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Topology of $\Sigma^{1,1}$-singular maps

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0. *Introduction*

Morin [**M**] gave a local normal form for singular maps having almost maximal rank, where almost maximal means maximal minus 1. The aim of the present paper is to give a global version of his normal form. We concentrate here on the case of $\Sigma^{1,1,0}$-singular maps. (For the definition see [**Bo**], [**A–G–V**], [**G–G**] and also here below.) The case of $\Sigma^{1,0}$ singular maps was considered by Haefliger in [**Ha**], see also [**Sz1**] and [**Sz2**]. For the motivation in finding such a global normal form see [**Sz1**], [**Sz2**], and the final remarks in this paper.

In order to formulate the result we recall some definitions and well-known facts, and also fix the notation.

Given a smooth map $f: M^n \to N^{n+k}$ of a smooth manifold M^n into a smooth manifold N^{n+k} of codimension $k > 0$, we shall denote by $\Sigma^i(f)$ the set of points of the source manifold M^n, where the kernel of the differential df is i dimensional, i.e.

$$\Sigma^i(f) = \{x \in M^n \mid \operatorname{rank} df(x) = n-i\}.$$

The points of $\Sigma^i(f)$ will be called the Σ^i-singular points of f.

For generic maps $\Sigma^i(f)$ is a submanifold of M^n and one can also define $\Sigma^{i,j}(f)$ as the set of Σ^j singular points of the restriction of f to $\Sigma^i(f)$, i.e.

$\Sigma^{i,j}(f) = \Sigma^j(f \mid \Sigma^i(f))$.

The points if $\Sigma^{i,j}(f)$ will be called the $\Sigma^{i,j}$-singular points of f.

Again for generic maps $\Sigma^{i,j}(f)$ is a submanifold in M^n and one can define $\Sigma^{i,j,k}(f) = \Sigma^k(f \mid \Sigma^{i,j}(f))$ etc. In this paper we shall not need more complicated singularities than those of type $\Sigma^{1,1}$. One has $\dim \Sigma^1(f) = n-(k+1)$, $\dim \Sigma^{1,1}(f) = n-2(k+1)$, (see for example [**A–G–V**]).

In the next definition we describe the type of maps we shall deal with in this paper (see also Remark 3 in Section 5).

Definition. Let M^n and N^{n+k} be smooth (not necessarily compact) manifolds of dimensions n and $n+k$ respectively and let $f: M^n \to N^{n+k}$ be a proper smooth map. We shall say that f is a simple $\Sigma^{1,1}$-singular map if

(a) f has only $\Sigma^{1,0}$ and $\Sigma^{1,1}$ singular points (it may not have them at all);

(b) the line bundle formed by the kernels of the differentials of f is a trivial bundle over $\Sigma^{1,1}(f)$;

(c) the $\Sigma^{1,1}$ singular points are not multiple (i.e. $x \in \Sigma^{1,1}(f) \Rightarrow f^{-1}(f(x)) = \{x\}$);

(d) the map is generic in the sense that at each $\Sigma^{1,1}$-singular point Morin's local

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normal form holds. (This is equivalent to some transversality condition for the 3-jet of the map.)

From now on the codimension k will be fixed and we shall consider throughout this paper simple $\Sigma^{1,1}$-singular maps of codimension k. Actually we shall consider these maps only in a tubular neighbourhood of the submanifold $\Sigma^{1,1}(f)$, and by a global description of the map f we mean a description of its restriction to such a neighbourhood. (Perhaps it would be more correct to call the description we are going to give not global but semi-global.)

Notation. Let us denote by $\tilde{\Sigma}^{1,1}(f) \subset N$ the image of $\Sigma^{1,1}(f)$ under f and let T and $\tilde{T}$ denote some tubular neighbourhoods of $\Sigma^{1,1}(f)$ in M and $\tilde{\Sigma}^{1,1}(f)$ in N respectively such that f maps T into $\tilde{T}$. So we have a commutative square

$$\begin{tikzcd} T \arrow[r, "f|T"] \arrow[d, "\pi"'] & \tilde{T} \arrow[d, "\tilde{\pi}"] \\ \Sigma^{1,1}(f) \arrow[r, "f|\Sigma^{1,1}(f)"] & \tilde{\Sigma}^{1,1}(f), \end{tikzcd} \tag{$*$}$$

where the vertical arrows are vector bundle projections with fibres R^{2k+2} and R^{3k+2} respectively, and the bottom horizontal arrow is a diffeomorphism (due to conditions (a) and (c)).

By giving a global description of simple $\Sigma^{1,1}$-singular maps we mean that:

(a) we show that there is a 'universal square' of the type $(*)$ (in the precise sense given in the theorem below), and

(b) we give a concrete construction of this universal square.

1. *Formulation of the theorem*

THEOREM 1. *There exist*

(i) *a space B,*

(ii) *vector bundles $\xi \to B$ and $\tilde{\xi} \to B$ with fibres R^{2k+2} and R^{3k+2} respectively and*

(iii) *a (nonlinear) fibrewise map $\Phi: \xi \to \tilde{\xi}$ such that for any simple $\Sigma^{1,1}$ map f of codimension k the square $(*)$ fits into the following commutative cube:*

$$\begin{tikzcd} \xi \arrow[rrr, "\Phi"] \arrow[ddd] & & & \tilde{\xi} \arrow[ddd] \\ & T \arrow[ul, "j"'] \arrow[r, "f|T"] \arrow[d] & \tilde{T} \arrow[ur, "\tilde{j}"] \arrow[d] & \\ & \Sigma^{1,1}(f) \arrow[r, "f|\Sigma^{1,1}(f)"] \arrow[dl] & \tilde{\Sigma}^{1,1}(f) \arrow[dr] & \\ B \arrow[rrr, "1"] & & & B\ . \end{tikzcd}$$

This means that there are fibrewise maps $j: T \to \xi$ and $\tilde{j}: \tilde{T} \to \tilde{\xi}$ which are linear isomorphisms on each fibre and $\Phi \circ j = \tilde{j} \circ (f|_T)$.

