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The A_∞-structures and differentials of the Adams spectral sequence

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Abstract. Using operad methods and functional homology operations, we obtain inductive formulae for the differentials of the Adams spectral sequence of stable homotopy groups of spheres.

The Adams spectral sequence was invented by Adams [1] almost fifty years ago for the calculation of stable homotopy groups of topological spaces (in particular, those of spheres). The calculation of the differentials of the Adams spectral sequence of homotopy groups of spheres is one of the most difficult problems of modern algebraic topology. Here we consider an approach to the solution of this problem based on the use of the A_∞-structures introduced by Stasheff [2], operad methods [3]–[6], and functional homology operations [7]–[9]. We apply our results to the Arf invariant problem [10], [11].

§ 1. The Adams and Bousfield–Kan spectral sequences

Let us recall that the E^1 term of the Adams spectral sequence of stable homotopy groups of a topological space Y with coefficients in $\mathbb{Z}/2$ is the complex

$$F(\mathcal{K}, Y_*)\colon Y_* \to \mathcal{K}\otimes Y_* \to \cdots \to \mathcal{K}^{\otimes n}\otimes Y_* \to \cdots,$$

where Y_* is the homology of Y and $\mathcal{K}$ is the Milnor coalgebra (dual to the Steenrod algebra), which is the algebra of polynomials in ξ_i of dimension 2^i-1. The comultiplication

$$\nabla\colon \mathcal{K}\to\mathcal{K}\otimes\mathcal{K}$$

is defined on the generators ξ_i by the formula

$$\nabla(\xi_i)=\sum_k \xi_{i-k}^{2^k}\otimes\xi_k,$$

and on other elements by the Hopf relations.

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In the case of stable homotopy groups of spheres (that is, $Z = S^0$), the E^1 term of the Adams spectral sequence can be written as

$$F(\mathcal{K})\colon \mathbb{Z}/2 \to \mathcal{K} \to \cdots \to \mathcal{K}^{\otimes n} \to \cdots .$$

Consider the Bousfield–Kan spectral sequence [12], which is the most general case of a spectral sequence of homotopy groups of topological spaces.

Let R be a field. If Z is a simplicial set, then we denote by RZ the free simplicial R-module generated by Z. There is a cosimplicial resolution

$$R^*Z\colon RZ \xrightarrow{\delta^0,\delta^1} R^2Z \to \cdots \to R^nZ \xrightarrow{\delta^0,\dots,\delta^n} R^{n+1}Z \to \cdots ,$$

which was used by Bousfield and Kan to construct a spectral sequence of homotopy groups of Z with coefficients in R.

The E^1 term of this spectral sequence is the complex

$$H_*(Z;R) \to H_*(RZ;R) \to \cdots \to H_*(R^{n-1}Z;R) \to H_*(R^nZ;R) \to \cdots .$$

The higher differentials are the homology operations

$$d_m\colon H_*(R^{n-1}Z;R) \to H_*(R^{n+m-1}Z;R).$$

In [7], [8] homology operations were defined as partial and multivalued maps. However, there is a general method that enables us to make them single-valued maps defined everywhere. The corresponding theory was developed in [9]. Let us recall the basic definitions.

If X is a chain complex, then we denote its homology by X_*, $X_* = H_*(X)$. Let us fix chain maps $\xi\colon X_* \to X$, $\eta\colon X \to X_*$ and a chain homotopy $h\colon X \to X$ satisfying the relations

$$\eta\circ\xi = \mathrm{Id}, \qquad d(h) = \xi\circ\eta - \mathrm{Id}, \qquad h\circ\xi = 0, \qquad \eta\circ h = 0, \qquad h\circ h = 0.$$

For any sequence $f^1\colon X^1 \to X^2, \dots, f^n\colon X^n \to X^{n+1}$ of maps of chain complexes we define functional homology operations

$$H_*(f^n,\dots,f^1)\colon X^1_* \to X^{n+1}_*$$

by the formula

$$H_*(f^n,\dots,f^1) = \eta\circ f^n\circ h\circ\cdots\circ f^1\circ\xi.$$

A direct calculation shows that

$$\sum_{i=1}^{n-1}(-1)^{n-i+1}H_*(f^n,\dots,f^{i+1}\circ f^i,\dots,f^1) = \sum_{i=1}^{n-1}(-1)^{n-i}H_*(f^n,\dots,f^{i+1})\circ H_*(f^i,\dots,f^1).$$

Functional homology operations can be defined not only for the category of chain complexes but also, for example, for the category of simplicial modules.

The definition of the higher differentials of the Bousfield–Kan spectral sequence implies that they can be expressed in terms of the functional homology operations

$$H_*(\delta,\dots,\delta)\colon H_*(R^{n-1}Z;R) \to H_*(R^{n+m-1}Z;R),$$

and we have the following theorem.

Theorem 1. *The differentials of the Bousfield–Kan spectral sequence are given by the functional homology operations*

$$H_*(\delta, \dots, \delta)\colon H_*(R^{n-1}Z, R) \to H_*(R^{n+m-1}Z, R).$$

These operations define on the E^1 term a new differential that makes E^1 into a complex whose homology is isomorphic to the E^∞ term of the sequence.

It was shown in [5] that instead of the Bousfield–Kan cosimplicial resolution one can consider the cosimplicial object

$$F^*(C, RZ)\colon RZ \xrightarrow{\delta^0, \delta^1} CRZ \to \cdots \to C^{n-1}RZ \xrightarrow{\delta^0, \dots, \delta^n} C^n RZ \to \cdots,$$

where CRZ is the free commutative simplicial coalgebra generated by the simplicial module RZ.

The E^1 term of the corresponding spectral sequence is the complex

$$H_*(Z; R) = \pi_*(RZ) \to \pi_*(CRZ) \to \cdots \to \pi_*(C^{n-1}RZ) \to \pi_*(C^n RZ) \to \cdots.$$

The definition of the higher differentials of this sequence implies that they can be expressed in terms of the functional homology operations

$$H_*[\delta, \dots, \delta]\colon \pi_*(C^n RZ) \to \pi_*(C^{n+m}RZ).$$

There is a map of cosimplicial objects

$$\begin{array}{ccccccccc} RZ & \longrightarrow & R^2Z & \longrightarrow & \cdots & \longrightarrow & R^{n+1}Z & \longrightarrow & \cdots \\ {\scriptstyle =}\downarrow & & \downarrow & & & & \downarrow & & \\ RZ & \longrightarrow & CRZ & \longrightarrow & \cdots & \longrightarrow & C^nRZ & \longrightarrow & \cdots \end{array}$$

that induces an isomorphism of the corresponding spectral sequences, and we have the following theorem.

Theorem 2. *The differentials of the Bousfield–Kan spectral sequence of the cosimplicial object*

$$F^*(C, RZ)\colon RZ \xrightarrow{\delta^0, \delta^1} CRZ \to \cdots \to C^{n-1}RZ \xrightarrow{\delta^0, \dots, \delta^n} C^n RZ \to \cdots$$

are given by the functional homology operations

$$H_*(\delta, \dots, \delta)\colon \pi_*(C^n RZ) \to \pi_*(C^{n+m}RZ).$$

These operations define on the E^1 term a new differential that makes E^1 into a complex whose homology is isomorphic to the E^∞ term of the sequence.

§ 2. E_∞-algebras and E_∞-coalgebras

Let us recall that an *operad in the category of chain complexes* is defined to be a family $E = \{E(j)\}_{j\geqslant 1}$ of chain complexes $E(j)$ on which the symmetric groups Σ_j act, and maps

$$\gamma\colon E(k)\otimes E(j_1)\otimes\dots\otimes E(j_k)\to E(j_1+\dots+j_k)$$

are defined in such a way that certain assumptions of associativity and compatibility with the action of the symmetric groups hold [3], [4].

An operad $E=\{E(j)\}$ is called an E_∞-*operad* if the complexes $E(j)$ are acyclic and the symmetric groups act freely on them.

A chain complex X is called *an algebra* (*a coalgebra*) *over the operad* E if there are maps

$$\mu\colon E(k)\otimes_{\Sigma_k} X^{\otimes k}\to X \qquad (\tau\colon X\to \operatorname{Hom}_{\Sigma_k}(E(k);X^{\otimes k}))$$

such that certain associativity relations hold.

Algebras (coalgebras) over an E_∞-operad are called E_∞-*algebras* (E_∞-*coalgebras*).

Every operad in the category of chain complexes defines a monad $\underline{E}$ and a comonad $\overline{E}$ by the formulae

$$\underline{E}(X)=\sum_j \underline{E}(j,X), \qquad \underline{E}(j,X)=E(j)\otimes_{\Sigma_j} X^{\otimes j},$$
$$\overline{E}(X)=\prod_j \overline{E}(j,X), \qquad \overline{E}(j,X)=\operatorname{Hom}_{\Sigma_j}(E(j),X^{\otimes j}).$$

Any operad structure γ induces transformations

$$\underline{\gamma}\colon \underline{E}\circ\underline{E}\to\underline{E}, \qquad \overline{\gamma}\colon \overline{E}\to\overline{E}\circ\overline{E}.$$

Any algebra (coalgebra) structure over an operad E on a complex X induces a map

$$\mu\colon \underline{E}(X)\to X \qquad (\tau\colon X\to\overline{E}(X)).$$

Hence, giving an algebra (coalgebra) structure over an operad E on a chain complex X is the same as giving an algebra (coalgebra) structure over the monad $\underline{E}$ (the comonad $\overline{E}$).

The singular cochain complex $C^*(Y;R)$ of a topological space Y is one of the main examples of an E_∞-algebra.

Dually, the singular chain complex $C_*(Y;R)$ of a topological space Y and the chain complex $N(RZ)$ of a simplicial set Z are examples of E_∞-coalgebras.

For E_∞-coalgebras there is a homotopy theory [5] and homotopy groups are defined. For the chain complex $N(RZ)$ of a simplicial set Z, these homotopy groups are isomorphic to the homotopy groups of Z with coefficients in R.

Using the cosimplicial resolution

$$F^*(\overline{E},\overline{E},X)\colon X\xrightarrow{\tau}\overline{E}(X)\to\dots\to\overline{E}^{\,n-1}(X)\to\overline{E}^{\,n}(X)\to\cdots,$$

one can construct a spectral sequence of homotopy groups of any E-coalgebra X (see [5]).

Let X_* be the homology of the complex X and $\overline{E}_*$ that of the comonad $\overline{E}$. There is a cosimplicial resolution

$$F^*(\overline{E}_*, \overline{E}_*, X_*) : \overline{E}_*(X_*) \to \overline{E}_*^2(X_*) \to \cdots \to \overline{E}_*^n(X_*) \to \overline{E}_*^{n+1}(X_*) \to \cdots .$$

The E^1 term of the spectral sequence obtained from this resolution by choosing primitive elements is a complex

$$F(\overline{E}_*, X_*)\colon X_* \to \overline{E}_*(X_*) \to \cdots \to \overline{E}_*^n(X_*) \to \overline{E}_*^{n+1}(X_*) \to \cdots .$$

The functional homology operations

$$H_*(\delta, \ldots, \delta)\colon \overline{E}_*^n(X_*) \to \overline{E}_*^{n+m}(X_*)$$

define the higher differentials of this sequence. We denote the corresponding complexes by $\widetilde{F}(\overline{E}_*, \overline{E}_*, X_*)$ and $\widetilde{F}(\overline{E}_*, X_*)$, and we have the following theorem.

