HOMOTOPY ASSOCIATIVITY OF H-SPACES. II

BY

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1. **Introduction.** This paper is a sequal to "Homotopy Associativity of H-spaces. I" [28], hereafter referred to as HAH I, in that it continues the study of the associative law from the point of view of homotopy theory, but knowledge of HAH I is assumed only in a few places. The essence of most of the results can be gathered from consideration of associative H-spaces (monoids); the remaining results are well represented by consideration of homotopy associative H-spaces. [Just remember that an A_2-space is an H-space, an A_2-form $M_2: X \times X \to X$ is a multiplication, while an A_3-space is a homotopy associative H-space and an A_3-form consists of a multiplication and an associating homotopy.] I have attempted to make the various sections which require different background knowledge as independent as possible.

§2 describes the tilde construction, a generalization of the bar construction [18, Exposé 3]. A few of the remarks assume familiarity with the bar construction, but the description of the tilde construction is self-contained. Theorems 2.3 and 2.7 involve the concepts of HAH I, but only the associative case as given in Corollary 2.8 will be needed in our applications and for this knowledge of the Dold and Lashof construction [3] or even the Milnor construction [8] is sufficient.

Based on §2, §3 presents the Yessam operations in homology, which resemble the Massey operations in cohomology [31; 23] but are in no way dependent on them.

§4 makes some slight mention of the A_n-spaces of HAH I, but the reader will find it sufficient to consider associative H-spaces. Following Sugawara [29], we define maps of such spaces, called A_n-maps, which are special kinds of H-maps; they are homotopy multiplicative in a strong sense. Making use of Sugawara's work, A_n-maps are related to maps of the Dold and Lashof construction [3].

In §5 the cohomology version of the spectral sequence in §2 is applied to analyze

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A_n-maps which represent cohomology classes. Some results are obtained about the "suspension" homomorphism $\sigma: H^{q+1}(X;\pi) \to H^q(\Omega X;\pi)$.

In §6, the techniques developed are applied to the Postnikov systems of A_n-spaces with particular attention to spaces with just two nontrivial homotopy groups.

In addition to the influences acknowledged in HAH I, which apply as well to the present paper, I would like to express my gratitude to the late Professor J. H. C. Whitehead, to Dr. M. G. Barratt and particularly to Dr. I. M. James who served in turn as supervisors of my thesis for Oxford University, which included the material on Yessam operations, §§2 and 3.

2. **The tilde construction and its spectral sequence.** The bar construction [18] is defined for any associative algebra and strong use is made of the associative law. However, as we shall see, this condition can be significantly weakened and it will still be possible to retain some of the important properties of the bar construction.

DEFINITION 2.1. Let Λ be a commutative ring with unit. An $A(n)$-algebra $(R, m_1, \cdots, m_n)$ over Λ consists of an augmented graded Λ-module $R = \sum R_q$ and maps $m_i: \otimes^i R \to R$ satisfying the following properties:

(1) m_i raises degree by $i-2$, i.e., $m_i[(\otimes^i R)_q] \subset R_{q+i-2}$ where $\otimes^i R$ has the usual grading of the tensor product;

(2) if $u = u_1 \otimes \cdots \otimes u_i \in \otimes^i R$, then

$$\sum_{\substack{r+s=i+1 \\ 1 \leqq k \leqq r}} \chi(k,s,u)\, m_r(u_1 \otimes \cdots \otimes m_s(u_k \otimes \cdots \otimes u_{k+s-1}) \otimes \cdots \otimes u_i) = 0$$

where $\chi(k,s,u)$ is ± 1 according to the parity of

$$\varepsilon(k,s,u) = (s+1)k + s\left(i + \sum_{j=1}^{k-1} \dim a_j \right).$$

An *$A(\infty)$-algebra* $(R, m_1, m_2, \cdots)$ consists of an augmented Λ-module R and maps $m_i: \otimes^i R \to R$ satisfying the above conditions, i running over all the positive integers.

Notice that any chain complex over Λ or DGA-Λ-module [18, Exposé 2] is an $A(1)$-algebra with $m_1 = \partial$. Any DGA-algebra over Λ is an $A(2)$-algebra.

PROPOSITION 2.2. *If (C, ∂, Δ) is an associative DGA-algebra over Λ, where $\Delta: C \otimes C \to C$, then $(C, \partial, \Delta, 0, 0, \cdots, 0)$ is an $A(n)$-algebra.*

Proof. Relation 2 reduces to:

$\partial\partial = 0$ for $i = 1$;

$\partial_\otimes \Delta = \Delta\partial$ for $i = 2$, where $\partial_\otimes$ is the usual tensor product differential,

i.e., $\partial_\otimes(u \otimes v) = \partial u \otimes v + (-1)^{\dim u} u \otimes \partial v$;

$$(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta \text{ for } i = 3, \text{ i.e., } \Delta \text{ is associative;}$$

$$0 = 0 \quad \text{for } i > 3.$$

Similarly examining the conditions, we see that an $A(3)$-algebra is, in the obvious sense, a chain homotopy associative DGA-algebra. For $n > 3$, the $A(n)$-algebras are again the algebraic image of the spaces which we have described in HAH I.

THEOREM 2.3. *If X admits an A_n-form $\{M_i\}$, then $C_*(X)$ admits the structure of an $A(n)$-algebra by defining $m_1 = \partial$ and for $i > 1$, $m_i(u_1 \otimes \cdots \otimes u_i) = M_{i\#}(\kappa_i \otimes u_1 \otimes \cdots \otimes u_i)$ where κ_i is a suitable generator of $C_*(K_i)$.*

Proof. It is easy to verify formula 2.1(2) up to sign. A little painstaking care with orientations will give the signs as well, for of course the signs in Definition 2.1 were chosen so as to make Theorem 2.3 true.

(2.4) *The tilde or homotopy-bar construction $\tilde{\mathscr{B}}(R)$ on an $A(n)$-algebra $(R, m_1, \cdots, m_n)$.*

We are guided by the bar construction. Define $\bar{R}$ as the kernel of the augmentation. Let $\tilde{\mathscr{B}}(R) = \sum_0^n \otimes^i \bar{R}$ with the convention $\otimes^0 \bar{R} = \Lambda$. We bigrade $\tilde{\mathscr{B}}(R)$ by the number of factors and the total dimension, that is an element $u_1 \otimes \cdots \otimes u_i \in \otimes^i \bar{R}$ is of *simplicial degree* i and *internal degree* $j = \sum_1^i \dim u_k$. We refer to $i + j$ as the *total degree.* We define

$$d : \tilde{\mathscr{B}}(R) \to \tilde{\mathscr{B}}(R),$$

which lowers the total degree by 1. Let $u \in (R)$ be represented by $u_2 \otimes \cdots \otimes u_{i-1} \in \otimes^{i-2} \bar{R}$, $i \leqq n + 2$ (notice the somewhat unusual indexing). Then $d(u)$ is represented by

$$\sum_{2 \leqq k \leqq k+s-1 \leqq i-1} \chi'(k, s, u)\, u_2 \otimes \cdots \otimes m_s(u_k \otimes \cdots \otimes u_{k+s-1}) \otimes \cdots \otimes u_{i-1} \tag{2.5}$$

where $\chi'(k, s, u)$ is ± 1 according to the parity of $(s + 1)k + s(i + \sum_2^{k-1} \dim u_j)$. Extend d to all of $\tilde{\mathscr{B}}(R)$ by linearity. It is straightforward, using 2.1(2), to compute that $dd = 0$.

REMARK. If (A, ∂, Δ) is an associative DGA-algebra, then $(A, \partial, \Delta, 0, \cdots)$ is an $A(\infty)$-algebra. The corresponding tilde construction $\tilde{\mathscr{B}}(A)$, is, up to sign, the bar construction $\bar{\mathscr{B}}(A)$. The difference in sign is quite unimportant; it is just a change in orientation. For a general $A(\infty)$-algebra, it is possible to give a differential on $A \otimes \tilde{\mathscr{B}}(A)$ to make it acyclic, much as in Cartan's theory of "constructions". It is probably possible to generalize all of that theory to cover the tilde construction, but the absence of exact associativity would certainly make the details very complicated.

