A CANONICAL LIFT OF FROBENIUS IN MORAVA E-THEORY

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Abstract. We prove that the $p$th Hecke operator on the Morava $E$-cohomology of a space is congruent to the Frobenius mod $p$. This is a generalization of the fact that the $p$th Adams operation on the complex $K$-theory of a space is congruent to the Frobenius mod $p$. The proof implies that the $p$th Hecke operator may be used to test Rezk’s congruence criterion.

1. Introduction

The $p$th Adams operation on the complex $K$-theory of a space is congruent to the Frobenius mod $p$. This fact plays a role in Adams and Atiyah’s proof [AA66] of the Hopf invariant one problem. It also implies the existence of a canonical operation $\theta$ on $K^0(X)$ satisfying

$$\psi^p(x) = x^p + p\theta(x),$$

when $K^0(X)$ is torsion-free. This extra structure was used by Bousfield [Bou96] to determine the $\lambda$-ring structure of the $K$-theory of an infinite loop space. There are several generalizations of the $p$th Adams operation in complex $K$-theory to Morava $E$-theory: the $p$th additive power operation, the $p$th Adams operation, and the $p$th Hecke operator. In this note, we show that the $p$th Hecke operator is a lift of Frobenius.

In [Rez09], Rezk studies the relationship between two algebraic structures related to power operations in Morava $E$-theory. One structure is a monad $T$ on the category of $E_0$-modules that is closely related to the free $E_{\infty}$-algebra functor. The other structure is a form of the Dyer-Lashof algebra for $E$, called $\Gamma$. Given a $\Gamma$-algebra $R$, each element $\sigma \in \Gamma$ gives rise to a linear endomorphism $Q_\sigma$ of $R$. He proves that a $\Gamma$-algebra $R$ admits the structure of an algebra over the monad $T$ if and only if there exists an element $\sigma \in \Gamma$ (over a certain element $\bar{\sigma} \in \Gamma/p$) such that $Q_\sigma$ is a lift of Frobenius in the following sense:

$$Q_\sigma(r) \equiv r^p \mod pR$$

for all $r \in R$.

We will show that $Q_\sigma$ may be taken to be the $p$th Hecke operator $T_p$, as defined by Ando in [And95 Section 3.6]. We prove this by producing a canonical element $\sigma_{can} \in \Gamma$ lifting the Frobenius class $\bar{\sigma} \in \Gamma/p$ [Rez09 Section 10.3] such that $Q_{\sigma_{can}} = T_p$. This provides us with extra algebraic structure on torsion-free algebras over the monad $T$ in the form of a canonical operation $\theta$ satisfying

$$T_p(r) = r^p + p\theta(r).$$

Let $G_{E_0}$ be the formal group associated to $E$, a Morava $E$-theory spectrum. The Frobenius $\phi$ on $E_0/p$ induces the relative Frobenius isogeny

$$G_{E_0/p} \to \phi^* G_{E_0/p}$$
over \( E_0/p \). The kernel of this isogeny is a subgroup scheme of order \( p \). By a theorem of Strickland, this corresponds to an \( E_0 \)-algebra map

\[
\tilde{\sigma} : E^0(B\Sigma_p)/I \rightarrow E_0/p,
\]

where \( I \) is the image of the transfer from the trivial group to \( \Sigma_p \). This map further corresponds to an element in the mod \( p \) Dyer-Lashof algebra \( \Gamma/p \). Rezk considers the set of \( E_0 \)-module maps \([\tilde{\sigma}] \subset \text{hom}(E^0(B\Sigma_p)/I, E_0)\) lifting \( \tilde{\sigma} \).

**Proposition 1.1.** There is a canonical choice of lift \( \sigma_{\text{can}} \in [\tilde{\sigma}] \).

The construction of \( \sigma_{\text{can}} \) is an application of the formula for the \( K(n) \)-local transfer (induction) along the surjection from \( \Sigma_p \) to the trivial group \([\text{Gan06, Section 7.3}]

Let \( X \) be a space and let

\[
P_p/I : E^0(X) \rightarrow E^0(B\Sigma_p)/I \otimes_{E_0} E^0(X)
\]

be the \( p \)th additive power operation. The endomorphism \( Q_{\sigma_{\text{can}}} \) of \( E^0(X) \) is the composite of \( P_p/I \) with \( \sigma_{\text{can}} \otimes 1 \).

