

Bordism of semi-free S^1 -actions

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Abstract. We calculate geometric and homotopical bordism rings associated to semi-free S^1 actions on complex manifolds, giving explicit generators for the geometric theory. The classification of semi-free actions with isolated fixed points up to cobordism complements similar results from symplectic geometry.

1. Introduction

In this paper we describe both the geometric and homotopical bordism rings associated to S^1 -actions in which only the two simplest orbit types, namely fixed points and free orbits, are allowed. Our work is of further interest in two different ways. To make the computation of geometric semi-free bordism, in Corollary 2.12 we prove the semi-free case of what we call the geometric realization conjecture, which if true in general would determine the ring structure of geometric S^1 -bordism from the ring structure of homotopical S^1 -bordism given in [21]. Additionally, we investigate semi-free actions with isolated fixed points as a first case, and that result is parallel to results from symplectic geometry. Let $\mathbb{P}(\mathbb{C} \oplus \rho)$ denote the space of complex lines in $\mathbb{C} \oplus \rho$ where ρ is the standard complex representation of S^1 (in other words, the Riemann sphere with S^1 action given by the action of the unit complex numbers.)

Theorem 1.1. *Let S^1 act semi-freely with isolated fixed points on M , compatible with a stable complex structure on M . Then M is equivariantly cobordant to a disjoint union of products of $\mathbb{P}(\mathbb{C} \oplus \rho)$.*

This result should be compared with the second main result of [19], which states that when M is connected a semi-free Hamiltonian S^1 action on M implies that M has a perfect Morse function which realizes the same Borel equivariant cohomology as a product of such $\mathbb{P}(\mathbb{C} \oplus \rho)$, as well as the same equivariant Chern classes. Our work also refines, in this case of isolated fixed points, results of Stong [25].

As Theorem 1.1 led us to the more general computation of bordism of semi-free actions given in Theorem 3.10, it would also be interesting to see if there is an analogue of Theorem 3.10 for Hamiltonian S^1 -actions. In general, the symplectic and cobordism approaches to transformation groups have considerable overlaps in language (for example, localization by inverting Euler classes of representations plays a key role in each theory), though the same words sometimes have different precise meanings. A synthesis of these techniques might be useful in addressing interesting questions within transformation groups such as classifying semi-free actions with isolated fixed points.

In section 2 of this paper we develop semi-free bordism theory and give a proof of Theorem 1.1. We will see that the main ingredients are the Conner-Floyd-tom Dieck exact sequences, which are standard. In section 3 we compute semi-free bordism theories. In the final section, we review what is known about S^1 -bordism and present a conjectural framework for the geometric theory.

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1.1. Notation

If X is a G -space, X_+ denotes X with a disjoint basepoint with trivial action added. If V is a representation of G equipped with a G -invariant inner product, let S^V denote its one-point compactification, let $D(V)$ be the unit disk in V , and let $S(V)$ be the boundary of $D(V)$, namely the unit sphere in V . Let $\Omega^V X$ denote the space of based maps from S^V to X , where S^1 acts by conjugation. Let X^{S^1} denote the fixed points of an S^1 action on X , so that $\text{Maps}(X, Y)^{S^1}$ denotes the equivariant maps from X to Y . Let $\bigoplus^n V = \bigoplus_{i=1}^n V$. Let ρ be the standard one-dimensional representation of S^1 and ρ^* its conjugate.

2. First computations and Theorem 1.1

The foundational results of this section are based on [16], and the computational results parallel those of [21].

Theorem 1.1 follows from little more than the computation of Conner-Floyd and tom Dieck exact sequences adapted for semi-free bordism. Because construction of these sequences is standard [6, 7, 3, 24, 16, 22, 23], we will be brief in our exposition.

Definition 2.1. Let Ω_*^{SF} denote the bordism theory represented by stably complex (in the sense of Definition 28.3.1 of [16]) semi-free S^1 -manifolds. Bordisms between the manifolds must also be semi-free (but see Remark 2.5 below). By equipping these manifolds and bordisms with equivariant maps to a space X we define an equivariant homology theory $\Omega_*^{SF}(X)$.

Bordism theory is approachable in general because of its relation to homotopy theory. We choose a definition of homotopical equivariant bordism with a relatively small amount of bookkeeping.

Definition 2.2. • Let $V^{SF} = \rho \oplus \mathbb{C} \oplus \rho^*$, with S^1 -invariant inner product defined through the standard inner products on ρ , \mathbb{C} and ρ^* , and let \mathcal{U}^{SF} be $\bigoplus^\infty V^{SF}$. Fix an isomorphism $\sigma : \mathcal{U}^{SF} \oplus V^{SF} \rightarrow \mathcal{U}^{SF}$ sending $(w_1, w_2, \dots) \oplus v$, with $w_i \in V^{SF}$, to (v, w_1, w_2, \dots) .

- Let $BU^{SF}(n)$ be the space of n -dimensional complex subspaces of \mathcal{U}^{SF} , topologized as the union over k of $BU^{SF}(n, k)$, the n -dimensional subspaces of $\bigoplus^{n+k}(\rho \oplus \mathbb{C} \oplus \rho^*)$.
- Let $\xi^{SF}(n)$ denote the total space of the tautological bundle over $BU^{SF}(n)$ with inner product inherited from \mathcal{U}^{SF} , and let $TU^{SF}(n)$ be its Thom space.
- Taking the direct sum of an n -dimensional subspace of \mathcal{U}^{SF} with V^{SF} defines a map $\xi^{SF}(n) \oplus V^{SF}$ to the total space of the tautological bundle of $(n+3)$ -dimensional subspaces of $\mathcal{U}^{SF} \oplus V^{SF}$, which through σ is isomorphic to $\xi^{SF}(n+3)$. Passing to Thom spaces we get $\beta : S^{V^{SF}} \wedge TU^{SF}(n) \rightarrow TU^{SF}(n+3)$.
- Let MU^{SF} denote the S^1 -spectrum with de-loopings by semi-free representations built from the prespectrum TU^{SF} whose V th entry is $TU^{SF}(\dim(V))$ and with bonding maps given by β . Explicitly, the $\bigoplus^k V^{SF}$ th de-looping of the infinite loop space associated to MU^{SF} is given by the direct limit $\text{colim}_n \Omega^{\bigoplus^{(n-k)} V^{SF}} TU^{SF}(3n)$, where the β serve as maps in this directed system.

Because any semi-free manifold can be embedded equivariantly in some $\bigoplus^k V^{SF}$ (a direct application of transversality results of [26] and the fact that ρ and ρ^* are the only representations which appear in the decomposition of the fiber of the normal bundle to a fixed set), there is a Pontryagin-Thom map from Ω_*^{SF} to $MU_*^{SF} = \pi_* MU^{SF}$. We will see that this map is not an isomorphism but that nonetheless MU_*^{SF} is essential in studying Ω_*^{SF} , in particular for proving Theorem 1.1.

The starting point in equivariant bordism is typically the use of a filtration which can be traced back to Conner and Floyd [6].

Definition 2.3. • Define $i : MU_*(BS^1) \rightarrow \Omega_*^{SF}$ by taking a representative M mapping to BS^1 and pulling back the canonical S^1 -bundle to get a principal S^1 -bundle over M , which is a free (and thus semi-free) S^1 manifold.

