$SU(n)/SO(n)$: Landweber's manifolds

Recently, Peter Landweber wrote to ask some questions about the manifolds $SU(n)/SO(n)$. These are quite interesting manifolds and I thought I would tell you a bit about them. At the same time, I will include $U(n)/O(n)$

To begin, one has fibrations

$$SU(n)/SO(n) \xrightarrow{i} BSO(n) \xrightarrow{\otimes\mathbb{C}} BSU(n)$$

$$U(n)/O(n) \xrightarrow{i} BO(n) \xrightarrow{\otimes\mathbb{C}} BU(n)$$

This gives n-plane bundles E_n over $U(n)/O(n)$ and $SU(n)/SO(n)$ with $E_n \otimes \mathbb{C}$ being a trivial complex bundle. These are the universal (oriented) n-plane bundles with trivial complexification.

If you consider the space of real n-dimensional subspaces $V \subset \mathbb{C}^n$ with the property that $iV = V^\perp$ is the orthogonal complement of V one has what is called the Lagrangian Grassmannian : $\Lambda_n = U(n)/O(n)$ and E_n consists of the pairs — a subspace and a vector in that subspace. $S\Lambda_n = SU(n)/SO(n)$ is called the special Lagrangian Grassmannian.

If you consider mod 2 cohomology for the fibrations one has

$$H^*(BSO(n)) = \mathbb{Z}_2[w_2, w_3, \dots, w_n] \longleftarrow H^*(BSU(n)) = \mathbb{Z}_2[c_2, c_3, \dots, c_n]$$

$$H^*(BO(n)) = \mathbb{Z}_2[w_1, w_2, \dots, w_n] \longleftarrow H^*(BU(n)) = \mathbb{Z}_2[c_1, c_2, \dots, c_n]$$

and one has $(\otimes\mathbb{C})^*(c_i) = w_i^2$ $\quad (c(E\otimes\mathbb{C}) = w(E\otimes\mathbb{C}) = w(E\oplus E) = w(E)^2)$

from which one has

$$H^*(U(n)/O(n)) = \frac{\mathbb{Z}_2[w_1, \dots, w_n]}{(w_1^2, \dots, w_n^2)} \quad \text{and} \quad H^*(SU(n)/SO(n)) = \frac{\mathbb{Z}_2[w_2, \dots, w_n]}{(w_2^2, \dots, w_n^2)}.$$

given as the exterior algebras on the Stiefel–Whitney classes of the bundles E_n.

One has $\dim U(n)/O(n) = 1 + 2 + \cdots + n = \frac{n(n+1)}{2} = \binom{n+1}{2}$ and $\dim SU(n)/SO(n) = 2 + \cdots + n = \binom{n+1}{2} - 1$.

One also has $\pi_1(U(n)/O(n)) = \mathbb{Z}$ and $\pi_1(SU(n)/SO(n)) = 0$, so $SU(n)/SO(n)$ is simply connected.

An interesting result is

Fact. (Eugenio Calabi) $SU(3)/SO(3)$ is a simply connected nonbounding 5-dimensional manifold.

Note. The cobordism group $\mathfrak{N}_5 = \mathbb{Z}_2$ so there is a unique cobordism class of nonbounding 5-manifolds. Thom observed that the generator was a manifold described by Wu which is actually the Dold manifold $P(1,2)$. An oriented manifold has $w_1 = 0$ and I think Wu's manifold was the first example of an oriented manifold having a nonzero odd dimensional Stiefel-Whitney class.

Proof. One has

i	0	1	2	3	4	5
H^i	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$
	1		w_2	w_3		$w_2 w_3$.

By Wu's formula $Sq^2 w_3 = w_2 w_3$ so the Wu class $v_2 = w_2$. Then $v = 1 + w_2$ and $\tilde{w} = w(SU(3)/SO(3)) = Sq\, v = 1 + w_2 + w_3$ so $\tilde{w}_2 \tilde{w}_3 [SU(3)/SO(3)] \neq 0$. $\square$

Note. To avoid confusion, I will let $\tilde{w}$ denote the Stiefel-Whitney class of the tangent bundle, using $w = w(E_n)$.

