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The homotopy groups $\pi_*(L_2S^0)$ at the prime 3

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Abstract

The homotopy groups $\pi_*(L_2S^0)$ of the L_2-localized sphere are determined by studying the Bockstein spectral sequence. The results also indicate the homotopy groups $\pi_*(L_{K(2)}S^0)$ and we observe that the fiber of the localization map $L_2S^0_3 \to L_{K(2)}S^0$ is homotopic to $\Sigma^{-2}L_1S^0_3$. Here S^0_3 denotes the 3-completed sphere. © 2002 Elsevier Science Ltd. All rights reserved.

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1. Introduction and statement of results

For each prime number p, there is the Bousfield localization functor $L_n : \mathscr{S}_{(p)} \to \mathscr{S}_{(p)}$ with respect to $v_n^{-1}BP$, where $\mathscr{S}_{(p)}$ denotes the stable homotopy category localized away from the prime p, BP the Brown–Peterson spectrum at p, and v_n the nth generator of the coefficient algebra BP_*. Consider the Morava K-theories $K(n)$ and the Johnson–Wilson spectra $E(n)$, where $K(n)_* = \mathbf{Z}/p[v_n^{\pm 1}]$ and $E(n)_* = \mathbf{Z}_{(p)}[v_1, v_2, \ldots, v_n, v_n^{-1}]$. Then L_n is also the localization with respect to $K(0) \vee K(1) \vee \cdots \vee K(n)$ or $E(n)$.

Hopkins and Ravenel present the homotopy equivalence $S^0_{(p)} \simeq \operatorname{holim}_{n \leftarrow} L_nS^0$, and so $\pi_*(L_nS^0)$ is an approximation of the homotopy groups of spheres. Actually, $\pi_*(L_0S^0) = \mathbf{Q}$ at any prime and $\pi_*(L_1S^0) = \mathbf{Z}_{(p)} \oplus A \oplus \mathbf{Q}/\mathbf{Z}_{(p)}\langle y \rangle$ at a prime $p > 2$ (cf. [4,5]), where A denotes the module

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generated by the generalized α-elements (see below) and $\boldsymbol{Q}/\boldsymbol{Z}_{(p)}\langle y\rangle \subset \pi_{-2}(L_1S^0)$ for the virtual generator y. The homotopy groups $\pi_*(L_2S^0)$ of $E(2)_*$-localized spheres are determined at a prime >3 in [10], which satisfies Hopkins' chromatic splitting conjecture [2]. In this paper, we determine $\pi_*(L_2S^0)$ at the prime 3.

Theorem A. *The homotopy groups $\pi_*(L_2S^0)$ at the prime* 3 *are a direct sum of three modules G_i's, which are described as follows*:

$$G_0 = \boldsymbol{Z}_{(3)} \oplus A_+ \oplus \Sigma^{-1}(A_- \oplus \boldsymbol{Q}/\boldsymbol{Z}_{(3)}\langle y\rangle)\zeta_2,$$

$$G_1 = B \oplus C \oplus CI \oplus B^* \oplus (B_1 \oplus C)\zeta_2,$$

$$G_2 = \hat{G} \oplus \hat{G}^* \oplus \widehat{GZ} \oplus \widehat{GZ}^*.$$

Here, the modules on the right-hand sides are as follows:

$$A = \sum_{i \geqslant 0} \boldsymbol{Z}/3^{i+1}\langle \alpha_{3^is/i+1} \mid 3 \nmid s \in \boldsymbol{Z}\rangle,$$

$$A_+ = \boldsymbol{Z}_{(3)}\{\alpha_{3^is/i+1} \mid \alpha_{3^is/i+1} \in A,\ s > 0\},$$

$$A_- = \boldsymbol{Z}_{(3)}\{\alpha_{3^is/i+1} \mid \alpha_{3^is/i+1} \in A,\ s < 0\}$$

for G_0,

$$B = \boldsymbol{Z}_{(3)}\{\beta_{3^ns/3^im,i+1} \mid n \geqslant 0,\ 3 \nmid s \in \boldsymbol{Z},\ i \geqslant 0,\ 1 \leqslant m < 4 \times 3^{n-2i-1}$$
$$\text{and } 3 \nmid m \text{ or } 4 \times 3^{n-2i-2} \leqslant m\},$$

$$B_1 = \boldsymbol{Z}_{(3)}\{\beta_{3^ns/3^im,i+1} \mid \beta_{3^ns/j,i+1} \in B,\ 3 \nmid (s+1),\ 3|m,$$
$$\text{or } i \neq k+1 \text{ if } s = 3^{k+2}t - 1 \text{ with } k \geqslant 0 \text{ and } 3 \nmid t\},$$

$$C = C_1 \oplus C_2,$$

$$C_1 = \boldsymbol{Z}_{(3)}\{\widetilde{\alpha_1\beta}_{3^n(3t+1)/3^im+1,i} \mid 0 < i \leqslant n, \text{ and } 2 \times 3^{n-i} < 3^im \leqslant 2 \times 3^{n-i+1}\},$$

$$C_2 = \boldsymbol{Z}_{(3)}\{\widetilde{\alpha_1\beta}_{3^n(9t-1)/3^im+1,i} \mid 0 < i \leqslant n, \text{ and}$$
$$2 \times 3^{n-i-1} < 3^im - 8 \times 3^n \leqslant 2 \times 3^{n-i}\},$$

$$CI = CI_0 \oplus CI_1 \oplus CI_2 \oplus CI_3,$$

$$CI_0 = \boldsymbol{Z}_{(3)}\{c_{3^ns} \mid n \geqslant 0,\ s = 3t+1 \text{ or } s = 9t-1\ (t \in \boldsymbol{Z})\},$$

$$CI_1 = \boldsymbol{Z}_{(3)}\{\widetilde{\alpha_1\beta}_{3^n(3t+1)/3^im+1,i+1} \mid 0 \leqslant i \leqslant n,\ 2 \times 3^{n-i-1} < 3^im \leqslant 2 \times 3^{n-i}, 3 \nmid m\},$$

$$CI_2 = \boldsymbol{Z}_{(3)}\{\widetilde{\alpha_1\beta}_{3^n(9t-1)/3^im+1,i+1} \mid 3 \nmid m \leqslant 8 \times 3^{n-i},\quad 0 \leqslant i < n,\quad k+1 > i \text{ if } 3^k|t\},$$

$$CI_3 = \mathbf{Z}_{(3)}\{\widetilde{\alpha_1\beta}_{3^n(9t-1)/3^n m+1,i} \mid i = \begin{cases} n+1 & \text{if } m = 1,3,4,6,7, \\ n+2 & \text{if } m = 2,8, \\ n+3 & \text{if } m = 5,\ k+1 > n \text{ if } 3^k | t\}, \end{cases}$$

$$B^* = \mathbf{Z}_{(3)}\{\beta(n)^*_{3^{n+l}s/3^i m, i+1} \mid 3 \nmid s \in \mathbf{Z},\ i,n \geqslant 0,\ 0 < 3^i m \leqslant 4 \times 3^n,$$
$$l > i \text{ and } l > i+1 \text{ if } 3|(s+1)\}$$

for G_1 and

$$\hat{G} = \sum_{t \in \mathbf{Z}} (B_5\{\beta_{9t+1}\} \oplus B_4\{\beta_{9t+1}\beta_{6/3}\} \oplus B_3\{\overline{\beta_{9t+7}\alpha_1}\}$$
$$\oplus B_2\{\beta_{9t+1}\alpha_1, [\beta_{9t+2}\beta'_1], [\beta_{9t+5}\beta'_1]\}) \oplus B_1\{[\beta_{9t-1/2}\beta'_1]\},$$

