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HOMOTOPY THEORIES OF ALGEBRAS OVER OPERADS

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Abstract

Homotopy theories over operads are defined. The corresponding spectral sequences for the homotopy groups are constructed. The calculations of the spectral sequences of the homotopy groups over the "n-dimensional little cubes" operads are produced.

There are two classical homotopy theories: the homotopy theory of topological spaces (the problem of calculating the homotopy groups of spheres is one of the most difficult problems of algebraic topology); the rational homotopy theory (the problem of calculating the homotopy groups of spheres is very simple).

In [**1**] it was shown that the rational homotopy theory of 1-connected topological spaces is equivalent to the homotopy theory of 1-connected commutative DGA-algebras. In [**2**], [**3**] it was shown that the singular chain complex $C_*(\mathcal{X})$ (cochain complex $C^*(\mathcal{X})$) of a topological space $\mathcal{X}$ possesses the structure of an E_∞-coalgebra (E_∞-algebra), and the homotopy theory of 1-connected topological spaces is equivalent to the homotopy theory of 1-connected E_∞-coalgebras (E_∞-algebras).

Here we consider the homotopy theories of algebras over operads and in particular over the "n-dimensional little cubes" operads E_n, $1 \leqslant n \leqslant \infty$, [**4**]. The ground ring will be assumed to be a field. We construct the spectral sequences for these homotopy theories and try to calculate the corresponding homotopy groups.

Recall that a family $\mathcal{E} = \{\mathcal{E}(j)\}_{j\geqslant 1}$ of chain complexes $\mathcal{E}(j)$ acted upon by the symmetric groups Σ_j is called an operad if there are given operaions

$$\gamma\colon \mathcal{E}(k) \otimes \mathcal{E}(j_1) \otimes \cdots \otimes \mathcal{E}(j_k) \to \mathcal{E}(j_1 + \cdots + j_k),$$

which are compatible with the actions of the symmetric groups and satisfy some associativity relations [**2**].

A chain complex X is called an algebra (coalgebra) over an operad $\mathcal{E}$ or simply $\mathcal{E}$-algebra ($\mathcal{E}$-coalgebra) if there is given a family of mappings

$$\mu(j)\colon \mathcal{E}(j) \otimes X^{\otimes j} \to X, \quad (\tau(j)\colon X \to Hom(\mathcal{E}(j); X^{\otimes j}),$$

which are compatible with the actions of the symmetric groups and satisfy some associativity relation [**2**].

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Denote the sum

$$\sum_j \mathcal{E}(j) \otimes_{\Sigma_j} X^{\otimes j}$$

by $\mathcal{E}(X)$. The correspondence $X \longmapsto \mathcal{E}(X)$ determines the functor in the categoty of chain complexes and an operad structure determines a natural transformation $\gamma\colon \mathcal{E} \circ \mathcal{E} \to \mathcal{E}$ of functors satisfying the associativity relation. It means that this functor is a monad in the category of chain complexes [**3**].

If X is an algebra over an operad $\mathcal{E}$ then there will be a chain mapping $\mu\colon \mathcal{E}(X) \to X$ and hence X will be an algebra over the monad $\mathcal{E}$.

If one wants to consider unitial algebras, the sum in the definition of $\mathcal{E}(X)$ must be modded out by the unit relation [**4**].

Dually denote

$$\overline{\mathcal{E}}(X) = \prod_j Hom_{\Sigma_j}(\mathcal{E}(j); X^{\otimes j}).$$

Then under suitable assumptions (for example if $\mathcal{E}$ is finitely generated) the correspondence $X \longmapsto \overline{\mathcal{E}}(X)$ determines the comonad in the category of chain complexes.

If X is a coalgebra over an operad $\mathcal{E}$ then it will be a coalgebra over the comonad $\overline{\mathcal{E}}$.

Operads and algebras over operads may be considered in the category of topological spaces (in this case we need instead of the tensor products $\otimes$ in the definition of the operation γ, the usual product $\times$) or other symmetric monoidal categories [**3**].

Consider some examples of operads and algebras (coalgebras) over operads.