More interesting than the mere existence of the universal mapping $\Phi: \xi \to \tilde{\xi}$ is that it can be constructed in very concrete terms, as follows: the space B is the Grassmann manifold $BO(k)$, which is the base space of the universal k dimensional vector bundle γ_k. Let us denote by ϵ^i the i dimensional trivial vector bundle. Then the bundles ξ and $\tilde{\xi}$ are $2\gamma_k \oplus \epsilon^2$ and $3\gamma_k \oplus \epsilon^2$ respectively.

Before giving the description of the map $\Phi: \xi \to \tilde{\xi}$ we first recall Morin's normal form for $\Sigma^{1,1}$-maps.

Let $\phi: (R^{2k+2}, 0) \to (R^{3k+2}, 0)$ be the map given by the following formulas:

Writing the coordinates in R^{2k+2} as $(t_1, t_2, \ldots, t_{2k+1}, x)$ and in R^{3k+2} as $(y_1, y_2, \ldots, y_{2k+1}, z_1, \ldots, z_k, z_{k+1})$ then

$$\begin{aligned} y_i \circ \phi &= t_i, \quad i = 1, \ldots, 2k+1, \\ z_1 \circ \phi &= xt_1 + x^2 \cdot t_2, \\ &\vdots \\ z_k \circ \phi &= xt_{2k-1} + x^2 \cdot t_{2k}, \\ z_{k+1} \circ \phi &= xt_{2k+1} + x^3, \end{aligned}$$

or equivalently

$$\phi(t_1, t_2, \ldots, t_{2k+1}, x) = (t_1, \ldots, t_{2k+1}, xt_1 + x^2 t_2, \ldots, xt_{2k-1} + x^2 t_{2k}, xt_{2k+1} + x^3).$$

THEOREM (Morin [**M**]). *If $g: M^n \to N^{n+k}$ is a generic $\Sigma^{1,1}$-singular map then for any $p \in \Sigma^{1,1}(g)$ there exist neighbourhoods of $p \in M^n$ and $g(p) \in N^{n+k}$ with local coordinates*

$$(t_1, t_2, \ldots, t_{2k+1}, x, u_1, u_2, \ldots, u_{n-2k-2})$$

and

$$(y_1, y_2, \ldots, y_{2k+1}, z_1, z_2, \ldots, z_1, \ldots, z_{k+1}, u_1, u_2, \ldots, u_{n-2k-2})$$

in which the map g has the form $\phi \times 1$, where 1 is the identity map of the space R^{n-2k-2} with the coordinates $(u_1, u_2, \ldots, u_{n-2k-2})$.

Now we define a $2k+2$ and a $3k+2$ dimensional representation of the group $O(k)$, closely related to the above map ϕ, as follows. We decompose the space R^{2k+2} with coordinates $t_1, t_2, \ldots, t_{2k+1}, x$ as

$$R^{2k+2} = R_1^k \oplus R_2^k \oplus R^2,$$

where R^2 is the coordinate space $\{(t_{2k+1}, x)\}$ (i.e. on R^2 all the other coordinates are zero), R_1^k is the 'odd' coordinate space $\{(t_1, t_3, \ldots, t_{2k-1})\}$ and R_2^k is the 'even' coordinate space $\{(t_2, t_4, \ldots, t_{2k})\}$.

Similarly we decompose the space $R^{3k+2} = \{(y_1, \ldots, y_{2k+1}, z_1, \ldots, z_k, z_{k+1})\}$ as:

$$R^{3k+2} = \tilde{R}_1^k \oplus \tilde{R}_2^k \oplus \tilde{R}_3^k \oplus \tilde{R}^2,$$

where $\tilde{R}^2 = \{(y_{2k+1}, z_{k+1})\}$, $\tilde{R}_1^k = \{(y_1, y_3, \ldots, y_{2k-1})\}$, $\tilde{R}_2^k = \{(y_2, y_4, \ldots, y_{2k})\}$ and $\tilde{R}_3^k = \{(z_1, z_2, \ldots, z_k)\}$.

Given an orthogonal $k \times k$ matrix $A \in O(k)$, let us denote by $\alpha(A)$ the map $R^{2k+2} \to R^{2k+2}$ acting on R_1^k and on R_2^k as A and as the identity on R^2 (i.e. $\alpha(A)$ is the blockdiagonal matrix $\langle A, A, 1, 1 \rangle$). Denote by $\beta(A)$ the map $R^{3k+2} \to R^{3k+2}$ acting on the spaces $\tilde{R}_1^k$, $\tilde{R}_2^k$, $\tilde{R}_3^k$ as A and as the identity on $\tilde{R}^2$, (i.e. $\beta(A) = \langle A, A, A, 1, 1 \rangle$).

Then $\alpha: O(k) \to O(2k+2)$, $A \to \alpha(A)$ and $\beta: O(k) \to O(3k+2)$, $A \to \beta(A)$ are representations of the group $O(k)$ of dimensions $2k+2$ and $3k+2$ respectively.

Now the description of the map $\Phi: \xi \to \tilde{\xi}$ is the following: let us consider the map $id \times \phi: EO(k) \times R^{2k+2} \to EO(k) \times R^{3k+2}$. Here $EO(k)$ is the usual contractible space with free $O(k)$ action, id is its identity map and ϕ is the Morin map we have just described. The map $id \times \phi$ is $O(k)$-equivariant with respect to the diagonal actions. (The actions on the euclidean factors are defined by the representations α and β respectively.) Factoring out by the $O(k)$ actions, we obtain the induced map of quotient spaces denoted Φ, i.e.

$$\Phi = id \times_{O(k)} \phi.$$

Note that the total space of the bundle $\xi = 2\gamma_k \oplus \epsilon^2$ coincides with the quotient space $EO(k) \times_{O(k)} R^{2k+2}$, and the total space of $\tilde{\xi} = 3\gamma_k \oplus \epsilon^2$ is the space $EO(k) \times_{O(k)} R^{3k+2}$, therefore the map Φ we have just defined is indeed a map from the bundle ξ into $\tilde{\xi}$ as promised.