**Proposition 1.2.** For any space \( X \), the following operations on \( E^0(X) \) are equal:

\[
Q_{\sigma_{\text{can}}} = (\sigma_{\text{can}} \otimes 1)(P_p/I) = T_p.
\]

This has the following immediate consequence:

**Corollary 1.3.** Let \( X \) be a space such that \( E^0(X) \) is torsion-free. There exists a canonical operation

\[
\theta : E^0(X) \rightarrow E^0(X)
\]

such that, for all \( x \in E^0(X) \),

\[
T_p(x) = x^p + p\theta(x).
\]

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2. Tools

Let \( E \) be a height \( n \) Morava \( E \)-theory spectrum at the prime \( p \). We will make use of several tools that let us access \( E \)-cohomology. We summarize them in this section.

For the remainder of this paper, let \( E(X) = E^0(X) \) for any space \( X \). We will also write \( E \) for the coefficients \( E^0 \) unless we state otherwise.

**Character theory** Hopkins, Kuhn, and Ravenel introduce character theory for \( E(BG) \) in \([\text{HKR00}] \). They construct the rationalized Drinfeld ring \( C_0 \) and introduce a ring of generalized class functions taking values in \( C_0 \):

\[
Cl_n(G, C_0) = \{C_0\text{-valued functions on conjugacy classes of map from } \mathbb{Z}_p^n \text{ to } G\}.
\]

They construct a map

\[
E(BG) \rightarrow Cl_n(G, C_0)
\]

and show that it induces an isomorphism after the domain has been base-changed to \( C_0 \) \([\text{HKR00, Theorem C}] \). When \( n = 1 \), this is a \( p \)-adic version of the classical character map from representation theory.

**Good groups** A finite group \( G \) is good if the character map

\[
E(BG) \rightarrow Cl_n(G, C_0)
\]
is injective. Hopkins, Kuhn, and Ravenel show that $\Sigma_p^k$ is good for all $k$ [HKR00, Theorem 7.3].

Transfer maps It follows from a result of Greenlees and Sadofsky [GS96] that there are transfer maps in $E$-cohomology along all maps of finite groups. In [Gan06, Section 7.3], Ganter studies the case of the transfer from $G$ to the trivial group and shows that there is a simple formula for the transfer on the level of class functions. Let

$$\text{Tr}_{C_0} : Cl_n(G, C_0) \to C_0$$

be given by the formula $f \mapsto \frac{1}{|G|} \sum_{[\alpha]} f([\alpha])$, where the sum runs over conjugacy classes of maps $\alpha : \mathbb{Z}_p^n \to G$. Ganter shows that there is a commutative diagram

$$\begin{array}{ccc}
E(BG) & \xrightarrow{\text{Tr}_E} & E \\
\downarrow & & \downarrow \\
Cl_n(G) & \xrightarrow{\text{Tr}_{C_0}} & C_0,
\end{array}$$

in which the vertical maps are the character map.

Subgroups of formal groups Let $G_E = \text{Spf}(E(BS^1))$ be the formal group associated to the spectrum $E$. In [Str98], Strickland produces a canonical isomorphism

$$\text{Spf}(E(B\Sigma_p^k)/I) \cong \text{Sub}_{p^k}(G_E),$$

where $I$ is the image of the transfer along $\Sigma_p^{k-1} \subset \Sigma_p^k$ and $\text{Sub}_{p^k}(G_E)$ is the scheme that classifies subgroup schemes of order $p^k$ in $G_E$. We will only need the case $k = 1$.

The Frobenius class The relative Frobenius is a degree $p$ isogeny of formal groups

$$G_{E/p} \to \phi^* G_{E/p},$$

where $\phi : E/p \to E/p$ is the Frobenius. The kernel of the map is a subgroup scheme of order $p$. Using Strickland’s result, there is a canonical map of $E$-algebras

$$\bar{\sigma} : E(B\Sigma_p)/I \to E/p$$

picking out the kernel. In [Rez09, Section 10.3], Rezk describes this map in terms of a coordinate and considers the set of $E$-module maps $[\bar{\sigma}] \subset \text{hom}(E(B\Sigma_p), E)$ that lift $\bar{\sigma}$.

Power operations In [GH04], Goerss, Hopkins, and Miller prove that the spectrum $E$ admits the structure of an $E_\infty$-ring spectrum in an essentially unique way. This implies a theory of power operations. These are natural multiplicative non-additive maps

$$P_m : E(X) \to E(B\Sigma_m) \otimes_E E(X)$$

for all $m > 0$. For $m = p^k$, they can be simplified to obtain interesting ring maps by further passing to the quotient

$$P_{p^k}/I : E(X) \to E(B\Sigma_{p^k}) \otimes_E E(X) \to E(B\Sigma_{p^k}/I \otimes_E E(X),$$

where $I$ is the transfer ideal that appeared above.