Fact. The complexification of the tangent bundle is trivial for both $U(n)/O(n)$ and $SU(n)/SO(n)$.

Proof. The tangent bundle of $U(n)/O(n)$ is known to be $S^2(E_n)$, the second symmetric power of E_n. Then $S^2(E_n) \otimes \mathbb{C} = S^2(E_n \otimes \mathbb{C}) = S^2(\text{trivial } \mathbb{C}^n \text{bundle}) = \text{trivial bundle.}$ Then $SU(n)/SO(n) \subset U(n)/O(n)$ is a codimension one submanifold and being simply connected, the normal line bundle is trivial. Thus for $SU(n)/SO(n)$, $\tau + 1 = S^2(E_n)$ and the complexification $\tau \otimes \mathbb{C} + 1_{\mathbb{C}}$ is trivial. Being in the stable range for complex vector bundles, $\tau \otimes \mathbb{C}$ is also trivial. $\square$

Note. For a manifold M^N with $\tau \otimes \mathbb{C}$ trivial, Gromov and Lees have shown that there is a Lagrangian immersion $\varphi : M^N \to \mathbb{C}^N$; i.e. an immersion with $i\varphi_* \tau_p M = (\varphi_* \tau_p M)^\perp$. This is also a totally real; i.e. $\varphi_* \tau_p M \cap i\varphi_* \tau_p M = \{0\}$. Audin has shown that $U(n)/O(n)$ has a Lagrangian imbedding of M^N in $\mathbb{C}^N$. Peter is interested in knowing whether $SU(n)/SO(n)$ has a totally real imbedding.

Once upon a time, Larry Smith and I calculated the cobordism group for manifolds for which the complexification of the stable tangent bundle was trivial. Our result was that the generators of $\mathfrak{N}_* = \mathbb{Z}_2[x_i \mid i \neq 2^s-1]$ can be chosen so that $\Omega_*^{U/O} = \mathbb{Z}_2[x_i \mid i = \text{odd}, i \neq 2^s-1]$.

Calabi's observation then shows that $SU(3)/SO(3)$ is the 5-dimensional generator and it would be interesting to know the cobordism classes of the other manifolds.

Fact. $SU(n)/SO(n) = \{X \in SU(n) \mid {}^tX = X\}$ and,
$U(n)/O(n) = \{X \in U(n) \mid {}^tX = X\}$

Proof. One considers the function $U(n) \to U(n)$ sending Y to $Y \cdot {}^tY$, where ${}^tY$ = transpose of Y. This maps onto the symmetric matrices $({}^t(Y \cdot {}^tY) = {}^t({}^tY) \cdot {}^tY = Y \cdot {}^tY)$ and sends $U(n)/O(n)$ diffeomorphically to the symmetric matrices ($O(n)$ is the matrices with $Y \cdot {}^tY = 1$). (This is proved by Mimura and Sugata). $\square$

Corollary. For n even, $SU(n)/SO(n)$ bounds and $U(n)/O(n)$ is always a boundary.

Proof. For n even, $-1 \in SU(n)$ is a central element and multiplication by -1 is a free involution on $\{X \in SU(n) \mid {}^tX = X\}$. For every n, the diagonal matrix with diagonal entry $z \in S^1$ is a central element of $U(n)$ an multiplication by these matrices defines a free circle action on $\{X \in U(n) \mid {}^tX = X\}$. $\square$

Note. $SU(2)/SO(2) = S^3/S^1 = S^2$ and the involution just described is the generalization of the antipodal involution. One notes that $SU(1)/SO(1) = \text{point}$ is nonbounding $(SU(1)=SO(1)=\text{unit group})$.

In order to study the cobordism class of $SU(n)/SO(n)$ for n odd, one would like to know the Stiefel-Whitney class $\tilde{w} = w(SU(n)/SO(n)) = w(S^2(E_n))$.