$$\hat{G}^* = \sum_{t \in \mathbf{Z}} (B_5\{\chi^1_{9t+7}\} \oplus B_4\{\chi^0_{9t+3}\} \oplus B_2\{\beta(0)^*_{9t+1}, \beta_{6/3}\beta(0)^*_{9t+1}, \beta_{6/3}\beta(0)^*_{9t+4}\}$$
$$\oplus \sum_{n \geqslant 1} (B_3\{\beta(0)^*_{3^{n+2}t+9u+3} \mid u \in \mathbf{Z} - I(n)\} \oplus B_2\{\beta(0)^*_{3^{n+2}t+9u+3} \mid u \in I(n)\})),$$

$$\widehat{GZ} = \sum_{t \in \mathbf{Z}} (B_5\{\zeta\beta_{9t+1}\} \oplus B_3\{\zeta\beta_{9t+1}\beta_{6/3}\}$$
$$\oplus B_2\{\overline{\zeta\beta_{9t+7}\alpha_1}, \zeta\beta_{9t+1}\alpha_1, \zeta[\beta_{9t+2}\beta'_1], \zeta[\beta_{9t+5}\beta'_1]\}),$$

$$\widehat{GZ}^* = \sum_{t \in \mathbf{Z}} (B_5\{\zeta_2\chi^1_{9t+7}\} \oplus B_4\{\zeta_2\chi^0_{9t+3}\} \oplus B_2\{\zeta_2\beta(0)^*_{9t+1}\}$$
$$\oplus B_1\{\zeta_2\beta_{6/3}\beta(0)^*_{9t+1}, \zeta_2\beta_{6/3}\beta(0)^*_{9t+4}\}$$
$$\oplus \sum_{n \geqslant 1} (B_3\{\zeta_2\beta(0)^*_{3^{n+2}t+9u+3} \mid u \in \mathbf{Z} - I(n)\}$$
$$\oplus B_2\{\zeta_2\beta(0)^*_{3^{n+2}t+9u+3} \mid u \in I(n)\}))$$

for G_2. Here, $B_k = \mathbf{Z}/3[\beta_1]/(\beta_1^k)$,

$$I(n) = \{x \in \mathbf{Z} \mid x = (3^{n-1} - 1)/2 \quad \text{or} \quad x = 5 \times 3^{n-2} + (3^{n-2} - 1)/2\},$$

$\bar{x}$ denotes a homotopy element detected by $x \in E_2^{*,*}(L_2S^0)$, the E_2-term of the Adams–Novikov spectral sequence converging to $\pi_*(L_2S^0)$, and $[x]$ for $x \in \pi_*(L_2V(0))$ is an element of $\pi_*(L_2S^0)$ such that $i_*([x]) = x$ for the inclusion $i : S^0 \to V(0) = S^0 \cup_3 e^1$. The generators are defined in Section 4 and degrees of them are

$$|\alpha_{a/b}| = 4a - 1, \quad |\beta'_1| = 11, \quad |\beta_{a/b,c}| = 16a - 4b - 2, \quad |c_a| = 16a - 7,$$

$$|\widetilde{\alpha_1\beta}_{a/b,c}| = 16a - 4b + 1, \quad |\beta(a)^*_{b/c,d}| = 16b - 8 \times 3^a - 4c - 4,$$

$$|\chi^0_a| = 16a + 7, \quad |\chi^1_a| = 16a + 15, \quad |\zeta_2| = -1$$

and orders of them are

$$o(\alpha_{a/b}) = 3^b, \quad o(\beta_{a/b,c}) = 3^c, \quad o(c_{3^n s}) = 3^{n+1} \quad \text{if } 3 \nmid s,$$

$$o(\widetilde{\alpha_1\beta}_{a/b,c}) = 3^c, \quad o(\beta(a)^*_{b/c,d}) = 3^d, \quad o(\chi^0_a) = 3, \quad o(\chi^1_a) = 3.$$

Furthermore, we abbreviate as follows:

$$\alpha_a = \alpha_{a/1}, \quad \beta_{a/b} = \beta_{a/b,1}, \quad \beta_a = \beta_{a/1}, \quad \widetilde{\alpha_1\beta}_{a/b} = \widetilde{\alpha_1\beta}_{a/b,1},$$

$$\widetilde{\alpha_1\beta}_a = \widetilde{\alpha_1\beta}_{a/1} \quad \beta(a)^*_{b/c} = \beta(a)^*_{b/c,1} \quad \text{and} \quad \beta(a)^*_b = \beta(a)^*_{b/1}.$$

Our computation of the differentials of the Bockstein spectral sequence $\pi_*(L_2V(0)) \Rightarrow \pi_*(L_2S^0)$ also works for the Bockstein spectral sequence associated to the cofiber sequence $L_{K(2)}S^0 \to L_{K(2)}S^0 \to L_{K(2)}V(0)$, and we obtain the homotopy groups $\pi_*(L_{K(2)}S^0)$ from the result on $\pi_*(L_{K(2)}V(0))$ given in [7].

Theorem B. *The homotopy groups $\pi_*(L_{K(2)}S^0)$ are the direct sum of $(\boldsymbol{Z}_3 \oplus A_+) \otimes \Lambda(\zeta_2)$, G_1 and G_2. Therefore, the homotopy fiber $F(L_1S^0, L_2S^0_3)$ of the localization map $L_2S^0_3 \to L_{K(2)}S^0$ of the 3-adic completed sphere S^0_3 is homotopic to $\Sigma^{-2}L_1S^0_3$.*

The second half of the theorem is observed in the same manner as the one at a prime > 3 [2]. This shows that there is only one summand in $F(L_1S^0, L_2S^0_3)$, while Hopkins' chromatic splitting conjecture denotes that it has three summands.

Theorem A is proved by using the Adams–Novikov spectral sequence. Let N^2 denote the spectrum such that $BP_*(N^2) = BP_*/(3^\infty, v_1^\infty)$. We denote the Adams–Novikov E_2-term for computing $\pi_*(L_2N^2)$ by $H^*M^2_0$. In the next section, we show that the E_2-term $H^*M^2_0$ is the direct sum of the three modules $\overline{A_i}$ for $i = 0, 1, 2$, and give the structures of $\overline{A_0}$ and $\overline{A_2}$. Section 3 is devoted to determine the structure of the module $\overline{A_1}$ using some results proven in the last section. The differentials of the Adams–Novikov spectral sequence are studied in [8], and we deduce the homotopy groups $\pi_*(L_2N^2)$ from these results, and then the chromatic spectral sequence shows Theorem A in Section 4.

2. Notations and the structure of $H^*M^2_0$

Consider the Hopf algebroid

$$(E(2)_*, E(2)_*(E(2))) = (\boldsymbol{Z}_{(3)}[v_1, v_2^{\pm 1}], E(2)_*[t_1, t_2, \dots] \otimes_{BP_*} E(2)_*)$$

associated to the Johnson–Wilson spectrum $E(2)$, where BP denotes the Brown–Peterson spectrum with $BP_* = \boldsymbol{Z}_{(3)}[v_1, v_2, \dots]$ and BP_* acts on $E(2)_*$ by moving v_i to v_i if $i \leqslant 2$, and to zero if $i > 2$. For an $E(2)_*(E(2))$-comodule M, H^*M denotes $\mathrm{Ext}^*_{E(2)_*(E(2))}(E(2)_*, M)$, which is given as a cohomology of the cobar complex Ω^*M of $E(2)_*(E(2))$-comodules (cf. Section 5). Then we have the Adams–Novikov spectral sequence

$$E_2^* = H^*E(2)_* \Rightarrow \pi_*(L_2S^0)$$

converging to the homotopy groups of $E(2)_*$-localized sphere spectrum.