The theorem can be reformulated briefly as follows:

THEOREM 1′. *Let f be a simple $\Sigma^{1,1}$-singular map, and let T and $\tilde{T}$ be as above* (*the tubular neighbourhoods of the set of $\Sigma^{1,1}$-singular points and its image respectively*). *Then the map*

$$f|T: T \to \tilde{T},$$

is a 'bundle of mappings with fibre $\phi: R^{2k+2} \to R^{3k+2}$, and with structure group $G \approx O(k)$'. Here G acts on R^{2k+2} by the above representation α and on R^{3k+2} by β.

In order to give a precise meaning to this brief formulation we define the notion of 'bundle of maps' with a given map as fibre and with a given structure group.

Definition. Let G be a topological group, let $P \to B$ be a principal G-bundle, let X and $\tilde{X}$ be G-spaces and let $\phi: X \to \tilde{X}$ be a G-equivariant map. Then the map

$$id_P \times \phi: P \times X \to P \times \tilde{X}$$

is G-equivariant if we let the group G act on $X \times P$ and $\tilde{X} \times P$ by the diagonal actions. The induced map of the quotient spaces

$$\Phi: P \times_G X \to P \times_G \tilde{X}$$

will be called the canonical bundle map with fibre ϕ associated to the principal bundle $P \to B$. A map $f: V \to \tilde{V}$ will be called a bundle of maps with fibre ϕ and structure group G if there exist:

(a) a principal G-bundle $P \to B$, and

(b) homeomorphisms $H: V \to P \times_G X$, $\tilde{H}: \tilde{V} \to P \times_G \tilde{X}$ such that $\Phi \circ H = \tilde{H} \circ f$.

A second reformulation of the theorem which we shall use is:

THEOREM 1″. *Let f, T and $\tilde{T}$ be as in Theorem* 1. *Then*

(i) *there are local coordinate systems $(U_1, \alpha_1), \ldots, (U_r, \alpha_r)$ in T, where $\{U_1, U_2, \ldots, U_r\}$ is an open covering of T, α_i is a diffeomorphism of U_i onto R^n and*

(ii) *there are local coordinate systems $(\tilde{U}_1, \tilde{\alpha}_1), (\tilde{U}_2, \tilde{\alpha}_2), \ldots, (\tilde{U}_r, \tilde{\alpha}_r)$, where $\{\tilde{U}_1, \ldots, \tilde{U}_r\}$ is an open covering of $\tilde{T}$, $\tilde{\alpha}_i$ is a diffeomorphism of $\tilde{U}_i$ onto R^{n+k} and the following conditions are satisfied*:

(a) $f(U_i) \subset \tilde{U}_i$,

(b) $\tilde{\alpha}_i \circ (f | U_i) \circ \alpha_i^{-1}$ *has the normal form given by Morin,*

(c) *these local coordinates define vector-bundle structures on T and $\tilde{T}$ with structure groups $\alpha(O(k)) = \{\langle A, A, 1, 1\rangle \mid A \in O(k)\}$ and $\beta(O(k)) = \{\langle A, A, A, 1, 1\rangle \mid A \in O(k)\}$ respectively.*

Here condition (c) means the following: decompositions of R^n and R^{n+k} are fixed into the products $R^n = R^{2k+2} \times R^{n-2k-2}$ and $R^{n+k} = R^{3k+2} \times R^{n-2k-2}$ respectively. For any $\underline{u} \in R^{n-2k-2}$ the products $R^{2k+2} \times \{\underline{u}\}$ and $R^{3k+2} \times \{\underline{u}\}$ can be identified with R^{2k+2} and R^{3k+2} respectively. Now the requirement is that the restrictions of the transition maps $\alpha_i \circ \alpha_j^{-1}$ and $\tilde{\alpha}_i \circ \tilde{\alpha}_j^{-1}$ to $R^{2k+2} \times \{\underline{u}\}$ and $R^{3k+2} \times \{\underline{u}\}$ are block diagonal matrices of the forms $\langle A, A, 1, 1\rangle$ and $\langle A, A, A, 1, 1\rangle$ respectively, where $A \in O(k)$.

2. *Preparations for the proof*

Definition. Let us denote by $\operatorname{Diff}(R^m, 0)$ the germ of diffeomorphisms of R^m at the origin. An automorphism of the map ϕ is a pair (a, b) where

$$a \in \operatorname{Diff}(R^{2k+2}, 0), \quad b \in \operatorname{Diff}(R^{3k+2}, 0),$$

and $\phi \circ a = b \circ \phi$. An automorphism (a, b) will be called reduced if the differential $da(0)$ keeps the orientation of the line $\ker d\phi$ at any point. We shall denote by $\mathscr{A}$ the group of reduced automorphisms of ϕ. Therefore

$$\mathscr{A} \subset \operatorname{Diff}(R^{2k+1}) \times \operatorname{Diff}(R^{3k+2})$$

$$\mathscr{A} = \{(a, b) \in \operatorname{Diff}(R^{2k+1}) \times \operatorname{Diff}(R^{3k+2}) \,|da| \ker df \text{ keeps the orientation}\}$$

and the index of $\mathscr{A}$ in the whole automorphism group $\operatorname{Aut}\phi$ is 2.

Notice that for any $A \in O(k)$ the pair $(\alpha(A), \beta(A))$ is a reduced automorphism of ϕ.

Remark 1. Let H be any subgroup of the reduced automorphism group $\mathscr{A}$. Then analogues of the bundles ξ, $\tilde{\xi}$ and of the map Φ can be defined as follows:

$$\xi_H = EH \times_H R^{2k+2}, \quad \tilde{\xi}_H = EH \times_H R^{3k+2}, \quad \text{and} \quad \Phi_H = id \times_H \phi.$$

Remark 2. Let us define the group G as follows: $G = \{(\alpha(A), \beta(A)) \mid A \in O(k)\}$. Obviously G is a subgroup of the reduced automorphism group $\mathscr{A}$ and it is isomorphic to the orthogonal group $O(k)$. Then $\Phi_G = \Phi$.

Let us denote by p_1 and p_2 the projections of $\operatorname{Diff}(R^{2k+2}, 0) \times \operatorname{Diff}(R^{3k+2}, 0)$ onto the first and second factors respectively.

