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Homology Comodules

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Let X be a topological space, $A_*(p)$ the dual of the mod p Steenrod algebra $A(p)$. Then the homology $H_*(X)=H_*(X;Z_p)$ of X is a comodule over $A_*(p)$; that is, there is a coaction

$$\mu_*\colon H_*(X)\to A_*(p)\otimes H_*(X).$$

μ_* is the dual of the action map

$$A(p)\otimes H^*(X)\to H^*(X).$$

In [3] Liulevicius gives some of the coefficients of μ_* for $X=BG$ and MG, $G=O(n)$, $U(n)$, $Sp(n)$, O, U, Sp, $p=2$. The coefficients were determined by using a computer to solve a recursion formula of Van de Velde. In this note we give a complete description of μ_* for the above cases.

§1. *O*, *U* and *Sp*

We recall some facts about $H_*(RP^\infty)$, etc.

i) $H^*(RP^\infty;Z_2)\cong Z_2[x]$, $x\in H^1(RP^\infty;Z_2)$;

ii) $H^*(CP^\infty;Z)\cong Z[y]$, $y\in H^2(CP^\infty;Z)$;

iii) $H^*(HP^\infty;Z)\cong Z[z]$, $z\in H^4(HP^\infty;Z)$.

Therefore in homology we have

iv) $\tilde H_*(RP^\infty;Z_2)$ has a Z_2-basis $x_1, x_2, \ldots$ with $x_i\in\tilde H_i(RP^\infty;Z_2)$ dual to $x^i\in H^i(RP^\infty;Z_2)$;

v) $\tilde H_*(CP^\infty;Z)$ has a Z-basis $y_1, y_2, \ldots$ with $y_i\in\tilde H_{2i}(CP^\infty;Z)$ dual to $y^i\in H^{2i}(CP^\infty;Z)$;

vi) $\tilde H_*(HP^\infty;Z)$ has a Z-basis $z_1, z_2, \ldots$ with $z_i\in\tilde H_{4i}(HP^\infty;Z)$ dual to $z^i\in H^{4i}(HP^\infty;Z)$.

If $\rho\colon H_*(\ ;Z)\to H_*(\ ;Z_p)$ denotes reduction mod p, then we shall also denote by y_i the element $\rho(y_i)\in\tilde H_{2i}(CP^\infty;Z_p)$ and by z_i the element $\rho(z_i)\in\tilde H_{4i}(HP^\infty;Z_p)$.

As Liulevicius remarks in [3], the coaction of $A_*(2)$ on $H_*(BO;Z_2)$, $H_*(BO(n);Z_2)$, $H_*(MO;Z_2)$ and $H_*(MO(n);Z_2)$ is completely determined once one knows $\mu_*(x_i)$, $i\geqq 1$. Similarly for any prime p the coaction of $A_*(p)$ on $H_*(BU;Z_p)$, $H_*(BU(n);Z_p)$, $H_*(MU;Z_p)$ and $H_*(MU(n);Z_p)$

is completely determined by a knowledge of $\mu_*(y_i)$, $i \geqq 1$. An analogous result holds for the symplectic case. We proceed to give $\mu_*(x_i)$, $\mu_*(y_i)$ and $\mu_*(z_i)$.

We recall that $A_*(2) \cong Z_2[\xi_1, \xi_2, \ldots]$ with $\xi_i \in A_{2^i-1}(2)$, while for an odd prime p

$$A_*(p) \cong E(\tau_0, \tau_1, \ldots) \otimes Z_p[\xi_1, \xi_2, \ldots]$$

with $\tau_i \in A_{2p^i-1}(p)$, $\xi_i \in A_{2p^i-2}(p)$. In both cases we write X for the formal sum $1 + \xi_1 + \xi_2 + \cdots$.

Theorem 1 i) *For $p=2$ we have*

$$\mu_*(x_i) = \sum_{j=0}^{i} (X^j)_{i-j} \otimes x_j.$$

ii)

$$\mu_*(y_i) = \begin{cases} \sum_{j=0}^{i} (X^{2j})_{2i-2j} \otimes y_j & p=2, \\ \sum_{j=0}^{i} (X^j)_{2i-2j} \otimes y_j & p \text{ odd}. \end{cases}$$

iii)

$$\mu_*(z_i) = \begin{cases} \sum_{j=0}^{i} (X^{4j})_{4i-4j} \otimes z_j & p=2, \\ \sum_{j=0}^{i} (X^{2j})_{4i-4j} \otimes z_j & p \text{ odd}. \end{cases}$$

Proof. First we consider the case $p=2$.

i) For any $\theta \in A$, $y \in H^*(X)$ and $u \in H_*(X)$ with $\mu_*(u) = \sum_i e_i \otimes u_i$, $e_i \in A_*$, $u_i \in H_*(X)$, we have the relation

$$\langle \theta y, u \rangle = \sum_i \langle \theta, e_i \rangle \langle y, u_i \rangle$$

(cf. [1]). Taking $X = RP^\infty$, $y = x^k$, $u = x_i$ we get

$$\langle \theta x^k, x_i \rangle = \sum_j \langle \theta, e_j^i \rangle \langle x^k, x_j \rangle = \langle \theta, e_k^i \rangle$$

if $\mu_*(x_i) = \sum_j e_j^i \otimes x_j$.