Hecke operators In [And95, Section 3.6], Ando produces operations

$$T_{p^k} : E(X) \to E(X)$$

by combining the structure of power operations, Strickland’s result, and ideas from character theory. Let $T = (\mathbb{Q}_p/\mathbb{Z}_p)^p$, let $H \subset T$ be a finite subgroup, and let $D_\infty$ be the Drinfeld ring
at infinite level so that $\text{Spf}(D_{\infty}) = \text{Level}(\mathbb{T}, G_E)$ and $\mathbb{Q} \otimes D_{\infty} = C_0$. Ando constructs an Adams operation depending on $H$ as the composite

$$\psi^H : E(X) \xrightarrow{\psi_H} E(B\Sigma_p)/I \otimes_E E(X) \xrightarrow{H \otimes I} D_{\infty} \otimes_E E(X).$$

He then defines the $p^k$th Hecke operator

$$T_{p^k} = \sum_{\substack{H \subseteq T \mid H = p^k}} \psi^H$$

and shows that this lands in $E(X)$.

3. A canonical representative of the Frobenius class

We construct a canonical representative of the set $[\bar{\sigma}]$. The construction is an elementary application of several of the tools presented in the previous section.

We specialize the transfers of the previous section to $G = \Sigma_p$. Let $\text{Tr}_E : E(B\Sigma_p) \to E$ be the transfer from $\Sigma_p$ to the trivial group and let $\text{Tr}_{C_0} : C_{\text{Cl}}(\Sigma_p, C_0) \to C_0$ be the transfer in class functions from $\Sigma_p$ to the trivial group. This is given by the formula

$$\text{Tr}_{C_0}(f) = \frac{1}{p!} \sum_{[\alpha]} f([\alpha]).$$

Recall that $\mathbb{T} = (\mathbb{Q}_p/\mathbb{Z}_p)^n$ and let $\text{Sub}_p(\mathbb{T})$ be the set of subgroups of order $p$ in $\mathbb{T}$.

**Lemma 3.1.** [Mar, Section 4.3.6] The restriction map along $\mathbb{Z}/p \subseteq \Sigma_p$ induces an isomorphism

$$E(B\Sigma_p) \xrightarrow{\cong} E(B\mathbb{Z}/p)^{\text{Aut}(\mathbb{Z}/p)}.$$ 

After a choice of coordinate $x$,

$$E(B\Sigma_p) \cong E[y]/(yf(y)),$$

where the degree of $f(y)$ is

$$|\text{Sub}_p(\mathbb{T})| = \frac{p^n - 1}{p - 1} = \sum_{i=0}^{n-1} p^i,$$

$f(0) = p$, and $y$ maps to $x^{p^{-1}}$ in $E(B\mathbb{Z}/p) \cong E[x]/[p](x)$.

**Lemma 3.2.** [Qui71, Proposition 4.2] After choosing a coordinate, there is an isomorphism

$$E(B\Sigma_p)/I \cong E[y]/(f(y)),$$

and the ring is free of rank $|\text{Sub}_p(\mathbb{T})|$ as an $E$-module.

After choosing a coordinate, the restriction map $E(B\Sigma_p) \to E$ sends $y$ to 0 and the map

$$E(B\Sigma_p) \to E(B\Sigma_p)/I$$

is the quotient by the ideal generated by $f(y)$.

**Lemma 3.3.** The index of the $E$-module $E(B\Sigma_p)$ inside $E \times E(B\Sigma_p)/I$ is $p$. 
Proof. This can be seen using the coordinate. There is a basis of $E(B\Sigma_p)$ given by the set \{1, y, \ldots, y^m\}, where $m = |\text{Sub}_p(T)|$, and a basis of $E \times E(B\Sigma_p)/I$ given by
\[\{(1,0), (0,1), (0,y), \ldots, (0,y^{m-1})\}.
\]
By Lemma 3.1, the image of the elements \{1, y, \ldots, y^{m-1}, p - f(y)\} in $E(B\Sigma_p)$ is the set
\[\{(1,1), (0,y), \ldots, (0,y^{m-1}), (0,p)\}
\]in $E \times E(B\Sigma_p)/I$. The image of $y^m$ is in the span of these elements and the submodule generated by these elements has index $p$. □

Lemma 3.4. [Rez09, Section 10.3] In terms of a coordinate, the Frobenius class $\bar{\sigma} : E(B\Sigma_p)/I \rightarrow E/p$ is the quotient by the ideal $(y)$.