- For a semi-free S^1 -manifold M , the normal bundle of M^{S^1} in M will have as the representation type of the fiber a direct sum of ρ 's and ρ^* 's. Because $BU(n)$ classifies n -dimensional complex bundles, $MU_*(BU(n))$ is the bordism module of stably complex manifolds with n -dimensional complex bundles over them. Let $F_*^{SF} = MU_*\left(\left(\bigsqcup_{n>0} BU(n)\right)^2\right)$ and define $\lambda : \Omega_*^{SF} \rightarrow F_*^{SF}$ as sending a semi-free bordism class to the bordism class of the normal bundle of its fixed set, split according to appearance of ρ and ρ^* in the fiber.
- Define $\partial : F_*^{SF} \rightarrow MU_{*-2}(BS^1)$ as taking a manifold with a direct sum of two bundles over it (classified by maps to $BU(i) \times BU(j)$) imposing S^1 action as ρ on the summand of the first factor and ρ^* on the second, imposing an equivariant Hermitian inner product, and then taking the unit sphere bundle of that S^1 -bundle.

Theorem 2.4. The following sequence is exact:

$$0 \rightarrow \Omega_*^{SF} \xrightarrow{\lambda} F_*^{SF} \xrightarrow{\partial} MU_*(BS^1) \rightarrow 0.$$

Note here that gradings are not preserved in the standard sense. The middle module must be graded so that M mapping to $BU(i) \times BU(j)$ has degree $dim(M) + 2(i + j)$. The map ∂ lowers degree by two.

Outline of proof. The maps i, λ and ∂ coincide with the maps in the families exact sequence for the family $\{S^1, 1\}$ consisting of S^1 and the trivial group (see chapter 15 of [16], or [22]). Exactness is straightforward and pleasant to verify. We claim that i is the zero map. It is well-known that $BS^1 = \mathbb{C}P^\infty$ and $MU_*(\mathbb{C}P^\infty)$ is generated by bordism representatives $\mathbb{C}P^n$ with their standard inclusions in $\mathbb{C}P^\infty$ (see for example Lemma 2.14 of part 2 of [1]). The principal S^1 -bundle over $\mathbb{C}P^n$ is equivariantly diffeomorphic to $S(\bigoplus^n \rho)$. But this class is zero in Ω_*^{SF} since it bounds $D(\bigoplus^n \rho)$. \square

Remark 2.5. If we let Ω_*^{SF1} denote the image of semi-free bordism in unrestricted S^1 bordism, thus allowing arbitrary bordism between semi-free representatives, we see that Ω_*^{SF1} also fits in the exact sequence of Theorem 2.4, and thus is isomorphic to Ω_*^{SF} by the five-lemma.

The space $\bigsqcup_{n>0} BU(n)$ has a product which corresponds to Whitney sum of bundles, through $BU(n)$'s role as the classifying space for complex vector bundles. Thus $MU_* \left(\left(\bigsqcup_{n>0} BU(n) \right)^2 \right)$ is a ring which we identify as follows.

Definition 2.6. Let $X_{n,\rho} \in MU_{2n}(BU(1) \times BU(0))$ be represented by \mathbb{P}^n mapping to $BU(1)$ by classifying the tautological line bundle. Let $X_{n,\rho^*} \in MU_{2n}(BU(0) \times BU(1))$ be defined similarly.

Proposition 2.7. $F_*^{SF} \cong MU_*[X_{n,\rho}, X_{n,\rho^*}]$, where $n \geq 0$.

The proof is standard, as in Lemma 4.14 of part two of [1], using collapse of the Atiyah-Hirzebruch spectral sequence for $MU_*(BU(n))$ and the corresponding computation for homology.

Corollary 2.8. Ω_*^{SF} is a free MU_* -module concentrated in even degrees.

Proof. Looking at the exact sequence of Theorem 2.4 we see that the middle and right terms are free modules over MU_* . The map ∂ is a split surjection, with one splitting given by sending the class represented by $\mathbb{C}P^n \hookrightarrow \mathbb{C}P^\infty$ to the class represented by $D(\bigoplus^{n+1} \rho)$, as in the outline of proof of Theorem 2.4. As a submodule of F_*^{SF} , Ω_*^{SF} is complementary to the image of this splitting, and thus is free. \square

We give one important example of computation of the map λ .

Proposition 2.9. $\lambda(\mathbb{P}(\mathbb{C}^n \oplus \rho)) = X_{n-1,\rho} + X_{0,\rho^*}^n$.

Proof. We use homogeneous coordinates on $\mathbb{P}(\mathbb{C}^n \oplus \rho)$. There are two possible components of the fixed sets. The points whose last coordinate is zero constitute a fixed \mathbb{P}^{n-1} , whose normal bundle is the tautological line bundle over which each fiber is isomorphic to ρ as a representation of S^1 . This manifold with (normal) bundle defines exactly $X_{n-1,\rho}$. There is also a fixed point in which all of the first n coordinates are zero, and its normal bundle is $\bigoplus^n \rho^*$. This fixed set contributes a summand of X_{0,ρ^*}^n . \square

Next we introduce the analogue of Theorem 2.4 for MU_*^{SF} , essentially the tom Dieck exact sequence. We first need to develop Euler classes, which play important roles in equivariant bordism. Consider $BU^{SF}(1)$, whose fixed set is three disjoint copies of $BU(1)$. The tautological bundle over $BU^{SF}(1)$ has fibers over these three fixed sets of ρ, \mathbb{C} and ρ^* .

Definition 2.10. *Let ι_ρ be the inclusion of a fiber isomorphic to ρ over a fixed point in the tautological bundle over $BU^{SF}(1)$, noting that all such inclusions are homotopic. Let $T(\iota_\rho)$ denote the induced map on Thom spaces, and let $e_\rho \in MU_{-2}^{SF}$ be the composite $S^0 \rightarrow S^\rho \xrightarrow{T(\iota_\rho)} TU^{SF}(1)$. Let e_{ρ^*} be defined similarly.*

The class e_ρ , when viewed as a class in $MU_{SF}^2(pt.)$ serves as the Euler class of ρ , viewed as a vector bundle over a point.

Next, we need to develop the analogue of F_*^{SF} . Let $\Phi_*^{SF} = MU_*[(BU \times \mathbb{Z})^2]$, where multiplication on $(BU \times \mathbb{Z})^2$ is the product of the standard Whitney sum multiplication on each factor of BU and addition on each factor of \mathbb{Z} . By inclusion of $\bigsqcup_{n>0} BU(n)$ in $BU \times \mathbb{Z}$ (which is a group completion map, though we will not need that here), F_*^{SF} maps to Φ_*^{SF} . The analogue of Proposition 2.7 is that

$$\Phi_*^{SF} \cong MU_*[X_{0,\rho}^{\pm 1}, X_{0,\rho^*}^{\pm 1}, X_{n,\rho}, X_{n,\rho^*} | n \geq 1],$$

where $X_{i,\rho}$ and X_{i,ρ^*} are the images of the classes of the same name under the map from F_*^{SF} . In particular, $X_{0,\rho}$ and X_{0,ρ^*} are the unit classes in $(BU \times 1) \times (BU \times 0)$ and $(BU \times 0) \times (BU \times 1)$ respectively.

