ON THE MODULI STACK OF COMMUTATIVE, 1-PARAMETER FORMAL GROUPS

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Abstract. We commence a general algebro-geometric study of the moduli stack of commutative, 1-parameter formal groups. We emphasize the pro-algebraic structure of this stack: it is the inverse limit, over varying n, of moduli stacks of n-buds, and these latter stacks are algebraic. Our main results pertain to various aspects of the height stratification relative to fixed prime p on the stacks of buds and formal groups.

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Introduction

The aim of this paper is to explicate some of the basic algebraic geometry of the moduli stack of commutative, 1-parameter formal Lie groups. Following tradition in algebraic topology and elsewhere, we abbreviate the term for such objects to *formal groups*. Our analysis focuses chiefly on two main aspects of this stack. The first is that it is not algebraic in the sense usually understood in algebraic geometry, but rather *pro-algebraic:* we exhibit it in a natural way as an inverse limit, in a 2-categorical or homotopy sense, of algebraic stacks of n-buds. The second aspect is its *height stratification* relative to a fixed prime, which is a canonical descending filtration of closed substacks. Notably, we obtain characterizations of the strata of the filtration, and we extend our analysis of the height stratification to the stacks of buds as well.

In a broad sense, this paper may be regarded as a re-expression of some aspects of the classical algebraic theory of formal group laws in a more global language, using modern-day algebraic geometry. Our approach is to develop the moduli theory largely from the ground up, beginning from the foundations of the classical literature. Accordingly, we rely heavily on the classical sources. In particular, a

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great deal of what we discuss ultimately traces to Lazard's seminal paper [L]. For example, our description of the stack of formal groups as the limit over n of the stacks of n-buds is already implicit in Lazard's construction of the universal formal group law. We give descriptions of the stack of formal groups and the stack of n-buds as certain quotient stacks; these are essentially reformulations of Lazard's characterization of the Lazard ring. And much of what we say about the height stratification is ultimately a translation of Lazard's classification of formal group laws over separably closed fields.

In summary, the contents of the paper are as follows. Section 1 serves chiefly to collect terminology surrounding the various objects at play.

In §2 we address the first properties of the stack of formal groups $\mathscr{M}$ and of the related stacks we consider. Unfortunately, as noted above, $\mathscr{M}$ is not algebraic in the sense traditional in algebraic geometry [LMB, 4.1]: for example, its diagonal is not of finite type [LMB, 4.2]. One remedy for this defect, due to Hopkins and followed in [Go1, 1.8; P, 3.15; N, 6], is to simply *redefine* the notion of algebraic stack to mean an "affine-ized" version of the usual one, using flat covers. See [Hol1] for an axiomatization of the idea. Then $\mathscr{M}$ is algebraic in the modified sense, and one can still do much of the algebraic geometry on such stacks that is available for usual algebraic stacks. However, this modified definition is ultimately awkward from the point of view of geometry, as many objects that ought to be algebraic are not,[1] even including all non-quasi-compact *schemes*. To express the reasonableness of $\mathscr{M}$ as a geometric object, then, we revert to the traditional definition of algebraic stack, as in [LMB], and observe that $\mathscr{M}$ is naturally described as a *pro-algebraic stack*. Namely, we consider the algebro-geometric classification of *n-bud laws*, as defined by Lazard [L]; see §1.5. Informally, these are just truncated formal group laws. The moduli stack $\mathscr{B}_n$ of n-buds is readily seen to be an algebraic stack (2.3.2), and $\mathscr{M}$ is naturally obtained as the 2-category limit $\varprojlim_n \mathscr{B}_n$ (2.7.7).

In §§3 and 4 we turn to the essential feature of the geometry of the stacks $\mathscr{B}_n$ and $\mathscr{M}$, respectively, namely the *height stratification* relative to a fixed prime p. These sections form the core of the paper. The height stratification on $\mathscr{M}$ consists of an infinite descending chain of closed substacks

$$\mathscr{M} = \mathscr{M}^{\geq 0} \supsetneq \mathscr{M}^{\geq 1} \supsetneq \cdots ,$$

and, for each n, the height stratification on $\mathscr{B}_n$ consists of a finite descending chain of closed substacks

$$\mathscr{B}_n = \mathscr{B}_n^{\geq 0} \supsetneq \mathscr{B}_n^{\geq 1} \supsetneq \cdots .$$

As n varies, the stratifications on $\mathscr{B}_n$ are compatible in a suitable sense, and their limit recovers the stratification on $\mathscr{M}$ (4.2.2). One of our main results is the following.

Theorem (3.4.8)**.** *$\mathscr{B}_n$ is smooth over* $\operatorname{Spec}\mathbb{Z}$ *of relative dimension* -1 *at every point, and, when it is defined, $\mathscr{B}_n^{\geq h}$ is smooth over* $\operatorname{Spec}\mathbb{F}_p$ *of relative dimension* $-h$ *at every point.*

Much of our subsequent effort is devoted to studying the strata $\mathscr{M}^h \subset \mathscr{M}$ and $\mathscr{B}_n^h \subset \mathscr{B}_n$ of height h formal groups and n-buds, respectively. By [L, Théorème IV], formal group laws over a separably closed field of characteristic p are classified

[1] A phenomenon already present in examples of interest to homotopy theorists, as noted in [Go1, footnote 5].

up to isomorphism by their height. We generalize Lazard's result in the following way. Let $H = H_h$ be a Honda formal group law of height h defined over $\mathbb{F}_p$ (3.5.8), and let $\mathscr{A}ut(H)\colon S \mapsto \operatorname{Aut}_{\Gamma(S,\mathscr{O}_S)}(H)$ denote its functor of automorphisms, defined on $\mathbb{F}_p$-schemes S.

Theorem (4.3.8)**.** *$\mathscr{M}^h$ is equivalent to the classifying stack $B\big(\mathscr{A}ut(H)\big)$ for the fpqc topology.*

There are two variants of the theorem worth mentioning. The first describes the stack $\mathscr{M}^h_{\mathrm{tr}}$ of formal groups of height h with trivialized conormal bundle. One has an exact sequence

$$1 \longrightarrow \mathscr{A}ut_{\mathrm{str}}(H) \longrightarrow \mathscr{A}ut(H) \longrightarrow \mathbb{G}_m,$$

where $\mathscr{A}ut_{\mathrm{str}}(H)$ is the sub-group functor of *strict* automorphisms of H. Then one obtains $\mathscr{M}^h_{\mathrm{tr}} \approx \mathscr{A}ut(H)\backslash\mathbb{G}_m$. Of course, here $\mathbb{G}_m$ acts naturally on the right-hand side; the action appears on the left-hand side as $\mathbb{G}_m$'s natural action on trivializations, and this action realizes the forgetful functor $\mathscr{M}^h_{\mathrm{tr}} \to \mathscr{M}^h$ as a $\mathbb{G}_m$-torsor. When K is a field containing $\mathbb{F}_{p^h}$, $\mathscr{A}ut(H)(K)$ is precisely the hth Morava stabilizer group studied in homotopy theory.

The second variant of (4.3.8) is a version for n-buds, which we give in (3.5.11); here one replaces H by $H^{(n)}$, the n-bud law obtained from H by discarding terms of degree $\geq n+1$.

The results (3.5.11) and (4.3.8) accord the groups $\mathscr{A}ut(H^{(n)})$ and $\mathscr{A}ut(H)$ important places in the theory. To investigate their structure, we observe that both groups carry natural descending filtrations by normal subgroups; see (3.6.2) and (4.4.4), respectively. In the case of $\mathscr{A}ut(H)$, this filtration recovers the usual topology on the Morava stabilizer group. We compute the successive quotients of the respective filtrations in (3.6.3) and (4.4.7). As a corollary, we deduce that $\mathscr{A}ut(H^{(n)})$ is a smooth group scheme over $\mathbb{F}_p$ of dimension h (3.6.4).

In addition to (4.3.8), we obtain another description of $\mathscr{M}^h$ via a classical theorem of Dieudonné [D, Théorème 3] and Lubin [Lu, 5.1.3]. Very roughly, their theorem characterizes $\operatorname{Aut}_{\mathbb{F}_{p^h}}(H)$ as the profinite group G of units in a certain p-adic division algebra; see (4.4.10) for a precise formulation.

Theorem (4.6.2)**.** *There is an equivalence of stacks over $\mathbb{F}_{p^h}$,*

$$\mathscr{M}^h \times_{\operatorname{Spec}\mathbb{F}_{p^h}} \operatorname{Spec}\mathbb{F}_{p^h} \approx \varprojlim B(G/N),$$

where the limit is taken over all open normal subgroups N of G.

The theorem is really a corollary of Dieudonné's and Lubin's theorem and of (4.6.1), where we show that $\mathscr{M}^h$ is a limit of certain classifying stacks of *finite étale* (but nonconstant) groups over $\mathbb{F}_p$. These groups all become constant after base change to $\mathbb{F}_{p^h}$.

In §5 we describe some aspects of the stacks $\mathscr{M}$ and $\mathscr{B}_n$ related to separatedness and properness.

Theorem (5.1)**.** *$\mathscr{B}_n$ is universally closed over $\operatorname{Spec}\mathbb{Z}$, and, when it is defined, $\mathscr{B}_n^{\geq h}$ is universally closed over $\operatorname{Spec}\mathbb{F}_p$.*

The stacks $\mathscr{B}_n$ and $\mathscr{B}_n^{\geq h}$ fail to be proper because they are not separated; see (5.2). The failure of separatedness prevents us from concluding in a formal way that $\mathscr{M}$ and $\mathscr{M}^{\geq h}$ also satisfy the valuative criterion of universal closedness. Nevertheless, these stacks do satisfy the valuative criterion in many cases; see (5.4). By contrast, we show that the *stratum* $\mathscr{M}^h$ does satisfy the valuative criterion of separatedness.

Theorem (5.5)**.** *Let $\mathscr{O}$ be a valuation ring and K its field of fractions. Then $\mathscr{M}^h(\mathscr{O}) \to \mathscr{M}^h(K)$ is fully faithful for all $h \geq 1$.*

When $\mathscr{O}$ is a discrete valuation ring, the theorem is a (very) special case of de Jong's general theorem [dJ, 1.2] that restriction of p-divisible groups from $\operatorname{Spec} \mathscr{O}$ to the generic point $\operatorname{Spec} K$ is fully faithful.

While formal group laws have found applications across a wide swath of mathematics, their moduli theory appears to be of greatest interest in stable homotopy theory. The importation of formal groups into topology began in earnest with work of Quillen [Q], and, following notably the influence of Morava [Mo], formal groups came to form a cornerstone of the *chromatic approach* to stable homotopy theory; see Ravenel's book [R1]. Morava also advocated for the importation of algebraic geometry into the subject as a means to gain conceptual insight; and more recently, owing notably to the influence of Hopkins, the algebraic geometry of the moduli stack of formal groups has emerged as a powerful way to understand the chromatic approach's impressive computational architecture.

Despite its intimate connections to homotopy theory, only fairly recently has material on the stack of formal groups begun to appear in earnest in the mathematical literature. Hopkins has covered a considerable amount of the theory in [Hop] and in other courses at MIT, and Pribble's thesis [P] has also covered some of the basic theory, including some aspects of the height stratification and an algebraic analog of the chromatic convergence theorem of Hopkins-Ravenel [R2, §8.6]. Naumann [N] has given the first published account of some of the basic moduli theory and has used it to prove generalizations of results of Hovey [Hov] and Hovey and Strickland [HS]. Our paper takes another step towards filling the gap in the literature, but we don't go so far as to study the important topics of quasi-coherent sheaves on $\mathscr{M}$ or its deformation theory: these are where the essential connections to topology are found. For a comprehensive account of the stack of formal groups and its relation to stable homotopy theory, we refer to Goerss's forthcoming book [Go2]. Hollander has also done some notable recent work: in [Hol2] she gives a simple proof of the Landweber exact functor theorem [La2] in terms of the geometry of $\mathscr{M}$, and in [Hol3] she uses this stack to give a proof of the Miller-Ravenel-Morava change of rings theorem and another proof of the algebraic chromatic convergence theorem.

Throughout, we assume that the reader is familiar with basic formal group law theory; where needed, we'll use [L] and [F] as our primary references, but other good sources, such as [Ha] and [R1, App. 2], abound.

This paper is a condensed version of the author's Ph.D. thesis [S].

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Notation and conventions. Except where noted otherwise, we adopt the following notation and conventions.

All rings are commutative with 1. We write (Sch) for the category of schemes and $(\mathrm{Sch})_{/S}$ for the category of schemes over a fixed scheme S. In case S is an affine scheme $\operatorname{Spec} A$, we also denote $(\mathrm{Sch})_{/S}$ by $(\mathrm{Sch})_{/A}$.

We write $\Gamma(S)$ for the global sections $\Gamma(S, \mathscr{O}_S)$ of the structure sheaf of the scheme S. For each integer $n \geq 0$ and indeterminates $T_1, \ldots, T_m$, we define the ring

$$\begin{aligned}\Gamma_n(S; T_1, \ldots, T_m) &:= \Gamma\big(S, \mathscr{O}_S[T_1, \ldots, T_m]/(T_1, \ldots, T_m)^{n+1}\big)\\ &\cong \Gamma(S)[T_1, \ldots, T_m](T_1, \ldots, T_m)^{n+1}.\end{aligned}$$

In particular, we have $\Gamma_0(S; T_1, \ldots, T_m) \cong \Gamma(S)$.

We relate objects in a category by writing $=$ for equal; $\cong$ for canonically isomorphic; $\simeq$ for isomorphic; and $\approx$ for equivalent or 2-isomorphic (e.g. for categories, fibered categories, stacks, etc.).

We abbreviate the term "category fibered in groupoids" to CFG. By default, "presheaf" means "presheaf of sets", and similarly for "sheaf".

We say that a diagram of fibered categories or stacks

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

is *Cartesian* if the two composites $A \to D$ are isomorphic and the choice of such an isomorphism induces an equivalence $A \xrightarrow{\approx} C \times_D B$.

We always take limits of diagrams of fibered categories or stacks in the sense of pseudofunctors. Given a category $\mathscr{I}$ and a pseudofunctor $F\colon \mathscr{I} \to (\mathrm{Cat})$ [SGA1, VI §8], where (Cat) denotes the 2-category of small categories, the limit $\varprojlim F$ of F is defined in [SGA1, VI 5.5] in terms of fibered categories. Up to equivalence, the category $\varprojlim F$ admits the following concrete description. An object is a family $(X_i, \varphi_\mu)_{i \in \operatorname{ob} \mathscr{I}, \mu \in \operatorname{mor} \mathscr{I}}$, where X_i is an object in Fi for every object i in $\mathscr{I}$, and φ_μ is an isomorphism $(F\mu)X_i \xrightarrow{\sim} X_j$ in Fj for every morphism $\mu\colon i \to j$ in $\mathscr{I}$, subject to a natural "cocycle condition" between φ_μ, φ_ν, $\varphi_{\nu\circ\mu}$, and the pseudofunctor data for every composition $\nu \circ \mu$. A morphism $(X_i, \varphi_\mu) \to (X_i', \varphi_\mu')$ is a family of morphisms $(\alpha_i)_{i \in \operatorname{ob} \mathscr{I}}$, where $\alpha_i\colon X_i \to X_i'$ for each $i \in \operatorname{ob} I$, compatible with the pseudofunctor data and the φ_μ's and φ_μ''s in a natural way. Given a diagram of fibered categories or stacks indexed by a pseudofunctor, the limit fibered category or stack, respectively, is the fibered category whose fiber over a given object S in the base category is the limit, in the sense just given, of the induced diagram of category fibers over S. See [S, Appendix] for more details. In the case of fibered products, this notion of the limit recovers the usual notion, as in [LMB, 2.2.2], up to equivalence.

In a certain sense, the pseudofunctor limit may be thought of as a kind of homotopy limit. But this is not true in the most literal sense, as homotopy limits are only defined for honest functors, not general pseudofunctors. While one can use formal tricks to replace a given pseudofunctor with an equivalent honest functor, the diagrams we're most interested in — chiefly inverse towers of stacks — are most naturally regarded as indexed by pseudofunctors. Hence we view the pseudofunctor approach to the limit as simpler and more natural.

For convenience, we use the terms "formal Lie variety", "n-germ", "formal group", and "n-bud" in a somewhat abusive way. We always understand the first two to be equipped with a distinguished section, and the second two to be commutative; neither requirement is necessary in general. And we always take all four in the 1-parameter sense, though more general notions allow for many parameters.

1. Definitions

In this section we collect some of the basic language and notation related to the objects we study in this paper.

1.1. **Formal Lie varieties.**

We begin by reviewing what are, in some sense, the basic geometric objects underlying the formal groups, namely the formal Lie varieties. Let S be a scheme.

Definition 1.1.1. A *(pointed,* 1*-parameter) formal Lie variety over* S is a sheaf X on $(\mathrm{Sch})_{/S}$ for the fppf topology equipped with a section $\sigma\colon S \to X$, such that, Zariski-locally on S, there is an isomorphism of pointed sheaves $X \simeq \operatorname{Spf} \mathscr{O}_S[[T]]$, where $\operatorname{Spf} \mathscr{O}_S[[T]]$ is pointed by the 0-section. A *morphism* of formal Lie varieties is a morphism of pointed sheaves.

In other words, a formal Lie variety is a pointed formal scheme over S Zariski-locally (on S) of the form $\operatorname{Spf} \mathscr{O}_S[[T]]$. This recovers the definition given in [G, VI 1.3] or [M, II 1.1.4] in the one-parameter case, as indicated in [M].

Example 1.1.2. The most basic and important example of a formal Lie variety over any base S is just $\operatorname{Spf} \mathscr{O}_S[[T]]$ itself, equipped with the 0 section. We denote this example by $\widehat{\mathbb{A}}_S$ or, when the base is clear from context, by $\widehat{\mathbb{A}}$. When S is an affine scheme $\operatorname{Spec} A$, we also denote $\widehat{\mathbb{A}}_S$ by $\widehat{\mathbb{A}}_A$.

Our notation is nonstandard. It is typical to write $\widehat{\mathbb{A}}^1_S$ for the formal line $\operatorname{Spf} \mathscr{O}_S[[T]]$, equipped with no distinguished section, obtained by completing $\mathbb{A}^1_S$ at the origin. But since our interest is almost exclusively in pointed, 1-parameter formal Lie varieties, we suppress the superscript ^{1} to reduce clutter, and we always understand $\widehat{\mathbb{A}}_S$ to be equipped with the zero section.

More generally, if T is any smooth scheme of relative dimension 1 over S and $S \to T$ is a section, then the completion of T along the section is a formal Lie variety over S.

1.2. **Formal groups.**

We recall the definition. Let S be a scheme.

Definition 1.2.1. A *(commutative,* 1*-parameter) formal group over* S is an fppf sheaf of commutative groups on $(\mathrm{Sch})_{/S}$ such that the underlying pointed sheaf of sets is a formal Lie variety (1.1.1).

Example 1.2.2. To make the formal Lie variety $\widehat{\mathbb{A}} = \widehat{\mathbb{A}}_S$ (1.1.2) into a formal group, one must define a multiplication map $\widehat{\mathbb{A}} \times_S \widehat{\mathbb{A}} \to \widehat{\mathbb{A}}$. Since

$$\widehat{\mathbb{A}} \times_S \widehat{\mathbb{A}} \cong \operatorname{Spf} \mathscr{O}_S[[T_1, T_2]],$$

it is equivalent to define a continuous $\mathscr{O}_S$-algebra map

$$\mathscr{O}_S[[T_1, T_2]] \longleftarrow \mathscr{O}_S[[T]].$$

Any such map is determined by the image $F(T_1, T_2)$ of T in the global sections $\Gamma(S)[[T_1, T_2]]$; and $\widehat{\mathbb{A}}$ becomes a formal group with the 0 section as identity exactly when F is a formal group law over $\Gamma(S)$ in the classical sense. Hence to give a formal group law is to give a formal group with a choice of coordinate. We write $\widehat{\mathbb{A}}^F = \widehat{\mathbb{A}}^F_S$ for the group structure on $\widehat{\mathbb{A}}$ obtained from F. Of course, Zariski-locally on S, every formal group is of the form $\widehat{\mathbb{A}}^F$ for some group law F.

Example 1.2.3. Let F and G be formal group laws over $\Gamma(S)$. Then a morphism of formal Lie varieties $f\colon \widehat{\mathbb{A}} \to \widehat{\mathbb{A}}$ is a morphism of formal groups $f\colon \widehat{\mathbb{A}}^F \to \widehat{\mathbb{A}}^G$ exactly when the diagram of maps on global sections

$$\Gamma(S)[[T]] \xleftarrow{f^\#} \Gamma(S)[[T]]$$

satisfies

$$f^\#\big(F(T_1, T_2)\big) = G\big(f^\#(T_1), f^\#(T_2)\big) \in \Gamma(S)[[T_1, T_2]],$$

that is, when $f^\#$ is a group law homomorphism $F \to G$ in the classical sense.

Example 1.2.4.