We now recall the definition of the chromatic comodules N_j^i and M_j^i. They are defined inductively by setting $N_j^0=E(2)_*/I_j$ for $I_0=0$, $I_1=(3)$ and $I_2=(3,v_1)$, $M_j^i=v_{i+j}^{-1}N_j^i$, and $N_j^{i+1}=M_j^i/N_j^i$. Note that $M_j^i=N_j^i$ if $i+j=2$ and $=0$ if $i+j>2$. Then we see that the Adams–Novikov E_2-term $H^*E(2)_*$ is obtained from $H^*M_0^i$ for $i\leqslant 2$ and the long exact sequences $H^sN_0^i \to H^sM_0^i \to H^sN_0^{i+1} \to H^{s+1}N_0^i$. Since the modules $H^*M_0^i$ for $i<2$ are determined in [4], we here determine $H^*M_0^2$. For this sake, we consider the comodule $M_1^1=\boldsymbol{Z}/3\{x/v_1^j \mid j>0,\ x\in K(2)_*\}$, where $K(2)_*=\boldsymbol{Z}/3[v_2^{\pm 1}]$, and the short exact sequence $0 \to M_1^1 \xrightarrow{1/3} M_0^2 \xrightarrow{3} M_0^2 \to 0$. Here, note that M_0^2 is described as $M_0^2=\boldsymbol{Z}_{(3)}\{x/3^iv_1^j \mid i,j>0,\ x\in K(2)_*\}$. Then we obtain $H^*M_0^2$ from $H^*M_1^1$ which is determined in [7], by using the lemma given in [4, Remark 3.11].

In order to describe the module $H^*M_1^1$, we set up some notations: $H^*M_2^0=H^*K(2)_*$ is determined (cf. [6]) to be $F\otimes K(2)_*[b_{10}]\otimes\Lambda(\zeta_2)$, where F is the $\boldsymbol{Z}/3$-vector space spanned by 1, h_{10}, h_{11}, b_{11}, ξ, ψ_0, ψ_1, and $b_{11}\xi$. These generators, ζ_2, h_{1i} and b_{1i}, are cohomology classes represented by cocycles $v_2^{-1}(t_2-t_1^4)+v_2^{-3}t_2^3$, $t_1^{3^i}$ and $-t_1^{3^i}\otimes t_1^{2\times 3^i}-t_1^{2\times 3^i}\otimes t_1^{3^i}$ of the cobar complex $\Omega^*K(2)_*$, respectively. Besides, ξ and ψ_i are the generators of $H^{2,8}K(2)_*$ and $H^{3,8(i+2)}K(2)_*$.

Put $k(1)_*=\boldsymbol{Z}/3[v_1]$, $K(1)_*=v_1^{-1}k(1)_*$, $PE=\boldsymbol{Z}/3[b_{10}]\otimes\Lambda(\zeta_2)$, $E(2,n)_*=\boldsymbol{Z}/3[v_1,v_2^{\pm 3^n}]$ and

$$F_{(h)}=\boldsymbol{Z}/3[v_2^{\pm 3}]\{v_2/v_1, v_2h_{10}/v_1, v_2^{-1}h_{11}/v_1, v_2b_{11}/v_1\},$$

$$F_{(t)}=\boldsymbol{Z}/3[v_2^{\pm 3}]\{v_2^{-1}/v_1, v_2h_{10}/v_1^2, v_2^{-1}h_{11}/v_1^2, v_2^{-1}b_{11}/v_1\},$$

$$F_{(h)}^*=\boldsymbol{Z}/3[v_2^{\pm 3}]\{\xi/v_1, \psi_0/v_1, v_2\psi_1/v_1, b_{11}\xi/v_1\},$$

$$F_{(t)}^*=\boldsymbol{Z}/3[v_2^{\pm 3}]\{\xi/v_1^2, v_2\psi_0/v_1, v_2^{-1}\psi_1/v_1, b_{11}\xi/v_1^2\},$$

$$F_n=E(2,n+2)_*\{v_2^{\pm 3^{n+1}}/v_1^{4\times 3^n-1}, v_2^{3^{n+1}}h_{10}/v_1^{2\times 3^{n+1}+1}, v_2^{8\times 3^n}h_{10}/v_1^{8\times 3^n+1}, v_2^{3^{n+1}(2\pm 1)}\xi_n/v_1^{4\times 3^n}\},$$

$$F_0'=E(2,2)_*\{v_2^{\pm 3}/v_1^2, v_2^3h_{10}/v_1^7, v_2^8h_{10}/v_1^9, v_2^{3(2\pm 1)}\xi_0/v_1^4\},$$

where $\xi_n=v_2^{-3^n+(3^n-1)/2}\xi$. Note that $F_0=F_0'\oplus F_0''$ for $F_0''=\boldsymbol{Z}/3[v_2^{\pm 9}]\{v_2^{\pm 3}/v_1^3\}$. Then, in [8], $H^*M_1^1$ is shown to be the direct sum of F_0'' and the three modules A_i, where

$$A_0=(K(1)_*/k(1)_*)\otimes\Lambda(h_{10},\zeta_2),$$

$$A_1=\left(F_0'\oplus\sum_{n>0}F_n\right)\otimes\Lambda(\zeta_2),$$

$$A_2=(F_{(h)}\oplus F_{(t)}\oplus F_{(h)}^*\oplus F_{(t)}^*)\otimes PE.$$

Once we know the behavior of connecting homomorphism $\delta: H^*M_1^1 \to H^{*+1}M_0^2$, we obtain $H^*M_0^2$ by Miller et al. [4, Remark 3.11]. In [4,8], it is shown that $\delta(A_i)\subset A_i$. Let A_i^s denote the submodule $A_i\cap H^sM_1^1$. We now define the submodule $\overline{A_i}^s$ to fit the commutative diagram of exact sequences

$$\begin{array}{ccccccc}
A_i^s & \xrightarrow{1/3} & \overline{A}_i^s & \xrightarrow{3} & \overline{A}_i^s & \xrightarrow{\delta} & A_i^{s+1} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^sM_1^1 & \xrightarrow{1/3} & H^sM_0^2 & \xrightarrow{3} & H^sM_0^2 & \xrightarrow{\delta} & H^{s+1}M_1^1.
\end{array} \tag{2.1}$$

We deduce the following lemma from [4, Remark 3.11]:

Lemma 2.1. *Suppose that we have a* 3*-torsion module* B^s *and a homomorphism* $f : B^s \to \overline{A_i}^s$ *such that the diagram of exact sequences*

$$\begin{array}{ccccccccc}
\overline{A}_i^{s-1} & \longrightarrow & A_i^s & \xrightarrow{1/3} & B^s & \xrightarrow{3} & B^s & \xrightarrow{\delta} & A_i^{s+1} \\
 & & \| & & \downarrow & & \downarrow & & \| \\
 & & A_i^s & \xrightarrow{1/3} & \overline{A}_i^s & \xrightarrow{3} & \overline{A}_i^s & \xrightarrow{\delta} & A_i^{s+1},
\end{array}$$

commutes. Then f *is an isomorphism.*

Note that [10, Proposition 7.2] is also valid for the prime 3, and for the elements y'_{3t}, V and G_1 there are $v_2^{3t+2}h_{11}/9v_1^2$, $-v_2^2h_{11}$ and $v_2^{-1}b_{10}$ in our notation, respectively. Therefore, we have

Proposition 2.2. *For each integer* t,

$$\delta(v_2^{3t+2}h_{11}/9v_1^2) = v_2^{3t+2}(-b_{10} + h_{11}\zeta_2)/v_1^2 + \cdots .$$

Since $\delta(v_2^{3s}/3v_1^3) = sv_2^{3s-1}h_{11}/v_1^2$ by Miller et al. [4, Proposition 6.9] and $\delta(v_2^{9t-1}h_{11}/9v_1^2) = v_2^{9t-1}(-b_{10} + h_{11}\zeta_2)/v_1^2 + \cdots$ by Proposition 2.2, we derive the following:

Proposition 2.3. $H^sM_0^2$ *is isomorphic to* $\sum_{i=0}^2 \overline{A_i}^s$ *if* $s \neq 1$, *and* $H^1M_0^2 \cong \sum_{i=0}^2 \overline{A_i}^s \oplus \sum_{t\in Z} \boldsymbol{Z}/9\{v_2^{9t-1}h_{11}/9v_1^2\}$.

