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Homotopy Theories of Algebras over Operads

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Abstract—Homotopy theories of algebras over operads, including operads over "little n-cubes," are defined. Spectral sequences are constructed and the corresponding homotopy groups are calculated.

Key words: *homotopy theory, algebra over an operad, little n-cubes operad, category of chain complexes, monad, natural transformation of functors, homotopy group of spheres.*

There are two classical homotopy theories:

(1) the homotopy theory of topological spaces, in which the calculation of the homotopy groups of spheres is one of the most difficult problems of algebraic topology;

(2) rational homotopy theory, in which the calculation of the homotopy groups of spheres is a fairly simple problem.

In [1], it was shown that rational homotopy theory is equivalent to the homotopy theory of commutative DGA-algebras. In [2, 3], it was proved that the singular chain complex $C_*(X)$ (cochain complex $C^*(X)$) of a topological space X has the natural structure of an E_∞-coalgebra (E_∞-algebra), and the homotopy theory of topological spaces is equivalent to the homotopy theory of E_∞-coalgebras (E_∞-algebras).

A natural problem is to find intermediate homotopy theories between the homotopy theories of DGA-algebras and E_∞-algebras and calculate the homotopy groups of spheres in these theories.

In this paper, we define homotopy theories of algebras over operads, in particular, the little n-cubes operads E_n over little n-cubes, where $1 \le n \le \infty$ (see [4]). We construct spectral sequences and calculate the corresponding homotopy groups.

Recall that a family $\mathcal{E} = \{\mathcal{E}(j)\}_{j\ge 1}$ of chain complexes $\mathcal{E}(j)$ on which the permutation groups Σ_j act is called an *operad* if it is endowed with operations

$$\gamma\colon \mathcal{E}(k) \otimes \mathcal{E}(j_1) \otimes \cdots \otimes \mathcal{E}(j_k) \to \mathcal{E}(j_1 + \cdots + j_k)$$

compatible with the permutation group actions and satisfying certain associativity relations [2].

A chain complex X with operations

$$\mu(j)\colon \mathcal{E}(j) \otimes X^{\otimes j} \to X \qquad \big(\tau(j)\colon X \to \mathrm{Hom}(\mathcal{E}(j)\,;\, X^{\otimes j})\big)$$

compatible with the permutation group actions and satisfying certain associativity relations [2] is called an *algebra* (respectively, a *coalgebra*) over an operad $\mathcal{E}$, or simply an $\mathcal{E}$-*algebra* (an $\mathcal{E}$-*coalgebra*).

Let us denote the sum

$$\sum_j \mathcal{E}(j) \otimes_{\Sigma_j} X^{\otimes j}$$

by $\underline{\mathcal{E}}(X)$. The correspondence $X \mapsto \underline{\mathcal{E}}(X)$ determines a functor on the category of chain complexes. The operad structure on $\mathcal{E}$ determines the natural transformation of functors $\gamma\colon \underline{\mathcal{E}} \circ \underline{\mathcal{E}} \to \underline{\mathcal{E}}$, which defines a monad structure on $\underline{\mathcal{E}}$ [3]. Moreover, a chain complex X is an algebra over the operad $\mathcal{E}$ if and only if it is an algebra over the monad $\underline{\mathcal{E}}$.

Dually, for

$$\overline{\mathcal{E}}(X) = \prod_j \operatorname{Hom}_{\Sigma_j}(\mathcal{E}(j)\,;\, X^{\otimes j}),$$

the correspondence $X \mapsto \overline{\mathcal{E}}(X)$ determines a comonad on the category of chain complexes. A chain complex X is a coalgebra over the operad $\mathcal{E}$ if and only if it is a coalgebra over the comonad $\overline{\mathcal{E}}$.

Operads and algebras over operads can also be defined in the category of topological spaces. In this case, in the definition of the operations γ, the tensor product $\otimes$ must be replaced by the usual Cartesian product $\times$ of topological spaces [3].