- The *additive formal group* $\widehat{\mathbb{G}}_a = \widehat{\mathbb{G}}_{a,S}$ *over* S is $\widehat{\mathbb{A}}^F_S$ for
$$F(T_1, T_2) = T_1 + T_2$$
the *additive group law.* $\widehat{\mathbb{G}}_a$ is the completion of $\mathbb{G}_a$ at the identity.
- The *multiplicative formal group* $\widehat{\mathbb{G}}_m = \widehat{\mathbb{G}}_{m,S}$ *over* S is $\widehat{\mathbb{A}}^F_S$ for
$$F(T_1, T_2) = T_1 + T_2 + T_1 T_2$$
the *multiplicative group law.* $\widehat{\mathbb{G}}_m$ is the completion of $\mathbb{G}_m$ at the identity.
- If E is an elliptic curve over S, then the completion of E at the identity is a formal group over S. This furnishes many examples of formal groups not admitting a global coordinate.
- More generally, completion at the identity of any smooth commutative group scheme of relative dimension 1 yields a formal group. When S is Spec of an algebraically closed field, then $\mathbb{G}_a$, $\mathbb{G}_m$, and elliptic curves are the *only* such connected group schemes.
- Our examples so far omit many formal groups; for instance, when S is Spec of a field of characteristic p, they only produce formal groups of heights (see 4.3.2) 1, 2, and ∞. Other heights may be obtained from higher dimensional groups: although completion at the identity of a smooth group scheme of relative dimension n yields an n-parameter formal group, certain groups equipped with additional structure allow for a 1-parameter summand to be canonically split off from the formal group. This idea is pursued in [BL] in the context of certain PEL Shimura varieties.

At this point, we could perfectly well begin to consider the moduli stack of formal groups. But, as noted in the introduction, this stack is not algebraic. So in the next subsection, we shall begin laying the groundwork to study the related "truncated" moduli problem of classifying *n-buds.* We shall return to the moduli stacks of formal Lie varieties and of formal groups in §2.6 and §2.7, respectively.

1.3. **Ind-infinitesimal sheaves I.** In this subsection we review and introduce notation for a few basic notions from infinitesimal geometry. Fix a base scheme S.

Let X be an fppf sheaf on $(\mathrm{Sch})_{/S}$ equipped with a section $S \to X$. We write $X^{(n)}$ for the nth infinitesimal neighborhood of S in X along the section, $n \geq 0$ [G, VI 1.1; M, II 1.01]. The map $X \mapsto X^{(n)}$ is compatible with base change on S [M, II 1.03] and is functorial in X.

Definition 1.3.1. X is *n-infinitesimal* (resp., *ind-infinitesimal*) if the natural arrow $X^{(n)} \to X$ (resp., $\varinjlim_n X^{(n)} \to X$) is an isomorphism. We denote by $(n\text{-Inf})(S)$ (resp., $(\infty\text{-Inf})(S)$) the category of n-infinitesimal (resp., ind-infinitesimal) pointed sheaves over S.

Example 1.3.2. The most important examples for us are that any formal Lie variety is ind-infinitesimal, and any nth infinitesimal neighborhood of a pointed sheaf is n-infinitesimal. When X is ind-infinitesimal, we often refer to $X^{(n)}$ as its *n-truncation.*

One verifies at once that the properties of being n-infinitesimal or ind-infinitesimal are stable under base change. Hence $(n\text{-Inf})$ and $(\infty\text{-Inf})$ define fibered categories over (Sch).

1.4. **Germs.** In this subsection we introduce the "truncated" analogs of the formal Lie varieties, namely the *germs.* Let $n \geq 0$, and let S be a scheme.

Definition 1.4.1. A *(pointed, 1-parameter) n-germ over S* is a pointed, n-infinitesimal (1.3.1) scheme X over S which is smooth to order n [G, VI 1.2; M, II 3.1.2] and whose conormal bundle $\omega_X := \Omega^1_{X/S}|_S$ is a line bundle on S.

In other words, an n-germ X over S is an S-scheme X equipped with a section, locally (for the Zariski topology on S) of the form $\operatorname{Spec} \mathscr{O}_S[T]/(T)^{n+1}$.

Remark 1.4.2. The line bundle ω_X appearing in (1.4.1) will play an important role later on when we consider the *height stratification.*

Example 1.4.3. The most basic and important example of an n-germ over S is just $\operatorname{Spec} \mathscr{O}_S[T]/(T)^{n+1}$ itself, equipped with the 0 section. We denote this by $\mathbb{T} := \mathbb{T}_S := \mathbb{T}_{n,S}$, suppressing the n or S when no confusion seems likely. When S is an affine scheme $\operatorname{Spec} A$, we also denote $\mathbb{T}_S$ by $\mathbb{T}_A$.

We say that an n-germ X over S is *trivial* if $X \simeq \mathbb{T}_S$ as pointed S-schemes.

Remark 1.4.4. For $m \geq n$, we have $\widehat{\mathbb{A}}_S^{(n)} \cong \mathbb{T}_{m,S}^{(n)} \cong \mathbb{T}_{n,S}$. More generally, any truncation of a formal Lie variety or of a germ is a germ.

Definition 1.4.5. We define $(n\text{-germs})(S)$ to be the full subcategory of pointed sheaves on $(\mathrm{Sch})_{/S}$ consisting of the n-germs. We define $\mathscr{G}_n(S)$ to be the groupoid of n-germs and their isomorphisms over S.

Remark 1.4.6. It is clear from the definition of germ that if X is an n-germ over S and $S' \to S$ is any base change, then $X \times_S S'$ is an n-germ over S'. Hence $(n\text{-germs})$ and $\mathscr{G}_n$ define a fibered category and a CFG, respectively, over (Sch).

Since every germ is locally trivial, the automorphisms of $\mathbb{T}_S$ will assume an important place in the theory.

Definition 1.4.7. We define $\mathscr{E}nd(\mathbb{T}_n)$ to be the presheaf of monoids on (Sch)

$$\mathscr{E}nd(\mathbb{T}_n)\colon S \longmapsto \mathrm{End}_{(n\text{-germs})(S)}(\mathbb{T}_{n,S})$$

and $\mathscr{A}ut(\mathbb{T}_n)$ to be the presheaf of groups on (Sch)

$$\mathscr{A}ut(\mathbb{T}_n)\colon S \longmapsto \mathrm{Aut}_{(n\text{-germs})(S)}(\mathbb{T}_{n,S}).$$

Now, to give an endomorphism of $\mathbb{T}_S$ is to give a map of augmented $\mathscr{O}_S$-algebras $\mathscr{O}_S[T]/(T)^{n+1} \to \mathscr{O}_S[T]/(T)^{n+1}$; this, in turn, is given by the image $a_1T+\cdots+a_nT^n$ of T in $\Gamma_n(S;T)$. The endomorphism of $\mathbb{T}_S$ is then an automorphism exactly when $a_1T+\cdots+a_nT^n$ is invertible under composition in $\Gamma_n(S;T)$, that is, when $a_1 \in \Gamma(S)^\times$. We have shown the following.

Proposition 1.4.8. *$\mathscr{E}nd(\mathbb{T}_n)$ is canonically represented by a monoid scheme structure on $\operatorname{Spec}\mathbb{Z}[a_1,a_2,\dots,a_n] = \mathbb{A}^n_{\mathbb{Z}}$, and $\mathscr{A}ut(\mathbb{T}_n)$ is canonically represented by a group scheme structure on the open subscheme $\operatorname{Spec}\mathbb{Z}[a_1,a_1^{-1},a_2,\dots,a_n]$ of $\mathbb{A}^n_{\mathbb{Z}}$.* □

Explicitly, the monoid structure on $\operatorname{Spec}\mathbb{Z}[a_1,\dots,a_n] = \mathbb{A}^n_{\mathbb{Z}}$ obtained from the identification with $\mathscr{E}nd(\mathbb{T}_n)$ is given by composition of polynomials $a_1T+\cdots+a_nT^n$ in the truncated polynomial ring $\mathbb{Z}[a_1,\dots,a_n][T]/(T)^{n+1}$.

Remark 1.4.9. $\mathscr{A}ut(\mathbb{T}_n)$ admits a decreasing filtration of closed sub-group schemes

$$\mathscr{A}ut(\mathbb{T}_n) =: \mathscr{A}_0^{\mathbb{T}_n} \supset \mathscr{A}_1^{\mathbb{T}_n} \supset \cdots \supset \mathscr{A}_{n-1}^{\mathbb{T}_n} \supset \mathscr{A}_n^{\mathbb{T}_n} := 1$$

defined on S-valued points by

$$\mathscr{A}_i^{\mathbb{T}_n}(S) = \{\, T + a_{i+1}T^{i+1} + \cdots + a_nT^n \mid a_{i+1},\dots,a_n \in \Gamma(S) \,\}, \quad 1 \le i \le n-1.$$

Said differently, $\mathscr{A}_i^{\mathbb{T}_n}$ is just the kernel of the homomorphism $\mathscr{A}ut(\mathbb{T}_n) \to \mathscr{A}ut(\mathbb{T}_i)$ induced by the identification $\mathbb{T}_n^{(i)} \cong \mathbb{T}_i$, $0 \le i \le n$. One verifies at once that the map on points

$$T + a_{i+1}T^{i+1} + \cdots + a_nT^n \longmapsto a_{i+1}$$

specifies an isomorphism of $\mathbb{Z}$-groups

$$\mathscr{A}_i^{\mathbb{T}_n}/\mathscr{A}_{i+1}^{\mathbb{T}_n} \xrightarrow{\sim} \begin{cases} \mathbb{G}_m, & i = 0; \\ \mathbb{G}_a, & 1 \le i \le n-1. \end{cases}$$

We'll return to the $\mathscr{A}_i^{\mathbb{T}_n}$'s in §3.6.

1.5. **Buds.** We now come to the "truncated" analogs of the formal groups, namely the *buds*; these are the algebro-geometric analogs of bud laws. Recall that for $n \ge 0$, an *n-bud law* over a ring A is an element

$$F(T_1,T_2) \in A[T_1,T_2]/(T_1,T_2)^{n+1}$$

satisfying

- (identity) $F(T,0) = F(0,T) = T$;
- (associativity) $F\big(F(T_1,T_2),T_3\big) = F\big(T_1,F(T_2,T_3)\big)$; and
- (commutativity) $F(T_1,T_2) = F(T_2,T_1)$

in the respective rings

$$A[T]/(T)^{n+1}, \quad A[T_1,T_2,T_3]/(T_1,T_2,T_3)^{n+1}, \quad \text{and} \quad A[T_1,T_2]/(T_1,T_2)^{n+1}.$$

The definition translates readily to algebraic geometry. Let S be a scheme and $n \ge 0$.

Definition 1.5.1. A *(commutative, 1-parameter) n-bud over S* consists of an n-germ $\pi\colon X \to S$ (1.4.1) equipped with a morphism of S-schemes, which we think of as a multiplication map,

$$F\colon (X \times_S X)^{(n)} \longrightarrow X,$$

satisfying the constraints

(I) (identity) the section $S \xrightarrow{\sigma} X$ is a left and right identity for F, that is, both compositions in the diagram

$$\begin{tikzcd} X \arrow[r, "(\sigma\pi, \mathrm{id}_X)"] \arrow[d, "(\mathrm{id}_X, \sigma\pi)"'] & (X \times_S X)^{(n)} \arrow[d, "F"] \\ (X \times_S X)^{(n)} \arrow[r, "F"] & X \end{tikzcd}$$

equal id_X;

(A) (associativity) F is associative on points of $(X \times_S X \times_S X)^{(n)}$, that is, the restrictions of $F \times \mathrm{id}_X$ and $\mathrm{id}_X \times F$ to $(X \times_S X \times_S X)^{(n)}$ yield a commutative diagram

$$\begin{tikzcd} (X \times_S X \times_S X)^{(n)} \arrow[r, "\mathrm{id}_X \times F"] \arrow[d, "F \times \mathrm{id}_X"'] & (X \times_S X)^{(n)} \arrow[d, "F"] \\ (X \times_S X)^{(n)} \arrow[r, "F"] & X; \end{tikzcd}$$

and

(C) (commutativity) F is commutative, that is, letting $\tau\colon X \times_S X \to X \times_S X$ denote the transposition map $(x, y) \mapsto (y, x)$ and restricting τ to $(X \times_S X)^{(n)}$, F is τ-equivariant, i.e. the diagram

$$\begin{tikzcd} (X \times_S X)^{(n)} \arrow[rr, "\tau"] \arrow[dr, "F"'] & & (X \times_S X)^{(n)} \arrow[dl, "F"] \\ & X & \end{tikzcd}$$

commutes.

Of course, the infinitesimal neighborhoods in the definition are all taken with respect to the sections induced by σ.

Remark 1.5.2. The multiplication map F in the definition does not define an S-monoid scheme structure on X, since F is defined only on points of a certain subfunctor of the product $X \times_S X$. On the other hand, $(X \times_S X)^{(n)}$ is the honest product of X with itself in the category $(n\text{-Inf})(S)$. Hence n-buds over S are commutative monoids in $(n\text{-Inf})(S)$. In fact, we'll see in §1.6 that the n-buds are precisely the n-germs endowed with a commutative *group* structure in $(n\text{-Inf})(S)$.

Example 1.5.3. Consider the n-germ $\mathbb{T}$ over S (1.4.3). Then, quite analogously to (1.2.2), to give an n-bud structure $(\mathbb{T} \times_S \mathbb{T})^{(n)} \to \mathbb{T}$ with the 0 section as identity is to give an n-bud law F over $\Gamma(S)$. Hence, to give an n-bud law is to give an n-bud with a choice of coordinate. We write $\mathbb{T}^F = \mathbb{T}_S^F$ for the bud structure on $\mathbb{T}$ obtained from F.

Example 1.5.4. Everything in (1.2.4) admits an obvious analog for buds. In particular, we mention the following.

- The *additive n-bud* $\mathbb{G}_a^{(n)} := \mathbb{G}_{a,S}^{(n)}$ *over* S is the nth infinitesimal neighborhood of $\mathbb{G}_a$ at the identity, that is, the n-bud $\mathbb{T}_{n,S}^F$ for $F(T_1,T_2) := T_1 + T_2$.
- The *multiplicative n-bud* $\mathbb{G}_m^{(n)} := \mathbb{G}_{m,S}^{(n)}$ *over* S is the nth infinitesimal neighborhood of $\mathbb{G}_m$ at the identity, that is, the n-bud $\mathbb{T}_S^F$ for $F(T_1,T_2) := T_1 + T_2 + T_1T_2$.

Remark 1.5.5. For F a group law (resp., m-bud law with $m \geq n$), let $F^{(n)}$ denote the n-bud law obtained from F by reducing modulo terms of degree $\geq n+1$. Then $(\widehat{\mathbb{A}}^F)^{(n)} \cong \mathbb{T}_n^{F^{(n)}}$ (resp., $(\mathbb{T}_m^F)^{(n)} \cong \mathbb{T}_n^{F^{(n)}}$). More generally, any truncation of a formal group or of a bud is a bud, since truncation $(\infty\text{-Inf})(S) \to (n\text{-Inf})(S)$ and $(m\text{-Inf})(S) \to (n\text{-Inf})(S)$, $m \geq n$, preserves finite (including empty) products.

There is an obvious notion of morphism.

Definition 1.5.6. A *morphism* $f \colon X \to Y$ of n-buds over S is a morphism of monoid objects in $(n\text{-Inf})(S)$, that is, a morphism of the underlying n-germs such that

$$\begin{array}{ccc} (X \times_S X)^{(n)} & \xrightarrow{(f\times f)^{(n)}} & (Y \times_S Y)^{(n)} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\quad f \quad} & Y \end{array}$$

commutes.

Example 1.5.7. Analogously to (1.2.3), to give a morphism of buds $f \colon \mathbb{T}_S^F \to \mathbb{T}_S^G$ over S is to give an element $f^\#(T) \in \Gamma_n(S;T)$ such that $f^\#\big(F(T_1,T_2)\big) = G\big(f^\#(T_1), f^\#(T_2)\big)$, that is, a homomorphism of bud laws $F \to G$ in the classical sense.

Definition 1.5.8. We define $(n\text{-buds})(S)$ to be the category of n-buds and bud morphisms over S. We define $\mathscr{B}_n(S)$ to be the groupoid of n-buds and bud isomorphisms over S.

Remark 1.5.9. Since the base change of an n-germ is an n-germ, and since infinitesimal neighborhoods and fibered products are compatible with base change, the base change of an n-bud is an n-bud. Hence (n-buds) and $\mathscr{B}_n$ define a fibered category and a CFG, respectively, over (Sch).

Note that when $X = \mathbb{T}_S^F$, one has $X' \cong \mathbb{T}_{S'}^{F'}$, where F' is the bud law over $\Gamma(S')$ obtained by applying $f^\#$ to the coefficients of F.

1.6. **Buds as group objects.** Fix a base scheme S. We remarked in (1.5.2) that n-buds over S are honest commutative monoids in the category $(n\text{-Inf})(S)$ (1.3.1). As promised, we'll now see that n-buds are honest *group* objects in $(n\text{-Inf})(S)$.

Proposition 1.6.1. *The n-buds over S are precisely the n-germs over S endowed with a commutative group structure in* $(n\text{-Inf})(S)$. *The n-bud morphisms over S are precisely the homomorphisms of group objects in* $(n\text{-Inf})(S)$.

Proof. All we need to show is that every n-bud X is automatically equipped with an inverse morphism $X \to X$. Since the inverse is unique if it exists, it suffices to find the inverse locally on S. Hence we may assume $X = \mathbb{T}_S^F$ for some n-bud law $F(T_1, T_2) \in \Gamma_n(S; T_1, T_2)$ (1.5.3). Now use (1.5.7) and the fact that every bud law has a unique inverse homomorphism [F, I §3 Proposition 1]. □

The following is a formal consequence.

Corollary 1.6.2. *For n-buds X and Y over S, the set of bud morphisms $X \to Y$ is naturally an abelian group. Moreover, composition of bud morphisms is bilinear.* □

Explicitly, bud morphisms $X \to Y$ are added as elements of $\mathrm{Hom}_{(n\text{-Inf})(S)}(X, Y)$ under the group structure coming from Y. The content of the corollary is that bud morphisms form a *subgroup* of $\mathrm{Hom}_{(n\text{-Inf})(S)}(X, Y)$.

Remark 1.6.3. The category of n-buds over S is not additive for $n \geq 1$, since the product of n-germs, whether taken in the category of pointed sheaves or in $(n\text{-Inf})(S)$, is not again an n-germ. But the problem is only that we've restricted to the 1-parameter case: commutative n-buds without constraint on the number of parameters do form an additive category.

2. Basic moduli theory

We now begin to consider the basic moduli theory of the stacks of n-germs, n-buds, formal Lie varieties, and formal groups.

2.1. The stack of n-germs.

In this subsection we show that the CFG of n-germs $\mathscr{G}_n$, $n \geq 1$, is an algebraic stack equivalent to the classifying stack $B\big(\mathscr{A}ut(\mathbb{T}_n)\big)$, with $\mathscr{A}ut(\mathbb{T}_n)$ as defined in (1.4.7).

Proposition 2.1.1. *$\mathscr{G}_n$ is a stack over* (Sch) *for the fpqc topology.*

Proof. We have to check that objects and morphisms descend. It is clear from the definitions that $\mathscr{G}_n$ is a stack for the Zariski topology. So we may restrict to the case of a base scheme S and a faithfully flat quasi-compact morphism $f\colon S' \to S$. The argument from here is a straightforward application of the descent theory in [SGA1, VIII].

Descent for morphisms of germs along f is an immediate consequence of descent for morphisms of schemes [SGA1, VIII 5.2]. To check descent for objects, let X' be an n-germ over S' equipped with a descent datum. Then X' is certainly affine over S'. So X' descends to a scheme $\pi\colon X \to S$ affine over S [SGA1, VIII 2.1]. By descent for morphisms, the section for X' descends to a section σ for X, say with associated ideal $\mathscr{I} \subset \pi_*\mathscr{O}_X$. Let $\omega_X \cong \mathscr{I}/\mathscr{I}^2$ denote the conormal sheaf. Since formation of the conormal sheaf associated to a section is compatible with base change [EGAIV$_4$, 16.2.3(ii)], and since line bundles descend [SGA1, VIII 1.10], we conclude that ω_X is a line bundle on S. By [G, VI 1.2; M, II 3.1.1], it remains to show that $\mathscr{I}^{n+1} = 0$ and that the natural map $\omega_X^{\otimes i} \to \mathscr{I}^i/\mathscr{I}^{i+1}$ is an isomorphism for $i = 1, \ldots, n$. But the analogous statements hold after faithfully flat base change to S'. So we're done by descent. □

Theorem 2.1.2. *$\mathscr{G}_n$ is equivalent to the classifying stack $B\big(\mathscr{A}ut(\mathbb{T}_n)\big)$. In particular, $\mathscr{G}_n$ is algebraic.*

Proof. By the previous proposition and the fact that every germ is locally trivial, $\mathscr{G}_n$ is a neutral gerbe (for any topology between the Zariski and fpqc topologies, inclusive) over $\operatorname{Spec}\mathbb{Z}$, with section given by $\mathbb{T}_{\mathbb{Z}}$ (1.4.3). The first assertion now follows from [LMB, 3.21].