Theorem 2.11. *There is a short exact sequence:*

$$0 \rightarrow MU_*^{SF} \xrightarrow{\lambda} \Phi_*^{SF} \rightarrow MU_{*-2}(BS^1) \rightarrow 0.$$

The exact sequence of Theorem 2.4 maps naturally to this exact sequence through Pontryagin-Thom maps. The Pontryagin-Thom map is the identity on $MU_(BS^1)$. On the middle terms, $X_{i,\rho}$ and X_{i,ρ^*} map to classes with the same names. Moreover, $\lambda(e_\rho) = X_{0,\rho}^{-1}$ and $\lambda(e_{\rho^*}) = X_{0,\rho^*}^{-1}$.*

Outline of proof. The proof of this theorem parallels the main results of [7] and section four of [21]. The sequence in question is the MU_*^{SF} long exact sequence associated to the cofiber sequence $ES_+^1 \rightarrow S^0 \rightarrow \widetilde{ES}^1$. The middle term is of course MU_*^{SF} . By either Adams' transfer argument [2] or the fact that transversality holds in the presence of free G -manifolds, $\widetilde{MU}_*^{SF}(ES_+^1)$ is isomorphic to $MU_{*-1}(BS^1)$. The map from $MU_*(BS^1)$ to MU_*^{SF} is zero since it factors through $i : MU_*(BS^1) \rightarrow \Omega_*^{SF}$, which was shown to be zero in Theorem 2.4, so this long exact sequence splits into short exact sequences.

To identify $MU_*^{SF}(\widetilde{ES}^1)$ as Φ_*^{SF} is a longer exercise. The basic fact one uses is that if X is semi-free and Y is contractible when forgetting S^1 -action (and both are CW-complexes) then $Maps(X, Y)^{S^1}$ is homotopy equivalent to $Maps(X^{S^1}, Y^{S^1})$ through the restriction map, since the fibers of this restriction map are spaces of (non-equivariant) maps into Y . In analyzing $MU_*^{SF}(\widetilde{ES}^1)$ one applies this fact to

$Maps(S^V, \widetilde{ES}^1 \wedge TU^{SF}(n))$ to reduce to computing the fixed sets of these Thom spaces. The fixed set $(TU^{SF}(n))^{S^1}$ is $\bigvee_{i+j+k=n} TU(i) \wedge (BU(j) \times BU(k))_+$ (see Lemma 4.7 of [21]). Careful bookkeeping of the passage to spectra leads to the identification

$$MU^{SF} \wedge \widetilde{ES}^1 \simeq \bigvee_{(i,j) \in \mathbb{Z} \times \mathbb{Z}} \Sigma^{2(i+j)} MU \wedge (BU \times BU)_+,$$

from which the isomorphism $MU_*(\widetilde{ES}^1) \cong \Phi_*^{SF}$ is immediate.

Identifying the Pontryagin-Thom map on the middle term with the inclusion map from F_*^{SF} to Φ_*^{SF} above is straightforward. What remains is analysis of the Euler class e_ρ and e_{ρ^*} . When one passes to fixed sets, e_ρ is represented by the inclusion $S^0 \hookrightarrow TU(0) \wedge (BU(1) \times BU(0))_+$. This class passes in the limit to the unit class in $MU_*((BU \times -1) \times (BU \times 0))$, which is the inverse of $X_{0,\rho}$. The analysis of e_{ρ^*} is similar. \square

In light of this theorem, we will usually express Φ_*^{SF} as $MU_*[e_\rho^{\pm 1}, e_{\rho^*}^{\pm 1}, X_{n,\rho}, X_{n,\rho^*} | n \geq 1]$. From this theorem we deduce the following, whose first part is an analogue of a theorem of Comezana (28.5.4 of [16]) and Löffler [14].

Corollary 2.12. *The Pontryagin-Thom map $\Omega_*^{SF} \rightarrow MU_*^{SF}$ is injective. The following diagram from Theorem 2.11 is a pullback square*

$$\begin{array}{ccc} \Omega_*^{SF} & \xrightarrow{\lambda} & F_*^{SF} \\ P-T \downarrow & & \downarrow \\ MU_*^{SF} & \longrightarrow & \Phi_*^{SF}. \end{array}$$

Proof. The horizontal maps are injective by Theorems 2.4 and 2.11, the right vertical map is injective by inspection, so the left vertical map is injective by commutativity.

The horizontal maps have isomorphic cokernels namely $MU_{*-2}(BS^1)$, so the square is a pull-back square through an elementary diagram chase. \square

We use the phrase “geometric realization” to refer to the fact that this square is a pull-back, since it implies that any fixed-set data which could be realized geometrically is so realized. Corollary 2.12 will be the first ingredient in computing Ω_*^{SF} in the next section.

Because homologically it is in negative degrees, e_ρ cannot be in the image of the Pontryagin-Thom map and thus might seem exotic to the eyes of someone unfamiliar with equivariant bordism. We will see now that Euler classes can nonetheless be of great use in proving geometric theorems.

Theorem 2.13. *The intersection of $\lambda(MU_*^{SF})$ with the subring $\mathbb{Z}[e_\rho^{-1}, e_{\rho^*}^{-1}]$ is the subring $\mathbb{Z}[e_\rho^{-1} + e_{\rho^*}^{-1}]$.*

Before proving this theorem, we deduce Theorem 1.1 from it.

Proof of Theorem 1.1. Let M be a stably complex semi-free S^1 -manifold with isolated fixed points. These isolated fixed points will have trivial normal bundles which are direct sums of ρ and ρ^* . Under λ , a fixed point with $\bigoplus^k \rho \oplus \bigoplus^l \rho^*$ for a normal bundle contributes $X_{0,\rho}^k X_{0,\rho^*}^l$. By Theorem 2.11, this term maps to $e_{\rho}^{-k} e_{\rho^*}^{-l}$. Therefore, $\lambda([M])$ lies in $\mathbb{Z}[e_{\rho}^{-1}, e_{\rho^*}^{-1}]$.

Applying Theorem 2.13, $\lambda([M])$ lies in $\mathbb{Z}[e_{\rho}^{-1} + e_{\rho^*}^{-1}]$, which by Proposition 2.9 is $\mathbb{Z}[\lambda(\mathbb{P}(\mathbb{C} \oplus \rho))]$. But by Theorem 2.4, λ is injective, so $[M]$ lies in $\mathbb{Z}[\mathbb{P}(\mathbb{C} \oplus \rho)]$ in MU_*^{SF} . Similarly, by Corollary 2.12, $[M]$ lies in $\mathbb{Z}[\mathbb{P}(\mathbb{C} \oplus \rho)]$ in Ω_*^{SF} , which means that M is equivariantly cobordant to a disjoint union of products of $\mathbb{P}(\mathbb{C} \oplus \rho)$. □

Our main tool in the proof of Theorem 2.13 is to use the augmentation map $\alpha : MU_*^{SF} \rightarrow MU_*$, which takes a map $S^V \rightarrow TU^{SF}(n)$ and forgets the S^1 action. Note that α is a map of rings.