1. An operad $E_0 = \{E_0(j)\}$, where $E_0(j)$ is the free module with one zero dimensional generator $e(j)$ and trivial action of the symmetric group Σ_j. So $E_0(j) \cong R$. The operation $\gamma\colon E_0 \times E_0 \to E_0$ is defined by the formula

$$\gamma(e(k) \otimes e(j_1) \otimes \cdots \otimes e(j_k)) = e(j_1 + \cdots + j_k).$$

It is easy to see that so defined, this operation is associative and compatible with the actions of the symmetric groups.

Algebras (coalgebras) over E_0 are simply commutative and associative algebras (coalgebras).

2. An operad $A = \{A(j)\}$, where $A(j)$ is the free Σ_j-module with one zero dimensional generator $a(j)$. So $A(j) \cong R(\Sigma_j)$. The operation $\gamma\colon A \times A \to A$ is defined by the formula

$$\gamma(a(k) \otimes a(j_1) \otimes \cdots \otimes a(j_k)) = a(j_1 + \cdots + j_k).$$

It is easy to see that the required relations are satisfied.

Algebras (coalgebras) over A are simply associative algebras (coalgebras).

3. For any chain complex X define operads $\mathcal{E}_X$, $\mathcal{E}^X$ by putting

$$\mathcal{E}_X(j) = Hom(X^{\otimes j}; X); \quad \mathcal{E}^X(j) = Hom(X; X^{\otimes j}).$$

The actions of the symmetric groups are determined by the permutations of factors of $X^{\otimes j}$ and operad structures are defined by the formulas

$$\gamma_X(f \otimes g_1 \otimes \cdots \otimes g_k) = f \circ (g_1 \otimes \cdots \otimes g_k), \quad f \in \mathcal{E}_X(k),\ g_i \in \mathcal{E}_X(j_i);$$
$$\gamma^X(f \otimes g_1 \otimes \cdots \otimes g_k) = (g_1 \otimes \cdots \otimes g_k) \circ f, \quad f \in \mathcal{E}^X(k),\ g_i \in \mathcal{E}^X(j_i).$$

A chain complex X is an algebra (coalgebra) over an operad $\mathcal{E}$ if and only if there is given an operad mapping $\xi\colon \mathcal{E} \to \mathcal{E}_X$ ($\xi\colon \mathcal{E} \to \mathcal{E}^X$).

4. For $n \geqslant 0$ denote by Δ^n the normalized chain complex of the standard n-dimensional simplex. Then $\Delta^* = \{\Delta^n\}$ is the cosimplicial object in the category of chain complexes. Denote the realization of the cosimplisial object $(\Delta^*)^{\otimes j} = \Delta^* \otimes \cdots \otimes \Delta^*$ as $E^\Delta(j)$, i.e.

$$E^\Delta(j) = Hom(\Delta^*; (\Delta^*)^{\otimes j}),$$

where Hom is considered in the category of cosimplicial objects.

So the elements of $E^\Delta(j)$ are the sequences $f = \{f^n\}$ of mappings $f^n\colon \Delta^n \to (\Delta^n)^{\otimes j}$ commuting the diagrams

$$\begin{array}{ccc} \Delta^n & \xrightarrow{f^n} & (\Delta^n)^{\otimes j} \\ \delta^i \uparrow \downarrow \sigma^i & & \delta^i \uparrow \downarrow \sigma^i \\ \Delta^{n-1} & \xrightarrow{f^{n-1}} & (\Delta^{n-1})^{\otimes j} \end{array}$$

The family $E^\Delta = \{E^\Delta(j)\}$ will be the operad for which the actions of the symmetric groups and the operad structure are defined similary to the corresponding structure for the above defined operad $\mathcal{E}^X$, where instead of X we take Δ^*.

Note that since the complexes Δ^n are acyclic then the operad E^Δ is also acyclic.

In [**3**] it was shown that on the chain complex $C_*(\mathcal{X})$ of a topological space $\mathcal{X}$ there exists a natural E^Δ-coalgebra structure. Dually, on the cochain complex $C^*(\mathcal{X})$ there exists a natural E^Δ-algebra structure.

5. The main examples of topological operads are the little n-cube operads E_n introduced by Boardman and Vogt [**5**] and studied by May [**4**]. Any n-fold loop space $\Omega^n\mathcal{X}$ is an algebra over the operad E_n.