As for algebraicity, we need note only that $\mathscr{A}ut(\mathbb{T}_n)$ is a smooth, separated group scheme of finite presentation over $\mathbb{Z}$ by (1.4.8), and that the quotient of any algebraic space by such a group scheme is algebraic [LMB, 4.6.1]. □

Remark 2.1.3. Of course, for an arbitrary group sheaf G on a site $\mathscr{C}$, the stack $B(G)$ depends on the topology on $\mathscr{C}$. By (2.1.1) and (2.1.2), $B\big(\mathscr{A}ut(\mathbb{T}_n)\big)$ is independent of the choice of topology on (Sch) between the Zariski and fpqc topologies, inclusive. In particular, every fpqc-torsor for $\mathscr{A}ut(\mathbb{T}_n)$ is in fact a Zariski-torsor.

2.2. **Bud structures on trivial germs.** In the next subsection we'll begin discussing the moduli stack of n-buds. Since every bud has locally trivial underlying germ, the classification of bud structures on $\mathbb{T}_S$ (1.4.3) assumes an important role in the theory. Let $n \geq 1$.

Definition 2.2.1. We define L_n to be the presheaf of sets on (Sch)

$$L_n\colon S \longmapsto \{n\text{-bud structures on } \mathbb{T}_{n,S}\}.$$

By (1.5.3), to give an n-bud structure on $\mathbb{T}_S$ over the scheme S is to give an n-bud law $F \in \Gamma_n(S; T_1, T_2)$. By Lazard's theorem — see [L, Théorèmes II and III and their proofs] or [Ha, I 5.7.3] — there exists a *universal* n-bud law defined over the polynomial ring $\mathbb{Z}[t_1, \dots, t_{n-1}]$. Hence for $n \geq 1$, L_n is (noncanonically) represented by $\mathbb{A}_{\mathbb{Z}}^{n-1}$. In the trivial case $n = 0$, we have $L_0 \cong \operatorname{Spec}\mathbb{Z}$.

Remark 2.2.2. The functor $\mathscr{A}ut(\mathbb{T}_n)$ (1.4.7) acts naturally on L_n as "changes of coordinate": given a bud structure $\mathbb{T}_S^F$ and a germ automorphism f of $\mathbb{T}_S$, transport of structure along f determines a bud structure $\mathbb{T}_S^G$, and f is tautologically a bud isomorphism $\mathbb{T}_S^F \xrightarrow{\sim} \mathbb{T}_S^G$. Explicitly, denoting by $f^\#$ the map on global sections of $\mathbb{T}_S$, we have $G(T_1, T_2) = f^\#\big[F\big(f^{\#-1}(T_1), f^{\#-1}(T_2)\big)\big]$.

2.3. **The stack of n-buds.** In this subsection we show that the moduli stack of n-buds $\mathscr{B}_n$, $n \geq 1$, is equivalent to the quotient algebraic stack $\mathscr{A}ut(\mathbb{T}_n)\backslash L_n$, with the schemes $\mathscr{A}ut(\mathbb{T}_n)$ and L_n as defined in (1.4.7) and (2.2.1), respectively, and with the action of $\mathscr{A}ut(\mathbb{T}_n)$ on L_n as described in (2.2.2).

Proposition 2.3.1. *$\mathscr{B}_n$ is a stack over* (Sch) *for the fpqc topology.*

Proof. The only new ingredient for n-buds, as compared to n-germs, is the multiplication map. This is handled using standard descent arguments, as in the proof of (2.1.1), along with the facts that fibered products and infinitesimal neighborhoods are compatible with base change. We leave the details to the reader. □

Theorem 2.3.2. *$\mathscr{B}_n$ is equivalent to the quotient stack $\mathscr{A}ut(\mathbb{T}_n)\backslash L_n$. In particular, $\mathscr{B}_n$ is algebraic.*

Proof. We'll apply [LMB, 3.8] to the tautological morphism $f\colon L_n \to \mathscr{B}_n$. Since every bud has Zariski-locally trivial underlying germ, f is locally essentially surjective for the Zariski topology, and hence for any finer topology. Moreover, it is clear from the definitions that the maps

$$\mathscr{A}ut(\mathbb{T}_n) \times L_n \underset{a}{\overset{\mathrm{pr}_{L_n}}{\rightrightarrows}} L_n, \tag{2.3.3}$$

where a denotes the action map described in (2.2.2), induce an isomorphism

$$\mathscr{A}ut(\mathbb{T}_n) \times L_n \xrightarrow{\sim} L_n \times_{\mathscr{B}_n} L_n.$$

The first assertion now follows, and, as in the proof of (2.1.2), the algebraicity assertion is immediate from [LMB, 4.6.1]. $\square$

Remark 2.3.4. As in (2.1.3), we deduce that the quotient stack $\mathscr{A}ut(\mathbb{T}_n)\backslash L_n$ is independent of the topology on (Sch) between the Zariski and fpqc topologies, inclusive.

2.4. **Equivalences of stacks of buds.** In this subsection we show that certain of the truncation functors between stacks of buds are equivalences. The main result is the following; we thank the referee for suggesting that we include it.

Theorem 2.4.1.

(i) *For any $n \geq 1$, $\mathscr{B}_n \to \mathscr{B}_{n-1}$ is locally essentially surjective for the Zariski topology.*
(ii) *Let Π be a set of primes (possibly empty or infinite), and let S be the multiplicative subset of $\mathbb{Z}$ generated by the primes* not *in Π. If $n > 1$ is not a power of an element of Π, then $\mathscr{B}_n \otimes_{\mathbb{Z}} S^{-1}\mathbb{Z} \to \mathscr{B}_{n-1} \otimes_{\mathbb{Z}} S^{-1}\mathbb{Z}$ is an equivalence of stacks.*

In particular, taking Π to be the set of all primes, we conclude that if n is not a prime power, then $\mathscr{B}_n \to \mathscr{B}_{n-1}$ is an equivalence; and, taking Π to consist of a single prime p, we conclude that if n is not a power of p, then $\mathscr{B}_n \otimes \mathbb{Z}_{(p)} \to \mathscr{B}_{n-1} \otimes \mathbb{Z}_{(p)}$ is an equivalence.

Proof of (2.4.1). By (2.3.2), $\mathscr{B}_n$ is the stackification, with respect to the Zariski topology, of the presheaf of groupoids associated to the diagram (2.3.3). So everything is immediate from the following theorem. $\square$

Theorem 2.4.2. *Let $n > 1$, let A be a ring, and consider the truncation functor*

$$(*) \qquad \{n\text{-bud laws over } A\} \longrightarrow \{(n-1)\text{-bud laws over } A\}.$$

Then $(*)$ is surjective on objects. In case n is a power of a prime p, suppose, in addition, that $p \in A^\times$. Then $(*)$ is an equivalence of categories.

The proof that $(*)$ is fully faithful under the stated hypothesis on n will require a couple preliminary lemmas on the algebra of bud laws and their homomorphisms.

Let F and G be n-bud laws over the ring A for $n \geq 1$. Let $f \in A[T]/(T)^{n+1}$ have 0 constant term. We define $\partial f \in A[T_1, T_2]/(T_1, T_2)^{n+1}$ to measure the failure of f to be a homomorphism $F \to G$,

$$(2.4.3) \qquad (\partial f)(T_1, T_2) := f\big(F(T_1, T_2)\big) - G\big(f(T_1), f(T_2)\big).$$

As an easy first lemma, we consider the effect of perturbations to f on ∂f. Following Lazard, let

$$B_m := (T_1 + T_2)^m - T_1^m - T_2^m \in \mathbb{Z}[T_1, T_2], \quad m \geq 1.$$

Lemma 2.4.4. *Let $g(T) := f(T) + aT^n$ for $a \in A$. Then $\partial g = \partial f + aB_n$.*

Proof. We just compute

$$g\big(F(T_1,T_2)\big) = f\big(F(T_1,T_2)\big) + aF(T_1,T_2)^n = f\big(F(T_1,T_2)\big) + a(T_1+T_2)^n$$

and

$$G\big(g(T_1),g(T_2)\big) = G\big(f(T_1)+aT^n, f(T_2)+aT^n\big) = G\big(f(T_1),f(T_2)\big) + aT_1^n + aT_2^n$$

and subtract. $\square$

The next lemma requires Lazard's polynomial

$$C_m := \tfrac{1}{\lambda(m)} B_m \in \mathbb{Z}[T_1,T_2], \quad m \geq 1, \tag{2.4.5}$$

where

$$\lambda(m) := \begin{cases} 1, & m \text{ is not a prime power;} \\ l, & m \text{ is a positive power of the prime } l. \end{cases} \tag{2.4.6}$$

In keeping with our notation for truncated bud and group laws (1.5.5), we write $f^{(n-1)}$ for the image of f in $A[T]/(T)^n$, and $(\partial f)^{(n-1)}$ for the image of ∂f in $A[T_1,T_2]/(T_1,T_2)^n$.

Lemma 2.4.7. *For $n > 1$, suppose that $f^{(n-1)}$ is a homomorphism $F^{(n-1)} \to G^{(n-1)}$. Then there exists a unique $a \in A$ such that $\partial f = aC_n$.*

Proof. Uniqueness of a follows because C_n is primitive. For existence, we show that ∂f satisfies the criterion in [L, Lemme 3] or [F, III §1 Theorem 1a] (note that the second line of the conditions (P) in [F] is redundant).

Since F and G are commutative, it is clear that $\partial f(T_1,T_2) = \partial f(T_2,T_1)$. Since $f^{(n-1)}$ is a homomorphism, we have $(\partial f)^{(n-1)} = 0$. Hence ∂f is homogenous of degree n. So we just have to show that ∂f satisfies the remaining "cocycle condition". We shall do so by exploiting the associativity of F and G.

Replacing T_2 by $F(T_2,T_3)$ in (2.4.3), we obtain an equality of elements in the ring $A[T_1,T_2,T_3]/(T_1,T_2,T_3)^{n+1}$,

$$(\partial f)\big(T_1,F(T_2,T_3)\big) = f\Big(F\big(T_1,F(T_2,T_3)\big)\Big) - G\Big(f(T_1), f\big(F(T_2,T_3)\big)\Big).$$

Replacing $f\big(F(T_2,T_3)\big)$ with $G\big(f(T_2),f(T_3)\big) + (\partial f)(T_2,T_3)$ in the display, we obtain

$$(\partial f)\big(T_1,F(T_2,T_3)\big) = f\Big(F\big(T_1,F(T_2,T_3)\big)\Big) \\ - G\Big(f(T_1), G\big(f(T_2),f(T_3)\big) + (\partial f)(T_2,T_3)\Big).$$

Since ∂f is homogenous of degree n, the left-hand side of this last display is

$$(\partial f)(T_1, T_2+T_3),$$

and the right-hand side is

$$f\Big(F\big(T_1,F(T_2,T_3)\big)\Big) - G\Big(f(T_1), G\big(f(T_2),f(T_3)\big)\Big) - (\partial f)(T_2,T_3).$$

Analogously, replacing T_1 by $F(T_1,T_2)$ and T_2 by T_3 in (2.4.3), one obtains a second equality in $A[T_1,T_2,T_3]/(T_1,T_2,T_3)^{n+1}$,

$$(\partial f)(T_1+T_2,T_3) = f\Big(F\big(F(T_1,T_2),T_3\big)\Big) \\ - G\Big(G\big(f(T_1),f(T_2)\big), f(T_3)\Big) - (\partial f)(T_1,T_2).$$

Now subtract equations and use the associativity of F and G. $\square$

Proof of (2.4.2). The surjectivity assertion is just Lazard's theorem. For the remaining assertion, we must show that the map $(*)$ is fully faithful. So let F and G be n-bud laws and $f\colon F^{(n-1)} \to G^{(n-1)}$ a homomorphism. The hypothesis on n implies that B_n and C_n are unit multiples of each other over A. So it is immediate from (2.4.4) and (2.4.7) that that f admits a unique extension to a homomorphism $F \to G$. $\square$

Remark 2.4.8. The hypotheses on n appearing in (2.4.1) and (2.4.2) are necessary for full faithfulness. For example, suppose n is a power of the prime p. Then over any $\mathbb{F}_p$-algebra, $T + aT^n$ with $a \neq 0$ is a nontrivial automorphism of the additive n-bud law that becomes trivial upon truncation.

2.5. **Ind-infinitesimal sheaves II.** In the next two subsections, we'll turn to the moduli stacks of formal Lie varieties and of formal groups. To help prepare, we now pause for a moment to formulate a general statement on the relationship between ind-infinitesimal sheaves (1.3.1) and their n-truncations (1.3.2) for varying n.

Since formation of infinitesimal neighborhoods is compatible with base change, truncation defines a morphism of fibered categories $(\infty\text{-Inf}) \to (n\text{-Inf})$. Since $(X^{(m)})^{(n)}$ is canonically isomorphic to $X^{(n)}$ for any ind-infinitesimal sheaf X and any $m \geq n$, we may form the limit $\varprojlim_n (n\text{-Inf})$ of the fibered categories $(n\text{-Inf})$ with respect to the truncation functors, and we obtain a natural arrow

$$(*) \qquad (\infty\text{-Inf}) \longrightarrow \varprojlim_n (n\text{-Inf}).$$

We emphasize that the limit $\varprojlim_n (n\text{-Inf})$ is taken in the sense of pseudofunctors.

Proposition 2.5.1. *The arrow* $(\infty\text{-Inf}) \to \varprojlim_n (n\text{-Inf})$ *in* $(*)$ *is an equivalence of fibered categories.*

Proof. We just specify a quasi-inverse $F\colon \varprojlim_n (n\text{-Inf}) \to (\infty\text{-Inf})$ and leave the needed verifications to the reader. An object in $\varprojlim_n (n\text{-Inf})$ over the scheme S is a family (X_n, φ_{mn}), where $X_n \in \operatorname{ob}(n\text{-Inf})(S)$ for all n, and $\varphi_{mn}\colon X_m^{(n)} \xrightarrow{\sim} X_n$ for all $m \geq n$, subject to a natural cocycle condition for every $l \geq m \geq n$. The cocycle condition says precisely that that the composites

$$X_n \xrightarrow[\sim]{\varphi_{mn}^{-1}} X_m^{(n)} \hookrightarrow X_m$$

for varying m and n, where the second arrow is the canonical monomorphism, form a diagram indexed on the totally ordered set $\mathbb{Z}_{\geq 0}$. We then define $F(X_n, \varphi_{mn})$ to be the colimit sheaf $\varinjlim_n X_n$. We define F on morphisms in the evident way. $\square$

2.6. **The stack of formal Lie varieties.** We now come to the moduli stack of formal Lie varieties. Let S be a scheme.

Definition 2.6.1. We define $\mathscr{FLV}(S)$ to be the groupoid of formal Lie varieties and isomorphisms over S.

Remark 2.6.2. It is clear that the base change of a formal Lie variety is again a formal Lie variety. Hence $\mathscr{FLV}$ defines a CFG over (Sch). Moreover, it is clear from the definition of formal Lie variety (1.1.1) that $\mathscr{FLV}$ is a stack for the Zariski topology. In fact, $\mathscr{FLV}$ is a stack for the fpqc topology; this is not hard to prove in a direct fashion, but we shall deduce it in (2.6.8) from the analogous statement for the stack of n-germs $\mathscr{G}_n$ (2.1.1).

Our first task for this subsection is to obtain the analog of (2.1.2) for $\mathscr{FLV}$. Recall the formal Lie variety $\widehat{\mathbb{A}}_S$ of (1.1.2).

Definition 2.6.3. We define $\mathscr{A}ut(\widehat{\mathbb{A}})$ to be the presheaf of groups on (Sch)

$$\mathscr{A}ut(\widehat{\mathbb{A}})\colon S \longmapsto \operatorname{Aut}_{\mathscr{FLV}(S)}(\widehat{\mathbb{A}}_S).$$

Quite analogously to (1.4.8), to give an automorphism of $\widehat{\mathbb{A}}_S$ is to give a power series $a_1T + a_2T^2 + \cdots \in \Gamma(S)[[T]]$ with a_1 a unit. So we have the following.

Proposition 2.6.4. *$\mathscr{A}ut(\widehat{\mathbb{A}})$ is canonically representable by a group scheme structure on the open subscheme* $\operatorname{Spec}\mathbb{Z}[a_1, a_1^{-1}, a_2, a_3 \dots]$ *of* $\mathbb{A}^{\infty}_{\mathbb{Z}}$. □

Theorem 2.6.5. *$\mathscr{FLV} \approx B\big(\mathscr{A}ut(\widehat{\mathbb{A}})\big)$, where the right-hand side denotes the classifying stack with respect to the Zariski topology.*

Proof. The proof is essentially the same as that of (2.1.2): $\mathscr{FLV}$ is plainly a gerbe over $\operatorname{Spec}\mathbb{Z}$ for the Zariski topology, and $\widehat{\mathbb{A}}_{\mathbb{Z}}$ specifies a section. □

Remark 2.6.6. Once we see in (2.6.8) that $\mathscr{FLV}$ is a stack for the fpqc topology, it will follow that $B\big(\mathscr{A}ut(\widehat{\mathbb{A}})\big)$ is independent of the choice of topology between the Zariski and fpqc topologies, inclusive. In particular, every fpqc-torsor for $\mathscr{A}ut(\widehat{\mathbb{A}})$ is in fact a Zariski-torsor.

We now turn to the relation between the stacks $\mathscr{FLV}$ and $\mathscr{G}_n$, $n \geq 0$. Recall that the truncation functors induce an equivalence $(\infty\text{-Inf}) \xrightarrow{\approx} \varprojlim_n (n\text{-Inf})$ (2.5.1). Since any truncation of a formal Lie variety is a germ, this equivalence restricts to an arrow

$$\mathscr{FLV} \longrightarrow \varprojlim_n \mathscr{G}_n.$$

Theorem 2.6.7. *The arrow $\mathscr{FLV} \to \varprojlim_n \mathscr{G}_n$ is an equivalence of stacks.*

Proof. The restriction to $\varprojlim_n \mathscr{G}_n$ of the functor F constructed in the proof of (2.5.1) is a quasi-inverse, as is readily checked. □

Corollary 2.6.8. *$\mathscr{FLV}$ is a stack over* (Sch) *for the fpqc topology.*

Proof. Since a limit of stacks is a stack, this follows from (2.1.1) and the theorem. □

Remark 2.6.9. The definition of formal Lie variety in (1.1.1) has a kind of built-in local triviality for the Zariski topology. Though one may consider formulating the local triviality condition with respect to other topologies, the corollary says that the notion of formal Lie variety is independent of the choice of topology for local triviality between the Zariski and fpqc topologies, inclusive.

2.7. **The stack of formal groups.** Now that we have discussed the stack of formal Lie varieties, we turn to the moduli stack of formal groups. Let S be a scheme.

Definition 2.7.1. We define $(\mathrm{FG})(S)$ to be the category of formal groups and homomorphisms over S. We define $\mathscr{M}(S)$ to be the groupoid of formal groups and isomorphisms over S.

Remark 2.7.2. Since formal Lie varieties are stable under base change, it is clear that formal groups are stable under base change. Hence (FG) defines a fibered category, and $\mathscr{M}$ a CFG, over (Sch). We shall verify in (2.7.6) that $\mathscr{M}$ is a stack for the fpqc topology.

In analogy with (1.5.9), given a group law F over $\Gamma(S)$ and a base change $f\colon S' \to S$, one has $\widehat{\mathbb{A}}_S^F \times_S S' \cong \widehat{\mathbb{A}}_{S'}^{F'}$, where F' is the group law over $\Gamma(S')$ obtained by applying $f^\#$ to the coefficients of F. Hence one recovers the usual notion of base change for group laws.

Our first goal in this subsection is to prove the analog of (2.3.2) for $\mathscr{M}$. Recall the formal Lie variety $\widehat{\mathbb{A}}_S$ of (1.1.2).

Definition 2.7.3. We define L to be the presheaf of sets on (Sch)

$$L\colon S \longmapsto \{\text{formal group structures on } \widehat{\mathbb{A}}_S\}.$$

In analogy with the situation for L_n (2.2.1), Lazard's theorem [L, Théorème II] yields a noncanonical isomorphism $L \simeq \operatorname{Spec} \mathbb{Z}[a_1, a_2, \dots] = \mathbb{A}_{\mathbb{Z}}^\infty$. Just as in (2.2.2), $\mathscr{A}ut(\widehat{\mathbb{A}})$ acts naturally on L as "changes of coordinate". Just as in (2.3.2), we deduce the following.

Theorem 2.7.4. $\mathscr{M} \approx \mathscr{A}ut(\widehat{\mathbb{A}})\backslash L$. □

Remark 2.7.5. It is an immediate consequence of (2.6.6) that the quotient stack $\mathscr{A}ut(\widehat{\mathbb{A}})\backslash L$ is independent of the choice of topology between the Zariski and fpqc topologies, inclusive.