Theorem 3. *The differentials of the spectral sequence of homotopy groups of the E-coalgebra X are defined by the functional homology operations*

$$H_*(\delta, \ldots, \delta)\colon \overline{E}_*^n(X_*) \to \overline{E}_*^{n+m}(X_*).$$

The homology of the corresponding complex $\widetilde{F}(\overline{E}_*, X_*)$ is isomorphic to the E^∞ term of the sequence.

If X is the normalized chain complex of the simplicial set Z, that is, $X = N(RZ)$, then there is a map of cosimplicial objects

$$\begin{array}{ccccccccc} N(RZ) & \longrightarrow & N(CRZ) & \longrightarrow & \cdots & \longrightarrow & N(C^nRZ) & \longrightarrow & \cdots \\ {\scriptstyle =}\downarrow & & \downarrow & & & & \downarrow & & \\ X & \longrightarrow & \overline{E}(X) & \longrightarrow & \cdots & \longrightarrow & \overline{E}^n(X) & \longrightarrow & \cdots \end{array}$$

that induces an isomorphism of the corresponding spectral sequences, and we have the following theorem.

Theorem 4. *The differentials of the Bousfield–Kan spectral sequence of homotopy groups of the simplicial set Z are defined by the functional homology operations*

$$H_*(\delta, \ldots, \delta)\colon \overline{E}_*^n(Z_*) \to \overline{E}_*^{n+m}(Z_*).$$

The homology of the corresponding complex $\widetilde{F}(\overline{E}_*, Z_*)$ is isomorphic to the E^∞ term of the sequence.

Let us note that the suspension SX over the E-coalgebra X is an SE-coalgebra and we have the following commutative diagrams.

$$\begin{array}{ccc} \overline{E} & \xrightarrow{\overline{\gamma}} & \overline{E} \circ \overline{E} \\ \downarrow & & \downarrow \\ \overline{SE} & \xrightarrow{\overline{S\gamma}} & \overline{SE} \circ \overline{SE} \end{array} \qquad \begin{array}{ccc} SX & \xrightarrow{\tau} & \overline{SE}(SX) \\ {\scriptstyle =}\downarrow & & \downarrow \\ SX & \xrightarrow{S\tau} & S(\overline{E}(X)) \end{array}$$

Moreover, the expression for the homology $\overline{E}_*$ of the comonad $\overline{E}$ implies that the maps $\xi\colon \overline{E}_* \to \overline{E}$, $\eta\colon \overline{E} \to \overline{E}_*$ and $h\colon \overline{E} \to \overline{E}$ can be chosen in such a way that they commute with the suspension homomorphism $\overline{SE} \to \overline{E}$. Then the resulting functional homology operations commute with the suspension homomorphism.

Stabilizing the Bousfield–Kan spectral sequence, we obtain the Adams spectral sequence of stable homotopy groups of a topological space, and we have the following theorem.

Theorem 5. *The functional homology operations defining the higher differentials of the Bousfield–Kan spectral sequence commute with the suspension homomorphism and induce the differentials of the Adams sequence.*

§ 3. A_∞-cosimplicial objects

Let us describe the higher differentials of the Bousfield–Kan spectral sequence using the definition of A_∞-cosimplicial objects.

A family of objects $X^* = \{X^n\}_{n\geqslant 0}$ of a category will be called a *precosimplicial object* if coface and codegeneracy operators

$$\begin{aligned} \delta^i\colon X^n &\to X^{n+1}, \qquad 1 \leqslant i \leqslant n+1, \\ \sigma^i\colon X^n &\to X^{n-1}, \qquad 0 \leqslant i \leqslant n-1, \end{aligned}$$

are given satisfying the relations

$$\begin{aligned} \delta^j\delta^i &= \delta^i\delta^{j-1}, \qquad i<j, \\ \sigma^j\sigma^i &= \sigma^i\sigma^{j+1}, \qquad i\leqslant j, \\ \sigma^j\delta^i &= \begin{cases} \delta^i\sigma^{j-1}, & i<j, \\ \mathrm{Id}, & i=j, \quad i=j+1, \\ \delta^{i-1}\sigma^j, & i>j+1. \end{cases} \end{aligned}$$

Hence, precosimplicial objects differ from cosimplicial objects only in the coface operator δ^0: cosimplicial objects have such operators, whereas precosimplicial objects do not.

A *map* $f^*\colon X \to Y$ *of precosimplicial objects* is a family $f^* = \{f^n\}_{n\geqslant 0}$ of maps $f^n\colon X^n \to Y^n$ commuting with the coface and codegeneracy operators:

$$\delta^i f_n = f^{n+1}\delta^i, \qquad \sigma^i f^n = f_{n-1}\sigma^i.$$

We shall define A_∞-cosimplicial objects (or A_∞-cosimplicial spaces) in the category of topological spaces.

Let I^n be the unit cube, $I^n = \{(t_1,\dots,t_n) \mid 0 \leqslant t_i \leqslant 1\}$. We denote by

$$\begin{aligned} u_i^\varepsilon\colon I^n &\to I^{n+1}, \qquad \varepsilon = 0,1, \quad 1\leqslant i\leqslant n+1, \\ v_i\colon I^n &\to I^{n-1}, \qquad 0\leqslant i\leqslant n, \end{aligned}$$

the maps defined by the formulae

$$u_i^\varepsilon(t_1,\dots,t_n) = (t_1,\dots t_{i-1},\varepsilon,t_i,\dots,t_n),$$
$$v_i(t_1,\dots,t_n) = \begin{cases} (t_2,\dots,t_n), & i=0,\\ (t_1,\dots,t_i * t_{i+1},\dots,t_n), & 1\leqslant i\leqslant n-1,\\ (t_1,\dots,t_{n-1}), & i=n,\end{cases}$$

where $t_i * t_{i+1} = t_i + t_{i+1} - t_i\cdot t_{i+1}$.

A precosimplicial object $X^* = \{X^n\}$ in the category of topological spaces will be called an A_∞-*cosimplicial object* (or an A_∞-*cosimplicial space*) if coface operators

$$\delta_m^0\colon X^n\times I^m\to X^{n+m+1}$$

are given satisfying the relations

$$\begin{aligned} \sigma^0\delta_0^0 &= \mathrm{Id}, && \\ \delta_m^0(1\times u_i^0) &= \delta^i\delta_{m-1}^0, && 1\leqslant i\leqslant m,\\ \delta_m^0(1\times u_i^1) &= \delta_{i-1}^0(\delta_{m-1}^0\times 1), && 1\leqslant i\leqslant m,\\ \delta_{m-1}^0(1\times v_i) &= \sigma^i\delta_m^0, && 0\leqslant i\leqslant m,\quad m\geqslant 1,\\ \delta_m^0(\delta^i\times 1) &= \delta^{i+m+1}\delta_m^0, && i\geqslant 1,\\ \delta_m^0(\sigma^i\times 1) &= \sigma^{i+m+1}\delta_m^0, && i\geqslant 0.\end{aligned}$$

It is clear that any cosimplicial object $X^* = \{X^n\}$ in the category of topological spaces can be regarded as an A_∞-cosimplicial object with the trivial operators $\delta_m^0\colon X^n\times I^m\to X^{n+m+1}$ for $m\geqslant 1$.

Let us note that the family $I^* = \{I^n\}$ can be regarded as an A_∞-cosimplicial object in the category of topological spaces for which

$$\begin{aligned} \delta^i &= u_i^0\colon I^n\to I^{n+1}, \quad 1\leqslant i\leqslant n+1,\\ \delta_m^0 &= u_{n+1}^1\colon I^n\times I^m = I^{n+m}\to I^{n+m+1},\\ \sigma^i &= v_i\colon I^n\to I^{n-1}, \quad 0\leqslant i\leqslant n.\end{aligned}$$

The A_∞-cosimplicial objects form a category whose morphisms are the maps $f^*\colon X^*\to Y^*$ preserving the A_∞-structure.

We can also define A_∞-maps. Let $X^* = \{X^n\}$ be an A_∞-cosimplicial space and $Y^* = \{Y^n\}$ a cosimplicial space. An A_∞-*map from* X^* *to* Y^* is a map $f^* = \{f^n\}$, $f^n\colon X^n\to Y^n$, of precosimplicial objects together with a family of maps

$$f_m^n\colon X^n\times I^m\to Y^{n+m}, \quad 0\leqslant m\leqslant n,$$

such that

$$\begin{aligned} f_0^n &= f^n, \\ f_m^n(1\times u_i^0) &= \delta^{i-1} f_{m-1}^n, && 1\leqslant i\leqslant m, \\ f_m^n(1\times u_i^1) &= f_i^{n+m-i-1}(\delta_{m-i}^0\times 1), && 1\leqslant i\leqslant m, \\ f_{n-1}^m(d_i\times 1) &= d_{i-m} f_n^m, && i>m, \\ f_{m-1}^n(1\times v_i) &= \sigma^i f_m^n, && 0\leqslant i<m, \\ f_m^n(\delta^i\times 1) &= \delta^{i+m} f_m^{n-1}, && i\geqslant 1, \\ f_{m-1}^n(\sigma^i\times 1) &= \sigma^{i+m} f_n^m, && i\geqslant 0. \end{aligned}$$

An A_∞-map will be called an A_∞-*homotopy equivalence* if the corresponding maps f_n are homotopy equivalences.

Of course, A_∞-cosimplicial objects can be defined not only for the category of topological spaces but also for the category of simplicial sets, the category of chain complexes, and so on. To make the corresponding definitions, one uses analogues of the n-dimensional cube I^n in these categories.

Let us consider A_∞-cosimplicial objects in the category of chain complexes in more detail using normalized chain complexes of n-dimensional cubes.

A_∞-cosimplicial objects and A_∞-maps can be defined as follows. A precosimplicial object $X^* = \{X^n\}$ in the category of chain complexes is an A_∞-cosimplicial object if there are maps

$$\delta_m^0\colon X^n \to X^{n+m+1}$$

that increase the dimension by m satisfy the relations

$$\begin{aligned} d(\delta_m^0) &= \sum_{i=1}^n (-1)^{i-1}(\delta^i\delta_{m-1}^0 - \delta_{i-1}^0\delta_{m-i}^0), \\ \delta_m^0\delta^i &= \delta^{i+m+1}\delta_m^0, \quad i\geqslant 1, \\ \sigma^i\delta_m^0 &= \begin{cases} 0, & 0\leqslant i\leqslant m, \quad m\geqslant 1, \\ \mathrm{Id}, & i=m=0, \\ \delta_m^0\sigma_{i-m-1}, & i>m. \end{cases} \end{aligned}$$

An A_∞-map from an A_∞-cosimplicial object X^* to a cosimplicial object Y^* in the category of chain complexes is a map of precosimplicial objects $f^* = \{f^n\}$, $f^n\colon X^n\to Y^n$, together with a family of maps $f_m^n\colon X^n\to Y^{n+m}$ that increase the dimension by m and satisfy the relations

$$\begin{aligned} f_0^n &= f^n, \\ d(f_m^n) &= \sum_{i=1}^n (-1)^{i-1}(\delta^{i-1} f_{m-1}^n - f_i^{n+m-i-1}\delta_{m-i}^0), \\ f_m^{n+1}\delta^i &= \delta^{i+m} f_m^n, \quad i\geqslant 1, \\ \sigma^i f_m^n &= \begin{cases} 0, & i<m, \\ f_m^{n+1}\sigma^{i-m}, & i\geqslant m. \end{cases} \end{aligned}$$

It is clear that any cosimplicial object $X^* = \{X^n\}$ in the category of chain complexes can be regarded as an A_∞-cosimplicial object with the trivial operators $\delta^0_m : X^n \otimes I^m \to X^{n+m+1}$ for $m \geqslant 1$.