There are inclusions $E_n \to E_{n+1}$ and its direct limit denoted as E_∞. It is acyclic operad with free actions of the symmetric groups.

Any acyclic operad with free action of the symmetric groups is called E_∞-operad. Any algebra (coalgebra) over E_∞-operad is called E_∞-algebra (E_∞-coalgebra).

6. It is easy to see that if $\mathcal{E} = \{\mathcal{E}(j)\}$ is an operad in the category of topological space then the family $C_*(\mathcal{E}) = \{C_*(\mathcal{E}(j))\}$ consisting of the corresponding chain complexes will be an operad in the category of chain complexes and if $\mathcal{E}$ is E_∞-operad then $C_*(\mathcal{E})$ is E_∞-operad.

7. An operad $\mathcal{E}$ is called a Hopf operad if there is given a coassociative operad mapping $\nabla\colon \mathcal{E} \to \mathcal{E} \otimes \mathcal{E}$. It is easy to see that E_0, A are Hopf operads.

The operad E^Δ is a Hopf operad. The Hopf structure $\nabla\colon E^\Delta \to E^\Delta \otimes E^\Delta$ is induced by the diagonal mapping $\Delta^* \to \Delta^* \otimes \Delta^*$.

If $\mathcal{E}$ is a toplogical operad then it's singular chain complexes operad $C_*(\mathcal{E})$ is a Hopf operad in which the Hopf structure is induced by the coalgebra structures on the $C_*(\mathcal{E}(j))$.

8. The singular chain complex $C_*(\mathcal{X})$ (cochain complex $C^*(\mathcal{X})$) is an E_∞-coalgebra (E_∞-algebra). Indeed, let E be an E_∞-operad. Consider the operad $E^\Delta \otimes E$. It is E_∞-operad and there is the projection of operads $p\colon E^\Delta \otimes E \to E^\Delta$. Then the composition

$$E^\Delta \otimes E \xrightarrow{p} E^\Delta \xrightarrow{\xi} \mathcal{E}^{C_*(\mathcal{X})} \quad (E^\Delta \otimes E \xrightarrow{p} E^\Delta \xrightarrow{\xi} \mathcal{E}_{C^*(\mathcal{X})}).$$

will give on $C_*(\mathcal{X})$ ($C^*(\mathcal{X})$) the structure of $E^\Delta \otimes E$-coalgebra ($E^\Delta \otimes E$-algebra).

Denote the operad $E^\Delta \otimes C_*(E_n)$ in the category of chain complex simply by E_n. Then $C_*(\mathcal{X})$ may be considered as an E_n-coalgebra. Dually, $C^*(\mathcal{X})$ may be considered as an E_n-algebra.

We will need the following general properties of algebras (coalgebras) over operads.

Proposition 1. *The category of $\mathcal{E}$-algebras ($\mathcal{E}$-coalgebras) over a Hopf operad $\mathcal{E}$ admits tensor products.*

Proof. Let X', X'' – $\mathcal{E}$-algebras, i.e. there are given operad mappings $\xi'\colon \mathcal{E} \to \mathcal{E}_{X'}$, $\xi''\colon \mathcal{E} \to \mathcal{E}_{X''}$. Defing the mapping $\xi\colon \mathcal{E} \to \mathcal{E}_{X'\otimes X''}$ as the composition

$$\mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes \mathcal{E} \xrightarrow{\xi'\otimes\xi''} \mathcal{E}_{X'} \otimes \mathcal{E}_{X''} \longrightarrow \mathcal{E}_{X'\otimes X''}.$$

This mapping will give on $X' \otimes X''$ the desired $\mathcal{E}$-algebra structure. □

Proposition 2. *If $X_* = \{X_n\}$ is a simplicial object in the category of algebras over an operad $\mathcal{E}$ then its realization $|X_*|$ will also be an $\mathcal{E}$-algebra. Dually, if $X^* = \{X^n\}$ is a cosimplicial object in the category of coalgebras over an operad $\mathcal{E}$ then its realization $|X^*|$ will be an $\mathcal{E}$-coalgebra.*