Below, we give examples of operads and algebras (coalgebras) over operads.

Example 1. Let $E_0(j)$ be the free module with one zero-dimensional generator $e(j)$ and trivial action of the permutation groups Σ_j (i.e., $E_0(j) \cong R$). Then $E_0 = \{E_0(j)\}$ is an operad. The operation $\gamma\colon E_0 \times E_0 \to E_0$ is defined by

$$\gamma(e(k) \otimes e(j_1) \otimes \dots \otimes e(j_k)) = e(j_1 + \dots + j_k).$$

It is easy to verify that it is associative and compatible with the actions of the permutation groups.

The algebras (coalgebras) over E_0 are simply commutative and associative algebras (coalgebras).

Example 2. Let $A(j)$ be the free Σ_j-module with one zero-dimensional generator $a(j)$ (i.e., $A(j) \cong R(\Sigma_j)$). Then $A = \{A(j)\}$ is an operad; the operation $\gamma\colon A \times A \to A$ is defined by

$$\gamma(a(k) \otimes a(j_1) \otimes \dots \otimes a(j_k)) = a(j_1 + \dots + j_k).$$

It is easy to verify that the required relations do hold.

The algebras (coalgebras) over the operad A are simply associative algebras (coalgebras).

Example 3. An arbitrary chain complex X determines the operads

$$\mathcal{E}_X(j) = \operatorname{Hom}(X^{\otimes j}\,;\, X), \qquad \mathcal{E}^X(j) = \operatorname{Hom}(X\,;\, X^{\otimes j}).$$

The actions of the permutation groups are permutations of factors in $X^{\otimes j}$, and the operad structure is defined by

$$\gamma_X(f \otimes g_1 \otimes \dots \otimes g_k) = f \circ (g_1 \otimes \dots \otimes g_k), \qquad f \in \mathcal{E}_X(k), \quad g_i \in \mathcal{E}_X(j_i)\,;$$
$$\gamma^X(f \otimes g_1 \otimes \dots \otimes g_k) = (g_1 \otimes \dots \otimes g_k) \circ f, \qquad f \in \mathcal{E}^X(k), \quad g_i \in \mathcal{E}^X(j_i).$$

A chain complex X is an algebra (coalgebra) over the operad $\mathcal{E}$ if and only if there is a map of operads $\xi\colon \mathcal{E} \to \mathcal{E}_X$ (respectively, $\xi\colon \mathcal{E} \to \mathcal{E}^X$).

Example 4. For $n \ge 0$, let Δ^n denote the normalized chain complex of the standard n-simplex. Then $\Delta^* = \{\Delta^n\}$ is a cosimplicial object in the category of chain complexes.

Let $E^\Delta(j)$ denote the realization of the cosimplicial object $(\Delta^*)^{\otimes j} = \Delta^* \otimes \dots \otimes \Delta^*$, i.e.,

$$E^\Delta(j) = \operatorname{Hom}(\Delta^*\,;\, (\Delta^*)^{\otimes j}),$$

where Hom is considered in the category cosimplicial objects.

The family $E^\Delta = \{E^\Delta(j)\}$ is an operad; the actions of permutation groups and the operad structure are similar to those for the operads $\mathcal{E}^X$ defined above (in the definition, Δ^* instead of X is taken).

Note that, since the chain complexes Δ^n are acyclic, the operad E^Δ is acyclic also.

In [3], it was shown that the singular chain complex $C_*(X)$ of a topological space X admits the natural structure of an E^Δ-coalgebra. Dually, the cochain complex $C^*(X)$ admits the natural structure of an E^Δ-algebra.

Example 5. The main examples of operads in the category of topological spaces are the little n-cubes operads $\mathcal{E}_n$, which were introduced by Boardman and Vogt [5] and studied by May [4]. In particular, May showed that any n-fold loop space $\Omega^n X$ is an algebra over the operad $\mathcal{E}_n$.