Proof of Theorem 2.13. Let R_* denote the subring $\mathbb{Z}[e_{\rho}^{-1}, e_{\rho^*}^{-1}]$ of Φ_* . Since R_* is graded and lies in non-negative degrees, we may proceed by induction on degree, focusing on homogeneous elements. Suppose that

$$a_0 e_{\rho}^{-n} + a_1 e_{\rho}^{-(n-1)} e_{\rho^*}^{-1} + \dots + a_n e_{\rho^*}^{-n} = \lambda(x)$$

for some x .

Consider $y = e_{\rho^*}(x - a_0[\mathbb{P}(\mathbb{C} \oplus \rho)]^n)$. The image $\lambda(y)$ is in R_* and is in degree $2(n - 1)$, thus by inductive hypothesis y is in $\mathbb{Z}[\mathbb{P}(\mathbb{C} \oplus \rho)]$. Hence $y = k[\mathbb{P}(\mathbb{C} \oplus \rho)]^{n-1}$ for some $k \in \mathbb{Z}$. Apply the augmentation map α to this equality. The image of e_{ρ^*} under α is zero since $MU_{-2} = 0$, thus so is $\alpha(y)$. It is well-known that $(\mathbb{P}^1)^{n-1}$ is non-zero in MU_* for any $n > 0$, so k must be zero. This implies $y = 0$, or since e_{ρ^*} is not a zero divisor, $x = a_0[\mathbb{P}(\mathbb{C} \oplus \rho)]^n$. The base case of this induction in degree zero is immediate since both R and $\mathbb{Z}[e_{\rho}^{-1} + e_{\rho^*}^{-1}]$ consist only of the integers in that degree. □

3. Computation of semi-free bordism

We turn our attention to homotopical semi-free bordism, following the example of [21]. Let $Z_{n,\rho} \in \Omega_*^{SF}$ be $[\mathbb{P}(\mathbb{C}^n \oplus \rho)]$, and by abuse let it also denote the image of this class under λ , which is equal to $X_{n-1,\rho} + X_{0,\rho^*}^n$ by Proposition 2.9. By further abuse, let $Z_{n,\rho}$ also denote its image under the Pontryagin-Thom map in MU_*^{SF} as well as its image in Φ_*^{SF} , namely $X_{n-1,\rho} + e_{\rho}^{-n}$. Let Z_{n,ρ^*} be defined (everywhere) similarly. We may use $Z_{n,\rho}$ and Z_{n,ρ^*} as generators of F_*^{SF} and Φ_*^{SF} . By Theorem 2.11 we have the following.

Proposition 3.1. *There is a sequence of inclusions*

$$MU_*[e_{\rho}, e_{\rho^*}, Z_{n,\rho}, Z_{n,\rho^*} | n \geq 2] \subset MU_*^{SF} \subset MU_*[e_{\rho}^{\pm 1}, e_{\rho^*}^{\pm 1}, Z_{n,\rho}, Z_{n,\rho^*} | n \geq 2].$$

Thus, to understand MU_*^{SF} is to understand divisibility by Euler classes, which is traditionally done as part of a Gysin sequence. Recall $\alpha : MU_*^{SF} \rightarrow MU_*$, the augmentation map which forgets S^1 action.

Theorem 3.2. *The sequences $0 \rightarrow MU_{*+2}^{SF} \xrightarrow{e_V} MU_*^{SF} \xrightarrow{\alpha} MU_* \rightarrow 0$, where V is either ρ or ρ^* , are exact.*

Proof. Apply \widetilde{MU}_{SF}^* to the cofiber sequence $S(\rho)_+ \xrightarrow{i} S^0 \xrightarrow{j} S^\rho$. The middle term is by definition MU_{SF}^* . Since $S(\rho)$ is a copy of the group S^1 , an equivariant map is determined by the image of one point so that $Maps(S(\rho), X)^{S^1} = X$, for any S^1 -space X (with action forgotten on the right-hand side), from which the similar statement follows for spectra (see [2]) and in particular MU^{SF} . The map i^* is thus the augmentation map.

The identification of the remaining term is through a Thom isomorphism for S^ρ . Note that if an equivariant cohomology theory has such Thom isomorphisms for all S^V with V complex it is said to be complex stable. We roughly follow the construction of Thom isomorphisms for unrestricted homotopical bordism given in section 10 of [9]. Unraveling definitions, we want to show that

$$\text{colim}_k \Omega^{\rho \oplus \bigoplus^k V^{SF}} TU^{SF}(3k) \simeq \text{colim}_k \Omega^{\mathbb{C} \oplus \bigoplus^k V^{SF}} TU^{SF}(3k). \tag{1}$$

We start by choosing linear isomorphisms. Choose coordinates on $U^{SF} = \bigoplus_{i=1}^\infty V^{SF}$ as $\bigoplus_{i=1}^\infty (v_i^\rho, v_i^\mathbb{C}, v_i^{\rho^*})$ where $v_i^\rho \in \rho$, $v_i^\mathbb{C} \in \mathbb{C}$ and $v_i^{\rho^*} \in \rho^*$. Recall $\sigma : U^{SF} \oplus V^{SF} \rightarrow U^{SF}$, chosen to define bonding maps for MU_{SF}^* , which in this notation sends $\bigoplus_{i=1}^\infty (v_i^\rho, v_i^\mathbb{C}, v_i^{\rho^*}) \oplus (u^\rho, u^\mathbb{C}, u^{\rho^*})$ to $\bigoplus_{i=1}^\infty (w_i^\rho, w_i^\mathbb{C}, w_i^{\rho^*})$, where $w_1^\rho = u^\rho$ and $w_i^\rho = v_{i-1}^\rho$ for $i > 1$. The vectors $w_i^\mathbb{C}$ and $w_i^{\rho^*}$ are defined similarly. Define $\sigma_1 : U^{SF} \oplus \rho^* \oplus \bigoplus^2 \mathbb{C} \xrightarrow{\cong} U^{SF}$ by

$$\bigoplus_{i=1}^\infty (v_i^\rho, v_i^\mathbb{C}, v_i^{\rho^*}) \oplus u^{\rho^*} \oplus u_1^\mathbb{C} \oplus u_2^\mathbb{C} \mapsto \bigoplus_{i=1}^\infty (w_i^\rho, w_i^\mathbb{C}, w_i^{\rho^*}),$$

where

$$w_i^\rho = v_i^\rho \quad w_i^\mathbb{C} = \begin{cases} u_i^\mathbb{C} & i \leq 2 \\ v_{i-2}^\mathbb{C} & i > 2 \end{cases} \quad w_i^{\rho^*} = \begin{cases} u_i^{\rho^*} & i = 1 \\ v_{i-1}^{\rho^*} & i > 1. \end{cases}$$