In the same manner as [4, Theorem 4.2], we obtain

Proposition 2.4. *The module* $\overline{A_0}$ *is given as follows*:

$$\overline{A_0} = (\overline{A_-} \oplus \boldsymbol{Q}/\boldsymbol{Z}_{(3)}) \otimes \Lambda(\zeta_2),$$

where

$$\overline{A_-} = \{1/3^{i+1}v_1^{3^i m} \mid i \geqslant 0,\ m > 0\}.$$

The module $\overline{A_2}$ is determined in [8] as follows:

Proposition 2.5. *The module $\overline{A_2}$ is the direct sum of $\boldsymbol{Z}/3[v_2^{\pm 3}]\{v_2^{-1}/3v_1, \xi/3v_1^2\}$ and*

$$(\overline{F_{(h)}} \oplus \overline{F^*_{(h)}}) \otimes PE.$$

Here, $\overline{F_{(h)}}$ and $\overline{F^*_{(h)}}$ are the images of $F_{(h)}$ and $F^*_{(h)}$ under the map $H^*M_1^1 \xrightarrow{\varphi} H^*M_0^2$ given by $\varphi(x) = x/3$.

3. Determination of $\overline{A_1}$

We divide A_1 into 14 pieces:

$$A_1 = ((X_1 \oplus X_2) \oplus (H \oplus HI \oplus H^*) \oplus (X_1^* \oplus X_2^*)) \otimes \Lambda(\zeta_2).$$

Here,

$$X_1 = X_{1,1} \oplus X_{1,2},$$

$$X_{1,1} = \boldsymbol{Z}/3\{v_2^{3^n(3t+1)}/v_1^j \mid n \geqslant 0,\ t \in \boldsymbol{Z},\ 0 < j < 4 \times 3^{n-1},$$
$$\text{such that } j > 4 \times 3^{n-i-1} - 1 \text{ if } 3^i | j\},$$

$$X_{1,2} = \boldsymbol{Z}/3\{v_2^{3^n(3^{k+2}u-1)}/v_1^{3^{k+1}m} \mid n \geqslant 0,\ 3 \nmid u \in \boldsymbol{Z},\ 0 < 3^{k+1}m < 4 \times 3^{n-1},$$
$$\text{such that } j > 4 \times 3^{n-i-1} - 1 \text{ if } 3^{i-k-1} | m\},$$

$$X_2 = X_{2,1} \oplus X_{2,2},$$

$$X_{2,1} = \boldsymbol{Z}/3\{v_2^{3^n(3t-1)}/v_1^{3^i m} \mid n \geqslant 0,\ 3 \nmid m,\ 1 \leqslant m \leqslant 4 \times 3^{n-2i-1}\},$$

$$X_{2,2} = \boldsymbol{Z}/3\{v_2^{3^n(3^{k+2}u-1)}/v_1^{3^{k+1}m} \mid 2 \leqslant 2k \leqslant n,\ 0 < m < 4 \times 3^{n-2k-1},\ 3 \nmid m\},$$

$$H = H_1 \oplus H_2,$$

$$H_1 = \boldsymbol{Z}/3\{v_2^{3^n(3t+1)}h_{10}/v_1^{j+1} \mid 0 < i < n,\ j \leqslant 2 \times 3^n,\ 2 \times 3^{n-i} \leqslant j \text{ if } 3^i | j\},$$

$$H_2 = \boldsymbol{Z}/3\{v_2^{3^n(9t-1)}h_{10}/v_1^j \mid n \geqslant 0,\ 8 \times 3^n < j \leqslant 10 \times 3^n + 1\},$$

$$HI = HI_1 \oplus HI_2,$$

$$HI_1 = \boldsymbol{Z}/3\{v_2^{3^n(3t+1)}h_{10}/v_1^{j+1} \mid 0 \leqslant i \leqslant n,\ 4\times 3^{n-i-1} \leqslant j \leqslant 2\times 3^{n-i}$$
$$\text{if } 3^i|j \text{ and } 3^{i+1} \nmid j\},$$

$$HI_2 = \boldsymbol{Z}/3\{v_2^{3^n(9t-1)}h_{10}/v_1^{j+1} \mid n \geqslant 0,\ 0 \leqslant j \leqslant 8\times 3^n,\ 3^{k+1} \nmid (j+4\times 3^n)$$
$$\text{if } t = 3^k u \text{ with } 3 \nmid u\},$$

$$H^* = H_1^* \oplus (H_{2,1}^* + H_{2,2}^*),$$

$$H_1^* = \boldsymbol{Z}/3\{v_2^{3^n(3t+1)}h_{10}/v_1^{j+1} \mid n \geqslant 0,\ j > 0 \text{ and } j < 4\times 3^{n-i-1} \text{ if } 3^{i+1} \nmid j\},$$

$$H_{2,1}^* = \boldsymbol{Z}/3\{v_2^{3^n(9t-1)}h_{10}/v_1^{j+1} \mid n \geqslant 0,\ j > 0 \text{ and } j \leqslant 4\times 3^{n-i-1} \text{ if } 3^{i+1} \nmid j\},$$

$$H_{2,2}^* = \boldsymbol{Z}/3\{v_2^{3^n(9t-1)}h_{10}/v_1^{j+1} \mid n \geqslant 0,\ j \geqslant 0 \text{ and } 3^{k+1}|(j+4\times 3^n) \leqslant 4\times 3^{n+1}$$
$$\text{if } t = 3^k u \text{ with } 3 \nmid u\},$$

$$X_1^* = X_{1,1}^* \oplus X_{1,2}^*,$$

$$X_{1,1}^* = \boldsymbol{Z}/3\{v_2^{3^{n+i}(3t+1)}\xi_n/v_1^{3^i k} \mid i > 0,\ 0 < 3^i k \leqslant 4\times 3^n\},$$

$$X_{1,2}^* = \boldsymbol{Z}/3\{v_2^{3^{n+i+1}(3t-1)}\xi_n/v_1^{3^i k} \mid i > 0,\ 0 < 3^i k \leqslant 4\times 3^n\},$$

$$X_2^* = \boldsymbol{Z}/3\{v_2^{3^{n+l}s}\xi_n/v_1^{3^i k} \mid 3 \nmid s,\ i,n \geqslant 0,\ 0 < 3^i k \leqslant 4\times 3^n,$$
$$l > i \text{ and } l > i+1 \text{ if } 3|(s+1)\}.$$

