivators K^c factors through $\mathscr{U}_a$ and so gives rise to a morphism of derivators

$$K^c : \mathscr{M}^{\mathrm{add}}_{\mathrm{dg}} \to \mathsf{HO}(\mathrm{Spt}).$$

We now prove that this morphism of derivators is corepresentable in $\mathscr{M}^{\mathrm{add}}_{\mathrm{dg}}$.

Let $\mathscr{A}$ be a small dg category.

Notation 15.8
We denote by $K(\mathscr{A})^c \in \mathsf{Sp}^{\mathbb{N}}(\mathsf{L}_{\widetilde{\mathscr{E}^s_{\mathrm{un}}}}\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}^{\mathrm{op}}_f, \mathrm{Sset}_\bullet))$ the spectrum such that

$$K(\mathscr{A})^c_n := |N.wS^{(n+1)}_\bullet \,\mathsf{rep}_{\mathrm{mor}}(?, \mathscr{A})|, \quad n \geq 0,$$

endowed with the natural structure morphisms

$$\beta_n : S^1 \wedge |N.wS^{(n+1)}_\bullet \,\mathsf{rep}_{\mathrm{mor}}(?, \mathscr{A})| \xrightarrow{\sim} |N.wS^{(n+2)}_\bullet \,\mathsf{rep}_{\mathrm{mor}}(?, \mathscr{A})|, \quad n \geq 0$$

(see [43]).

Note that $\mathscr{U}_a(\mathscr{A})$ identifies in $\mathsf{Ho}(\mathscr{M}^{\mathrm{add}}_{\mathrm{dg}})$ with the suspension spectrum given by

$$\big(\Sigma^\infty|N.w\,\mathsf{rep}_{\mathrm{mor}}(?,\mathscr{A})|\big)_n := S^n \wedge |N.w\,\mathsf{rep}_{\mathrm{mor}}(?,\mathscr{A})|.$$

Now, Proposition 14.11 and the fact that the morphism of derivators φ commutes with homotopy colimits imply that we have an isomorphism

$$\mathscr{U}_a(\mathscr{A})[1] \xrightarrow{\sim} p_!\mathscr{U}_a(S_\bullet\mathscr{A}).$$

In particular, we have a natural morphism

$$\eta : \mathscr{U}_a(\mathscr{A})[1] \to K(\mathscr{A})^c$$

in $\mathsf{Sp}^{\mathbb{N}}(\mathsf{L}_{\widetilde{\mathscr{E}^s_{\mathrm{un}}}}\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet))$ induced by the identity in degree zero.

THEOREM 15.9
The morphism η is a fibrant resolution of $\mathscr{U}_a(\mathscr{A})[1]$.

Proof
We prove first that $K(\mathscr{A})^c$ is a fibrant object in $\mathsf{Sp}^{\mathbb{N}}(\mathsf{L}_{\widetilde{\mathscr{E}^s_{\mathrm{un}}}}\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet))$. By [16] and [36], we need to show that $K(\mathscr{A})^c$ is an Ω-spectrum, that is, that $K(\mathscr{A})^c_n$ is a fibrant object in $\mathsf{L}_{\widetilde{\mathscr{E}^s_{\mathrm{un}}}}\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)$ and that the induced map

$$K(\mathscr{A})^c_n \to \Omega K(\mathscr{A})^c_{n+1}$$

is a weak equivalence. By Waldhausen's additivity theorem (see [43]), we have weak equivalences

$$K(\mathscr{A})^c_n \xrightarrow{\sim} \Omega K(\mathscr{A})^c_{n+1}$$

in $\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)$. Now, observe that for every integer n, $K^c(\mathscr{A})_n$ satisfies conditions (1)–(3) of Proposition 14.10. Condition (4) follows from Waldhausen's fibration theorem (as in the proof of Proposition 14.11) applied to the $S_\bullet$-construction. This shows that $K(\mathscr{A})^c$ is an Ω-spectrum.