Consider a general n-plane bundle γ_n. One then has $\gamma_n \otimes \gamma_n = S^2(\gamma_n) \oplus \Lambda^2(\gamma_n)$, where Λ^2 is the second exterior power. $(a\otimes b = \frac{1}{2}(a\otimes b + b\otimes a) + \frac{1}{2}(a\otimes b - b\otimes a))$. One may then apply the splitting principle to write γ_n as a sum of line bundles $\ell_1 \oplus \cdots \oplus \ell_n$ with $w(\ell_i) = 1+x_i$ and one has

$$w(\gamma_n\otimes\gamma_n) = \prod_{i,j}(1+x_i+x_j) = \prod_i (1+x_i+x_i)\cdot\prod_{i<j}(1+x_i+x_j)\prod_{i>j}(1+x_i+x_j)$$

$$= \Big\{\prod_{i<j}(1+x_i+x_j)\Big\}^2$$

$$w(\Lambda^2(\gamma_n)) = \prod_{i<j}(1+x_i+x_j)$$

from which one has $w(S^2(\gamma_n)) = w(\Lambda^2(\gamma_n)) = \prod_{i<j}(1+x_i+x_j)$. Unfortunately, there is no known formula to describe $w(\Lambda^2(\gamma_n))$ in terms of $w(\gamma_n)$. However, one has

<u>Lemma</u>. For any n-plane bundle γ_n, one has
$$w(\Lambda^s(\gamma_n)) = 1 + \binom{n-1}{s-1} w_1 + \binom{n-2}{s-1}\{w_2 + w_3 + \cdots + w_n\}$$
modulo decomposables.

<u>Proof</u>. Using the splitting principle to write $w(\gamma_n) = \prod_1^n (1+x_i)$ one has $w(\Lambda^s(\gamma_n)) = \prod_{1\le i_1<i_2<\cdots<i_s\le n}(1+x_{i_1}+x_{i_2}+\cdots+x_{i_s})$.

To establish the formula it is sufficient to find α with $w_{2^t}(\Lambda^s(\gamma_n)) = \alpha w_{2^t}$ modulo decomposables. For $2^t \le k = 2^t+i < 2^{t+1}$ one has, working modulo decomposables, $w_k(\Lambda^s(\gamma_n)) \equiv Sq^i w_{2^t}(\Lambda^s(\gamma_n)) \equiv Sq^i \alpha w_{2^t} \equiv \alpha w_k$ since $Sq^i w_j = \binom{j-1}{i} w_{i+j}$ modulo decomposables and Sq^i takes decomposables to decomposables.

To find $\tilde{w}_{2^t} = w_{2^t}(\Lambda^s(\gamma_n))$ one considers the bundle $\gamma_n = 2^t\ell + (n-2^t)$ over RP^∞ where ℓ is the standard line bundle with $w(\ell) = 1+x$. Then $w = w(\gamma_n) = (1+x)^{2^t} = 1+x^{2^t}$

so that $w_{2^t} = x^{2^t}$ and all decomposable classes of degree 2^t are zero. The set of classes x_i is then $\{\underbrace{x,\dots,x}_{2^t},\underbrace{0,\dots,0}_{n-2^t}\}$. Choosing s of these classes will choose p of the x's and $s-p$ zeros for some p with $p \le s$, $p \le 2^t$ which can be done in $\binom{2^t}{p}\binom{n-2^t}{s-p}$ ways. Thus

$$\tilde{w} = \prod (1+px)^{\binom{2^t}{p}\binom{n-2^t}{s-p}}.$$

For p even, $px = 0$, so this becomes

$$\tilde{w} = \prod_{p\,\mathrm{odd}} (1+x)^{\binom{2^t}{p}\binom{n-2^t}{s-p}} = (1+x)^{\sum_{p\,\mathrm{odd}} \binom{2^t}{p}\binom{n-2^t}{s-p}}$$

where the product and sum are taken for p odd, $1 \le p \le 2^t, s$.

