THE HOMOTOPY GROUPS $\pi_*(L_2S^0)$

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1. INTRODUCTION

To determine the homotopy groups $\pi_*(S^n)$ of spheres is one of the main problems in homotopy theory, and several methods have been found to reach it. One of them is the one using a generalized Adams spectral sequence based on a ring spectrum $E$. In many examples, it converges to the homotopy groups of the localized spheres with respect to $E$, not to those of the unlocalized one. Consider the Brown–Peterson spectrum $BP$ at a prime $p$. Then the localized sphere with respect to $BP$ is the one localized at the prime $p$. We call the spectral sequence based on it the Adams–Novikov spectral sequence. It converges to the homotopy groups of the $p$-localized spheres and its $E_2$-term is expressed by the Ext group, which is computable object. For odd prime $p$, it seems more powerful than the Adams spectral sequence that is based on the Eilenberg–MacLane spectrum. When we use the spectral sequence, we have to compute the $E_2$-term. The $E_2$-term $E_2^{p,q}$ of the Adams–Novikov spectral sequence based on $BP$ for the sphere is computed for $t < p^3q$ by Ravenel, and for $s < 3$ by Miller, Ravenel and Wilson using the chromatic spectral sequences. The $E_1$-term of the chromatic spectral sequence does not only converge to the $E_2$-term of the Adams–Novikov spectral sequence but also is itself the $E_2$-term of the Adams–Novikov spectral sequence for computing homotopy groups of a spectrum whose existence is shown by Ravenel [12].

Now we fix a prime $p > 3$. The $E_1$-term of the chromatic spectral sequence is denoted by $H^* M^*$ and converging to $H^{*+1} N^0$ (see section 4). In this paper, we study it for the Johnson–Wilson spectrum $E(2)$ whose homotopy group $E(2)_*$ is the polynomial ring $\mathbb{Z}[v_1, v_2, v_3^{-1}]$ on the generators of $BP_*$. Here note that $E(n)$ is not proved to be a ring spectrum. Since $E(n)$ is a spectrum representing the homology theory $E(n)_* = E(n)_* \otimes_{BP_*, BP_*} (X)$, we can construct a generalized Adams spectral sequence based on $E(n)$ similarly to the original one. So we use $E(2)$ here. Then we may put $N^0 = E(2)_*, N^1 = E(2)_*/(p)$ and $M^*_v = v_0^{-1} N^*_v$ ($v_0 = p$) for the $E_1$-term of the chromatic spectral sequence. Moreover, $M^*_0 = 0$ if $n + s > 2$. So far, for this case, the $E_1$-terms of the chromatic spectral sequence are computed for all $n$ and $s$ but $n = 0$ and $s = 2$. Here we obtain this case $(n, s) = (0, 2)$ (Theorem 2.3). This module $H^* M^*_0$ seems to have many applications. One of them is the one for the Greek letter elements in the stable homotopy groups of spheres. As is remarked in [6], it gives complete information on products of $a$'s and $p$'s and decomposability of the $y$'s. Let $L_2$ denote the Bousfield localization functor with respect to the spectrum $E(2)$ [2, 10]. Before our computation, we only know about the homotopy groups $\pi_*(L_2 M)$ for the mod $p$ Moore spectrum $M$. Our computation on $H^* M^*_0$ gives rise to the homotopy groups of $\pi_*(L_2 S^0)$ (Theorem 2.4) by the mod $p$ Bockstein
spectral sequence. This has also much information on the products of the homotopy elements. In fact, the localization map $\eta:S^0 \to L_2S^0$ induces the homomorphism $\pi_\ast(S^0) \to \pi_\ast(L_2S^0)$, by which we can tell some information. On this map, we have the relating exact sequence $H^{*+3}N_0^{3}\to \pi_\ast(S^0) \to H^{*+3}N_0^{3}$ of the $E_2$-terms given in [3], where $N_0^{3} = BP_\ast/(p^n, u^n, v^n)$, and $Gr$ denotes the universal Greek letter map introduced in [6]. Thus the map $\eta_\ast$ maps the elements of $\pi_\ast(S^0)$ whose filtration degree is less than 3 monomorphically to those of $\pi_\ast(L_2S^0)$. Furthermore, M. Hopkins and D. Ravenel show that $L_2X$ is homotopic to $L_2S^0 \wedge X$ (cf. [13]). So our computation will be a grip to understand the $L_2$-localization. These applications will be discussed in the forthcoming papers.

2. STATEMENT OF RESULTS

Let $E(2)$ denote the Johnson–Wilson spectrum at a prime $p > 3$ with the homotopy groups $E(2)_\ast = \mathbb{Z}(p)[v_1, v_2, v_2']$. Then it is known (cf. [2, 10]) that we have the Adams–Novikov spectral sequence converging to $\pi_\ast(L_2S^0)$ with the $E_2$-term

$$H^{*+4}E(2)_\ast = \text{Ext}_E^{E(2)}(E(2)_\ast, E(2)_\ast).$$

Here $L_2$ denotes the Bousfield localization functor with respect to $E(2)$. Consider the comodules $N_i^k$ and $M_i^k$ for $i + n \leq 2$ such that $M_i^k = v_i^{n-i}N_i^k$, $N_i^{2-n} = M_i^{2-n}$ and

$$N_0^k = E(2)_\ast, \quad N_0^{1} = E(2)_\ast/(p), \quad N_0^{2} = E(2)_\ast/(p, v_1), \quad N_1^{1} = E(2)_\ast/(p, v_1'), \quad N_2^{0} = E(2)_\ast/(p^n, v_1').$$

Then we have the chromatic spectral sequence converging to our target $H^{*+4}E(2)_\ast$ with the $E_1$-term $H^1M_0^1$. The $E_1$-terms for $s < 2$ are determined in [6]. In order to determine $H^1M_0^1$, we have the $v_1$- and the mod $p$-Bockstein spectral sequences converging to $H^1M_0^1$ with the $E_1$-term $H^1M_0^1$ for $n = 0, 1$. Ravenel shows the following result.

**Theorem 2.1** (Ravenel [9]). $H^1M_0^1 = F_p[v_2, v_2'^{-1}](1, h_0, h_1, g_0, g_1, h_0h_1) \otimes E(\zeta)$.

Here $F_p$ denotes the prime field of characteristic $p$, which is identified with $\mathbb{Z}/p$, $R\{x\}$ denotes the $R$-module generated by $x$, and $F(x)$ the exterior algebra generated by $x$. By the $v_1$-Bockstein spectral sequence, we compute $H^1M_0^1$ from Theorem 2.1. For simplicity, we denote a cocycle by its leading term. Put

$$X = F_p[v_1]\{v_2^mp/v_1^n : n \geq 0, s \in \mathbb{Z} - p\mathbb{Z}\}$$

$$X_\infty = F_p[\{1/v_1^j : j > 0\}] \cong F_p[v_1, v_1^{-1}]/F_p[v_1]$$

$$Y_0 = F_p[v_1]\{v_2^mp_1/v_1^n + A_1 : m \in \mathbb{Z}(0), n = v_p(m)\}$$

$$Y_1 = F_p[v_1]\{v_2^mp_1/v_1^n + A_1 : m \in \mathbb{Z}(2), n = v_p(m)\}$$

$$Y = F_p[v_1]\{v_2^m, v_2'^{m-1}h_1/v_1^{n-1}, 1 \in \mathbb{Z}\}$$

$$Y_\infty = F_p[\{h_0/v_1^j : j > 0\}] \cong F_p[v_1, v_1^{-1}]/F_p[v_1]$$

$$G = F_p[v_1]\{v_2^m, v_2'^{m-1}g_1/v_1^{n-1}, v_2g_0/v_1 : n \geq 1, s + 1 \in \mathbb{Z} - p\mathbb{Z}\}.$$
by
\[ a_0 = 1, \quad a_n = p^n + p^{n-1} - 1 \]
\[ A_n = (p + 1)(p^n - 1)/(p - 1) \]
\[ A_n' = (p + 1)(p^{n+1} - p^n + (p^n - 1)/(p - 1)) \]
and we use the subsets of integers
\[ \mathbb{Z}(0) = \{ m : m = sp^n \text{ with } p \mid s(s + 1) \} \]
\[ \mathbb{Z}(2) = \{ m : m = (sp^2 - 1)p^n \}. \]
Then we have the structure of \( H^*M_1 \) shown as follows.

**Theorem 2.2.** (Miller et al. [6], Shimomura and Tamura [16] and Shimomura [14]).

\[ H^*M_1 = (X \oplus X_\infty \oplus Y_0 \oplus Y_1 \oplus Y \oplus Y_\infty \oplus G) \otimes E(\zeta). \]

In this paper, we use the mod \( p \)-Bockstein spectral sequence, and obtain the following theorem.

**Theorem 2.3.** The module \( H^*M_0 \) is isomorphic to
\[ (X_\infty \oplus Y_{\infty, c} \oplus G^\infty) \otimes E(\zeta) \otimes X^\infty \oplus X_{\infty, c} \oplus Y_{0, c} \oplus Y_{1, c} \oplus Y^\infty \oplus G^\infty. \]

Here the modules are defined by
\[ X^\infty = Z\{v^p/p^{i+1}v^j : n \geq 0, s \in \mathbb{Z} - p\mathbb{Z}, i \geq 0, j \geq 1 \text{ with } p^i j \leq a_{n-1} \text{ and either } p^{i+1} \nmid j \text{ or } a_{n-1} - 1 < j \} \]
\[ X_{\infty, c} = Z\{v^p/p^{i+1}v^j : i = v_p(j) \geq 0 \} \]
for dimension 0,
\[ X_{\infty, c} = Z\{v^2p^{p+1}/p^{i+1}v^j : s \in \mathbb{Z} - p\mathbb{Z}, j > 0, p^i j \leq a_{n-1} \]
\[ \text{either } p^{i+1} \nmid j \text{ or } j > a_{n-2} \text{, and } p^{i+1} \mid j \text{ if } p^{i+1} \mid j \text{ for } s = tp^{i+1} - 1 \text{ with } k \geq 0 \} \]
\[ Y_{0, c} = Z\{v^p h_0/p^{i+1}v^j : p \mid s(s + 1), \text{ for } k = 0, i = n, \text{ and for } k > 0, \]
\[ kp^i + 1 \leq A_{n-i} + 2, kp^i + 1 > a_{n-i} \text{ if } p \nmid k, \text{ and } > A_{n-i-1} + 2 \text{ otherwise} \}
\[ Y_{1, c} = Z\{v^{(p^2-1)p}h_0/p^{i+1}v^j : l = n + 1 \text{ if } k = 0; \text{ for } k > 0 \text{ with } kp^i > a_{n-i}, \]
\[ l = i > 0 \text{ for } p^{n+2} < kp^i < p^{n+2} - p^n + A_{n-i+1} + 2 \text{ and } \]
\[ p^{n+2} - p^n + A_{n-i} + 2 \leq kp^i \text{ if } p \mid k \]
\[ l = i + 1 \text{ for } i = 0 \text{ and } p \mid (k + p^{n-1}), \text{ for } kp^i = (p^2 - 1)p^n \text{ or } \]
\[ kp^i < p^{n+2} - p^n, \text{ for } p \mid (k + p^{n-1}) \text{ and } 0 < i \leq n \]
\[ l = n + 2 \text{ for } i = n, k \leq p^2 - 1, p \mid (k + 1) \text{ and } k \neq p^2 - p - 1; \text{ and } \]
\[ l = n + 3 \text{ if } i = n \text{ and } k = p^2 - p - 1 \}
\[ Y_{\infty, c} = Z\{v^{p-1}h_0/p^{i+1}v^j : l = 1 \text{ if } j < p - 1, \text{ and } l = 2 \text{ if } p \mid t \text{ and } j = p - 1 \}
\[ Y_{\infty, c} = \mathbb{Q}/\mathbb{Z}(p) \text{ generated by the set } \{h_0/p^iv^j, j > 0 \} \]
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for dimension 1, and

\[ G^\infty = G_c^\infty \oplus Y_c^\infty \]

\[ Y_c^\infty = (Y_{0,c}^\infty \oplus Y_{1,c}^\infty) \otimes \mathbb{Z}_{(p)}(\zeta) \]

for

\[ Y_{0,c}^\infty = \mathbb{Z}_{(p)}(v_2^{p\infty}h_0/p^{i+1}v_1^{p\infty+1}; p \not| s(s + 1), \]

\[ k \neq 0, A_{n-i-1} + 1 < kp^{i+1} \leq A_{n-i} + 1 \text{ for } i \geq 0 \}

\[ Y_{1,c}^\infty = \mathbb{Z}_{(p)}(v_2^{p\infty+1}v_1^{p\infty+1}; k \neq 0, \]

\[ p^{s+2} - p^a + A_{n-i-1} + 1 < kp^{i+1} \leq p^{s+2} - p^a + A_{n-i} + 1 \text{ for } i \geq 0 \}

\[ G_c^\infty = \mathbb{Z}_{(p)}(v_2^{p\infty}g_0/p^{i+1}v_1^{p\infty+1}, v_2^{p\infty-(p^n-1)(p^n-1)}g_0/p^{i+1}v_1^{p\infty+1}; \]

\[ p \not| (s + 1), 0 < j \leq a_n, p^{i+1} \not| (j + A_{n-i-1} + 1) \text{ if } s = up^{i} \in \mathbb{Z}(0), \]

\[ p^{i+1} \not| (j + A_{n-i} + 1) \text{ if } s = up^{i} \in \mathbb{Z}(2), \text{ and } l = i + 1 \text{ if } n \geq 1 \text{ and } v_p(s) = i; \]

\[ l = i + 1 \text{ if } n \geq 1 \text{ and } v_p(s) = i; \]

\[ G_0^\infty = \mathbb{Q}/\mathbb{Z}_{(p)} \text{ generated by the set } \{g_0/p^{i+1}v_1^{j}; j > 0\}. \]

As a corollary of this theorem, we have the \( E_2 \)-term \( H^*(L^2S^0)_\bullet \) of the Adams–Novikov spectral sequence. Furthermore it collapses since \( E_2^s = 0 \) for \( s > 4 \), and so the \( E_2 \)-term is isomorphic to the homotopy groups of \( L^2S^0 \). Thus we have our main theorem.