We now prove that η is a (componentwise) weak equivalence in the model structure $\mathsf{Sp}^{\mathbb{N}}(\mathsf{L}_{\widetilde{\mathscr{E}^s_{\mathrm{un}}}}\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet))$. For this, we prove first that the structural morphisms

$$\beta_n : S^1 \wedge N.wS^{(n+1)}_\bullet\,\mathsf{rep}_{\mathrm{mor}}(?,\mathscr{A}) \xrightarrow{\sim} |N.wS^{(n+2)}_\bullet\,\mathsf{rep}_{\mathrm{mor}}(?,\mathscr{A})|, \quad n \geq 0$$

(see Notation 15.8), are weak equivalences in $\mathsf{L}_{\widetilde{\mathscr{E}^s_{\mathrm{un}}}}\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)$. By considering the same argument as in the proof of Proposition 14.11 and using $S^{(n+1)}_\bullet\mathscr{A}$

instead of $\mathscr{A}$, we obtain the homotopy co-Cartesian square

$$\begin{array}{ccc} K(\mathscr{A})_n^c & \longrightarrow & p_!\big(\mathscr{U}_u(PS_\bullet^{(n+2)}\mathscr{A})\big) \\ \downarrow & & \downarrow \\ * & \longrightarrow & K(\mathscr{A})_{n+1}^c \end{array}$$

in $\mathsf{L}_{\widetilde{\mathscr{E}^s_{\mathrm{un}}}}\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)$ with $p_!(\mathscr{U}_u(PS_\bullet^{(n+2)}\mathscr{A}))$ contractible. Since $K(\mathscr{A})_{n+1}^c$ is fibrant, [43, Proposition 1.5.3] implies that the previous square is also homotopy Cartesian, and so the canonical morphism

$$\beta_n^\sharp : K(\mathscr{A})_n^c \to \Omega K(\mathscr{A})_{n+1}^c$$

is a weak equivalence in $\mathsf{L}_{\widetilde{\mathscr{E}^s_{\mathrm{un}}}}\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet)$. We now show that the structure morphism β_n, which corresponds to $\beta_n^\sharp$ by adjunction (see [43]), is also a weak equivalence. The derived adjunction $(S^1 \wedge -, \mathbb{R}\Omega(-))$ induces the commutative diagram

$$\begin{array}{ccc} S^1 \wedge K(\mathscr{A})_n^c & \longrightarrow & K(\mathscr{A})_{n+1}^c \\ & \searrow^{S^1\wedge^{\mathbb{L}}\beta_n^\sharp} & \uparrow \sim \\ & & S^1 \wedge^{\mathbb{L}} \Omega K(\mathscr{A})_{n+1}^c \end{array}$$

in $\mathsf{Ho}(\mathsf{L}_{\widetilde{\mathscr{E}^s_{\mathrm{un}}}}\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet))$, where the vertical arrow is an isomorphism since the previous square is homotopy bi-Cartesian. This shows that the induced morphism

$$S^1 \wedge K(\mathscr{A})_n^c \longrightarrow K(\mathscr{A})_{n+1}^c$$

is an isomorphism in $\mathsf{Ho}(\mathsf{L}_{\widetilde{\mathscr{E}^s_{\mathrm{un}}}}\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet))$, and so β_n is a weak equivalence.

Now, to prove that η is a componentwise weak equivalence, we proceed by induction. Observe that the zero component of the morphism η is the identity. Now, suppose that the n-component of η is a weak equivalence. The $(n+1)$-component of η is the composition of β_{n+1}, which is a weak equivalence, with the suspension of the n-component of η, which by Proposition 4.1 is also a weak equivalence.

This proves the theorem. □

Let $\mathscr{A}$ and $\mathscr{B}$ be small dg categories with $\mathscr{A} \in \mathsf{dgcat}_f$. We denote by $\mathsf{Hom}^{\mathsf{Sp}^{\mathbb{N}}}(-,-)$ the spectrum of morphisms in $\mathsf{Sp}^{\mathbb{N}}(\mathsf{L}_{\widetilde{\mathscr{E}^s_{\mathrm{un}}}}\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet))$.