(2.6) *The spectral sequence canonically associated with an $A(n)$-algebra $(R, m_1, \cdots, m_n)$.*

The bar construction, filtered by the number of bars, gives rise to an interesting spectral sequence. Similarly, filter $\tilde{\mathscr{B}}(R)$ by the simplicial degree:

$$F^p(\tilde{\mathscr{B}}(R)) = \sum_{i \leqq p} \otimes^i \bar{R}.$$

This filtration is compatible with the differential d, hence [22, §10.3] there results a spectral sequence

$$\{\mathbf{E}^i(R, \{m_j\}), d^i\}, \qquad i \geqq 1.$$

Since both the simplicial and internal degrees are non-negative, the spectra sequence converges; the term $\mathbf{E}^\infty$ is just $\mathbf{E}^0(H_*(\tilde{\mathscr{B}}(R)))$, where $H_*(\tilde{\mathscr{B}}(R))$ has the induced filtration. In the case in which the $A(n)$-algebra is derived from an A_n-space as in Theorem 2.3, the terms $\mathbf{E}^1$ and $\mathbf{E}^\infty$ can be identified in terms of the geometry.

Consider an A_n-form on a space X and the corresponding projective spaces $XP(i)$. The largest, $XP(n)$, is filtred by its subspaces $XP(i)$; the chain group $C_*(XP(n))$ is filtered by $C_*(XP(i))$, and the filtration is compatible with the usual boundary. Let $\{\mathbf{E}^i(XP(n)), d^i\}$ denote the corresponding spectral sequence. (This is much the setup of the spectral sequence Milnor mentions in connection with his universal bundle [8]. The corollary to the following theorem can be regarded as stating that the Milnor or Dold-Lashof construction is a geometric realization of the bar construction.)

The n-fold smashed product $X^{(n)}$ is the space obtained from the pair $(X, e)^n$ by identifying the subspace to a point; that is, $X^n/X^{[n]}$ where $X^{[n]}$ is the subset consisting of all points with at least one coordinate being e.

THEOREM 2.7. *Let $\{M_i\}$ be an A_n-form on an arc-connected space, and let the induced $A(n)$-algebra be $(C_*(X), \{m_i\})$. Let $\{\mathbf{E}^i, d^i\}$ be the spectral sequence canonically associated with $(C_*(X), \{m_i\})$.*

A. *The term $\mathbf{E}^1_{p,q}$ is isomorphic to $\bar{H}_q(X^{(p)})$ and $d^1: H_q(X^{(p+1)}) \to H_q(X^{(p)})$ is given by $\sum(-1)^k(1 \times \cdots \times M_2 \times \cdots \times 1)_*$ (where M_2 is in the kth place in the general term).*

B. *There exists a filtration preserving chain map $\vartheta: \tilde{\mathscr{B}}(C_*(X)) \to C_*(XP(n))$ which induces isomorphisms*

$$\vartheta^i: \mathbf{E}^i \to \mathbf{E}^i(XP(n))$$

and hence is a chain equivalence.

Proof of A. The identification of the term $\mathbf{E}^1$ follows by using the chain equivalence between $C(X^{(p)})$ and $\otimes^p \bar{C}(X)$, and applying certain facts about spectral sequences [19, p. 331, ¶. 2–3]. F^p/F^{p-1} may be identified with $\otimes^p \bar{C}(X)$ and the differential d^0 induced from d can be identified with the product differential. The homology with respect to this differential is the (reduced) homology of the smashed product, $\bar{H}_*(X^{(p)})$. The form of d^1 follows directly from the definition of d.

Proof of B. ϑ is defined by

$$\vartheta(u_2 \otimes \cdots \otimes u_{i-1}) = \beta_{i-1\#}(\kappa_i \otimes u_2 \otimes \cdots \otimes u_{i-1}).$$

Being careful about orientations, it is straightforward to check that ϑ is a filtration preserving map and that ϑ^1 is an isomorphism. The theorem now follows by a standard theorem on spectral sequences [22, 10.3. 13, p. 410].

If X is an associative H-space, our construction reduces to that of Dold and Lashof [3]; $XP(i)$ has the homotopy type of the Dold and Lashof base space B_{i+1}. If we replace X by a topological group G, then $XP(i)$ has the homotopy type of the Milnor base space X_{i+1} [8]. In both of the above cases, the tilde construction reduces to the bar construction. Thus we have

Corollary 2.8. *If X is an associative H-space, there is a filtration preserving chain equivalence $\vartheta:\bar{\mathscr{B}}(C_*(X)) \to C_*(B_\infty)$. If G is a topological group, there is a filtration preserving chain equivalence $\vartheta:\bar{\mathscr{B}}(C_*(G)) \to C_*(X_\infty)$.*

Thus Milnor's spectral sequence is equivalent to that derived from the bar construction. (This has also been proved by M. Ginsburg in his thesis(2) for M.I.T.) In fact on the chain level, Milnor's construction looks more like the bar construction as far as signs are concerned because he regards $G * i * G$ as formed from a subset of $I \times G \times \cdots \times I \times G$, while it is implicit in what we have said that the signs in $\tilde{\mathscr{B}}(C_*(G))$ correspond to starting with $K_{i+1} \times G^i$, as a subset of $I^{i-1} \times G^i$.

3. **Yessam operations.** The differentials of the spectral sequence $\mathbf{E}^i$ can be interpreted as homology operations. We will concentrate on d^p restricted to the subset of $\otimes^{p+1}\bar{H}_*(X)$ in $\mathbf{E}^p_{p+1,q}$ (where X has a given A_n-form, $n > p$). It can be regarded as a pth order homology operation of $(p+1)$-variables, analogous to the Massey operations in cohomology [31;23]; hence we call them Yessam operations. If defined, we denote $d^p(u_1 \otimes \cdots \otimes u_{p+1})$ by $[u_1, \cdots, u_{p+1}]$. For example, $d^1 \,|\, H_*(X) \otimes H_*(X)$ can be identified with the Pontryagin product. The triple Yessam product defined by d^2 includes the following case:

Suppose $u \in H_p(X)$, $v \in H_q(X)$, $w \in H_r(X)$ and that the Pontryagin products uv and vw are zero. Let $\bar{u}, \bar{v}, \bar{w}$ be chains representing u, v, w respectively. Then there exist $x \in C_{p+q+1}(X)$ and $y \in C_{q+r+1}(X)$ such that $\partial x = \bar{u}\bar{v}$ and $\partial y = \bar{v}\bar{w}$. The triple product $[u, v, w]$ is the coset of $H_{p+q+r+1}(X)$ modulo $M_{2*}H_{p+q+r+1}(X^{(2)})$ determined by $(-1)^p\, uy - xw - M_{3\#}(\kappa_3 \otimes \bar{u} \otimes \bar{v} \otimes \bar{w})$. [If the A_n-form is trivial, the last term drops out. In general d^p is much easier to describe and to calculate if the A_n-form is trivial, i.e., $(C_*(X), M_{2\#})$ is an associative algebra.]

As for nontrivial examples of these operations, we have

Application 3.1. *Let $CP(n)$ denote complex projective space of dimension n and let u generate $H_1(\Omega CP(n))$. $H_{2n}(\Omega CP(n))$ is generated by the $(n+1)$-fold Yessam operation $[u, \cdots, u]$.*

Proof. $\Omega CP(n)$ has the homotopy type of $S^1 \times \Omega S^{2n+1}$ [11, Proposition 6] so

(2) *Added in proof.* This has appeared as *On the Lusternik-Schnirelman category*, Ann. of Math. (2) 77 (1963), 538–551. The Yessam operations are referred to there (p. 550) and Application 3.1 is proved.

that $H_q(\Omega CP(n))=0$ for $1<q<2n$. Thus $\mathbf{E}^r_{p,q}=0$ for $q<p$ and $p<q<2n+p-1$. It follows that $\mathbf{E}^1_{n+1,n+1}\approx\mathbf{E}^2_{n+1,n+1}\approx\cdots\approx\mathbf{E}^n_{n+1,n+1}$ so that the $(n+1)$-fold Yessam product $[u,\cdots,u]$ is defined. Now $\mathbf{E}^{n+1}_{n+1,n+1}\approx\cdots\approx\mathbf{E}^\infty_{n+1,n+1}$ which is trivial since $H_{2n+2}(CP(n))=0$. By a similar argument, $\mathbf{E}^1_{1,2n}\approx\mathbf{E}^n_{1,2n}$ but $\mathbf{E}^{n+1}_{1,2n}\approx\mathbf{E}^\infty_{1,2n}=0$ so that $d^n:\mathbf{E}^n_{n+1,n+1}\to\mathbf{E}^n_{1,2n}$ must be an isomorphism. Since $\mathbf{E}^1_{1,2n}=H_{2n}(\Omega CP(n))$, the latter group must be generated by $[u,\cdots,u]$. (For low values of n, $[u,\cdots,u]$ can be calculated directly and fairly easily using the cobar representation for $C_*(\Omega CP(n))$ [17]. This method was suggested by W. Browder.)