Proof. Consider a simplicial object $X_* = \{X_n\}$ in the category of $\mathcal{E}$-algebras, $\mu_n\colon \mathcal{E}(X_n) \to X_n$, the $\mathcal{E}$-algebra structure on X_n. The Eilenberg-Zilber mappings

$$\psi\colon |X_*| \otimes \cdots \otimes |X_*| \to |X_* \otimes \cdots \otimes X_*|$$

commute with the actions of the symmetric groups and hence induce mappings

$$\psi\colon \mathcal{E}(j) \otimes_{\Sigma_j} |X_*|^{\otimes j} \to |\mathcal{E}(j) \otimes_{\Sigma_j} X_*^{\otimes j}|.$$

These mappings give us the mapping $\psi\colon \mathcal{E}(|X_*|) \to |\mathcal{E}(X_*)|$ and desired mapping $\mathcal{E}(|X_*|) \to |X_*|$ is the composition

$$\mathcal{E}(|X_*|) \xrightarrow{\psi} |\mathcal{E}(X_*)| \xrightarrow{\mu_*} |X_*|.$$

□

Corollary. *The realization $B(\mathcal{E}, \mathcal{E}, X)$ of the simplicial resolution*

$$B_*(\mathcal{E}, \mathcal{E}, X) : \mathcal{E}(X) \longleftarrow \mathcal{E}^2(X) \longleftarrow \cdots \longleftarrow \mathcal{E}^n(X) \longleftarrow \cdots$$

over an $\mathcal{E}$-algebra X is an $\mathcal{E}$-algebra with chain equivalence $\eta\colon B(\mathcal{E},\mathcal{E},X) \to X$. Dually, the realization $F(\mathcal{E},\mathcal{E},X)$ of the cosimplicial resolution

$$F^*(\mathcal{E},\mathcal{E},X) : \overline{\mathcal{E}}(X) \longrightarrow \overline{\mathcal{E}}^2(X) \longrightarrow \cdots \longrightarrow \overline{\mathcal{E}}^n(X) \longrightarrow \cdots$$

over an $\mathcal{E}$-coalgebra X is an $\mathcal{E}$-coalgebra with chain equivalence $\xi\colon X \to F(\mathcal{E},\mathcal{E},X)$.

Pass now to the homotopy theories. Let $\mathcal{E}$ be a Hopf operad for which there is given operad mapping $\mathcal{E} \to \mathcal{E}^{\Delta}$. It means that the chain complexes Δ^n possess $\mathcal{E}$-coalgebra structures compatible with the coface and codegeneracy operators. In particular, the unit segment $I = \Delta^1$ possesses $\mathcal{E}$-coalgebra structure.

Denote $\mathcal{A}_{\mathcal{E}}$ ($\mathcal{K}_{\mathcal{E}}$) the category in which objects are $\mathcal{E}$-algebras ($\mathcal{E}$-coalgebras) and morphisms are $\mathcal{E}$-algebra mappings ($\mathcal{E}$-coalgebra mappings).

In [**6**] there are given sufficient conditions for the existence of a closed model structure on the category of operads in an arbitrary symmetric monoidal category. In particular chain operads carry a closed model structure.

Here we prove that the category $\mathcal{A}_{\mathcal{E}}$ ($\mathcal{K}_{\mathcal{E}}$) possesses a closed model structure [**7**].

Define a map in $\mathcal{A}_{\mathcal{E}}$ to be a weak equivalence if it induces isomorphism on homology, a fibration if it is surjective and a cofibration if it has the left lifting property with respect to all trivial fibrations.

Theorem 1. *The category $\mathcal{A}_{\mathcal{E}}$ is a closed model category.*

Proof. As in the case of usual algebras [**7**] the only nontrivial part of the theorem to prove is that any map $f\colon X \to Y$ of $\mathcal{E}$-algebras may be factored into the composition $f = p \circ i$, where i is a cofibration and p is a trivial fibration.

The idea of the proof repeats the corresponding proof for usual algebras. Namely, let $f\colon X \to Y$ be a mapping of $\mathcal{E}$-algebras. Define an $\mathcal{E}$-algebra $\mathcal{E}(X,Y)$, putting $\mathcal{E}(X,Y) = X + \mathcal{E}(Y)$. An $\mathcal{E}$-algebra structure is induced by $\mathcal{E}$-algebra structures on X and $\mathcal{E}(Y)$.