As well as for the category of topological spaces, the family of chain complexes $I^* = \{I^n\}$ can be regarded as an A_∞-cosimplicial object in the category of chain complexes.

If X'^* and X''^* are A_∞-cosimplicial objects, then their tensor product $X'^* \otimes X''^*$ is an A_∞-cosimplicial object X^* such that

$$\begin{aligned} X^n &= X'^n \otimes X''^n, \\ \delta^i &= \delta'^i \otimes \delta''^i, \quad i \geqslant 1, \\ \sigma^i &= \sigma'^i \otimes \sigma''^i, \quad i \geqslant 0, \end{aligned}$$

and the operations $\delta^0_m : X^n \to X^{n+m+1}$ are the composites

$$X'^n \otimes X''^n \otimes I^m \xrightarrow{1\otimes 1\otimes \nabla} X'^n \otimes X''^n \otimes I^m \otimes I^m \xrightarrow{T}$$
$$\xrightarrow{T} X'^n \otimes I^m \otimes X''^n \otimes I^m \xrightarrow{\delta'^0_m \otimes \delta''^0_m} X'^{n+m+1} \otimes X''^{n+m+1},$$

where $\nabla : I^m \to I^m \otimes I^m$ is the comultiplication in the coalgebra I^m.

If X^* is an A_∞-cosimplicial object in the category of chain complexes, then we define its realization $|X^*|$ by the formula

$$|X^*| = \mathrm{Hom}(I^*; X^*),$$

where Hom is taken in the category of A_∞-cosimplicial objects.

For A_∞-cosimplicial objects X'^* and X''^* there is a chain equivalence

$$|X'^* \otimes X''^*| \simeq |X'^*| \otimes |X''^*|.$$

Let us transfer the perturbation theory of chain complexes [13] to A_∞-cosimplicial objects. We shall say that a chain complex $\widetilde{X}$ is a deformation retract of the chain complex X if there are chain maps $\xi : \widetilde{X} \to X$, $\eta : X \to \widetilde{X}$ and a chain homotopy $h : X \to X$ such that

$$\eta \circ \xi = \mathrm{Id}, \qquad d(h) = \mathrm{Id} - \xi \circ \eta.$$

It turns out [13] that in this case we can assume that $h \circ \xi = 0$, $\eta \circ h = 0$ and $h \circ h = 0$.

A precosimplicial object $\widetilde{X}^*$ in the category of chain complexes will be called a *deformation retract of the precosimplicial object* X^* if for every n the chain complex $\widetilde{X}^n$ is a deformation retract of the chain complex X^n and the corresponding chain maps and chain homotopies commute with the precosimplicial structure.

The following theorem is a cosimplicial analogue of the main lemma in the perturbation theory of chain complexes [13].

Theorem 6. *Let $X^* = \{X^n\}$ be a cosimplicial object in the category of chain complexes, and assume that the precosimplicial object $\widetilde{X}^* = \{\widetilde{X}^n\}$ is a deformation retract of X^*. Then $\widetilde{X}^*$ has the structure of an A_∞-cosimplicial object and there is an A_∞-cosimplicial homotopy equivalence between X^* and $\widetilde{X}^*$.*

Proof. Let $\xi^*\colon \widetilde{X}^* \to X^*$, $\eta^*\colon X^* \to \widetilde{X}^*$ and $h^*\colon X^* \to X^*$ be the corresponding maps. We define operators $\delta^0_m\colon \widetilde{X}^n \to \widetilde{X}^{n+m+1}$ by the formula

$$\delta^0_m = \eta\delta^0 h^{n+m} \dots \delta^0 h^{n+1} \delta_0 \xi.$$

A direct calculation shows that the desired relations hold.

We also define maps $\xi^n_m\colon \widetilde{X}^n \to X^{n+m}$ by the formula

$$\xi^n_m = h^{n+m}\delta^0 \dots h^{n+1}\delta^0\xi.$$

They give an A_∞-cosimplicial homotopy equivalence between $\widetilde{X}^*$ and X^*.

Theorem 7. *Let $f^*\colon X^* \to Y^*$ be a map of cosimplicial objects in the category of chain complexes, and let precosimplicial objects $\widetilde{X}^*$ and $\widetilde{Y}^*$ be deformation retracts of X^* and Y^*. Then for the above-defined structures of A_∞-cosimplicial objects on $\widetilde{X}^*$ and $\widetilde{Y}^*$ there is an A_∞-map $\widetilde{f}^*\colon \widetilde{X}^* \to \widetilde{Y}^*$ such that the following diagram is homotopy commutative.*

$$\begin{array}{ccc} X^* & \xrightarrow{f^*} & Y^* \\ {\scriptstyle \xi^*}\uparrow & & \uparrow{\scriptstyle \xi^*} \\ \widetilde{X}^* & \xrightarrow{\widetilde{f}^*} & \widetilde{Y}^* \end{array}$$

The proof is similar to that of the preceding theorem.

Since the E^1 term of the Bousfield–Kan spectral sequence, regarded as a precosimplicial object, is a deformation retract of the original cosimplicial object, Theorem 6 implies the following result.

Theorem 8. *The functional homology operations defining the higher differentials of the Bousfield–Kan spectral sequence of the space Y can be chosen in such a way that they form the structure of an A_∞-cosimplicial object on the complex $F^*(\overline{E}_*, Y_*)$.*

Passing to the stable case, we obtain the following theorem.

Theorem 9. *The functional homology operations defining the higher differentials of the Adams spectral sequence of stable homotopy groups of a topological space Y can be chosen in such a way that they form the structure of an A_∞-cosimplicial object.*

§ 4. An E_∞-structure on the Bousfield–Kan spectral sequence

Our purpose is to define an E_∞-structure on the Bousfield–Kan spectral sequence. To do this, we consider the following additional property of the E_∞-operad E.

Theorem 10. *For the E_∞-operad E there is a permutation map*

$$T\colon \underline{E}\circ\overline{E}\to\overline{E}\circ\underline{E}$$

such that the following diagrams are commutative.

$$\begin{array}{ccc} \underline{E}^2\circ\overline{E} & \xrightarrow{T\underline{E}\circ\overline{E}T} & \overline{E}\circ\underline{E}^2 \\ {\scriptstyle \underline{\gamma}\overline{E}}\downarrow & & \downarrow{\scriptstyle \overline{E}\underline{\gamma}} \\ \underline{E}\circ\overline{E} & \xrightarrow{T} & \overline{E}\circ\underline{E} \end{array} \qquad \begin{array}{ccc} \underline{E}\circ\overline{E} & \xrightarrow{T} & \overline{E}\circ\underline{E} \\ {\scriptstyle \underline{E}\overline{\gamma}}\downarrow & & \downarrow{\scriptstyle \overline{\gamma}\underline{E}} \\ \underline{E}\circ\overline{E}^2 & \xrightarrow{\overline{E}T\circ T\overline{E}} & \overline{E}^2\circ\underline{E} \end{array}$$

Proof. Since the E_∞-operad E is acyclic and free, we can construct a map $\nabla\colon E\to E\otimes E$ of operads that is a family of maps $\nabla(j)\colon E(j)\to E(j)\otimes E(j)$ defining a Hopf operad structure on E. We denote by

$$\nabla(j,i)\colon E(j)\to E(j)^{\otimes i}$$

the iterates of these maps, $\nabla(j,2)=\nabla(j)$. They are Σ_j-maps, that is, $\nabla(j)(x\sigma)=\nabla(j)(x)\sigma^{\otimes j}$, $\sigma\in\Sigma_j$, but they do not commute with permutations of the factors in $E(j)^{\otimes i}$. However, since E is acyclic and Σ-free, they can be extended to maps

$$\nabla(j,i)\colon E(i)\otimes E(j)\to E(j)^{\otimes i}$$

that are compatible with the action of the symmetric groups Σ_i and Σ_j. We write these maps as

$$\nabla(j,i)\colon E(j)\to\overline{E}(i)\otimes E(j)^{\otimes i}.$$

If the operad E is free, then we can assume that these maps are compatible with the operad structure.

Passing to dual maps, we obtain maps

$$\overline{\nabla}(j,i)\colon E(i)\otimes\overline{E}(j)^{\otimes i}\to\overline{E}(j).$$

Now we define

$$T(j,i)\colon E(j)\otimes\overline{E}(i)^{\otimes j}\to E(i)\otimes\overline{E}(j)^{\otimes i}$$

to be the composites

$$\begin{aligned} E(j)\otimes\overline{E}(i)^{\otimes j} &\xrightarrow{\nabla(j)\otimes 1^{\otimes j}} E(j)\otimes E(j)\otimes\overline{E}(i)^{\otimes j}\to E(j)\otimes\overline{E}(i)^{\otimes j}\otimes E(j) \\ &\xrightarrow{\overline{\nabla}(j,i)\otimes 1}\overline{E}(i)\otimes E(j)\xrightarrow{1\otimes\nabla(j,i)}\overline{E}(i)\otimes\overline{E}(i)\otimes E(j)^{\otimes i} \\ &\xrightarrow{\overline{\nabla}(i)\otimes 1^{\otimes i}}\overline{E}(i)\otimes E(j)^{\otimes i}. \end{aligned}$$

This family of maps $T(j,i)$ defines the desired permutation map $T\colon\underline{E}\circ\overline{E}\to\overline{E}\circ\underline{E}$.

A chain complex X is called an E_∞-Hopf algebra if an E_∞-algebra structure $\mu\colon \underline{E}(X) \to X$ and an E_∞-coalgebra structure $\tau\colon X \to \overline{E}(X)$ are given on it and these structures are compatible, that is, the following diagram is commutative.

$$\begin{array}{ccccc} \underline{E}(X) & \xrightarrow{\mu} & X & \xrightarrow{\tau} & \overline{E}(X) \\ \big\downarrow{\scriptstyle \underline{E}(\tau)} & & & & \big\downarrow{\scriptstyle =} \\ \underline{E}\overline{E}(X) & \xrightarrow{T} & \overline{E}\underline{E}(X) & \xrightarrow{\overline{E}(\mu)} & \overline{E}(X) \end{array}$$

In the case when the topological space Y is an E_∞-space, its singular chain complex $C_*(Y;R)$ is an E_∞-Hopf algebra. For example, the singular chain complexes of iterated loop spaces are E_∞-Hopf algebras.

If X is an E_∞-Hopf algebra, then there is a map of augmented cosimplicial objects

$$\begin{array}{ccccccccc} \underline{E}(X) & \longrightarrow & \underline{E}\overline{E}(X) & \longrightarrow & \cdots & \longrightarrow & \underline{E}\overline{E}^n(X) & \longrightarrow & \cdots \\ \big\downarrow & & \big\downarrow & & & & \big\downarrow & & \\ X & \longrightarrow & \overline{E}(X) & \longrightarrow & \cdots & \longrightarrow & \overline{E}^n(X) & \longrightarrow & \cdots, \end{array}$$

that is, $F^*(\overline{E}, \overline{E}, X)$ is a cosimplicial object in the category of E_∞-algebras. Therefore, its realization $F(\overline{E}, \overline{E}, X)$ is an E_∞-algebra. Passing to homology, we obtain the following theorem.

Theorem 11. *If X is an E_∞-Hopf algebra, then there is an E_∞-algebra structure on the complex $\widetilde{F}(\overline{E}_*, \overline{E}_*, X_*)$.*

We shall use this structure to calculate the higher differentials of the Adams spectral sequence.

Let us note that there is no natural E_∞-algebra structure on the complex $\widetilde{F}(\overline{E}_*, X_*)$.