Lemma 1. *The projection p_2 restricted to the group of reduced automorphisms of ϕ is monomorphic.*

Proof. The set of non-multiple points of ϕ in R^{2k+2} forms a dense subset, because the codimension of ϕ is positive. Suppose then that $\theta \in \mathscr{A}$ is an automorphism of the form $\theta = (a, id)$, where $a \in \operatorname{Diff}(R^{2k+2}, 0)$ and $id \in \operatorname{Diff}(R^{3k+2}, 0)$ is the identity. Then $\phi \circ a = \phi$. If x is a non-multiple point of ϕ, i.e. $\phi^{-1}(\phi(x)) = \{x\}$, then $\phi(a(x)) = \phi(x)$ implies $a(x) = x$, i.e. a is fixed on the set of non-multiple points of ϕ. Since this set is dense and a is continuous the map a is the identity.

Corollary. *For any subgroup σ of the group $\mathscr{A}$ we have $p_2(\sigma) \approx \sigma$.*

3. *Proof of the theorem*

In [**Sz3**] we have shown that the local coordinate systems can be chosen to satisfy (a) and (b) of Theorem 1″. This implies that the following weaker form of (c) is also satisfied: (c_{weak}) the restrictions of the transition maps $\alpha_i \circ \alpha_j^{-1}$ and $\tilde{\alpha}_i \circ \tilde{\alpha}_j^{-1}$ to the spaces $R^{2k+2} \times \{\underline{u}\}$ and $R^{3k+2} \times \{\underline{u}\}$ respectively form a pair that defines an automorphism of the map $\phi: R^{2k+2} \to R^{3k+2}$.

Lemma 2. *If $f: M^n \to N^{n+k}$ is a simple $\Sigma^{1,1}$-singular map then the local coordinate systems above can be chosen in such a way that the above defined automorphisms of ϕ are reduced.*

Proof of Lemma 2. Suppose the transition map $\alpha_j \circ \alpha_i^{-1}$ changes the orientation of $\ker \mathrm{d}f$. Then we can change the signs of the variables $x, t_1, t_3, \dots, t_{2k-1}$ in the source and of $y_1, y_3, \dots, y_{2k-1}, z_{k+1}$ to get coordinate systems satisfying the requirement. More precisely: we start with sets of local coordinate systems $\{(U_1, \alpha_1), \dots, (U_r, \alpha_r)\}$ and $\{(\tilde{U}_1, \tilde{\alpha}_1), \dots, \tilde{U}_r, \tilde{\alpha}_r)\}$ satisfying (a) and (b) in Theorem 1″, and take the first pair (U_1, α_1), ($\tilde{U}_1, \tilde{\alpha}_1$), then we take the second pair and check whether the transition map $\alpha_1 \circ \alpha_2^{-1}$ keeps the orientation of the kernel or not. If it does then we do not change anything, but go to the third pair of local coordinates. If it does change the orientation of $\ker df$ then we replace (U_2, α_2) and $(\tilde{U}_2, \tilde{\alpha}_2)$ by (U_2, α_2') and $(\tilde{U}_2, \tilde{\alpha}_2')$ respectively, where α_2' is the composition of α_2 with the map $s: R^n \to R^n$ given by

$$s(t_1, t_2, \dots, t_{2k+1}, x, u_1, u_2, \dots, u_{n-2k-2}) \\ = (-t_1, t_2, -t_3, \dots, -t_{2k-1}, t_{2k}, t_{2k+1}, -x, u_1, u_2, \dots, u_{n-2k-2})$$

and $\tilde{\alpha}_2'$ is the composition of $\tilde{\alpha}_2$ with the map $\tilde{s}: R^{n+k} \to R^{n+k}$ given by

$$\tilde{s}(y_1, y_2, \dots, y_{2k+1}, z_1, z_2, \dots, z_{k+1}, u_1, u_2, \dots, u_{n-2k-2}) \\ = (-y_1, y_2, -y_3, \dots, -y_{2k-1}, y_{2k}, y_{2k+1}, z_1, z_2, \dots, -z_{k+1}, u_1, u_2, \dots, u_{n-2k-2}).$$

Then we go to the third pair of local coordinate systems and do the same alteration if necessary, etc. ∎

Now a theorem of Jänich and Wall (see [**J**] and [**W**]) says that the automorphism group of a finitely determined stable germ has a maximal compact subgroup, moreover any compact subgroup is contained in a maximal one, and finally a maximal compact subgroup is homotopically equivalent to the whole automorphism group in a suitable sense, in particular any fibre bundle whose structure group is the whole automorphism group can be reduced to any maximal compact subgroup. It is not hard to see that these claims also remain true for the index 2 subgroup of reduced automorphisms.

This means that the coordinate systems (U_i, α_i) and $(\tilde{U}_i, \tilde{\alpha}_i)$ can be chosen in such a way that the transition maps $\alpha_i \circ \alpha_j^{-1}$ and $\tilde{\alpha}_i \circ \tilde{\alpha}_j^{-1}$ restricted to $R^{2k+2} \times \{\underline{u}\}$ and $R^{3k+2} \times \{\underline{u}\}$ respectively give a pair of diffeomorphisms forming an element of any given maximal compact subgroup σ of $\mathscr{A}$.

Therefore in order to prove the theorem it remains to show that the structure group σ can be further reduced to the group $G = \{(\alpha(A), \beta(A)) \mid A \in O(k)\}$. Let us choose σ as a maximal compact subgroup containing G.

Lemma 3. *G itself is a maximal compact subgroup of the group of reduced automorphisms of the Morin map $\phi: R^{2k+2} \to R^{3k+2}$, i.e. $\sigma = G$.*

Proof. It is enough to show that the inclusion $G \subset \sigma$ is a homotopy equivalence. (A compact Lie group σ cannot be homotopy equivalent to a proper subgroup G. Indeed, since their top dimensional homology groups are isomorphic they have the same dimension. Therefore, the space σ/G is 0-dimensional. Because of the isomorphism on the 0-dimensional homologies this space consists of one single point and so $\sigma = G$.)