Define $\sigma_2 : U^{SF} \oplus \rho^* \oplus \bigoplus^2 \rho \xrightarrow{\cong} U^{SF}$ analogously so that the following diagram, in which the leftmost arrows are the obvious isomorphisms which reorder coordinates, commutes:

$$\begin{array}{ccccc} & & U^{SF} \oplus (\rho^* \oplus \bigoplus^2 \mathbb{C}) & \xrightarrow{\sigma_1 \oplus id} & U^{SF} \oplus (\rho^* \oplus \bigoplus^2 \mathbb{C}) & \xrightarrow{\sigma_2} & U^{SF} \\ & \nearrow & \oplus (\rho^* \oplus \bigoplus^2 \rho) & & & & \\ U^{SF} \oplus \bigoplus^2 V^{SF} & \xrightarrow{\sigma \oplus id} & U^{SF} \oplus V^{SF} & \xrightarrow{\sigma} & & & \\ & \searrow & U^{SF} \oplus (\rho^* \oplus \bigoplus^2 \rho) & \xrightarrow{\sigma_2 \oplus id} & U^{SF} \oplus (\rho^* \oplus \bigoplus^2 \rho) & \xrightarrow{\sigma_1} & U^{SF} \\ & & \oplus (\rho^* \oplus \bigoplus^2 \mathbb{C}) & & & & \end{array}$$

On passage to Thom spaces, σ_1 defines a map $S^{(\rho^* \oplus \bigoplus^2 \mathbb{C})} \wedge TU^{SF}(3k)$ to $TU^{SF}(3k + 3)$, or by adjointness $T(\sigma_1)^\dagger : TU_{3k}^{SF} \rightarrow \Omega^{(\rho^* \oplus \bigoplus^2 \mathbb{C})} TU^{SF}(3k + 3)$. Define

$$\beta_1 : \Omega^{(\rho \oplus \bigoplus^k V^{SF})} TU^{SF}(3k) \rightarrow \Omega^{(\mathbb{C} \oplus \bigoplus^{k+1} V^{SF})} TU^{SF}(3k + 3)$$

as sending an $f : S^{(\rho \oplus \bigoplus^k V^{SF})} \rightarrow TU^{SF}(3k)$ to its composite with $T(\sigma_1)^\dagger$, using adjointness and the standard isomorphism $(\rho \oplus \bigoplus^k V^{SF}) \oplus (\rho^* \oplus \bigoplus^2 \mathbb{C}) \cong \mathbb{C} \oplus \bigoplus^{k+1} V^{SF}$ to get an element of the range.

Define $\beta_2 : \Omega^{(\mathbb{C} \oplus \bigoplus^k V^{SF})} TU^{SF}(3k) \rightarrow \Omega^{(\rho \oplus \bigoplus^{k+1} V^{SF})} TU^{SF}(3k + 3)$ similarly by using σ_2 . By the commutativity of the diagram involving σ , σ_1 and σ_2 above, as well as standard facts about associativity of smash products and adjointness, $\beta_2 \circ \beta_1$ and $\beta_1 \circ \beta_2$ coincide with the structure maps in the colimits of Equation 1, so that β_1 and β_2 give rise to the isomorphism of Equation 1. Moreover, the map defining j^* at the prespectrum level composed with β_1 coincides with the definition of e_ρ , so that j^* is multiplication by e_ρ .

Finally, MU_* is concentrated in even degrees, as is MU_*^{SF} since by Theorem 2.11 it is a sub-algebra of Φ_*^{SF} which is so. Therefore this long exact sequence breaks up into short exact sequences, as stated. \square

We introduce operations in MU_*^{SF} which are essentially division by Euler classes. We will see below that these operations have a geometric representation.

Definition 3.3. • Let σ be the canonical (up to homotopy) splitting of the augmentation map α , defined by taking some $S^m \rightarrow TU(n)$, suspending it by $\bigoplus^m (\rho \oplus \rho^*)$ to get a map from $S^{\bigoplus^m V^{SF}}$ to a Thom space which is chosen as a subspace of $TU^{SF}(n)$.

- Let $\Gamma_\rho : MU_*^{SF} \rightarrow MU_{*+2}^{SF}$ (respectively Γ_{ρ^*}) be the splitting of multiplication by e_ρ (respectively e_{ρ^*}) which arises from the canonical splitting of α through Theorem 3.2.
- If I is a sequence of ρ and ρ^* , let $\Gamma_I(x)$ be the composite of the corresponding Γ_ρ and Γ_{ρ^*} applied to x . For example, $\Gamma_{\rho\rho^*}(x) = \Gamma_\rho\Gamma_{\rho^*}(x)$.
- For $x \in MU_*^{SF}$ let $\bar{x} = \sigma \circ \alpha(x)$.

The following lemma is immediate from the fact that $e_\rho\Gamma_\rho(x) = x - \bar{x}$.

Lemma 3.4. $\lambda(\Gamma_\rho(x)) = e_\rho^{-1}(\lambda(x) - \alpha(x))$ and similarly $\lambda(\Gamma_{\rho^*}(x)) = e_{\rho^*}^{-1}(\lambda(x) - \alpha(x))$.

We are now ready for our first computation.

Definition 3.5. Let B be the set of MU_*^T elements $\{e_\rho, e_{\rho^*}, Z_{n,\rho}, \text{ and } Z_{n,\rho^*}\}$ where $n \geq 2$. Order B as follows

$$e_\rho < e_{\rho^*} < Z_{2,\rho} < Z_{2,\rho^*} < Z_{3,\rho} < Z_{3,\rho^*} < \dots$$

Theorem 3.6. MU_*^{SF} is generated as a ring by classes $\Gamma_\rho^i \Gamma_{\rho^*}^j(x)$ where $x \in B$. Relations are

- (1) $e_\rho\Gamma_\rho(x) = x - \bar{x} = e_{\rho^*}\Gamma_{\rho^*}(x)$,
- (2) $\Gamma_V(x)(y - \bar{y}) = (x - \bar{x})\Gamma_V(y)$, where V is ρ or ρ^* ,
- (3) $\Gamma_V(e_V \cdot x) = x$, where V is ρ or ρ^* .
- (4) $\Gamma_{\rho^*}\Gamma_\rho(x) = \Gamma_\rho\Gamma_{\rho^*}(x) + \overline{\Gamma_\rho(x)}\Gamma_\rho\Gamma_{\rho^*}(e_\rho)$,
- (5) $\overline{e_V} = 0$.

MU_*^{SF} is free as a module over MU_* with an additive basis given by monomials $\Gamma_\rho^i \Gamma_{\rho^*}^j(x)m$, where $x \in B$, m is a monomial in the $y \geq x$ in B and with the following restrictions: if $x = e_{\rho^*}$ then $j = 0$; if $x = e_\rho$ and $j \neq 0$ then no positive power of e_{ρ^*} occurs in m ; if $i \neq 0$ then $j \neq 0$ and no positive power of e_ρ occurs in m .

Proof. Proposition 3.1 implies that if $y \in MU_*^{SF}$ then for some i and j , the product $x = e_\rho^i e_{\rho^*}^j y$ is in the subalgebra of MU_*^{SF} generated by B . Then $\Gamma_\rho^i \Gamma_{\rho^*}^j(x) = y$. By linearity, $\Gamma_\rho^i \Gamma_{\rho^*}^j(x)$ is a sum of $\Gamma_\rho^i \Gamma_{\rho^*}^j(m)$ for some monomials m in B . There is a product formula

$$\Gamma_\rho(wz) = \Gamma_\rho(w)z + \overline{w}\Gamma_\rho(z),$$

and similarly for Γ_{ρ^*} , as can be verified by applying λ , which is injective, to both sides using Lemma 3.4. Thus, $\Gamma_\rho^i \Gamma_{\rho^*}^j(m)$ is a sum of products of $\Gamma_\rho^i \Gamma_{\rho^*}^j(b)$ for $b \in B$, which means these classes generate.