§ 5. The homology of the E_∞-operad and the Milnor coalgebra

It is well known (see, for example, [6]) that for an E_∞-operad E and a graded module M (over the field $\mathbb{Z}/2$) the homology $\underline{E}_*(M)$ of the complex $\underline{E}(M)$ is the algebra of polynomials in the generators of the form $e_{i_1} \dots e_{i_k} x_m$, $1 \leqslant i_1 \leqslant \dots \leqslant i_k$, $x_m \in M$, of dimension $i_1 + 2i_2 + \dots + 2^{k-1} i_k + 2^k m$.

The elements $e_{i_1} \dots e_{i_k} x_m$ of $\underline{E}_*(M)$ can be written as

$$Q^{j_1} \dots Q^{j_k} \otimes x_m, \qquad j_1 \leqslant 2j_2, \dots, j_{k-1} \leqslant 2j_k, \qquad m \leqslant j_k,$$

where

$$\begin{gathered} j_k = i_k + m, \\ j_{k-1} = i_{k-1} + i_k + 2m, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ j_1 = i_1 + i_2 + 2i_3 + \dots + 2^{k-2} i_k + 2^{k-1} m. \end{gathered}$$

The sequences $Q^{j_1} \dots Q^{j_k}$ are elements of the Dyer–Lashof algebra $\mathcal{R}$ [14], [15].

If M is a graded module, then we denote by $\mathcal{R} \times M$ the factor module of the tensor product $\mathcal{R} \otimes M$ by the submodule generated by the elements of the form $Q^{j_1} \dots Q^{j_k} \otimes x_m$ with $j_k < m$. The correspondence $M \longmapsto \mathcal{R} \times M$ defines a monad in the category of graded modules.

A graded module M is called an *unstable module over the Dyer–Lashof algebra* if it is an algebra over the corresponding monad.

Dually, the homology $\overline{E}_*(M)$ of the complex $\overline{E}(M)$ is the free commutative coalgebra with the generators

$$e_{i_1} \dots e_{i_k} x^m, \qquad 1 \leqslant i_1 \leqslant \dots \leqslant i_k, \quad x^m \in M,$$

of dimension $2^k m - (i_1 + 2i_2 + \dots + 2^{k-1} i_k)$.

By regrading elements of $\overline{E}_*(M)$, we obtain the Milnor coalgebra $\mathcal{K}$. We define a grading $\deg(x)$, $x \in \mathcal{K}$, by putting $\deg(\xi_i) = 1$ and assuming that the degree of any product is equal to the sum of the degrees of its factors.

If M is a graded module, then we denote by $\mathcal{K} \times M$ the submodule of the tensor product $\mathcal{K} \otimes M$ generated by the elements $x \otimes y$ with $\deg(x) = \dim(y)$. The correspondence $M \longmapsto \mathcal{K} \times M$ defines a comonad $\mathcal{K}$ in the category of graded modules.

A graded module M is called an *unstable comodule over the Milnor coalgebra* if it is a coalgebra over the corresponding comonad. If M is an unstable comodule over the Milnor coalgebra, then there is a cosimplicial resolution

$$F^*(\mathcal{K}, \mathcal{K}, M) \colon M \to \mathcal{K} \times M \to \dots \to \mathcal{K}^{\times n-1} \times M \to \mathcal{K}^{\times n} \times M \to \cdots .$$

If the space Y is "good" (in the sense of Massey–Peterson), then the Bousfield–Kan spectral sequence becomes the Massey–Peterson spectral sequence and the E^1 term of this sequence can be written as

$$F^*(\mathcal{K}, Y_*) \colon Y_* \to \mathcal{K} \times Y_* \to \dots \to \mathcal{K}^{\times n} \times Y_* \to \cdots ,$$

where $Y_* = H_*(Y; \mathbb{Z}/2)$. Therefore, the following theorem holds.

Theorem 12. *If Y is a good space, then the functional homology operations defining the higher differentials of the Bousfield–Kan spectral sequence can be chosen in such a way that they induce the structure of an A_∞-cosimplicial object on $F^*(\mathcal{K}, \mathcal{K}, Y_*)$. The homology of the corresponding complex $\widetilde{F}(\mathcal{K}, Y_*)$ is isomorphic to the E^∞ term of the Massey–Peterson spectral sequence.*

Theorem 13. *If Y is an E_∞-space, then $\widetilde{F}(\mathcal{K}, \mathcal{K}, Y_*)$ is an E_∞-algebra.*

Let us note that there is no natural E_∞-algebra structure on the complex $\widetilde{F}(\mathcal{K}, Y_*)$.

Along with the Milnor coalgebra $\mathcal{K}$ we shall consider the stable Milnor coalgebra $\mathcal{K}_s$ for which $\xi_0 = 1$.

If M is a comodule over the stable Milnor coalgebra, then there is a cosimplicial resolution

$$F^*(\mathcal{K}_s, \mathcal{K}_s, M) \colon \mathcal{K}_s \otimes M \to \mathcal{K}_s^{\otimes 2} \otimes M \to \dots \to \mathcal{K}_s^{\otimes n} \otimes M \to \cdots .$$

By stabilizing the Bousfield–Kan spectral sequence, we obtain the Adams spectral sequence of stable homotopy groups of a topological space. The E^1 term of this sequence can be written as a cosimplicial object:

$$F^*(\mathcal{K}_s, Y_*)\colon Y_* \to \mathcal{K}_s^{\otimes} Y_* \to \cdots \to \mathcal{K}_s^{\otimes n} \otimes Y_* \to \cdots .$$

Therefore, the following theorem holds.

Theorem 14. *The functional homology operations defining the higher differentials of the Adams spectral sequence of stable homotopy groups of a topological space Y can be chosen in such a way that they form the structure of an A_∞-cosimplicial object on $F^*(\mathcal{K}_s, \mathcal{K}_s, Y_*)$. The homology of the corresponding complex $\widetilde{F}(\mathcal{K}_s, Y_*)$ is isomorphic to the E^∞ term of the Adams sequence.*

Let us calculate the E_∞-algebra structure on the Milnor coalgebra. As mentioned above, for the E_∞-operad E there is a permutation map

$$T\colon \underline{E} \circ \overline{E} \to \overline{E} \circ \underline{E}.$$

It induces a permutation map

$$T_*\colon \underline{E}_* \circ \overline{E}_* \to \overline{E}_* \circ \underline{E}_*$$

such that the following diagrams are commutative.

$$\begin{array}{ccc} \underline{E}_*^2 \circ \overline{E}_* & \xrightarrow{T_*\underline{E}_* \circ \overline{E}_* T_*} & \overline{E}_* \circ \underline{E}_*^2 \\ \downarrow{\scriptstyle \underline{\gamma}_* \overline{E}_*} & & \downarrow{\scriptstyle \overline{E}_* \underline{\gamma}_*} \\ \underline{E}_* \circ \overline{E}_* & \xrightarrow{T_*} & \overline{E}_* \circ \underline{E}_* \end{array} \qquad \begin{array}{ccc} \underline{E}_* \circ \overline{E}_* & \xrightarrow{T_*} & \overline{E}_* \circ \underline{E}_* \\ \downarrow{\scriptstyle \underline{E}_* \overline{\gamma}_*} & & \downarrow{\scriptstyle \overline{\gamma}_* \underline{E}_*} \\ \underline{E}_* \circ \overline{E}_*^2 & \xrightarrow{\overline{E}_* T_* \circ T_* \overline{E}_*} & \overline{E}_*^2 \circ \underline{E}_* \end{array}$$

The permutation map T_* induces an action $\mu_*\colon \underline{E}_* \circ \overline{E}_* \to \overline{E}_*$ and a dual coaction $\tau_*\colon \underline{E}_* \to \overline{E}_* \circ \underline{E}_*$.

We denote by $e_i\colon \mathcal{K} \to \mathcal{K}$ the operations on the Milnor coalgebra induced by μ_* on the e_i. The relations for the permutation map T_* imply that the following theorem holds.

Theorem 15. *The following relations hold for the operations $e_i\colon \mathcal{K} \to \mathcal{K}$:*

(i) $e_0(x) = x^2$,

(ii) $e_i(xy) = \sum e_k(x) e_{i-k}(y)$,

(iii) $\nabla e_i(x) = \sum \xi_0^{-k} e_{i-k}(\xi_0^k x') \otimes e_k(x'')$, *where* $\sum x' \otimes x'' = \nabla(x)$.

These relations enable us to calculate the e_i, since $e_1(\xi_0)$ can be calculated directly: $e_1(\xi_0) = \xi_1\xi_0$. Using relation (iii), we obtain the following theorem.

Theorem 16.

$$e_i(\xi_k) = \begin{cases} \xi_{m+k}\xi_k, & i = 2^{m+k} - 2^k, \\ \xi_{m+k}\xi_{k-1}, & i = 2^{m+k} - 2^k - 2^{k-1}, \\ \dots\dots\dots\dots\dots\dots & \\ \xi_{m+k}\xi_0, & i = 2^{m+k} - 2^k - \cdots - 1, \\ 0 & \textit{otherwise}. \end{cases}$$

Using relation (ii) in Theorem 15, we can obtain formulae for the operations e_i on products of the ξ_k.

Passing from e_i to the elements of the Dyer–Lashof algebra, we obtain that the action of the latter on the Milnor coalgebra on the ξ_i is defined by the formula

$$Q^{i+2^k-1}(\xi_k) = \begin{cases} \xi_{m+k}\xi_k, & i = 2^{m+k} - 2^k, \\ \xi_{m+k}\xi_{k-1}, & i = 2^{m+k} - 2^k - 2^{k-1}, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \xi_{m+k}\xi_0, & i = 2^{m+k} - 2^k - \dots - 1, \\ 0 & \text{otherwise}. \end{cases}$$

On other elements this action is defined using the Hopf relations

$$Q^i(xy) = \sum Q^k(x)Q^{i-k}(y).$$

In addition to the action of the Dyer–Lashof algebra, we can define $\cup_i$-products and an E_∞-algebra structure on the Milnor coalgebra $\mathcal{K}$ by putting

$$x \cup_i x = e_i(x)$$

for $x \in \mathcal{K}$. Let us note that on the stable Milnor coalgebra $\mathcal{K}_s$ there is no action of the Dyer–Lashof algebra and no E_∞-algebra structure.

§ 6. Functorial homology operations

Let $\Delta^* = \{\Delta^n\}$ be a cosimplicial object in the category of chain complexes consisting of chain complexes of standard n-dimensional simplexes, and let F be a functor on the category of chain complexes such that there are transformations

$$\Delta^n \otimes F(X) \to F(\Delta^n \otimes X)$$

commuting with the coface and codegeneracy operators. Such a functor F is called a *chain functor*.

A *transformation* $\alpha\colon F' \to F''$ *of chain functors* is defined to be a transformation of functors such that the following diagram is commutative.

$$\begin{array}{ccc} \Delta^n \otimes F'(X) & \longrightarrow & F'(\Delta^n \otimes X) \\ {\scriptstyle 1\otimes\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ \Delta^n \otimes F''(X) & \longrightarrow & F''(\Delta^n \otimes X) \end{array}$$

If F is a chain functor, then we can consider the maps

$$F(f)\colon F(X) \to F(Y)$$

induced by the chain maps $f\colon X \to Y$ of dimension n (not only zero). This can be done by representing the map $f\colon X \to Y$ of dimension n as the restriction of the chain map $\widetilde{f}\colon \Delta^n \otimes X \to Y$ to the n-dimensional generator $u_n \in \Delta^n$. Then the desired map

$$F(f)\colon F(X) \to F(Y)$$

of dimension n is the restriction of the composite

$$\Delta^n \otimes F(X) \to F(\Delta^n \otimes X) \xrightarrow{F(\widetilde{f})} F(Y)$$

to the n-dimensional generator $u_n \in \Delta^n$.