Corollary 2.7.6. *$\mathscr{M}$ is a stack for the fpqc topology.* □

In analogy with the previous subsection, we now turn to the relation between the stacks $\mathscr{M}$ and $\mathscr{B}_n$, $n \geq 0$. By (1.5.5), truncation induces arrows between $\mathscr{M}$ and the various $\mathscr{B}_n$'s, and between the various $\mathscr{B}_n$'s themselves, and these arrows are compatible up to canonical isomorphism. Hence we may form the limit $\varprojlim_n \mathscr{B}_n$, and we obtain an arrow

$$\mathscr{M} \longrightarrow \varprojlim_n \mathscr{B}_n.$$

Theorem 2.7.7. *The arrow $\mathscr{M} \to \varprojlim_n \mathscr{B}_n$ is an equivalence of stacks.*

Proof. As in the proofs of (2.5.1) and (2.6.7), a quasi-inverse is specified by sending $(X_n, \varphi_{mn}) \mapsto \varinjlim_n X_n$. Note that the bud structures on the various X_n's endow the colimit with a commutative group structure. □

3. The height stratification: buds

Fix a prime number p once and for all. We shall now begin to study the algebraic geometry of the classical notion of *height* for formal group laws and bud laws over rings of characteristic p. The essential feature of the theory is a *stratification*, relative to p, on the stack of formal groups and on the stacks of n-buds for varying n. We'll begin by working with buds in this section; in the next, we'll turn to formal groups.

In order to reduce clutter, we won't embed p in the notation, though one certainly obtains a different stratification for each choice of p.

3.1. **Multiplication by p.** Let X be an n-bud or formal group over the scheme S. Then, using (1.6.2) in the bud case, the endomorphisms of X form an abelian group. So we may make the following definition.

Definition 3.1.1. We define $[p]_X$ to be the endomorphism $p\,\mathrm{id}_X$ of X.

Remark 3.1.2. When $X = \mathbb{T}_S^F$ for an n-bud law $F \in \Gamma_n(S; T_1, T_2)$ (1.5.3), the morphism $[p]_X\colon X \to X$ corresponds to the $\mathscr{O}_S$-algebra map

$$\begin{aligned} \mathscr{O}_S[T]/(T)^{n+1} &\longleftarrow \mathscr{O}_S[T]/(T)^{n+1} \\ [p]_F(T) &\longleftarrow\!\!\!\shortmid\, T, \end{aligned}$$

where for any positive integer m,

$$[m]_F := F(\cdots F(F(\underbrace{T, T), T), \cdots, T}_{m\ T\text{'s}}).$$

There is an obvious analogous definition of $[m]_F$ and statement when F is a group law.

Remark 3.1.3. One checks at once that truncation functors are additive functors on the category of formal groups and on the various categories of buds. Hence truncation preserves $[p]$: if X is a formal group or an m-bud with $m \geq n$, then $[p]_X^{(n)} = [p]_{X^{(n)}}$.

Remark 3.1.4. Similarly, consider the category of n-buds or of formal groups over S. Then for any morphism $f\colon S' \to S$, the base change functor $- \times_S S'$ is additive, hence preserves $[p]$. When $X = \mathbb{T}_S^F$ or $X = \widehat{\mathbb{A}}_S^F$, recall that the multiplication law on $X \times_S S'$ is given by the law F' obtained by applying $f^\#$ to the coefficients of F (1.5.9, 2.7.2). Hence $[p]_{F'}$ is obtained by applying $f^\#$ to the coefficients of $[p]_F$.

3.2. **Zero loci of line bundles.** In the next subsection we'll describe the height stratification on $\mathscr{B}_n$ as arising from a succession of *zero loci of sections of line bundles*. Our aim in this subsection is to dispense with a few of the basic preliminaries. The material we shall discuss is general in nature and is independent of our earlier discussion.

Let (Vect_1) denote the fibered category on (Sch) that assigns to each scheme S the category of locally free $\mathscr{O}_S$-modules of rank 1 and all module homomorphisms (with the usual pullback functors). Then (Vect_1) is an fpqc stack [SGA1, VIII 1.12]. Note that the underlying moduli stack of (Vect_1), obtained by discarding the non-Cartesian morphisms, is just $B(\mathbb{G}_m)$. Let $\mathscr{F}$ be a fibered category over (Sch).

Definition 3.2.1. A *line bundle* on $\mathscr{F}$ is a 1-morphism $\mathscr{L}\colon \mathscr{F} \to (\mathrm{Vect}_1)$ between fibered categories over (Sch). A *morphism* $\mathscr{L} \to \mathscr{L}'$ of line bundles on $\mathscr{F}$ is a 2-morphism $\mathscr{L} \to \mathscr{L}'$ between 1-morphisms of fibered categories.

When $\mathscr{F}$ is an algebraic stack, one recovers the usual notion of line bundle on $\mathscr{F}$; see [LMB, 13.3] (though, strictly speaking, [LMB] would take $(\mathrm{Vect}_1)(S)$ to be the *opposite* of the category of locally free $\mathscr{O}_S$-modules of rank 1.)

Example 3.2.2. For any fibered category $\mathscr{F}$, we denote by $\mathscr{O}_{\mathscr{F}}$ the line bundle that assigns to each $X \in \mathrm{ob}\,\mathscr{F}$ over the scheme S the trivial line bundle $\mathscr{O}_S$, and to each morphism $\mu\colon Y \to X$ over $f\colon T \to S$ the Cartesian morphism in (Vect_1) corresponding to the canonical isomorphism $\mathscr{O}_T \xrightarrow{\sim} f^*\mathscr{O}_S$.

Let $\mathscr{L}$ be a line bundle on the fibered category $\mathscr{F}$.

Definition 3.2.3.

(a) A *global section*, or just *section, of* $\mathscr{L}$ is a morphism $a\colon \mathscr{O}_{\mathscr{F}} \to \mathscr{L}$.

(b) Given a section a of $\mathscr{L}$, the *zero locus of* a is the full subcategory $V(a)$ of $\mathscr{F}$ whose objects X over the scheme S are those for which $a_X\colon \mathscr{O}_S \to \mathscr{L}_X$ is the 0 map.

Let a be a section of $\mathscr{L}$. The basic result we'll need is the following. The proof is straightforward.

Proposition 3.2.4.

(i) *$V(a)$ is a sub-fibered category of $\mathscr{F}$, and the inclusion functor $V(a) \to \mathscr{F}$ is a closed immersion.*

(ii) *If $\mathscr{F}$ is a CFG, or stack, or algebraic stack, then so is $V(a)$.* $\square$

3.3. **The height stratification on the stack of n-buds.** In this subsection, we translate the classical notion of height for bud laws to the geometric setting by defining the *height stratification* on the stack $\mathscr{B}_n$. We shall define the analogous stratification on the stack of formal Lie groups in §4.1.

Let X be an n-bud over the scheme S with section $\sigma\colon S \to X$. Let $\mathscr{I}_X \subset \mathscr{O}_X$ denote the sheaf of ideals associated to σ. Since the endomorphism $[p]_X$ (3.1.1) of X is compatible with σ, it determines an endomorphism $[p]_{\mathscr{I}_X}$ of $\mathscr{I}_X$.

Now let h be a nonnegative integer, and assume $n \geq p^h$.

Definition 3.3.1. We say X has *height* $\geq h$ if the endomorphism $[p]_{\mathscr{I}_X}\colon \mathscr{I}_X \to \mathscr{I}_X$ has image in $\mathscr{I}_X^{p^h}$. We denote by $\mathscr{B}_n^{\geq h}$ the full subcategory of $\mathscr{B}_n$ of n-buds of height $\geq h$.

Example 3.3.2. Let $X = \mathbb{T}_S^F$ for some n-bud law F over $\Gamma(S)$ (1.5.3). Then we see from (3.1.2) that X has height $\geq h \iff [p]_F$ is of the form $a_{p^h}T^{p^h} +$ (higher order terms) for some $a_{p^h} \in \Gamma(S)$. In this case we say F has height $\geq h$.

Remark 3.3.3. Recall that $[p]$ (3.1.4) is compatible with base change. Hence height $\geq h$ is stable under base change. Hence $\mathscr{B}_n^{\geq h}$ is fibered over (Sch).

Remark 3.3.4. Similarly, height $\geq h$ is stable under truncation, provided we don't truncate below $n = p^h$. More precisely, X has height $\geq h \iff X^{(p^h)}$ has height $\geq h$.

Remark 3.3.5. Of course, for fixed n, there are only finitely many values of h for which height $\geq h$ makes sense. So we get a finite decreasing chain $\mathscr{B}_n = \mathscr{B}_n^{\geq 0} \supsetneqq \mathscr{B}_n^{\geq 1} \supsetneqq \mathscr{B}_n^{\geq 2} \supsetneqq \cdots$.

We shall see in a moment that the inclusion $\mathscr{B}_n^{\geq h} \hookrightarrow \mathscr{B}_n$ is a *closed immersion.* First, some notation.

By definition of n-germ (1.4.1) for $n \geq 1$, the conormal sheaf $\mathscr{I}_X/\mathscr{I}_X^2|_S$ is a line bundle on S. Moreover, since the conormal sheaf associated to a section is compatible with base change on S [EGAIV$_4$, 16.2.3(ii)], formation of the conormal bundle defines a line bundle on $\mathscr{B}_n$.

Definition 3.3.6. We denote by ω the line bundle on $\mathscr{B}_n$ associating to each bud its associated conormal sheaf. We define $\omega_h := \omega|_{\mathscr{B}_n^{\geq h}}$.

Remark 3.3.7. Clearly, the same construction defines a line bundle ω' on the stack $\mathscr{G}_n$ of n-germs, and ω is just the pullback of ω' along the natural forgetful morphism $\mathscr{B}_n \to \mathscr{G}_n$.

Remark 3.3.8. Strictly speaking, ω and ω_h depend on n. But the dependence on n is largely superficial: since the conormal sheaf of an immersion depends only on the 1st infinitesimal neighborhood, formation of ω is compatible with truncation $\mathscr{B}_m \to \mathscr{B}_n$; and similarly for ω_h. In other words, up to canonical isomorphism, we may construct ω on $\mathscr{B}_n$ by first constructing ω on $\mathscr{B}_1$ and then pulling back along $\mathscr{B}_n \to \mathscr{B}_1$; and similarly for ω_h, replacing $\mathscr{B}_1$ by $\mathscr{B}_{p^h}$. So, to avoid clutter, we shall abuse notation and suppress the n when writing ω and ω_h.

When X has height $\geq h$, the map $[p]_{\mathscr{I}_X} : \mathscr{I}_X \to \mathscr{I}_X^{p^h}$ induces a map of sheaves

$$\mathscr{I}_X/\mathscr{I}_X^2 \longrightarrow \mathscr{I}_X^{p^h}/\mathscr{I}_X^{p^h+1}.$$

But plainly $\mathscr{I}_X^{p^h}/\mathscr{I}_X^{p^h+1} \cong (\mathscr{I}_X/\mathscr{I}_X^2)^{\otimes p^h}$. So, restricting the displayed map to S, we get

$$(\omega_h)_X \longrightarrow (\omega_h)_X^{\otimes p^h},$$

or, equivalently, a section

$$(*) \qquad (v_h)_X : \mathscr{O}_S \longrightarrow (\omega_h)_X^{\otimes p^h - 1}.$$

Since $[p]$ is compatible with pullbacks, we may make the following definition.

Definition 3.3.9. We denote by v_h the section of $\omega_h^{\otimes p^h - 1}$ defined by $(*)$.

Example 3.3.10. Of course, we've taken pains to express v_h in a coordinate-free way, so that it is, in some sense, canonical. But when X admits a coordinate, ω_X is trivial and v_h can be understood more concretely. Precisely, suppose $X = \mathbb{T}_S^F$ for some n-bud law F over $\Gamma(S)$. Then $\mathscr{I}_X = T \cdot \mathscr{O}_S[T]/(T)^{n+1}$, and there is an obvious trivialization

$$\begin{aligned} \mathscr{O}_S &\xrightarrow{\sim} \mathscr{I}_X/\mathscr{I}_X^2 = \omega_X \\ 1 &\longmapsto \text{image of } T. \end{aligned}$$

The displayed trivialization induces a natural trivialization of $\omega_X^{\otimes p^h - 1}$.

Now suppose X has height $\geq h$, so that $[p]_F$ is of the form

$$a_{p^h} T^{p^h} + (\text{higher order terms}) \qquad (3.3.2).$$

Then, under our trivialization of $(\omega_h)_X^{\otimes p^h - 1} = \omega_X^{\otimes p^h - 1}$, $(v_h)_X$ corresponds exactly to $a_{p^h} \in \Gamma(S)$.

In particular, since any bud law F satisfies $F(T_1, T_2) \equiv T_1 + T_2 \bmod (T_1, T_2)^2$, we have $[p]_F(T) \equiv pT \bmod (T)^2$, and v_0 is just the section p of $\mathscr{O}_{\mathscr{B}_n}$.

Remark 3.3.11. Just as for ω_h, v_h implicitly depends on n. But v_h is essentially independent of n in the same sense as ω_h is (3.3.8).

Proposition 3.3.12. *Assume $h \geq 1$. Then $\mathscr{B}_n^{\geq h}$ is the zero locus* (3.2.3) *in $\mathscr{B}_n^{\geq h-1}$ of the section v_{h-1}.*

Before proving the proposition, we recall a classical lemma which we'll also put to use later in §3.6.

Lemma 3.3.13. *Suppose A is a ring of characteristic p. Let $f\colon F \to G$ be a homomorphism of bud laws or formal group laws over A. Then f is 0 or of the form $a_{p^h}T^{p^h} + a_{2p^h}T^{2p^h} + \cdots$ for some nonnegative integer h and some nonzero $a_{p^h} \in A$. In particular, $[p]_F$ is 0 or of the form $a_{p^h}T^{p^h} + a_{2p^h}T^{2p^h} + \cdots$, $a_{p^h} \neq 0$.*

Proof. [F, I §3 Theorem 2(ii)], which works for bud laws as well as formal group laws. Note that $[p]_F$ is plainly a homomorphism $F \to F$. □

Proof of (3.3.12). Let $X \to S$ be an n-bud of height $\geq h-1$. We claim

$$\begin{aligned} X \in \operatorname{ob} V(v_{h-1}) &\iff (v_{h-1})_X = 0 \\ &\iff [p]_{\mathscr{I}_X} \text{ carries } \mathscr{I}_X \text{ into } \mathscr{I}_X^{p^{h-1}+1} \\ &\overset{(\dagger)}{\iff} [p]_{\mathscr{I}_X} \text{ carries } \mathscr{I}_X \text{ into } \mathscr{I}_X^{p^h} \\ &\iff X \in \operatorname{ob} \mathscr{B}_n^{\geq h}. \end{aligned}$$

Only the implication $\Longrightarrow$ in the "$\iff$" marked (†) requires proof. For this, the assertion is local on S, so we may assume X is of the form $\mathbb{T}_S^F$ (1.5.3). By (3.3.10), S must have characteristic p. Now use (3.1.2) and the lemma. □

Corollary 3.3.14. *$\mathscr{B}_n^{\geq h}$ is an algebraic stack for the fpqc topology, and the inclusion functor $\mathscr{B}_n^{\geq h} \to \mathscr{B}_n$ is a closed immersion.*

Proof. Apply (2.3.1), (2.3.2), (3.2.4), and the proposition. □

Remark 3.3.15. Combining (3.3.10) and (3.3.12), we see that $\mathscr{B}_n^{\geq 1}$ is precisely the stack of n-buds over $\mathbb{F}_p$-schemes.

Remark 3.3.16. The proposition says that the property of height $\geq h$ depends only on a bud's p^{h-1}-truncation. So we could extend the notion of height $\geq h$ to n-buds for $n \geq p^{h-1}$, but this added bit of generality offers no real advantage to us.

3.4. **The stack of height $\geq h$ n-buds.** Let $h \geq 1$ and $n \geq p^h$. Our aims in this subsection are to describe $\mathscr{B}_n^{\geq h}$ (3.3.1) as a quotient stack in a way analogous to the description (2.3.2) of $\mathscr{B}_n$, to show that $\mathscr{B}_n^{\geq h}$ is smooth, and to compute its dimension.

As a warm-up, the case $h = 1$ is simply base change to $\mathbb{F}_p$: $\mathscr{B}_n^{\geq 1} \approx \mathscr{B}_n \otimes \mathbb{F}_p$ by (3.3.15), so $\mathscr{B}_n^{\geq 1} \approx \mathscr{A}ut(\mathbb{T}_n)_{\mathbb{F}_p} \backslash (L_n)_{\mathbb{F}_p}$ by (2.3.2), where

$$(L_n)_{\mathbb{F}_p} := L_n \otimes \mathbb{F}_p \simeq \mathbb{A}_{\mathbb{F}_p}^{n-1}$$

and

$$\mathscr{A}ut(\mathbb{T}_n)_{\mathbb{F}_p} := \mathscr{A}ut(\mathbb{T}_n) \otimes \mathbb{F}_p$$

is an open subscheme of $\mathbb{A}_{\mathbb{F}_p}^n$.

To treat the case of general h, we'll make use of the following sharp version of Lazard's theorem. It describes the universal group law constructed by Lazard in the proofs of [L, Théorèmes II and III].

Theorem 3.4.1 (Lazard)**.** *There exists a universal formal group law $U(T_1, T_2)$ over the polynomial ring $\mathbb{Z}[t_1, t_2, \dots]$ such that for all $n \geq 1$,*

(i) *the coefficients of the truncation $U^{(n)}$ involve only $t_1, \dots, t_{n-1}$;*
(ii) *$U^{(n)}$, regarded as defined over $\mathbb{Z}[t_1, \dots, t_{n-1}]$ by* (i)*, is a universal n-bud law; and*

(iii) *there is an equality of elements in* $\mathbb{Z}[t_1, \ldots, t_{n-1}, s][T_1, T_2]/(T_1, T_2)^{n+1}$,

$$U^{(n)}(t_1, \ldots, t_{n-2}, t_{n-1} + s)(T_1, T_2) - U^{(n)}(t_1, \ldots, t_{n-1},)(T_1, T_2) = sC_n(T_1, T_2),$$

where $C_n(T_1, T_2)$ *is Lazard's polynomial* (2.4.5).

Using the theorem, we now construct a convenient presentation of $\mathscr{B}_n^{\geq h}$. Let A be the polynomial ring $\mathbb{Z}_{(p)}[t_1, \ldots, t_{n-1}]$, and let F be a universal (for $\mathbb{Z}_{(p)}$-algebras) n-bud law over A such that the truncation $F^{(n')}$ satisfies (i)–(iii) in (3.4.1) for all $n' \leq n$. Let a_h denote the coefficient of T^{p^h} in $[p]_F(T) \in A[T]/(T)^{n+1}$ (3.1.2). For $h \geq 1$, we define

$$A_h := A/(p, a_1, \ldots, a_{h-1}), \quad F_h := \text{the reduction of } F \text{ over } A_h, \quad X_h := \mathbb{T}_{A_h}^{F_h}.$$

By (3.3.2) and (3.3.13), X_h has height $\geq h$. So, up to isomorphism, X_h specifies a classifying map

$$(*) \qquad \operatorname{Spec} A_h \xrightarrow{X_h} \mathscr{B}_n^{\geq h}.$$

Theorem 3.4.2. $\mathscr{A}ut(\mathbb{T}_n)_{\mathbb{F}_p}$ *acts naturally on* $\operatorname{Spec} A_h$*, and the map* $(*)$ *identifies*

$$\mathscr{A}ut(\mathbb{T}_n)_{\mathbb{F}_p} \backslash \operatorname{Spec} A_h \approx \mathscr{B}_n^{\geq h}.$$

Proof. The theorem is clear for $h = 1$, since plainly X_1 identifies $\operatorname{Spec} A_1 \xrightarrow{\sim} (L_n)_{\mathbb{F}_p}$. Now, for any h, the theorem is equivalent by [LMB, 3.8] to

- $(*)$ is locally essentially surjective;
- $\mathscr{A}ut(\mathbb{T}_n)_{\mathbb{F}_p}$ acts on $\operatorname{Spec} A_h$; and
- the $\mathscr{A}ut(\mathbb{T}_n)_{\mathbb{F}_p}$-action induces

$$\mathscr{A}ut(\mathbb{T}_n)_{\mathbb{F}_p} \times \operatorname{Spec} A_h \cong \operatorname{Spec} A_h \times_{\mathscr{B}_n^{\geq h}} \operatorname{Spec} A_h.$$

So we need note just that the diagram

$$\begin{tikzcd}
\operatorname{Spec} A_{h+1} \arrow[r, hook] \arrow[d, "X_{h+1}"'] & \operatorname{Spec} A_h \arrow[d, "X_h"] \\
\mathscr{B}_n^{\geq h+1} \arrow[r, hook] & \mathscr{B}_n^{\geq h}
\end{tikzcd}$$

is Cartesian by (3.3.10) and (3.3.12), and everything follows by induction on h. □

Remark 3.4.3. Though we didn't need it explicitly for the proof of the theorem, the bud law F_h over the ring A_h admits an obvious modular interpretation: namely, F_h is *universal* amongst n-bud laws of height $\geq h$. On the other hand, universality of F_h easily yields an alternative proof of the theorem quite along the lines of (2.3.2), without appealing to induction.