Note that it is just for simplicity that each direct summand of A_1 is presented as the direct sum of two modules. We also consider the following submodules of $H^*M_0^2$:

$$\tilde{X} = \boldsymbol{Z}_{(3)}\{v_2^{3^n s}/3^{i+1}v_1^j \mid n \geqslant 0,\ 3 \nmid s \in \boldsymbol{Z},\ i \geqslant 0,\ j > 0,$$
$$\text{with } 3^i|j < 4\times 3^{n-i-1} \text{ and either } 3^{i+1} \nmid j \text{ or } 4\times 3^{n-i-2} < j\}$$
$$= \widetilde{X_1} \oplus \widetilde{X_2},$$

$$\widetilde{X_1} = \boldsymbol{Z}_{(3)}\{v_2^{3^n s}/3^{i+1}v_1^{3^i m} \mid v_2^{3^n s}/3^{i+1}v_1^{3^i m} \in \tilde{X},\ 3 \nmid (s+1),\ 3|m$$
$$\text{or } i \neq k+1 \text{ if } s = 3^{k+2}t - 1 \text{ with } k \geqslant 0 \text{ and } 3 \nmid t\},$$

$$\widetilde{X_2} = \boldsymbol{Z}_{(3)}\{v_2^{3^n s}/3^{i+1}v_1^{3^i m} \mid v_2^{3^n s}/3^{i+1}v_1^{3^i m} \in \tilde{X},\ 3|(s+1),\ 3 \nmid m$$
$$\text{and } i = k+1 \text{ if } s = 3^{k+2}t - 1 \text{ with } k \geqslant 0 \text{ and } 3 \nmid t\},$$

$$\tilde{H} = \widetilde{H_1} \oplus \widetilde{H_2},$$

$$\widetilde{H_1} = \boldsymbol{Z}_{(3)}\{v_2^{3^n(3t+1)}h_{10}/3^i v_1^{3^i m+1} \mid 0 < i \leqslant n \text{ and } 2\times 3^{n-i} < 3^i m \leqslant 2\times 3^{n-i+1}\},$$

$$\widetilde{H_2} = \boldsymbol{Z}_{(3)}\{v_2^{3^n(9t-1)}h_{10}/3^i v_1^{3^i m+1} \mid 0 < i \leqslant n \text{ and } 2\times 3^{n-i} < 3^i m - 8\times 3^n \leqslant 2\times 3^{n-i+1}\},$$

$$\widetilde{HI} = \widetilde{HI_0} \oplus \widetilde{HI_1} \oplus \widetilde{HI_2} \oplus \widetilde{HI_3},$$

$$\widetilde{HI_0} = \boldsymbol{Z}_{(3)}\{v_2^{3^n s}h_{10}/3^{n+1}v_1 \mid n \geqslant 0,\ s = 3t+1 \text{ or } s = 9t-1\ (t \in \boldsymbol{Z})\},$$

$$\widetilde{HI_1} = \boldsymbol{Z}_{(3)}\{v_2^{3^n(3t+1)}h_{10}/3^{i+1}v_1^{3^i m+1} \mid 0 \leqslant i \leqslant n,\ 2\times 3^{n-i-1} < 3^i m \leqslant 2\times 3^{n-i},\ 3 \nmid m\},$$

$$\widetilde{HI_2} = \boldsymbol{Z}_{(3)}\{v_2^{3^n(9t-1)}h_{10}/3^{i+1}v_1^{3^i m+1} \mid 3 \nmid m < 8\times 3^{n-i},\ 0 \leqslant i < n,\ k+1 > i \text{ if } 3^k | t\},$$

$$\widetilde{HI_3} = \boldsymbol{Z}_{(3)}\{v_2^{3^n(9t-1)}h_{10}/3^i v_1^{3^n m+1} \mid i = \begin{cases} n+1 & \text{if } m = 1,3,4,6,7, \\ n+2 & \text{if } m = 2,8, \\ n+3 & \text{if } m = 5, \end{cases}$$
$$k+1 > n \text{ if } 3^k | t\},$$

$$\widetilde{X_2}^* = \boldsymbol{Z}_{(3)}\{v_2^{3^{n+l}s}\xi_n/3^{i+1}v_1^{3^i m} \mid 3 \nmid s \in \boldsymbol{Z},\ i,n \geqslant 0,\ 0 < 3^i m \leqslant 4\times 3^n,$$
$$l > i \text{ and } l > i+1 \text{ if } 3|(s+1)\}.$$

The propositions of Section 5 below show the behavior of the connecting homomorphism $\delta : \overline{A_1}^s \to A_1^{s+1}$ as follows:

Proposition 3.1. *The connecting homomorphism* $\delta : \overline{A_1}^s \to A_1^{s+1}$ *maps* $\widetilde{X_1}$, $\widetilde{X_2}$, $\widetilde{H}$, $\widetilde{HI}$, $\widetilde{X_1}\zeta_2\widetilde{X_2}^*$ *and* $\widetilde{HI}\zeta_2$ *to* H^*, $X_2\zeta_2$, X_1^*, $HI\zeta_2$, $X_2^*\zeta_2$ *and* $X_1^*\zeta_2$, *respectively. Furthermore, the images of generators under* δ *are linearly independent.*

We now use Lemma 2.1 to obtain our main theorem:

Theorem 3.2. $\overline{A_1}^s$ *is isomorphic to* $\widetilde{X_1} \oplus \widetilde{X_2}$ *if* $s = 0$, $\widetilde{H} \oplus \widetilde{HI} \oplus \widetilde{X_1}\zeta_2$ *if* $s = 1$, $\widetilde{X_2}^* \oplus \widetilde{H}\zeta_2$ *if* $s = 2$, *and* 0 *otherwise.*

4. The homotopy groups $\pi_*(L_2S^0)$

Let $E_r(X)$ denote the E_r-term of the Adams–Novikov spectral sequence converging to the homotopy groups $\pi_*(X)$. We start with a general result on the spectral sequence, which is well known and proved in the same manner as [3, Theorem 4.1].

Lemma 4.1. *Let* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ *be the cofiber sequence with* $BP_*(h) = 0$. *Then we have the induced maps* $E_r^s(X) \xrightarrow{f_*} E_r^s(Y) \xrightarrow{g_*} E_r^s(Z) \xrightarrow{\delta} E_r^{s+1}(X)$. *Suppose that* $g_*(\bar{y}) = \bar{z}$ *for non-zero elements* $y \in E_r^s(Y)$ *and* $z \in E_r^{s+r}(Z)$. *Here,* $\bar{a}$ *denotes a homotopy element that is detected by an element* a *of the* E_r*-term. Then if* $y = f_*(x)$ *for some* $x \in E_r^s(X)$, *then* $d_r(x) = \delta(z)$.

Let N^1 and N^2 denote the cofibers of the localization maps $S^0 \to L_0S^0$ and $N^1 \to L_1N^1$, respectively. Then we have the Adams–Novikov spectral sequence $E_2(L_2N^2) = H^*M_0^2 \Rightarrow \pi_*(L_2N^2)$. The differentials of the spectral sequence are determined in [8], and we have the following.