**Theorem 2.4.** \( 
\pi_\ast(L^2S^0) \) is isomorphic to \( H^*(L^2S^0)_\bullet \), which is isomorphic to

\[ \mathbb{Z}_{(p)} \oplus \mathbb{Z}_{(p)}(v_1^{p\infty/p^{i+1}}; i \geq 0, s \geq 0, p \not| s) \oplus X^\infty \]

\[ \oplus Y_{0,c}^\infty \oplus Y_{1,c}^\infty \oplus Y_c^\infty \oplus Xc^\infty \oplus (X^\infty \otimes \mathbb{Z}_{(p)}(\zeta)) \]

\[ \oplus Y_{0,c}^\infty \oplus G_c^\infty \oplus (Y_{0,c}^\infty \otimes \mathbb{Z}_{(p)}(\zeta)) \oplus G_0^\infty \]

\[ \oplus (G_0^\infty \otimes \mathbb{Z}_{(p)}(\zeta)). \]

The degrees of the elements are read off from Theorem 10.1 as follows. Here a homotopy element \( \xi \in \pi_\ast(L^2S^0) \) has degree \( r \) if \( \xi \in \pi_r(L^2S^0) \), and we denote \( |\xi| = r \). Then all elements in the first factor are 0. If we identify the elements in the theorem with the corresponding homotopy elements under the isomorphism, we have degrees:

\[ |v_1^{i+1}| = jq - 1 \]

\[ |v_1^{p+1}v_1^{p+1}| = m(p + 1)q + jq - 2 \]

\[ |v_1^{p+1}h_0/p^{i+1}v_1^{p+1}| = m(p + 1)q + q - jq - 3 \]

\[ |v_2^{p-1}h_1/p^{i+1}v_1^{p+1}| = tp(p + 1)q - q - jq - 3 \]

\[ |v_2^{p+1}g_0/p^{i+1}v_1^{p+1}| = m(p + 1)q + q - jq - 4 \]

\[ |v_2^{p+1}g_1/p^{i+1}v_1^{p+1}| = m(p + 1)q - q - jq - 4 \]

and for the elements of the form \( z \otimes \zeta \),

\[ |z \otimes \zeta| - |z| - 1. \]
3. HOPF ALGEBROIDS

Let $E$ be a ring spectrum, and denote $E_\ast = E_\ast(S^0)$. If the homology $E_\ast(E)$ of $E$ is flat over $E$, then the pair $(E_\ast, E_\ast(E))$ becomes a Hopf algebroid in the usual way (cf. [1, 11]), and we can do homological algebra in the category of $E_\ast(E)$-comodules (cf. [11, A1]).

Among such spectra $E$, at each prime number $p$, we have the Brown–Peterson spectrum $BP$ and the Johnson–Wilson spectrum $E(n)$ for a nonnegative integer $n$. Here we note that, although we do not know whether or not $E(n)$ is a ring spectrum, we have the Hopf algebroid $(E(n)_\ast, (E(n)_\ast(E(n))))$ whose structure is induced from that of $BP_\ast(BP)$, since $E(n)_\ast(X) = E(n)_\ast \otimes_{BP_\ast} BP_\ast(X)$ for any spectrum $X$. Here the action of $BP_\ast$ to $E(n)_\ast$ is given by sending $v_k$ ($k > n$) to 0, in which $v_k$ is the Hazewinkel’s generator of the coefficient rings $E(0)_\ast = \mathbb{Q}$.

$$BP_\ast = \mathbb{Z}(p)[v_1, v_2, \ldots] \quad \text{and} \quad E(n)_\ast = \mathbb{Z}(p)[v_1, \ldots, v_n, v_n^{-1}] \quad (3.1)$$

for $n > 0$ (cf. [11]). Their self-homologies are

$$BP_\ast(BP) = BP_\ast[t_1, t_2, \ldots] \quad \text{and} \quad E(n)_\ast(E(n)) = E(n)_\ast \otimes_{BP_\ast} BP_\ast(BP) \otimes_{BP_\ast} E(n)_\ast. \quad (3.2)$$

We obtain the formulae of the structure maps of the Hopf algebroids associated to these spectra by [7] (cf. [11]). The structure of the Hopf algebroid associated to $E(n)$ is induced from that of $BP_\ast$. So we give here the formulae for $BP_\ast$. The left unit $\eta_L : BP_\ast \to BP_\ast(BP)$ is the inclusion $BP_\ast \subset BP_\ast(BP)$. Then $BP_\ast(BP)$ is a left $BP_\ast$-module by $\eta_L$. The right unit $\eta_R : BP_\ast \to BP_\ast(BP)$, which also gives $BP_\ast(BP)$ a right $BP_\ast$-module structure, sustains Landweber’s formula

$$\eta_R(v_n) = v_n + v_{n-1} t_{p^{n-1}} - v_{p-1} t_1 \quad (3.3)$$

mod $I_{n-1}$ for the prime ideal $I_n = (p, v_1, \ldots, v_{n-1})$ of $BP_\ast$. We also have

$$\eta_R(v_1) = v_1 + pt_1 \quad (3.4)$$

$$\eta_R(v_2) = v_2 + v_1 t_2^p + pt_2 - t_1(v_1 + pt_1)(p + 1)v_1^2 t_1 - p^{-1}((v_1 + pt_1)^p - v_1^p) \equiv v_2 + v_1 t_2^p + pt_2 - (p + 1)v_1^2 t_1 \mod (p^2) \quad (3.5)$$

$$\eta_R(v_3) = v_3 + v_2 t_3^p + v_1 t_2^p - t_1 \eta_R(v_2)^p + v_1^2 V \mod (p, v_1^p) \quad (3.6)$$

where we use the same notation $V$ as that of [16] defined by

$$pv_1 V = v_2^p + v_1^p t_1^p - v_1^2 t_1^p - (v_2 + v_1 t_1^p - v_1 t_1^p)^p. \quad (3.7)$$

For the diagonal $\Delta : BP_\ast(BP) \to BP_\ast(BP) \otimes_{BP_\ast} BP_\ast(BP)$, we have

$$\Delta(t_1) = t_1 \otimes 1 + 1 \otimes t_1 \quad (3.8)$$

where $g, T \in BP_\ast(BP) \otimes_{BP_\ast} BP_\ast(BP)$ denote the elements

$$g = t_1 \otimes t_1 + t_2 \otimes t_1^p \quad (3.9)$$

$$pT = t_1^p \otimes 1 + 1 \otimes t_1^p - (t \otimes 1 + 1 \otimes t_1)^p. \quad (3.10)$$
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Turn now to the structure of \( E(n) \). Noticing that \( E(n)(BP) = E(n) \otimes_{BP} \), \( BP \otimes \) \( E(n)(BP) = E(n) \otimes_{BP} E(n) \), we have

\[
E(n)(E(n)) = E(n)(BP) \otimes_{BP} E(n) = E(n) \otimes_{BP} E(n)
\]

In this paper, we consider only for the case \( n = 2 \). Then the formula on \( v_3 \) in (3.4) gives rise to the relation in \( E(2) \):

\[
v_{2t} t_{2}^{p^{2}} \equiv t_{1} \eta_{c}(v_{2}) p - v_{1} t_{2}^{p^{2}} - v_{1}^{2} V \mod (p, v_{1}^{p^{2}}).
\]

Furthermore, we have the following result.

\[
(3.8)
\]

(Shimomura and Tamura [16, Lemma 3.2]). In the \( E(2) \), we have the relations

\[
v_{2t} t_{n}^{p^{2}} \equiv v_{2} t_{n} - v_{1} t_{n+1} \mod (p, v_{1})
\]

and

\[
v_{2}^{2} t_{n}^{p^{2}} \equiv v_{2}^{2} T \mod (p, v_{1}).
\]

Let \( (A, \Gamma) \) denote one of the Hopf algebroids \( (BP, \), \( BP \) \( (BP) \)) and \( (E(2), \), \( E(2) \) \( (E(2)) \)). Then the Ext group

\[
H^{*}M = \text{Ext}^{*}_{A}(M)
\]

of a comodule \( M \) can be computed by the homology of the cobar complex \( (\Omega_{\star} M, d_{\star}) \). It is shown in [5] that there is an isomorphism

\[
\text{Ext}_{BP}(BP, M) \overset{\sim}{\longrightarrow} \text{Ext}_{E(2)}(E(2), E(2) \otimes_{BP} M)
\]

for a \( v_{2} \)-local \( BP \)-comodule \( M \). Thus there would be no confusion if we write \( H^{*}M \) for those Ext groups, as long as we consider \( v_{2} \)-local comodules. A cobar complex of a comodule \( M \) is a pair \( (\Omega_{\star}^{*}M, d_{\star}) \) of graded \( Z_{\Gamma_{\star}} \)-modules

\[
\Omega_{\star}^{*}M = M \otimes_{A} \Gamma \otimes_{A} \cdots \otimes_{A} \Gamma \quad (s \text{ copies of } \Gamma)
\]

for \( s \geq 0 \) and the differentials \( d_{s}: \Omega_{\star}^{*}M \rightarrow \Omega_{\star}^{*+1}M \) in the sense \( d_{s+1} d_{s} = 0 \) for \( s \geq 0 \) which are defined inductively by

\[
d_{0}(m) = \psi(m) - m \otimes 1
\]

\[
d_{1}(m \otimes x) = \psi(m) \otimes x - m \otimes \Delta(x) + m \otimes x \otimes 1
\]

\[
d_{s}(m \otimes x_{s-1}) = d_{s}(m \otimes x) \otimes x_{s-1} - m \otimes x \otimes d_{s-1}(x_{s-1})
\]

for \( m \in M \), \( x \in \Gamma \) and \( x_{s} \in \Omega_{\star}^{*}A = \Gamma \otimes_{A} \cdots \otimes_{A} \Gamma \) (\( s \) copies). Here \( \psi: M \rightarrow M \otimes_{A} \Gamma \) is the comodule structure of \( M \).

We note that in the following sections, we mainly treat comodules with structure maps induced from \( \eta_{c} \), and so the comodule structure \( \psi \) is computable by using the formulae (3.3) and (3.4). We further use the notation \( \eta_{c} \) for such a structure map \( \psi \). For example, by (3.4) and (3.10), we obtain the following lemma.

**Lemma 3.11.** In the cobar complex \( \Omega_{\star}^{*}A \), we have \( d_{0}(v_{2}) = v_{2}^{2} t_{p^{2}} - v_{1}^{2} t_{p^{2}} - pv_{1}

\[
V + p^{2}(v_{2} + v_{1} t_{2}^{p} - v_{1}^{2} t_{1})^{p-1}(t_{2} - v_{1} t_{1}) \mod (p, v_{1}^{p^{2}}). \]

In particular, \( d_{0}(v_{2}) \equiv pv_{1} v_{2}^{p-1} t_{p}^{2} + (p^{3})

\[
v_{2}^{2} v_{2}^{-2} t_{p^{2}} + p^{2} v_{2}^{p-1} t_{2} - p^{2} v_{1} v_{2}^{p-2} t_{2}^{p} t_{2} + (p^{3}/2) v_{1} v_{2}^{p-3} v_{2}^{-2} t_{2} \mod (p, v_{1}^{p^{2}}). \]

We have more formulae on the differential shown immediately from the definition (3.10).

\[ d_0(uv) = d_0(u)\eta_R(v) + ud_0(v) \]
\[ d_1(xy) = d_1(x)\Delta(y) + (x \otimes 1 + 1 \otimes x)d_1(y) - x \otimes y - y \otimes x \]
\[ d_1(uv) = d_0(u) \otimes y + ud_1(y) \]
\[ d_1(x\eta_R(v)) = d_1(x)(1 \otimes \eta_R(v)) - x \otimes d_0(v). \]

Note that once we give an element in \( \Omega^2_1A \), then we have the corresponding elements in \( \Omega^2_1A/I \) for any ideal \( I \) of \( A \). In this case we use the same notation for those elements. First define the elements \( g_0 \) and \( g_1 \in \Omega^2_1A \):

\[ g_0 = v_2^{-2}g \quad \text{and} \quad g_1 = v_2^{-2}r_1 - 1 - v_2^{-1}g_0 \quad (3.13) \]

for \( g \) given in (3.7). The element \( \zeta_2 \) in \( \Omega^1_1A \) is defined in [6] by

\[ \zeta_2 = v_2^{-1}\tau_2 + v_2^{-2}(t_2^2 - t_1^2 + 1) - v_2^{-1}v_3t_1^2. \]

**Lemma 3.14.** In the coobar complex \( \Omega^2_1A/(p, v_1) \), we have

\[ d_1(t_1t_2) = v_2t_1 \otimes \zeta_2 - v_2g_0 - 2t_1 \otimes t_2 - t_1^2 \otimes t_1^p \]
\[ d_1(t_1^2t_2) = v_2\zeta_2 \otimes t_1^2 - v_2^2g_1 - 2t_2 \otimes t_2 - t_1 \otimes t_1^2 + 1 \]
\[ d_1(t_1t_2^2) = v_2^2\zeta_2 \otimes t_1 - v_2^2g_0 - 2t_2 \otimes t_1 - t_1 \otimes t_1^2 + 1. \]

This follows from a direct calculation by definition with the help of (3.6) and (3.12).

### 4. THE CHROMATIC SPECTRAL SEQUENCE

In this section we also consider the ring spectra \( BP \) and \( E(2) \), the Brown–Peterson and the Johnson–Wilson spectra, respectively, and denote those spectra by \( E \). Then we have the Adams–Novikov spectral sequence converging to the homotopy groups of \( \pi_*(L_EX) \) of the Bousfield localization of \( X \) with respect to \( E \) if \( X \) is connected (cf. [1, 2]). Note that \( L_{BP}X - X \) for a connected \( p \)-local spectrum \( X \). The \( E_2 \)-term is \( H^*E_*(X) = \text{Ext}^*_{E_*(E)}(E_*, E_*(X)) \). By virtue of the Landweber filtration theorem [4], the \( E_2 \)-term can be lead by computing \( H^*E_*/I_s \)'s. Here \( I_s \) denotes the invariant prime ideal \( (p, v_1, \ldots, v_{k-1}) \) of \( E_\ast \), and \( k \leq 2 \) if \( E = E(2) \). The Ext groups \( H^*E_*/I_s \)'s also are the \( E_2 \)-term of the Adams–Novikov spectral sequence for computing the homotopy \( \pi_*(L_EV(k - 1)) \) of the Toda–Smith spectra \( V(k - 1) \) when they exist. Miller et al. [6] introduced the chromatic spectral sequence for computing the Ext groups \( H^*E_*/I_s \)'s.

We now give the definition of the chromatic spectral sequence. Put first \( N_0^k = E_*/I_k \) and inductively suppose that \( N_i^k \) is defined. Then define \( M_i^k = v_{k+1}^{-1}N_i^k \), which has the comodule structure induced from that of \( N_i^k \) by [5]. Now \( N_i^{k+1} \) is the cokernel of the inclusion \( N_i^k \subset M_i^k \), which also has the induced comodule structure. In other words, we have the short exact sequence of comodules

\[ 0 \to N_i^k \to M_i^k \to N_i^{k+1} \to 0. \quad (4.1) \]

As usual, we will denote an element \( \xi \) of \( M_i^k \) by a linear combination of the elements of the form

\[ x/v_{k+1}^0 \cdots v_{k+1}^{n-1} \quad (v_0 = p) \]
for $e_i > 0$ with $k \leq i \leq k + s - 1$ and for $x \in M_{k,s}$. Furthermore, the element $x \eta^s \cdots \eta^{k+s-1}$ is killed by $\nu^i$ for each $i$.