For $2^t = 1$, this is $\tilde{w} = (1+x)^{\binom{n-1}{s-1}} = 1 + \binom{n-1}{s-1}x + \cdots$ giving the desired result.

For $2^t > 1$, with p odd, one has $\binom{2^t}{p} = \frac{2^t}{p}\frac{(2^t-1)!}{(p-1)!(2^t-p)!} = \frac{2^t}{p}\binom{2^t-1}{p-1}$

and $$\tilde{w} = (1+x)^{2^t \sum_{p\,\mathrm{odd}} \frac{1}{p}\binom{2^t-1}{p-1}\binom{n-2^t}{s-p}} = (1+x^{2^t})^{\sum_{p\,\mathrm{odd}} \frac{1}{p}\binom{2^t-1}{p-1}\binom{n-2^t}{s-p}}$$

$$= 1 + \sum_{p\,\mathrm{odd}} \frac{1}{p}\binom{2^t-1}{p-1}\binom{n-2^t}{s-p} x^{2^t} + \cdots$$

where the coefficient of x^{2^t} is taken modulo 2. Now $\binom{2^t-2}{p-1} = \binom{2^t-1}{p-1}$ for p odd and $\binom{2^t-2}{p-1} = 0$ for p even, so this coefficient becomes

$$\sum_p \binom{2^t-2}{p-1}\binom{n-2^t}{s-p}$$

where the sum is over all p, and that sum is $\binom{n-2}{s-1}$ giving the desired result. $\square$

<u>Corollary</u>. For n odd, $SU(n)/SO(n)$ is always nonbounding. In particular, $\tilde{w}_2\tilde{w}_3\cdots\tilde{w}_n[SU(n)/SO(n)] \neq 0$.

<u>Proof</u>. For n odd and $s=2$, $\binom{n-1}{s-1} = 0$ and $\binom{n-2}{s-1} \neq 0$ so one has $\tilde{w} = 1 + w_2 + \cdots + w_n$ modulo decomposables. Then $\tilde{w}_2\tilde{w}_3\cdots\tilde{w}_n = w_2 w_3 \cdots w_n +$ terms having more than $n-1$ factors. Since every product of n factors in $H^*(SU(n)/SO(n);\mathbb{Z}_2)$ is zero, $\tilde{w}_2\tilde{w}_3\cdots\tilde{w}_n = w_2 w_3 \cdots w_n$ which is the nonzero class of top degree. $\square$

<u>Fact</u>. $SU(n)/SO(n)$ is indecomposable in $\mathfrak{N}_*$ only for $n=3$.

<u>Proof.</u> A d-dimensional manifold M^d is indecomposable if and only if the characteristic number $s_d[M^d] \neq 0$, where writing $w(M) = \prod(1+x_i)$ $s_d = \sum x_i^d$. One now lets $M^d = SU(n)/SO(n)$, where $d = \binom{n+1}{2} - 1$. Since $s_{2j} = s_j^2$ and squares are zero in $H^*(SU(n)/SO(n))$, $SU(n)/SO(n)$ is decomposable if d is even.

One has a general formula (for any bundle) that the total s-class, $s = s_1 + s_2 + s_3 + \cdots$ is equal to $\frac{w_{odd}}{w} = \frac{w_1 + w_3 + w_5 + \cdots}{1 + w_1 + w_2 + w_3 + \cdots}$ and since $\tilde{w}_i^2 = 1$ in $H^*(SU(n)/SO(n))$, $s = \tilde{w}_{odd} \cdot \tilde{w} = \tilde{w}_{odd}\tilde{w}_{even} + \tilde{w}_{odd}^2$ $= \tilde{w}_{odd}\tilde{w}_{even}$, since $\tilde{w}_{odd}^2 = 0$.