The theorem leads us to consider closely the ring A_h. The essential observation is the following result on A.

Proposition 3.4.4. *The map of* $\mathbb{Z}_{(p)}$*-polynomial rings* $\mathbb{Z}_{(p)}[u_1, \ldots, u_{n-1}] \to A$ *determined by*

$$(**) \qquad u_i \longmapsto \begin{cases} a_h & i = p^h - 1 \text{ for } h = 1,\, 2, \ldots; \\ t_i & \text{otherwise} \end{cases}$$

is an isomorphism. In particular, $A/(a_1, \ldots, a_{h-1})$ *is a polynomial ring over* $\mathbb{Z}_{(p)}$ *on* $n - h$ *variables.*

Proof. By [L, proof of Lemme 6] or [F, III §1 Lemma 4] and the form of F described in (3.4.1), we have

$$[p]_{F^{(p^h)}(t_1,\dots,t_{p^h-2},t_{p^h-1}+s)}(T) = [p]_{F^{(p^h)}(t_1,\dots,t_{p^h-1})}(T) + \big(p^{p^h-1}-1\big)sT^{p^h}.$$

Hence

$$a_h(t_1,\dots,t_{p^h-2},t_{p^h-1}+s) - a_h(t_1,\dots,t_{p^h-1}) = (p^{p^h-1}-1)s$$

in $\mathbb{Z}_{(p)}[t_1,\dots,t_{p^h-1},s]$. Hence

$$a_h(t_1,\dots,t_{p^h-1}) = (p^{p^h-1}-1)t_{p^h-1} + (\text{terms involving } t_1,\dots,t_{p^h-2}).$$

But, for $h \geq 1$, $p^{p^h-1}-1$ is a *unit* in any $\mathbb{Z}_{(p)}$-algebra. The proposition now follows easily. $\square$

Corollary 3.4.5. $\operatorname{Spec} A_h \simeq \mathbb{A}^{n-h}_{\mathbb{F}_p}$. $\square$

Remark 3.4.6. There are natural analogs of (3.4.4) and (3.4.5) in the group law setting: if U is a universal (for $\mathbb{Z}_{(p)}$-algebras) formal group law over $\mathbb{Z}_{(p)}[t_1,t_2,\dots]$ of the form described in (3.4.1), and we again denote by a_h the coefficient of T^{p^h} in $[p]_U(T)$, then

- the map $\mathbb{Z}_{(p)}[u_1,u_2,\dots] \to \mathbb{Z}_{(p)}[t_1,t_2,\dots]$ specified by $(**)$ is an isomorphism of polynomial rings; and
- $\mathbb{Z}_{(p)}[t_1,t_2,\dots]/(a_1,\dots,a_{h-1})$ is a polynomial ring over $\mathbb{Z}_{(p)}$ on the images of the t_i for $i \neq p^1-1$, $p^2-1,\dots$, $p^{h-1}-1$.

Moreover, in analogy with (3.4.3), the reduction of U over

$$\mathbb{Z}_{(p)}[t_1,t_2,\dots]/(p,a_1,\dots,a_{h-1}) \cong \mathbb{F}_p[t_1,t_2,\dots]/(\overline{a}_1,\dots,\overline{a}_{h-1}),$$

where $\overline{a}_i$ denotes the reduction of a_i mod p, is plainly of height $\geq h$, and indeed is *universal* amongst group laws of height $\geq h$.

Remark 3.4.7. Alternatively, it is not hard to obtain (3.4.5) essentially from Landweber's classification of invariant prime ideals in MU_* [La1, 2.7], or by considering *p-typical* group laws over $\mathbb{Z}_{(p)}$. The approach we've taken above places more direct emphasis on elementary properties of $[p]$.

We now conclude the subsection by turning to smoothness and dimension properties of the algebraic stacks $\mathscr{B}_n$ and $\mathscr{B}_n^{\geq h}$. We shall use freely the language of [LMB], but let us state explicitly the notion of *relative dimension* of a morphism. We will not (and [LMB] does not) attempt to define the relative dimension of an arbitrary locally finite type morphism of algebraic stacks $f\colon \mathscr{X} \to \mathscr{Y}$. We can, however, give a satisfactory definition when f is smooth. Indeed, if ξ is a point of $\mathscr{X}$ [LMB, 5.2], then let $\operatorname{Spec} L \to \mathscr{Y}$ be any representative of $f(\xi)$, set $\mathscr{X}_L := \operatorname{Spec} L \times_{\mathscr{Y}} \mathscr{X}$, and let $\widetilde{\xi}$ be any point of $\mathscr{X}_L$ lying over ξ. Then $\mathscr{X}_L$ is a locally Noetherian algebraic stack, and the *relative dimension of f at ξ* is the integer $\dim_\xi f := \dim_{\widetilde{\xi}} \mathscr{X}_L$ [LMB, 11.14]. It is straightforward to verify that the definition is independent of the choices made.

Theorem 3.4.8. *For $n \geq 1$, $\mathscr{B}_n$ is smooth over $\operatorname{Spec}\mathbb{Z}$ of relative dimension -1 at every point. For $h \geq 1$ and $n \geq p^h$, $\mathscr{B}_n^{\geq h}$ is smooth over $\operatorname{Spec}\mathbb{F}_p$ of relative dimension $-h$ at every point.*

Proof. The assertion for $\mathscr{B}_n$ is immediate from the definitions and from the equivalence $\mathscr{B}_n \approx \mathscr{A}ut(\mathbb{T}_n)\backslash L_n$ (2.3.2), since $L_n \simeq \mathbb{A}^{n-1}_{\mathbb{Z}}$ and $\mathscr{A}ut(\mathbb{T}_n)$ is an open subscheme of $\mathbb{A}^n_{\mathbb{Z}}$ (1.4.8). The assertion for $\mathscr{B}_n^{\geq h}$ is similarly immediate from (3.4.2) and (3.4.5). □

3.5. **The stratum of height h n-buds.** In this subsection we consider the *strata* of the height stratification on $\mathscr{B}_n$, or, in other words, the notion of (exact) height for buds. Let $h \geq 1$ and $n \geq p^{h+1}$.

Definition 3.5.1. We denote by $\mathscr{B}_n^h$ the algebraic stack obtained as the open complement of $\mathscr{B}_n^{\geq h+1}$ in $\mathscr{B}_n^{\geq h}$. We call the objects of $\mathscr{B}_n^h$ the n-buds of *height* h, or sometimes of *exact height* h.

Remark 3.5.2. Since formation of infinitesimal neighborhoods is compatible with base change, it's clear from the definitions that the property height h is stable under truncation, provided we don't truncate below $n = p^{h+1}$. More precisely, an n-bud X has height $h \iff X^{(p^{h+1})}$ has height h.

Let S be a scheme. We can give a more concrete description of the n-buds of height h over S as follows. Since height h is a local condition, the essential case to consider is the n-bud $\mathbb{T}_S^F$ (1.5.3) for some bud law F over $\Gamma(S)$.

Proposition 3.5.3. *$\mathbb{T}_S^F$ has height $h \iff [p]_F$ (3.1.2) is of the form*

$$[p]_F(T) = a_{p^h}T^{p^h} + a_{2p^h}T^{2p^h} + \cdots, \quad a_{p^h} \in \Gamma(S)^\times.$$

Proof. Let $X := \mathbb{T}_S^F$. By (3.3.2) and (3.3.13), X has height $\geq h \iff [p]_F$ is of the asserted form, only with no constraint on a_{p^h}. Now, (3.3.10) and (3.3.12) say that the closed subscheme $Z := \operatorname{Spec} \mathscr{O}_S/a_{p^h}\mathscr{O}_S$ of S is *universal* amongst all S-schemes S' with the property that $S' \times_S X$ is an n-bud of height $\geq h+1$. So X is of height $h \iff \operatorname{Spec} \mathscr{O}_S/a_{p^h}\mathscr{O}_S = \emptyset \iff a_{p^h}$ is a unit. □

Remark 3.5.4. The proposition recovers the now-accepted notion of height for a bud or group law F over a ring of characteristic p. But some older sources in the literature use a weaker notion of height h, requiring only that $[p]_F$ be of the form $a_{p^h}T^{p^h} + a_{2p^h}T^{2p^h} + \cdots$ for some *nonzero* a_{p^h}; see e.g. [L, p. 266] or [F, p. 27].

Remark 3.5.5. There is a natural notion of *principal open substack* associated to a section of a line bundle which serves as a sort-of complement to the notion of zero locus discussed in §3.2. Then (3.5.3) shows, in essence, that $\mathscr{B}_n^h$ is the principal open substack in $\mathscr{B}_n^{\geq h}$ associated to v_h (3.3.9). We leave the details to the reader.

Remark 3.5.6. Analogously to (3.3.16), (3.5.3) allows us to extend the notion of exact height h to n-buds for $n \geq p^h$. But the added bit of generality again offers no real advantage to us.

Remark 3.5.7. Potential confusion lurks in definitions (3.3.1) and (3.5.1): to say that a bud has "height $\geq h$" is *not* to say that it has "height h' for some $h' \geq h$". For example, if $[p]_F(T) = a_{p^h}T^{p^h} + a_{2p^h}T^{2p^h} + \cdots$ with a_{p^h} a nonzero nonunit, then $\mathbb{T}_S^F$ will have height $\geq h$ but will not have an exact height.

Our goal for the remainder of the subsection is to obtain a characterization of $\mathscr{B}_n^h$. We'll first need the following.

Notation 3.5.8. For $h \geq 1$, we denote by H_h a fixed Honda group law over $\mathbb{F}_p$ of height h.

Recall that a Honda group law is a group law satisfying $[p]_{H_h} = T^{p^h}$. It is well-known that such laws exist over $\mathbb{F}_p$ for every h; see e.g. [F, III §2 Theorem 1]. Fixing h, we abbreviate H_h to H and consider its n-truncation $H^{(n)}$ (1.5.5). By (3.5.3), the n-bud $\mathbb{T}_S^{H^{(n)}}$ has height h.

Definition 3.5.9. We define $\mathscr{A}ut(H^{(n)})$ to be the presheaf of groups on $(\mathrm{Sch})_{/\mathbb{F}_p}$

$$\mathscr{A}ut(H^{(n)})\colon S \longmapsto \mathrm{Aut}_{\Gamma(S)}(H^{(n)}) \cong \mathrm{Aut}_{(n\text{-buds})(S)}\big(\mathbb{T}_{n,S}^{H^{(n)}}\big).$$

The theorem we're aiming for will assert that $\mathscr{B}_n^h$ is the classifying stack of $\mathscr{A}ut(H^{(n)})$. The key algebraic input is the following.

Proposition 3.5.10. *Let F be an n-bud law (resp., formal group law) of height h over A. Then there exists a finite étale extension ring (resp., a countable ascending union of finite étale extension rings) B of A such that $F \simeq H^{(n)}$ (resp., $F \simeq H$) over B.*

Proof. We'll proceed by extracting some arguments from the proofs of the statements leading up to the proof of Theorem 2 in [F, III §2]. There is also a somewhat cleaner version of the proof sketched in [Mi, 10.4].

It suffices to consider the bud law case; the group law case then follows by considering the various truncations $F^{(n)}$ for higher and higher n. To begin, the proof of [F, III §2 Lemma 3] shows that there exists a finite étale extension ring B of A such that F is isomorphic over B to an n-bud law G for which $[p]_G(T) = T^{p^h}$. A bit more explicitly, whereas [F] takes A to be a separably closed field and proceeds by finding solutions to certain (separable) equations in A, one can proceed in our case by formally adjoining solutions to certain (separable) equations to A, that is, one can obtain B as an iterated extension ring of (finitely many) rings of the form $A[X]$ modulo a separable polynomial.

The remaining step is to show that over any ring, any two bud laws for which $[p](T) = T^{p^h}$ are isomorphic. This is probably best and most simply seen via a direct argument, but it can be gleaned from [F, III §2] by combining arguments (suitably adapted to the bud case) in the proofs of Lemma 2, Proposition 3, and Theorem 2. □

In particular, the ring B in the proposition is faithfully flat over A.

Now, up to this point, the classifying stacks we've encountered have been essentially independent of the choice of topology; see (2.1.3) and (2.6.6). But our theorem below would fail if we only considered $\mathscr{A}ut(H^{(n)})$-torsors for, for example, the Zariski topology. The proposition leads us to formulate the theorem in terms of the finite étale topology [SGA3$_\mathrm{I}$, IV 6.3] instead. Quite generally, given a group G over $\operatorname{Spec}\mathbb{F}_p$, we write $B_{\text{fét}}(G)$ for the stack over $(\mathrm{Sch})_{/\mathbb{F}_p}$ of G-torsors for the finite étale topology.

Theorem 3.5.11. $\mathscr{B}_n^h \approx B_{\text{fét}}\big(\mathscr{A}ut(H^{(n)})\big)$.

Proof. By (3.5.10) and the fact that every bud has Zariski-locally trivial underlying germ, $\mathscr{B}_n^h$ is a neutral gerbe over $\operatorname{Spec}\mathbb{F}_p$ for the finite étale topology, with section provided by $\mathbb{T}_{\mathbb{F}_p}^{H^{(n)}}$. So apply [LMB, 3.21]. □

Remark 3.5.12. Since $\mathscr{B}_n$ is a stack for the fpqc topology (2.3.1), so is its locally closed substack $\mathscr{B}_n^h$. Hence we deduce that $B\big(\mathscr{A}ut(H^{(n)})\big)$ is independent of the topology on $(\mathrm{Sch})_{/\mathbb{F}_p}$ between the finite étale and fpqc topologies, inclusive. In particular, every fpqc-torsor for $\mathscr{A}ut(H^{(n)})$ is in fact a finite-étale-torsor.

3.6. **Automorphisms and endomorphisms of buds of height h.** Let $h \geq 1$ and let S be a scheme. For $n \geq p^{h+1}$, by (3.5.11), every n-bud of height h over S is isomorphic finite-étale locally to $\mathbb{T}_S^{H^{(n)}}$, with $H = H_h$ the formal group law of (3.5.8) and $H^{(n)}$ its n-truncation (1.5.5). So we are naturally led to consider closely the group $\mathscr{A}ut(H^{(n)})$ (3.5.9). We shall devote this subsection to investigating some aspects of its structure. It will be convenient, especially in later sections, to work as much as possible with regard to any $n \geq 1$; but our main results here will require $n \geq p^{h+1}$ (or at least $n \geq p^h$, granting (3.5.6)), so that height h makes sense.

To begin, let $n \geq 1$, and recall the schemes $\mathscr{A}ut(\mathbb{T}_n)_{\mathbb{F}_p}$ and $(L_n)_{\mathbb{F}_p}$ from §3.4.

Lemma 3.6.1. *$\mathscr{A}ut(H^{(n)})$ is canonically represented by a closed sub-group scheme of $\mathscr{A}ut(\mathbb{T}_n)_{\mathbb{F}_p}$.*

Proof. The point is just that $\mathscr{A}ut(H^{(n)})$ naturally sits inside $\mathscr{A}ut(\mathbb{T}_n)_{\mathbb{F}_p}$ as the stabilizer of $H^{(n)}$. More precisely, $\mathscr{A}ut(\mathbb{T}_n)_{\mathbb{F}_p}$ acts naturally on $(L_n)_{\mathbb{F}_p}$ by (2.2.2), and we have a Cartesian square

$$\begin{tikzcd} \mathscr{A}ut(H^{(n)}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathscr{A}ut(\mathbb{T}_n)_{\mathbb{F}_p} \arrow[d] \\ \operatorname{Spec}\mathbb{F}_p \arrow[r, "H^{(n)}"'] & (L_n)_{\mathbb{F}_p}, \end{tikzcd}$$

where the lower horizontal arrow is the map classifying $H^{(n)}$ and the right vertical arrow is defined on points by $f \mapsto f \cdot H^{(n)}$. $\square$

By (3.4.8) and (3.5.11), for $n \geq p^{h+1}$, the classifying stack $B\big(\mathscr{A}ut(H^{(n)})\big)$ is an open substack of a smooth stack of relative dimension $-h$ over $\operatorname{Spec}\mathbb{F}_p$. Hence $B\big(\mathscr{A}ut(H^{(n)})\big)$ is itself smooth of relative dimension $-h$ over $\operatorname{Spec}\mathbb{F}_p$. Hence it natural to ask if $\mathscr{A}ut(H^{(n)})$ is smooth of dimension h over $\operatorname{Spec}\mathbb{F}_p$.

We shall answer the question in the affirmative in (3.6.4) below. To prepare, let $n \geq 1$, and recall the $\mathscr{A}_\bullet^{\mathbb{T}_n}$-filtration on $\mathscr{A}ut(\mathbb{T}_n)$ from (1.4.9). Let $(\mathscr{A}_\bullet^{\mathbb{T}_n})_{\mathbb{F}_p}$ denote the filtration on $\mathscr{A}ut(\mathbb{T}_n)_{\mathbb{F}_p}$ obtained by base change to $\mathbb{F}_p$.

Definition 3.6.2. We define $\mathscr{A}_\bullet^{H^{(n)}}$ to be the intersection filtration on $\mathscr{A}ut(H^{(n)})$,

$$\mathscr{A}_i^{H^{(n)}} := \mathscr{A}ut(H^{(n)}) \times_{\mathscr{A}ut(\mathbb{T}_n)_{\mathbb{F}_p}} (\mathscr{A}_i^{\mathbb{T}_n})_{\mathbb{F}_p}, \qquad i = 0,\ 1, \ldots,\ n.$$

Concretely, $\mathscr{A}_0^{H^{(n)}} = \mathscr{A}ut(H^{(n)})$, and for $i \geq 1$, $\mathscr{A}_i^{H^{(n)}}$ is given on points by

$$\mathscr{A}_i^{H^{(n)}}(S) := \left\{ f \in \operatorname{Aut}_{\Gamma(S)}(H^{(n)}) \;\middle|\; \begin{array}{c} f(T) \text{ is of the form} \\ T + a_{i+1}T^{i+1} + a_{i+2}T^{i+2} + \cdots + a_nT^n \end{array} \right\}.$$

By (1.4.9) and (3.6.1), $\mathscr{A}_0^{H^{(n)}}/\mathscr{A}_1^{H^{(n)}}$ embeds as a closed subscheme of $\mathbb{G}_m = \mathbb{G}_{m,\mathbb{F}_p}$, and $\mathscr{A}_i^{H^{(n)}}/\mathscr{A}_{i+1}^{H^{(n)}}$ embeds as a closed subscheme of $\mathbb{G}_a = \mathbb{G}_{a,\mathbb{F}_p}$ for $i = 1,\ 2, \ldots,$ $n-1$.

Our main result for the subsection is following calculation of the successive quotients of the $\mathscr{A}_\bullet^{H^{(n)}}$-filtration for $n \geq p^{h+1}$. Let l be the nonnegative integer such that $p^l \leq n < p^{l+1}$.

Theorem 3.6.3. *We have an identification of presheaves*

$$\mathscr{A}_i^{H^{(n)}}/\mathscr{A}_{i+1}^{H^{(n)}} \cong \begin{cases} \mu_{p^h-1}, & i = 0; \\ \mathbb{G}_a^{\mathrm{Fr}_{p^h}}, & i = p-1,\ p^2-1, \dots,\ p^{l-h}-1; \\ \mathbb{G}_a, & i = p^{l-h+1}-1,\ p^{l-h+2}-1, \dots,\ p^l - 1; \\ 0, & \textit{otherwise}; \end{cases}$$

where $\mu_{p^h-1} \subset \mathbb{G}_m$ is the sub-group scheme of (p^h-1)th roots of unity, and $\mathbb{G}_a^{\mathrm{Fr}_{p^h}} \subset \mathbb{G}_a$ is the sub-group scheme of fixed points for the p^hth-power Frobenius operator.

In other words, for any $\mathbb{F}_p$-scheme S,

$$\mu_{p^h-1}(S) = \{\, a \in \Gamma(S)^\times \mid a^{p^h-1} = 1 \,\} \quad \text{and} \quad \mathbb{G}_a^{\mathrm{Fr}_{p^h}}(S) = \{\, a \in \Gamma(S) \mid a^{p^h} = a \,\}.$$

Hence μ_{p^h-1} and $\mathbb{G}_a^{\mathrm{Fr}_{p^h}}$ are represented, respectively, by

$$\operatorname{Spec} \mathbb{F}_p[T]/(T^{p^h-1} - 1) \quad \text{and} \quad \operatorname{Spec} \mathbb{F}_p[T]/(T^{p^h} - T).$$

Hence both μ_{p^h-1} and $\mathbb{G}_a^{\mathrm{Fr}_{p^h}}$ are finite étale groups over $\operatorname{Spec} \mathbb{F}_p$.

Before proceeding to the proof of the theorem, we first signal an immediate consequence. We continue with $n \geq p^{h+1}$.