Now in order to show that the inclusion map $G \subset \sigma$ is a homotopy equivalence it is enough to show that it induces a homotopy equivalence of the classifying spaces $i: BG \to B\sigma$, since $G \cong \Omega BG \cong \Omega B\sigma \cong \sigma$, where $\cong$ means 'homotopically equivalent'. It remains to construct a homotopy inverse $j: B\sigma \to BG$ of the map $i: BG \to B\sigma$.

We shall construct j on each finite dimensional approximation of $B\sigma$. Given a natural number q let us consider the q times join of σ with itself: $E^q\sigma = \sigma * \sigma * \ldots * \sigma$ provided with the diagonal σ-action. Let $B^q\sigma$ be the quotient space $E^q\sigma/\sigma$. According to Milnor's construction the space $B\sigma$ is the limit of the spaces $B^q\sigma$:

$$B\sigma = \lim_{q\to\infty} B^q\sigma.$$

Let us denote by ξ^q_σ the space $E^q\sigma \times_\sigma R^{2k+2}$, by $\tilde{\xi}^q_\sigma$ the space $E^q\sigma \times_\sigma R^{3k+2}$, and by $\Phi^q_\sigma: \xi^q_\sigma \to \tilde{\xi}^q_\sigma$ the map induced by the σ-equivariant map $id(E^q\sigma) \times \phi$, where $id(E^q\sigma)$ denotes the identity of the space $E^q\sigma$.

Lemma 4. *For any codimension k simple $\Sigma^{1,1}$-singular map g denote by $\nu(g)$ and $\tilde{\nu}(g)$ the normal bundles of the submanifold of $\Sigma^{1,1}$ singular points $\Sigma^{1,1}(g)$ in the source and of its image $\tilde{\Sigma}^{1,1}(g)$ in the target manifold respectively. Then*

(*a*) *there exists a k dimensional vector-bundle $\tilde{\eta}$ over $\tilde{\Sigma}^{1,1}(g)$ such that $\tilde{\nu}(g) \approx \epsilon^2 \oplus 3\tilde{\eta}$, and*

(*b*) *if we denote by η the pull-back $(g \mid \Sigma^{1,1}(g))^*(\tilde{\eta})$ then $\nu(g) \approx \epsilon^2 \oplus 2\eta$.*

The proof of this lemma will be given in Section 4.

Key remark. Note that each map Φ^q_σ itself is a simple $\Sigma^{1,1}$-map. Therefore we can apply Lemma 4 to the map $g = \Phi^q_\sigma$.

In this case $\Sigma^{1,1}(g)$ is the zero section of ξ^q_σ and $\tilde{\Sigma}^{1,1}(g)$ is the zero section of $\tilde{\xi}^q_\sigma$. Therefore the normal bundles $\nu(g)$ and $\tilde{\nu}(g)$ are the bundles ξ^q_σ and $\tilde{\xi}^q_\sigma$ respectively. By the statement of Lemma 4 there is a bundle map

$$\begin{array}{ccc} \tilde{\xi}^q_\sigma & \longrightarrow & 3\gamma_k \oplus \epsilon^2 \\ \downarrow & & \downarrow \\ B^q\sigma & \xrightarrow{j_q} & BO(k) \end{array}$$

which is an isomorphism on each fibre.

The bundle $3\gamma_k \oplus \epsilon^2$ is the same as $\tilde{\xi}_G \to BG = BO(k)$. (See Remark 1 in Section 2 with $H = G$.) In Lemma 5 below we will prove that the maps j_q can be chosen in such a way that $j_{q+1} \mid B^q\sigma = j_q$. Therefore they define a map $j: B\sigma \to BG$, which is similarly covered by a bundle map $\tilde{\xi}_\sigma \to \tilde{\xi}_G$ that is a linear isomorphism on each fibre.

To finish the proof of the statement $G = \sigma$ we show that i and j are each other's homotopy inverses.

Notice that by construction the bundles $\tilde{\xi}_G$ and $\tilde{\xi}_\sigma$ are the universal $3k+2$-dimensional bundles with structure groups $p_2(G)$ and $p_2(\sigma)$ respectively and $p_2(G) \approx G$; $p_2(\sigma) \approx \sigma$ by the Corollary in Section 1. Universality of the bundles $\tilde{\xi}_G$ and $\tilde{\xi}_\sigma$ means in particular that if a bundle (with fibre R^{3k+2}) over a space X can be induced from $\tilde{\xi}_G$ or $\tilde{\xi}_\sigma$ by a map $X \to BG$ or $X \to B\sigma$ respectively, then the homotopy class of this map is uniquely defined. The composition $j \circ i$ induces the bundle $\tilde{\xi}_G$ from itself (since the map $j \circ i$ is covered by a mapping of the total space $\tilde{\xi}_G \to \tilde{\xi}_G$, which is a linear isomorphism on each fibre). Since the identity map also obviously induces the bundle $\tilde{\xi}_G$ from itself, by the universality of $\tilde{\xi}_G$ the map $j \circ i$ is homotopic to the identity. Similar reasoning using the universality of the bundle $\tilde{\xi}_\sigma$ gives $i \circ j \cong 1$. ∎

LEMMA 5. *The maps* $j_q : B^q\sigma \to BG$ *can be defined in such a way that* $j_{q+1} | B^q\sigma = j_q$ *and therefore they define a map* $j : B\sigma \to BG$.

Proof. The space $B\sigma$ is well-defined only up to homotopy. We shall replace $B\sigma = \lim_{q\to\infty} (B^q\sigma)$ by the so called telescope construction, a homotopically equivalent space $B\sigma'$ for which we shall obtain a map $j' : B\sigma' \to BG$ covered by a bundle map of the bundles.

Let $i^q : B^q\sigma \subset B^{q+1}\sigma$ be the natural inclusion. Let $C(i^q)$ be the cylinder of this map. This cylinder contains both $B^q\sigma$ and $B^{q+1}\sigma$. Attach $C(i^{q+1})$ to $C(i^q)$ by identifying $B^{q+1}\sigma \subset C(i^q)$ with $B^{q+1}\sigma \subset C(i^{q+1})$. After having done this for each q we get a ‘telescope’-space homotopically equivalent to $B\sigma$. Now we define the map j on this telescope space as j_q on $B^q\sigma$ and as the homotopy between j_q and $j_{q+1} | B^q\sigma$ on the qth cylinder. ∎

It remains to show Lemma 4. We shall do this by performing a rather laborious analysis of the local normal form and finding some invariantly defined subsets in the source and their images in the target.