The inclusions $\mathcal{E}_n \to \mathcal{E}_{n+1}$ hold; we denote the direct limit determined by these inclusions by $\mathcal{E}_\infty$. The operad $\mathcal{E}_\infty$ is an acyclic operad with free actions of permutation groups.

Any acyclic operad with free actions of permutation groups is called an E_∞-*operad*, and any algebra (coalgebra) over an E_∞-operad is called an E_∞-*algebra* (an E_∞-*coalgebra*).

Example 6. It is easy to see that if $\mathcal{E} = \{\mathcal{E}(j)\}$ is an operad in the category of topological spaces, then the family of chain complexes $C_*(\mathcal{E}) = \{C_*(\mathcal{E}(j))\}$ is an operad in the category of chain complexes. If $\mathcal{E}$ is an E_∞-operad, then $C_*(\mathcal{E})$ is an E_∞-operad.

Let us show that any singular chain complex $C_*(X)$ (singular cochain complex $C^*(X)$) is an E_∞-coalgebra (respectively, an E_∞-algebra).

Let E be an E_∞-operad. Consider the operad $E^\Delta \otimes E$. It is an E_∞-operad. Consider the projection of operads $p\colon E^\Delta \otimes E \to E^\Delta$. The composition

$$E^\Delta \otimes E \xrightarrow{p} E^\Delta \xrightarrow{\xi} \mathcal{E}^{C_*(X)} \qquad \big(E^\Delta \otimes E \xrightarrow{p} E^\Delta \xrightarrow{\xi} \mathcal{E}_{C^*(X)}\big)$$

determines the structure of an $E^\Delta \otimes E$-coalgebra (respectively, of an $E^\Delta \otimes E$-algebra) on $C_*(X)$ (on $C^*(X)$).

We denote the operad $E^\Delta \otimes C_*(\mathcal{E}_n)$ simply by E_n and call it the little n-cubes operad. The complex $C_*(X)$ can be regarded as an E_n-coalgebra, and $C^*(X)$ can be regarded as an E_n-algebra.

We need the following general property of algebras (coalgebras) over operads.

Theorem 1. *If $X_* = \{X_n\}$ is a simplicial object in the category of algebras over an operad $\mathcal{E}$, then its realization $|X_*|$ is an $\mathcal{E}$-algebra also. Dually, if $X^* = \{X^n\}$ is a cosimplicial object in the category of coalgebras over an operad $\mathcal{E}$, then its realization $|X^*|$ is an $\mathcal{E}$-coalgebra also.*

Proof. Suppose that $X_* = \{X_n\}$ be a simplicial object in the category of $\mathcal{E}$-algebras, and let $\mu_n\colon \mathcal{E}(X_n) \to X_n$ be an $\mathcal{E}$-algebra structure on X_n. The Eilenberg–Zilber maps

$$\psi\colon |X_*| \otimes \cdots \otimes |X_*| \to |X_* \otimes \cdots \otimes X_*|$$

commute with the actions of permutation groups and, therefore, induce maps

$$\psi\colon \mathcal{E}(j) \otimes_{\Sigma_j} |X_*|^{\otimes j} \to |\mathcal{E}(j) \otimes_{\Sigma_j} X_*^{\otimes j}|.$$

These maps determine a map $\psi\colon \mathcal{E}(|X_*|) \to |\mathcal{E}(X_*)|$, and the required map $\mathcal{E}(|X_*|) \to |X_*|$ is defined as the composition

$$\mathcal{E}(|X_*|) \xrightarrow{\psi} |\mathcal{E}(X_*)| \xrightarrow{\mu_*} |X_*|. \quad \square$$