We now compute θx^k; let

$$\Delta^k \colon RP^\infty \to \underbrace{RP^\infty \times \cdots \times RP^\infty}_{k}$$

be the k-fold diagonal $\Delta^k(x)=(x, x, \ldots, x)$. Let $u_j \in H^1(RP^\infty \times \cdots \times RP^\infty)$ be $\pi_j^* x$, where

$$\pi_j \colon RP^\infty \times \cdots \times RP^\infty \to RP^\infty$$

is the projection on the j-th factor. Then $x^k = \Delta^{k*}(u_1 u_2 \ldots u_k)$ and hence

$$\begin{aligned} \theta x^k &= \theta \Delta^{k*}(u_1 u_2 \ldots u_k) = \Delta^{k*} \theta(u_1 u_2 \ldots u_k) \\ &= \Delta^{k*}\big(\sum_\alpha \langle \theta, \xi_\alpha\rangle u_1^{2^{\alpha_1}} u_2^{2^{\alpha_2}} \ldots u_k^{2^{\alpha_k}}\big) \\ &= \sum_\alpha \langle \theta, \xi_\alpha\rangle x^{2^{\alpha_1}+2^{\alpha_2}+\cdots+2^{\alpha_k}} \\ &= \sum_{j \geqq k} \langle \theta, (X^k)_{j-k}\rangle x^j. \end{aligned}$$

Thus

$$\langle \theta, e_k^i\rangle = \langle \theta x^k, x_i\rangle = \sum_{j \geqq k} \langle \theta, (X^k)_{j-k}\rangle \langle x^j, x_i\rangle = \langle \theta, (X^k)_{i-k}\rangle.$$

Statement i) follows.

ii) and iii): It follows from [2, Prop. II.1.] that there is a commutative diagram

$$\begin{array}{ccc} H_*(RP^\infty) & \xrightarrow{\mu_*} & A_* \otimes H_*(RP^\infty) \\ \downarrow f & & \downarrow \alpha\otimes f \\ H_*(CP^\infty) & \xrightarrow{\mu_*} & A_* \otimes H_*(CP^\infty) \\ \downarrow g & & \downarrow \alpha\otimes g \\ H_*(HP^\infty) & \xrightarrow{\mu_*} & A_* \otimes H_*(HP^\infty) \end{array}$$

where f is given by $f(x_i)=y_i$, g by $g(y_i)=z_i$ and $\alpha\colon A_* \to A_*$ is the squaring homomorphism $\alpha(a)=a^2$, $a \in A_*$. Statements ii) and iii) follow immediately.

Next we treat the case of odd primes p.

ii) For the admissible monomials $P^I \in A(p)$ we have

$$P^I y = \begin{cases} y^{p^s} & I=(0, p^{s-1}, 0, p^{s-2}, 0, \ldots, 1, 0), \quad s \geqq 0, \\ 0 & \text{otherwise.} \end{cases}$$

We can express this by writing

$$P^I y = \sum_i \langle P^I, \xi_i\rangle y^{p^i},$$

or since the P^I form a basis for $A(p)$ over Z_p

$$\theta y = \sum_i \langle \theta, \xi_i\rangle y^{p^i}, \qquad \theta \in A(p).$$

By a calculation analogous to the one for θx^k we now find

$$\theta y^k = \sum_i \langle \theta, (X^k)_{2i-2k} \rangle\, y^i.$$

Hence from the formula $\langle \theta y, u \rangle = \sum_j \langle \theta, e_j \rangle \langle y, u_j \rangle$ follows the statement ii) for odd p.

iii) The maps $BU(n) \to BSp(n)$ classifying $\xi_n \otimes H$ induce a map $q_*: H_*(BU) \to H_*(BSp)$ which satisfies $q_*(y_{2i}) = z_i$, $q_*(y_{2i-1}) = 0$, $i \geqq 1$. If we apply $1 \otimes q_*$ to the equation

$$\mu_*(y_{2i}) = \sum_{j=0}^{2i} (X^j)_{4i-2j} \otimes y_j,$$

we get

$$\mu_*(z_i) = \sum_{j=0}^{i} (X^{2j})_{4i-4j} \otimes z_j.$$

Remark. If $\mu_*(x_i) = \sum_{j=0}^{i} e^i_j \otimes x_j$, $e^i_j \in A_{i-j}(2)$, then the recursion relation of Van de Velde for the e^i_j is the following: for each i and each pair of integers s, t with $s+t=j$, we have

$$e^i_j = \sum_k e^k_s\, e^{i-k}_t.$$

One readily checks that

$$e^i_j = (X^j)_{i-j}$$

satisfies this relation.