Applying the functor $H^* -$ to the exact sequence (4.1), we have the long exact sequence

$$0 \to H^0 N^0_k \to H^0 M^k_0 \to H^0 N^1_k \to H^1 N^0_k \to H^1 M^k_0 \to H^1 N^1_k \to \cdots$$

(4.2)

for each nonnegative integers $k$ and $s$. These exact sequences are the exact couple that gives rise to the chromatic spectral sequence. The $E_1$-term is $E_1^{p,q} = H^p M^{q+1}_k$ and the abutting module is the desired Ext group $H^{s+t} N^0_k = H^{s+t} E_\ast / I_k$. To compute the $E_1$-term, Miller et al. [6] further introduced the Bockstein spectral sequence that is defined by the exact couple obtained by applying the functor $H^* -$ to the short exact sequence

$$0 \to M^{k+1}_{k+1} \overset{\varphi}{\to} M^k_0 \xrightarrow{\varepsilon^k} M^k_0 \to 0$$

where $\varphi$ is the comodule map defined by $\varphi(x) = x / \nu_k$. The Bockstein spectral sequence has the $E_1$-term $H^* M^{k+1}_k$ and abuts to $H^* M^k_0$. Thus we can compute inductively the $E_1$-term of the chromatic spectral sequence. When we work on the Bockstein spectral sequence, $(k, s) = (0, 2)$ in our case, we mainly use the following result.

**Lemma 4.3** (Miller et al. [6, Remark 3.11]). Consider a map of exact couples

$$0 \to \cdots \overset{\delta_{t-1}}{\to} H^t M^1_0 \overset{\varphi}{\to} B^t \overset{p}{\to} B^t \overset{\delta_t}{\to} H^{t+1} M^1_0 \overset{g_t}{\to} \cdots$$

$$0 \to \cdots \overset{\delta_{t-1}}{\to} H^t M^1_0 \overset{\varphi}{\to} H^t M^0_0 \overset{p}{\to} H^t M^2_0 \overset{\delta_t}{\to} H^{t+1} M^1_0 \overset{g_t}{\to} \cdots$$

If $B^t$ is $p$-torsion, then $f$ is an isomorphism.

The first step of the induction is Morava’s theorem.

(4.4) (Ravenel [9]). If $p > 2$, then $H^* M^0_1 = F_p[v_1, v_1^{-1}] \otimes E(t_1)$. If $p > 3$, then $H^* M^2_2 = F_p[v_2, v_2^{-1}] \{1, t_1^2, g_0, g_1, g_0 t_1^2\} \otimes E(t_2)$.

Here $E(x)$ and $F\{b_i\}$ denote the exterior algebra over the generators $\{x\}$ and the $F$-vector space with basis $\{b_i\}$, respectively, in which $F$ denotes a field.

Turn to the second step.

(4.5) (Miller et al. [6]). If $p > 2$, then $H^t M^1_0 = 0$ for $t > 1$, $H^1 M^1_0 = \mathbb{Q}/\mathbb{Z}(p)$ whose subgroup of order $p^t$ is generated by

$$y_{1,j} = \sum_{k \geq 0} \frac{(-1)^k v_1^{k+1} t_1^k}{kp^{j+1-k}}$$

and

$$H^0 M^1_0 = \mathbb{Q}/\mathbb{Z}(p) \oplus \sum_{i \geq 0 \text{, } (p, s) = 1} (\mathbb{Z}/p^{i+1}) \langle v^i p / p^{i+1} \rangle.$$

Here $G\langle x \rangle$ denotes the group isomorphic to $G$ whose generator is $x$. 
For stating the results on $H^*M_4$, we introduce some more elements. From here on we
work on $E(2)$ not on $BP$, and the Hopf algebroid $(A, \Gamma)$ denotes $(E(2)_*, E(2)_*(E(2)))$. The
elements $x_n \in \Omega^2_1 A$ are inductively defined by

\[ x_0 = v_2, \quad x_1 = v_2^2, \quad x_2 = x_1^p - v_1 v_2^{p^2 - p + 1}, \]
\[ x_n = x_{n-1} - 2 v_1^{a_n} v_2^{p - 1} v_1^{a_n + 1} \quad \text{for } n > 2 \tag{4.6} \]

for the integer $a_n$ with $a_0 = 1$ and

\[ a_n = p^n + p^{n-1} - 1 \tag{4.7} \]

for $n > 0$. For the differential $d_0$ of the cobar complex, we have the following result.

(4.8) (Miller et al. [6]). \mod (p, v_1^{2\cdot a_n}),

\[ d_0(x_i) = v_1 t_1^{p-1} \quad i = 0 \]
\[ = v_1 t_1^{p-1} (t_1 + v_1 (v_2^{-1} (t_2 - t_1^{p+1}) - \zeta_2)), \quad i = 1 \]
\[ = 2 v_1 a_1 (p - 1) v_1^{p-1} \sigma_{i-1}, \quad i > 1. \]

The above element $\sigma_n$ is given by

\[ \sigma_n = t_1 - \frac{1}{2} v_1 \zeta_2^n. \]

The element $\zeta_2$ satisfies the following.

(4.9) (Miller et al. [6]). $\zeta_2 = v_2^{-1} t_2 + v_2^{-1} (t_2 - t_1^{p+1})$ is homologous to $\zeta_2^p$ for $i \geq 0$ in $
Omega^1_1 A/(p, v_1)$.

(4.10) (Shimomura [15]). We have a cocycle $\zeta$ in each cobar complex $\Omega^1_1 A/(p^{i+1}, v_1^{mp})$ such that $\zeta$ is homologous to $\zeta_2$ in $\Omega^1_1 A/(p, v_1)$. By virtue of this, we will use the notation $\zeta$ for a cocycle of $\Omega^1_1 A/(p^{i+1}, v_1^{mp})$ such that $\zeta$ is homologous to $\zeta_2$ in $\Omega^1_1 A/(p, v_1)$ including $\zeta^p$. We also use the notation

\[ \sigma = t_1 - \frac{1}{2} v_1 \zeta. \tag{4.11} \]

Then $\sigma$ is homologous to $\sigma_n$ for any $n$ in $\Omega^1_1 A/(p, v_1)$.

Divide the set $Z - pZ$ of integers into three parts:

\[ Z_0 = \{ s \in Z \mid p \nmid s(s + 1) \} \]
\[ Z_1 = \{ sp - 1 : s \in Z \mid p \nmid s \} \tag{4.12} \]
\[ Z_2 = \{ sp^2 - 1 : s \in Z \} \]

and $Z - \{ 0 \}$ into

\[ Z(i) = \{ m : m = sp^i \text{ with } n \geq 0 \text{ and } s \in Z_i \} \tag{4.13} \]

for $i = 0, 1$ and 2. We then introduce the element $y_m = v_2^m t_1^i v_1 y_m$ of $\Omega^1_1 A$ for

\[ d_1(y_m) = - s_m v_1^{4(n^2 + 1)} v_1^{a_2} \mod (p, v_1^{4(n^2 + 1)}). \tag{4.14} \]
for \( g_1 = v_2^{p^2 - 1}(t_1^p \otimes t_2^p \mid t_1^p \otimes t_2^p) \) in (3.13). Here \( s_m \) for \( m = s p^n \) with \( p \neq s \) equals
\[
\begin{align*}
&\left(\frac{s+1}{2}\right) \quad \text{if } n = 0 \text{ and } m \in \mathbb{Z}(0) \\
&\frac{(-1)^n}{2} \left(\frac{s+1}{2}\right) \quad \text{if } n > 0 \text{ and } m \in \mathbb{Z}(0) \\
&1 \quad \text{if } n = 0 \text{ and } m \in \mathbb{Z}(2) \\
&\frac{(-1)^n}{4} \quad \text{if } n > 0 \text{ and } m \in \mathbb{Z}(2).
\end{align*}
\]
(4.15)

We define the integers \( e(m) \) and \( A(m) \) for \( m = s p^n \) with \( p \neq s \) by
\[
\begin{align*}
e(m) &= m - (p^n - 1)/(p - 1) \quad \text{if } m \in \mathbb{Z}(0) \\
e(m) &= m - p^n(p - 1) - (p^n - 1)/(p - 1) \quad \text{if } m \in \mathbb{Z}(2)
\end{align*}
\]
(4.16)

and
\[
\begin{align*}
A_n &= (p + 1)(p^n - 1)/(p - 1) \\
A'_n &= (p + 1)(p^{n+1} - p^n + (p^n - 1)/(p - 1)) \\
A(m) &= A_n + 2 \quad \text{if } m = s p^n \text{ and } m \in \mathbb{Z}(0) \\
&= A'_n + 2 \quad \text{if } m = s p^n \text{ and } m \in \mathbb{Z}(2) \\
&= \infty \quad \text{if } m = 0.
\end{align*}
\]
(4.17)

Now we define inductively the elements \( y_m \) for \( m \in \mathbb{Z}(0) \cup \mathbb{Z}(2) \) [16]. Let \( m = s p^n \) with \( p \neq s \). Then, for \( s \in \mathbb{Z}_o \), we put
\[
\begin{align*}
y_s &= v_2^s t_1 + s v_1 v_2^{s-1}(t_1^{p^2} - t_2) + \frac{s-1}{2} v_1 v_2^{s-1} \\
&\quad + \left(\frac{s}{2}\right) v_1^2 v_2^{s-2} t_1(t_1^{p^2} - t_2 + v_2) + s v_1^2 v_2^{s-1-p^2} t_2 \\
y_{s p} &= v_2^{s p} - \frac{s}{2} v_1^2 Z_s
\end{align*}
\]
(4.18)

and, for \( s = tp^2 - 1 \in \mathbb{Z}_s \),
\[
\begin{align*}
y_s &= W_p t_1 + v_1^{p^2 - p - 2} v_2^{p^2} X \\
&\quad + 2v_1^2 y_{sp} = v_1 y_p^s - d_0(v_2^{p+1}) + v_1^{p^2 - 2} W_{tp^2 - p}.
\end{align*}
\]
(4.19)

Once \( y_m \) for \( p | m \) is defined, \( y_{mp} \) is given by
\[
y_{mp} = v_1 y_m^p - d_0(v_2^{p+1}) + s_m v_1^p A(m-p^2) W_{e(m)}.
\]
(4.20)

Here the elements \( W_s, Z_s \) and \( X \) are defined in [16] so that they satisfy
\[
\begin{align*}
d_1(W_s) &= v_1^{p^2 - 1} v_2^{p^2} g_1^p - \frac{s-1}{2} v_1^{p^2+1} v_2^{p^2 - 1} g_1 \mod (p, v_1^{p^2+2}) \\
&= v_1^{p^2 - 1} v_2^{p^2 - 1} r_1^{p^2} \otimes \sigma - \frac{(s+1)}{2} v_1^{p^2+1} v_2^{p^2 - 1} g_1 \mod (p, v_1^{p^2+2}) \\
d_1(X) &= - v_1^2 g_1^{p^2} - v_1^{p^2+3} v_2^{p^2 - 1} g_1 \mod (p, v_1^{p^2+4})
\end{align*}
\]
(4.21)
and

\[ W_s = -v_2^{p-1}v^2 + v_1^{p-1}v_2^{-1}t_2^p - \frac{s-1}{2}v_1^p(v_3^2(t_2^p + 1) - t_2^p)(2 - v_1v_2^{-1}t_2^p) + v_2^{-1}v_3^p \]

\[ Z_s \equiv W_s \mod (p, v_1^{p-2}). \quad (4.22) \]

We need other cocycles \( G_\bullet \) of \( \Omega^2_A/(p, v_\bullet^*) \) for \( n \geq 0 \) introduced in [14] such that

\[ G_0 \equiv g_0 \quad \text{and} \quad G_n \equiv v_2^{(p-1)n-1}(p-1)g_1 \mod (p, v_1) \quad (4.23) \]

where the elements \( g_0 \) and \( g_1 \) are given in (3.13). We prepare some notation here:

\[ k(l)_\bullet = F_p[v_\bullet] \]

\[ K(l)_\bullet = v_1^{-1}k(l)_\bullet = F_p[v_1, v_1^{-1}] \]

\( k(l)_\bullet \{x/v_\bullet: x \in \Lambda\} \) denotes the direct sum of the cyclic \( k(l)_\bullet \)-modules isomorphic to \( k(l)_\bullet/(v_\bullet) \) generated by \( x/v_\bullet \), and

\( k(l)_\bullet \{x/v_\bullet: x \in \Lambda\} \) denotes the direct sum of the modules isomorphic to \( K(l)_\bullet/k(l)_\bullet \) with \( F_p \)-basis \( \{x/v_\bullet: j > 0\} \).

Now consider the following \( k(l)_\bullet \)-modules:

\[ X = k(l)_\bullet \{x_\bullet^s/v_\bullet^p: n \geq 0, s \in \mathbb{Z} - p\mathbb{Z}\} \]

\[ X_\infty = k(l)_\bullet \{1/v_\bullet^p\} \]

\[ Y_0 = k(l)_\bullet \{y_m/v_\bullet^m: m \in \mathbb{Z}(0), n = v_\bullet(m)\} \]

\[ Y_1 = k(l)_\bullet \{y_m/v_\bullet^m: m \in \mathbb{Z}(2), n = v_\bullet(m)\} \]

\[ Y = k(l)_\bullet \{v_2^pV/v_1^{p-1}: t \in \mathbb{Z}\} \]

\[ Y_\infty = k(l)_\bullet \{t_1/v_1^p\} \]

\[ G = k(l)_\bullet \{x_\bullet^sG_s/v_\bullet^p: n \geq 0, s + 1 \in \mathbb{Z} - p\mathbb{Z}\}. \]

Here \( v_\bullet(m) \) denotes the maximal power of \( p \) dividing \( m \). Then we have the structure of \( H^*M_1 \) obtained in [6, 16, 14]:

\[ H^*M_1 = (X \oplus X_\infty \oplus Y_0 \oplus Y_1 \oplus Y \oplus Y_\infty \oplus G) \oplus E(\zeta). \quad (4.24) \]

We will end this section with rewriting the element \( y_m \) as follows.

**Lemma 4.25.** Let \( s \) and \( n \) be integers with \( n > 0 \). Then we have

\[ y_{sp^n} = v_2^{p^n}(t_1 - \frac{1}{2}v_1^p) + \frac{s}{2}v_1^{p^n-1}v_2^{(s-1)p^n}V^{p^n-1} \]

\mod \( (p, v_1^{(p-1)p^n-1}) \) and moreover

\[ 2v_1^{p^n-1}y_{(tp^n-1)p^n} \equiv -v_2^{(p-1)p^n-3}V^{p^n-1} \]

\mod \( (p, v_1^{p^n-1 + F(n)}) \) for \( F(n) = p^{n+2} - p^{n+1} - 3p^n + 1 \), up to homology.

**Remark.** We can define \( y_m \) so that the congruence (4.26) holds \( \mod (p, v_1^{p^n}) \) after replacing \( v_2^p \) by \( x_\bullet^s \).
Proof. We first prove (4.26). By the definition of the elements (4.18) and (4.22),
\[ y_s = y^p_s - \frac{s}{2} (v_1^2 Z_s) \]
if \( s \in \mathbb{Z}_0 \) and
\[ Z_s \equiv W_s \equiv - v_2^{p-1} V \mod (p, v_1^{p-2}). \]
Therefore we have the case for \( n = 1 \) and \( s \in \mathbb{Z}_0 \).