4. **A_n-maps.** Before discussing the limited naturality of these operations, we will have to consider maps of A_n-spaces.

DEFINITION 4.1. Let $\{M_i\}$ be an A_n-form on X, $\{N_i\}$ an A_n-form on W. A map $f: X\to W$ is an *A_n-homomorphism* if $f\circ M_i=N_i(1\times f^i)$ for $i\leqq n$.

THEOREM 4.2. *If $f:X\to W$ is an A_n-homomorphism, there exist maps $f_E:\mathscr{E}_n(X)\to\mathscr{E}_n(W)$ and $f_B:\mathscr{B}_n(X)\to\mathscr{B}_n(W)$ such that*

$$f_B(\mathscr{B}_i(X))\subset\mathscr{B}_i(W), \tag{1}$$

$$\mathfrak{p}_n f_E=f_B\mathfrak{p}_n. \tag{2}$$

Proof. If $f:X\to W$ is an A_n-homomorphism, then the maps $1\times f^i:K_{i+1}\times X^i\to K_{i+1}\times W^i$ and $1\times f^{i-1}:K_{i+1}\times X^{i-1}\to K_{i+1}\times W^{i-1}$ respect the identifications imposed by α_i and β_i respectively; hence they induce maps of the derived A_n-structures.

COROLLARY 4.3. *If $f:X\to W$ is an A_n-homomorphism, then the induced map ${}^1f$: $\mathbf{E}^1(\tilde{\mathscr{B}}(C_*(X)))\to\mathbf{E}^1(\tilde{\mathscr{B}}(C_*(W)))$ is given by ${}^1f\,|\,\mathbf{E}^1_{p,q}=f^{(p)}_*:H_q(X^{(p)})\to H_q(W^{(p)})$. The Yessam operations of p variables are natural with respect to A_n-homomorphisms, $p\leqq n$.*

Though useful, the concept of an A_n-homomorphism is too limited. For $n=2$, it does not include even H-maps. To give an appropriate extension of the concept of an H-map to maps of A_n-spaces (intuitively, we want a map which respects A_n-forms "up to homotopy") is extremely complicated. Only because the weight of notation and detail is so indigestible in the general case, we will restrict ourselves for a while to associative H-spaces (X,m) (otherwise called monoids), i.e., $m:X\times X\to X$ and $m(m\times 1)=m(1\times m)$.

DEFINITION 4.4. Let (X,m) and (Y,n) be associative H-spaces. A map $f:X\to Y$ is an *A_n-map* if there exist *sputnik homotopies* $h_i:I^{i-1}\times X^i\to Y$ for $i\leqq n$ such that $h_1=f$,

$$\begin{aligned} &h_i(t_1,\cdots,t_{i-1},x_1,\cdots,x_i)\\ &\quad= h_{i-1}(t_1,\cdots,\hat{t}_k,\cdots,t_{i-1},x_1,\cdots,x_kx_{k+1},\cdots,x_i) && \text{for } t_k=0\\ &\quad= h_k(t_1,\cdots,t_{k-1},x_1,\cdots,x_k)h_{i-k}(t_{k+1},\cdots,t_{i-1},x_{k+1},\cdots,x_i) && \text{for } t_k=1. \end{aligned}$$

If the maps h_i exist for all positive integers i, then f is *strongly homotopy-multiplicative* [29, p. 259]. (Note that an A_2-map is just an H-map.)

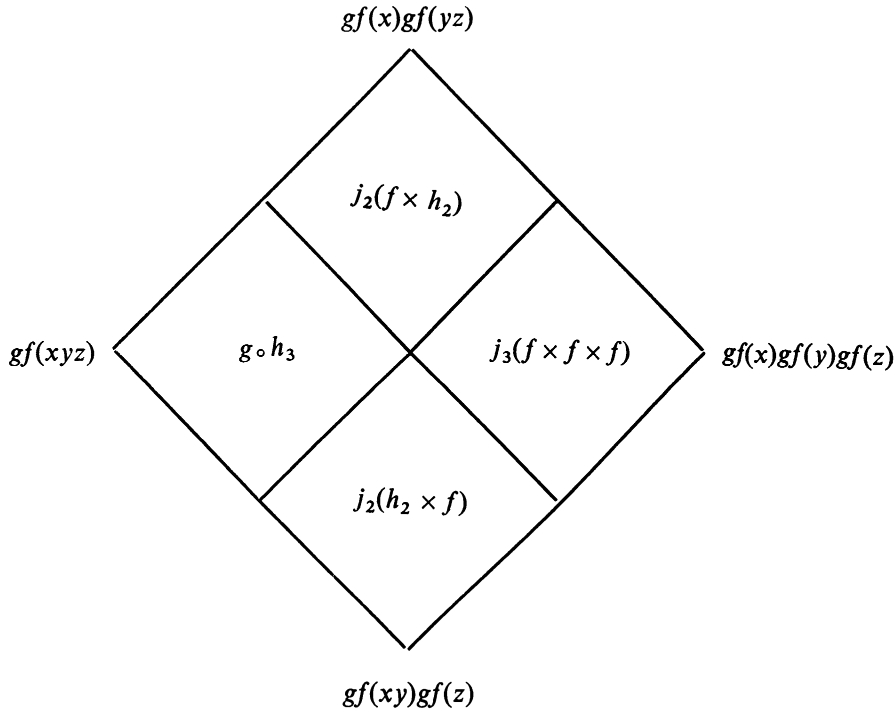
It is complicated but straightforward to verify that the composition of A_n-maps is an A_n-map. For example, suppose $f: X \to Y$ is an A_n-map with $h_i: I^{i-1} \times X^i \to Y$ and $g: Y \to Z$ is an A_n-map with $j_i: I^{i-1} \times Y^i \to Z$. For $n = 2$, define $k^2: I \times X^2 \to Z$ by

$$k_2(t,x,y) = g \circ h_2(2t,x,y), \qquad 0 \leqq 2t \leqq 1,$$

$$k_2(t,x,y) = j_2(2t-1, f(x), f(y)), \qquad 1 \leqq 2t \leqq 2.$$

For $n = 3$, define $k_3: I^2 \times X^3 \to Z$ by

$$\begin{aligned} k_3(t,s,x,y,z) &= g \circ h_3(2t,2s,x,y,z) \\ &= j_2(2t-1, f(x), h_2(2s,y,z)) \\ &= j_2(2s-1, h_2(2t,x,y), f(z)) \\ &= j_3(2t-1, 2s-1, f(x), f(y), f(z)). \end{aligned}$$



For monoids, our construction (8 in HAH I) of A_n-structures $\mathfrak{p}_i: \mathscr{E}_i \to \mathscr{B}_i$ reduces to that of Dold and Lashof $p_i: E_i \to B_i$ [3] as remarked at the end of HAH I. As one might expect, A_n-maps induce maps of th corresponding A_n-structure.

THEOREM 4.5. *A map $f: X \to Y$ is an A_n-map if and only if there exist maps $f_E: E_n(X) \to E_n(Y)$ and $f_B: B_n(X) \to B_n(Y)$ such that*

(1) $f_E | X = f$,

(2) $f_B(B_i(X)) \subset B_i(Y)$,

(3) $f_B p_n = p_n f_E$,

(4)
$$\begin{array}{ccc} CE_{n-1}(X) & \xrightarrow{Cf_E|} & CE_{n-1}(Y) \\ p_n k \downarrow & & \downarrow p_n h \\ B_n(X) & \xrightarrow[f_B]{} & B_n(Y) \end{array}$$

(*where k and h are the contractions*) *is homotopy commutative* rel $E_{n-1}(X)$, *keeping $CE_{i-1}(X)$ in $B_i(Y)$.*

The construction of f_E and f_B from h_i is given by Sugawara [29, Lemma 22] who verifies (1)–(3). Condition (4) is easy to verify since $p_n h \circ Cf_E \mid (t, y) = (t, f_E(y))$ for $y \in E_{n-1}(X)$ while

$$f_B p_n k(t, y) = \begin{cases} (2t, f_E(y)), & 0 \leqq 2t \leqq 1, \\ p_n f_E(y), & 1 \leqq 2t \leqq 2. \end{cases}$$

We postpone the construction of h_i from f_E and f_B.