Now we modify $\text{Tr}_{C_0}$ to construct a map $\sigma_\text{can} : E(B\Sigma_p)/I \rightarrow E$.

By Ganter’s result [Gan06, Section 7.3] and the fact that $\Sigma_p$ is good, the restriction of $\text{Tr}_{C_0}$ to $E(B\Sigma_p)$ is equal to $\text{Tr}_E$. It makes sense to restrict $\text{Tr}_{C_0}$ to
\[E \times E(B\Sigma_p)/I \subset \text{Cl}_n(\Sigma_p, C_0).
\]
Lemma 3.3 implies that this lands in $\frac{1}{p}E$. Thus we see that the target of the map $p!\text{Tr}_{C_0} \big|_{E \times E(B\Sigma_p)/I}$ can be taken to be $E$. We may further restrict this map to the subring $E(B\Sigma_p)/I$ to get
\[p!\text{Tr}_{C_0} \big|_{E(B\Sigma_p)/I} : E(B\Sigma_p)/I \rightarrow E.
\]
From the formula for $\text{Tr}_{C_0}$, for $e \in E \subset E(B\Sigma_p)/I$, we have
\[p!\text{Tr}_{C_0} \big|_{E(B\Sigma_p)/I}(e) = |\text{Sub}_p(T)|e.
\]
Note that $|\text{Sub}_p(T)|$ is congruent to 1 mod $p$ (and therefore a $p$-adic unit). We set
\[\sigma_\text{can} = p!\text{Tr}_{C_0} \big|_{E(B\Sigma_p)/I}.
\]
Remark 3.5. One may also normalize $\sigma_\text{can}$ by dividing by $|\text{Sub}_p(T)|$ so that $e$ is sent to $e$.

We now show that $\sigma_\text{can}$ fits in the diagram

\[
\begin{array}{ccc}
E & \xrightarrow{\sigma_\text{can}} & E \\
\downarrow \sigma & & \downarrow \text{id} \\
E(B\Sigma_p)/I & \rightarrow & E/p,
\end{array}
\]

where $\sigma$ picks out the kernel of the relative Frobenius.

Proposition 3.6. The map $\sigma_\text{can} : E(B\Sigma_p)/I \rightarrow E$ is a representative of Rezk’s Frobenius class.
Proof. We may be explicit. Choose a coordinate so that the quotient map
\[ q: E(B\Sigma_p) \rightarrow E(B\Sigma_p)/I \]
is given by
\[ q: E[y]/(yf(y)) \rightarrow E[y]/(f(y)). \]
We must show that
\[ E(B\Sigma_p)/I \xrightarrow{\sigma_{\text{can}}} E \mod p \rightarrow E/p \]
is the quotient by the ideal \((y) \subset E(B\Sigma_p)/I\).
There is a basis of \(E(B\Sigma_p)\) (as an \(E\)-module) given by \(\{1, y, \ldots, y^m\}\), where \(m = |\text{Sub}_p(T)|\). We will be careful to refer to the image of \(y^i\) in \(E(B\Sigma_p)/I\) as \(q(y^i)\). For the basis elements of the form \(y^i\), where \(i \neq 0\), the restriction map \(E \rightarrow E(B\Sigma_p)/I\) sends \(y^i\) to 0. Thus
\[ \text{Tr}_E(y^i) = \text{Tr}_{C^0} \mid_{E(B\Sigma_p)/I}(q(y^i)) \in E. \]
Now the definition of \(\sigma_{\text{can}}\) implies that \(\sigma_{\text{can}}(q(y^i))\) is divisible by \(p\). So
\[ \sigma_{\text{can}}(q(y^i)) \equiv 0 \mod p. \]
It is left to show that, for \(e\) in the image of \(E \rightarrow E(B\Sigma_p)/I\),
\[ \sigma_{\text{can}}(e) \equiv e \mod p. \]
We have already seen that
\[ p! \text{Tr}_{C^0} \mid_{E(B\Sigma_p)/I}(e) = |\text{Sub}_p(T)|e. \]
The result follows from the fact that \(|\text{Sub}_p(T)| \equiv 1 \mod p. \]

4. The Hecke operator congruence

We show that the \(p\)th additive power operation composed with \(\sigma_{\text{can}}\) is the \(p\)th Hecke operator. This implies that the Hecke operator satisfies a certain congruence.
The two maps in question are the composite
\[ E(X) \xrightarrow{P_p/I} E(B\Sigma_p)/I \otimes_E E(X) \xrightarrow{\sigma_{\text{can}} \otimes 1} E(X) \]
and the Hecke operator \(T_p\) described in Section \(\mathbb{2}\).