Corollary 3.6.4. *$\mathscr{A}ut(H^{(n)})$ is smooth of dimension h over* $\operatorname{Spec} \mathbb{F}_p$.

Proof. By the theorem, $\mathscr{A}ut(H^{(n)})$ is an iterated extension of smooth groups, so is smooth. Moreover, the $\mathscr{A}_\bullet^{H^{(n)}}$-filtration has precisely h successive quotients of dimension 1, and all other successive quotients of dimension 0. So the dimension assertion follows from [DG, III §3 5.5(a)]. □

We'll devote the rest of the subsection to the proof of (3.6.3). One can extract the proof from a careful analysis of some of the statements and arguments in [L, §IV] or in [F, I §3, III §2]. But, for sake of clarity, we shall give a reasonably self-contained proof here. Our presentation has profited significantly from notes we received from Spallone on a course of Kottwitz.

To prove (3.6.3), it is somewhat more convenient to translate the problem into one concerning endomorphisms of $H^{(n)}$, as opposed to automorphisms. Let $n \geq 1$.

Definition 3.6.5. We define $\mathscr{E}nd(H^{(n)})$ to be the presheaf of (noncommutative) rings on $(\mathrm{Sch})_{/\mathbb{F}_p}$

$$\mathscr{E}nd(H^{(n)})\colon S \longmapsto \operatorname{End}_{\Gamma(S)}(H^{(n)}) \cong \operatorname{End}_{(n\text{-buds})(S)}\big(\mathbb{T}_{n,S}^{H^{(n)}}\big).$$

Recall that the elements of $\operatorname{End}_{\Gamma(S)}(H^{(n)})$ are the truncated polynomials $f(T) \in \Gamma_n(S;T)$ that "commute" with $H^{(n)}$ in the sense of (1.5.7). Quite generally, for any group or bud law F over the ring A, the product and sum in $\operatorname{End}_A(F)$ are given explicitly by

$$(f \cdot_F g)(T) := f\big(g(T)\big) \quad \text{and} \quad (f +_F g)(T) := F\big(f(T), g(T)\big),$$

respectively. The multiplicative identity in $\mathrm{End}_A(F)$ is just $\mathrm{id}(T) := T$. We denote by $i(T) := i_F(T)$ the additive inverse of id, that is, the unique endomorphism satisfying $F\big(i(T), T\big) = F\big(T, i(T)\big) = 0$.

Definition 3.6.6. For $i = 0, 1, \dots, n$, we denote by $\mathscr{I}_i^{H^{(n)}}$ the subpresheaf of $\mathscr{E}nd(H^{(n)})$ defined on points by

$$\mathscr{I}_i^{H^{(n)}}(S) := \left\{ f \in \mathrm{End}_{\Gamma(S)}(H^{(n)}) \;\middle|\; \begin{matrix} f(T) \text{ is of the form} \\ a_{i+1}T^{i+1} + a_{i+2}T^{i+2} + \cdots + a_n T^n \end{matrix} \right\}.$$

It is immediate that $\mathscr{I}_i^{H^{(n)}}$ is a presheaf of 2-*sided ideals* in $\mathscr{E}nd(H^{(n)})$ for all i, and we have a decreasing filtration

$$\mathscr{E}nd(H^{(n)}) = \mathscr{I}_0^{H^{(n)}} \supset \mathscr{I}_1^{H^{(n)}} \supset \cdots \supset \mathscr{I}_{n-1}^{H^{(n)}} \supset \mathscr{I}_n^{H^{(n)}} = 0.$$

For any $n \geq 1$, we can relate the $\mathscr{I}_\bullet^{H^{(n)}}$-filtration of $\mathscr{E}nd(H^{(n)})$ to the $\mathscr{A}_\bullet^{H^{(n)}}$-filtration of $\mathscr{A}ut(H^{(n)})$ as follows. One verifies immediately that the map on points

$$f \longmapsto \mathrm{id} +_{H^{(n)}} f,$$

where $\mathrm{id}(T) = T$ is the identity endomorphism of $H^{(n)}$, defines a morphism of *set-valued* presheaves

$$(*) \qquad \mathscr{I}_i^{H^{(n)}} \longrightarrow \mathscr{A}_i^{H^{(n)}}, \qquad 1 \leq i \leq n.$$

Lemma 3.6.7. *The arrow* $\mathscr{I}_i^{H^{(n)}} \to \mathscr{A}_i^{H^{(n)}}$ *in* $(*)$ *is an isomorphism of presheaves of sets.*

Proof. The inverse is given by addition with $i_{H^{(n)}}$. □

In a moment, we shall exploit the lemma to express the successive quotients of the $\mathscr{A}_\bullet^{H^{(n)}}$-filtration in terms of the successive quotients of the $\mathscr{I}_\bullet^{H^{(n)}}$-filtration. But we first need another lemma. Quite generally, let R be a possibly noncommutative ring with unit, and let $I \subset R$ be a 2-sided ideal such that $1 + I \subset R^\times$.

Lemma 3.6.8.

(i) *The natural map* $R^\times/(1 + I) \to (R/I)^\times$ *is an isomorphism of groups.*

(ii) *Let* J *be a 2-sided ideal such that* $I^2 \subset J \subset I$. *Then the map*

$$(**) \qquad i \longmapsto 1 + i \mod 1 + J$$

induces an isomorphism of groups $I/J \xrightarrow{\sim} (1 + I)/(1 + J)$.

Proof. (i) Immediate.

(ii) It plainly suffices to show that $(**)$ defines a group homomorphism $I \to (1 + I)/(1 + J)$. That is, given i and $i' \in I$, we must find $j \in J$ such that $(1 + i + i')(1 + j) = (1 + i)(1 + i')$. Take $j := (1 + i + i')^{-1}ii'$. □

The two previous lemmas yield the following as an immediate consequence.

Lemma 3.6.9. *The natural arrow*

$$\mathscr{A}_0^{H^{(n)}}/\mathscr{A}_1^{H^{(n)}} = \mathscr{A}ut(H^{(n)})/\mathscr{A}_1^{H^{(n)}} \longrightarrow \big(\mathscr{E}nd(H^{(n)})/\mathscr{I}_1^{H^{(n)}}\big)^\times = (\mathscr{I}_0^{H^{(n)}}/\mathscr{I}_1^{H^{(n)}})^\times$$

is an isomorphism of presheaves of abelian groups. For $1 \leq i \leq n-1$, *the arrow* $(*)$ *induces an isomorphism of presheaves of abelian groups*

$$\mathscr{I}_i^{H^{(n)}}/\mathscr{I}_{i+1}^{H^{(n)}} \xrightarrow{\sim} \mathscr{A}_i^{H^{(n)}}/\mathscr{A}_{i+1}^{H^{(n)}}.$$ □

This last lemma reduces (3.6.3) to the calculation of the successive quotients of the $\mathscr{I}_\bullet^{H^{(n)}}$-filtration. For this, we will make use of some of the material in §2.4, as well as the following general lemma.

Lemma 3.6.10. *Let F and G be m-bud laws over a ring A, $m \geq 2$. Suppose that $f \in A[T]/(T)^{m+1}$ determines a homomorphism $f^{(m-1)} \colon F^{(m-1)} \to G^{(m-1)}$, so that $\partial f = aC_m$ for a unique $a \in A$ (2.4.7). Then for any $k \geq 1$,*

$$(f^{(m)} \circ [k]_{F^{(m)}})(T) = ([k]_{G^{(m)}} \circ f^{(m)})(T) + \frac{k^m - k}{\lambda(m)} \cdot aT^m.$$

In particular, for $k = p$, m of the form p^j, and A of characteristic p, we have

$$(f^{(p^j)} \circ [p]_{F^{(p^j)}})(T) = ([p]_{G^{(p^j)}} \circ f^{(p^j)})(T) - aT^{p^j}.$$

Proof. Entirely similar to that given in [L, Lemme 6] or in [F, III §1 Lemma 4]. $\square$

We are now ready to compute the $\mathscr{I}_i^{H^{(n)}}/\mathscr{I}_{i+1}^{H^{(n)}}$'s. We denote by $\mathbb{O} := \mathbb{O}_{\mathbb{F}_p}$ the tautological ring scheme structure on $\mathbb{A}^1_{\mathbb{F}_p}$, and by $\mathbb{O}^{\mathrm{Fr}_{p^h}}$ the sub-ring scheme of $\mathbb{O}$ of fixed points for the p^hth-power Frobenius operator,

$$\mathbb{O}^{\mathrm{Fr}_{p^h}}(S) := \{\, a \in \Gamma(S) \bigm| a^{p^h} = a \,\}.$$

Quite as in (1.4.9), the map on points

$$(\dagger) \qquad a_{i+1}T^{i+1} + \cdots + a_nT^n \longmapsto a_{i+1}$$

specifies a monomorphism of presheaves of rings

$$\mathscr{I}_0^{H^{(n)}}/\mathscr{I}_1^{H^{(n)}} \hookrightarrow \mathbb{O}, \qquad i = 0,$$

and a monomorphism of presheaves of abelian groups

$$\mathscr{I}_i^{H^{(n)}}/\mathscr{I}_{i+1}^{H^{(n)}} \hookrightarrow \mathbb{G}_a, \qquad 1 \leq i \leq n-1.$$

Assume $n \geq p^{h+1}$, and again let l be the nonnegative integer such that $p^l \leq n < p^{l+1}$.

Theorem 3.6.11. *For $0 \leq i \leq n-1$, $(\dagger)$ induces an identification of presheaves*

$$\mathscr{I}_i^{H^{(n)}}/\mathscr{I}_{i+1}^{H^{(n)}} \cong \begin{cases} \mathbb{O}^{\mathrm{Fr}_{p^h}}, & i = 0; \\ \mathbb{G}_a^{\mathrm{Fr}_{p^h}}, & i = p-1,\ p^2-1, \dots,\ p^{l-h}-1; \\ \mathbb{G}_a, & i = p^{l-h+1}-1,\ p^{l-h+2}-1, \dots,\ p^l - 1; \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. Let A be a ring of characteristic p, and let $I_i := \mathscr{I}_i^{H^{(n)}}(A)$, $0 \leq i \leq n$. For $i \neq 0$, $p-1$, $p^2-1, \dots, p^l-1$, we have $I_i/I_{i+1} = 0$ by (3.3.13). So we are left to compute the quotients for i of the form $p^j - 1$.

As a first step, let $f(T) = a_1T + \cdots a_nT^n$ be any endomorphism of $H^{(n)}$ over A. Let $A^{\mathrm{Fr}_{p^h}} := \mathbb{O}^{\mathrm{Fr}_{p^h}}(A)$. Since f must commute with $[p]_{H^{(n)}}(T) = T^{p^h}$, we deduce $a_i \in A^{\mathrm{Fr}_{p^h}}$ for $ip^h \leq n$. In particular, the map $(\dagger)$ carries I_i/I_{i+1} into $A^{\mathrm{Fr}_{p^h}}$ for $i = 0, p-1, \dots, p^{l-h}-1$, as asserted. So we are reduced to showing the following: given $a_{p^j}T^{p^j} \in A[T]/(T)^{n+1}$, with $a_{p^j} \in A^{\mathrm{Fr}_{p^h}}$ in case $j \leq l-h$ and no constraint on a_{p^j} in case $j > l-h$, we can add terms of degree $> p^j$ to obtain an endomorphism of $H^{(n)}$.

We shall proceed by induction on the degree of the term to be added. To get started, let $f(T) := a_{p^j}T^{p^j}$. Then in $A[T]/(T)^{p^j+1}$, we have

$$f\big(H^{(p^j)}(T_1,T_2)\big) - H^{(p^j)}\big(f(T_1), f(T_2)\big) = a_{p^j}(T_1+T_2)^{p^j} - a_{p^j}T_1^{p^j} - a_{p^j}T_2^{p^j} = 0.$$

Hence f defines an endomorphism of $H^{(p^j)}$. We must now show that if the polynomial $g(T) = a_{p^j}T^{p^j} + \cdots + a_{m-1}T^{m-1} \in A[T]$ specifies an endomorphism of $H^{(m-1)}$, $p^j + 1 \le m \le n$, then we can always add a term of degree m to g such that the result specifies an endomorphism of $H^{(m)}$.

If m is not a power of p, then (2.4.2) is the end of the story: there is a unique $a_m \in A$ such that $g(T) + a_m T^m$ does the job. Note that if $a_{p^j}, \dots, a_{m-1} \in A^{\mathrm{Fr}_{p^h}}$, then $a_m \in A^{\mathrm{Fr}_{p^h}}$ too, since H is defined over $\mathbb{F}_p$, $A^{\mathrm{Fr}_{p^h}}$ is a subring of A, and a_m is uniquely determined.

If m is a power of p, then we claim that g already specifies an endomorphism of $H^{(m)}$. By (2.4.7) and (3.6.10), it suffices to show that $g^{(m)} \circ [p]_{H^{(m)}} = [p]_{H^{(m)}} \circ g^{(m)}$ in $A[T]/(T)^{m+1}$. In fact, the stronger statement $g^{(n)} \circ [p]_{H^{(n)}} = [p]_{H^{(n)}} \circ g^{(n)}$ in $A[T]/(T)^{n+1}$ holds: by induction, referring to the above case that m is not a power of p, all terms in g of degree $< p^{l+1-h}$ must have coefficients in $A^{\mathrm{Fr}_{p^h}}$, so that $g^{(n)}$ commutes with $[p]_{H^{(n)}} = T^{p^h}$. □

At last we obtain the proof of (3.6.3).

Proof of (3.6.3). Clear from (3.6.9) and (3.6.11), noting for the $i = 0$ case that μ_{p^h-1} sits naturally inside $\mathbb{O}^{\mathrm{Fr}_{p^h}}$ as the subfunctor of units. □

Remark 3.6.12. One verifies immediately that the maps

$$\mathscr{A}_i^{H^{(n)}}/\mathscr{A}_{i+1}^{H^{(n)}} \longrightarrow \begin{cases} \mathbb{G}_m, & i = 0; \\ \mathbb{G}_a, & 1 \le i \le n-1 \end{cases}$$

induced by (1.4.9) and the maps

$$\mathscr{I}_i^{H^{(n)}}/\mathscr{I}_{i+1}^{H^{(n)}} \longrightarrow \begin{cases} \mathbb{O}, & i = 0; \\ \mathbb{G}_a, & 1 \le i \le n-1 \end{cases}$$

of the previous theorem are compatible with the identifications of (3.6.9).

4. The height stratification: formal groups

We continue working with respect to a fixed prime p.

4.1. **The height stratification on the stack of formal Lie groups.** In this subsection we introduce the height stratification on the stack of formal groups, quite in analogy with the height stratification on $\mathscr{B}_n$, $n \ge 1$.

Let $h \ge 0$. We denote by $\mathscr{M}^{\ge h}$ the full sub-fibered category of $\mathscr{M}$ rendering the diagram

$$(*) \qquad \begin{array}{ccc} \mathscr{M}^{\ge h} & \longrightarrow & \mathscr{M} \\ \big\downarrow & & \big\downarrow \\ \mathscr{B}_{p^h}^{\ge h} & \longrightarrow & \mathscr{B}_{p^h} \end{array}$$

Cartesian; here, as usual, the right vertical arrow denotes truncation. Abusing notation, we denote again by ω the pullback to $\mathscr{M}$ of the line bundle ω on $\mathscr{B}_1$

(3.3.6). Similarly, we abusively denote by ω_h the restriction of ω to $\mathscr{M}^{\geq h}$; then ω_h is canonically isomorphic to the pullback to $\mathscr{M}^{\geq h}$ of the line bundle ω_h on $\mathscr{B}_{p^h}^{\geq h}$ (3.3.6). We abusively denote again by v_h the section $\mathscr{O}_{\mathscr{M}^{\geq h}} \to \omega_h^{\otimes p^h-1}$ over $\mathscr{M}^{\geq h}$ obtained by pulling back the section $v_h \colon \mathscr{O}_{\mathscr{B}_{p^h}^{\geq h}} \to \omega_h^{\otimes p^h-1}$ from $\mathscr{B}_{p^h}^{\geq h}$ (3.3.9).

Remark 4.1.1. As for ordinary group schemes, the conormal bundle ω_X associated to a formal group X over S may be interpreted as the sheaf on S of invariant differentials of X [SGA3$_{\mathrm{I}}$, II 4.11], suitably understood in the formal setting.

The fibered category $\mathscr{M}^{\geq h}$ and the various sections v_i are related in the following simple way. Let X be a formal group over the base scheme S.

Proposition 4.1.2. *The following are equivalent.*

(i) *X is an object in $\mathscr{M}^{\geq h}$.*
(ii) *The p^h-bud $X^{(p^h)}$ has height $\geq h$.*
(iii) *For any $n \geq p^h$, the n-bud $X^{(n)}$ has height $\geq h$.*
(iv) *X is an object in each of the successive zero loci* (3.2.3) *$V(v_0)$, $V(v_1), \ldots,$ $V(v_{h-1})$.*

Proof. (3.3.4) and (3.3.12). □

Definition 4.1.3. X has *height* $\geq h$ if it satisfies the equivalent conditions of (4.1.2).

Example 4.1.4. Quite as for buds (3.3.2), given a formal group law F over $\Gamma(S)$, the formal group $\widehat{\mathbb{A}}_S^F$ (1.2.2) has height $\geq h \iff [p]_F \in T^{p^h} \cdot \Gamma(S)[[T]]$, in which case we say F has height $\geq h$.

Remark 4.1.5. Many of the above definitions are independent of particular choices we've made. For example, the proposition says that we could have just as well defined $\mathscr{M}^{\geq h}$ by replacing the diagram $(*)$ with one in which p^h is everywhere replaced by any $n \geq p^h$. Up to canonical isomorphism, (3.3.8) says we could have defined ω as the pullback to $\mathscr{M}$ of the line bundle ω on $\mathscr{B}_n$, for any $n \geq 1$; and similarly for ω_h, for any $n \geq p^h$. Analogously, (3.3.11) says that we could have defined v_h as the pullback of the section v_h over $\mathscr{B}_n^{\geq h}$, for any $n \geq p^h$.

Remark 4.1.6. Just as for buds, $\mathscr{M}^{\geq 0} = \mathscr{M}$, and $\mathscr{M}^{\geq 1}$ is the stack of formal groups over $\mathbb{F}_p$-schemes.

Proposition 4.1.7. *$\mathscr{M}^{\geq h}$ is a stack for the fpqc topology, and the inclusion functor $\mathscr{M}^{\geq h} \to \mathscr{M}$ is a closed immersion.*

Proof. The diagram $(*)$ is Cartesian. So the first assertions follows because $\mathscr{B}_{p^h}$ (2.3.1), $\mathscr{M}$ (2.7.6), and $\mathscr{B}_{p^h}^{\geq h}$ (3.3.14) are fpqc stacks. And the second assertion follows because $\mathscr{B}_{p^h}^{\geq h} \to \mathscr{B}_{p^h}$ is a closed immersion (3.3.14). □

Remark 4.1.8. As for buds, we obtain a decreasing filtration of closed substacks

$$\mathscr{M} = \mathscr{M}^{\geq 0} \supsetneq \mathscr{M}^{\geq 1} \supsetneq \mathscr{M}^{\geq 2} \supsetneq \cdots.$$

In contrast with the bud case (3.3.4), the filtration for $\mathscr{M}$ is of infinite length.

4.2. **The stack of height $\geq h$ formal groups.** In this subsection we collect some characterizations of $\mathscr{M}^{\geq h}$ analogous to previous results on $\mathscr{B}_n^{\geq h}$ and $\mathscr{M}$.

We first consider an analog to the description of $\mathscr{B}_n^{\geq h}$ as a quotient stack in (3.4.2). Let U be a universal (for $\mathbb{Z}_{(p)}$-algebras) formal group law over $\mathbb{Z}_{(p)}[t_1, t_2, \dots]$ as in (3.4.6). Recall that, in the notation of (3.4.6), the reduction of U over the ring

$$B_h := \mathbb{Z}_{(p)}[t_1, t_2, \dots]/(p, a_1, \dots, a_{h-1}) \cong \mathbb{F}_p[t_1, t_2, \dots]/(\overline{a}_1, \dots, \overline{a}_{h-1})$$

is a universal group law of height $\geq h$, and B_h is a polynomial ring over $\mathbb{F}_p$ on the images of the t_i for $i \neq p^1 - 1$, $p^2 - 1, \dots, p^{h-1} - 1$. Let $\mathscr{A}ut(\widehat{\mathbb{A}})_{\mathbb{F}_p} := \mathscr{A}ut(\widehat{\mathbb{A}}) \otimes \mathbb{F}_p$, with $\mathscr{A}ut(\widehat{\mathbb{A}})$ as in (2.6.3). Just as in (3.4.2), we deduce the following.