COROLLARY 4.6. *If $f: X \to Y$ is an A_n-map, there is an induced map of spectral sequences $f^i: \mathbf{E}(X) \to \mathbf{E}^i(Y)$ for $i \leqq n$ such that f^1 is induced by $f_*^{(p)}: H_q(X^{(p)}) \to H_q(Y^{(p)})$. The Yessam operations of p variables are natural with respect to A_n-maps, $p \leqq n$.*

It follows easily that a homotopy equivalence between ΩCP^n and $S^1 \times \Omega S^{2n+1}$ cannot be realized by an A_{n+1}-map. For $n = 2$, ΩCP^3 and $S^1 \times \Omega S^7$ do have the same A_2-type or H-type [13, Corollary 1.19].

5. **Cohomology classes represented by A_n-maps.** A particularly interesting and significant class of candidates for A_n-maps are maps of a monoid (X, m) into an Eilenberg-MacLane space $K(\pi, n)$. It is a familiar result of obstruction theory that homotopy classes of such maps correspond to cohomology classes of $H^n(X; \pi)$. To analyze cohomology classes, a convenient tool is the dual spectral sequence $\{\mathbf{E}_i, d_i\}$ derived from $\mathrm{Hom}(\tilde{\mathscr{B}}(C(X)), \pi)$. $\mathbf{E}_1^{p,q}$ can be identified with $H^q(X^{(p)})$, coefficients in π being understood hereafter.

THEOREM 5.1. *A class $u \in H^q(X)$ is represented by an A_n-map if and only if regarding $H^q(X)$ as $\mathbf{E}_1^{1,q}$ we have $d_j(u) = 0$ for all $j < n$.*

All connected monoids are essentially loop spaces, for a connected monoid (X, m) has the homotopy type of ΩB_∞ [15, Lemma 10]. When working with a loop space ΩY, we will always be considering loop addition, which is associative [10; 17].

DEFINITION 5.2. A map $f:\Omega Y \to \Omega Z$ is a *loop map* if there exists a map $g: Y \to Z$ such that f is homotopic to Ωg. A class $u \in H^q(\Omega Y)$ is a *loop class* if it is represented by a loop map $f:\Omega Y \to K(\pi,q) = \Omega K(\pi, q+1)$.

REMARK. Note that a loop class is what is usually referred to as a class in the image of suspension $\sigma: H^{q+1}(X;\pi) \to H^q(\Omega X;\pi)$. We have revised the terminology to increase the conceptual content of the label and to relieve the burden on the overworked word "suspension".

Our main result concerning loop classes is:

THEOREM 5.3. *A loop class $u \in H^q(\Omega Y)$ is represented by an A_n-map with respect to the A_n-structure derived from loop addition. The condition is also sufficient for a class to be a loop class if $q < (n+1)(s+1) - 2$ where Y is s-connected.*

For $n \leqq 2$, this can be rephrased in more familiar terms. Any map is an A_1-map, so for $n = 1$ we have the statement: If Y is s-connected, then $\Omega: H^{q+1}(Y,\pi) \to H^q(\Omega Y;\pi)$ is an isomorphism for $q < 2s$ [30, p. 94 or 26]. For $n = 2$, Theorem 5.1 reduces to: A class u is represented by an H-map if and only if it is primitive. Thus we have: If Y is s-connected, then $\Omega(H^{q+1}(Y;\pi))$ is precisely the set of primitive elements of $H^q(\Omega Y;\pi)$ for $q \leqq 3s$.

Theorem 5.3 provides us with a whole spectrum of A_n-maps.

APPLICATION 5.4. *The generator of $H^{2n}(\Omega CP(n))$ is represented by an A_n-map but not by an A_{n+1}-map.*

Proof. We proceed essentially as in the proof that H_{2n} is generated by the Yessam product. Let $w \in H^{2n}(\Omega CP(n))$ be a generator. Since $H^{2n-j+1}(\Omega CP(n)^{(j+1)}) = 0$ unless $j = n$, we have $d_j(w) = 0$ for $j < n$ so that w is represented by an A_n-map. Should w be represented by an A_{n+1}-map, it would be a loop class by Theorem 5.3. This is impossible since $H^{2n+1}(CP(n)) = 0$. (Apparently it has been realized for some time that these classes are primitive but not suspensions; cf. [33].)

6. **A_n-fibrings and k-invariants.** One way to build new spaces from old is by means of fibrings. If both fibre and base are monoids, is it always possible to put a multiplication on the total space? The answer is: No, only certain fibrings are "nice" enough.

THEOREM 6.1. *Let (X,m) and (W,n) be monoids. If $f: X \to W$ is an A_n-map, then the fibre space Y over X induced by f from $\Omega W \to \mathscr{L}W \to W$ admits an A_n-form.*

The theorem is known in the case $n = 2$ [21, p. 294 or 20]. A partial converse is given below, by analyzing the k-invariants of A_n-spaces. First we have

THEOREM 6.2. *If a space Y admits an A_n-form, then each stage Y of the*

Postnikov system has an A_n-form such that the projections $\rho_q\colon Y_q \to Y_{q-1}$ are A_n-homomorphisms.

Now if Y_{q-1} has in fact an associative multiplication, we can ask if the k-invariant $k \in H^{q+1}(Y_{q-1};\pi_q(X))$ is represented by an A_n-map and in fact the answer is yes. (Of course, this discussion could be carried out if Y_{q-1} admits only an A_n-form, but again the notation and details are intolerably complex.) We state the theorem only in its neatest case.

THEOREM 6.3. *Suppose a space Y has only two nontrivial homotopy groups, $\pi_p(Y) = \pi$, $\pi_q(Y) = G$, $q > r$. Then Y admits an A_n-form if and only if the k-invariant $k \in H^{q+1}(\pi, n; G)$ is represented by an A_n-map.*

We do not have to specify the A_n-form on $K(\pi, n)$ because, up to homotopy, there is only one, as can readily be seen using obstruction theory. In the case $n = 2$, the theorem was first stated by Copeland [20] in slightly different terms. In this case, the multiplication on Y can described very neatly and explicitly [27]. However, it is not true that all the k-invariants considered are "suspensions", as can be seen from the following corollary of Theorem 6.3 in the case $n = 3$.

APPLICATION 6.4. *Let α be a generator of $H^{n+1}(Z_p, n; Z)$ with n odd and p an odd prime. The space Y with k-invariant α^p admits a multiplication but not a homotopy associative multiplication.*

Application 6.4 gives an alternative proof of Moore's statement [24] that α^p is not a suspension, for if it were, Y would be of the homotopy type of a space of loops [30] and so *would* admit a homotopy associative multiplication. Conversely, Moore's statement can be used to prove Application 6.4 in the case $p=3$, for if α^3 were represented by an A_3-map then by Theorem 5.3 it would be a loop class (a suspension).

7. **Proof of Theorem 4.5.** Given a monoid (X,m), Dold and Lashof constructed quasifibrings $p_n\colon E_n \to B_n$ as follows (we have changed a few details such as indexing and ordering of coordinates): $E_1 = X$, $B_1 = *$, a point, p_1 is the canonical map. $E_n = X \times CE_{n-1} \cup_{\mu_{n-1}} E_{n-1}$, $B_n = CE_{n-1} \cup_{p_{n-1}} B_{n-1}$, and μ_n and p_n are defined by

$$\begin{array}{ccc} X \times X \times CE_{n-1} & \xrightarrow{m \times 1} & X \times CE_{n-1} \\ \downarrow & & \downarrow \\ X \times E_n & \xrightarrow[\mu_n]{} & E_n \end{array} \qquad \begin{array}{ccc} X \times CE_{n-1} & \xrightarrow{\pi} & CE_{n-1} \\ \downarrow & & \downarrow \\ E_n & \xrightarrow[p_n]{} & B_n \end{array}$$

where the vertical maps correspond to the identifications and π is projection onto the second factor. Thus a point of E_n can be represented by $(x_1, t_1, x_2, \cdots, t_{n-1}, x_n)$ and a point of B_n by $(t_1, x_2, \cdots, t_{n-1}, x_n)$. At the end of HAH I, we showed that we can alter the Dold and Lashof construction without changing its homotopy type by identifying $(x_1, t_1, \cdots, t_{i-1}, e, t_i, \cdots, x_n)$ in E_n with $(x_1, t_1, \cdots, x_{i-1}, t_{i-1} + t_i - t_{i-1}t_i, x_{i+1}, \cdots, x_n)$ for any i, and similarly in B_n. Assume this done, and from now on let $I = [0,2]$. We first define $j_i: I^{i-1} \times X^i \to E_i(Y)$ by

$$j_i(t_1, \cdots, t_{i-1}, x_1, \cdots, x_i)$$
$$= [f_E(x_1, 1 - t_1, x_2, \cdots, x_{k_1}),\ t_{k_1} - 1, f_E(x_{k_1+1}, 1 - t_{k_1+1}, \cdots, x_{k_2}), \cdots$$
$$\cdots, t_{k_s} - 1, f_E(x_{k_s+1}, \cdots, x_i)]$$

for $1 \leqq t_{k_r} \leqq 2$, $1 \leqq r \leqq s$ and $0 \leqq t_m \leqq 1$ otherwise. Notice that

$$i_i(t_1, \cdots, x_i) = \begin{cases} j_{i-1}(t_1, \cdots, \hat{t}_r, \cdots, t_{i-1}, \cdots, x_1, \cdots, x_r x_{r+1}, \cdots, x_i) & \text{when } t_r = 0, \\ (j_r(t_1, \cdots, t_{r-1}, x_1, \cdots, x_r), 1, j_{i-r}(t_{r+1}, \cdots, t_{i-1}, x_{r+1}, \cdots, x_i)) & \text{when } t_r = 1. \end{cases}$$