Next suppose that \( s = tp^2 - 1 \). Then the definitions (4.19) and (4.22) give
\[ y_s = W_1^p \mod (p, v_1^p) \]
\[ = - v_2^{p^2-p^2} V \mod (p, v_1^p) \]
\[ = v_2^{tp^2-p} t_1^p \mod (p, v_1^p) \]
which is congruent to \( v_2^{p^2-p} t_1 - v_1 v_2^{p^2-p-1} t_1^p \mod (p, v_1^p) \) by (3.8). Thus we have
\[ y_1^p \equiv y_2^p - v_1^p v_2^{p-1} V \mod (p, v_1^p). \]
Furthermore, the formula (3.4) gives us
\[ d_0(v_1^p + (p^2-1)p^p) \equiv (v_2 + v_1 t_1^p - v_1^p t_1)(v_2^{p^2-1} - v_2^{p^2-2} t_1^p) - v_2^{1+(p^2-1)p^p} \]
\[ = v_1 v_2^{p^2-1} t_1^p - v_2^{p^2-1} t_1 \]
\[ - v_2^{2(p^2-2)} t_1^p - v_1^p v_2^{p^2-1} t_1 \]
\[ \mod (p, v_1^p) \]
which turns out to be congruent, again by (3.8), to
\[ v_2^{p^2-1} t_1^p - 2v_1^p v_2^{p^2-1} t_1 + v_1^p v_2^{p^2-2} t_1^p + v_1^p v_2^{p^2-2} t_1^p V = v_1^p v_2^{p^2-1} t_1^p + r. \]
Put these into the congruence
\[ 2v_1^p y_{sp} \equiv v_1 y_s - d_0(v_2^{p^2-1} p^p) \mod (p, v_1^p) \]
of (4.19), and we have the same result as the case for \( s \in \mathbb{Z}_0 \).

Since \( d_0(v_2^{mp^n}) \equiv v_2^{mp^n} (v_2 t_1^p - v_1^p t_1) \mod (p, v_1^{mp^n}) \) for \( m = sp^n \) with \( n \geq 2 \), we have
\[ y_s^{mp^n} \equiv v_1 v_2^{mp^n} (t_1^p - \frac{1}{2} v_1^p t_1^p) + \frac{s}{2} v_1^{p+1} p^p v_2^{mp^2-p^2} V p \]
\[ = v_1^{mp^n} (v_1 t_1^p - v_1^p t_1) \mod (p, v_1^{mp^n}) \]
by the definition (4.20) under the inductive hypothesis. Thus the case for \( n \geq 2 \) immediately follows from the induction on \( n \).

Now turn to (4.27). By the definitions (4.19) and (4.22), we have
\[ y_s \equiv - v_2^{(p^2-1)p^p} V \]
mod \( (p, v_1^{p^2-p-2}) \) for \( n = 0 \), and \( 2v_1^{p^n-1} y_{sp} \equiv - v_2^{(p^2-1)p^p} V p \mod (p, v_1^{p^2-p^2}) \) for \( n = 1 \) up to homology. Suppose that
\[ 2v_1^{p^n-1} y_{sp} \equiv - v_2^{(p^2-1)p^n+1} V p \]
mod \( (p, v_1^{p^n-1} + F(n)) \) for \( F(n) = p^{n+2} - p^{n+1} - 3p^n + 1 \) up to homology. Then the definition (4.20) leads us to
\[ 2v_1^{p^n-1} y_{sp} \equiv 2v_1^{p^n-1} y_{sp} \mod (p, v_1^{p^n-1} + F(n+1)) \]
up to homology. Hence we have the desired congruence. Q.E.D.
COROLLARY 4.30. Let $m$ be an integer in $\mathbb{Z}(0) \cup \mathbb{Z}(2)$ with $p|m$. In the complex $\Omega \cdot A/(p, v_i^2)$, $t_1 \otimes y_m$ is homologous to $-\frac{1}{2}v_1y_m \otimes \zeta$.

Proof. By (4.26), we see that $t_1 \otimes y_m = t_1 \otimes v_2^2(t_1 - \frac{1}{2}v_1 \zeta)$. Since $p|m$, the first term is $t_1 \otimes v_2^2t_1 = v_2^2t_1 \otimes t_1$ in our complex, which equals $d_0(-\frac{1}{2}v_1^2t_1^2)$. Thus $t_1 \otimes y_m$ is homologous to $-\frac{1}{2}v_1v_2^2t_1 \otimes \zeta$. Q.E.D.

5. THE COKERNEL OF $\delta_0$

In this section, we consider the connecting homomorphism $\delta_0: H^0M_0^2 \to H^1M_1^1$ associated to the short exact sequence $0 \to M_0^2 \to M_1^1 \to M_1^1 \to 0$. The image of $\delta_0$ is given in [6, Prop. 6.9]. First we rewrite the result by using the bases of $H^1M_1^1$. By Lemma 4.25, we obtain the following lemma.

LEMMA 5.1. The connecting homomorphism $\delta_0: H^0M_0^2 \to H^1M_1^1$ sends an element $x_i^j k/p^{i+1}v_i^j$ for $s \in \mathbb{Z} - p\mathbb{Z}$ and $j = mp^i$ with $0 \leq i \leq k$ and $1 \leq m \leq a_{k-i}$ to

\begin{align*}
y_{s/v_i^j} & \quad (s) v_2^2 \zeta/v_1 \quad \text{if } k = 0 \\
-mv_{s/p^{i+1}v_i^j} + mx_i^{j+1}k/v_2^j - su_{2}^{p_i+1-p}v_i^j & \quad \text{if } k = 1 \\
-mv_{s/p^{i+1}v_i^j} + mx_i^{j+2}k/v_2^j + s_{v_{2}}^{p_i+1-p}v_i^j & \quad \text{if } k = 2 \\
my_{s/p^{i+1}v_i^j} + mx_i^{j+2}k/v_2^j + 2s_{v_{2}}^{p_i+1-p}v_i^j & \quad \text{if } k > 2.
\end{align*}

For the generator $h_0/v_i^j \in H^1M_1^1$, which is represented by $t_1/v_i^j$, we also have Lemma 5.2.

LEMMA 5.2. Put $j = kp^i > 0$ for $k \neq k$. Then, $\delta_0(1/p^{i+1}v_i^j) = -kh_0/v_i^j$.

Proof. This follows immediately from the definition of $\delta_0$ and the formula $d_0(v_i^j) = jpv_i^{j-1}t_1 \mod (p^{i+2})$. Q.E.D.

We read off the following proposition on the cokernel of $\delta_0$ from Lemma 5.1 and the structures of $H^0M_0^2$ given in [6] and $H^1M_1^1$ in (4.24). We prepare the following notation:

\begin{align*}
Z_{s/p^i}(x/p^i v_i^j; & \begin{array}{c} x \in \Lambda, j \in J, i \in I \end{array})
\end{align*}

denotes the direct sum of the cyclic $\mathbb{Z}_{s/p^i}$-modules isomorphic to $\mathbb{Z}/(p^i)$ generated by the elements $x/p^i v_i^j$ subject to the conditions $x \in \Lambda$, $j \in J$, and $i \in I$.

\begin{align*}
X^\infty = Z_{s/p^i}(x_n/p^{i+1} v_i^j; n \geq 0, s \in \mathbb{Z} - p\mathbb{Z}, i \geq 0, j \geq 1, \text{such that } p^i | j \leq a_{n-i} \text{ and either } p^{i+1} | j \text{ or } a_{n-i-1} < j) \end{align*}

and

\begin{align*}
X^\infty = Z_{s/p^i}(1/p^{i+1} v_i^j; i = \nu_{p}(j) \geq 0). \tag{5.3}
\end{align*}

Under these notations, we have the following result.

(5.4) (Miller et al. [6, Theorem 6.1]). $H^0M_0^2 = X^\infty \oplus X^\infty$.

We also have

\begin{align*}
H^1M_1^1 = (X \oplus X_n) \oplus F_p(\zeta) \oplus Y_0 \oplus Y_1 \oplus Y_\infty \oplus Y \tag{5.5}
\end{align*}
by (4.24). Recall the notation (4.12) and (4.13):
\[ Z_0 = \{ s \in \mathbb{Z} \text{ with } p \not| s(s + 1) \} \]
\[ Z_1 = \{ sp - 1 \in \mathbb{Z} \text{ with } p \not| s \} \]
\[ Z_2 = \{ sp^2 - 1 \in \mathbb{Z} \} \]
\[ Z(i) = \{ m : m = sp^n \text{ with } n \geq 0 \text{ and } s \in Z_i \} \quad (i \in \{ 0, 1, 2 \}). \]

We further define a subset of \( Z(2) \) by
\[ Z_2 = \{ tp^{i+2} - 1 : t \in \mathbb{Z} - p\mathbb{Z} \} \quad (5.6) \]
for each nonnegative integer \( i \). Then \( \bigcup_i Z_2^i = Z_2 \), which is a disjoint union.

**Proposition 5.7.** The cokernel of \( \delta_0 : H^0M_0^2 \to H^1M_1^1 \) is a vector space spanned by the bases represented by the cocycles:

(i) \( t_1/v_1 \) and \( \zeta_1/v_1^j \) for \( j \geq 1 \);
(ii) \( y_{sp}v_1^j/v_1^j \) for \( s \in Z_0 \cup Z_2, n \geq 0, j \leq A(sp^n) \) such that \( j = 1 \) or \( j - 1 > a_n-1 if p'|\,(j - 1), \)
subject to \( p^{k+1}j + a_{n+1} \) or \( j > a_{n+1} \) if \( s \in Z_2^2 \);
(iii) \( v_1^jv_{s,j}^2 \) for \( s \in T - p\mathbb{Z}, n \geq 0 \) and \( 1 \leq j \leq a_n \) such that \( j > a_n \) if \( p'|\,j \) for either \( s \in Z_2 \) or \( s \in Z_1^2 \) and \( p^{k+1}j \);
(iv) \( v_1^jv_{s,j}^p + v_{s,j}^{p-1} \) for \( s \in \mathbb{Z} \) and \( 1 \leq j \leq p - 1 \) with a condition that \( p|\,(s + 1) \) if \( j = p - 1 \).

**Proof.** By Lemma 5.1, the leading terms of \( \delta_0 \)-images of the generators \( x_1^j/p^{i+1}v_1^j \) of \( H^0M_0^2 \) are rewritten to be:

1. \( y_{sp}v_1^j/v_1^j \), \( s \in Z_0 \cup Z_2, n \geq 0, j \geq 1 \) and \( j \leq a_n-1 if p^{i+1}j \).
2. \( x_{s,j}^0/v_1^j \), \( s \in Z_1, n \geq 0, j \geq 1 \), and \( j \leq a_{n-1} if p^{i+1}j \).
3. \( x_{s,j}^j/v_1^j \), \( s \in Z_2, n \geq 0, j \geq 1 \), and \( j \leq a_{n-1} if p^{i+1}j \), subject to \( p^{k+1}j + a_{n+1} \) if \( s \in Z_2^2 \).
4. \( v_1^jv_{s,j}^{p-1} \), \( s \in \mathbb{Z} - p\mathbb{Z} \).
5. \( y_{sp}v_1^j/v_1^j \), \( s \in Z_2, n \geq 0, j \geq 1 \), and \( p^{k+1}j + a_{n+1} \) if \( s \in Z_2^2 \).

Here we make a note on the elements of (3). Since \( s \in Z_2^2 \), we may put \( s = tp^{k+2} - 1 \) for \( t \) with \( p \not| t \). Lemma 5.1 says that if \( p^{i+1}j + a_{n+1} + 1 \leq a_{n+2} \),
\[ \delta_0(x_{s+2,k}/p^{i+1}v_1^j + v_{s+2,k}/p^{i+1}v_1^j + \cdots) = \epsilon \bar{y}_{s+2,k}/(p^{i+1}v_1^j + \cdots) \]
where \( \epsilon = 1 \) if \( n = 0 \), and \( \epsilon = 2 \) otherwise. Thus we have the elements in both of (1) and (5).
Therefore, the second terms of the last two equations of Lemma 5.1 turn into the leading terms and give the elements of (3).

The cokernel of \( \delta_0 \) is expressed by the generators of \( H^1M_1^1 \) other than those that appear above as the leading terms. For the generator of the form \( v_1^jv_{s,j}^p + v_1^jv_{s,j}^{p-1} \) with \( p \not| \,(t + 1) \) die in the cokernel by (4). Thus we have (iv).

Considering the negative statements, we deduce (ii) from (1) and (5), and (iii) from (2) and (3).

Lemma 5.2 gives the first half of (i), and the second half follows from Lemma 5.1.

Q.E.D.

6. SOME LEMMAS ON \( \delta_1 \)

Consider the short exact sequence
\[ 0 \to M_1^1 \xrightarrow{\delta_1} M_0^2 \xrightarrow{\delta} M_0^0 \to 0 \]
and apply the functor $H^*$ to it to obtain the Bockstein spectral sequence. The differentials $d_i$ of the spectral sequence depend on the computation of the connecting homomorphism $\delta_i : H'M_0^2 \to H'^{i+1}M_1^1$.

As we have remarked in (4.10), we have a cocycle $\zeta$ in each cobar complex $\Omega^1_{L}(p^i+1, v_1^r)$. So we have Lemma 6.1.

**Lemma 6.1.** Let $x \in H'^{i-1}M_0^2$, and $\delta_i : H'M_0^2 \to H'^{i+1}M_1^1$, the connecting homomorphism. Then

$$\delta_i(x \otimes \zeta) = \delta_{i-1}(x) \otimes \zeta.$$ 

By the definition of $\delta_i$, we obtain the following lemma.

**Lemma 6.2.** Let $w$ be an element of $E(2)_*$ such that $d_0(w) \equiv 0 \mod (p^i+1, v_1^r)$. Then we have

$$w\delta_i(x/p^i v_1^r) = \delta_i(wx/p^i v_1^r)$$

for $x/p^i v_1^r \in H'M_0^2$.

**Proof.** This follows from the definition of $\delta_i$ and the calculation

$$d_1(wx/p^i+1 v_1^r) = w d_i(x/p^i+1 v_1^r).$$

Q.E.D.

Since $d_0(v_1^r) \equiv 0 \mod (p^i+1, v_1^r)$ even if $p^i < j$, we have the following corollary.

**Corollary 6.3.** Let $\delta_i : H'M_0^2 \to H'^{i+1}M_1^1$ be the connecting homomorphism. Then we have

$$v_1^r \delta_i(x/p^i v_1^r) = \delta_i(x/p^i v_1^{r-p})$$

for $x/p^i v_1^r \in H'M_0^2$. In particular,

$$v_1^r \delta_i(x/p v_1^r) = \delta_i(x/p v_1^{r-p})$$

for $x/p v_1^r \in H'M_0^2$.

In order to state the next corollary, we define integers $p(i, j)$ depending on integers $i$ and $j$ by

$$p(i, j) = \min \{n \in \mathbb{Z} : p^i \mid n \text{ and } j \leq n\}.$$ 

**Corollary 6.4.** For a cocycle $x/p^i v_1^r \in H'M_0^2$, suppose that

$$\delta_i(x/p^i v_1^r) = \sum \zeta_i/v_1^r \neq 0$$

for the elements $\zeta_i/v_1^r \in H'^{i+1}M_1^1$ with $\zeta_i/v_1^r \neq 0$. Then $k \leq p(i, j)$.