4. *The proof of Lemma* 4

4.1. *Sets and bundles in the source manifold of* g. Below we give the equations of certain sets intrinsically associated with the map g in Morin's normal coordinates. In this section we shall denote the coordinates u_i from Section 1 by t_{i+2k+1} for $i = 1, \dots, n-2k-2$. The source and target manifolds of g will be denoted by M^n and N^{n+k} respectively (like those for f, but f will not occur anymore).

1. The set of singular points

$$\Sigma^1(g) = \{p \in M \mid \operatorname{rank} dg(p) = n-1\}$$
$$= \{p = (t_1, \dots, t_{n-1}, x) \mid t_1 + 2xt_2 = \dots = t_{2k-1} + 2xt_{2k} = t_{2k+1} + 3x^2 = 0\}.$$

Since the map g will be fixed from now on it will be omitted from the notation of the subsets defined here. For example the set $\Sigma^1(g)$ will be denoted simply as Σ^1.

2. The lines $\ker dg$ are the tangent lines of the x-curves ($\forall i\ t_i = \text{constant}$).

3. The set of the double points is:

$$\Delta = \{p \in M \mid \exists p' : p \neq p' \quad \text{and} \quad g(p) = g(p')\}$$
$$= \{(t_1, \dots, t_{n-1}, x) \mid \exists x^* : x + x^* = -t_1/t_2 = -t_3/t_4$$
$$= \dots = -t_{2k-1}/t_{2k} \quad \text{and} \quad -t_{2k+1} = x^2 + xx^* + x^{*2}\}.$$

4. The set of points that are both double and singular is

$$\Sigma^1 \cap \Delta = \{p \in M \mid \operatorname{rank} dg(p) = n-1 \text{ and } \exists p' \neq p \text{ such that } g(p) = g(p')\}$$
$$= \{(t_1, \dots, t_{n-1}, x) \mid t_1 = t_2 = \dots = t_{2k} = 0 \text{ and } t_{2k+1} = -3x^2\}.$$

5. The set of those nonsingular points which have the same image as a singular point will be denoted by $\Delta * \Sigma^1$.

$$\Delta * \Sigma^1 = \{p \in M \mid \operatorname{rank} dg(p) = n$$

and

$$\exists p' \neq p \text{ such that } g(p) = g(p') \text{ and } \operatorname{rank} dg(p') = n-1\}$$
$$= \{(t_1, \dots, t_{n-1}, x) \mid t_1 = t_2 = \dots = t_{2k} = 0 \text{ and } t_{2k+1} = -3/4 \cdot x^2\}.$$

6. The triple points will be denoted by $\mathfrak{T}$. The closure of this set $\mathfrak{T}$ will be denoted by $\bar{\mathfrak{T}}$. It is the union

$$\bar{\mathfrak{T}} = \mathfrak{T} \cup (\Sigma^1 \cap \Delta) \cup (\Delta * \Sigma^1)$$

In the local normal coordinates the set $\bar{\mathfrak{T}}$ is given by the following equalities and inequalities:

$$\bar{\mathfrak{T}} = \{p \in M \mid \exists p', p'' : p' \neq p'' \quad \text{and} \quad p \neq p', p''$$

such that

$$g(p) = g(p') = g(p'') \quad \text{or} \quad p \in \Sigma^1 \cap \Delta\}$$
$$= \{(t_1, t_2, \dots, t_{n-1}, x) \mid t_1 = t_2 = \dots = t_{2k} = 0 \text{ and } \exists x^* : -t_{2k+1} = x^2 + xx^* + x^{*2}\}$$
$$= \{(t_1, t_2, \dots, t_{n-1}, x) \mid t_1 = t_2 = \dots = t_{2k} = 0 \text{ and } 3x^2 \leqslant -4t_{2k+1}\}.$$

A point (t, x) of $\bar{\mathfrak{T}}$ belongs to the set $\mathfrak{T}$ if

$$-(x + x^*) \neq x, x^*$$

7. The set of $\Sigma^{1,1}$ singular points is:

$$\Sigma^{1,1} = \Sigma^1(g \mid \Sigma^1) = \{(t_1, \dots, t_{n-1}, x) \mid x = 0 \text{ and } t_1 = t_2 = \dots = t_{2k} = t_{2k+1} = 0\}.$$

Consequences

(1) $\Sigma^{1,1}$ is an $n-2k-2$ dimensional closed manifold with local coordinates

$$(t_{2k+2}, \dots, t_{n-1}).$$

(2) $\Sigma^1 \cap \Delta$ is an $n-2k-1$ dimensional manifold containing $\Sigma^{1,1}$. The normal bundle of $\Sigma^{1,1}$ in $\Sigma^1 \cap \Delta$ is the kernel of the differential dg.

(2′) $\Delta * \Sigma^1$ is also an $n-2k-1$ dimensional manifold containing $\Sigma^{1,1}$. The two manifolds $\Delta * \Sigma^1$ and $\Sigma^1 \cap \Delta$ are tangent along the manifold $\Sigma^{1,1}$ and have no further common points.

(3) The set of triple points $\mathfrak{T}$ is a manifold with boundary $\Delta \cap \Sigma^1$.

(4) At every point of $\mathfrak{T}$, Δ has two branches. These two branches get closer and closer to each other as the point tends to the boundary and finally they coincide at the points of the boundary. Let us denote by ξ_1 and ξ_2 the normal bundle of $\mathfrak{T}$ in the first and the second branch respectively. (These are k dimensional bundles over $\mathfrak{T}$.) We shall denote by ν_2 the restriction of ξ_1 (or ξ_2) over $\Sigma^{1,1}$. In Morin's normal coordinates the fibres of ν_2 are parallel to the coordinate space $(t_1, t_3, t_5, \dots, t_{2k-1})$.