Except for relation 4, verification of the relations is straightforward. In each case one checks the equality after λ , which is injective, using Lemma 3.4 as needed. For example, for relation 2, the image of both sides under λ is $e_V^{-1}(x - \bar{x})(y - \bar{y})$. For relation 4 we also need that $\overline{\Gamma_\rho(x)} = -\Gamma_\rho(x)$, which we derive as follows. Take relation 1 that $x = e_{\rho^*} \Gamma_{\rho^*}(x)$ and apply the product formula with $w = e_{\rho^*}$ and $z = \Gamma_{\rho^*}(x)$ to get that

$$\Gamma_\rho(x) = \Gamma_\rho(e_{\rho^*})\Gamma_{\rho^*}(x),$$

noting that the second term in the product formula vanishes since $\overline{e_{\rho^*}} = 0$. If we apply the augmentation map to both sides, $\overline{\Gamma_\rho(x)} = -\Gamma_\rho(x)$ will follow from computing that $\overline{\Gamma_\rho(e_{\rho^*})} = -1$. Represent $\Gamma_\rho(e_{\rho^*})$ as the composite $S^\rho \rightarrow S^{\rho^*} \rightarrow TU^{SF}(1)$, where the first map is through complex conjugation and the second is the unit map, which includes S^ρ as the Thom space of a fiber of the tautological bundle. This composite represents -1 when the S^1 action is forgotten.

To show that the members of the additive basis $\Gamma_\rho^i \Gamma_{\rho^*}^j(x)m$ are linearly independent over MU_* we apply λ , after which the verification is straightforward by looking at the leading terms $e_{\rho^*}^{-i} e_{\rho^*}^{-j} xm$.

To complete the proof we show that one can use the relations to reduce to the additive basis. Consider a product $\Gamma_{I_1}(x_1)\Gamma_{I_2}(x_2) \dots \Gamma_{I_k}(x_k)$ where x_1 is minimal among the x_i in order within B . We may use relation 2, rewritten as $\Gamma_\rho(x)y = x\Gamma_\rho(y) - \bar{x}\Gamma_\rho(y) + \bar{y}\Gamma_\rho(x)$ (and similarly for ρ^*) to perform a reduction. Choose y to be $\Gamma_{I_1}(x_1)$ and x to be $\Gamma_{I'_2}(x_2)$ where I'_2 is I_2 with the first ρ or ρ^* removed, to decrease either the number of operations Γ_V which are applied to non-minimal generators, in the cases of $x\Gamma_\rho(y)$ and $\bar{x}\Gamma_\rho(y)$, or the number of non-minimal generators, in the case of $\bar{y}\Gamma_\rho(x)$. Inductively, we reduce to a sum of $\Gamma_I(b)m$, where m is a monomial in B and b is less than any generator which appears in m . Finally, consider some $\Gamma_{I_1 \rho^* \rho I_2}(b)m$. We decrease the number of ρ and ρ^* which are out of order by applying relations 4 to get $\Gamma_{I_1 \rho \rho^* I_2}(x)m + \overline{\Gamma_{\rho I_2}(x)}\Gamma_{I_1 \rho \rho^*}(e_\rho)m$. Note each of these monomials still has Γ_V applied only to a minimal element of B . Inductively, we reduce to monomials in which Γ_ρ is applied after Γ_{ρ^*} . \square

We now turn our attention to Ω_*^{SF} , adding to the short list of geometric bordism theories which have been computed [3, 11, 23]. By the geometric realization Corollary 2.12 we can deduce the structure of Ω_*^{SF} algebraically from Theorem 3.6 and understanding of the localization map λ . We choose, in addition, to find explicit geometric representatives.

We start by making geometric constructions of Γ_ρ and Γ_{ρ^*} on classes represented by manifolds. These constructions follow ones made by Conner and Floyd.

Definition 3.7. Define $\gamma(M)$ for any stably complex S^1 -manifold to be the stably complex S^1 -manifold

$$\gamma(M) = M \times_{S^1} S^3 \sqcup (-\overline{M}) \times \mathbb{P}(\mathbb{C} \oplus \rho),$$

where S^3 has the standard Hopf S^1 -action and the S^1 -action on $M \times_{S^1} S^3$ is given by

$$\zeta \cdot [m, z_1, z_2] = [\zeta \cdot m, z_1, \zeta z_2]. \tag{2}$$

Define $\gamma^*(M)$ similarly with the quotient of $M \times S^3$ by the S^1 action in which τ sends $m, (z_1, z_2)$ to $\tau m, (\tau z_1, \tau^{-1} z_2)$ and with induced S^1 action on the quotient given by

$$\zeta \cdot [m, z_1, z_2] = [\zeta \cdot m, z_1, \zeta^{-1} z_2]. \tag{3}$$

Proposition 3.8. Let M be a stably complex S^1 -manifold. Then $\Gamma_\rho[M] = [\gamma(M)]$ and $\Gamma_{\rho^*}[M] = [\gamma^*(M)]$ in $MU_*^{S^1}$.

Proof. By Lemma 3.4 and the injectivity of λ , it suffices to check the fixed sets and normal data of $\gamma(M)$ and $\tilde{\gamma}(M)$. One type of fixed points of $\gamma(M)$ are points $[m, z_1, z_2]$ such that m is fixed in M and $z_2 = 0$. This fixed set is diffeomorphic to M^G , and its normal bundle is that of M^G in M crossed with the representation ρ . Crossing with ρ coincides with multiplying by e_ρ^{-1} in F_*^{SF} . The second set of fixed points are $[m, z_1, z_2]$ such that $z_1 = 0$. This set of fixed points is diffeomorphic to M , and its normal bundle is the trivial bundle ρ^* .

Hence, if $x = \lambda([M])$, then the image of $[\gamma(M)]$ is $x e_\rho^{-1} + \overline{M} e_{\rho^*}^{-1}$. By subtracting the image of $\overline{M} \times \mathbb{P}(\mathbb{C} \oplus \rho)$ we obtain $x e_\rho^{-1} - \overline{M} e_\rho^{-1}$. By Lemma 3.4, this is $\lambda(\Gamma_\rho([M]))$. The analysis is similar for $\gamma^*(M)$. \square

The classes $\Gamma_I(Z_{n,\rho})$ and $\Gamma_I(Z_{n,\rho^*})$ can thus be realized geometrically, as $Z_{n,\rho}$ and Z_{n,ρ^*} are represented by linear actions on projective spaces. Additionally we have the following.

Lemma 3.9. $\Gamma_{\rho\rho^*}(e_\rho) = \mathbb{P}(\mathbb{C} \oplus \rho)$.

Proof. The equality of these classes also follows from computation of their image under λ . Proposition 2.9 states that $\lambda(\mathbb{P}(\mathbb{C} \oplus \rho)) = e_\rho^{-1} + e_{\rho^*}^{-1}$. To show that this is also $\lambda(\Gamma_\rho \Gamma_{\rho^*}(e_\rho))$, by applying Lemma 3.4 twice it suffices to know that $\overline{\Gamma_\rho(e_{\rho^*})} = -1$, which was shown in the proof of Theorem 3.6. \square

Given the general complexities of equivariant bordism, in particular for the geometric theories, Ω_*^{SF} has a remarkably simple form.