Theorem 4.2.1. *$\mathscr{A}ut(\widehat{\mathbb{A}})_{\mathbb{F}_p}$ acts naturally on $\operatorname{Spec} B_h$, and we have*

$$\mathscr{M}^{\geq h} \approx \mathscr{A}ut(\widehat{\mathbb{A}})_{\mathbb{F}_p} \backslash \operatorname{Spec} B_h. \qquad \square$$

In analogy with (2.7.7), we next consider the relation between the stacks $\mathscr{M}^{\geq h}$ and $\mathscr{B}_n^{\geq h}$, $n \geq p^h$. By (3.3.4), we may form the limit $\varprojlim_{n \geq p^h} \mathscr{B}_n^{\geq h}$ of the $\mathscr{B}_n^{\geq h}$'s with respect to the truncation functors. By (4.1.2), truncation determines an arrow

$$\mathscr{M}^{\geq h} \longrightarrow \varprojlim_{n \geq p^h} \mathscr{B}_n^{\geq h}.$$

Theorem 4.2.2. *The arrow $\mathscr{M}^{\geq h} \to \varprojlim_{n \geq p^h} \mathscr{B}_n^{\geq h}$ is an equivalence of stacks.*

Proof. Combine (2.7.7) and (4.1.2). $\square$

4.3. **The stratum of height h formal groups I.** In this subsection, in analogy with §3.5, we begin to study the *strata* of the height stratification on $\mathscr{M}$, or, in other words, the notion of (exact) height for formal groups. Let X be a formal group over the base scheme S.

Proposition 4.3.1. *The following are equivalent.*

(i) *The p^{h+1}-bud $X^{(p^{h+1})}$ has height h.*
(ii) *For any $n \geq p^{h+1}$, the n-bud $X^{(n)}$ has height h.*
(iii) *X is an object in the open complement of $\mathscr{M}^{\geq h+1}$ in $\mathscr{M}^{\geq h}$.*

Proof. (3.5.2). $\square$

Definition 4.3.2. X has *height h*, or *exact height h*, if it satisfies the equivalent conditions of (4.3.1). We denote by $\mathscr{M}^h$ the substack of $\mathscr{M}$ of formal groups of height h.

Example 4.3.3. Quite as for buds (3.5.4), if $X = \widehat{\mathbb{A}}_S^F$ for the formal group law F over $\Gamma(S)$ (1.2.2), then the notion of height h for X recovers precisely that for F.

Remark 4.3.4. The caution of (3.5.7) still applies: to say that a formal group has "height $\geq h$" is *not* to say that it has "height h' for some $h' \geq h$".

Remark 4.3.5 (Relation to p-Barsotti-Tate groups)**.** Our notion of height for formal groups is related to, but *not* strictly compatible with, the notion of height for p-Barsotti-Tate, or p-divisible, groups. In rough form, the difference is that (exact) height for formal groups is a *locally closed* condition, whereas height for Barsotti-Tate groups is a *fiberwise* condition. For example, if X is a formal group of height h in the sense of (4.3.2), then X is an ind-infinitesimal Barsotti-Tate group of height h in the sense of Barsotti-Tate groups. But the converse can easily fail. For example, $\widehat{\mathbb{G}}_m$ is simultaneously a Barsotti-Tate group of height 1, and a formal group, over any base scheme on which p is locally nilpotent. But $\widehat{\mathbb{G}}_m$ has exact height 1 as a formal group exactly when p is honestly 0. Similar examples exist for any height $h > 1$.

Remark 4.3.6 (Relation to p-typical formal group laws)**.** We now digress for a moment to discuss BP-theory and p-typical formal group laws. We refer to [R1] for general background, and especially to [R1, App. 2] for the relevant group law theory. Recall that BP_* and the ring $W := BP_*[t_0, t_0^{-1}, t_1, t_2, \dots]$ admit a natural Hopf algebroid structure such that the associated internal groupoid in the category of affine $\mathbb{Z}_{(p)}$-schemes

$$(*) \qquad \operatorname{Spec} W \rightrightarrows \operatorname{Spec} BP_*$$

represents p-typical formal group laws and the isomorphisms between them. In particular, letting $\mathscr{X}$ denote the stackification of $(*)$, there is a natural morphism $f\colon \mathscr{X} \to \mathscr{M} \otimes \mathbb{Z}_{(p)}$, and one verifies just as in [N, 33(ii)] that f is an equivalence. Hence the height stratification on $\mathscr{M}$ induces a stratification on $\mathscr{X}$, or in other words, a stratification on $\operatorname{Spec} BP_*$ by *invariant* closed subschemes.

Now, recall that $BP_* \simeq \mathbb{Z}_{(p)}[u_1, u_2, \dots]$, where for convenience we take the u_i's to be the Araki generators and set $u_0 := p$. Recall also Landweber's ideals $I_0 := 0$ and $I_h := (u_0, u_1, \dots, u_{h-1})$, $h > 0$, in BP_*. Then for all $h \geq 0$, *the closed substack $\mathscr{M}^{\geq h} \otimes \mathbb{Z}_{(p)}$ in $\mathscr{M} \otimes \mathbb{Z}_{(p)} \approx \mathscr{X}$ corresponds to the ideal $I_h \subset BP_*$*; one may deduce this essentially from Landweber's classification of invariant prime ideals in BP_* [La1, 2.7; La2, 6.2], or in a more direct fashion from the formula [R1, A2.2.4] (this formula is the only point where our particular choice of the Araki generators enters). In particular, our notion of (exact) height agrees with Pribble's [P, 4.5]. The identification of the height stratification and the I_h-stratification on $\mathscr{X}$ is also noted in [N, §6 p. 597]; one verifies immediately that Naumann's definition of the height stratification agrees with ours.

This said, we note that our notion of height is not completely compatible with the notion of height for BP_*-algebras in [HS, 4.1]. Namely, given a BP_*-algebra A, consider the composite

$$\operatorname{Spec} A \longrightarrow \operatorname{Spec} BP_* \longrightarrow \mathscr{M}.$$

From the point of view of this paper, it would be reasonable to say that A is a BP_*-algebra of height h if the displayed composite factors through $\mathscr{M}^h$. But, as noted in [N, 24], A has height h in the sense of [HS] if it satisfies the strictly weaker condition that h is the smallest nonnegative integer for which the composite factors through the open substack $\mathscr{M} \smallsetminus \mathscr{M}^{\geq h+1}$ of $\mathscr{M}$.

We next formulate a characterization of $\mathscr{M}^h$ analogous to (3.5.11). Recall our fixed Honda formal group law $H = H_h$ (3.5.8).

Definition 4.3.7. We define $\mathscr{A}ut(H)$ to be the presheaf of groups on $(\mathrm{Sch})_{/\mathbb{F}_p}$

$$\mathscr{A}ut(H)\colon S \longmapsto \operatorname{Aut}_{\Gamma(S)}(H) \cong \operatorname{Aut}_{(\mathrm{FG})(S)}\big(\widehat{\mathbb{A}}_S^H\big).$$

Whereas in (3.5.11) we were led to consider torsors for the finite étale topology, we'll now need to consider $\mathscr{A}ut(H)$-torsors for the fpqc topology. Given a group

G over $\operatorname{Spec} \mathbb{F}_p$, we write $B_{\mathrm{fpqc}}(G)$ for the stack over $(\mathrm{Sch})_{/\mathbb{F}_p}$ of G-torsors for the fpqc topology.

Theorem 4.3.8. $\mathscr{M}^h \approx B_{\mathrm{fpqc}}\big(\mathscr{A}ut(H)\big)$.

Proof. Essentially identical to the proof of (3.5.11). □

We shall study the group $\mathscr{A}ut(H)$ and its relation to the groups $\mathscr{A}ut(H^{(n)})$ for varying $n \geq p^{h+1}$ in the next subsection.

Remark 4.3.9. The statement of the theorem is not entirely sharp: by (3.5.10), it would suffice to replace the fpqc topology by the topology on $(\mathrm{Sch})_{/\mathbb{F}_p}$ generated by the Zariski topology and all surjective maps $\operatorname{Spec} B \to \operatorname{Spec} A$ between affine schemes obtained as a limit of finite étale maps $\cdots \to \operatorname{Spec} B_2 \to \operatorname{Spec} B_1 \to \operatorname{Spec} A$.

Remark 4.3.10. It is natural to say that a formal group X has *height* ∞ if $[p]_X = 0$. The stack of formal groups of height ∞ is a closed substack of $\mathscr{M}^{\geq h}$ for all h, and it follows from classical formal group law theory that this stack is the classifying stack, with respect to the Zariski topology, of the automorphism scheme of $\widehat{\mathbb{G}}_a$. As we won't have occasion to consider this stack further, we leave the details to the reader.

We conclude this subsection by formulating another characterization of the stack $\mathscr{M}^h$, this time the obvious analog of (4.2.2). By (3.5.2), we may form the limit $\varprojlim_{n \geq p^{h+1}} \mathscr{B}_n^h$ of the $\mathscr{B}_n^h$'s with respect to the truncation functors. By (4.3.1), truncation determines an arrow

$$(**) \qquad \mathscr{M}^h \longrightarrow \varprojlim_{n \geq p^{h+1}} \mathscr{B}_n^h.$$

As in (4.2.2), only replacing the reference to (4.1.2) with (4.3.1), we obtain the following.

Theorem 4.3.11. *The arrow* $\mathscr{M}^h \to \varprojlim_{n \geq p^{h+1}} \mathscr{B}_n^h$ *in* $(**)$ *is an equivalence of stacks.* □

4.4. **Automorphisms and endomorphisms of formal groups of height h.** Let $h \geq 1$. Our result $\mathscr{M}^h \approx B_{\mathrm{fpqc}}\big(\mathscr{A}ut(H)\big)$ (4.3.8), with $\mathscr{M}^h$ the stratum in $\mathscr{M}$ of formal groups of height h, leads us to consider the $\mathbb{F}_p$-group scheme $\mathscr{A}ut(H)$ (4.3.7). We shall devote this subsection to investigating some aspects of its structure and of its relation to the group schemes $\mathscr{A}ut(H^{(n)})$ (3.5.9), $n \geq 1$. We shall ultimately apply our final result of this subsection, (4.4.11), to obtain another characterization of $\mathscr{M}^h$ in §4.6.

Let us begin with the analog of (3.6.1) for $\mathscr{A}ut(H)$. Recall the $\mathbb{Z}$-group scheme $\mathscr{A}ut(\widehat{\mathbb{A}})$ (2.6.3), and let $\mathscr{A}ut(\widehat{\mathbb{A}})_{\mathbb{F}_p} := \mathscr{A}ut(\widehat{\mathbb{A}}) \otimes \mathbb{F}_p$. Quite as in (3.6.1), we obtain the following.

Lemma 4.4.1. $\mathscr{A}ut(H)$ *is canonically represented by a closed sub-group scheme of* $\mathscr{A}ut(\widehat{\mathbb{A}})_{\mathbb{F}_p}$. □

Quite as in §3.6, although we will ultimately be interested in automorphisms of H, we shall accord the endomorphisms of H a more fundamental role.

Definition 4.4.2. We define $\mathscr{E}nd(H)$ to be the presheaf of (noncommutative) rings on $(\mathrm{Sch})_{/\mathbb{F}_p}$

$$\mathscr{E}nd(H)\colon S \longmapsto \operatorname{End}_{\Gamma(S)}(H) \cong \operatorname{End}_{(\mathrm{FG})(S)}\big(\widehat{\mathbb{A}}_S^H\big).$$

The $\mathscr{I}_\bullet^{H^{(n)}}$-filtration on $\mathscr{E}nd(H^{(n)})$ (3.6.6) admits a natural analog for $\mathscr{E}nd(H)$, as follows.

Definition 4.4.3. For $i = 0, 1, 2, \ldots$, we denote by $\mathscr{I}_i^H$ the subpresheaf of $\mathscr{E}nd(H)$ defined on points by

$$\mathscr{I}_i^H(S) := \left\{ f \in \mathrm{End}_{\Gamma(S)}(H) \;\middle|\; \begin{array}{c} f(T) \text{ is of the form} \\ a_{i+1}T^{i+1} + (\text{higher order terms}) \end{array} \right\}.$$

Quite as for $\mathscr{I}_i^{H^{(n)}}$, one verifies immediately that $\mathscr{I}_i^H$ is a presheaf of 2-*sided ideals* in $\mathscr{E}nd(H)$ for all i, and we have a decreasing filtration

$$\mathscr{E}nd(H) = \mathscr{I}_0^H \supset \mathscr{I}_1^H \supset \mathscr{I}_2^H \supset \cdots,$$

this time of infinite length.

We now wish to introduce the analog for $\mathscr{A}ut(H)$ of the $\mathscr{A}_\bullet^{H^{(n)}}$-filtration on $\mathscr{A}ut(H^{(n)})$ (3.6.2). We could do so by mimicking the definition of the $\mathscr{A}_\bullet^{H^{(n)}}$-filtration in the obvious way: there is a natural filtration on $\mathscr{A}ut(\widehat{\mathbb{A}})$ in plain analogy with (1.4.9), hence an induced filtration on $\mathscr{A}ut(\widehat{\mathbb{A}})_{\mathbb{F}_p}$, hence an intersection filtration on $\mathscr{A}ut(H)$. Instead, we will just use directly the $\mathscr{I}_\bullet^H$-filtration on $\mathscr{E}nd(H)$.

Definition 4.4.4. We define $\mathscr{A}_i^H$ to be the subpresheaf of $\mathscr{E}nd(H)$

$$\mathscr{A}_i^H := \begin{cases} \mathscr{A}ut(H), & i = 0 \\ T +_H \mathscr{I}_i^H, & i = 1,\ 2, \ldots. \end{cases}$$

Concretely, analogously to (3.6.7), $\mathscr{A}_i^H$ is given on points by

$$\mathscr{A}_i^H(S) := \left\{ f \in \mathrm{Aut}_{\Gamma(S)}(H) \;\middle|\; \begin{array}{c} f(T) \text{ is of the form} \\ T + a_{i+1}T^{i+1} + (\text{higher order terms}) \end{array} \right\}.$$

It is immediate that $\mathscr{A}_i^H$ is a *normal subgroup* in $\mathscr{A}ut(H)$ for all i, and we have a decreasing filtration

$$\mathscr{A}ut(H) = \mathscr{A}_0^H \supset \mathscr{A}_1^H \supset \mathscr{A}_2^H \supset \cdots.$$

Let us now turn to the relation between $\mathscr{E}nd(H)$ and the $\mathscr{E}nd(H^{(n)})$'s, and between $\mathscr{A}ut(H)$ and the $\mathscr{A}ut(H^{(n)})$'s. For any $m \geq n \geq 1$, truncation of H induces a commutative diagram of presheaves of rings

$$(*) \qquad \begin{tikzcd} & \mathscr{E}nd(H) \arrow[dl] \arrow[dr] & \\ \mathscr{E}nd(H^{(m)}) \arrow[rr] & & \mathscr{E}nd(H^{(n)}). \end{tikzcd}$$

Proposition 4.4.5. *For all $i \geq 0$, the diagram $(*)$ induces*

(i) $\mathscr{I}_i^H \xrightarrow{\sim} \varprojlim_{n \geq 1} \mathscr{I}_i^{H^{(n)}}$, *where we take* $\mathscr{I}_i^{H^{(n)}} := 0$ *for* $i \geq n$*; and*

(ii) $\mathscr{A}_i^H \xrightarrow{\sim} \varprojlim_{n \geq 1} \mathscr{A}_i^{H^{(n)}}$, *where we take* $\mathscr{A}_i^{H^{(n)}} := 1$ *for* $i \geq n$.

Moreover,

(iii) $\mathscr{I}_i^H \xrightarrow{\sim} \varprojlim_{n \geq i} \mathscr{I}_i^H / \mathscr{I}_n^H$*; and*

(iv) $\mathscr{A}_i^H \xrightarrow{\sim} \varprojlim_{n \geq i} \mathscr{A}_i^H / \mathscr{A}_n^H$.

In particular, $\mathscr{E}nd(H)$ (resp. $\mathscr{A}ut(H)$) is complete and separated with respect to the $\mathscr{I}_\bullet^H$- (resp. $\mathscr{A}_\bullet^H$-) topology.

Proof. Before anything else, it is clear from the definitions that truncation carries $\mathscr{I}_i^H$ and $\mathscr{I}_i^{H^{(m)}}$ into $\mathscr{I}_i^{H^{(n)}}$, $m \geq n$, so that the limit and arrow in (i) are well-defined; and analogously for (ii).

(i) First consider the case $i = 0$. As in (1.5.5), the formal group $\widehat{\mathbb{A}}_{\mathbb{F}_p}^H$ (1.2.2) truncates to the n-bud $\mathbb{T}_{n,\mathbb{F}_p}^{H^{(n)}}$ (1.5.3) for all n. Hence the equivalence $\mathscr{M} \approx \varprojlim_n \mathscr{B}_n$ of (2.7.7) identifies $\mathscr{E}nd(\widehat{\mathbb{A}}_{\mathbb{F}_p}^H) \cong \mathscr{I}_0^H$ with $\varprojlim_n \mathscr{E}nd(\mathbb{T}_{n,\mathbb{F}_p}^{H^{(n)}}) \cong \varprojlim_n \mathscr{I}_0^{H^{(n)}}$, as desired. The case $i > 0$ is then clear because, for all $n \geq i$, the inverse image of $\mathscr{I}_i^{H^{(n)}}$ in $\mathscr{E}nd(H)$ is $\mathscr{I}_i^H$.

(ii) Immediate from (i) and, when $i > 0$, from (3.6.7).

(iii) Immediate from (i), since for $n \geq i$, $\mathscr{I}_i^H/\mathscr{I}_n^H$ identifies with the image of $\mathscr{I}_i^H$ in $\mathscr{I}_i^{H^{(n)}}$.

(iv) Immediate from (ii), since for $n \geq i$, $\mathscr{A}_i^H/\mathscr{A}_n^H$ identifies with the image of $\mathscr{A}_i^H$ in $\mathscr{A}_i^{H^{(n)}}$. □

As a consequence of the proposition and of our earlier calculation of the the successive quotients of the $\mathscr{I}_\bullet^{H^{(n)}}$-filtration (3.6.11), we now obtain the successive quotients of the $\mathscr{I}_\bullet^H$-filtration. For any i and any $n \geq i+1$, we have monomorphisms

$$(**) \qquad \mathscr{I}_i^H/\mathscr{I}_{i+1}^H \hookrightarrow \mathscr{I}_i^{H^{(n)}}/\mathscr{I}_{i+1}^{H^{(n)}} \hookrightarrow \begin{cases} \mathbb{O}, & i = 0; \\ \mathbb{G}_a, & i > 0; \end{cases}$$

plainly the composite is independent of the choice of n.

Corollary 4.4.6. *The diagram* $(**)$ *induces an identification of presheaves*

$$\mathscr{I}_i^H/\mathscr{I}_{i+1}^H \cong \begin{cases} \mathbb{O}^{\mathrm{Fr}_{p^h}}, & i = 0; \\ \mathbb{G}_a^{\mathrm{Fr}_{p^h}}, & i = p-1,\ p^2-1,\ p^3-1, \dots; \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. Fix i. For any $n \geq i+1$, we have an exact sequence of presheaves

$$0 \longrightarrow \mathscr{I}_{i+1}^{H^{(n)}} \longrightarrow \mathscr{I}_i^{H^{(n)}} \longrightarrow \mathscr{I}_i^{H^{(n)}}/\mathscr{I}_{i+1}^{H^{(n)}} \longrightarrow 0.$$

It follows from (3.6.11) that

- $(\mathscr{I}_{i+1}^{H^{(n)}})_{n \geq i+1}$ satisfies the Mittag-Leffler condition as a diagram of presheaves of abelian groups; and
- as n increases, $\mathscr{I}_i^{H^{(n)}}/\mathscr{I}_{i+1}^{H^{(n)}}$ is eventually constant of the asserted value.

Now take the limit over n and use (4.4.5). □

In an entirely similar fashion, using (3.6.3) in place of (3.6.11), and using the Mittag-Leffler condition for not-necessarily-abelian groups, we obtain the successive quotients of the $\mathscr{A}_\bullet^H$-filtration.

Corollary 4.4.7. *We have an identification of presheaves*

$$\mathscr{A}_i^H/\mathscr{A}_{i+1}^H \cong \begin{cases} \mu_{p^h-1}, & i = 0; \\ \mathbb{G}_a^{\mathrm{Fr}_{p^h}}, & i = p-1,\ p^2-1,\ p^3-1, \dots; \\ 0, & \textit{otherwise.} \end{cases}$$ □

In the rest of the subsection we shall study the following quotient groups, which appear in (4.4.5), and their relation to the $\mathscr{E}nd(H^{(n)})$'s and $\mathscr{A}ut(H^{(n)})$'s.

Definition 4.4.8. We define $\mathscr{E}_n^H$ to be the presheaf quotient ring $\mathscr{E}nd(H)/\mathscr{I}_n^H$, and $\mathscr{U}_n^H$ to be the subpresheaf of units in $\mathscr{E}_n^H$.

In other words, by (3.6.8), $\mathscr{U}_n^H \cong \mathscr{A}ut(H)/\mathscr{A}_n^H$.