Proposition 4.2. *The E_∞-term of the Adams–Novikov spectral sequence for $\pi_*(L_2N^2)$ is the direct sum of the three modules $\overline{A_0}$, $\overline{A_1}$ and $\widetilde{A_2}$. Here, $\overline{A_0}$ and $\overline{A_1}$ are determined in the previous sections, and*

$$\widetilde{A_2} = \tilde{G} \oplus \tilde{G}^* \oplus \widetilde{GZ} \oplus \widetilde{GZ}^*,$$

where these four modules are determined in [8]:

$$\begin{aligned}\tilde{G} = {} & B_5(2,2)_*\{v_2/3v_1\} \oplus B_4(2,2)_*\{v_2^4b_{11}/3v_1\} \oplus B_3(2,2)_*\{v_2^7h_{10}/3v_1\}\\ & \oplus B_2(2,2)_*\{v_2h_{10}/3v_1, v_2^2h_{11}/3v_1, v_2^5h_{11}/3v_1\} \oplus B_1(2,2)\{v_2^{-1}h_{11}/3v_1^2\},\end{aligned}$$

$$\begin{aligned}\tilde{G}^* = {} & B_5(2,2)_*\{v_2^7\psi_1/3v_1\} \oplus B_4(2,2)_*\{v_2^3\psi_0/3v_1\} \oplus B_2(2,2)_*\{\xi/3v_1, v_2^3b_{11}\xi/3v_1, v_2^6b_{11}\xi/3v_1\}\\ & \oplus \sum_{n\geqslant 1} (B_3(2,n+2)_*\{v_2^{9u+3}\xi/3v_1 \mid u \in \boldsymbol{Z} - I(n)\}\\ & \qquad \oplus B_2(2,n+2)_*\{v_2^{9u+3}\xi/3v_1 \mid u \in I(n)\}),\end{aligned}$$

$$\begin{aligned}\widetilde{GZ} = {} & B_5(2,2)_*\{v_2\zeta_2/3v_1\}\\ & \oplus B_3(2,2)_*\{v_2^4b_{11}\zeta_2/3v_1\}\\ & \oplus B_2(2,2)_*\{v_2h_{10}\zeta_2/3v_1, v_2^2h_{11}\zeta_2/3v_1, v_2^5h_{11}\zeta_2/3v_1, v_2^7h_{10}\zeta_2/3v_1\},\end{aligned}$$

$$\begin{aligned}\widetilde{GZ}^* = {} & B_5(2,2)_*\{v_2^7\psi_1\zeta_2/3v_1\} \oplus B_4(2,2)_*\{v_2^3\psi_0\zeta_2/3v_1\}\\ & \oplus B_2(2,2)_*\{\xi\zeta_2/3v_1\}\\ & \oplus B_1(2,2)_*\{v_2^3b_{11}\xi\zeta_2/3v_1, v_2^6b_{11}\xi\zeta_2/3v_1\}\\ & \oplus \sum_{n\geqslant 1} (B_3(2,n+2)_*\{v_2^{9u+3}\xi\zeta_2/3v_1 \mid u \in \boldsymbol{Z} - I(n)\}\\ & \qquad \oplus B_2(2,n+2)_*\{v_2^{9u+3}\xi\zeta_2/3v_1 \mid u \in I(n)\})\end{aligned}$$

for $B_k(2,n)_* = (\boldsymbol{Z}/3)[v_2^{\pm 3^n}, b_{10}]/(b_{10}^k)$ and $I(n)$ given in Section 1.

Lemma 4.3. *There is no extension problem in the spectral sequence for $\pi_*(L_2N^2)$.*

Proof. Let $M(i,\infty)$ be a cofiber of the localization map $M(i) \to L_1M(i)$ of the mod 3^i Moore spectrum $M(i)$. Then we have the cofiber sequence $M(i,\infty) \xrightarrow{\varphi} N^2 \xrightarrow{3^i} N^2$. If there are non-zero elements $x \in E_\infty^{s,*}(L_2N^2)$ and $y \in E_\infty^{s+r-1,*}(L_2N^2)$ for integers $s \geqslant 0$ and $r > 1$ such that $3^i\bar{x} = \bar{y}$ in $\pi_*(L_2N^2)$, then there exists an element $\tilde{x} \in E_r^{s,*}(L_2M(i,\infty))$ such that $\varphi_*(\tilde{x}) = x$ and $d_r(\tilde{x}) = \delta(y)$ in $E_r^{s+r,*}(L_2M(i,\infty))$ by Lemma 4.1. Consider the commutative diagram

$$\begin{array}{ccccccc}
E_2^*(L_2N^2) & \xrightarrow{\delta} & E_2^*(L_2M(i,\infty)) & \xrightarrow{\varphi_*} & E_2^*(L_2N^2) & \xrightarrow{3^i} & E_2^*(L_2N^2) \\
\Big\downarrow {=} & & \Big\downarrow pr & & \Big\downarrow 3^{i-1} & & \Big\downarrow {=} \\
E_2^*(L_2N^2) & \xrightarrow{\delta} & E_2^*(L_2M(1,\infty)) & \xrightarrow{\varphi_*} & E_2^*(L_2N^2) & \xrightarrow{3} & E_2^*(L_2N^2)
\end{array}$$

Then the relation $d_r(\tilde{x}) = \delta(y)$ in $E_r^{s+r,*}(L_2M(i,\infty))$ is the one in $E_r^{s+r,*}(L_2M(1,\infty))$. Note that $M(1,\infty)$ is denoted by W in [7]. We observe in [8] that the differentials on $E_r^*(L_2N^2)$ are obtained by sending those on $E_r^*(L_2W)$ by the map $\varphi_* : E_r^*(L_2W) \to E_r^*(L_2N^2)$, and so y cannot be an image of the connecting homomorphism δ. This means that there are no non-zero elements $x, y \in E_\infty^{*,*}(L_2N^2)$ such that $3^i\bar{x} = \bar{y}$ in $\pi_*(L_2N^2)$. □

Corollary 4.4. *The homotopy groups $\pi_*(L_2N^2)$ are the direct sum of the three modules $\overline{A_0}$, $\overline{A_1}$ and $\widetilde{A_2}$.*

Proof of Theorem A. Consider the exact sequences $\cdots \to \pi_*(L_2S^0) \to \pi_*(L_0S^0) \to \pi_*(L_2N^1) \to \cdots$ and $\cdots \to \pi_*(L_2N^1) \to \pi_*(L_1N^1) \xrightarrow{v_1} \pi_*(L_2N^2) \to \cdots$ associated to the cofiber sequence $S^0 \to L_0S^0 \to N^1$ and $N^1 \to L_1N^1 \to N^2$. They also induce the connecting homomorphisms $\delta : E_2^s(L_2N^1) \to E_2^{s+1}(L_2S^0)$ and $\delta' : E_2^s(L_2N^2) \to E_2^{s+1}(L_2N^1)$ of E_2-terms. Now define the elements of the E_2-term $E_2^*(L_2S^0)$ by

$$\alpha_{a/b} = \delta(v_1^a/3^b), \quad \beta_1' = h_{11} - v_1^2h_{10},$$

$$\beta_{a/b,c} = \delta\delta'(v_2^a/3^cv_1^b),$$

$$c_{3^ns} = \delta\delta'(v_2^{3^ns}h_{10}/3^{n+1}v_1) \quad \text{for } 3 \nmid s,$$

$$\widetilde{\alpha_1\beta_{a/b,c}} = \delta\delta'(v_2^ah_{10}/3^cv_1^b),$$

$$\beta(a)^*_{b/c,d} = \delta\delta'(v_2^b\xi_a/3^dv_1^c),$$

$$\chi_a^0 = \delta\delta'(v_2^a\psi_0/3v_1)$$

and

$$\chi_a^1 = \delta\delta'(v_2^a\psi_1/3v_1).$$

Then $\overline{A_1}$ and $\overline{A_2}$ are isomorphic to G_1 and G_2, respectively. Since $\pi_*(L_0S^0) = \boldsymbol{Q}$ and $\pi_*(L_1N^1) = \boldsymbol{Q}/\boldsymbol{Z}_{(3)} \otimes \Lambda(y) \oplus Al$, an easy diagram chasing with Corollary 4.4 enables us to obtain G_0 from $\overline{A_0}$, and proves Theorem A. Here, Al is the $\boldsymbol{Z}_{(3)}$-module generated by $v_1^{3^is}/3^{i+1}$ for $i \geqslant 0$ and $3 \nmid s \in \boldsymbol{Z}$. □