**Proposition 4.1.** The \(p\)th additive power operation composed with the canonical representative of the Frobenius class is equal to the \(p\)th Hecke operator:
\[ (\sigma_{\text{can}} \otimes 1)(P_p/I) = T_p. \]
**Proof.** This follows in a straightforward way from the definitions. Unwrapping the definition of the character map, the map \(\sigma_{\text{can}}\) is the sum of a collection of maps
\[ E(B\Sigma_p)/I \rightarrow C_0, \]
one for each subgroup of order \(p\) in \(T\). These are the maps induced by the canonical isomorphism
\[ C_0 \otimes \text{Sub}_p(G_E) \cong \text{Sub}_p(T), \]
in other words, they classify the subgroups of order \(p\) in \(T.\):
\[ \square \]
Since $\sigma_{\text{can}} \in [\bar{\sigma}]$, the following diagram commutes:

$$
\begin{array}{cccc}
E(X) & \xrightarrow{P_p} & E(B\Sigma_p) \otimes_E E(X) & \xrightarrow{\sigma_{\text{can}} \otimes 1} & E(X) \\
\text{Res} \otimes 1 & & \sigma \otimes 1
\end{array}
$$

and this implies that

$$(\sigma_{\text{can}} \otimes 1)(P_p/I)(x) \equiv x^p \mod p.$$

**Corollary 4.2.** For $x \in E(X)$, there is a congruence

$$T_p(x) \equiv x^p \mod p.$$

Let $X$ be a space with the property that $E(X)$ is torsion-free. The corollary above implies the existence of a canonical function

$$\theta: E(X) \rightarrow E(X)$$

such that

$$T_p(x) = x^p + p\theta(x).$$

**Example 4.3.** When $n = 1$, $G_E$ is a height 1 formal group, $E(B\Sigma_p)/I$ is a rank one $E$-module, and $\sigma_{\text{can}}$ is an $E$-algebra isomorphism. The composite

$$E(X) \xrightarrow{P_p/I} E(B\Sigma_p)/I \otimes_E E(X) \xrightarrow{\sigma_{\text{can}} \otimes 1} E(X)$$

is the $p$th unstable Adams operation. In this situation, the function $\theta$ is understood by work of Bousfield [Bou96].

**Example 4.4.** At arbitrary height, we may consider the effect of $T_p$ on $z \in \mathbb{Z}_p \subset E$. Since $T_p$ is a sum of ring maps

$$T_p(z) = |\text{Sub}_p(T)|z.$$

This is congruent to $z^p \mod p$.

**Example 4.5.** At height 2 and the prime 2, Rezk constructed an $E$-theory associated to a certain elliptic curve [Rez]. He calculated $P_2/I$, when $X = \ast$. He found that, after choosing a particular coordinate $x$,

$$E(B\Sigma_2)/I \cong \mathbb{Z}_2[[u_1]][x]/(x^3 - u_1x - 2)$$

and

$$P_2/I: \mathbb{Z}_2[[u_1]] \rightarrow \mathbb{Z}_2[[u_1]][x]/(x^3 - u_1x - 2)$$

sends $u_1 \mapsto u_1^2 + 3x - u_1x^2$. In [Dri74, Section 4B], Drinfeld explains how to compute the ring that corepresents $\mathbb{Z}/2 \times \mathbb{Z}/2$-level structures. Note that in the ring

$$\mathbb{Z}_2[[u_1]][y, z]/(y^3 - u_1y - 2),$$

$y$ is a root of $z^3 - u_1z - 2$ and

$$\frac{z^3 - u_1z - 2}{z - y} = z^2 + yz + y^2 - u_1.$$

Drinfeld’s construction gives

$$D_1 = \Gamma \text{Level}(\mathbb{Z}/2 \times \mathbb{Z}/2, G_E) \cong \mathbb{Z}_2[[u_1]][y, z]/(y^3 - u_1y - 2, z^2 + yz + y^2 - u_1).$$
The point of this construction is that $x^3 - u_1 x - 2$ factors into linear terms over this ring. In fact,

$$x^3 - u_1 x - 2 = (x - y)(x - z)(x + y + z).$$

The three maps $E(B\Sigma_2) / I \to D_1 \subset C_0$ that show up in the character map are given by sending $x$ to these roots. A calculation shows that

$$\sigma_{can}(x) = 0$$

and that

$$T_p(u_1) = (\sigma_{can} \otimes 1)(P_2 / I)(u_1) = u_1^2.$$

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