If F is a chain functor, then we denote by F_* the functor assigning to the chain complex X the graded homology module $F_*(X) = H_*(F(X))$. The functor F_* is an A_∞-functor, that is, there are operations assigning to a sequence of maps of chain complexes $f^1\colon X^1 \to X^2, \dots, f^n\colon X^n \to X^{n+1}$ the map

$$F_*(f^n, \dots, f^1) = H_*(F(f^n), \dots, F(f^1))\colon F_*(X^1) \to F_*(X^{n+1})$$

of dimension $n-1$.

Theorem 17. *If F is a chain functor and $f^1\colon X^1 \to X^2, \dots, f^n\colon X^n \to X^{n+1}$ is a sequences of maps of chain complexes, then*

$$H_*(F(f^n), \dots, F(f^1)) = \sum (-1)^\varepsilon F_*(H_*(f^n, \dots, f^{n_m+1}), \dots, H_*(f^{n_1}, \dots, f^1)),$$

where the sum is taken over all m and $n_1, \dots, n_m$ such that $1 \leqslant n_1 < \dots < n_m < n$.

Proof. For any chain complex X the map of choice of representatives $F_*(X_*) \to F(X)$ can be given as the composite of the maps

$$\xi(F)\colon F_*(X_*) \to F(X_*), \qquad F(\xi)\colon F(X_*) \to F(X).$$

The projection $F(X) \to F_*(X_*)$ can be put equal to the composite of the maps

$$F(\eta)\colon F(X) \to F(X_*), \qquad \eta(F)\colon F(X_*) \to F_*(X_*).$$

The homotopy $H\colon F(X) \to F(X)$ can be put equal to the sum

$$F(\xi) \circ h(F) \circ F(\eta) + F(h).$$

Substituting these maps into the formula for the functional homology operations, we obtain the desired identity.

Any transformation $\alpha\colon F' \to F''$ of chain functors induces an A_∞-transformation of the A_∞-functor F'_* into an A_∞-functor F''_*, that is, there are operations assigning to any sequence

$$f^1\colon X^1 \to X^2, \quad \dots, \quad f^n\colon X^n \to X^{n+1}$$

of maps of chain complexes the map

$$\alpha_*(f^n, \dots, f^1)\colon F'_*(X^1) \to F''_*(X^{n+1})$$

of dimension n.

The proof of the following theorem is similar to that of Theorem 17.

Theorem 18. *If $\alpha\colon F' \to F''$ is a transformation of chain functors and $f^1\colon X^1 \to X^2, \dots, f^n\colon X^n \to X^{n+1}$ is a sequence of maps of chain complexes, then*

$$\alpha_*(f^n,\dots,f^1) = \sum (-1)^\varepsilon \alpha_*\bigl(H_*(f^n,\dots,f^{n_m+1}),\dots,H_*(f^{n_1},\dots,f^1)\bigr),$$

where the sum is taken over all m and $n_1,\dots,n_m$ such that $1 \leqslant n_1 < \dots < n_m < n$.

A functor F is said to be *formal* if the homotopy $h(F)$ can be chosen in such a way that $h''(F)\circ F(f) = F(f)\circ h'(F)$ for any map $f\colon M' \to M''$ of graded modules.

This relation implies, in particular, that

$$\eta''(F)\circ F(f) = F_*(f)\circ\eta'(F), \qquad F(f)\circ\xi'(F) = \xi''(F)\circ F_*(f).$$

The definition of functorial homology operations implies that for a formal functor F restricted to the category of graded modules the A_∞-functor structure on F_* is degenerate. In this case the following formula holds for the maps $f^1,\dots,f^n$ of chain complexes:

$$F_*(f^n,\dots,f^1) = F_*(H_*(f^n,\dots,f^1)).$$

A transformation $\alpha\colon F' \to F''$ of chain functors is said to be *formal* if the homotopies $h(F')$ and $h(F'')$ can be chosen in such a way that

$$h''(F)\circ\alpha = \alpha\circ h'(F).$$

If $\alpha\colon F' \to F''$ is a formal transformation, then the structure of the A_∞-transformation from F'_* to F''_* is degenerate. In this case the following formula holds for the maps $f^1,\dots,f^n$ of chain complexes:

$$\alpha_*(f^n,\dots,f^1) = \alpha_*(H_*(f^n,\dots,f^1)).$$

§ 7. Homology operations for the E_∞-operad

We claim that the functors $\underline{E}$ and $\overline{E}$ corresponding to the E_∞-operad E are chain functors. To prove this, we define a family of maps

$$\Delta^n \otimes \underline{E}(j,X) \to \underline{E}(j,\Delta^n\otimes X)$$

to be the composites

$$\Delta^n \otimes \underline{E}(j,X) = \Delta^n\otimes E(j)\otimes_{\Sigma_j} X^{\otimes j} \xrightarrow{1\otimes\nabla\otimes 1} \Delta^n\otimes E(j)\otimes E(j)\otimes_{\Sigma_j} X^{\otimes j}$$
$$\xrightarrow{\tau\otimes 1\otimes 1} \Delta^{n\otimes j}\otimes E(j)\otimes_{\Sigma_j} X^{\otimes j} \to E(j)\otimes_{\Sigma_j}(\Delta^n\otimes X)^{\otimes j} = \underline{E}(j,\Delta^n\otimes X),$$

where $\tau\colon \Delta^n\otimes E(j) \to \Delta^{n\otimes j}$ is the E-coalgebra structure on the complex Δ^n. A direct verification shows that the desired relations hold.

In a similar way, we define the maps

$$\Delta^n\otimes\overline{E}(j,X) \to \overline{E}(j,\Delta^n\otimes X),$$

or, which is the same, the maps

$$E(j)\otimes\Delta^n\otimes \operatorname{Hom}_{\Sigma_j}(E(j),X^{\otimes j}) \to (\Delta^n\otimes X)^{\otimes j}$$

to be the composites

$$E(j)\otimes\Delta^n\otimes\operatorname{Hom}_{\Sigma_j}(E(j),X^{\otimes j}) \xrightarrow{\nabla\otimes 1\otimes 1} E(j)\otimes E(j)\otimes\Delta^n\otimes\operatorname{Hom}_{\Sigma_j}(E(j),X^{\otimes j})$$
$$\to E(j)\otimes\Delta^{n\otimes j}\otimes\operatorname{Hom}_{\Sigma_j}(E(j),X^{\otimes j}) \to (\Delta^n\otimes X)^{\otimes j}.$$

A direct verification shows that the desired relations hold, and we have the following theorem.

Theorem 19. *$\underline{E}_*$ and $\overline{E}_*$ are A_∞-functors.*

Our purpose is to calculate the functional homology operations for $\underline{E}_*$ and $\overline{E}_*$. This means that for every sequence

$$f^1\colon X^1 \to X^2, \quad \dots, \quad f^n\colon X^n \to X^{n+1}$$

of maps of chain complexes we have to calculate the maps

$$\underline{E}_*(f^n,\dots,f^1)\colon \underline{E}_*(X^1) \to \underline{E}_*(X^{n+1}), \qquad \overline{E}_*(f^n,\dots,f^1)\colon \overline{E}_*(X^1) \to \overline{E}_*(X^{n+1}).$$

We first consider the functor $\underline{E}(2,-)$ that assigns to any complex X the complex

$$\underline{E}(2,X) = E(2) \otimes_{\Sigma_2} X \otimes X,$$

where $E(2)$ is a Σ_2-free and acyclic complex with the generators e_i of dimension i and the differential is defined by the formula

$$d(e_i) = e_{i-1} + e_{i-1}T, \qquad T \in \Sigma_2.$$

The ground ring is the field $\mathbb{Z}/2$.

As mentioned above, the homology $\underline{E}_*(2,-)$ of this functor is not only a functor but an A_∞-functor, that is, to every sequence

$$f^1\colon X^1 \to X^2, \quad \dots, \quad f^n\colon X^n \to X^{n+1}$$

of maps of chain complexes a map

$$\underline{E}_*(2,f^n,\dots,f^1)\colon \underline{E}_*(2,X^1) \to \underline{E}_*(2,X^{n+1})$$

is assigned. Let us calculate these functional operations.

Note that for the chain complex X there is an isomorphism

$$\underline{E}_*(2,X) \cong \underline{E}_*(2,X_*).$$

If X_* is a graded module, then $\underline{E}_*(2,X_*)$ is the direct sum of the factor module $X_* \cdot X_*$ of the tensor product $X_* \otimes X_*$ with respect to permutations of the factors of elements and the module generated by the elements of the form $e_i \times y_n$, $i \geqslant 1$, of dimension $i+2n$. We shall denote $y_n \cdot y_n \in X_* \cdot X_*$ by $e_0 \times y_n$.

Let $\xi\colon X_* \to X$, $\eta\colon X \to X_*$ and $h\colon X \to X$ be the maps that realize the chain equivalence between X and X_*. We denote by

$$E(\xi)\colon E(2,X_*) \to E(2,X), \qquad E(\eta)\colon E(2,X) \to E(2,X_*),$$
$$E(h)\colon E(2,X) \to E(2,X)$$

the maps defined by the formulae

$$E(\xi)(e_i \otimes y_1 \otimes y_2) = e_i \otimes \xi(y_1) \otimes \xi(y_2),$$
$$E(\eta)(e_i \otimes x_1 \otimes x_2) = e_i \otimes \eta(x_1) \otimes \eta(x_2),$$
$$E(h)(e_i \otimes x_1 \otimes x_2) = e_i \otimes (x_1 \otimes h(x_2) + h(x_1) \otimes \xi\eta(x_2)) + e_{i-1} \otimes h(x_1) \otimes h(x_2).$$

It is clear that they realize a chain equivalence $\underline{E}(2,X) \simeq \underline{E}(2,X_*)$.

We shall now define maps

$$\xi(E)\colon \underline{E}_*(2,X_*) \to \underline{E}(2,X_*), \qquad \eta(E)\colon \underline{E}(2,X_*) \to \underline{E}_*(2,X_*),$$
$$h(E)\colon \underline{E}(2,X_*) \to \underline{E}(2,X_*).$$

For this, we choose an ordered basis $\{y\}$ of X_* (that is, for every n we choose a basis of the module $H_n(X)$). We define the map $\xi(E)$ by the formula

$$\xi(E)(e_i \times y) = e_i \otimes y \otimes y, \qquad \xi(E)(y_1 \cdot y_2) = e_0 \otimes (y_1 \otimes y_2), \qquad y_1 \leqslant y_2.$$

We define the map $\eta(E)$ by the formula

$$\eta(E)(e_i \otimes y_1 \otimes y_2) = \begin{cases} e_i \times y_1, & y_1 = y_2, \\ y_1 \cdot y_2, & y_1 < y_2, \quad i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

We define the map $h(E)$ by the formula

$$h(E)(e_i \otimes y_1 \otimes y_2) = \begin{cases} e_{i+1} \otimes y_2 \otimes y_1, & y_1 > y_2, \\ 0 & \text{otherwise.} \end{cases}$$

Direct calculations show that the desired relations hold.

The maps

$$E(\xi) \circ \xi(E)\colon E_*(2,X_*) \to E(2,X), \qquad \eta(E) \circ E(\eta)\colon E(2,X) \to E_*(2,X_*),$$
$$E(\xi) \circ h(E) \circ E(\eta) + E(h)\colon E(2,X) \to E(2,X)$$

realize a chain equivalence between $E(2,X)$ and $E_*(2,X_*)$.