There is a projection $p\colon \mathcal{E}(X,Y) \to Y$, induced by the mapping $f\colon X \to Y$ and the $\mathcal{E}$-algebra structure $\mu\colon \mathcal{E}(Y) \to Y$, $p(x+y) = f(x) + \mu(y)$. Besides that there are an injection $i\colon X \to \mathcal{E}(X,Y)$ and a chain mapping $j\colon Y \to \mathcal{E}(X,Y)$ such that $p \circ i = f$, $p \circ j = Id$. More over p is a fibration and i is a cofibration. However p is not a trivial fibration. To improve this fibration we construct a simplicial resolution $\mathcal{E}_*(X,Y)$, putting

$$\mathcal{E}_0(X,Y) = \mathcal{E}(X,Y), \qquad \mathcal{E}_{n+1}(X,Y) = \mathcal{E}(X, \mathcal{E}_n(X,Y)).$$

The face and degeneracy mappings are defined by the inductive formulas. Namely,

$$d_0 = p\colon \mathcal{E}(X,Y) \to Y, \quad s_0 = \mathcal{E}(-,j)\mathcal{E}(X,Y) \to \mathcal{E}_1(X,Y) = \mathcal{E}(X,\mathcal{E}(X,Y)).$$

Similary there are defined

$$d_0 = p\colon \mathcal{E}_{n+1}(X,Y) = \mathcal{E}(X,\mathcal{E}_n(X,Y)) \to \mathcal{E}_n(X,Y);$$
$$s_0 = \mathcal{E}(-,j)\colon \mathcal{E}_n(X,Y) \to \mathcal{E}_{n+1}(X,Y) = \mathcal{E}(X,\mathcal{E}_n(X,Y)).$$

Finally, define

$$d_{i+1} = \mathcal{E}(-, d_i)\colon \mathcal{E}_{n+1}(X, Y) \to \mathcal{E}_n(X, Y);$$
$$s_{i+1} = \mathcal{E}(-, s_i)\colon \mathcal{E}_n(X, Y) \to \mathcal{E}_{n+1}(X, Y).$$

Note that if X is a trivial then $\mathcal{E}_*(X, Y)$ is isomorphic to $B_*(\mathcal{E}, \mathcal{E}, Y)$.

The realization $|\mathcal{E}_*(X, Y)|$ is an $\mathcal{E}$-algebra which is chain equivalent to Y. Moreover there are the surjective mapping $p\colon |\mathcal{E}_*(X, Y)| \to Y$ and an injective mapping $i\colon X \to |\mathcal{E}_*(X, Y)|$ such that $f = p \circ i$. If X is a trivial we have the isomorphism $|\mathcal{E}_*(X, Y)| \cong B(\mathcal{E}, \mathcal{E}, Y)$.

We prove that the mapping $i\colon X \to |\mathcal{E}_*(X, Y)|$ is a cofibration. Let $u\colon U \to V$ be a trivial fibration. It means that u is an surjective and induces an isomorphism of homologies. Then there is a chain mapping $v\colon V \to U$ and a chain homotopy $w\colon U \to U$ such that

$$u \circ u = Id; \quad d(w) = v \circ u - Id; \quad u \circ w = 0; \quad w \circ v = 0; \quad w \circ w = 0.$$

Further, let $g\colon X \to U$, $h\colon |\mathcal{E}_*(X, Y)| \to V$ be $\mathcal{E}$-algebra mappings commuting the diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & U \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle u} \\ |\mathcal{E}_*(X, Y)| & \xrightarrow{h} & V \end{array}$$

We need to construct an $\mathcal{E}$-algebra mapping $\widetilde{h}\colon |\mathcal{E}_*(X, Y)| \to U$ preserving commutativity of the diagram.