Using condition (3) to pair off portions of $p_i j_i$, we can easily extend $p_i j_i$ to a map $g_i: I^i \times X^i \to B_i(Y)$ such that

$$g_i(t_0, \cdots, t_{i-1}, x_1, \cdots, x_i)$$
$$(7.1) = \begin{cases} * & \text{if } t_0 = 0, \\ p_i j_i(t_1, \cdots, x_i) & \text{if } t_0 = 2, \\ g_{i-1}(t_0, \cdots, \hat{t}_r, \cdots, t_{i-1}, x_1, \cdots, x_r x_{r+1}, \cdots, x_i) & \text{if } t_r = 0, \\ (g_r(t_0, \cdots, t_{r-1}, x_1, \cdots, x_r), 1, g_{i-r}(t_0, t_{r+1}, \cdots, t_{i-1}, x_{r+1}, \cdots, x_i)) & \text{if } t_r = 2. \end{cases}$$

Proceeding by induction, we cover (replace p_n by a true fibring or use (A_1) of [15]) each g_i by a deformation $\bar{g}_i: I^i \times X^i \to E_i(Y)$ such that

$$\bar{g}_i \text{ lies in } Y \qquad \text{if } t_0 = 0,$$
$$\bar{g}_i \text{ is given by } j_i \qquad \text{when } t_0 = 2,$$

and (7.1) is valid for $t_r = 0$ or 2 with g barred throughout. Thus if we define $h_i(t_1, \cdots, x_i) = \bar{g}_i(0, t_1, \cdots, x_i)$, we will have the desired sputnik homotopies.

8. **Obstructions to A_n-maps.** A class $u \in H^q(X; \pi)$ is represented by a homotopy class of maps $f: X \to K(\pi, q)$. Assume by induction that some such f is an A_{n-1}-map so that we have sputnik homotopies $h_i: I^{i-1} \times X^i \to K(\pi, q)$ for $i < n$. Using these maps, we can define $\bar{h}_n = h_n \mid \partial I^{n-1} \times X^n$ even though we may not be able to construct all of h_n. The obstruction to extending to all of $I^{n-1} \times X^n$ is a class in $H^q((I^{n-1}, \partial I^{n-1}) \times X^n; \pi)$, which is isomorphic to $H^{q-n+1}(X^n; \pi)$. We would like to show that this obstruction lies in $H^{q-n+1}(X^{(n)}; \pi)$. Then we can show that this obstruction represents $d_{n-1}(u)$.

LEMMA 8.1. *Let (X, m) and (W, n) be monoids. If $f: X \to W$ is an A_n-map, then there exist sputnik homotopies h_i: $I^{n-1} \times X^n \to W$ such that if $x_j = e$, $1 < j < i$, then*

$$h_i(t_1, \cdots, x_i)$$

$$= \begin{cases} h_{i-1}(t_1, \cdots, t_{j-1} + t_j, \cdots, t_{i-1}, x_1, \cdots, \hat{x}_j, \cdots, x_i) & \textit{for } 0 \leqq t_{j-1} + t_j \leqq 1 \\ h_{j-1}(t_1, \cdots, t_{j-2}, x_1, \cdots, x_{j-1}) h_{i-j}(t_{j-1} + t_j - 1, \cdots, t_{i-1}, x_{j+1}, \cdots, x_i) & \\ & \textit{for } 1 \leqq t_{j-1} + t_j \leqq 2 \end{cases}$$

while

$$h_i(t_1, \cdots, x_{i-1}, e) = h_{i-1}(t_1, \cdots, t_{i-2}, x_1, \cdots, x_{i-1})$$

and

$$h_i(t_1, \cdots, t_{i-1}, e, x_2, \cdots, x_i) = h_{i-1}(t_2, \cdots, t_{i-1}, x_2, \cdots, x_i).$$

Proof. By interpreting h_i suitably as a homotopy, we find as in the Appendix to HAH I that it can be specified at will on $X^{[i]}$, the subset of X^i consisting of points with at least one coordinate e.

Therefore, let us assume that the sputnik homotopies satisfy the conditions of Lemma 8.1. It is then possible to extend $\bar{h}_n$ to $\partial I^{n-1} \times X^n \cup I^{n-1} \times X^{[n]}$. The remaining obstruction to extending to all of $I^{n-1} \times X^n$, now lies in $H^q((I^{n-1}, \partial I^{n-1}) \times (X^n, X^{[n]}); \pi)$ which is isomorphic to $H^{q-n+1}(X^{(n)}; \pi)$. Let us identify that obstruction.

For $i > 1$, we can identify I^{i-1} with CS^{i-2}, the cone on the sphere. Corresponding to the maps $h_i: I^{i-1} \times X^i \to K = K(\pi, q)$, we have maps $j_i: X^i \to \mathscr{L}\Omega^{i-2}K$ defined by

$$j_i(x_1, \cdots, x_i)(t, \tau) = h_i(t, \tau, x_1, \cdots, x_i) - f(x_1) - f(x_2) - \cdots - f(x_i).$$

Let $g_i: X^i \to \Omega^{i-2}K$ be defined by

$$\begin{array}{ccc} & \overset{j_i}{\nearrow} & \mathscr{L}\Omega^{i-2}K \\ & & \downarrow \pi \\ X^i & \xrightarrow[g_i]{} & \Omega^{i-2}K \end{array}$$

so that g_i corresponds to $h_i \big| \partial I^{i-1} \times X^i$. Using obstruction theory, we have corresponding chains $\bar{b}_i \in C^{q-i+1}(X^i)$ and $\bar{a}_i \in C^{q-i+2}(X^i)$ such that $\delta \bar{b}_i = \bar{a}_i$, for $i > 1$. For $i = 1$, we have $\bar{b}_1 = h_1^{\#}(\iota)$ which represents $u \in H^q(X)$. Let the components in $C^*(X^{(i)})$ be b_i and a_i respectively. By repeated application of the homotopy addition theorem, we are able to write

$$\text{(8.2)} \qquad \begin{aligned} \delta b_i &= \sum_{1 \leqq k \leqq i-1} (-1)^k (1 \times \cdots \times m \times \cdots \times 1)^{\#} b_{i-1}, \ i > 1; \\ \delta b_1 &= 0. \end{aligned}$$

The other terms which correspond to

$$h_r(t_1, \cdots, t_{r-1}, x_1, \cdots, x_r) \cdot h_s(t_{r+1}, \cdots, t_{i-1}, x_{r+1}, \cdots, x_i)$$

drop out in $C^*(X^{(i)})$ since for the multiplication in K or $\Omega^{i-2}K$ we have $m^{\#}(\iota) = p_1(\iota) + p_2(\iota)$ where ι is the fundamental class. By the same reasoning, we see that the component in $C^*(X^{(n)})$ of $g_n(\iota)$, which is the obstruction to extending $\bar{h}_n$, is

$$\sum_k (-1)^k(1 \times \cdots \times m \times \cdots \times 1)^{\#} b_{n-1}.$$

Now $\sum_{i<n} b_i$ represents u in $\mathbf{E}^1_{1,q}$. Consider $d\sum_{i<n} b_i \bmod F^{n+1}$, the part of $\tilde{\mathscr{B}}^*(C_*(X))$ of filtration $n+1$. We have

$$d\sum_{i\leqq n} b_i = \sum_{i<k}\left[(-1)^i\delta b_i - \sum_k (-1)^k(1 \times \cdots \times m \times \cdots \times 1)^{\#} b_i\right]$$

which by (8.2) reduces mod F^{n+1} to $\sum_k(-1)^k(1 \times \cdots \times m \times \cdots \times 1)^{\#} b_{n-1}$, precisely the obstruction to extending $\bar{h}_n$. Therefore the obstruction represents $d_{n-1}(u)$, which thus must vanish if f is an A_n-map. Conversely, if $d_j(u) = 0$ for $j < n$, then there exist chains $b_i \in C^{q-i+1}(X^{(i)})$ for $i \leqq n$ such that $d\sum b_i = 0 \bmod F_{n+1}$ and b_1 represents u. It follows easily that (8.2) holds. If we assume by induction that sputnik homotopies $h_i: I^{i-1} \times X^i \to K(\pi, q)$ exist for $i < n$ and give rise to b_i as above, then (8.2) for $i = n$ tells us that the obstruction vanishes, i.e., h_n can also be constructed and f is an A_n-map.