The general formula for the functional homology operations for the chain functor implies that the following formula holds for the functor $\underline{E}(2,-)$:

$$\underline{E}_*(2,f^n,\dots,f^1) = \sum \underline{E}_*(2,H_*(f^n,\dots,f^{n_m+1}),\dots,H_*(f^{n_1},\dots,f^1)),$$

where the sum is taken over those m and $n_1,\dots,n_m$ for which $1 \leqslant n_1 < \dots < n_m < n$.

If X is a graded module with a fixed ordered basis $\{x_i\}$, then we define maps $p\colon X \otimes X \to X$, $q\colon X \to X \otimes X$ and $r\colon X \otimes X \to X \otimes X$ by the formulae $q(x_i) = x_i \otimes x_i$ and

$$p(x_i \otimes x_j) = \begin{cases} x_i, & i = j, \\ 0, & i \neq j, \end{cases} \qquad r(x_i \otimes x_j) = \begin{cases} x_j \otimes x_i, & i > j, \\ 0, & i \ngeqslant j. \end{cases}$$

If $f^1\colon X^1 \to X^2, \dots, f^n\colon X^n \to X^{n+1}$ is a sequence of maps of graded modules with fixed ordered bases, then we define a map $(f^n,\dots,f^1)\colon X^1 \to X^{n+1}$ by the formula

$$(f^n,\dots,f^1) = p \circ (f^n)^{\otimes 2} \circ r \circ (f^{n-1})^{\otimes 2} \circ \dots \circ r \circ (f^1)^{\otimes 2} \circ q.$$

The definition of homology operations implies that the following theorem holds.

Theorem 20. *If $f^1\colon X^1 \to X^2, \dots, f^n\colon X^n \to X^{n+1}$ is a sequence of maps of graded modules, then*

$$\underline{E}_*(2, f^n, \dots, f^1)(e_i \times x) = e_{i+n-1} \times (f^n, \dots, f^1)(x).$$

We can calculate the functional homology operations for the whole functor $\underline{E}_*$ using the monad structure $\underline{\gamma}_*\colon \underline{E}_* \circ \underline{E}_* \to \underline{E}_*$ and the formula

$$\underline{E}_*(f^n, \dots, f^1) \circ \gamma_* = \sum \underline{\gamma}_* \circ \underline{E}_*(\underline{E}_*(f^n, \dots, f^{n_m+1}), \dots, \underline{E}_*(f^{n_1}, \dots, f^1)),$$

where the sum is taken over those m and $n_1, \dots, n_m$ for which $1 \leqslant n_1 < \dots < n_m < n$.

Passing to the Dyer–Lashof algebra $\mathcal{R}$, we obtain operations

$$\mathcal{R}(f^n, \dots, f^1)\colon \mathcal{R} \times X^1 \to \mathcal{R} \times X^{n+1},$$

which can be calculated on the generators Q^i by the formulae

$$\mathcal{R}(f^n, \dots, f^1)(Q^i \otimes x) = Q^{i+n-1} \otimes (f^n, \dots, f^1)(x).$$

Dually, the following theorem holds for the functor $\overline{E}_*(2, -)$.

Theorem 21. *If $f^1\colon X^1 \to X^2, \dots, f^n\colon X^n \to X^{n+1}$ is a sequence of maps of graded modules, then*

$$\overline{E}_*(2, f^n, \dots, f^1)(\overline{e}_i \times x) = \overline{e}_{i-n+1} \times (f^n, \dots, f^1)(x).$$

We can calculate the functional homology operations for the whole functor $\overline{E}_*$ using the comonad structure $\overline{\gamma}_*\colon \overline{E}_* \to \overline{E}_* \circ \overline{E}_*$ and the formula

$$\overline{\gamma}_* \circ \overline{E}_*(f^n, \dots, f^1) = \sum \overline{E}_*(\overline{E}_*(f^n, \dots, f^{n_m+1}), \dots, \overline{E}_*(f^{n_1}, \dots, f^1)) \circ \overline{\gamma}_*,$$

where the sum is taken over those m and $n_1, \dots, n_m$ for which $1 \leqslant n_1 < \dots < n_m < n$.

Passing to the Milnor coalgebra $\mathcal{K}$, we obtain operations

$$\mathcal{K}(f^n, \dots, f^1)\colon \mathcal{K} \times X^1 \to \mathcal{K} \times X^{n+1},$$

which can be calculated by the formulae

$$\mathcal{K}(f^n, \dots, f^1)(y \otimes x) = y \cdot \xi_1^{n-1} \otimes (f^n, \dots, f^1)(x).$$

For the permutation transformation

$$T(X)\colon \underline{E}\overline{E}(X) \to \overline{E}\underline{E}(X)$$

there are homology operations assigning to every sequence of maps $f^1\colon X^1 \to X^2, \dots, f^n\colon X^n \to X^{n+1}$ of chain complexes the maps

$$T_*(f^n, \dots, f^1)\colon \underline{E}_*\overline{E}_*(X^1_*) \to \overline{E}_*\underline{E}_*(X^{n+1}_*).$$

Consider, for example, the operations connected with the comultiplication in the Milnor coalgebra $\mathcal{K}$. We put

$$\nabla(n) = \nabla \otimes 1 \otimes \cdots \otimes 1 - \cdots + (-1)^{n-1} 1 \otimes \cdots \otimes 1 \otimes \nabla \colon \mathcal{K}^{\times n} \to \mathcal{K}^{\times (n+1)}.$$

Direct calculations show that the operations

$$(\nabla(n), \ldots, \nabla) \colon \mathcal{K} \to \mathcal{K}^{\times (n+1)}, \quad n \geqslant 2,$$

are trivial on the $\xi_i^{2^k}$ but can be non-trivial on other elements. For example,

$$(\nabla(2), \nabla)(\xi_i \xi_j) = \xi_{j-i}^{2^i} \xi_0^{2^i} \otimes \xi_i \xi_0^{2^i} \otimes \xi_i, \quad i < j.$$

We denote by $\widetilde{\nabla}$ the comultiplication in the tensor product $\mathcal{K} \otimes \mathcal{K}$,

$$\widetilde{\nabla} = (1 \otimes T \otimes 1)(\nabla \otimes \nabla),$$

and put

$$\widetilde{\nabla}(n) = \widetilde{\nabla} \otimes 1 \otimes \cdots \otimes 1 - \cdots + (-1)^{n-1} 1 \otimes \cdots \otimes 1 \otimes \widetilde{\nabla} \colon (\mathcal{K} \otimes \mathcal{K})^{\times n} \to (\mathcal{K} \otimes \mathcal{K})^{\times (n+1)}.$$

Consider the operation $\bigl(\pi^{\times (n+1)}, \widetilde{\nabla}(n), \ldots, \widetilde{\nabla}\bigr) \colon \mathcal{K}^{\otimes 2} \to \mathcal{K}^{\times (n+1)}$. Its restriction to the $x \otimes x \in \mathcal{K} \otimes \mathcal{K}$ will be denoted by $\Psi^n \colon \mathcal{K} \to \mathcal{K}^{\times (n+1)}$.

The formula for the comultiplication in the Milnor coalgebra implies that

$$\Psi^1(\xi_n) = \sum_{i<j} \xi_{n-i}^{2^i} \xi_{n-j}^{2^j} \otimes \xi_i \xi_j,$$

or, more generally,

$$\Psi^1(\xi_n^{2^m}) = \sum_{i<j} \xi_{n-i}^{2^{i+m}} \xi_{n-j}^{2^{j+m}} \otimes \xi_i^{2^m} \xi_j^{2^m}.$$

In particular, we have the following formula for the primitive elements $\xi_1^{2^m} \in \mathcal{K}$:

$$\Psi^1(\xi_1^{2^m}) = \xi_1^{2^m} \xi_0^{2^m} \otimes \xi_1^{2^m}.$$

The following formula for the operation Ψ^2 can be proved likewise:

$$\Psi^2(\xi_n^{2^m}) = \sum_{\substack{i<j \\ k>l}} \xi_{n-i}^{2^{i+m}} \xi_{n-j}^{2^{j+m}} \otimes \xi_{i-k}^{2^{k+m}} \xi_{j-l}^{2^{l+m}} \otimes \xi_k^{2^m} \xi_l^{2^m}.$$

In particular, we have the following formula for the primitive elements $\xi_1^{2^m} \in \mathcal{K}$:

$$\Psi^2(\xi_1^{2^m}) = 0.$$

§ 8. $\cup_\infty$-A_∞-Hopf algebras

To calculate the higher differentials of the Adams spectral sequence we need the action of the Dyer–Lashof algebra and the E_∞-structure. Since the latter is too large and complicated, we shall use only the part consisting of $\cup_i$-products.

A chain complex A is called a $\cup_\infty$-*algebra* if there are operations $\cup_i\colon A\otimes A\to A$, $i\geqslant 0$, called $\cup_i$-*products*, increasing the dimension by i and such that

$$d(x\cup_i y)=d(x)\cup_i y+x\cup_i d(y)+x\cup_{i-1}y+y\cup_{i-1}x.$$

A differential coalgebra K will be called a $\cup_\infty$-*Hopf algebra* if there are $\cup_i$-products $\cup_i\colon K\otimes K\to K$ such that the following distributivity relation holds:

$$\nabla(x\cup_i y)=\sum_k (x'\cup_{i-k}T^k y')\otimes(x''\cup_k y''),$$

where $\nabla(x)=\sum x'\otimes x''$, $\nabla(y)=\sum y'\otimes y''$, $T\colon K\otimes K\to K\otimes K$ is the permutation map, and T^k is its kth iterate.

Theorem 22. *The cobar construction FK over the $\cup_\infty$-Hopf algebra K is a $\cup_\infty$-algebra. Moreover, the $\cup_i$-products $\cup_i\colon FK\otimes FK\to FK$ are defined unambiguously by the formula*

$$[x]\cup_i[y]=\begin{cases}[x\cup_{i-1}y], & i\geqslant 1,\\ [x,y], & i=0,\end{cases}$$

and the two relations

$$\begin{aligned}(x_1x_2)\cup_i[y]&=(x_1\cup_i[y])x_2+x_1(x_2\cup_i[y]),\\ (x_1x_2)\cup_i(y_1y_2)&=\sum_k(x_1\cup_{i-k}T^k y_1)(x_2\cup_k y_2)\\ &\quad+(x_1\cup_i(y_1y_2))x_2+x_1(x_2\cup_i(y_1y_2))\\ &\quad+((x_1x_2)\cup_i y_1)y_2+y_1((x_1x_2)\cup_i y_2)\\ &\quad+x_1(x_2\cup_i y_1)y_2+y_1(x_1\cup_i y_2)x_2,\end{aligned}$$

where $x_1,x_2,y_1,y_2\in FK$, $y\in K$, $i\geqslant 1$.

Proof. The products $[x_1,\dots,x_n]\cup_i[y]$ are defined by the first relation:

$$[x_1,\dots,x_n]\cup_i[y]=\sum_{k=1}^n[x_1,\dots,x_k\cup_i y,\dots,x_n].$$

The second relation in the theorem implies that the $\cup_i$-products

$$[x_1,\dots,x_n]\cup_i[y_1,\dots,y_m]$$

are defined in the general case if we know the $\cup_i$-products $[x]\cup_i[y_1,\dots,y_m]$.