THEOREM 15.10
We have the weak equivalence of spectra

$$\mathsf{Hom}^{\mathsf{Sp}^{\mathbb{N}}}\big(\mathscr{U}_a(\mathscr{A}), \mathscr{U}_a(\mathscr{B})[1]\big) \stackrel{\sim}{\to} K^c\big(\mathsf{rep}_{\mathrm{mor}}(\mathscr{A}, \mathscr{B})\big),$$

where $K^c(\mathsf{rep}_{\mathrm{mor}}(\mathscr{A}, \mathscr{B}))$ *denotes Waldhausen's connective* K*-theory spectrum of* $\mathsf{rep}_{\mathrm{mor}}(\mathscr{A}, \mathscr{B})$.

In particular, we have the weak equivalence of simplicial sets

$$\mathsf{Map}\big(\mathscr{U}_a(\mathscr{A}), \mathscr{U}_a(\mathscr{B})[1]\big) \stackrel{\sim}{\to} |N.wS_\bullet\, \mathsf{rep}_{\mathrm{mor}}(\mathscr{A}, \mathscr{B})|,$$

and so the isomorphisms

$$\pi_{i+1}\mathsf{Map}\big(\mathscr{U}_a(A), \mathscr{U}_a(\mathscr{B})[1]\big) \stackrel{\sim}{\to} K_i\big(\mathsf{rep}_{\mathrm{mor}}(\mathscr{A}, \mathscr{B})\big), \quad \forall i \geq 0.$$

Proof
Notice that $\mathscr{U}_a(\mathscr{A})$ identifies with the suspension spectrum

$$\Sigma^\infty |N.w\, \mathsf{rep}_{\mathrm{mor}}(?, \mathscr{A})|$$

which is cofibrant in $\mathsf{Sp}^{\mathbb{N}}(\mathsf{L}_{\widetilde{\mathscr{E}^s_{\mathrm{un}}}}\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, \mathrm{Sset}_\bullet))$. By Theorem 15.9, we have the equivalences

$$\begin{aligned}\mathsf{Hom}^{\mathsf{Sp}^{\mathbb{N}}}\big(\mathscr{U}_a(\mathscr{A}), \mathscr{U}_a(\mathscr{B})[1]\big) &\simeq \mathsf{Hom}^{\mathsf{Sp}^{\mathbb{N}}}\big(\mathscr{U}_a(\mathscr{A}), K^c(\mathscr{B})\big)\\ &\simeq K^c(\mathscr{B})(\mathscr{A})\\ &\simeq K^c\big(\mathsf{rep}_{\mathrm{mor}}(\mathscr{A}, \mathscr{B})\big).\end{aligned}$$

This proves the theorem. □

Remark 15.11
Note in Theorem 15.10 that if we consider $\mathscr{A} = k$, we have

$$\mathsf{Hom}^{\mathsf{Sp}^{\mathbb{N}}}\big(\mathscr{U}_a(k), \mathscr{U}_a(\mathscr{B})[1]\big) \stackrel{\sim}{\to} K^c(\mathscr{B}).$$

This shows that Waldhausen's connective K-theory spectrum becomes corepresentable in $\mathscr{M}^{\mathrm{add}}_{\mathrm{dg}}$. To the best of the author's knowledge, this is the first conceptual characterization of Quillen and Waldhausen's K-theory (see [34], [43]) since its definition in the early 1970s. This result gives us a completely new way to think about algebraic K-theory.

16. Higher Chern characters

In this section, we apply our main corepresentability throrem, Theorem 15.10, in the construction of the higher Chern characters (see [26, Section 11.4.3]).

Let $\mathscr{A}$ and $\mathscr{B}$ be small dg categories with $\mathscr{A} \in \mathsf{dgcat}_f$.