(5) The normal bundle of Δ in M^n along $\Sigma^{1,1}$ will be denoted by ν_3. (In Morin's normal coordinates the fibres are parallel to the coordinate space $(t_2, t_4, \ldots, t_{2k})$.) So we have the following sequence of embeddings

$$\Sigma^{1,1} \subset \Delta * \Sigma^1 = \partial\mathfrak{T} \subset \mathfrak{T} \subset \Delta \subset M^n.$$

The normal bundles of these embeddings restricted to $\Sigma^{1,1}$ will be denoted by

$$\nu_0, \nu_1, \nu_2, \nu_3$$

respectively and the corresponding coordinate directions are

$$(x), (t_{2k+1}), (t_1, t_3, \ldots, t_{2k-1}) \quad \text{and} \quad (t_2, t_4, \ldots, t_{2k})$$

respectively.

Remarks

(1) The line bundle ν_0 is trivial because it coincides with the bundle $\ker dg$ restricted to $\Sigma^{1,1}$, and g is a simple $\Sigma^{1,1}$ map. (See condition (b) in the definition of simple $\Sigma^{1,1}$ singular map.)

(2) The bundle ν_1 is the restriction to $\Sigma^{1,1}$ of the normal bundle of the boundary of $\mathfrak{T}$ in $\mathfrak{T}$, and so this bundle is also trivial. Fix an orientation on the trivial bundle $\ker dg$. We can choose Morin's local coordinates $(t_1, t_2, \ldots, t_{2k+1}, x, u_1, u_2, \ldots, u_{n-2k-2})$ in a neighbourhood of any point in $\Sigma^{1,1}$ in such a way that the derivative of the coordinate x in the positive direction of $\ker dg$ will be always positive. Such a coordinate system will be called admissible. (Notice that if for a certain Morin local coordinate system this does not hold, then we can change the sign of x if we change the signs of all the t_i coordinates with odd i as well, and also change the signs of the corresponding y coordinates and of z_{k+1} in the target.)

Now the set $\Sigma^1 \cap \Delta$ can be decomposed into the union of $\Sigma^1 \cap \Delta_+$ and $\Sigma^1 \cap \Delta_-$, according to the sign of the x coordinate in an admissible Morin's local coordinate system. (This decomposition will help us to show the triviality of a certain line bundle (named ζ) in the target manifold.)

Lemma 6. *The k-dimensional vector bundles $\nu_2 \to \Sigma^{1,1}$ and $\nu_3 \to \Sigma^{1,1}$ are isomorphic.*

Proof. Let us consider a short nonzero inward normal vector field v of $\partial\mathfrak{T}$ in $\mathfrak{T}$ along $\Sigma^{1,1}$. Given a real number ϵ such that $0 \leqslant \epsilon \leqslant 1$ – and having identified the tubular neighbourhood of $\Sigma^{1,1}$ in M^n with the corresponding normal bundle – the endpoints of the vector field ϵv will be identified with a submanifold of $\mathfrak{T}$ diffeomorphic to $\Sigma^{1,1}$. Let us denote this submanifold by $\Sigma^{1,1}_{\epsilon v}$. Then the bundles ξ_1 and ξ_2 restricted to $\Sigma^{1,1}_{\epsilon v}$ are isomorphic. (Recall that ξ_1 and ξ_2 are the normal bundles of the set of triple points $\mathfrak{T}$ in the first and the second branch of the set of double points respectively.)

Indeed, on decreasing ϵ continuously from 1 to 0 both restrictions tend to the bundle $\nu_2 \to \Sigma^{1,1}$. (Recall that ν_2 was the normal bundle of $\mathfrak{T}$ in the single branch of the double point-set Δ over $\Sigma^{1,1}$. As we have mentioned above, the two branches of Δ coincide over $\partial\mathfrak{T}$, so in particular over $\Sigma^{1,1}$ too.)

We can suppose that for $\epsilon = 1$ the restrictions of the bundles ξ_1 and ξ_2 are orthogonal. Therefore, if we denote by $\xi_i^\perp$ the orthogonal complement of ξ_i, $i = 1, 2$, in the normal bundle of $\mathfrak{T}$ in M^n, then we have

$$\xi_1 \,|\, \Sigma^{1,1}_v \approx \xi_2^\perp \,|\, \Sigma^{1,1}_v \quad \text{and} \quad \xi_2 \,|\, \Sigma^{1,1}_v \approx \xi_1^\perp \,|\, \Sigma^{1,1}_v.$$

Now as ϵ tends to zero the orthogonal complements will tend to the bundle ν_3. Therefore $\nu_2 \approx \nu_3$. ∎

This implies the first part of Lemma 4 (i.e. $\nu(g) \approx \epsilon^2 \oplus 2\eta^k$).

4.2. *Sets and bundles in the target manifold.*

1. The restriction of f to $\Sigma^{1,1}$ is a diffeomorphism onto $\tilde{\Sigma}^{1,1}$. In Morin's normal coordinates a point $(y_1, \dots, y_{n-1}, z_1, \dots, z_{k+1})$ belongs to $\tilde{\Sigma}^{1,1}$ if and only if $y_1 = \dots = y_{2k+1} = z_1 = \dots = z_{k+1} = 0$.

2. The image of the set of triple points, which will be denoted by $\tilde{\mathfrak{T}}$, belongs to the $n-2k$ dimensional subspace of R^{n+k} defined by the equations:

$$y_1 = y_2 = \dots = y_{2k} = z_1 = z_2 = \dots = z_k = 0$$

and it forms a topological manifold with non-smooth ('cuspidal') boundary.

3. This cuspidal boundary is the image of the set $\Sigma^1 \cap \Delta$ and it is given by the union of 'semicubic parabolas' which have the following parametric equation:

$$z_{k+1} = -2 \cdot x^3, \quad y_{2k+1} = -3 \cdot x^2 \quad (\text{and } y_1 = \dots = y_{2k} = z_1 = \dots = z_k = 0).$$

The cuspidal boundary of $\tilde{\mathfrak{T}}$ consists of two $n-2k-1$ dimensional manifolds with boundary $g(\Sigma^1 \cap \Delta)_+$ and $g(\Sigma^1 \cap \Delta)_-$ respectively. Notice that these two manifolds can not interchange by going to another local coordinate system if we restrict ourselves to admissible Morin local coordinates. They have the common boundary $\tilde{\Sigma}^{1,1} = g(\Sigma^{1,1})$ and their tangent spaces coincide at the common boundary.