Corollary. *The realization $B(\mathcal{E}, \mathcal{E}, X)$ of a simplicial resolution*

$$B_*(\mathcal{E}, \mathcal{E}, X)\colon \mathcal{E}(X) \leftarrow \mathcal{E}^2(X) \leftarrow \cdots \leftarrow \mathcal{E}^n(X) \leftarrow \cdots$$

over an $\mathcal{E}$-algebra X is an $\mathcal{E}$-algebra. Moreover, the augmentation $\eta\colon B(\mathcal{E}, \mathcal{E}, X) \to X$ is a chain equivalence. Dually, the realization $F(\mathcal{E}, \mathcal{E}, X)$ of a cosimplicial resolution

$$F^*(\mathcal{E}, \mathcal{E}, X)\colon \overline{\mathcal{E}}(X) \to \overline{\mathcal{E}}^2(X) \to \cdots \to \overline{\mathcal{E}}^n(X) \to \cdots$$

over an $\mathcal{E}$-coalgebra X is an $\mathcal{E}$-coalgebra. Moreover, the augmentation $\xi\colon X \to F(\mathcal{E}, \mathcal{E}, X)$ is a chain equivalence.

We proceed to construct the corresponding homotopy theories. Suppose that $\mathcal{E}$ is an operad and $\mathcal{E} \to \mathcal{E}^\Delta$ is a map of operads. This means that the chain complexes Δ^n have $\mathcal{E}$-coalgebra structures compatible with the coface and codegeneracy operators. This requirement is quite natural for homotopy theories. In particular, it allows us to define homotopy groups in these theories.

Let $\mathcal{A}_{\mathcal{E}}$ denote the category whose objects are $\mathcal{E}$-algebras and morphisms are maps of $\mathcal{E}$-algebras.

The category $\mathcal{A}_{\mathcal{E}}$ is a closed model category [6] in which the fibrations are surjective maps $p\colon X \to Y$, the weak equivalences are maps inducing isomorphisms in homology, and the cofibrations are maps $i\colon A \to B$ having the left lifting property with respect to the trivial fibrations. This means that, for any commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array},$$

there exists a diagonal map $f\colon B \to X$ preserving commutativity.

Dually, let $\mathcal{K}_{\mathcal{E}}$ denote the category whose objects are $\mathcal{E}$-coalgebras and morphisms are maps of $\mathcal{E}$-coalgebras.

The category $\mathcal{K}_{\mathcal{E}}$ is a closed model category in which cofibrations are injective maps $i\colon A \to B$, weak equivalences are maps inducing isomorphisms in homology, and fibrations are maps $p\colon X \to Y$ having the right lifting property with respect to the trivial cofibrations. This means that, for any commutative diagram of the above form, there exists a diagonal map $f\colon B \to X$ preserving commutativity.

Theorem 2. *For any trivial fibration $p\colon X \to Y$ in the category $\mathcal{A}_{\mathcal{E}}$, there exists a map of $\mathcal{E}$-algebras $\tilde{q}\colon B(\mathcal{E}, \mathcal{E}, Y) \to X$ such that*

$$p \circ \tilde{q} = \eta\colon B(\mathcal{E}, \mathcal{E}, Y) \to Y.$$

Proof. Let $p\colon X \to Y$ be a trivial fibration. This means that p is surjective and induces an isomorphism in homology. Hence there exists a chain map $q\colon Y \to X$ and a chain homotopy $h\colon X \to X$ such that

$$p \circ q = \mathrm{Id}, \qquad d(h) = q \circ p - \mathrm{Id}, \qquad p \circ h = 0, \quad h \circ q = 0, \quad h \circ h = 0.$$

Let us construct the required map of $\mathcal{E}$-algebras $\tilde{q}\colon B(\mathcal{E}, \mathcal{E}, Y) \to X$.