PROPOSITION 16.1
We have the isomorphisms of abelian groups

$$\mathsf{Hom}_{\mathscr{M}_{\mathrm{dg}}^{\mathrm{add}}(e)}\big(\mathscr{U}_a(\mathscr{A}), \mathscr{U}_a(\mathscr{B})[-n]\big) \xrightarrow{\sim} K_n\big(\mathsf{rep}_{\mathrm{mor}}(\mathscr{A}, \mathscr{B})\big), \quad \forall n \geq 0.$$

Proof
In the first place, note that the abelian group

$$\mathsf{Hom}_{\mathscr{M}_{\mathrm{dg}}^{\mathrm{add}}(e)}\big(\mathscr{U}_a(\mathscr{A}), \mathscr{U}_a(\mathscr{B})[-n]\big)$$

identifies with

$$\pi_0 \mathsf{Map}\big(\mathscr{U}_a(\mathscr{A}), \mathscr{U}_a(\mathscr{B})[-n]\big),$$

where Map denotes the mapping space in $\mathsf{Sp}^{\mathbb{N}}(\mathsf{L}_{\widetilde{\mathscr{E}^s_{\mathrm{un}}}}\mathsf{L}_{\Sigma,P}\mathsf{Fun}(\mathsf{dgcat}_f^{\mathrm{op}}, Sset_\bullet))$. By Theorem 15.9, the morphism

$$\eta : \mathscr{U}_a(\mathscr{B})[1] \longrightarrow K(\mathscr{B})^c$$

is a fibrant resolution of $\mathscr{U}_a(\mathscr{B})[1]$. This implies that in $\mathscr{M}_{\mathrm{dg}}^{\mathrm{add}}(e)$, $\mathscr{U}_a(\mathscr{B})[-n]$ identifies with the spectrum $\Omega^{n+1}(K^c(\mathscr{B}))$. Since $\mathscr{U}_a(\mathscr{A})$ is cofibrant and

$$\Omega^{n+1}\big(K^c(\mathscr{B})\big)_0 = \Omega^{n+1}|N.wS_\bullet \,\mathsf{rep}_{\mathrm{mor}}(?, \mathscr{B})|,$$

we conclude that

$$\pi_0 \mathsf{Map}\big(\mathscr{U}_a(\mathscr{A}), \mathscr{U}_a(\mathscr{B})[-n]\big) \simeq \pi_0 \Omega^{n+1}|N.wS_\bullet \,\mathsf{rep}_{\mathrm{mor}}(\mathscr{A}, \mathscr{B})|.$$

Finally, note that

$$\begin{aligned}\pi_0 \Omega^{n+1}|N.wS_\bullet \,\mathsf{rep}_{\mathrm{mor}}(\mathscr{A}, \mathscr{B})| &\simeq \pi_{n+1}|N.wS_\bullet \,\mathsf{rep}_{\mathrm{mor}}(\mathscr{A}, \mathscr{B})| \\ &\simeq K_n\big(\mathsf{rep}_{\mathrm{mor}}(\mathscr{A}, \mathscr{B})\big).\end{aligned}$$

This proves the proposition. □

Remark 16.2
Note that if, in Proposition 16.1, we consider $\mathscr{A} = k$, then we have the isomorphisms

$$\mathsf{Hom}_{\mathscr{M}_{\mathrm{dg}}^{\mathrm{add}}(e)}\big(\mathscr{U}_a(k), \mathscr{U}_a(\mathscr{B})[-n]\big) \xrightarrow{\sim} K_n(\mathscr{B}), \quad \forall n \geq 0.$$

This shows that the algebraic K-theory groups $K_n(-)$, $n \geq 0$, are corepresentable in the triangulated category $\mathscr{M}_{\mathrm{dg}}^{\mathrm{add}}(e)$.