**Proof.** Corollary 6.3 implies that $v_1^{p(i, j)} \delta_i(x/p^i v_1^r) = 0$ since $j \leq p(i, j)$. Thus we have $\zeta_i/v_1^{p(i, j)} = 0$. If $k > p(i, j)$, then $\zeta_i/v_1^{k-p(i, j)} = 0$, which is a contradiction. Q.F.D.

For the next section we introduce some more elements defined by

$$P_n = v_1^n v_1^{r-1} - v_1^{r+1} t_1^{r-r} - d_0(v_1^n) \quad (6.5)$$

for $n \geq 1$, in which the right-hand side is divisible by $p$ since $\eta_p(v_2) = v_2 + v_1 t_1^{r-r} - v_1 t_1^{r}$ mod $(p, v_1)$. Then we have Lemma 6.6.
LEMMA 6.6. In the complex $\Omega^p_f A$, 
\[
V_{n+1} = v_1^{p^n} V^p \mod (p)
\]
\[
= -v_1^{p^n} v_2^{(p-1)p^n t_1^{n-1}} \mod (p, v_1)
\]
d_1(V_n) = 0 \mod (p^n, v_1^{p^n}).

Here $V$ is the element of (3.5).

Proof. The first two are seen by direct calculations and the definition of $V$. We deduce the last one by the definition (6.5) and the following facts: $d_1 d_0 = 0$, $d_1(v_1^{p^n} t_1^{n-1}) \equiv 0$ and $d_1(v_1^{p^n} t_1^{n-1}) \equiv 0 \mod (p^{n+1}, v_1^{p^n})$, the map $p: \Omega^p_f A/(p^n, v_1^{p^n}) \to \Omega^p_f A/(p^{n+1}, v_1^{p^n})$ is monomorphic and $(p^{n+1}, v_1^{p^n})$ is an invariant ideal. Q.E.D.

Noticing that $d_0(v_1^{p^n+1}) = v_1^{p^n+1} t_1^{n+1} - p V_{n+2} \mod (v_1^{p^n+1})$ by the definition (6.5), we obtain Lemma 6.7.

LEMMA 6.7. Let $u, n$ and $l \geq 0$ be integers such that $p \not| u$ and $n, l \leq 0$. Then in the complex $\Omega^q_f A$,
\[
d_0(v_1^{p^{n-1}+1}) \equiv -u p^l v_2^{(p-1)p^n} V_{n+3} \mod (p^{n+1}, v_1^{p^n})
\]
\[
= u p^l v_2^{(p-1)p^n} (v_1^{p^n+1} t_1^{n+1} - p V_{n+2}) \mod (p^{n+2}, p^{n+1} + p, v_1^{p^n+2}).
\]

We have some relations on the elements $V_n$'s.

LEMMA 6.8. In the cobar complex $\Omega^q_f A/(p, v_1^k)$, $2t_1 \otimes V$ is homologous to 
\[-v_1 v_2^2 g_1 - v_1 v_2^{p-1} \zeta \otimes t_1^p.
\]

Proof. By the definition of $V$, mod $(p, v_1^k)$,
\[
2t_1 \otimes V \equiv 2t_1 \otimes (v_2^{p-1} t_1^p) + v_1 v_2^{p-2} t_1 \otimes t_1^p
\]
\[
= -2v_2^{p-1} t_1 \otimes t_1^p + 2v_1 v_2^{p-2} t_1^{p+1} \otimes t_1^p + v_1 v_2^{p-2} t_1 \otimes t_1^p.
\]

Direct computation by the formulae (3.6) and (3.10) brings us
\[
d_1(-2v_2^{p-1} t_1) \equiv 2v_1 v_2^{p-2} t_1 \otimes t_2 + 2v_2^{p-1} t_1 \otimes t_1^p + 2v_1 v_2^{p-1} T \mod (p, v_1^k).
\]
The last term is homologous to $-2v_1 v_2^2 g_1$ by (3.6), (3.9) and (3.13). We also see the following by Lemma 3.14:
\[
d_1(v_1 v_2^p t_1^p t_2) \equiv v_1 v_2^{p-2} (v_2 \zeta \otimes t_1^p - v_2^2 g_1 - 2t_2 \otimes t_1^p - t_2 \otimes t_1^p)
\]
\[
\mod (p, v_1^k). These together with the equation $t_1^p \otimes t_2 + t_1^{p+1} \otimes t_1^p - t_2 \otimes t_1^p = v_2^2 g_1 - v_2 \zeta \otimes t_1^p$ show
\[
2t_1 \otimes V \equiv -v_1 v_2^2 g_1 - v_1 v_2^{p-1} \zeta \otimes t_1^p \mod (p, v_1^k)
\]
up to homology. Q.E.D.

LEMMA 6.9. In the cobar complex $\Omega^q_f A/(p, v_1^k)$, $2t_1 \otimes V^p$ is homologous to 
\[-v_1 v_2^{p-1} (g_0 - \zeta \otimes t_1).
\]
**Proof.** We see that 
\[ 2t_1 \otimes \tau \equiv 2v_2^{p-1} t_1 \otimes t_1 \quad 2v_1 v_2^{p-2} t_1^{p+1} \otimes t_1 - 2v_1 v_2^{p-1} t_1 \otimes t_1^p \] 
by (3.5), (3.8) and (3.4). We also compute

\[ d_1(v_2^{p-1} t_1^p) = -v_1 v_2^{p-2} t_1^p \otimes t_1^2 - 2v_2^{p-1} t_1 \otimes t_1 \]
\[ d_1(v_2 v_2^{p-1} t_1 t_1^p) = -v_1 v_2^{p-2} t_1^{p+1} (t_1^{p+1} \otimes t_1^p + t_1^p \otimes t_1^{p+1} + t_1 \otimes t_1^p + t_1^p \otimes t_1) \]

by (3.4), (3.6) and (3.10). Since \( v_1 t_1^p = v_1 v_2^{p-1} t_1 \), these computations show that 
\( 2t_1 \otimes \tau \) is homologous to 
\( -v_1 v_2^{p-2} t_1 t_1^{p+1} + 2t_1^p \otimes t_1 \). Thus Lemma 3.14 implies the lemma.

Q.E.D.

For the last lemma of this section, we prepare the following.

**Lemma 6.10** (Shimomura and Yabe [17, Lemma 3.4]). We have the elements 
\( w(s, n) \in v^{-1} \Gamma \) such that \( v^n w(s, n) \in \Gamma \), and

\[ d_1(w(s, 0)) = 2v_2^{p-1} \otimes \sigma - v_1 v_2^{p-1} g_1 \mod (p, v_2^2) \]
\[ d_1(w(s, n)) = 2v_1^{(2s-1)p-1} v_2^{p-1} \otimes \sigma - \frac{(1)}{2} v_1^{n-1} + A_{n-1} + 2v_2^{p-1} v_1^{(n+1)} g_1 \quad (n \geq 1) \mod (p, v_1^{n-1} + A_{n-1} + 3) \]

Here \( a_n = p^n + p^{n-1} - 1 \) and \( e(n, s) = sp^n - (p^n - 1)/(p - 1) \).

**Lemma 6.11.** In the cobar complex \( \Omega_2 A/(p, v_1^{n-1} + A_{n-1} + 3) \), \( 2t_1 \otimes v_2^{(s-1)p} \tau \sigma \) for \( n \geq 2 \) is homologous to

\[ -2v_1^{p-2} \otimes y_{(sp^n - 1)p - 2} - \frac{(1)}{2} v_1^{n-2} + A_{n-2} + 2 \chi_n^2 G_n. \]

**Proof.** Since \( 2t_1 \otimes v_2^{(s-1)p} \tau \sigma = 2 \sigma \otimes v_2^{(s-1)p} \tau \sigma + v_1 \zeta \otimes v_2^{(s-1)p} \tau \sigma - \chi_{(sp^n - 1)p - 2} \), Lemma 4.25 gives

\[ 2t_1 \otimes v_2^{(s-1)p} \tau \sigma = 2 \sigma \otimes v_2^{(s-1)p} \tau \sigma - 2v_1 \zeta \otimes v_1^{n-2} y_{(sp^n - 1)p - 2}. \]

Now apply Lemma 6.10 to it, and we have the result by the property (4.23) of \( G_n \).

Q.E.D.

**7. Computation of \( \delta \)**

We will compute the \( \delta_1 \)-image of the generators of \( H^1 M_1 \). Let \( x/v \) denote one of the generators of the cokernel of \( \delta_1 : H^1 M_1 \to H^1 M_1^0 \) that are given in Proposition 5.7. Then \( x(j, 1) = x/pv \) gives a nonzero element of \( H^1 M_3^0 \). Suppose inductively that there exists a nonzero element \( x(j, l) \in H^1 M_3^0 \) such that \( px(j, l) = x(j, l - 1) \). If \( \delta_1 x(j, l) = 0 \), then there exists a cochain \( \rho \) such that \( d_1(x(j, l)/p) = d_1(p) \). Put now

\[ x(j, l + 1) = x(j, l)/p - \rho \]

and we see that \( x(j, l + 1) \) is a coycle and

\[ px(j, l + 1) = x(j, l). \]

Thus this proceeds until we have an integer \( i \) (maybe infinity) such that

\[ \delta_1(x(j, i)) \neq 0. \]

We will find such an integer \( i \) for each generator \( x/v \) of the cokernel.
We first consider the elements of $X$ of (4.24). Notice that an element $x^i \zeta / v_1^i$ is bounded by $-\frac{1}{2} x^i \zeta^2 / v_1^i$ as long as $j \leq a_n$. By Lemmas 5.1 and 6.1, we then have the following result.

**Proposition 7.1.** The connecting homomorphism $\delta_1 : H^1 \mathbb{M} \to H^2 \mathbb{M}$ sends an element $x^i \zeta^j (a, b + 1) = x^i \zeta^j / v_1^{j+1}$ for $s \in \mathbb{Z} - p \mathbb{Z}$, $0 \leq i \leq k$ and $1 \leq m \leq a_{k-1}$ for $j = mp^4$ to

- $y_4 \otimes \zeta / v_1^4$ if $k = 0$
- $my_{sp^{i+1}} \otimes \zeta / v_1^{i+1} - sv_1^{p^{i+1}} \otimes \zeta / v_1^{i+1}$ if $k = 1$
- $my_{sp^{i+2}} \otimes \zeta / v_1^{i+2} + sy_{sp^{i+1}} \otimes \zeta / v_1^{i+2} + sv_1^{sp^{i+2}} V \otimes \zeta / v_1^{i+2}$ if $k = 2$
- $my_{sp^{i+3}} \otimes \zeta / v_1^{i+3} + 2sy_{sp^{i+2}} \otimes \zeta / v_1^{i+3}$ if $k > 2$.

Next we study the elements in $Y$ of (4.24).

**Proposition 7.2.** For each integer $t$, we have the cocycles $y_{sp}(j, 1) = v_1^t V / p v_1^4$ and $y_{sp}(p - 1, 2) = v_1^t V / p v_1^4 - \frac{1}{2} v_1^{t+1} t_1^p / p^2 v_1^4$ and

$$\delta_1 (y_{sp}(j, 1)) = \frac{j+1}{2} (x^{j+1} \zeta / v_1^j + v_1^t V \otimes \zeta / v_1^j) + \cdots$$

Proof: We get the congruences $d_0(y_{sp}) = v_1^t V / p v_1^4$, and $d_0(y_{sp}) = -p v_1 V \mod (p^2, v_1^4)$ by Lemma 3.11. These congruences lead us to the equation $v_1^t V / p v_1^4 = -d_0(y_{sp} + 1)/p$ in our cobar complex if $j+1 \leq p$. Therefore the definition of the connecting homomorphism shows that

$$\delta_1 (v_1^t V / p v_1^4) = \varphi^{-1}(\xi_j)$$

for $\xi_j = d_1 (-d_0(v_1^{t+1}))/((t+1)/p^2 v_1^4)$. Put $t+1 = sp^n$ for $s$ and $u$ with $p \mid s$ and $u \geq 0$. Then $\xi_j$ equals $-d_1(v_1^{sp^{n+1}})/((t+1)/p^3 v_1^{sp-n})$, and we have

$$\xi_j = -\left( j+1 \right) v_1^t t_1^p \otimes V / p v_1^4$$

by the equation $d_1 d_0 = 0$. Now applying Lemma 6.8, we have the case for $j < p - 1$ by definition of the elements. The above computation is applied for the case $j = p - 1$ by setting $\xi_{p-1} = d_1 (-d_0(v_1^{t+1}))/((t+1)/p^3 v_1^4)$, and we have

$$\xi_{p-1} = (t_1/p v_1^4) \otimes v_1^t V - (t_1/p v_1^4) \otimes v_1^t t_1^p.$$

We also compute $d_1 (v_1^{t+1} t_1^p / p^2 v_1^4) = -\left( \eta_p (v_1^{t+1} t_1^p / p) \otimes t_1^p + (v_1^{t+1} t_1^p / p v_1^4) \otimes t_1^p - 2(v_1^{t+1} t_1^p / p v_1^4) \otimes v_1^t t_1^p - v_1^{t+1} t_1^p / p^2 v_1^4 \right)$ in (3.4). Since the last congruence of (3.4) also gives $v_1^{t+1} t_1^p / p^2 v_1^4 - t_1^p / p v_1^4 - v_1^{t+1} t_1^p / p v_1^4$, the second term of $\xi_{p-1}$ is homologous to an element $X / p v_1^4$ for some $X \in BP_\bullet (p, v_1^4)$, which is denoted by $\cdots$ in the proposition.

Q.E.D.

For the case $j = 0$, we also have Proposition 7.3.

**Proposition 7.3 (Shimomura and Yabe [17, Prop. 4.4]).** For each $sp^n \in \mathbb{Z}(0) \cup \mathbb{Z}(2)$ with $n \geq 0$ and $p \mid s$ we have a cocycle $y_{sp^n}(1, n + 1)$ such that

$$\delta_1 (y_{sp^n}(1, n + 1)) = \frac{1}{2} (x_0^{sp^n} G_0 / v_1 - y_{sp^n} \otimes \zeta / v_1).$$
We introduce here elements:
\[ \sigma_{j,l} = y_{j,l} - \frac{1}{2} \zeta/p^j v_1^{l-1} \]
for positive integers \( j \) and \( l \). Here
\[ y_{j,l} = \sum_{k>0} \left( \frac{k+j-2}{k-1} \right) \frac{(-1)^{k-1}}{kp^{l+1-k}} v_1^{l-1+k} \]
is given in [6] to satisfy
\[ p^{l-1} y_{j,l} = t_1/p v_1^l, \quad \text{and} \quad d_1(y_{j,l}) = 0 \]
for any \( j \) and \( l \). Then we see that
\[ p^{l-1} \sigma_{j,l} = \sigma/p v_1^l, \]
and the following lemma holds.