Theorem 3.10. Ω_*^{SF} is generated as an algebra over MU_* by classes $\gamma^i(\gamma^*)^j \mathbb{P}(\mathbb{C}^n \oplus \rho)$ for $n \geq 1$ and $\gamma^i(\gamma^*)^j \mathbb{P}(\mathbb{C}^n \oplus \rho^*)$ where $n \geq 2$. Relations are

- (1) $\gamma(x)(y - \bar{y}) = (x - \bar{x})\gamma(y)$, and similarly for γ_* ,
- (2) $\gamma^* \gamma(x) = \gamma \gamma^*(x) + \gamma(x) \mathbb{P}(\mathbb{C} \oplus \rho)$,

where x and y can be any stably complex S^1 -manifolds, in particular those in the generating set above. An additive basis is given by monomials $\gamma^i(\gamma^*)^j(x)m$ where m is a monomial in $\mathbb{P}(\mathbb{C}^n \oplus \rho)$ and $\mathbb{P}(\mathbb{C}^n \oplus \rho^*)$ and x is such a projective space of smaller dimension than those appearing in m .

Proof. We start with Corollary 2.12, which at the level of coefficients looks like

$$\begin{array}{ccc} \Omega_*^{SF} & \longrightarrow & F_*^{SF} = MU_*[e_\rho^{-1}, e_{\rho^*}^{-1}, Z_{n,\rho}, Z_{n,\rho^*} | n \geq 2] \\ \downarrow & & \downarrow \\ MU_*^{SF} & \longrightarrow & \Phi_*^{SF} = MU_*[e_\rho^{\pm 1}, e_{\rho^*}^{\pm 1}, Z_{n,\rho}, Z_{n,\rho^*} | n \geq 2], \end{array}$$

where $n > 0$. All maps are inclusions, so we are looking to characterize the elements in MU_*^{SF} which map to F_*^{SF} . Observe that F_*^{SF} is an MU_* -direct summand of Φ_*^{SF} . A complementary submodule C_* is the submodule generated by reduced monomials in which a strictly positive power of e_ρ or e_{ρ^*} appears. We analyze the image under λ of each additive basis element from Theorem 3.6 in terms of the $F_*^{SF} \oplus C_*$ decomposition of Φ_*^{SF} .

Consider the basis element $y = \Gamma_\rho^i \Gamma_{\rho^*}^j(x)m$ in which x is an element of the generating set B of Theorem 3.6 and m is a monomial in the elements of B , each greater than or equal to x in the ordering on B and with additional provisions of $x = e_\rho$ or e_{ρ^*} . This y maps to F_*^{SF} if $x = Z_{i,\rho}$ or Z_{i,ρ^*} because by Lemma 3.4 $\lambda(\Gamma_I(Z_{i,\rho}))$ and $\lambda(\Gamma_I(Z_{i,\rho^*}))$ are polynomials over MU_* in $e_\rho^{-1}, e_{\rho^*}^{-1}$ and $Z_{i,\rho}$ or respectively Z_{i,ρ^*} and m is a monomial in $Z_{n,\rho}$ and Z_{n,ρ^*} for some $n \geq i$ by the ordering on B . Next we focus on when $x = e_\rho$. By applying Lemma 3.4 we see that $\lambda(\Gamma_{\rho^*}^j(e_\rho)) = e_{\rho^*}^{-j} e_\rho + P$, where P is a polynomial in $e_{\rho^*}^{-1}$ over MU_* . Continuing we see $\lambda(\Gamma_\rho^i \Gamma_{\rho^*}^j(e_\rho)) = e_\rho^{1-i} e_{\rho^*}^{-j} + Q$, where $Q \in MU_*[e_\rho^{-1}, e_{\rho^*}^{-1}]$. Recall that for the basis element $y = \Gamma_\rho^i \Gamma_{\rho^*}^j(e_\rho)m$ with $i, j > 0$, the generators e_ρ and e_{ρ^*} do not appear in m . We deduce that $\lambda(y)$ is in F_*^{SF} since both $\lambda(\Gamma_\rho^i \Gamma_{\rho^*}^j(e_\rho))$ and $\lambda(m)$ are.

There are three classes of basis elements remaining, namely $e_\rho^i e_{\rho^*}^j m$ with i or $j > 0$, $\Gamma_{\rho^*}^i(e_\rho) e_{\rho^*}^j m$ with $i > 0$ and $\Gamma_\rho^i(e_{\rho^*}) e_{\rho^*}^j m$ with $i > 0$, where m is a monomial in $MU_*[Z_{n,\rho}, Z_{n,\rho^*} | n \geq 2]$. We take the image under λ and project onto C_* to get $e_\rho^i e_{\rho^*}^j m$, $e_\rho^{j+1} e_{\rho^*}^{-i} m$ and $e_\rho^{-i} e_{\rho^*}^{j+1} m$ respectively. These three kinds of classes are linearly independent taken all together in C_* (in fact, they form a basis as m varies over all possible monomials).

Summarizing, we have shown that the additive basis elements for MU_*^{SF} fall into two groups, one group which maps to F_*^{SF} and one group whose projections

onto C_* is linearly independent. Therefore, the only elements of MU_*^{SF} which can map to F_*^{SF} are in the span of the first group. By Corollary 2.12 the first group serves as an additive basis for Ω_*^{SF} .

We will verify the additive basis stated in the theorem only after we use the current additive basis to check that Ω_*^{SF} is generated as an algebra by classes $\gamma^i(\gamma^*)^j \mathbb{P}(\mathbb{C}^n \oplus \rho)$ and $\gamma^i(\gamma^*)^j \mathbb{P}(\mathbb{C}^n \oplus \rho^*)$. By Proposition 3.8, $\gamma^i(\gamma^*)^j \mathbb{P}(\mathbb{C}^n \oplus \rho)$ represents $\Gamma_\rho^i \Gamma_{\rho^*}^j(Z_{n,\rho})$. These generate the additive basis elements of the form $\Gamma_\rho^i \Gamma_{\rho^*}^j(x)m$ where $x = Z_{i,\rho}$ or Z_{i,ρ^*} . To see that $\Gamma_\rho^i \Gamma_{\rho^*}^j(e_\rho)$ where $i, j > 0$ is in this subalgebra, first note that it is true for $i, j = 1$ by Lemma 3.9. We apply relation 4 from Theorem 3.6 to reduce to this case as follows

$$\begin{aligned} \Gamma_\rho^i \Gamma_{\rho^*}^j(e_\rho) &= \Gamma_\rho^{i-1} \Gamma_{\rho^*} \Gamma_\rho \Gamma_{\rho^*}^{j-1}(e_\rho) - \overline{\Gamma_\rho^i \Gamma_{\rho^*}^j(e_\rho)} \Gamma_\rho \Gamma_{\rho^*}(e_\rho) \\ &= \dots = \Gamma_\rho^{i-1} \Gamma_{\rho^*}^{j-1}(\Gamma_\rho \Gamma_{\rho^*}(e_\rho)) + Q, \end{aligned}$$

where $Q \in MU_*[\Gamma_\rho \Gamma_{\rho^*}(e_\rho)]$. We see that Q is in our subalgebra by Lemma 3.9, which along with Proposition 3.8 implies that $\Gamma_\rho^{i-1} \Gamma_{\rho^*}^{j-1}(\Gamma_\rho \Gamma_{\rho^*}(e_\rho)) = \gamma^{i-1}(\gamma^*)^{j-1} \mathbb{P}(\mathbb{C} \oplus \rho)$. We deduce that $\Gamma_\rho^i \Gamma_{\rho^*}^j(e_\rho)$ is in our subalgebra, so that all additive basis elements for Ω_*^{SF} are generated by the classes as stated.