Now, let

$$K_n(-) : \mathsf{Ho}(\mathsf{dgcat}) \longrightarrow \text{Mod-}\mathbb{Z}, \quad n \geq 0,$$

be the nth K-theory group functor (see Theorem 10.7), and let

$$HC_j(-) : \mathsf{Ho}(\mathsf{dgcat}) \longrightarrow \text{Mod-}\mathbb{Z}, \quad j \geq 0,$$

be the jth cyclic homology group functor (see Theorem 10.9).

THEOREM 16.3

The corepresentability theorem, Theorem 15.10, furnishes us with the higher Chern characters (see [26, Section 11.4.3])

$$\mathrm{ch}_{n,r} : K_n(-) \longrightarrow HC_{n+2r}(-), \quad n, r \geq 0.$$

Proof

By Theorem 10.9, the morphism of derivators

$$C : \mathsf{HO}(\mathsf{dgcat}) \longrightarrow \mathsf{HO}(\Lambda\text{-Mod})$$

is an additive invariant and so descends to $\mathscr{M}^{\mathrm{add}}_{\mathrm{dg}}$ and induces a functor (still denoted by C)

$$C : \mathscr{M}^{\mathrm{add}}_{\mathrm{dg}}(e) \longrightarrow \mathscr{D}(\Lambda).$$

By [18, Theorem 1.3], the cyclic homology functor $HC_j(-)$, $j \geq 0$, is obtained by composing C with the functor

$$H^{-j}(k \overset{\mathbb{L}}{\otimes}_{\Lambda} -) : \mathscr{D}(\Lambda) \longrightarrow \text{Mod-}\mathbb{Z}, \quad j \geq 0.$$

Now, by Proposition 16.1 and Remark 16.2, the functor

$$K_n(-) : \mathscr{M}^{\mathrm{add}}_{\mathrm{dg}}(e) \longrightarrow \text{Mod-}\mathbb{Z}$$

is corepresented by $\mathscr{U}_a(k)[n]$. This implies, by the Yoneda lemma, that

$$\mathsf{Nat}\big(K_n(-), HC_j(-)\big) \simeq HC_j\big(\mathscr{U}_a(k)[n]\big).$$

Since we have the isomorphisms

$$\begin{aligned} HC_j\big(\mathscr{U}_a(k)[n]\big) &\simeq H^{-j}\big(k \overset{\mathbb{L}}{\otimes}_\Lambda C(\mathscr{U}_a(k)[n])\big) \\ &\simeq H^{-j}\big(k \overset{\mathbb{L}}{\otimes}_\Lambda C(k)[n]\big) \\ &\simeq H^{-j+n}\big(k \overset{\mathbb{L}}{\otimes}_\Lambda C(k)\big) \\ &\simeq HC_{j-n}(k) \end{aligned}$$

and since

$$HC_*(k) \simeq k[u], \quad |u| = 2,$$

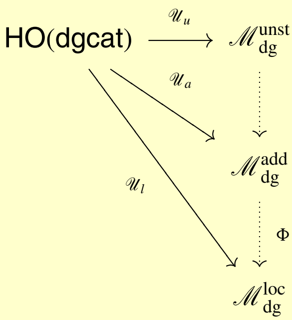
we conclude that

$$HC_j\big(\mathscr{U}_a(k)[n]\big) = \begin{cases} k & \text{if } j = n + 2r,\ r \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, note that the canonical element $1 \in k$ furnishes us with the higher Chern characters, and so the theorem is proved. □

17. Concluding remarks

By the universal properties of $\mathscr{U}_u$, $\mathscr{U}_a$, and $\mathscr{U}_l$, we obtain the following diagram:



Note that Waldhausen's connective K-theory is an example of an additive invariant which is *not* a derived one (see [21]). Waldhausen's connective K-theory becomes corepresentable in $\mathscr{M}_{dg}^{add}$ by Theorem 15.10.

An analogous result should be true for nonconnective K-theory, and the morphism

$$\Phi : \mathscr{M}_{dg}^{add} \dashrightarrow \mathscr{M}_{dg}^{loc}$$

should be thought of as corepresenting the "passage from additivity to localization."

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