**Lemma 7.4.** \( d_1(\sigma_{k+1,l}) = 0 \) if \( l \leq i+1 \), and \[ \frac{1}{2} k t_1 \otimes \zeta/p v_1^{k+1} \] if \( l = i + 2 \).

**Proof.** Note that \( d_1(y_{j,l}) = 0 \) and \( d_1(\zeta) \equiv 0 \mod J \) for any ideal \( J = (p', \nu) \) by the convention on \( \zeta \). We further see that \( d_0(1/p^j \nu^{k-1}) = -k t_1/p^{j-1} \nu^{k-1} \) in \( \Omega^2 M^2_{\nu} \) if \( l \leq i + 2 \). Thus the lemma follows. Q.E.D.

For a while we abuse the notation: For an integer \( s \in \mathbb{Z} - p \mathbb{Z} \), \( y_{m}(j,l) \) denotes a cocycle whose leading term is \( v_2^{s} t_1/p v_1^l \), if such a cocycle exists. It would be possible that the cocycle is a coboundary.

**Proposition 7.5.** Let \( s, n, i \) and \( k \) be integers with \( s \in \mathbb{Z} - p \mathbb{Z} \), \( k \geq 1 \), \( n \geq 0 \) and \( kp < A_{n+1} + 2 \) and put \( m = sp^n \). Then we have cocycles \( y_m(kp^i + 1, l) \) for \( 0 \leq l \leq i + 1 \), whose \( \delta_1 \)-images are given by \( \delta_1(y_{m}(kp^i + 1, l)) = 0 \) for \( l \leq i + 1 \) and
\[ \delta_1(y_{m}(kp^i + 1, i + 1)) = \frac{k}{2} y_m \otimes \zeta/v_1^{kp^i+1} + s \lambda_{n-1,x_1}^{kp^i} G_{n-1}/v_1^{kp^i - A_{n+1} - 1} \]
\[ + \left( -\frac{ks}{2} \lambda_n x_n G_{n}/v_1^{k+1} - A_{n+1} - 1 \quad \text{if} \quad l = 0 \right) + \cdots \]
where \( \lambda_n = (-1)^n/4 \) for \( n > 1 \), \( - - \frac{1}{2} \) for \( n = 1 \), and \( \cdots \) denotes an element killed by \( v_1^{-2p^n-1} \).

**Proof.** For \( i = 0 \), we see that \( y_m(k + 1, 1) = \phi(y_m/v_1^{k+1}) \) is a cocycle, and consider \( \omega(m; k + 1, 2) = \eta_{R}(v_2^m) \sigma_{k+1, 2} - \frac{1}{2} d_0(v_2^m)/p^2 \nu_1^{l} \). Then \( p\omega(m; k + 1, 2) = y_{m}(k + 1, 1) \) for \( k \leq (p-1) p^{n-1} \) with \( n = v_p(m) \) by Lemma 6.25, and \( \delta_1(y_{m}(k + 1, 1)) = \phi^{-1}(d_1(\omega(m; k + 1, 2))) \). Use (3.12) and Lemma 7.4 to get
\[ d_1(\sigma_{k+1, 2}) = \frac{k}{2} t_1 \otimes \zeta \eta_{R}(v_2^m)/pv_1^{k+1} - \sigma_{k+1, 2} \otimes d_0(v_2^m) \]
\[ d_1(d_0(v_2^m)/p^2 \nu_1^{l}) = -kt_1 \otimes d_0(v_2^m)/pv_1^{k+1} - \frac{k + 1}{2} t_1 \otimes d_0(v_2^m)/pv_1^{k+2} \].

Lemma 6.7 says that \( d_0(v_2^m) \equiv -sp v_2^{p^n-n} V_\nu \mod (p, v_2^m) \) for \( n = v_p(m) \), in which \( V_\nu = v_2^{n+1} \nu v_2^{n+1} \mod (p) \) by Lemma 6.6. Substituting this to the above equations gives the...
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following:

\[ d_1(\omega(m; k + 1, 2)) = \frac{k}{2} v_2^m t_1 \otimes \xi/p v_1^{k+1} + sv_2^{m-p} \sigma \otimes V v_1^{m-1} + t_1 \otimes \xi/p v_1^{k+1-n} \]

which is homologous to \( \frac{1}{2} k y_m \otimes \xi/p v_1^{k+1} + (1 - k/2) v_2^{m-p} \sigma \otimes V v_1^{p-1} + t_1 \otimes \xi/p v_1^{k+1-n} \) by Lemma 4.25, noticing that \( v_2^{m-p} \sigma \otimes V v_1^{p-1} \) is homologous to zero. This gives \( \delta_1 (y_m(k + 1, 1)) \) for \( k \leq (p - 1)p^{n-1} \). For \( k > (p - 1)p^{n-1} \) with \( k > p \) or \( n > 1 \), use Corollary 6.3 to find the leading term. In fact, we have an integer \( e \) such that \( 0 < k - ep \leq (p - 1)p^{n-1} \) and consider \( v_1^{ep} \delta_1 (y_m(k + 1, 1)) = \delta_1 (y_m(k + 1 - ep, 1)) \). A direct calculation also gives the case \( n = 1 \) and \( k = p \). Thus we have the case \( i = 0 \). For the additional case for \( m \in \mathbb{Z}(2) \) is also seen in the same manner.

Now turn to the case \( i > 0 \). Assume that \( kp^i + 1 \leq A_n - i + 2 \) and put \( y_m(kp^i + 1, l) = \eta(k/p_{i+1}) \sigma_{kp^i+1} \), for \( l > 0 \). Then we obtain that \( d_1(y_m(kp^i + 1, l)) = 0 \) for \( l \leq i \). We further see that \( y_m(kp^i + 1, i + 1) \) is also a cocycle for \( kp^i < a_n \), if we put \( y_m(kp^i + 1, i + 1) = \eta(k/p_{i+1}) \sigma_{kp^i+1} \). Inductively, we have a cocycle \( y_m(kp^i + 1, i + 1) \) for \( kp^i + 1 \leq A_n - i \) such that \( p y_m(kp^i + 1, i + 1) = y_m(kp^i + 1, i + 1) \) and \( v_2^{kp^i} y_m(kp^i + 1, i + 1) = y_m((k - a)p^i + 1, i + 1) \) for \( a \) with \( A_n - i + 2 < (k - a)p^i + 1 \leq p^{n-1} + 1 \). Then to tell the leading term of \( \delta_1 (y_m(kp^i + 1, i + 1)) \) suffices to show the one only for \( kp^i \leq p^{n-1} \) by virtue of Corollary 6.3. Now compute \( d_1 (\eta(k/p_{i+1}) \sigma_{kp^i+1, i+2}) \) for \( kp^i \leq p^{n-1} \) by (3.12) and Lemma 7.4 as above, and obtain \( \frac{1}{2} k y_m \otimes \xi/p v_1^{kp^i+1} + sv_2^{m-p} \sigma \otimes V v_1^{n-1}/p v_1^{kp^i+1} + \xi/p v_1 \). The proposition then follows from Lemmas 4.25 and 6.10.

Q.E.D.

**Proposition 7.6.** Let \( t \) and \( j \) be integers with \( 1 \leq j \leq p^2 + 1 \). Then there exist cocycles \( y_{tp^2-1}(j, l) \) for \( 1 \leq l \leq i + 1 \) such that

\[
\delta_1(y_{tp^2-1}(j, 1)) = \frac{j}{2} y_{tp^2-1} \otimes \xi/v_1^{i+1} + \cdots
\]

\[
\delta_1(y_{tp^2-1}(p^2, 1)) = x_0^{(p-1)p} G_0/v_1 \quad (p^2 \neq t)
\]

\[
\delta_1(y_{tp^2-1}(kp, 2)) = \frac{k + 1}{2} y_{tp^2-1} \otimes \xi/v_1^{kp}
\]

\[
\delta_1(y_{tp^2-1}(p^2 - p, 3)) = \frac{1}{2} y_{tp^2-1} \otimes \xi/v_1^{p^2-p}.
\]

Proof. First consider the case \( j = p^2 \), where we put

\[
y_{tp^2-1}(p^2, 1) = d_0(x_p)/t p_1^{p^2+p}
\]

and obtain the desired equation from (4.8) and Proposition 7.3.

By the definition (6.5), we see that \( d_0(v_2^{p^2}) \equiv - t p v_2^{(p-1)p} V_2 \mod (p^2, v_2^{p^2}) \). So if we put

\[
y_{tp^2-1}(kp^i - p, i + 1) = d_0(v_2^{p^2+1})/t p^2 v_1^{p^i+1} \quad (+ v_2^{p^2+p-2} p_2/v_1) \text{ if } i = 2
\]

then \( p y_{tp^2-1}(kp^i - p, i + 1) = d_0(v_2^{p^2+1})/t p^2 v_1^{p^i+1} = - v_2^{(p-1)^{p^i+1} V_2/P v_1^{p^i+1}} = y_{tp^2-1}(p^2, 3) \) by Lemma 6.6 and (4.28). Besides, if \( i < 2 \),

\[
d_1(y_{tp^2-1}(kp^i - p, i + 1))/p = - k t_1 \otimes d_0(v_2^{p^2+1})/t p^2 v_1^{p^i+1}
\]
which equals $kt_1 \otimes v_\ell^{q-1)p_1^p} V_{p_1^p/p_1^p} v_\ell^{p-1-p_1^p}$. Use now Lemma 6.9 to obtain the proposition, noticing that some element bounds the element corresponding to $g_0$ in Lemma 6.9, which can be read off from the structure of $H^2 M_1$ of (4.24). Comparing degrees, the structure of $H^2 M_1$ also induces the other fact that the $\delta_1$-image has no lower term. A similar argument also shows the case $i = 2$. Q.E.D.

Thus we have computed the $\delta_1$-images of the elements associated to the generators $y_m/v_1^2$ of $Y_{1,c}$ with $p \not\mid m$ or $p \not\mid (j + 1)$.

**Lemma 7.7.** Let $n, t, k$ and $i$ be integers such that $n, k, i > 0$ and $kp_1^p + 1 \leq A_n + 2 - p_1^{n+2} - p_1^n + A_n + 2$. Then $y_{(tp_{i-1}^p)}(kp_1^p + 1, 1)$ is redefined to be a sum of $\frac{1}{2}d_0(x_{\ell}^{l+2})/tp_{i}^{2}v_1^{kp_1^p+p_1^{n+1}+p_1^n}$ and $- y_{(v_{1}^{n+1})}(kp_1^p + p_1^n - p_1^{n+2} + 1, 2)$.

**Proof.** In this proof, we put $m = (tp_{i} - 1)p_1^{n+1}$ and $jp_1^p = kp_1^p + p_1^n - p_1^{n+2}$. We read off that $y_m(jq_1^p + 1, 2) = y_{(tp_{i-1}^p)}(kp_1^p + p_1^n - p_1^{n+2} + 1, 2)$ is a cocycle from Proposition 7.5, since $kp_1^p + 1 \leq p_1^{n+2} - p_1^2 + A_n + 2$. Furthermore, $y_m(jq_1^p + 1, 2)$ has the leading term $\eta_1(v_1^n)\sigma_{jp_1^p+1,2}$ by the proof of Proposition 7.5. We compute

$$d_0(x_{\ell}^{l+2})/tp_{i}^{2}v_1^{kp_1^p+p_1^{n+1}+p_1^n} \equiv 2tp_{i}^{n+1}v_1^{(p_{i-1}^p-1)p_1^n} \sigma \mod (p_{i+1}^{n+2}, v_1^{p_1^n+p_1^n})$$

by (4.8) and the binomial theorem, where $l = v_1(t)$.

Put now

$$\zeta = \frac{1}{2} d_0(x_{\ell}^{l+2})/tp_{i}^{2}v_1^{kp_1^p+p_1^{n+1}+p_1^n} - y_m(jq_1^p + 1, 2).$$

Then the above statements say that $p \zeta = 0$, $\xi$ is a cocycle and $\xi$ has the leading term $v_1^{(p_{i-1}^p-1)p_1^n} \sigma v_1^{kp_1^p+p_1^n}$. These properties are those of $y_{(tp_{i-1}^p)}(kp_1^p + 1, 1)$, and so we redefine $y_{(tp_{i-1}^p)}(kp_1^p + 1, 1) = \xi$. Q.E.D.

**Proposition 7.8.** Consider an integer $m = (tp_{i} - 1)p_1^{n+1} \in \mathbb{Z}(2)$ for $t, n \in \mathbb{Z}$ with $n > 0$. If $i$ and $k$ are positive integers with $kp_1^p < p_1^{n+2} - p_1^n + A_n - 1 + 2$, then we have cocycles $y_m(kp_1^p + 1, 1)$ for $0 \leq i \leq i$, whose $\delta_i$-images are given by $\delta_i(y_m(kp_1^p + 1, 1)) = 0$ for $l \leq i$ and

$$\delta_1(y_m(kp_1^p + 1, 1)) = \lambda \kappa x_{\ell}^{l+1} G_{n+1}/v_1^{kp_1^p+p_1^{n+1}+p_1^n} - y_m(kp_1^p + 1, 2).$$

Furthermore, if $kp_1^p \leq p_1^{n+2} - p_1^n$, then we have more cocycles such that

$$\delta_1(y_m(kp_1^p + 1, n + 1)) = \frac{k + p_1^{n+1}}{2} y_m \otimes \zeta/v_1^{kp_1^p+p_1^n+1} + \cdots$$

$$\delta_1(y_m(p_1^{n+2} - p_1^n + 1, n + 1)) = \frac{1}{2} x_{\ell}^{l+1} G_0/v_1 (p_1^{n+2} \not\mid t)$$

$$\delta_1(y_m(p_1^{n+1} - p_1^n + 1, n + 2)) = \frac{k + 1}{2} y_m \otimes \zeta/v_1^{kp_1^p+p_1^{n+1}+p_1^n+1} + \cdots$$

$$\delta_1(y_m(p_1^{n+2} - p_1^n + 1, n + 3)) = \frac{1}{2} y_m \otimes \zeta/v_1^{p_1^{n+2} - p_1^n - p_1^{n+1}}.$$  

In the above equations, $\cdots$ denotes a lower term.

**Proof.** For the case $kp_1^p \leq p_1^{n+2} - p_1^n$, we put

$$y_m(kp_1^p + 1, 1) = \frac{1}{2} d_0(v_1^{p_1^{n+2}}/tp_1^{n+1} v_1^{kp_1^p+p_1^{n+1}+p_1^n}.$$
if \( kp' \neq p^{n+2} - p^n \), and

\[ y_m(p^{n+2} - p^n + 1, l) = \frac{1}{2} d_0(x'_{n+2})/tp^{l+1}v_1^{p^{n+2}+p^n} \cdot \]

This is guaranteed by Lemmas 4.25 and 6.7. Note that this element \( y \) may differ from the one \( y \) in Proposition 7.5 and we will denote the latter by \( \tilde{y} \). Thus if \( kp' < p^{n+2} - p^n \), we see that

\[ \delta_1(y_m(kp' + 1, l)) = 0 \text{ for } l \leq i \] as we have seen above. For \( l = i + 1 \), we deduce the results from (3.12) and Lemmas 6.7 and 6.11 using the formula

\[ d_0(1/p^{l+2}v_1^{p^l+p^{n+1}+p^n}) = -(k + p^{n-1})t_1/p^{l+1}v_1^{p^{n+1}+p^n-1}. \]

Even in the case \( kp' = p^{n+2} - p^n \), we have the same results as above by (4.8) and Proposition 7.3. Furthermore, a similar computation gives the case \( i = n \) and \( p(k + 1) + 1 \) and the case \( i = n \) and \( p^2(k + p + 1) \). Note that the last condition \( i = n \) and \( p^2(k + p + 1) \) is equivalent to \( k = p^2 - p - 1 \).