It is easy to see that to define such a map is the same thing as to define a family of maps of $\mathcal{E}$-algebras $q^n\colon \mathcal{E}^{n+1}(Y) \to \mathrm{Hom}(\Delta^n; X)$ for which the diagrams

$$\begin{array}{ccc} \mathcal{E}^n(Y) & \xrightarrow{q^{n-1}} & \mathrm{Hom}(\Delta^{n-1}; X) \\ {\scriptstyle s_i}\big\downarrow\big\uparrow{\scriptstyle d_i} & & {\scriptstyle s_i}\big\downarrow\big\uparrow{\scriptstyle d_i} \\ \mathcal{E}^{n+1}(Y) & \xrightarrow{q^n} & \mathrm{Hom}(\Delta^n; X) \end{array}$$

are commutative.

Defining a map of $\mathcal{E}$-algebras $q^n\colon \mathcal{E}^{n+1}(Y) \to \mathrm{Hom}(\Delta^n; X)$ is equivalent to defining a chain map $\bar{q}^n\colon \mathcal{E}^n(Y) \to \mathrm{Hom}(\Delta^n; X)$; thus, defining a map of $\mathcal{E}$-algebras $\tilde{q}\colon B(\mathcal{E}, \mathcal{E}, Y) \to X$ is equivalent to defining a family of chain maps $\bar{q}^n\colon \mathcal{E}^n(A) \to \mathrm{Hom}(\Delta^n; X)$ for which the corresponding maps f^n of $\mathcal{E}$-algebras give the commutative diagrams specified above.

We set $\bar{q}^0 = q\colon Y \to X$ and $\bar{q}^n = h \circ \mu \circ \mathcal{E}(h) \circ \mathcal{E}(\mu) \circ \dots \circ \mathcal{E}^{n-1}(\mu) \circ \mathcal{E}^n(q)$. A direct calculation shows that these maps satisfy the required relations. $\square$

Corollary. *If A is an $\mathcal{E}$-algebra, then the $\mathcal{E}$-algebra $B(\mathcal{E},\mathcal{E},A)$ is a cofibered object in the category $\mathcal{A}_{\mathcal{E}}$.*

Indeed, suppose that $p\colon X\to Y$ is a trivial fibration and $f\colon B(\mathcal{E},\mathcal{E},A)\to Y$ is a map of $\mathcal{E}$-algebras. Let us construct a map of $\mathcal{E}$-algebras $\tilde f\colon B(\mathcal{E},\mathcal{E},A)\to X$ for which $p\circ\widetilde{f}=f$.

Theorem 2 gives a map of $\mathcal{E}$-algebras $\tilde q\colon B(\mathcal{E},\mathcal{E},Y)\to X$. Since $\eta\colon B(\mathcal{E},\mathcal{E},A)\to A$ is a trivial fibration, there exists a map of $\mathcal{E}$-algebras $\psi\colon B(\mathcal{E},\mathcal{E},A)\to B(\mathcal{E},\mathcal{E},B(\mathcal{E},\mathcal{E},A))$. We define $\widetilde{q}$ as the composition

$$B(\mathcal{E},\mathcal{E},A)\xrightarrow{\psi}B(\mathcal{E},\mathcal{E},B(\mathcal{E},\mathcal{E},A))\xrightarrow{B(\mathcal{E},\mathcal{E},f)}B(\mathcal{E},\mathcal{E},Y)\xrightarrow{\tilde q}X.$$

The dual assertion is the following theorem.

Theorem 2′. *For any trivial cofibration $i\colon A\to B$ in the category $\mathcal{K}_{\mathcal{E}}$, there exists a map of $\mathcal{E}$-coalgebras $\tilde j\colon B\to F(\mathcal{E},\mathcal{E},A)$ for which*

$$\tilde j\circ i=\xi\colon A\to F(\mathcal{E},\mathcal{E},A).$$

Corollary. *If X is an $\mathcal{E}$-coalgebra, then the $\mathcal{E}$-coalgebra $F(\mathcal{E},\mathcal{E},X)$ is a fibered object in the category $\mathcal{K}_{\mathcal{E}}$.*

Consider the category $\widetilde{\mathcal{K}}_{\mathcal{E}}$ whose objects are $\mathcal{E}$-coalgebras and morphisms $f\colon X\to Y$ are maps of $\mathcal{E}$-coalgebras $\tilde f\colon X\to F(\mathcal{E},\mathcal{E},Y)$.