Remark 4.4.9. By (4.4.6) and (4.4.7), $\mathscr{E}_n^H$ and $\mathscr{U}_n^H$ can be obtained from finitely many iterated extensions of finite étale groups. Hence both are finite étale over $\operatorname{Spec}\mathbb{F}_p$. In fact, it is easy to write down explicit representing schemes. To fix ideas, consider $\mathscr{E}_n^H$. For all $i \geq 0$, the exact sequence of presheaves

$$0 \longrightarrow \mathscr{I}_{i+1}^H \longrightarrow \mathscr{I}_i^H \xrightarrow{\text{can}} \mathscr{I}_i^H/\mathscr{I}_{i+1}^H \longrightarrow 0$$

has representable cokernel. Hence the quotient map "can" admits a section in the category of *set-valued* presheaves. Hence $\mathscr{I}_i^H \simeq \mathscr{I}_{i+1}^H \times (\mathscr{I}_i^H/\mathscr{I}_{i+1}^H)$ as presheaves of sets. Now, the possible nontrivial values of $\mathscr{I}_i^H/\mathscr{I}_{i+1}^H$, namely $\mathbb{O}^{\mathrm{Fr}_{p^h}}$ and $\mathbb{G}_a^{\mathrm{Fr}_{p^h}}$, both have underlying scheme $\operatorname{Spec}\mathbb{F}_p[T]/(T^{p^h}-T)$. Hence, letting l denote the integer such that $p^l \leq n < p^{l+1}$, we deduce that $\mathscr{E}_n^H$ is representable by

$$(\sharp) \qquad \operatorname{Spec}\mathbb{F}_p[T_0,\dots,T_l]/(T_0^{p^h}-T_0,\dots,T_l^{p^h}-T_l).$$

We can even specify a natural representation: S-points of $(\sharp)$ are canonically identified with ordered $(l+1)$-tuples of elements $a \in \Gamma(S)$ satisfying $a^{p^h} = a$, and we can take the map from $\mathscr{E}_n^H$ to $(\sharp)$ specified on points by sending the class of $f(T)$ to the coefficients of $T, T^p, \dots, T^{p^l}$.

Similarly, $\mathscr{U}_n^H$ is representable by

$$\operatorname{Spec}\mathbb{F}_p[T_0,T_0^{-1},T_1,\dots,T_l]/(T_0^{p^h}-T_0,\dots,T_l^{p^h}-T_l).$$

Remark 4.4.10. Let us digress for a moment to make a remark on the $\mathscr{U}_n^H$'s. Let D denote the central division algebra over $\mathbb{Q}_p$ of dimension h^2 and Hasse invariant $\frac{1}{h}$. Let $\mathscr{O}_D$ denote the maximal order in D. Then a classical theorem of Dieudonné [D, Théorème 3] and Lubin [Lu, 5.1.3] in the theory of formal group laws asserts that $\mathscr{O}_D \simeq \operatorname{End}_{\mathbb{F}_{p^h}}(H)$ as topological rings, where $\operatorname{End}_{\mathbb{F}_{p^h}}(H)$ has the $\mathscr{I}_\bullet^H(\mathbb{F}_{p^h})$-topology; precisely, one has $p^r\mathscr{O}_D \simeq \mathscr{I}_{p^{rh}-1}^H(\mathbb{F}_{p^h})$ for all $r \geq 0$. Hence

$$(\natural) \qquad \mathscr{O}_D^\times \simeq \operatorname{Aut}_{\mathbb{F}_{p^h}}(H) \cong \varprojlim_n \mathscr{U}_n^H(\mathbb{F}_{p^h})$$

as pro-finite groups.

The finite algebraic group $\mathscr{U}_n^H$ and the abstract finite group $\mathscr{U}_n^H(\mathbb{F}_{p^h})$ are closely related: indeed, the former is a *twist* over $\operatorname{Spec}\mathbb{F}_p$ of the latter. Precisely, for any abstract group G and ring A, write G_A for the corresponding constant group scheme over $\operatorname{Spec}A$. Then $\mathscr{U}_n^H$ is not constant over $\operatorname{Spec}\mathbb{F}_p$, but it becomes isomorphic to the group scheme $\mathscr{U}_n^H(\mathbb{F}_{p^h})_{\mathbb{F}_{p^h}}$ after the base change $\operatorname{Spec}\mathbb{F}_{p^h} \to \operatorname{Spec}\mathbb{F}_p$, as we see very explicitly from (4.4.9).

As pointed out by the referee, the isomorphisms in $(\natural)$ afford an explicit description of the affine algebra of $\mathscr{A}ut(H)$ upon base change to $\mathbb{F}_{p^h}$. Indeed, the affine algebra of the constant group scheme $\mathscr{U}_n^H(\mathbb{F}_{p^h})_{\mathbb{F}_{p^h}}$ is $\operatorname{Fun}\big(\mathscr{U}_n^H(\mathbb{F}_{p^h}),\mathbb{F}_{p^h}\big)$, the Hopf algebra of functions (of sets) $\mathscr{U}_n^H(\mathbb{F}_{p^h}) \to \mathbb{F}_{p^h}$. Hence $\mathscr{A}ut(H) \otimes \mathbb{F}_{p^h} \simeq \varprojlim_n \mathscr{U}_n^H(\mathbb{F}_{p^h})_{\mathbb{F}_{p^h}}$ has affine algebra

$$\varinjlim_n \operatorname{Fun}\big(\mathscr{U}_n^H(\mathbb{F}_{p^h}),\mathbb{F}_{p^h}\big) \simeq \operatorname{Fun}_{\mathrm{cts}}(\mathscr{O}_D^\times,\mathbb{F}_{p^h}),$$

the Hopf algebra of continuous functions $\mathscr{O}_D^\times \to \mathbb{F}_{p^h}$. The $\mathbb{F}_{p^h}$-linear dual of this last display is essentially given in [R1, 6.2.3]; strictly speaking, [R1] works the closed subscheme of $\mathscr{A}ut(H)$ of *strict* isomorphisms.

There are a number of immediate relations between the varying $\mathscr{E}_n^H$'s and the $\mathscr{E}nd(H^{(n)})$'s, and between the $\mathscr{U}_n^H$'s and the $\mathscr{A}ut(H^{(n)})$'s. To fix ideas, consider the $\mathscr{E}_n^H$'s and the $\mathscr{E}nd(H^{(n)})$'s. For all $n \geq 1$, $\mathscr{E}_n^H$ is identified with the image of $\mathscr{E}nd(H)$ in $\mathscr{E}nd(H^{(n)})$. And by (4.4.5), the $\mathscr{E}_n^H$'s and the $\mathscr{E}nd(H^{(n)})$'s have the same limit, namely $\mathscr{E}nd(H)$, endowed with the same topology. Our final goal for the subsection is to show that a yet stronger statement holds: namely, that the $\mathscr{E}_n^H$'s and the $\mathscr{E}nd(H^{(n)})$'s determine *isomorphic pro-objects* [SGA4$_1$, I §8.10]; and similarly for the $\mathscr{U}_n^H$'s and the $\mathscr{A}ut(H^{(n)})$'s.

Precisely, let "$\varprojlim_n$" $\mathscr{E}_n^H$ be the pro-ring scheme obtained from the diagram

$$\cdots \longrightarrow \mathscr{E}_3^H \longrightarrow \mathscr{E}_2^H \longrightarrow \mathscr{E}_1^H,$$

and let "$\varprojlim_n$" $\mathscr{E}nd(H^{(n)})$ be the pro-ring scheme obtained from the diagram

$$\cdots \longrightarrow \mathscr{E}nd(H^{(3)}) \longrightarrow \mathscr{E}nd(H^{(2)}) \longrightarrow \mathscr{E}nd(H^{(1)}).$$

The natural inclusions $\mathscr{E}_n^H \hookrightarrow \mathscr{E}nd(H^{(n)})$ for $n \geq 1$ plainly determine a morphism of pro-objects

$$\alpha\colon \text{“}\varprojlim_n\text{”}\, \mathscr{E}_n^H \longrightarrow \text{“}\varprojlim_n\text{”}\, \mathscr{E}nd(H^{(n)}).$$

We shall show that α is an isomorphism by exhibiting an explicit inverse β. To define β, we must define $\beta_n\colon$ "$\varprojlim_m$" $\mathscr{E}nd(H^{(m)}) \to \mathscr{E}_n^H$ for each $n \geq 1$. For this, let l be the integer such that $p^l \leq n < p^{l+1}$, and take any $m \geq p^{l+h}$. Consider the natural map

$$\mathscr{E}nd(H^{(m)}) \longrightarrow \mathscr{E}nd(H^{(n)}) \tag{$\flat$}$$

induced by truncation. By (3.6.11), (4.4.6), and choice of m, the image of ($\flat$) in $\mathscr{E}nd(H^{(n)})$ identifies with $\mathscr{E}_n^H$. Hence ($\flat$) induces "$\varprojlim_m$" $\mathscr{E}nd(H^{(m)}) \to \mathscr{E}_n^H$, which we take as the desired β_n. It is clear that the β_n's are compatible as n varies, so that we obtain the desired β.

Analogously, we may form the pro-algebraic groups

$$\text{“}\varprojlim_n\text{”}\, \mathscr{U}_n^H \quad \text{and} \quad \text{“}\varprojlim_n\text{”}\, \mathscr{A}ut(H^{(n)}),$$

and we obtain morphisms

$$\text{“}\varprojlim_n\text{”}\, \mathscr{U}_n^H \underset{\beta'}{\overset{\alpha'}{\rightleftarrows}} \text{“}\varprojlim_n\text{”}\, \mathscr{A}ut(H^{(n)}).$$

Theorem 4.4.11. *The morphisms α and β (resp., α' and β') are inverse isomorphisms of pro-objects.*

Proof. Everything is elementary from what we've already said. $\square$

4.5. **Some abstract nonsense.** In the next subsection we'll wish to interpret (4.4.11) in terms of the classifying stacks $B(\mathscr{U}_n)$ and $B\big(\mathscr{A}ut(H^{(n)})\big)$. To do so, we'll make use of a couple pieces of abstract nonsense which we now pause to record. Let $\mathscr{C}$ be a site.

The first statement is that if we let $\mathscr{P}$ denote the category of pairs (G, X), where X is a sheaf on $\mathscr{C}$ and G is a group sheaf on $\mathscr{C}$ acting on X (on the left, say), then passing to the quotient stack $(G, X) \mapsto G\backslash X$ defines a pseudofunctor $\mathscr{P} \to \mathrm{St}(\mathscr{C})$. In particular, taking X to be the sheaf with constant value $\{*\}$, the map $G \mapsto B(G)$ defines a morphism from group sheaves on $\mathscr{C}$ to stacks.

The second statement is that, given a category $\mathscr{D}$ and a pseudofunctor $F\colon \mathscr{D} \to \mathrm{St}(\mathscr{C})$, "taking the limit" determines a pseudofunctor from the category of pro-objects pro-$\mathscr{D}$ to $\mathrm{St}(\mathscr{C})$, "$\varprojlim_i$" $D_i \mapsto \varprojlim_i FD_i$.

The verifications of both statements are straightforward, and we leave them to the reader.

4.6. **The stratum of height h formal groups II.** In this subsection we apply the work of the previous two subsections to give another characterization of the stack $\mathscr{M}^h$ of formal groups of height h, $h \geq 1$. Recall the algebraic groups $\mathscr{U}_n^H$, $n \geq 1$, of (4.4.8).

Theorem 4.6.1. *$\mathscr{M}^h \approx \varprojlim_n B_{\text{fét}}(\mathscr{U}_n^H)$.*

Proof. The proof mostly consists of stringing together some of our previous results. By §4.5, the isomorphism of pro-objects

$$“\varprojlim_n” \mathscr{A}ut(H^{(n)}) \xrightarrow{\sim} “\varprojlim_n” \mathscr{U}_n^H$$

from (4.4.11) induces an equivalence of stacks

$$\varprojlim_n B_{\text{fét}}\big(\mathscr{A}ut(H^{(n)})\big) \xrightarrow{\approx} \varprojlim_n B_{\text{fét}}(\mathscr{U}_n^H).$$

By (3.5.11), we have an equivalence $\mathscr{B}_n^h \approx B_{\text{fét}}\big(\mathscr{A}ut(H^{(n)})\big)$ for $n \geq p^{h+1}$, plainly compatible with truncation on the $\mathscr{B}_n^h$ side and with the transition maps induced by "$\varprojlim_n$" $\mathscr{A}ut(H^{(n)})$ on the $B_{\text{fét}}\big(\mathscr{A}ut(H^{(n)})\big)$ side. Now use (4.3.11). $\square$

Remark 4.6.2. One may consider the equivalences

$$B_{\text{fpqc}}\big(\mathscr{A}ut(H)\big) \approx \mathscr{M}^h \approx \varprojlim_n B_{\text{fét}}(\mathscr{U}_n^H)$$

combined from (4.3.8) and (4.6.1) to be a stack analog of the theorem $\mathscr{O}_D^\times \simeq \mathrm{Aut}_{\mathbb{F}_q}(H)$ discussed in (4.4.10). Indeed, $\mathscr{U}_n^H$ becomes constant after the base change $\operatorname{Spec} \mathbb{F}_{p^h} \to \operatorname{Spec} \mathbb{F}_p$, and we obtain equivalences over $\mathbb{F}_{p^h}$

$$B_{\text{fpqc}}\big(\mathscr{A}ut(H)_{\mathbb{F}_{p^h}}\big) \approx \varprojlim_n B\big((\mathscr{U}_n^H)_{\mathbb{F}_{p^h}}\big) \approx \varprojlim B(\mathscr{O}_D^\times/N),$$

where the limit on the right runs through the open normal subgroups N of $\mathscr{O}_D^\times$.

5. Valuative criteria

In this section we'll conduct a basic investigation of some properties of the stacks $\mathscr{M}$ and $\mathscr{B}_n$, $n \geq 1$, related to valuative criteria. As in previous sections, we work with the notion of height relative to a fixed prime p.

Theorem 5.1. *$\mathscr{B}_n$ is universally closed over $\operatorname{Spec}\mathbb{Z}$, and for all $h \geq 1$ and $n \geq p^h$, $\mathscr{B}_n^{\geq h}$ is universally closed over $\operatorname{Spec}\mathbb{F}_p$.*

Proof. The proof is the same in all cases, so let's just consider $\mathscr{B}_n$ over $\operatorname{Spec}\mathbb{Z}$. We apply the valuative criterion in [LMB, 7.3]. Let $\mathscr{O}$ be a valuation ring and K its field of fractions. Let X be an n-bud over K. Then X admits a coordinate, so we may assume X is given by a bud law

$$F(T_1,T_2) = T_1 + T_2 + \sum_{2\leq i+j\leq n} a_{ij}T_1^iT_2^j, \qquad a_{ij} \in K.$$

For changes of coordinate of the form $f(T) = \lambda T$ for nonzero $\lambda \in K$, we obtain

$$f\bigl[F\bigl(f^{-1}(T_1), f^{-1}(T_2)\bigr)\bigr] = T_1 + T_2 + \sum_{2\leq i+j\leq n} a_{ij}\lambda^{1-i-j}T_1^iT_2^j.$$

So, by taking λ of sufficiently negative valuation, we see that F is K-isomorphic to a bud law defined over $\mathscr{O}$. $\square$

Remark 5.2. $\mathscr{B}_n$ is not *proper* over $\mathbb{Z}$ because it is not separated. Indeed, let $\mathscr{O}$ be a valuation ring with fraction field K. Then the natural functor

$$\mathscr{B}_n(\mathscr{O}) \longrightarrow \mathscr{B}_n(K)$$

is faithful but not full. For example, for the additive n-bud $\mathbb{G}_a^{(n)}$ (1.5.4) we have

$$\operatorname{Aut}_{\mathscr{O}}(\mathbb{G}_a^{(n)}) \subsetneq \operatorname{Aut}_K(\mathbb{G}_a^{(n)}),$$

since the latter contains automorphisms of the form $f(T) = \lambda T$ for λ of nonzero valuation.

Similarly, $\mathscr{B}_n^{\geq h}$ is not separated over $\mathbb{F}_p$.

Example 5.3. The following may be taken as an exhibition of the non-separatedness of $\mathscr{B}_n$ and of $\mathscr{M}$. Let $\mathscr{O}$ be a DVR with uniformizing element π and residue field of positive characteristic. Then the group law $F(T_1,T_2) := T_1 + T_2 + \pi T_1T_2$ determines a formal Lie group over $\operatorname{Spec}\mathscr{O}$. Let $f(T) := \pi T$. Then, over the generic point η, we have

$$f\bigl[F\bigl(f^{-1}(T_1), f^{-1}(T_2)\bigr)\bigr] = T_1 + T_2 + T_1T_2.$$

Hence f specifies an isomorphism $\widehat{\mathbb{A}}_\eta^F \xrightarrow{\sim} \widehat{\mathbb{G}}_m$. But $\widehat{\mathbb{G}}_m$ is certainly not isomorphic to $\widehat{\mathbb{A}}^F$ over $\operatorname{Spec}\mathscr{O}$, since $\widehat{\mathbb{A}}_{\mathscr{O}}^F$ reduces to $\widehat{\mathbb{G}}_a$ at the closed point. Hence $\widehat{\mathbb{G}}_m$ admits nonisomorphic extensions from the generic point to $\operatorname{Spec}\mathscr{O}$.

The failure of $\mathscr{B}_n$ and of $\mathscr{B}_n^{\geq h}$ to be separated prevents one from concluding formally that the valuative criterion used in the proof of (5.1) holds for $\mathscr{M}$ and for $\mathscr{M}^{\geq h}$, respectively. Nevertheless, these stacks do satisfy a kind of "formal universal closedness", in the following sense.

Theorem 5.4. *Let $\mathscr{O}$ be a valuation ring with field of fractions K.*

(i) *If K has characteristic 0, then the map $\mathscr{M}(\mathscr{O}) \to \mathscr{M}(K)$ is essentially surjective.*

(ii) *If K has characteristic p and is* separably closed*, then the map* $\mathscr{M}^{\geq h}(\mathscr{O}) \to \mathscr{M}^{\geq h}(K)$ *is essentially surjective.*

Proof. (i) As is well-known, over a $\mathbb{Q}$-algebra, every formal group law is isomorphic to the additive law.

(ii) By Lazard's theorem [L, Théorème IV], formal group laws over separably closed fields of characteristic p are classified up to isomorphism by their height. Now use that group laws of every height are defined over $\mathbb{F}_p$, hence over $\mathscr{O}$. □

Our remarks in (5.2) suggest that the failure of $\mathscr{B}_n^{\geq h}$ to be separated is tied to the additive n-bud, which has "height ∞". So it is natural to ask if the *stratum* $\mathscr{B}_n^h$ is separated. But the answer here is also negative: by (3.5.11), (3.6.1), and (3.6.4), $\mathscr{B}_n^h$ is the classifying stack of a group $\mathscr{A}ut(H^{(n)})$ which is positive dimensional and affine, so that $\mathscr{A}ut(H^{(n)})$ is not proper, so that $B\big(\mathscr{A}ut(H^{(n)})\big)$ is not separated [LMB, 7.8.1(2)]. There is, however, a positive result when we take the limit over n.

Theorem 5.5. *Let $\mathscr{O}$ be a valuation ring and K its field of fractions. Then $\mathscr{M}^h(\mathscr{O}) \to \mathscr{M}^h(K)$ is fully faithful for all $h \geq 1$.*

Proof. Of course, the assertion only has content when $\operatorname{char} K = p$, since otherwise $\mathscr{M}^h(\mathscr{O}) = \mathscr{M}^h(K) = \emptyset$. So assume $\operatorname{char} K = p$. By (4.6.1), $\mathscr{M}^h \approx \varprojlim_n B_{\text{fét}}(\mathscr{U}_n^H)$, where $\mathscr{U}_n^H$ is the finite étale group scheme over $\mathbb{F}_p$ of (4.4.8). In particular, $\mathscr{U}_n^H$ is proper. Hence $B_{\text{fét}}(\mathscr{U}_n^H)$ is a separated algebraic stack over $\mathbb{F}_p$ [LMB, 7.8.1(2)]. Hence $B(\mathscr{U}_n^H)(\mathscr{O}) \to B(\mathscr{U}_n^H)(K)$ is fully faithful. Now use that a limit of fully faithful maps is fully faithful. □

Remark 5.6. As noted in the introduction, when $\mathscr{O}$ is a *discrete* valuation ring, (5.5) is a special case of de Jong's theorem that, when $\operatorname{char} K = p$, the base change functor

$$(*) \qquad \left\{ \begin{matrix} p\text{-divisible groups and} \\ \text{homomorphisms over } \mathscr{O} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} p\text{-divisible groups and} \\ \text{homomorphisms over } K \end{matrix} \right\}$$

is fully faithful [dJ, 1.2]. (Tate proved that $(*)$ is fully faithful when $\operatorname{char} K = 0$ [T, Theorem 4].) Note that (5.5) only asserts bijections between Isom sets of objects, not Hom sets, as in de Jong's theorem. But it appears that the methods used to prove (5.5) extend to give bijections between Hom sets, provided one considers stacks of *categories*, not just stacks of groupoids.

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