It is easy to see that giving an $\mathcal{E}$-algebra mapping $\widetilde{h}\colon |\mathcal{E}_*(X, Y)| \to U$ is equivalent to giving a family of $\mathcal{E}$-algebra mappings $h^n\colon \mathcal{E}_n(X, Y) \to Hom(\Delta^n; U)$ such that the following diagrams are commutative

$$\begin{array}{ccc} \mathcal{E}_n(X, Y) & \xrightarrow{h^n} & Hom(\Delta^n; U) \\ {\scriptstyle s_i}\downarrow\uparrow{\scriptstyle d_i} & & {\scriptstyle s_i}\downarrow\uparrow{\scriptstyle d_i} \\ \mathcal{E}_{n+1}(X, Y) & \xrightarrow{h^{n+1}} & Hom(\Delta^{n+1}; U) \end{array}$$

Note that to giving $\mathcal{E}$-algebra mapings $h^n\colon \mathcal{E}_n(X, Y) \to Hom(\Delta^n; U)$ is equivalent to giving a mapping on X (determined by g) and a chain mapping

$$\overline{h}^n\colon \mathcal{E}_{n-1}(X, Y) \to Hom(\Delta^n; U).$$

So we conclude that to give an $\mathcal{E}$-algebra mapping $\widetilde{h}\colon |\mathcal{E}_*(X, Y) \to U$ is equivalent to give a family of chain mappings $\overline{h}^n\colon \mathcal{E}_{n-1}(X, Y) \to Hom(\Delta^n; U)$ such that the corresponding mappings h^n are $\mathcal{E}$-algebra mappings commuting the above diagram.

We put $\overline{h}^0 = v \circ h\colon Y \to U$ and $\overline{h}^n = w \circ \mu \circ \mathcal{E}(g, \overline{h}^{n-1})$. Straight verifications show that the required relations are satisfied. □

Corollary. *For any $\mathcal{E}$-algebra Y the $\mathcal{E}$-algebra $B(\mathcal{E}, \mathcal{E}, Y)$ is a cofibrant object in the category $\mathcal{A}_{\mathcal{E}}$.*

It follows from the fact that for trivial X there is the isomorphism $|\mathcal{E}_*(X,Y)| \cong B(\mathcal{E},\mathcal{E},Y)$.

Dually consider the category $\mathcal{K}_{\mathcal{E}}$. Define a map of this category to be a weak equivalences if it induces the isomorphism on homology a cofibration if it is injective and fibration if it has the right lifting property with respect to all trivial cofibrations.

Theorem (1'). *The category $\mathcal{K}_{\mathcal{E}}$ is a closed model category.*

Denote by $Ho\mathcal{K}_{\mathcal{E}}$ the localization of the category $\mathcal{K}_{\mathcal{E}}$ with respect to the class of weak equivalences, i.e. morphisms induce the isomorphisms of homologies.

For an $\mathcal{E}$-coalgebra X the tensor product $X\otimes\Delta^1$ will be a cylinder object, and $\mathcal{E}$-coalgebra mappings $f_0, f_1 \colon X \to Y$ will be left homotopic if there exists a mapping $h\colon X\otimes\Delta^1 \to Y$ such that $h\circ\delta^0 = f_0$, $h\circ\delta^1 = f_1$.

Let $\widetilde{\mathcal{K}}_{\mathcal{E}}$ denote the category, whose objects are $\mathcal{E}$-coalgebras and morphisms $f\colon X\to Y$ are $\mathcal{E}$-coalgebra mappings $\widetilde{f}\colon X \to F(\mathcal{E},\mathcal{E},Y)$.

Denote by $\pi\mathcal{K}_{\mathcal{E}}$ the category whose objects are $\mathcal{E}$-coalgebras and morphisms are the homotopy classes of morphisms in $\widetilde{\mathcal{K}}_{\mathcal{E}}$. From general homotopy theory [**7**] it follows

Theorem 2. *There is an equivalence of categories*

$$Ho\mathcal{K}_{\mathcal{E}} \cong \pi\mathcal{K}_{\mathcal{E}}.$$

Dually, for $\mathcal{E}$-algebras we have

Theorem (2'). *There is an equivalence of categories*

$$Ho\mathcal{A}_{\mathcal{E}} \cong \pi\mathcal{A}_{\mathcal{E}}.$$

Consider now the problem of calculating the homotopy groups of $\mathcal{E}$-coalgebras. $\mathcal{E}$ will be assumed to satisfy some suitable assumptions, for example $\mathcal{E}$ is finitely generated.