The reduction to the additive basis given in the statement of the theorem, and thus the proof that relations are complete, is similar to that given in Theorem 3.6. Given a monomial in $\gamma^i(\gamma^*)^j \mathbb{P}(\mathbb{C}^n \oplus \rho)$ and $\gamma^i(\gamma^*)^j \mathbb{P}(\mathbb{C}^n \oplus \rho^*)$ we use relation 1 to reduce to monomials in which the operations γ and γ^* are applied to only the projective space of the smallest dimension, and then use relation 2 to reorder the operations. □

4. Further directions in geometric bordism

We are led to ask about geometric bordism for unrestricted S^1 actions or for actions by other groups. Bordism which is equivariant with respect to \mathbb{Z}/p behaves similarly to semi-free bordism, as expected. The Conner-Floyd and tom Dieck exact sequences are well-known in those cases (indeed, it is for \mathbb{Z}/p that these sequences first appeared in [6] and [7]), and the theories were computed in [11, 13, 20], though the description is complicated by the classes which are not restrictions from Ω_*^{SF} . As in Corollary 2.12, these theories fit in a pullback square

$$\begin{array}{ccc} \Omega_*^{U, \mathbb{Z}/p} & \longrightarrow & F_*^{\mathbb{Z}/p} \\ \downarrow & & \downarrow \\ MU_*^{\mathbb{Z}/p} & \longrightarrow & \Phi_*^{\mathbb{Z}/p}, \end{array}$$

which follows because the kernels and cokernels of the horizontal maps are the even and odd degrees, respectively, of $MU_*(B\mathbb{Z}/p)$. From this one can recover the Kosniowski generators from those of [20]. Kriz in [13] gave the first computation of $MU_*^{\mathbb{Z}/p}$, but the relationship with the Kosniowski generators of geometric bordism is not clear in Kriz's approach.

Less is known about Ω_*^{U,S^1} , but we give a conjectural framework as follows. In [21], $MU_*^{S^1}$ was computed, and it has the following prominent features, much as we have seen for semi-free bordism:

- Basic classes include Euler classes e_V and linear actions on projective spaces $Z_{n,V} = [\mathbb{P}(\mathbb{C}^n \oplus V)]$ for all irreducible representations V .
- There is a sequence of inclusions $MU_*[e_V, Z_{n,V} | n \geq 2] \subset MU_*^{S^1} \subset \Phi_* = MU_*[e_V^{\pm 1}, Z_{n,V} | n \geq 2]$, where V ranges over all irreducibles.
- There are operations Γ_V such that $e_V \Gamma_V(x) = x - \beta_V(x)$, where $\beta_V(x)$ is restriction to $MU_*^{K(V)}$ followed by a splitting map back to $MU_*^{S^1}$. Here $K(V)$ is the kernel of $V : S^1 \rightarrow \mathbb{C}^\times$. Note that β_V is not canonical if $V = \rho$ or ρ^* .
- $MU_*^{S^1}$ is generated over the operations Γ^V by e_V and $Z_{n,V}$.

There are also the following facts about the geometric theory:

- (Comezaña and Löffler) The Pontryagin-Thom map $\Omega_*^{U,S^1} \rightarrow MU_*^{S^1}$ is injective.
- Under the inclusion $MU_*^{S^1} \rightarrow \Phi_*$, the geometric theory Ω_*^{U,S^1} maps to $F_* = MU_*[e_V^{-1}, Z_{n,V}]$.

A first important step towards understanding Ω_*^{U,S^1} would be to establish the analogue of Corollary 2.12, for which there are isolated computations, as well as Corollary 2.12, as evidence.

Conjecture 4.1. *The square*

$$\begin{array}{ccc}
 \Omega_*^{U,S^1} & \longrightarrow & F_* \\
 \downarrow & & \downarrow \\
 MU_*^{S^1} & \longrightarrow & \Phi_*
 \end{array}$$

is a pull-back.

This conjecture is likely to be approachable through the families filtration, perhaps with S^1 replaced by $\mathbb{Z}/(p^2)$ as a starting point. There would be two more steps needed to parallel our computation of Ω_*^{SF} .

Question 4.2. Is there a version of the construction γ for representations other than ρ and ρ^* ? In other words, given some M can one find a manifold which represents $\Gamma_V(M)$?

There is some doubt as to whether such a construction should even exist, given that embedded in such a construction would be a construction of splitting maps $MU_*^{\mathbb{Z}/n} \rightarrow MU_*^{S^1}$, which are non-canonical and chosen with some effort in [21]. A concrete starting point would be to search for a manifold whose fixed sets are $D(\rho^2)$ crossed with the fixed sets of $\mathbb{P}(\mathbb{C}^n \oplus \rho^3)$ and $\mathbb{P}(\mathbb{C}^n \oplus \rho)$ with its orientation reversed.

We should add that even Γ_ρ deserves more attention. For example, what are the relationships between the equivariant characteristic numbers (in both cohomology

and K -theory) of M and $\gamma(M)$? How might Γ_ρ be used to construct familiar classes in MU_* ? For example, in Proposition 6.5 of [21] we show that $(\Gamma_\rho)^k(e_{\rho^n})$ form the coefficients of the n -series.

Finally, to compute $\Omega_*^{S^1}$ it would be helpful to understand the analogue of Theorem 1.1, which promises to be much more difficult in the general setting. Lemma 3.9 that $\Gamma_\rho \Gamma_{\rho^*}(e_\rho) = [\mathbb{P}(\mathbb{C} \oplus \rho)]$ is surprising at first, since Euler classes seem unrelated to geometric ones. But in fact all manifolds with framed fixed sets, in particular those with isolated fixed sets, must arise within the description of $MU_*^{S^1}$ of [21] as $\Gamma_I(x)$ where x is a polynomial in e_V . These constructions seem to be the most difficult part of describing geometric classes within the homotopical setting, so once we proved Theorem 1.1 we knew Theorem 3.10 would be possible. To provide a framework for such investigation, we make the following.

Conjecture 4.3. *Stably complex S^1 actions with isolated fixed points up to bordism are generated by linear actions on projective spaces $\mathbb{P}(V_1 \oplus V_2 \oplus \cdots \oplus V_k)$, where the weights of the V_i are relatively prime.*

See Theorem 1.6 of [21] for an example. Taken all together, these questions and conjectures point to the following.

Conjecture 4.4. *Ω^{S^1} is generated over geometric versions of the operations Γ_V by linear actions on projective spaces.*

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