We have

$$\begin{aligned} d([y_1,\dots,y_m]\cup_{i+1}[x]) = \sum_{k=1}^{m}&[y_1,\dots,d(y_k),\dots,y_m]\cup_{i+1}[x] \\ &+\sum_{k=1}^{m}[y_1,\dots,y'_k,y''_k,\dots,y_m]\cup_{i+1}[x] \\ &+[y_1,\dots,y_m]\cup_{i+1}([d(x)]+[x',x'']) \\ &+[y_1,\dots,y_m]\cup_i[x]+[x]\cup_i[y_1,\dots,y_m]. \end{aligned}$$

Hence, the product $[x]\cup_i[y_1,\dots,y_m]$ is expressed in terms of the products defined above and those of elements of smaller dimension. Therefore, the $\cup_i$-products are defined by induction.

This theorem provides formulae for the $\cup_i$-products in the cobar constructions. However, these inductive formulae are not simple even in the case when the higher $\cup_i$-products ($i\geqslant 1$) on K are trivial, that is, in the case when K is a commutative Hopf algebra.

A $\cup_\infty$-Hopf algebra K is said to be *commutative* if the comultiplication $\nabla\colon K\to K\otimes K$ is commutative.

Theorem 23. *The cobar construction FK over a commutative $\cup_\infty$-Hopf algebra K is a commutative $\cup_\infty$-Hopf algebra. Hence, in this case the cobar construction can be iterated.*

Proof. We define a comultiplication $\nabla\colon FK\to FK\otimes FK$ by the formula

$$\nabla[x_1,\dots,x_n]=\sum[x_{i_1},\dots,x_{i_p}]\otimes[x_{j_1},\dots,x_{j_q}],$$

where the sum is taken over the (p,q)-shuffles of the set $1,2,\dots,n$. A direct verification shows that the desired relations hold.

What structures are there on the homology of the $\cup_\infty$-Hopf algebra K? It is clear that K has a $\cup_\infty$-algebra structure consisting of the operations $\cup_i\colon K_*\otimes K_*\to K_*$ and an A_∞-coalgebra structure consisting of the operations $\nabla_n\colon K_*\to K_*^{\otimes n+2}$, but there are other operations of the form

$$\Psi_{i,n}\colon K_*\otimes K_*\to K_*^{\otimes n+2}.$$

To describe these operations, we define the notion of a $\cup_\infty$-A_∞-Hopf algebra.

An A_∞-coalgebra K will be called a *$\cup_\infty$-A_∞-Hopf algebra* if there is a $\cup_\infty$-algebra structure on the cobar construction $\widetilde{F}K$ such that

$$\begin{aligned} (x_1x_2)\cup_i[y] &= (x_1\cup_i[y])x_2+x_1(x_2\cup_i[y]), \\ (x_1x_2)\cup_i(y_1y_2) &= \sum_k(x_1\cup_{i-k}T^ky_1)(x_2\cup_k y_2) \\ &\quad+(x_1\cup_i(y_1y_2))x_2+x_1(x_2\cup_i(y_1y_2)) \\ &\quad+((x_1x_2)\cup_i y_1)y_2+y_1((x_1x_2)\cup_i y_2) \\ &\quad+x_1(x_2\cup_i y_1)y_2+y_1(x_1\cup_i y_2)x_2, \end{aligned}$$

where $x_1,x_2,y_1,y_2\in FK$, $y\in K$, and $i\geqslant 1$.

Theorem 24. *If K is a $\cup_\infty$-Hopf algebra, then on its homology $K_* = H_*(K)$ there is a $\cup_\infty$-A_∞-Hopf algebra structure, and there is a chain equivalence $\widetilde{F}K_* \simeq FK$ of $\cup_\infty$-algebras.*

Proof. It is well known [16] that the homology K_* of the differential coalgebra K is an A_∞-coalgebra and there are maps of algebras $\xi\colon \widetilde{F}K_* \to FK$ and $\eta\colon FK \to \widetilde{F}K_*$, and an algebra chain homotopy $h\colon FK \to FK$ such that $\eta\circ\xi = \mathrm{Id}$ and $d(h) = \xi\circ\eta - \mathrm{Id}$. It remains to define the $\cup_i$-products. We put

$$[x]\cup_i [y] = \eta(\xi[x]\cup_i \xi[y])$$

on the generators. The $\cup_i$-products are defined on other elements by the above relations.

Applying this theorem to the unstable Milnor coalgebra, we obtain the following theorem.

Theorem 25. *The Milnor coalgebra $\mathcal{K}$ has a $\cup_\infty$-A_∞-Hopf algebra structure.*

§ 9. The differentials of the Adams sequence

Let us apply the methods developed above to the calculation of the higher differentials of the Adams spectral sequence of stable homotopy groups of spheres. To do this, we need to calculate the differential in the complex $\widetilde{F}\mathcal{K}$.

Since every element of $\mathcal{K}$ can be obtained from ξ_0 using $\cup_i$-products, the following theorem holds.

Theorem 26. *The formulae for the $\cup_i$-products in the Milnor coalgebra and the $\cup_\infty$-algebra structure in the cobar construction $\widetilde{F}\mathcal{K}$ completely define the differential in $\widetilde{F}\mathcal{K}$.*

However, these inductive formulae for the differential are very complicated. To simplify them, we replace the Milnor coalgebra $\mathcal{K}$ and the cobar construction $\widetilde{F}\mathcal{K}$ by certain simpler objects.

Note that $\widetilde{F}\mathcal{K}$ is an unreduced cobar construction. To obtain a reduced cobar construction whose homology is isomorphic to the E_∞ term of the Adams sequence, we have to factor the Milnor coalgebra $\mathcal{K}$ by ξ_0.

We define a filtration in $\widetilde{F}\mathcal{K}$ by letting the filter degree of $\xi_{i_1}\dots\xi_{i_n} \in K$ be equal to n. Then the first term of the corresponding spectral sequence will be isomorphic to the polynomial algebra $PS^{-1}X$ over the desuspension over the module X generated by the $\xi_m^{2^n}$. We shall use the same notation for the elements of $PS^{-1}X$ and those of the cobar construction: $[x_1,\dots,x_n]$, $x_i \in X$.

There is a map of algebras $\eta\colon F\mathcal{K} \to PS^{-1}X$ given by the formula

$$\eta[x] = \begin{cases} [\xi_m^{2^n}], & x = \xi_m^{2^n}, \\ 0 & \text{otherwise.} \end{cases}$$

The inverse map $\xi\colon PS^{-1}X \to \mathcal{F}K$ can be given by the formula

$$\xi([x_1,\dots,x_n]) = [x_1,\dots,x_n], \quad x_1 \leqslant \dots \leqslant x_n.$$

The next theorem follows from perturbation theory.

Theorem 27. *On the polynomial algebra $PS^{-1}X$ there is a $\cup_\infty$-algebra structure defined on the generators by the formula*

$$[x] \cup_i [y] = \begin{cases} 0, & i \geqslant 1, \quad x < y, \\ [x \cup_{i-1} x], & i \geqslant 1, \quad x = y, \\ [x, y], & i = 0, \quad x \leqslant y, \end{cases}$$

and such that

$$\begin{aligned} (x_1 x_2) \cup_i [y] &= (x_1 \cup_i [y]) x_2 + x_1 (x_2 \cup_i [y]), \\ (x_1 x_2) \cup_i (y_1 y_2) &= \sum_k (x_1 \cup_{i-k} T^k y_1)(x_2 \cup_k y_2) \\ &\quad + (x_1 \cup_i (y_1 y_2)) x_2 + x_1 (x_2 \cup_i (y_1 y_2)) \\ &\quad + ((x_1 x_2) \cup_i y_1) y_2 + y_1 ((x_1 x_2) \cup_i y_2) \\ &\quad + x_1 (x_2 \cup_i y_1) y_2 + y_1 (x_1 \cup_i y_2) x_2, \end{aligned}$$

where $x_1, x_2, y_1, y_2 \in PS^{-1}X$, $y \in X$, $i \geqslant 1$.

Since every element of $\widetilde{P}S^{-1}X$ can be obtained from $[\xi_0]$ using $\cup_i$-products, the following theorem holds.

Theorem 28. *The formulae for the $\cup_i$-products in the module X and the relations for the $\cup_i$-products in $\widetilde{P}S^{-1}X$ completely define the differential in $\widetilde{P}S^{-1}X$.*

As before, these formulae are inductive. Using Adams' notation, we put $h_n = [\xi_1^{2^n}]$. The formula $h_n \cup_1 h_n = h_{n+1}$ enables us to prove the following theorem by induction.

Theorem 29. *The differential in $\widetilde{P}S^{-1}X$ can be expressed by the formula*

$$d(h_n) = \sum_{i=0}^{n-1} [\xi_i] h_{n-i}^{2^i}.$$

on the h_n.

Indeed, it is clear that $d(h_1) = h_{-1} h_1$. For h_2 we have

$$\begin{aligned} d(h_2) &= d(h_1 \cup_1 h_1) = (h_{-1} h_1 \cup_1 h_1 + h_1 \cup_1 (h_{-1} h_1) \\ &= (h_{-1} h_1) \cup_2 (h_{-1} h_1) = h_{-1} h_2 + h_0 h_1^2. \end{aligned}$$

Assume that this formula holds for n. We claim that it holds for $n+1$. We have

$$\begin{aligned} d(h_{n+1}) &= d(h_n \cup_1 h_n) = d(h_n) \cup_1 h_n + h_n \cup_1 d(h_n) \\ &= \left(\sum_{i=0}^{n-1} [\xi_i] h_{n-i}^{2^i} \right) \cup_2 \left(\sum_{i=0}^{n-1} [\xi_i] h_{n-i}^{2^i} \right) \\ &= \sum_{i=0}^{n} [\xi_i] h_{n+1-i}^{2^i}. \end{aligned}$$

§ 10. The Dyer–Lashof operations and the Arf invariant

Let Y be an E_∞-space. Then the Dyer–Lashof algebra R acts on its homology. This action and the functional homology operations induce on the cobar construction $\widetilde{F}(K, K, Y_*)$ an action of the Dyer–Lashof algebra,

$$\mu\colon R \otimes \widetilde{F}(K, K, Y_*) \to \widetilde{F}(K, K, Y_*),$$

that commutes with the differential. The corresponding action $\mu\colon R \otimes \widetilde{F}(K, Y_*) \to \widetilde{F}(K, Y_*)$ does not commute with the differential. Hence, there is no natural action of the Dyer–Lashof algebra on the homology of the complex $\widetilde{F}(K, Y_*)$, which is isomorphic to the E^∞ term of the Adams spectral sequence of homotopy groups of Y. Nevertheless, the action of the Dyer–Lashof algebra on $\widetilde{F}(K, Y_*)$ can be extended to an action

$$\mu\colon \widetilde{F}(K, R) \otimes \widetilde{F}(K, Y_*) \to \widetilde{F}(K, Y_*)$$

that commutes with the differential. It therefore induces an action on the homology:

$$\mu_*\colon H_*(\widetilde{F}(K, R)) \otimes H_*(\widetilde{F}(K, Y_*)) \to H_*(\widetilde{F}(K, Y_*)).$$

Hence, the algebra $H_*(\widetilde{F}(K, R))$ acts on the E^∞ term of the Adams spectral sequence of homotopy groups of the E_∞-space Y. This algebra is the E^∞ term of the Adams spectral sequence of stable homotopy groups of spheres. However, this algebra is seldom used, since it is very difficult to calculate.

We shall use the action of the Dyer–Lashof algebra on the complex $\widetilde{P}S^{-1}X$, whose homology is isomorphic to the E^∞ term of the Adams spectral sequence. For brevity we denote the complex $\widetilde{P}S^{-1}X$ by H and its n-dimensional homology by H_n.