5. Computations in the cobar complex

In this section, we work on the cobar complex (cf. [3]) based on the Hopf algebroid $(E(2)_*, E(2)_*(E(2)))$ in order to study the connecting homomorphism $\delta : \overline{A_1}^s \to A_1^{s+1}$. The structure maps $\eta_R : E(2)_* \to E(2)_*(E(2))$ and $\Delta : E(2)_*(E(2)) \to E(2)_*(E(2)) \otimes_{E(2)_*} E(2)_*(E(2))$ behave as follows:

$$\eta_R(v_1) = v_1 + 3t_1,$$

$$\eta_R(v_2) = v_2 + v_1 t_1^3 - t_1 \eta_R(v_1)^3 - 3v_1 t_1(v_1^2 + 3v_1 t_1 + 3t_1^2),$$

$$\Delta(t_1) = t_1 \otimes 1 + 1 \otimes t_1,$$

$$\Delta(t_2) = t_2 \otimes 1 + t_1 \otimes t_1^2 + v_1 b_0,$$

$$\Delta(t_3) \equiv t_3 \otimes 1 + t_2 \otimes t_1^9 + t_1 \otimes t_2^3 + 1 \otimes t_3 + v_2 b_1 - v_1 b_{20} \bmod (9, v_1^2),$$

where $3b_i = t_1^{3^{i+1}} \otimes 1 + 1 \otimes t_1^{3^{i+1}} - (t_1 \otimes 1 + 1 \otimes t_1)^{3^{i+1}}$ and $3b_{20} = (t_2^3 \otimes 1 + t_1^3 \otimes t_1^9 + 1 \otimes t_2^3) - (t_2 \otimes 1 + t_1 \otimes t_1^3 + 1 \otimes t_2)^3$. Furthermore, we have the relations in $E(2)_*(E(2))$ by setting $\eta_R(v_i) = 0$ in $BP_*(BP)$ for $i > 2$ such as $v_2 t_{i-2}^9 \equiv v_2^{3^i} t_{i-2} \bmod (3, v_1)$ and $v_2 t_1^9 \equiv v_2^3 t_1 - v_1 t_2^3 \bmod (3, v_1^2)$. For an $E(2)_*(E(2))$-comodule M with structure map induced from η_R, the cobar complex is a family of $E(2)_*$-modules $\Omega^s M = M \otimes_{E(2)_*} E(2)_*(E(2)) \otimes_{E(2)_*} \cdots \otimes_{E(2)_*} E(2)_*(E(2))$ (s factors) with differential $d : \Omega^s M \to \Omega^{s+1} M$ defined by $d(m \otimes x) = \eta_R(m) \otimes x + \sum_{i=1}^s (-1)^i m \otimes \Delta_i(x) - (-1)^s m \otimes x \otimes 1$ for $m \in M$ and $x \in \Omega^s M$, where $\Delta_i(x_1 \otimes \cdots \otimes x_n) = x_1 \otimes \cdots \otimes \Delta(x_i) \otimes \cdots \otimes x_n$.

Lemma 5.1. *In the cobar complex $\Omega^* E(2)_*/(3, v_1^3)$, put $t_{31} = v_2^{-6} t_3^3 - t_1^3 t_2^3$, and we obtain*

$$d(t_{31}) = t_1^6 \otimes t_1^9 - t_1^3 \otimes t_2^3 - v_2^3 t_1^3 \otimes z^3 - v_2^{-3} b_{11}^3.$$

Here $z = v_2^{-1}(t_2 - t_1^4) + v_2^{-3} t_2^3$.

Proof. This follows immediately from the computation

$$d(v_2^{-6} t_3^3) = -v_2^{-6} t_1^3 \otimes t_2^9 - v_2^{-6} t_2^3 \otimes t_1^{27} - v_2^{-3} b_{11}^3,$$

$$\begin{aligned} d(-t_1^3 t_2^3) &= t_1^6 \otimes t_1^9 + t_1^3 \otimes t_1^{12} + t_1^3 \otimes t_2^3 + t_2^3 \otimes t_1^3 \\ &= t_1^6 \otimes t_1^9 - v_2^3 t_1^3 \otimes z^3 + v_2^{-6} t_1^3 \otimes t_2^9 - t_1^3 \otimes t_2^3 + t_2^3 \otimes t_1^3. \qquad \square \end{aligned}$$

Since $\eta_R(v_2) \equiv v_2 + v_1 t_1^3 - v_1^3 t_1 \bmod (3)$, we note that $\eta_R(v_2^3) \equiv v_2^3 + v_1^3 t_1^9 - v_1^9 t_1^3 \bmod (9, 3v_1)$. We then define an element V by the congruence

$$3v_1 V \equiv v_2^3 + v_1^3 t_1^9 - v_1^9 t_1^3 - \eta_R(v_2^3) \bmod (9).$$

Lemma 5.2. *There exists an element Y_n such that*

$$d(Y_n) = -v_1^{4 \times 3^{n-1} - 1} \sigma_{n+1} \otimes V^{3^n} - v_1^{7 \times 3^{n-1}} v_2^{2 \times 3^n} x^{3^n} \bmod (3, v_1^8)$$

for each $n > 0$. Here $\sigma_n = t_1 + v_1 z^{3^n}$ and x is a cocycle whose leading term is $-v_2^{10} t_3^3 \otimes t_1^3 - v_2^{-3} t_1 \otimes t_3$.

Proof. First note that $V \equiv -v_2^2 t_1^3 - v_1 v_2 t_1^6 \bmod(3, v_1^2)$. Recall the element Y of $\Omega^1 E(2)_*$ ([1, Theorem 4.8]) such that $Y \equiv \sigma_2 \eta_R(v_2^3) - v_1^2 V + v_1^3 v_2^{-2} t_1^{18} + v_1^4 v_2^{-27}(t_1^9 t_2^{27} - t_3^9) \bmod(3, v_1^5)$ and

$$d(Y) \equiv v_1^7 x^3 \bmod(3, v_1^8).$$

Here, x^3 denotes a cocycle whose leading term is $-v_2^{30} t_3^9 \otimes t_1^9 - v_2^{-9} t_1^3 \otimes t_3^3$, which represents $v_2 \xi \in H^{2,8} M_2^0$. We define elements Y_i inductively by

$$Y_1 = \eta_R(v_2^6) Y + v_1^5 v_2^5 t_2^3 - v_1^6 v_2^4 t_{31} + v_1^5 v_2^8 z^3,$$

$$Y_n = Y_{n-1}^3 + v_1^{4\times 3^{n-1}-4} v_2 V^{3^n}.$$

Since $d(\eta_R(v)t) = d(t)\Delta\eta_R(v) - t \otimes d(v)$ for $v \in E(2)_*$ and $t \in E(2)_* E(2)$, we compute $\bmod(3, v_1^8)$,

$$\begin{aligned}
d(\eta_R(v_2^6)Y) &= v_1^7 v_2^6 x^3 - Y \otimes (-v_1^3 v_2^3 t_1^9 + v_1^6 t_1^{18}) \\
&= v_1^7 v_2^6 x^3 - v_1^3 Y \otimes v_2^{-3} V^3 + v_1^6 Y \otimes t_1^{18} \\
&= v_1^7 v_2^6 x^3 + v_1^6 \sigma_2 \eta_R(v_2^3) \otimes t_1^{18} \\
&\quad - v_1^3(\sigma_2 \eta_R(v_2^3) - v_1^2 V + v_1^3 v_2^{-2} t_1^{18} + v_1^4 v_2^{-27}(t_1^9 t_2^{27} - t_3^9)) \otimes v_2^{-3} V^3 \\
&= v_1^7 v_2^6 x^3 + v_1^6(\underline{v_2 t_1^9}_4 + v_1 t_2^3 + v_1 z^9 \eta_R(v_2^3)) \otimes t_1^{18} \\
&\quad - v_1^3 \sigma_2 \otimes V^3 - v_1^5(\underline{-v_2^2 t_1^3}_1 - \underline{v_1 v_2 t_1^6}_2 + v_1^2 v_2^2 t_1) \otimes v_2^3 t_1^9 \\
&\quad + \underline{v_1^6 v_2^{-2} t_1^{18} \otimes v_2^3 t_1^9}_4 + v_1^7 v_2^{-27}(t_1^9 t_2^{27} - t_3^9)) \otimes v_2^3 t_1^9,
\end{aligned}$$