One then has

$$s_d = \begin{cases} (\tilde{w}_0 \tilde{w}_{4p+1} + \tilde{w}_1 \tilde{w}_{4p}) + (\tilde{w}_2 \tilde{w}_{4p-1} + \tilde{w}_3 \tilde{w}_{4p-2}) + \cdots + (\tilde{w}_{2p-2}\tilde{w}_{2p+3} + \tilde{w}_{2p-1}\tilde{w}_{2p+2}) + \tilde{w}_{2p}\tilde{w}_{2p+1} \\ \qquad \text{if } d = 4p+1 \\ (\tilde{w}_0 \tilde{w}_{4p+3} + \tilde{w}_1 \tilde{w}_{4p+2}) + (\tilde{w}_2 \tilde{w}_{4p+1} + \tilde{w}_3 \tilde{w}_{4p}) + \cdots + (\tilde{w}_{2p}\tilde{w}_{2p+3} + \tilde{w}_{2p+1}\tilde{w}_{2p+2}) \\ \qquad \text{if } d = 4p+3. \end{cases}$$

and $Sq^1(\tilde{w}_{2i}\tilde{w}_{2j}) = Sq^1\tilde{w}_{2i} \cdot \tilde{w}_{2j} + \tilde{w}_{2i} Sq^1 \tilde{w}_{2j} = (\tilde{w}_{2i+1} + \tilde{w}_1\tilde{w}_{2i}) \cdot \tilde{w}_{2j} + \tilde{w}_{2i}(\tilde{w}_{2j+1} + \tilde{w}_1 \tilde{w}_{2j})$

$= \tilde{w}_{2i+1}\tilde{w}_{2j} + \tilde{w}_{2i}\tilde{w}_{2j+1}$

and for $2i + 2j + 1 = d$, $Sq^1(\tilde{w}_{2i}\tilde{w}_{2j}) = v_1 \tilde{w}_{2i}\tilde{w}_{2j} = \tilde{w}_1 \tilde{w}_{2i}\tilde{w}_{2j}$ which is zero in $SU(n)/SO(n)$, since $\tilde{w}_1 = 0$.

Thus $s_d[M^d] = 0$ if $d = 4p+3$, and if $d = 4p+1$,

$s_d[M^d] = \tilde{w}_{2p}\tilde{w}_{2p+1} = \tilde{w}_{2p} Sq^1 \tilde{w}_{2p}$.

Now, if $\tilde{w}_{2p}$ is decomposable, $\tilde{w}_{2p} = \sum z_{i_1} \cdots z_{i_r}$ then

$\tilde{w}_{2p} Sq^1 \tilde{w}_{2p} = \sum z_{i_1} \cdots z_{i_r} (\sum Sq^1(z_{j_1} \cdots z_{j_s})) = \sum \{z_{i_1} \cdots z_{i_r}(Sq^1 z_{j_1} \cdots z_{j_s}\}$

and for $\{i_1 \cdots i_r\} \neq \{j_1, \cdots, j_s\}$, $z_{i_1} \cdots z_{i_r} Sq^1 z_{j_1} \cdots z_{j_s} + Sq^1 z_{i_1} \cdots z_{i_r} \cdot z_{j_1} \cdots z_{j_s}$

$= Sq^1(z_{i_1} \cdots z_{i_r} z_{j_1} \cdots z_{j_s}) = v_1 (z_{i_1} \cdots z_{i_r} z_{j_1} \cdots z_{j_s}) = 0$, so

$\tilde{w}_{2p} Sq^1 \tilde{w}_{2p} = \sum z_{i_1} \cdots z_{i_r} Sq^1 z_{i_1} \cdots z_{i_r} = \sum z_{i_1} \cdots z_{i_r} \cdot \sum_k z_{i_1} \cdots (Sq^1 z_{i_k}) \cdots z_{i_r}$

and every term here is zero because it has a factor z_i^2.