4. The image of the vector field v under the differential dg will be denoted by η. This is an inward normal vector field along the boundary $\tilde{\Sigma}^{1,1}$ in $g(\Sigma^1 \cap \Delta)_+$ (and also in $g(\Sigma^1 \cap \Delta)_-$). Having identified the normal bundle of $\tilde{\Sigma}^{1,1}$ with its tubular neighbourhood the endpoints of the vector field η will define a submanifold of $\tilde{\mathfrak{T}}$ which we denote by $\tilde{\Sigma}^{1,1}_\eta$.

5. Let $\{q_i\}$ be a sequence of points of $\tilde{\mathfrak{T}}$ converging to a point $q \in \tilde{\Sigma}^{1,1}$. Then the tangent spaces of $\tilde{\mathfrak{T}}$ at the points q_i will converge to a well-defined subspace θ_q of $T_q N^{n+k}$ ($\dim \theta_q = n-2k$). Let us denote by θ the $n-2k$ dimensional vector bundle over $\tilde{\Sigma}^{1,1}$ formed by the vector spaces θ_q, $q \in \tilde{\Sigma}^{1,1}$. The bundle θ contains the vector field η.

6. Notice that $T\tilde{\Sigma}^{1,1} \subset \theta$, where $T\tilde{\Sigma}^{1,1}$ is the tangent space of $\tilde{\Sigma}^{1,1}$.

Denote by ζ the orthogonal complement of $T\tilde{\Sigma}^{1,1} \oplus \eta$ in θ.

Notice that the vector field η and the globally defined sets $g(\Sigma^1 \cap \Delta)_+$ and $g(\Sigma^1 \cap \Delta)_-$ define an orientation of the line bundle ζ and so this line bundle is trivial. (Indeed, η and θ define at each point $p \in \tilde{\Sigma}^{1,1}$ a plane, which is divided by the line of η into a positive half whose germ at p intersects the set $\Sigma^1 \cap \Delta_+$ and a negative half. Now ζ is clearly trivial.)

7. Any interior point Q of $\tilde{\mathfrak{T}}$ has exactly three preimage points

$$g^{-1}(Q) = \{P_1(Q), P_2(Q), P_3(Q)\}.$$

Let $\xi_1^{P_i}$ and $\xi_2^{P_i}$ be the fibres of the bundles ξ_1 and ξ_2 at $P_i = P_i(Q)$, $i = 1, 2, 3$.

The six vector spaces $\xi_1^{P_i}$, $\xi_2^{P_i}$, $i = 1, 2, 3$ are mapped by dg pairwise into the same subspace of $T_Q N^{n+k}$:

$$dg(\xi_1^{P_1}) = dg(\xi_2^{P_2}); \quad dg(\xi_2^{P_1}) = dg(\xi_1^{P_3}); \quad dg(\xi_2^{P_3}) = dg(\xi_1^{P_2}).$$

We obtain three vector spaces in $T_Q N^{n+k}$ and we shall denote them by $\tilde{\xi}_1^Q, \tilde{\xi}_2^Q, \tilde{\xi}_3^Q$. These vector spaces are k-dimensional and together they span the $3k$-dimensional normal space of $\tilde{\mathfrak{T}}$ in $T_Q N^{n+k}$. Now let the point Q run over the manifold $\tilde{\Sigma}^{1,1}_\eta$. Then the points P_1, P_2, P_3 will run over three submanifolds of $\mathfrak{T}$ naturally diffeomorphic to $\Sigma^{1,1}$. Let us consider the restrictions of the bundles ξ_1 and ξ_2 to these submanifolds and denote them by ξ^j_i, where $i = 1, 2$ and $j = 1, 2, 3$. The union of the vector spaces $\tilde{\xi}^Q_j$ considered for each $Q \in \tilde{\Sigma}^{1,1}_\eta$ forms a vector bundle $\tilde{\xi}_j$, for each $j = 1, 2, 3$. These three vector bundles $\tilde{\xi}_1$, $\tilde{\xi}_2$ and $\tilde{\xi}_3$ restricted to $\tilde{\Sigma}^{1,1}_\eta$ are isomorphic to each other since they are isomorphic images of the isomorphic bundles ξ^j_i; $i = 1, 2$; $j = 1, 2, 3$.

Therefore the normal bundle of $\tilde{\Sigma}^{1,1}_\eta$ (which can be naturally identified with the normal bundle of $\tilde{\Sigma}^{1,1}$) is isomorphic to the direct sum of the trivial 2-dimensional bundle (= the normal bundle in $\tilde{\mathfrak{T}}$) and the direct sum of three isomorphic k-dimensional bundles ($\tilde{\xi}_j$, $j = 1, 2, 3$).

This proves Lemma 4.

5. *Final remarks*

1. The analogue of Theorem 1 has been proved for $\Sigma^{1,0}$ singular maps in [**Sz1**] and [**Sz2**].

2. Theorem 1 has been formulated with a sketch of the proof in [**Sz3**].

3. Theorem 1 is one of the main ingredients for the construction of the classifying space for the cobordism groups of those $\Sigma^{1,1}$ singular maps, which are projections of immersions (or embeddings) in euclidean spaces of one dimension higher. (It is not hard to see that for such a map f the kernels of the differentials at the singular points form a trivial line bundle over the set of singular points, and hence f is simple $\Sigma^{1,1}$-singular maps if $n \leqslant 3k$.) These latter cobordism groups have been used in the computations of the cobordism groups of immersions and embeddings in [**Sz4**] and [**Sz5**].

4. If the source and the target manifolds are oriented then the group $O(k)$ can be replaced by the group $SO(k)$.

5. If we drop condition (b) in the definition of simple $\Sigma^{1,1}$-singular map (see the Introduction), then the theorem still holds if we replace the group G with its suitable Z_2 extension.

6. The method of this paper applies more generally than the formulated result. For example this method works for any Morin singularities. This fact will allow us to construct models for loop-spaces of Thom spaces, analogous to the 'James product' model for the loop space of a suspension.

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