9. **Loop classes.** Before proving Theorem 5.3, we need to study the classifying space for ΩY more closely. Let $B_\infty(\Omega Y)$ be constructed using loop addition.

THEOREM 9.1. *There is a commutative diagram*

$$\begin{array}{ccc}
\Omega Y & = & \Omega Y \\
\downarrow & & \downarrow \\
E_\infty(\Omega Y) & \to & \mathscr{L}Y \\
{\scriptstyle p_\infty}\downarrow & & \downarrow{\scriptstyle \pi} \\
B_\infty(\Omega Y) & \xrightarrow{\phi} & Y
\end{array}$$

which respects the action of ΩY *on* p_∞ *and on* π.

Proof. By induction, define $\phi_i: B_i \to Y$, $\psi_i: E_i \to \mathscr{L}Y$ and $\theta_i: CE_i \to \mathscr{L}Y$ as follows: ψ_1 is the inclusion of ΩY in $\mathscr{L}Y$. $\theta_i(t,z)(s) = \psi_i(z)(ts)$ for $z \in E_i$, $\psi_{i+1}(x,t,z) = x + \theta_i(t,z)$ where $+$ denotes the usual action of ΩY on $\mathscr{L}Y$. $\phi_i(t,z) = \pi\theta_i(t,z)$. It is easy to verify that these maps are well defined, (in particular, $\psi_i(x+y,t,z) = x + \psi_i(y,t,z)$), and that $\pi\psi_i = \phi_i p_i$.

COROLLARY 9.2. ϕ *is a homotopy equivalence.*

COROLLARY 9.3. *Let* $i: S\Omega Y \to B_\infty(\Omega Y)$ *be the inclusion of* $B_2(\Omega Y)$. *A class* $u \in H^q(\Omega Y)$ *is a loop class if and only if there exists* $v \in H^{q+1}(B_\infty(\Omega Y))$ *such that* $i^*v = Su \in H^{q+1}(S\Omega Y)$.

Proof. $\phi \mid S\Omega Y$ is given by $\phi(t,\lambda) = \lambda(rt)$ where $\lambda: [0,r] \to Y$. Let χ be an inverse for ϕ. It is easy to check that $i^*v = S\Omega(\chi^*v)$. [Recall that $S: H^q(Z) \to H^{q+1}(SZ)$

is defined in terms of maps as follows: If $f: Z \to \Omega K(\pi, q+1)$, then $Sf: SZ \to K(\pi, q+1)$ is given by $Sf(t,z) = f(z)(t)$.] Thus $u = \Omega(\chi^* v)$ and is a loop class.

Proof of Theorem 5.3. As is well known, a loop class is a homomorphism for with $\lambda: [0,r] \to Y$, $\mu: [0,s] \to Y$, we have

$$[(\Omega f)(\lambda + \mu)](t) = \begin{cases} f \circ \lambda(t), & 0 \leqq t \leqq r, \\ f \circ \mu(t), & 0 \leqq t + r \leqq s \end{cases}$$
$$= [(\Omega f)\lambda + (\Omega f)\mu](t).$$

A homomorphism of monoids is clearly an A_n-map.

Conversely, suppose $u \in H^q(\Omega Y)$ is represented by an A_n-map so that $d_j(u) = 0$ for $j < n$. But then $d_j(u) = 0$ for all j since $H^{q-j+1}(\Omega Y^{(j+1)}) = 0$ for $q - j + 1 < s(j+1)$, i.e., $q < (n+1)(s+1) - 2$ for $j \geqq n$, so that u survives to $\mathbf{E}_\infty$. Since $\mathbf{E}_\infty = E_0(H^*(B_\infty))$ as filtered by the subspaces B_i, we know that $\mathbf{E}_\infty^{1,q}$ is a subgroup of $H^{q+1}(SX)$; in fact this subgroup is the image of $H^{q+1}(B_\infty(\Omega Y))$ under the inclusion map. Since $\mathbf{E}_r^{0,q} = 0$ for $q > 0$, $\mathbf{E}_\infty^{1,q}$ can be identified with a subset of $\mathbf{E}_1^{1,q}$, and the following diagram is commutative:

$$\begin{array}{ccc} H^q(X) = \mathbf{E}_1^{1,q} \supset & \mathbf{E}_\infty^{1,q} = i^* H^*(B_\infty(\Omega Y)) \\ {\scriptstyle S}\searrow & \cap \\ & H^{q+1}(SX) \end{array}$$

where S is the suspension isomorphism. This can be seen by considering $C(X)$ as an $A(1)$-algebra; the composition $C(X) \to \tilde{\mathscr{B}}(C(X)) \to C(SX)$ clearly induces the homology suspension isomorphism.

Since u survives to $\mathbf{E}_\infty$, there exist $v \in H^{q+1}(B_\infty(\Omega Y))$ such that $i^* v = Su$. By Corollary 9.3, u is a loop class.

10. Proofs for §6.

Proof of Theorem 6.1. We have monoids (X, m) and (W, n) and a fibring $\Omega W \to Y \to X$ induced by an A_n-map $f: X \to W$, i.e.,

$$\begin{array}{ccccc} \Omega W & \to & Y & \to & X \\ \| & & \downarrow & & \downarrow f \\ \Omega W & \to & \mathscr{L} W & \to & W. \end{array}$$

We wish to construct an A_n-structure on Y. Since f is an A_n-map, we have a commutative diagram of n-tuples

$$\begin{array}{ccc} E_i(X) & \xrightarrow{d_i} & E_i(W) \\ p_i \downarrow & & \downarrow q_i \\ B_i(X) & \xrightarrow{b_i} & B_i(W) \end{array} \qquad \text{for } i \leqq n.$$

Define $E_i(Y)$ as the space induced by d_i from $\Omega E_i(W) \to \mathscr{L}E_i(W) \to E_i(W)$ and similarly define $B_i(Y)$ in terms of b_i.

$$\begin{array}{ccccc} \Omega E_i(W) \to & E_i(Y) & \to & E_i(X) \\ \| & \downarrow & & \downarrow d_i \\ \Omega E_i(W) \to & \mathscr{L}E_i(W) & \to & E_i(W) \end{array} \qquad \begin{array}{ccccc} \Omega B_i(W) & \to & B_i(Y) & \to & B_i(X) \\ \| & & \downarrow & & \downarrow b_i \\ \Omega B_i(W) & \to & \mathscr{L}B_i(W) & \to & B_i(W). \end{array}$$

Represent $E_i(Y)$ as pairs (z,λ) with $z \in E_i(X)$, $\lambda \in \mathscr{L}E_i(W)$, $\lambda(\bar{1}) = d_i(z)$ and represent $B_i(Y)$ similarly. Define r_i: $E_i(Y) \to B_i(Y)$ by $r_i(z,\lambda) = (p_i(z), \mathscr{L}q_i(\lambda))$ where $[\mathscr{L}q_i(\lambda)](t) = q_i \circ \lambda(t)$. Let $B_i(X)$ and $B_i(W)$ have base points $*$ and $\bar{*}$ respectively.

LEMMA 10.1. $Y = r_i^{-1}(*,\bar{*})$.

Proof. We can write $r_i^{-1}(*,\bar{*})$ as the set

$$\{(z,\lambda) \mid z \in E_i(X), \lambda \in \mathscr{L}E_i(W), d(z) = \lambda(\bar{1}), p_i(z) = *, q_i \circ \lambda(t) = \bar{*}\},$$

which is clearly the same as $Y = \{(x,\mu) \mid x \in X, \mu \in \mathscr{L}W, f(x) = \mu(\bar{1})\}$ since X is the fibre of p_i, W that of q_i and $d_i \mid X = f$.

LEMMA 10.2. $r_{i*}: \pi_q(E_i(Y), Y) \to \pi_q(B_i(Y))$ *is an isomorphism for all* q.