Turn now to the case \( kp' > p^{n+2} - p^n \). Then Lemma 7.7 enables us to define

\[ y_m(kp' + 1, l) = \frac{1}{2} d_0(v_2^{p^{n+2}})/tp^{l+1}v_1^{p^l+p^{n+1}+p^n} - \tilde{y}(1/p^{l+1}p^{n+1})(kp' + p^n - p^{n+2} + 1, l + 1) \]

where \( \tilde{y} \) denotes the element \( y \) in Proposition 7.5 as we noted above. We use the notation \( \tilde{y} \) here in order to distinguish these \( y \)'s appearing in both of Propositions 7.5 and 7.8. Then the first term is a cocycle for \( l \leq i \) and mapped to \( \frac{1}{2} k y_{(1/p^{l+1}p^{n+1})/p^{n+1}+p^n-1} \) for \( l = i + 1 \) by \( d_1 \) as we have seen above. For the second term, use Proposition 7.5 to see that it is a cocycle for \( l \leq i \) and

\[ \delta_1(y_m(kp' + 1, i)) = \frac{k}{2} y_{(1/p^{l+1}p^{n+1})/p^{n+1}+p^n-1} - \frac{k}{2} y_{(1/p^{l+1}p^{n+1})/p^{n+1}+p^n-1} \]

as desired.

Q.E.D.

8. \( H^*M^n_0 \)

Let \( \delta_i: H^M^n_0 \to H^{i+1}M^n_0 \) be the connecting homomorphism. Then we introduce some notation:

For a submodule \( M \) of \( H^{i+1}M^n_1 \), \( M_C \) (resp. \( M_I \)) denotes the intersection of \( M \) and cokernel (resp. image) of \( \delta_i \) up to isomorphism. \( M^\infty \) denotes the submodule of \( H^{i+1}M^n_0 \) consisting of \( x \in H^{i+1}M^n_0 \) such that \( p^nx \in \varphi(M) \) for some \( n \).

We also denote

\[ X\zeta = X \otimes Z_{(i)} \zeta \] and \( X_{\omega}\zeta = X_{\omega} \otimes Z_{(i)} \zeta \).

Then Proposition 5.7 gives

\[ X\zeta_C = F_p\{x'p\zeta/v_1^1:s \in Z - pZ, n \geq 0, 1 \leq j \leq a_n \text{ such that} \]

\[ j > a_{n-1} \text{ if } p^j | i \text{ for either } s \in Z_1, \text{ or } s \in Z_2 \text{ and } p^{k+1} | j \}

\[ X_{\omega}\zeta_C = X_{\omega}\zeta \]

\[ Y_{0, c} = F_p\{y_{(p^n)}v_1^j:s \in Z_0, n \geq 0, j \leq A_n + 2, \text{ such that} \]

\[ j = 1 \text{ or } j - 1 > a_{n-1} \text{ if } p^j | (j - 1) \}

\[ Y_{1, c} = F_p\{y_{(p^n)}v_1^j:s \in Z_1, k \geq 0, n \geq 0, j \leq A_n + 2, \text{ such that} \]

\[ j = 1, j - 1 > a_{n-1} \text{ if } p^j | (j - 1) \text{ and } p^{k+1} \not{|} (j + a_{n-1}), \text{ or } j > a_{n+2} - a_{n+1} \}

\[ Y_C = F_p\{v_1^pY/v_1^j:s \in Z, 1 \leq j \leq p - 1, p | (s + 1) \text{ if } j = p - 1 \}

\[ Y_{\omega, c} = F_p\{t_1/v_1 \} \].
Since \( \Xi_c \subset X^\infty \otimes \mathbb{Z}_{(p)} \{ \zeta \} \), we have

\[
X_c \Xi_c = \mathbb{Z}_{(p)} \{ x_i^{a_i} / p^{i+1} v^1 \mid s \in \mathbb{Z} - p \mathbb{Z}, j > 0, p^i | j \leq a_{n-i} \}
\]

either \( p^{i+1} | j \) or \( j > a_{n-i-1} \), and

\[
p^{i+1} | j \text{ if either } s \in \mathbb{Z}_1 \text{ or } s \in \mathbb{Z}_1^k \text{ and } p^{k+1} | j \}
\]

by Proposition 7.1. We also see that

\[
X_c \Xi_c = X_c \otimes \mathbb{Z}_{(p)} \{ \zeta \}.
\]

We have more modules:

\[
Y^0_{0,c} = \mathbb{Z}_{(p)} \{ y_{kp'} + l, i + 1 \mid y_{kp'} / v^1 \} \in Y_{0,c},
\]

for \( k = 0 \), \( i = n \), and

for \( k > 0 \), \( kp^i + 1 \leq A_{n-i} + 2 \),

\[
k^p + 1 > a_{n-i} \text{ if } p \nmid k, \text{ and } > A_{n-i-1} + 2 \text{ otherwise}.\}
\]

\[
Y^0_{1,c} = \mathbb{Z}_{(p)} \{ y_{(kp^2 - 1)p^i + 1, l} \mid y_{(kp^2 - 1)p^i / v^1} \} \in Y_{1,c},
\]

\[
l = n + 1 \text{ if } k = 0.
\]

For \( k > 0 \),

\[
l = i > 0 \text{ for } p^i < kp^i \leq A_{n-i} + 2 \text{ and } p^i - p^* - A_{n-i-1} + 2 \text{ if } p \nmid k;
\]

\[
l = i + 1 \text{ for } i = 0 \text{ and } p \nmid (k + p^{*i}) \text{ or } \text{ for } kp^i \leq p^i - p^* - A_{n-i} \text{ and } 0 < i \leq n;
\]

\[
l = n + 2 \text{ for } i = n, k \leq p^2 - 1, p((k + 1) \text{ and } k \neq p^2 - p - 1 \text{ and } n + 3 \text{ if } i = n \text{ and } k = p^2 - p - 1}.
\]

\[
Y^0_{c} = \mathbb{Z}_{(p)} \{ y_{ip}(j, l) \mid y_{ip} / v^1 \} \in Y_c,
\]

\[
l = 1 \text{ if } j < p - 1, \text{ and } l = 2 \text{ if } j = p - 1}.
\]

\[
Y^0_{0,c} = \mathbb{Z}_{(p)} \{ y_{ip}(j, l) \mid y_{ip} / v^1 \} \in Y_{0,c},
\]

Moreover, by Propositions 7.2, 7.3, 7.5, 7.6 and 7.8, we divide \( Y^0_{0,c} \) and \( Y^0_{1,c} \) into two submodules, respectively:

\[
Y^0_{0,c}^G = \mathbb{Z}_{(p)} \{ y_{ip}(kp^i + 1, i + 1) \mid y_{ip} / v^1 \} \in Y_{0,c}, k \neq 0,
\]

\[
A_{n-i-1} + 1 < kp^i + 1 \leq A_{n-i} + 1 \text{ for } i \geq 0}.
\]

\[
Y^0_{0,c}^G = \{ 0 \} \cup Y^0_{0,c} - Y^0_{0,c}^G.
\]

\[
Y^0_{1,c}^G = \mathbb{Z}_{(p)} \{ y_{ip}(kp^i + 1, i + 1) \mid y_{ip} / v^1 \} \in Y_{1,c}, k \neq 0,
\]

\[
p^i + 2 - p^* + A_{n-i-1} + 1 < kp^i + 1 < p^i + 2 - p^* + A_{n-i} + 1 \text{ for } i \geq 0}.
\]

\[
Y^0_{1,c}^G = \{ 0 \} \cup Y^0_{1,c} - Y^0_{1,c}^G.
\]

We summarize the results of the previous section as follows.

**Proposition 8.1.** The connecting homomorphism \( \delta_1 \) sends an element \( y / p^{l} v^1 \) of \( Y^0_{0,c} \oplus Y^0_{1,c} \oplus Y^0_c \) to the element \( y \otimes \zeta / v^1 \) of \( H^2 \mathcal{M}_1 \). An element \( y / p v^1 \) of \( Y^0_{0,c} \oplus Y^0_{1,c} \).
is mapped to an element of $G = H^2M^1_0$ by $\delta_1$. Furthermore, $X^\infty$ and $X_{n,c}^\infty$ are sent to $Y_{0,t} \oplus Y_{1,t} \oplus Y_t$ and $Y_n$, respectively, and $Y_{n,c}^\infty$ to 0.

Now we have the following theorem.

**THEOREM 8.2.** $H^1M^1_0$ is a $\mathbb{Z}_{(p)}$-module isomorphic to
\[ Y_{0,c}^\infty \oplus Y_{1,c}^\infty \oplus Y_c^\infty \oplus Y_{n,c}^\infty \oplus X_{c,c}^\infty \oplus (X^\infty \otimes \mathbb{Z}_{(p)}[\zeta]). \]
(8.3)

**Proof.** We will prove this by Lemma 4.3. Let $B^1$ be the module (8.3) and the map $f: B^1 \to H^1M^1_0$ the inclusion. Since the cokernel of $\delta_0$ is isomorphic to the image of $\varphi$, $\varphi$ induces the map $\tilde{\varphi}: H^1M^1_0 \to B^1$ by the definition of the modules $M_c$. It is easy to see that $pB^1 \subset B^1$. Thus we have the commutative diagram of Lemma 4.3.

Now it is sufficient to show that the sequence including $B^1$ is exact. It follows from the exact couple of the Bockstein spectral sequence that the sequence $H^1M^1_0 \to B^1 \to B^1$ is exact. To see that the sequence $B^1 \to B^1 : H^2M^1_0$ is exact, we assume that a linear combination $c$ of the elements of $B^1$ maps to zero by $\delta_1$. If $\delta_1(\xi) = 0$, then there exists $\xi' \in B^1$ such that $\xi = p\xi'$ by the definition of $B^1$. Furthermore, if the sum of $\xi'$'s with $\delta_1(\xi) \neq 0$ is null, then there is some nontrivial relation between these elements, which is a contradiction to Proposition 8.1. In fact, the generators in $H^2M^1_0$ are linearly independent. Thus the linear combination does not have a term $\xi$ such that $\delta_1(\xi) \neq 0$, and so it is in the image of $p$.

Q.E.D.

9. $H^2M^2_0$

In order to state the structure of $H^2M^2_0$, we divide the module $G$ into two parts: one is $G_C$ and the other is $G_l$. Propositions 7.3, 7.5, 7.6 and 7.8 show that
\[
G_C = \mathbb{Z}_{(p)}[x_{n,j}^c G_0/v_1, x_{n}^\infty G_0/v_1^1 : k \geq 0, n > 0, s + 1 \in \mathbb{Z} - p\mathbb{Z}, \\
1 \leq j \leq a_n, \text{ and for } n > 0, \\
p^{i+1} \lambda(j + A_n - i + 1) \text{ if } s = u p_i \in \mathbb{Z}(0), \text{ or} \\
p^{i+1} \lambda(j + A_n - i + 1) \text{ if } s = u p_i \in \mathbb{Z}(2) \text{ and } i > 0).
\]

Now we compute the connecting homomorphism $\delta_2 : H^2M^2_0 \to H^3M^1_0$.

**PROPOSITION 9.1** (Shimomura and Yabe [17, Prop. 4.1 and 4.3]).
\[
\delta_2(x_{j}^t G_1/p v_1^t) = - j + \frac{1}{2} x_{j}^t G_1 \otimes \zeta/v_1^t \\
\delta_2(x_{j}^t G_1/p^2 v_1^t) = - \frac{1}{2} x_{j}^t G_1 \otimes \zeta/v_1^{p-1}.
\]

In order to generalize the results of [17, Prop. 4.1], we redefine the generators $x_{s}^t G_n/v_1^t$ of $H^2M^2_0$ as for the generator $y_{n,i}^t v_1^t$ in Section 7.

Recall that the generator $x_{s}^t G_n/v_1^t$ with $n > 0$ is characterized by the two conditions:
\[
v_1^{t-1} x_{s}^t G_n/v_1^t = v_2^{s p_n - (p^{n-1} - 1)/(p-1)} g_1/v_1 \text{ and } d_1(x_{s}^t G_n/v_1^t) = 0. \]

Put now
\[
x_{s}^t G_n(j, 1) = \frac{\lambda_n}{u} d_0(v_2^{p_n}) \otimes \sigma_{j + A_n - 1 + 2. e + 2} v_1^{t-1} v_1^{p-1}
\]
for $s = u p^e$ with $p \not| u$, where $\lambda_n = (-1)^{n+1}4$ if $n > 1$ and $=2$ if $n = 1$. Then we have Lemma 9.2.
LEMMA 9.2. Let \( n, s \) and \( j \) be integers such that \( n > 0, 0 < j \leq a_n \) and \( p|s(j + A_n - 1 + 1) \).
Then the element \( x^s_n G_j(j, 1) \) satisfies the following: \( px^s_n G_j(j, 1) \) and \( v^l_1 x^s_n G_j(j, 1) \) are homologous to zero and \( v^l_1 p^{p^n - (p^n - 1)/(p - 1)} g_1/p^v \), respectively, and \( x^s_n G_j(j, 1) \) is a cocycle in the cobar complex \( \Omega^2 M^n_0 \). Therefore, we have a generator \( x^s_n G_n/v^l_1 \) of \( H^2 M^n_1 \) such that \( \varphi(x^s_n G_n/v^l_1) = x^s_n G_j(j, 1) \).

Proof. Since \( x^s_n G_n/v^l_1 \) is a generator, we have \( j \leq a_n \), and so \( j + A_n - 1 + 2 \leq A_n + 1 \). Note that

\[
d_0(v^l_1 p^n) = -sv^l_1 p^n v^{p^n - p^n - 1} v^l_1 p - psv^l_1 p^{p^n - p^n - 1} v^l_1 p \quad \text{(9.3)}
\]

mod \((p^{n+2}, p^{n+1}, p^n, v^l_1, p^2)\) for \( e = v_p(s) \) by Lemma 6.7. Then we see that \( px^s_n G_j(j, 1) \) is homologous to zero by Lemma 6.10, and \( v^l_1 x^s_n G_j(j, 1) = \lambda x^s_n G_j(j, 1) \) is homologous to \( v^l_1 p^{p^n - (p^n - 1)/(p - 1)} g_1/p^v \) by Lemma 6.10. We also see that \( x^s_n G_j(j, 1) \) is a cocycle by Lemma 7.4. Q.E.D.