Since the chain complexes Δ^n of the standard n-dimensional simplexes are $\mathcal{E}$-coalgebras, the chain complexes S^n of the n-dimensional spheres will be $\mathcal{E}$-coalgebras. Define the homotopy groups $\pi_n^{\mathcal{E}}(X)$ of an $\mathcal{E}$-coalgebra X by putting $\pi_n^{\mathcal{E}}(X) = [S^n; F(\mathcal{E},\mathcal{E},X)]$, the set of homotopy classes of $\mathcal{E}$-coalgebra mappings $f\colon S^n \to F(\mathcal{E},\mathcal{E},X)$.

Theorem 3. *For any $\mathcal{E}$-coalgebra X there is the spectral sequence of the homotopy groups $\pi_*^{\mathcal{E}}(X)$ in which the E^1 term is isomorphic to the cobar construction $F(\mathcal{E}_*, X_*)$, where $\mathcal{E}_*$, X_* denotes the homologies of $\mathcal{E}$ and X correspondingly.*

Proof. Consider the filtration

$$F(\mathcal{E},\mathcal{E},X) \supset F^1(\mathcal{E},\mathcal{E},X) \supset \cdots \supset F^m(\mathcal{E},\mathcal{E},X) \supset \ldots,$$

where $F^m(\mathcal{E},\mathcal{E},X) : \overline{\mathcal{E}}^m(X) \longrightarrow \overline{\mathcal{E}}^{m+1}(X) \longrightarrow \cdots$.

This filtration induces the spectral sequence. Exact sequences

$$0 \to F^{m+1}(\mathcal{E},\mathcal{E},X) \to F^m(\mathcal{E},\mathcal{E},X) \to \overline{\mathcal{E}}^{m+1}(X) \to 0$$

induce the isomorphisms

$$E^1_{n,m} = [S^n, \overline{\mathcal{E}}^{m+1}(X)] \cong H_n(\overline{\mathcal{E}}^m(X))$$

and hence the isomorphism $E^1 \cong F(\mathcal{E}_*, X_*)$. □

If S^n is a trivial $\mathcal{E}$-coalgebra then the differentials of the spectral sequence are determined only by the differentials of the cobar construction $F(\mathcal{E}, X)$ and thus we have

Theorem 4. *If S^n is a trivial $\mathcal{E}$-coalgebra then for any $\mathcal{E}$-coalgebra X there is an isomorphism*

$$\pi^{\mathcal{E}}_n(X) \cong H_n(F(\mathcal{E}, X)).$$

Now let E_n be the little n-cube operad. Note that if $m \geqslant n$ then the homology of $\overline{E}_n(S^m)$ is trivial up to the dimension $2m-n+1 > m$. From here it follows that S^m has trivial E_n-coalgebra structure and hence we have

Theorem 5. *If $\mathcal{X}$ is a topological space, $m \geqslant n$ then there is an isomorphism*

$$\pi^{E_n}_m(\mathcal{X}) \cong H_m(F(E_n, C_*(\mathcal{X}))).$$

The E^1-term of the spectral sequence is expressed through the Dyer-Lashof algebra [**8**], [**9**] and the result is the following

Theorem 6. *The E^1-term of the spectral sequence of the homotopy groups $\pi^{E_n}_*(\mathcal{X})$ of a topological space $\mathcal{X}$ is isomorphic to the module $S^n T_s R_{n-1} L_{n-1} S^{-n} H_*(\mathcal{X})$, where T_s is the free commutative algebra, R_{n-1} is the submodule of the Dyer-Lashof algebra generated by allowable sequence of excess less then n, L_{n-1} is the free $(n-1)$-Lie algebra.*

If $\mathcal{X} - n$-connected topological space then the homology of the cobar construction $F(E_n, C_*(\mathcal{X}))$ is isomorphic to the n-fold suspension over the homology of iterated loop space $\Omega^n\mathcal{X}$ [**9**]. Hence we have

Theorem 7. *If $\mathcal{X}$ is an n-connected topological space then there is the isomorphism*

$$\pi^{E_n}_*(X) \cong S^n H_*(\Omega^n \mathcal{X}).$$

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