This follows from the more general

PROPOSITION 10.3. *Suppose we have a commutative diagram*

$$\begin{array}{ccccc} F & \to & E & \xrightarrow{p} & B \\ \downarrow & & \downarrow d & & \downarrow b \\ F' & \to & E' & \xrightarrow[p']{} & B' \end{array}$$

where $p_*: \pi_q(E,F) \approx \pi_q(B)$ *and* $p'_*: \pi_q(E',F') \approx \pi_q(B')$ *for all* q. *Let* d *and* b *induce fibrings* σ *and* ρ *as follows*:

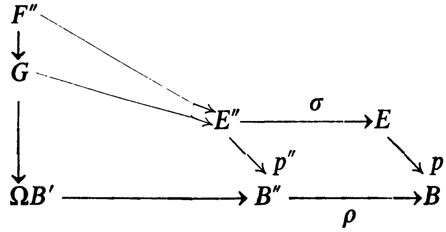
$$\begin{array}{ccccc} \Omega E' & \to & E'' & \xrightarrow{\sigma} & E \\ \| & & \downarrow & & \downarrow d \\ \Omega E' & \to & \mathscr{L}E' & \to & E' \end{array} \qquad \begin{array}{ccccc} \Omega B' & \to & B'' & \xrightarrow{\rho} & B \\ \| & & \downarrow & & \downarrow b \\ \Omega B' & \to & \mathscr{L}B' & \to & B'. \end{array}$$

Define $p'': E'' \to B''$ *by* $p''(z,\lambda) = (p(z), \mathscr{L}p'(\lambda))$. *Then* $p''_*: \pi_q(E'',F'') \to \pi_q(B'')$ *is an isomorphism for all* q.

Proof. The fibre $F'' = p''^{-1}(*)$ (where $*$ is the base point of B'') is a fibre space as shown in Lemma 10.1:

$$\begin{array}{ccccc} \Omega F' & \to & F'' & \to & F \\ \| & & \downarrow & & \downarrow d \mid F \\ \Omega F' & \to & \mathscr{L}F' & \to & F'. \end{array}$$

Let G denote $(p\sigma)^{-1}(*) = (\rho p'')^{-1}(*)$ so that $G = \sigma^{-1}(F) = p''^{-1}(\Omega B')$.



Since $p''(F'')$ is a point, $F'' \subset G$. Since $\rho p''(G)$ is a point, $p'' \mid G$ lies in $\Omega B'$. Consider the diagram

$$\begin{array}{ccccc} (\Omega E', \Omega F') & \rightarrow & (G, F'') & \rightarrow & (F, F) \\ \Omega p' \downarrow & & \downarrow p'' & & \\ \Omega B' & \longrightarrow & \Omega B' & & \end{array}$$

where $\Omega E'$ is, as a fibre of σ, contained in G. We see that $p''_* : \pi_n(G, F'') \rightarrow \pi_n(\Omega B')$ is an isomorphism. Now consider the exact sequences

$$\begin{array}{ccccccccc} \cdots & \rightarrow & \pi_n(G, F'') & \rightarrow & \pi_n(E'', F'') & \rightarrow & \pi_n(E'', G) & \rightarrow & \cdots \\ & & \downarrow p''_* & & \downarrow p''_* & & \downarrow p_* \sigma_* & & \\ \cdots & \rightarrow & \pi_n(\Omega B') & \rightarrow & \pi_n(B'') & \rightarrow & \pi_n(B) & \rightarrow & \cdots . \end{array}$$

The first vertical arrow is an isomorphism. The last is also because of the assumption on p and the fact that σ is a fibring. Thus the middle one is an isomorphism.

LEMMA 10.4. *$E_{i-1}(Y)$ is contractible in $E_i(Y)$.*

Proof. Consider the diagram

$$\begin{array}{ccccc} & & E_{i-1}(Y) & \rightarrow & E_{i-1}(X) \\ & & \downarrow & & \downarrow \\ \Omega E_i(W) & \rightarrow & E_i(Y) & \longrightarrow & E_i(X). \end{array}$$

Since $E_{i-1}(X)$ is contractible in $E_i(X)$, $E_{i-1}(Y)$ can be deformed into the fibre $\Omega E_i(W)$. Explicitly representing points of $E_{i-1}(Y)$ as pairs (z, λ) with $z \in E_{i-1}(X)$, $\lambda \in \mathscr{L}E_{i-1}(W)$, we can deform $j : E_{i-1}(Y) \rightarrow E_i(Y)$ to the map $j_1 : E_{i-1}(Y) \rightarrow \Omega E_i(W)$ given by $j_1 : (z, \lambda) \rightarrow \lambda - v_1$ where $v_1 \in \mathscr{L}E_i(W)$ and $v_1(t) = dk(t,z)$ where $k : CE_{i-1}(X) \rightarrow E_i(X)$ is the contraction.

Consider the projection of j_1 into $\Omega B_i(W)$. Since f is an A_n-map, the diagram

$$\begin{array}{ccc} CE_{i-1}(X) & \xrightarrow{Cd_{i-1}} & CE_{i-1}(W) \\ p_i k \downarrow & & \downarrow q_i h \\ B_i(X) & \xrightarrow{b_i} & B_i(W) \end{array}$$

is homotopy commutative rel $E_{i-1}(X)$ (cf. Theorem 4.5). From this we see that

$\Omega q_i \circ j_1$ is homotopic to $\Omega q_i \circ j_2$ where $j_2(z,\lambda)=\lambda - v_2$ and $v_2(t)=h_t d(z)$. In turn j_2 is null-homotopic since $(z,\lambda)\to\lambda$ is homotopic to $(z,\lambda)\to v_2$ by $\mu(s,t)=h_t(\lambda(s))$. (Notice that $\mu(1,t)=v_2(t)$ and $\mu(s,1)=\lambda(s)$.) Thus $\Omega q_i \circ j_1$ is null-homotopic and hence j_1 can be deformed into the fibre ΩW which is contractible in $\Omega E_i(W)$; $E_{i-1}(Y)$ is contractible in $E_i(Y)$. Thus we have exhibited an A_n-structure on Y. Theorem 5 of HAH I then guarantees the existence of an A_n-form on Y.

Now we turn to the proof of Theorem 6.2. We will need the following proposition:

PROPOSITION 10.5. *Suppose given*

(1) *an A_n-form $\{N_i\}$ on a space Y,*

(2) *an A_{n-1}-form $\{M_i\}$ on a space X,*

(3) *a fibring $\pi: Y\to X$ which is an A_{n-1}-homomorphism.*

Provided that

(4) *X is $(r-1)$-connected and*

(5) *the fibre is $(s-1)$-connected, $s>r$, while*

(6) *$\pi_q(X)=0$ for $q \geqq (n-1)(r+1)+s-1$*

then there exists an A_n-form $\{M_i\}$ and N_n', a deformation of N_n rel $L_n\times Y^n \cup K_n\times Y^n$, such that π is an A_n-homomorphism.

Proof. Since the fibre is $(s-1)$-connected, we can construct a cross section over the s-skeleton Z of X. Hence we can define M_n at least on $K_n\times Z^n$. (For $n=2$, we have the following diagram:

$$\begin{array}{ccc} Y\times Y & \xrightarrow{N_2} & Y \\ \uparrow & & \downarrow \pi \\ Z\times Z & \xrightarrow{M_2'} & X.) \end{array}$$

The obstructions to extending M_n' to all of $K_n\times X^n$ will appear as classes in $H^{q+1}(K_n\times X^n, L_n\times X^n\cup K_n\times(X^{[n]}\cup Z^n);\pi_q(X))$. Since (K_n,L_n) is homeomorphic to $(I^{n-2},\dot{I}^{n-2})$, this group is isomorphic to $H^{q-n+3}(X^n,X^{[n]}\cup Z^n;\pi_q(X))$ which is trivial for any coefficient group unless $q-n+3\geqq(n-1)r+s+1$, i.e., $q\geqq(n-1)(r+1)+s-1$. Hence, by condition (6), we can extend to all of $K_n\times X^n$, obtaining M_n. Similarly the obstructions to π being a homomorphism, at least up to homotopy rel $L_n\times Y^n\cup K_n\times(Y^{[n]}\cup Z^n)$, are classes in $H^{q-n+2}(Y^n,Y^{[n]}\cup Z^n;\pi_q(X))$ which again are trivial groups for all q. Now π is a fibring. Hence if we have $\pi N_n\cong M_n(1\times\pi^n)$ rel $L_n\times Y^n\cup K_n\times(Y^{[n]}\cup Z^n)$, N_n can be deformed to N_n' rel this subset so that $\pi N_n'=M_n(1\times\pi^n)$, which proves the proposition. (This argument was obtained from W. Browder for $n=2$.)