We say that two morphisms $f_0,f_1\colon X\to Y$ in the category $\widetilde{\mathcal{K}}_{\mathcal{E}}$ are *homotopic* and write $f\simeq g$ if there exists a morphism $h\colon\Delta^1\otimes X\to Y$, which is called a *homotopy*, such that

$$h|_{0\otimes X}=f_0,\qquad h|_{1\otimes X}=f_1.$$

Let $Ho\mathcal{K}_{\mathcal{E}}$ be the localization of the category $\mathcal{K}_{\mathcal{E}}$ with respect to the weak equivalences (i.e., morphisms inducing isomorphisms in homology).

By $\pi\mathcal{K}_{\mathcal{E}}$ we denote the category whose objects are $\mathcal{E}$-coalgebras and morphisms are the homotopy classes of morphisms of the category $\widetilde{\mathcal{K}}_{\mathcal{E}}$. The general theory of homotopy in categories [6] gives the following result.

Theorem 3. *The following equivalence of categories hold*:

$$Ho\mathcal{K}_{\mathcal{E}}\cong\pi\mathcal{K}_{\mathcal{E}}.$$

The dual assertion for $\mathcal{E}$-algebras is the following theorem.

Theorem 3′. *The following equivalence of categories hold*:

$$Ho\mathcal{A}_{\mathcal{E}}\cong\pi\mathcal{A}_{\mathcal{E}}.$$

Now, consider the problem of calculating homotopy groups of $\mathcal{E}$-coalgebras. Since the chain complexes Δ^n of the standard n-simplices are $\mathcal{E}$-coalgebras, it follows that the chain complexes S^n of the n-spheres are $\mathcal{E}$-coalgebras as well.

We define the homotopy groups $\pi_n^{\mathcal{E}}(X)$ of a $\mathcal{E}$-coalgebra X by $\pi_n^{\mathcal{E}}(X)=[S^n;F(\mathcal{E},\mathcal{E},X)]$, i.e., as the sets of homotopy classes of maps $f\colon S^n\to F(\mathcal{E},\mathcal{E},X)$ of $\mathcal{E}$-coalgebras.

Theorem 4. *For any $\mathcal{E}$-coalgebra X, there is a spectral sequence of homotopy groups $\pi_*^{\mathcal{E}}(X)$ in which the term E^1 is isomorphic to the cobar construction $F(\mathcal{E}_*,X_*)$, where $\mathcal{E}_*$ and X_* denote the homology of $\mathcal{E}$ and X, respectively.*

Proof. Consider the filtration

$$F(\mathcal{E},\mathcal{E},X)\supset F^1(\mathcal{E},\mathcal{E},X)\supset\cdots\supset F^m(\mathcal{E},\mathcal{E},X)\supset\cdots,$$

where $F^m(\mathcal{E},\mathcal{E},X)\colon \overline{\mathcal{E}}^m(X)\to\overline{\mathcal{E}}^{m+1}(X)\to\cdots$.

It induces a spectral sequence. The exact sequences

$$0\to F^{m+1}(\mathcal{E},\mathcal{E},X)\to F^m(\mathcal{E},\mathcal{E},X)\to\overline{\mathcal{E}}^{m+1}(X)\to 0$$

induce the isomorphisms

$$E^1_{n,m}=[S^n,\overline{\mathcal{E}}^{m+1}(X)]\cong H_n(\overline{\mathcal{E}}^m(X))$$

and, therefore, the isomorphism $E^1\cong F(\mathcal{E}_*,X_*)$. $\square$

If S^n is the trivial $\mathcal{E}$-coalgebra, then the differentials of this spectral sequence are determined by the differentials in the cobar construction $F(\mathcal{E},X)$; thus, we obtain the following result.