Thus $s_d[M^d] = 0$ if $\tilde{w}_{2p}$ is decomposable. For $M^d = SU(n)/SO(n)$, M^d bounds if n is even, and if n is odd, $n = 2q+1$, $\tilde{w}_{2j}$ is decomposable for $2j > 2q$. Thus if $s_d[M^d] \neq 0$ then $d = 2 + 3 + \cdots + 2q + 2q + 1 \leq 2q + 2q + 1$ and $2q + 1 = 3$. $\square$

Combining the results one has

<u>Fact.</u> $SU(n)/SO(n)$ bounds if n is even and is nonbounding for n odd. Also $SU(n)/SO(n)$ is indecomposable only for $n=3$.

<u>Note.</u> To prove that $S = w_{odd}/w$ one has

$$\frac{w_{odd}(E+F)}{w(E+F)} = \frac{w_{odd}(E)\, w_{even}(F) + w_{odd}(E)\, w_{odd}(F) + w_{odd}(E)\, w_{odd}(F) + w_{even}(E)\cdot w_{odd}}{w(E)\cdot w(F)}$$

$$= \frac{w_{odd}(E)\cdot w(F)}{w(E)\, w(F)} + \frac{w_{odd}(F)\, w(E)}{w(F)\, w(E)}$$

$$= \frac{w_{odd}(E)}{w(E)} + \frac{w_{odd}(F)}{w(F)}$$

and if L is a line bundle with $w(L) = 1+x$ then

$$\frac{w_{odd}(L)}{w(L)} = \frac{x}{1+x} = x + x^2 + x^3 + \cdots = S(L).$$

The result then follows from the splitting principle.

<u>Comment.</u> One has $\Lambda^2(\gamma_n + 2) = \Lambda^2(\gamma_n) \otimes \Lambda^0(2) + \Lambda^1(\gamma_n) \otimes \Lambda^1(2) + \Lambda^0(\gamma_n) \otimes \Lambda^2(2)$

$$= \Lambda^2(\gamma_n) + 2\gamma_n + 1$$

and under the inclusion of $SU(n)/SO(n)$ in $SU(n+2)/SO(n+2)$ the bundle E_{n+2} restricts to E_n. Thus $w(SU(n+2)/SO(n+2))$ restricts to $SU(n)/SO(n)$ to become $w(SU(n)/SO(n)) \cdot w(E_n)^2 = w(SU(n)/SO(n))$. Thus there is a formula for $w(SU(n)/SO(n))$ which is universal, depending only on n modulo 2. Using the formula for $\Lambda^2(\gamma_n + 1)$ one sees that $w(SU(\text{even})/SO(\text{even})) = w(SU(\text{odd})/SO(\text{odd})) \cdot w$, relating the two formulae.

<u>Comment.</u> One also has a formula

$w(S^s(\gamma_n)) = 1 + \binom{n+s-1}{s-1} w_1 + \binom{n+s}{s-1}\{w_2 + \cdots + w_n\}$ modulo decomposables.

One can also ask for similar formulae for complex vector bundles, working over the integers. It appears that

$$c_k(\Lambda^s(\gamma_n^{cx})) = \left\{\binom{n}{s-1} - \binom{n}{s-2}2^{k-1} + \binom{n}{s-3}3^{k-1} + \cdots + (-1)^{s-2}\binom{n}{1}(s-1)^{k-1} + (-1)^{s-1}s^{k-1}\right\}c_k$$

modulo decomposables. This formula is correct for $s=2$ and 3, is correct for $k=1$ and 2 (the coefficients reduce to $\binom{n-1}{s-1}$ and $\binom{n-2}{s-1}$), and has the correct mod 2 reduction. For $s=2$, the formula is

$$c(\Lambda^2(\gamma_n^{cx})) = 1 + (n-1)c_1 + (n-2)c_2 + (n-4)c_3 + \cdots + (n-2^{k-1})c_k + \cdots + (n-2^{n-1})c_n$$

modulo decomposables, and one has

$$c(S^2(\gamma_n^{cx})) = 1 + (n+1)c_1 + (n+2)c_2 + (n+4)c_3 + \cdots + (n+2^{n-1})c_n$$

modulo decomposables.