LEMMA 9.4. Suppose that \( x^s_n G_j(j, 1) \) is a cocycle, \( k' = j + A_n - 1 + 1 \leq A_n + 1 \) and \( l \leq i + 1 \). Then \( \delta_2(x^s_n G_j(j, 1)) = \lambda x^s_n G_n \otimes \zeta/v^l_1 \).

Proof. Since \( H^3 M^n_1 \) is generated by the elements \( \gamma/v^l_1 = x^s_n G_m \otimes \zeta/v^l_1 \) with \( p\gamma(t + 1) \) and \( a < p^{n - 1}(p + 1) \), we may put

\[
\delta_2(x^s_n G_j(j, 1)) = \sum k \gamma/v^l_1
\]

for \( k \in F_p \). In the summation, we see that \( a \leq p^{n + 1} \) by Corollary 6.4 since \( j < p^{n + 1} \). Furthermore, the above equation is homogeneous, and so the internal degree of \( \gamma/v^l_1 \) is the same as that of \( x^s_n G_j(j, 1) \).

As is stated in [14, (4.3.3)], \( |x^s_n G_j/j| = (p^n - (p^n - 1)/(p - 1))(p + 1) - 1 - j \). Thus we have an equation and an inequality

\[
sp^n(p + 1) - kp^l = (tp^n - (p^n - 1)/(p - 1))(p + 1) - 1 - a
\]

\[
0 < a < \min\{p^{n + 1}, p^{n - 1}(p + 1)\}
\]

since \( kp^l = j + 1 + (p + 1)(p - 1)/(p - 1) \). Here we note that \( a \neq p^{n + 1} \). In fact, if so, we deduce that \( i = n \) and \( k = 1 \) and the equation does not hold even if we consider it modulo \( p \).

Now we solve these. First suppose that \( m \geq n + 1 \). Then \( a < p^{n + 1} \). Note that \( A_n - 1 + 1 = kn \leq A_n + 1 \), and the above equations give us

\[
(\text{tp}^n - sp^n - (p^n - 1)/(p - 1))(p + 1) - 1 < a < p^{n + 1}
\]

\[
0 < a < \left(\text{tp}^n - sp^n - (p^{n - 1} - p^n)/(p - 1)\right)(p + 1) - 1.
\]

This gives \( tp^n = sp^n + (p^n - 1)/(p - 1) + p^n \) and then deduce the contradiction \( a > p^{n + 1} \). Next consider the case that \( m \leq n \). Then \( a < p^{n - 1}(p + 1) \) and similarly to the above, we have inequalities

\[
(\text{tp}^n - sp^n + (p^{n - 1} - p^{n - 1})/(p - 1))(p + 1) < a < p^{n - 1}(p + 1)
\]

\[
0 < a < \left(\text{tp}^n - sp^n + (p^n - p^{n - 1})/(p - 1)\right)(p + 1) - 1.
\]
If \( m = n \), then we have the trivial solution: \( t - s \) and \( a - j \). For the case \( m < n \), we obtain that 
\[
 t = sp^{n-m} - (p^n - m - 1)(p - 1) + a \\
 a = \frac{-(p^n - 1)(p - 1) + p^{n+1} + a}{p + 1} - 1 + kp' 
\]
for \( 0 \leq a < p^{n-m-1} \). We further see that the inequality \( a > 0 \) indicates \( \alpha \geq p^{n-m-1} \). This is a contradiction. Therefore we have no solution in this case, either. Hence the above summation has only a term
\[
k_{j}g/v_1^2 = \lambda x^*_n G_n \otimes \zeta/v_1^j.
\]
Q.E.D.

Now we have the generalization of [17, Prop. 4.1].

**PROPOSITION 9.5.** Let \( n, s, i, j \) and \( k \) be integers such that \( n > 1, i \geq 0, j > 0, \) and \( kp' = j + A_{n-1} + 1 \). Then we have cocycles \( x^*_n G_n(j, l) \) for \( 0 < l < i + 1 \), and

\[
\delta_2(x^*_n G_n(j, i + 1)) = -\frac{k}{2} x^*_n G_n \otimes \zeta/v_1^j.
\]

**Proof.** We show first that \( x^*_n G_n(j, l) \) is a cocycle for \( 0 < l < i + 1 \), inductively. For \( l = 1 \), it is trivial since \( x^*_n G_n(j, 1) = \phi(x^*_n G_n/v_1^j) \). Assume now that \( x^*_n G_n(j, l) \) is a cocycle for \( l < i \).

Lemma 9.4 says that

\[
\delta_2(x^*_n G_n(j, l)) = \lambda x^*_n G_n \otimes \zeta/v_1^j
\]

for some \( \lambda \in \mathbb{F}_p \). By virtue of Lemma 6.10, we may put

\[
x^*_n G_n(j, l) = \lambda^{n} v^{(i+1-j) - 1} - V_0 \otimes p^{k+1, l+1} = -\frac{\lambda^n}{s} d_0 (v^{p^n}) \otimes p^{k+1, l+1} \quad \text{(by Lemma 6.7)}
\]

for \( kp' \leq p^n \), and for \( kp' > p^n \), some lower terms would be added. Here \( \lambda^n = (-1)^{n+1} / 4 \) for \( n > 1 \). Lemma 6.3 tells us that \( x^*_n G_n(j, l + 1) \) is a cocycle for \( j < p^n \). For \( j > p^n \), use Corollary 6.3 to find that \( \lambda \) in (9.6) is null, which means that \( x^*_n G_n(j, l + 1) \) is a cocycle.

We compute \( d_2(x^*_n G_n(j, i + 2)) = -\frac{1}{k} \lambda^{n} v^{(i+1-j) - 1} - V_0 \otimes t_i \otimes \zeta/pv_1^{2k+1} \) for a small value \( j \) by Lemma 7.4. Then the proposition follows from the definition of \( \delta_2 \) and (9.7) for a small value of \( j \). For a larger value of \( j \), again use Corollary 6.3, and we have \( \lambda = -\frac{1}{k} \) in (9.6) by comparing internal degrees.

We also have in [17, Prop. 4.4 and Lemma 4.5] the following:

\[
\delta_1(y_{sp^n}/p^{n+1} v_1) = \frac{s}{2} v^{p^n}(g_0 - t_1 \otimes \zeta)/v_1
\]

and

\[
\delta_2(y_m \otimes \zeta/p^{n+1} v_1) = \frac{s}{2} x^*_m G_0 \otimes \zeta/v_1.
\]

Putting these together, we obtain Proposition 9.8.

**PROPOSITION 9.8.** For the integers \( s \) and \( n \) with \( p \not| s(s + 1) \), we have

\[
\delta_2(x^*_n G_0/p^{n+1} v_1) = \frac{s}{2} x^*_n G_0 \otimes \zeta/v_1
\]

In fact, the first equation gives \( x^*_n G_0/pv_1 = y_{sp^n} \otimes \zeta/pv_1 \) and then use the second one.
Corollary 9.9. For any $j > 0$, we have

$$\delta_2(G_0/p^jv_1) = 0.$$ 

Proof. Suppose that there exists a positive integer $j$ such that $\delta_2(G_0/p^jv_1) \neq 0$. By Lemma 6.2 and Proposition 9.8, we see that $v_1^j \delta_2(G_0/p^jv_1) = 0$. Since $v_1^j$ acts monomorphically on the submodule $\langle x_1^j G_0/v_1 : s + 1 \in \mathbb{Z} - p \mathbb{Z} \rangle$ of $H^2 M_1$, the above two statements produce a contradiction. Q.E.D.

Now define

$$G_{i^j} = \mathbb{Z}_{(p)} \{ x_1^i G_n(j, l) : x_1^i G_n/v_1^j \in G_C - \{G_0/v_1\},$$

$$l = i + 1 \text{ if } n = 0 \text{ and } v_1(s) = i,$$

$$l = i + 1 \text{ if } n \geq 1 \text{ and } v_1(j + A_{n-1} + 1) = i \}$$

$$G_0^e = \mathbb{Z}_{(p)} \{ G_0/p^jv_1 : j > 0 \}.$$ 

From Propositions 7.5 and 7.8, we further have $Y^e G = ((Y_0, C \oplus Y_1, C) \otimes F_r(\zeta))_C$ that is an $F_r$-vector space over the basis

$$\{ y_{sp^n} \otimes \zeta/s^{p^{k+1}+1} : y_{sp^n}/v_1^{p^{k+1}+1} \in Y_0, C \oplus Y_1, C, k = 0, \text{ or}$$

$$p \not| s \text{ and } A_{n-1} + 2 < kp^{i+1} + 1 \leq A_n - 2 \text{ if } s \in \mathbb{Z}_0, \text { and}$$

$$kp^{i+1} > p^{n+2} - p^n \text{ for } i > 0 \text{ and } kp > A_{n-1} + 1 \text{ for } i = 0 \text{ if } s \in \mathbb{Z}_2 \}.$$ 

Note that we have an isomorphism.

Remark 9.10. $Y^e G \cong G_1$ as $F_r$-vector spaces.

In fact, the correspondence can be read off from Propositions 7.5 and 7.8. Note also that $Y^e G$ produces the submodule $(Y_0, C \oplus Y_1, C) \otimes Z_{(p)}(\zeta)_C$ of $H^2 M_1$. Thus we introduce another notation:

$$Y^e G = (Y_0, C \oplus Y_1, C) \otimes Z_{(p)}(\zeta)_C.$$ 

Now we have the following result.

Theorem 9.11.

$$H^2 M_1^e = Y^e G \oplus G_0^e \oplus (Y_\infty, C \otimes Z_{(p)}(\zeta)_C) \oplus G_0^e.$$ 

Proof. First we study the cokernel of $\delta_1$. By the results of Section 6, we see that the submodule of $H^2 M_1$ of the form $M \otimes Z_{(p)}(\zeta)_C$ is in the image of $\delta_1$ except for $Y^e G$ and $Y_\infty \otimes Z_{(p)}(\zeta)$. For the submodule $G$, $G = G_C \oplus G_1$ and $G_1$ is in the image of $\delta_1$. Now the theorem follows in the same way as the proof of Theorem 8.2. Q.E.D.

Note that $\delta_2$ maps $Y^e G$ isomorphically to $G_1 \otimes Z_{(p)}(\zeta)_C$, which is deduced from Lemma 6.1 and Propositions 7.5 and 7.8, and $G_0^e$ to $(G_C - \{G_0/v_1\}) \otimes F_r(\zeta)$. Besides, $\delta_2(Y^e \otimes Z_{(p)}(\zeta)_C) = 0$. Therefore we have Lemma 9.12.

Lemma 9.12. The cokernel of $\delta_2$ is the submodule generated by $G_0 \otimes \zeta/v_1$.

Using Lemma 4.3, the following is now a corollary of this lemma and Corollary 9.9.
Theorem 9.13. The module $H^3M_0$ is isomorphic to $\mathbb{Q}/\mathbb{Z}_{(p)}$ generated by $G_0 \otimes \zeta/p^lv_1$.

Summarizing these, we have the following result.

Theorem 9.14. The module $H^*M_0$ is isomorphic to

$$(X_{\infty} \oplus Y_{\infty,K} \oplus G_0^x) \otimes E(\zeta) \oplus X_{\infty} \oplus X_{\infty} \oplus Y_{\infty,K} \oplus Y_{\infty,K} \oplus Y_{\infty,K} \oplus G^x.$$ 

Here $G^x = G_0^x \oplus Y_{\infty,K}$.

Since we see that $Y_{\infty,K}$ is isomorphic to $G_0$, the notation $G^x$ is reasonable.

10. $\pi_*(L,S)$

Consider the Adams–Novikov spectral sequence based on $E(2)$ converging to the homotopy groups $\pi_*(L,S)$ of the Bousfield localization of the sphere $S^0$ ([1, 2], cf. [10]). Then the $E_2$-term of the spectral sequence is

$$H^{s,t}A = \text{Ext}^{s,t}_I(A, A)$$

where $(A, I)$ denotes the Hopf algebroid $(E(2), E(2), E(2))$ associated to the spectrum $E(2)$. We have the long exact sequence (4.2)

$$0 \to H^0N_0^0 \to H^0M_0^0 \to H^0N_1^0 \xrightarrow{d_0} H^1N_0^0 \to \cdots$$

$$\to H^1N_0^0 \to H^1M_0^0 \to H^1N_1^0 \xrightarrow{d_1} H^{1+1}N_0^0 \to \cdots$$

and

$$0 \to H^0N_0^1 \to H^0M_0^1 \to H^0M_0^1 \xrightarrow{d_0} H^1N_0^1 \to \cdots$$

$$\to H^1N_0^1 \to H^1M_0^1 \to H^1M_0^1 \xrightarrow{d_1} H^{1+1}N_0^1 \to \cdots$$

In these long exact sequences, $H^*N_0^0 = H^*A$, and the modules $H^*M_0^0, H^*M_0^0$ and $H^*M_0^0$ are known now. Since $H^1M_0^0 = 0$ for $t > 1$, $d_2: H^1M_0^0 \to H^{1+1}N_0^0$ is isomorphic for $t > 1$ and epimorphic for $t = 1$. The kernel of $d_2$ is $Y_{\infty,K}$; since $H^1M_0^0 = Y_{\infty,K}$ by (4.5). This further means that the map $H^1M_0^0 \to H^1M_0^2$ in the above sequence is a monomorphism, and we have the exact sequences

$$0 \to H^0N_0^1 \to H^0M_0^1 \xrightarrow{d_0} \cdots$$

and

$$0 \to H^1N_0^1 \to H^1M_0^1 \xrightarrow{d_1} H^2N_0^1 \to 0.$$ 

By the structures (4.5) and Theorem 9.14, we see that

$$\ker f = Z_{(p)} \{v_i^{s\cdot p^i}/p^i+1; i \geq 0, s \geq 0, p \not| s\} \oplus \mathbb{Q}/\mathbb{Z}_{(p)}$$

$$\text{Im } f = X_{\infty,K}.$$ 

Furthermore $H^1M_0^0 = 0$ if $t > 0$, and $= \mathbb{Q}$ at the internal degree 0 if $t = 0$. Therefore we have the following theorem.
Theorem 10.1. The $E_2$-term $E_2^s$ of the Adams-Novikov spectral sequence for $\pi_* (S^0)$ is given by

\begin{enumerate}
  \item $E_2^s \cong \mathbb{Z}_{(p)}$.
  \item $E_2^s \cong X^\infty$.
  \item $E_2^s \cong Y_{\infty,C}^C \oplus Y_{\infty,C}^C \oplus \tilde{X}_{\infty} \oplus (Y_{\infty,C}^C \oplus \mathbb{Z}_{(p)}[\xi])$.
  \item $E_2^s \cong G_{\infty,0}^C \oplus (G_{\infty,0}^C \oplus \mathbb{Z}_{(p)}[\xi]) \oplus \mathbb{Q}$.
  \item $E_2^s = 0$ for $t > 5$.
\end{enumerate}

Since the prime $p$ is greater than 3, the Adams-Novikov spectral sequence for $\pi_* (S^0)$ collapses from the $E_2$-term and so Theorem 10.1 gives the structure of the homotopy groups as well.

REFERENCES