We may take $BO(1) = RP^\infty$; then we have the inclusion

$$RP^\infty = BO(1) \to BO$$

and we denote the image of x_i in $H_*(BO; Z_2)$ with x_i again. We also have the Thom isomorphism

$$\Phi_*: \; H_*(MO; Z_2) \to H_*(BO; Z_2)$$

and we denote $\Phi_*^{-1}(x_i)$ by a_i. Then the composite

$$f: \; RP^\infty \simeq MO(1) \to \Sigma MO$$

induces

$$f_*: \; H_i(RP^\infty; Z_2) \to H_{i-1}(MO; Z_2)$$

which satisfies

$$f_*(x_i) = a_{i-1}, \qquad i \geqq 1.$$

Analogous results hold if we replace O by U or Sp and Z_2 by Z. Then we get the following.

Theorem 2.

i) $H_*(BO;Z_2)\cong Z_2[x_1,x_2,\ldots]$ *and*

$$\mu_*(x_i)=\sum_{j=0}^{i}(X^j)_{i-j}\otimes x_j.$$

$H_*(MO;Z_2)\cong Z_2[a_1,a_2,\ldots]$ *and*

$$\mu_*(a_i)=\sum_{j=0}^{i}(X^{j+1})_{i-j}\otimes a_j.$$

ii) $H_*(BU;Z_p)\cong Z_p[y_1,y_2,\ldots]$ *and*

$$\mu_*(y_i)=\begin{cases}\sum_{j=0}^{i}(X^{2j})_{2i-2j}\otimes y_j & p=2,\\ \sum_{j=0}^{i}(X^{j})_{2i-2j}\otimes y_j & p\text{ odd}.\end{cases}$$

$H_*(MU;Z_p)\cong Z_p[b_1,b_2,\ldots]$ *and*

$$\mu_*(b_i)=\begin{cases}\sum_{j=0}^{i}(X^{2j+2})_{2i-2j}\otimes b_j & p=2,\\ \sum_{j=0}^{i}(X^{j+1})_{2i-2j}\otimes b_j & p\text{ odd}.\end{cases}$$

iii) $H_*(BSp;Z_p)\cong Z_p[z_1,z_2,\ldots]$ *and*

$$\mu_*(z_i)=\begin{cases}\sum_{j=0}^{i}(X^{4j})_{4i-4j}\otimes z_j & p=2,\\ \sum_{j=0}^{i}(X^{2j})_{4i-4j}\otimes z_j & p\text{ odd}.\end{cases}$$

$H_*(MSp;Z_p)\cong Z_p[q_1,q_2,\ldots]$ *and*

$$\mu_*(q_i)=\begin{cases}\sum_{j=0}^{i}(X^{4j+4})_{4i-4j}\otimes q_j & p=2,\\ \sum_{j=0}^{i}(X^{2j+2})_{4i-4j}\otimes q_j & p\text{ odd}.\end{cases}$$

Now $H_*(BO(n);Z_2)$ is the subgroup of $H_*(BO;Z_2)$ spanned by monomials $x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_r^{\alpha_r}$ with at most n factors. Hence the comodule structure of $H_*(BO(n);Z_2)$ over $A_*(2)$ is also determined by Theorem 2i). The inclusion $MO(n)\to\Sigma^n MO$ maps $H_*(MO(n);Z_2)$ monomorphically onto the subgroup spanned by all monomials $a_1^{\alpha_1}a_2^{\alpha_2}\ldots a_r^{\alpha_r}$ with at most n factors. Hence the comodule structure of $H_*(MO(n);Z_2)$ over $A_*(2)$ is also determined by Theorem 2i). Analogous results hold for $BU(n)$, $MU(n)$, $BSp(n)$, $MSp(n)$ and all primes p.

§2. *SO* and Spin

Let p be an odd prime. The map $r: BU \to BSO$ which "forgets complex structure" induces $r_*: H_*(BU; Z_p) \to H_*(BSO; Z_p)$ with $r_*(y_{2i-1}) = O$ for $i \geqq 1$ and

$$H_*(BSO; Z_p) \cong Z_p[z'_1, z'_2, \dots],$$

where

$$z'_i = r_*(y_{2i}) \in H_{4i}(BSO; Z_p) \quad \text{and} \quad \mu_*(z'_i) = \sum_{j=0}^{i} (X^{2j})_{4i-4j} \otimes z'_j.$$

Similarly for Mr_*: $H_*(MU; Z_p) \to H_*(MSO; Z_p)$ we have $Mr_*(b_{2i-1}) = 0$ for $i \geqq 1$ and

$$H_*(MSO; Z_p) \cong Z_p[q'_1, q'_2, \dots],$$

where $q'_i = Mr_*(b_{2i})$ in $H_{4i}(MSO; Z_p)$ and $\mu_*(q'_i) = \sum_{j=0}^{i} (X^{2j+1})_{4i-4j} \otimes q'_j$. Since $H_*(B\,\mathrm{Spin}; Z_p) \cong H_*(BSO; Z_p)$ and $H_*(M\,\mathrm{Spin}; Z_p) \cong H_*(MSO; Z_p)$, we have also described the comodule structures of $H_*(B\,\mathrm{Spin}; Z_p)$ and $H_*(M\,\mathrm{Spin}; Z_p)$.

The difficulty with SU is that we do not have a convenient description of $H_*(BSU; Z_p)$.

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