COROLLARY 10.6. *Suppose $\pi: Y\to X$ is a fibring, that X is $(r-1)$-connected for some $s>r$, the fibre is $(s-1)$-connected and $\pi_q(X)=0$ for $q\geqq r+s$. If Y*

admits an A_n-form, then there are A_n-forms on X and Y with respect to which π is an A_n-homomorphism.

In fact, the form on Y is a deformation of the one given. Repeated application of the theorem yields the corollary. Repeated application of the corollary to the various stages of a Postnikov system yields Theorem 6.2.

Proof of Theorem 6.3. Let Y be a space with only two nontrivial homotopy groups, $\pi_r(Y) = \pi$, $\pi_q(Y) = G$ with $q > r$, so that Y is a fibre space over $K(\pi, r)$ induced by a map $f: K(\pi, r) \to K(G, q+1)$. The k-invariant $u \in H^{q+1}(\pi, r; G)$ is just $f^*(\iota)$.

Let $X = K(\pi, r)$ and $W = K(G, q+1)$. If Y admits an A_n-form, then it can be deformed to one such that the projection $\pi: Y \to X$ is an A_n-homomorphism. Thus there is an induced map $\pi_i: \mathbf{E}_i(X) \to \mathbf{E}_i(Y)$ for $r \leqq n$ and $\pi_1 \mid \mathbf{E}_1^{p,q}$ is just $\pi^{(p)*}: H^q(X^{(p)}) \to H^q(Y^{(p)})$. Since $f \circ \pi$ is null-homotopic, $\pi^*(u) = 0$ and hence $d_i(\pi^*(u)) = 0$ for all i. We can conclude that $d_i(u) = 0$ for $i < n$ if $\pi^{(j+1)*} \mid \mathbf{E}_1^{j+1, q-j+1}$ is an isomorphism for $j \leqq i$. Since ΩW is $(q-2)$-connected, $\pi^*: H^s(X) \to H^s(Y)$ is an isomorphism for $s < q$; hence $\pi^{(j+1)*} \mid \mathbf{E}_1^{j+1, q-j+2}$ is an isomorphism for $q - j + 2 \leqq jr + q$, i.e., for $r > 0$, and f is therefore an A_n-map.

Conversely, if f is an A_n-map, Theorem 6.1 shows that Y admits an A_n-form.

11. Calculation of $d_2(\alpha^p)$.

Proof of Application 6.4. We must show that $d_1(\alpha^p) = 0$ and $d_2(\alpha^p) \neq 0$. Since we can assume $C_*(X)$ finitely generated in each dimension, we have $\tilde{\mathscr{B}}^* = \mathrm{Hom}(\tilde{\mathscr{B}}(C_*(X)), \Lambda) = \Sigma \otimes {}^i\bar{C}^*(X)$ where $\bar{C}^*(X) = \mathrm{Hom}(\bar{C}_*(X), \Lambda)$. The differential $d^*: \tilde{\mathscr{B}}^* \to \tilde{\mathscr{B}}^*$ starts as follows: $d^*(u) = -\delta u + m^\#(u)$ and $d^*(u \otimes v) = \delta u \otimes v + (-1)^p u \otimes \delta v + m^\#(u) \otimes v - u \otimes m^\#(v)$ for $u \in C^p(X)$ and $v \in C^q(X)$. Thus the differential $d_1: H^p(X) \to H^p(X \wedge X)$ is induced by m^*. In particular for α^p where α generates $H^{n+1}(Z_p, n; Z)$, we have $d_1(\alpha^p) = 0$ since taking a pth power is a homomorphism mod p and $m^*(\alpha) = p_1^*(\alpha) + p_2^*(\alpha)$ where $p_i: X_1 \times X_2 \to X_i$ is the projection. Explicitly, writing $\alpha_i = p_i^*(\alpha)$ we have $m^*(\alpha^p) = (m^*(\alpha))^p = (\alpha_1 + \alpha_2)^p = \sum \binom{p}{i} \alpha_1^i \alpha_2^{p-i} = p_1^*(\alpha^p) + p_2^*(\alpha^p)$ since α is of order p and p divides $\binom{p}{i}$ for $0 < i < p$. Now for $d_2(\alpha^p)$.

Because the cohomology of $K(Z_p, n)$ consists entirely of p-torsion, it is convenient to use as coefficient ring Z^p, the ring of p-adic integers [32, Chapter 10]. For any space X with finitely generated homology, the Universal Coefficient Theorem gives us $H^i(X; Z^p) = H^i(X) \otimes Z^p + \mathrm{Tor}(H^{i-1}(X), Z^p)$. Since Z^p is torsion free and divisible by all primes q other than p, we can write $H^i(X; Z^p) = H^i(X)_\infty \otimes Z^p + H^i(X)_p$ where $H^i(X)_\infty$ denotes the torsion free component and $H^i(X)_p$ the p-primary component. In particular for $i > 0$, $H^i(Z_p, n; Z^p) \approx H^i(Z_p, n)$.

The cohomology of $K(\pi, n)$ has been extensively analyzed by Cartan using his theory of "constructions". The Cartan Seminar 1954–55 is our basic reference; we will assume familiarity with the basic notions involved. We consider the following graded algebras over a commutative ring Λ:

$E(\bar{y}, n)$, the exterior algebra with generator $\bar{y}$ in dimension n odd,

$\Gamma(\bar{x}, n)$, the divided polynomial algebra with generator $\bar{x}$ in dimension n even, i.e., $\Gamma(\bar{x}, n)$ has additive basis $\bar{x}_i$ in dimension in with relations:

$$\bar{x}_i\bar{x}_j = (i,j)\bar{x}_{i+j} \quad \left[(i,j) = \frac{(i+j)!}{i!j!} \right].$$

Cartan has obtained $H_*(Z_p, n; Z_p)$ as a Hopf algebra by an iterated construction starting with $C_*(Z_p, 1) = E(\bar{y}, 1) \otimes \Gamma(\bar{x}, 2)$ with $d\bar{y} = 0$, $d\bar{x}_i = p\bar{y}\bar{x}_{i-1}$. For n odd, let $C = E(\bar{y}, n) \otimes \Gamma(\bar{x}, n+1)$ with $d\bar{x}_i = p\bar{y}\bar{x}_{i-1}$, $d\bar{y} = 0$. Using Cartan's methods, Moore [24] has shown that for each odd n there exists a Hopf algebra D_n and a multiplicative chain equivalence $\phi_n: C \otimes D_n \to C_*(Z_p, n; Z^p)$. If x is the cocycle dual to $\bar{x}, \phi_n^*(\alpha^p)$ is represented by x^p. If $d_2(x^p)$ is nonzero, so must $d_2(\alpha^p)$ be since ϕ_n is multiplicative.

Since C is trivial in dimensions between 0 and n, $m^*(x) = 1 \otimes x + x \otimes 1$. [If $n = 1$, we use the fact that x is a cocycle as well.] As for α^p, we have

$$m^*(x^p) = \sum_{i+j=p} (i,j)x^i \otimes x^j.$$

Since p divides $(i, p-i)$ for $0 < i < p$ and $\delta y = px$, we have

$$m^*(x^p) = 1 \otimes x^p + x^p \otimes 1 + \delta B$$

with

$$B = 1/p \sum_{i+j=p;\, i,j>0} (i,j)x^i \otimes x^{j-1}y.$$

A simple calculation shows that $d_2(x^p)$ is represented by

$$(1/p) \sum (i,j+k)(j-1,k)[x^i \otimes x^{j-1}y \otimes x^k - x^i \otimes x^j \otimes x^{k-1}y]$$

where the sum is now over $i+j+k = p$, $i,j,k > 0$. [Hint: The calculation uses the relation $(i,j+k)(j-1,k) = (i,j+k)(j,k-1) - (i+j,k)(i,j)$.] Since $x^i \otimes x^j \otimes x^{k-1}y - x^i \otimes x^{j-1}y \otimes x^k$ represents a generator of $H^{p(n+1)-1}(\bar{C} \otimes \bar{C} \otimes \bar{C}; Z^p)$ modulo $[(1 \otimes m) - {}^*(m \otimes 1)^*]$ $H^{p(n+1)-1}(\bar{C} \otimes \bar{C}; Z^p)$, the theorem follows from the fact that for some $0 < i < p$, p^2 does not divide $(i,j+k)(j,k-1)$. (This statement is not applicable for $p = 2$, since there are no values of i,j and k such that $i+j+k = 2$, $i,j,k > 0$.) The proof also applies, mutatis mutandis, for α^{p^j}, including the case $p = 2$ if $j > 1$.

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(continued from HAH I [28])

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