Theorem 5. *If S^n is a trivial $\mathcal{E}$-coalgebra, then, for any $\mathcal{E}$-coalgebra X, the following isomorphism holds*:

$$\pi_n^{\mathcal{E}}(X)\cong H_n(F(\mathcal{E},X)).$$

Now, suppose that X is a topological space and E_n is the little n-cubes operad. Note that if $m\ge n$, then S^m has the trivial structure of an E_n-coalgebra, which implies the following theorem.

Theorem 6. *If X is a topological space and $m\ge n$, then*

$$\pi_m^{E_n}(X)\cong H_m(F(E_n,C_*(X))).$$

The term E^1 of this spectral sequence can be expressed via the homology of the operad E_n and, therefore, of the Dyer–Lashof algebra [7, 8].

Theorem 7. *If X is a topological space, then the term E^1 of the spectral sequence of homotopy groups $\pi_*^{E_n}(X)$ is isomorphic to the module $S^nT_sR_{n-1}L_{n-1}S^{-n}H_*(X)$, where T_s is a free commutative algebra, R_{n-1} is the submodule of the Dyer–Lashof algebra generated by the admissible sequences of redundancy less than n, and L_{n-1} is the free Lie $(n-1)$-algebra.*

If X is an n-connected topological space, then the cobar construction $F(E_n,C_*(X))$ is chain equivalent to the n-fold suspension over the chain complex of the iterated loop space Ω^nX [8]. Thus, the following theorem is valid.

Theorem 8. *If X is an n-connected topological space, then*

$$\pi_*^{E_n}(X)\cong S^nH_*(\Omega^nX).$$

This theorem generalizes the result of Quillen [1, 6] asserting that the rational homotopy groups of a simply connected topological space can be expressed in terms of the homology of its loop space. It determines the upper bound on those m for which the homotopy groups (over the operad E_m) of an n-connected space X are related directly to the homology of the m-fold loop space over X.

The method suggested above makes it possible to reduce the very difficult problem of calculating the homotopy groups of spheres over the operad E_∞ to calculating the homotopy groups of spheres over the operads E_n.

In particular, it follows from the theorem proved in this paper that if $m\le n$, then the homotopy groups (over the operad E_m) of the sphere S^n are isomorphic to the m-fold suspension over the homology of the m-fold loop space over S^n. In this author's opinion, it would be interesting to calculate the homotopy groups of the sphere S^n over the operads E_n, E_{n+1}, etc.

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REFERENCES

1. D. Quillen, "Rational homotopy theory," *Ann. of Math.*, **90** (1969), no. 2, 205–295.
2. V. A. Smirnov, "On cochain complexes of topological spaces," *Mat. Sb.* [*Math. USSR-Sb.*], **115** (1981), 146–158.
3. V. A. Smirnov, "The homotopy theory of coalgebras," *Izv. Akad. Nauk SSSR Ser. Mat.* [*Math. USSR-Izv.*], **49** (1985), 1302–1321.
4. J. P. May, *The Geometry of Iterated Loop Spaces*, Lecture Notes in Math., vol. 271, Springer-Verlag, New York, 1972.
5. J. M. Boardman and R. M. Vogt, *Homotopy Invariant Algebraic Structures on Topological Spaces*, Lecture Notes in Math., vol. 347, Springer-Verlag, New York, 1973.
6. D. Quillen, *Homotopical Algebra*, Lecture Notes in Math., vol. 43, Springer-Verlag, New York, 1967.
7. E. Dyer and R. Lashof, "Homology of iterated loop spaces," *Amer. J. Math.*, **84** (1962), 35–88.
8. V. A. Smirnov, "The homology of iterated loop spaces," *Forum Mathematicum*, **14** (2002), 345–381.

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