It is clear that if u_n is a cycle in H, then $e_0(u_n) = u_n \cup_0 u_n$ also is a cycle. Therefore, there is a squaring operation

$$e_0 = Q^n\colon H_n \to H_{2n}.$$

Consider the $\cup_1$-product $e_1(u_n) = u_n \cup_1 u_n$. If u_n is a cycle, then $e_1(u_n)$, generally speaking, is not. In fact, the following formula holds for the cycle u_n:

$$d(e_1(u_n)) = \binom{n}{1} h_0 e_0(u_n).$$

If n is even, that is, $n = 2k$, then $e_1(u_n)$ is a cycle. Therefore, the correspondence $u_n \longmapsto e_1(u_n)$ defines an operation

$$e_1 = Q^{2k+1}\colon H_{2k} \to H_{4k+1}.$$

If n is odd, that is, $n = 2k+1$, then $d(e_1(u_n)) = h_0 e_0(u_n)$. Assume that $e_0(u_n)$ is homologous to zero, that is, there is a y_{2n+1} such that $d(y_{2n+1}) = e_0(u_n)$. Then

$\widetilde{e}_1(u_n) = e_1(u_n) + h_0 y_{2n+1}$ is a cycle. Hence, the correspondence $u_n \longmapsto \widetilde{e}_1(u_n)$ defines an operation

$$\widetilde{e}_1 = \widetilde{Q}^{2k+2}\colon \operatorname{Ker}(e_0) \subset H_{2k+1} \to H_{4k+3}.$$

Some values of this operation can lie outside $\operatorname{Ker}(e_0)$. Therefore, this operation cannot be iterated in general. Let us determine when the values of the operation $\widetilde{e}_1$ belong to $\operatorname{Ker}(e_0)$.

Consider the element $e_0(y_{2n+1})$. It is a cycle, since $d(e_0(y_{2n+1})) = e_1(e_0(u_n)) = 0$. Its homology class defines the value of the secondary operation with respect to the operation e_0 on u_n. We denote this operation by $e_0^{(2)}: H_n \to H_{4n+2}$.

For example, let us calculate $e_0^{(2)}(h_1^2)$. We have $h_1^4 = d(h_1[\xi_2]^2 + h_0^2[\xi_2^2])$. The homology class of $(h_1[\xi_2]^2 + h_0^2[\xi_2^2])^2$ is equal to $h_1 P_1(h_1)$, whence

$$e_0^{(2)}(h_1^2) = h_1 P_1(h_1) \neq 0.$$

We claim that the operation $e_0^{(2)}$ gives obstructions for the existence of the double composite of the operation $\widetilde{e}_1$. Assume that $e_0^{(2)}(u_n) = 0$. This means that $e_0(y_{2n+1})$ is homologous to zero, that is, there is a z_{4n+3} such that $d(z_{4n+3}) = e_0(y_{2n+1})$. In this case we have

$$\begin{aligned} e_0(\widetilde{e}_1(u_n)) &= e_0(e_1(u_n) + h_0 y_{2n+1}) = e_2(e_0(u_n)) + h_0^2 e_0(y_{2n+1}) \\ &= d(e_1(y_{2n+1}) + h_0 z_{4n+3} + h_0^2 z_{4n+3}). \end{aligned}$$

We put $\widetilde{y}_{4n+3} = e_1(y_{2n+1}) + h_0 z_{4n+3} + h_0^2 z_{4n+3}$. Then the element

$$\widetilde{e}_1\widetilde{e}_1(u_n) = e_1(\widetilde{e}_1(u_n)) + h_0 \widetilde{y}_{4n+3}$$

is a cycle. Hence, the double composite of $\widetilde{e}_1$ is well defined. Higher-order obstructions and composites for the operation $\widetilde{e}_1$ can be defined likewise.

Now consider the $\cup_2$-product $e_2(u_n) = u_n \cup_2 u_n$. If u_n is a cycle, then $e_2(u_n)$, generally speaking, is not. In fact, the following formula holds for the cycle u_n:

$$d(e_2(u_n)) = \binom{n+1}{1} h_0 e_1(u_n) + \binom{n}{2} h_1 e_0(u_n).$$

If $n = 4k+1$, then $e_2(u_n)$ is a cycle. Therefore, the correspondence $u_n \longmapsto e_2(u_n)$ defines an operation

$$e_2 = Q^{4k+3}\colon H_{4k+1} \to H_{8k+4}.$$

If $n = 4k+3$, then $d(e_2(u_n)) = h_1 e_0(u_n)$. Assume that $e_0(u_n)$ is homologous to zero, that is, there is a y_{2n+1} such that $d(y_{2n+1}) = e_0(u_n)$. Then $\widetilde{e}_2(u_n) = e_2(u_n) + h_1 y_{2n+1}$ is a cycle. Therefore, the correspondence $u_n \longmapsto \widetilde{e}_2(u_n)$ defines an operation

$$\widetilde{e}_2 = \widetilde{Q}^{4k+5}\colon \operatorname{Ker}(e_0) \subset H_{4k+3} \to H_{8k+8}.$$

The iteration of this operation and an obstruction for its existence can be defined as before.

If $n = 4k$, then $d(e_2(u_n)) = h_0e_1(u_n)$. Assume that $e_1(u_n)$ is homologous to zero, that is, there is a y_{2n+2} such that $d(y_{2n+2}) = e_1(u_n)$. Then $\widetilde{e}_2(u_n) = e_2(u_n) + h_0y_{2n+2}$ is a cycle. Therefore, the correspondence $u_n \longmapsto \widetilde{e}_2(u_n)$ defines an operation

$$e_2 = Q^{4k+2}\colon \operatorname{Ker}(e_1) \subset H_{4k} \to H_{8k+2}.$$

The iteration of this operation and an obstruction for its existence can be defined as before.

The case when $n = 4k + 2$ is the most interesting for us. In this case the formula $d(e_2(u_n)) = h_0e_1(u_n)+h_1e_0(u_n)$ holds. Assume that $e_0(u_n)$ and $e_1(u_n)$ are homologous to zero, that is, there are y_{2n+1} and y_{2n+2} such that $d(y_{2n+1}) = e_0(u_n)$ and $d(y_{2n+2}) = e_1(u_n)$.

Then $\widetilde{e}_2(u_n) = e_2(u_n) + h_0y_{2n+2} + h_1y_{2n+1}$ is a cycle. Therefore, the correspondence $u_n \longmapsto \widetilde{e}_2(u_n)$ defines an operation

$$\widetilde{e}_2 = \widetilde{Q}^{4k+4}\colon \operatorname{Ker}(e_0) \cap \operatorname{Ker}(e_1) \subset H_{4k+2} \to H_{8k+6}.$$

Let us apply the formulae obtained above to the calculation of the higher differentials on the h_n^2.

Theorem 30. *The following equalities hold for the differentials d_i of the Adams spectral sequence*: $d_i(h_n^2) = 0,\ \ 1 \leqslant i \leqslant 6$.

Proof. The above implies that the total differential on the h_n^2 is given by the formula

$$d(h_n^2) = e_1(d(h_n)) = e_1(h_0h_{n-1}^2 + [\xi_2]h_{n-2}^4 + \cdots) = h_1h_{n-1}^4 + F^9,$$

where F^9 denotes elements of filter degree 9. This implies, in particular, that $d_2(h_n^2) = 0$.

Put $x_n = h_n^2$ and $y_{n-1} = e_1(h_{n-2}[\xi_2^{2^{n-2}}]) = h_{n-1}[\xi_2^{2^{n-2}}]^2 + h_{n-2}^2[\xi_2^{2^{n-1}}]$. We have

$$d(y_{n-1}) = e_2(h_{n-2}^2h_{n-1} + F^4) = h_{n-1}^4 + F^6.$$

Therefore, $d(x_n + h_1y_{n-1}) \in F^7$. Hence, the differentials d_2, d_3 and d_4 vanish on the elements h_n^2 of the Adams spectral sequence.

To calculate the differentials of higher dimensions we consider the elements $\widetilde{x}_{n-1} = x_{n-1} + h_1y_{n-2}$. We have

$$\begin{aligned} e_2(x_{n-1}) &= h_n^2 + h_2e_1(y_{n-2}) + h_1^2e_2(y_{n-2}),\\ d(e_2(\widetilde{x}_{n-1})) &= e_3(d(\widetilde{x}_{n-1})) + h_0e_1(\widetilde{x}_{n-1}) + h_1e_0(\widetilde{x}_{n-1})\\ &= e_3(F^7) + h_0(h_2e_0(y_{n-2}) + h_1^2e_1(y_{n-2})) + h_1(h_n^4 + h_1^2e_0(y_{n-2})). \end{aligned}$$

We add $h_1e_3(y_{n-2})$ and $h_2e_1(y_{n-2})$ to $e_2(\widetilde{x}_{n-1})$ and consider the element

$$z_n = e_2(x_{n-1}) + h_1e_3(y_{n-2}) + h_2e_1(y_{n-2}).$$

We have

$$\begin{aligned} d(h_1e_3(y_{n-2})) &= h_1h_n^4 + h_1^2e_1(y_{n-2}) + h_1[\xi_2]e_0(y_{n-2}) + F^9,\\ d(h_2e_1(y_{n-2})) &= h_2h_0e_0(y_{n-2}) + h_0h_1^2e_1(y_{n-2}) + F^{11}. \end{aligned}$$

Therefore,

$$d(z_n) = h_1^2 e_1(y_{n-2}) + h_1[\xi_2]e_0(y_{n-2}) + F^9.$$

We denote by g_{n-4} the homology class represented by the cycle

$$[\xi_2^{2^{n-3}}]^4 + h_{n-3}[\xi_2^{2^{n-4}}]^2 h_n + h_{n-4}^2[\xi_2^{2^{n-3}}]h_n + h_{n-2}^3[\xi_2^{2^n}].$$

Then

$$e_1(y_{n-2}) = h_{n-1}[\xi_2^{2^{n-3}}]^4 + h_{n-3}^4[\xi_2^{2^{n-1}}] + h_{n-2}^3 h_{n-1}[\xi_2^{2^{n-2}}] = g_{n-4}h_{n-1}.$$

It is well known that in the E^1 term of the Adams spectral sequence $g_0 h_3 \sim 0$, whence $e_1(y_{n-2}) = g_{n-4}h_{n-1} \sim 0$, that is, there are u_{n-1} such that $d_1(u_{n-1}) = e_1(y_{n-2})$. The formula for d_2 on the $[\xi_i^{2^n}]$ implies that $d_2(u_{n-1})$ can be written as $d_2(u_{n-1}) = h_0 v_{n-1}$.

Since $d(e_1(y_{n-2})) = h_0 e_0(y_{n-2})$, we have $d_1(v_{n-1}) = e_0(y_{n-2})$, whence $d(h_1^2 u_{n-1} + h_1[\xi_2]v_{n-1}) = h_1^2 e_1(y_{n-2}) + h_1[\xi_2]e_0(y_{n-2}) + F^9$.

Hence, for $t_n = z_n + h_1^2 u_{n-1} + h_1[\xi_2]v_{n-1}$ we have $d(t_n) \in F^9$. Therefore, $d_i(h_n^2) = 0$ if $1 \leqslant i \leqslant 6$.

We can deduce from this theorem the result of Mahowald concerning the survival of h_4^2 and h_5^2 until the E^∞ term of the Adams sequence. Indeed, h_3^4 is homologous to zero, that is, there is a y such that $d(y) = h_3^4$. Therefore, $h_4^2 + h_1 y$ is a cycle. Hence, h_4^2 survives until the E^∞ term. Repeating the proof of Theorem 30 for $x_4 = h_4^2$, we obtain that $t_5 = z_5 + h_1^2 u_4 + h_1[\xi_2]v_4$ is a cycle. Therefore, h_5^2 survives until the E^∞ term of the Adams sequence.

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