$$d(v_1^5 v_2^5 t_2^3) = -\underline{v_1^6 v_2^4 t_1^3 \otimes t_2^3}_3 - \underline{v_1^5 v_2^5 t_1^3 \otimes t_1^9}_1,$$

$$d(-v_1^6 v_2^4 t_{31}) = -v_1^7 v_2^3 t_1^3 \otimes t_{31} - v_1^6 v_2^4(\underline{t_1^6 \otimes t_1^9}_2 - \underline{t_1^3 \otimes t_2^3}_3 - \underline{v_2^3 t_1^3 \otimes z^3}_5 - \underline{v_2^{-3} b_{11}^3}_4),$$

$$d(v_1^5 v_2^8 z^3) = -\underline{v_1^6 v_2^7 t_1^3 \otimes z^3}_5 + v_1^7 v_2^6 t_1^6 \otimes z^3,$$

in which the underlined terms with the same subscript cancel out. So we redefine the cocycle $-x^3$ by the cocycle that appears in the sum of the above congruences to satisfy

$$d(Y_1) \equiv -v_1^3 \sigma_2 \otimes V^3 - v_1^7 v_2^6 x^3.$$

Here, x^3 has the same leading term $-v_2^{30} t_3^9 \otimes t_1^9 - v_2^{-9} t_1^3 \otimes t_3^3$ as the above cocycle x^3.

Now turn to the case n. We assume the case for $n-1$. Then

$$d(Y_{n-1}^3) \equiv -v_1^{4\times 3^{n-1}-3} \sigma_n^3 \otimes V^{3^n} - v_1^{7\times 3^{n-1}} v_2^{2\times 3^n} x^{3^n},$$

$$d(v_1^{4\times 3^{n-1}-4} v_2 V^{3^n}) = v_1^{4\times 3^{n-1}-3}(t_1^3 - v_1^2 t_1) \otimes V^{3^n}.$$

Note that $\sigma_n^3 - (t_1^3 - v_1^2 t_1) = v_1^2 \sigma_{n+1}$, and we have the case for n. □

The following is also shown in [9, Proposition 4.4] which also holds for the prime 3. Here, the elements $y_{3^n s}$, $t_1 \otimes \zeta$, and g_0 are our $v_2^{3^n s} h_{10}$, $h_{10}\zeta_2$ and $v_2^{-2} b_{11}$, respectively.

Proposition 5.3. *For $n \geqslant 0$ and $s \in I$,*

$$\delta(v_2^{3^ns}h_{10}/3^{n+1}v_1) = v_2^{3^ns}h_{10}\zeta_2/v_1 + v_2^{3^ns-2}b_{11}/v_1.$$

We have similar results to [10, Propositions 7.5, 7.6 and 7.8]:

Proposition 5.4. *Let s,n,i,k be integers with $3 \nmid s$, $k > 0$ and $0 \leqslant i \leqslant n$. Then the Bockstein differential on $v_2^{3^ns}h_{10}/v_1^{3^ik+1}$ is given as follows*:

1. *If $3^ik \leqslant 2 \times 3^{n-i}$, then*
$$\delta(v_2^{3^ns}h_{10}/3^{i+1}v_1^{3^ik+1}) = -kv_2^{3^ns}h_{10}\zeta_2/v_1^{3^ik+1} - (-1)^{n-i}sv_2^{3^ns}\xi_{n-i-1}/v_1^{3^ik-2\times 3^{n-i-1}} + \cdots .$$
2. *If $s = 9t-1$ and $3^ik \leqslant 8 \times 3^n + 2 \times 3^{n-i+1}$, then*
$$\delta(v_2^{3^ns}h_{10}/3^iv_1^{3^ik+1}) = (-1)^{n-i}v_2^{3^{n+1}(3t-1)}\xi_{n-i}/v_1^{3^ik-8\times 3^n-2\times 3^{n-i}} + \cdots .$$
3. *If $s = 9t-1$, $3^{i+1} \nmid 3^ik \leqslant 8 \times 3^n$ and $i < n$, then*
$$\delta(v_2^{3^ns}h_{10}/3^{i+1}v_1^{3^ik+1}) = -kv_2^{3^ns}h_{10}\zeta_2/v_1^{3^ik+1} + \cdots .$$
4. *If $s = 9t-1$, $3^nk \leqslant 8 \times 3^n$ and $3 \nmid (k+1)$, then*
$$\delta(v_2^{3^ns}h_{10}/3^{n+1}v_1^{3^nk+1}) = -(k+1)v_2^{3^ns}h_{10}\zeta_2/v_1^{3^nk+1} + \cdots .$$
5. *If $s = 9t-1$, $3^nk \leqslant 8 \times 3^n$ and $3 \mid (k+1)$ (i.e. $k = 2,5,8$), then*
$$\delta(v_2^{3^ns}h_{10}/3^{n+2}v_1^{2\times 3^n+1}) = -v_2^{3^ns}h_{10}\zeta_2/v_1^{2\times 3^n+1} + \cdots ,$$
$$\delta(v_2^{3^ns}h_{10}/3^{n+2}v_1^{8\times 3^n+1}) = v_2^{3^ns}h_{10}\zeta_2/v_1^{8\times 3^n+1} + \cdots ,$$
$$\delta(v_2^{3^ns}h_{10}/3^{n+3}v_1^{5\times 3^n+1}) = -v_2^{3^ns}h_{10}\zeta_2/v_1^{5\times 3^n+1} + \cdots .$$

Here $\cdots$ denotes an element killed by a lower power of v_1 than is shown.

Proof. Let $\tilde{z}$ denote the element given in [8] such that $\tilde{z} \equiv v_2^{-1}(t_2 - t_1^4) + v_2^{-3}t_2^3 \bmod (3, v_1)$ and $d(\tilde{z}) \equiv 0 \bmod (3^i, v_1^{3^{i-1}k})$ for any $i,k > 0$, and denote $\sigma = t_1 + v_1\tilde{z}$. We also consider a cocycle

$$y_{j,l} = \sum_{k>0} \binom{k+j-2}{k-1} \frac{-(-t_1)^k}{3^{l-k+1}kv_1^{j+k-1}}$$

of $\Omega^1 M_0^2$ ([4]). Put $\sigma_{a,b} = y_{a,b} + \tilde{z}/3^bv_1^{a-1}$, and we note that $3^{b-1}\sigma_{a,b} = \sigma/3v_1^a$ and

$$d(\sigma_{3^ik+1,i+2}) = kt_1 \otimes \tilde{z}/3v_1^{3^ik+1}.$$

Note that $v_2^{3^ns}h_{10}/v_1^{3^ik+1}$ is represented by a cocycle $c(3^ns/3^ik+1) = \eta_R(v_2^{3^ns})\sigma/v_1^{3^ik+1} + w/v_1^{3^ik-3^n}$ for some $w \in E(2)_*(E(2))$.

For $i = 0$ and $k < 3^n - 1$, we define $c(3^ns/k+1, l) = \eta_R(v_2^{3^ns})\sigma_{k+1,l}$ for an integer $l > 0$, and replace the generator $v_2^{3^ns}h_{10}/v_1^{k+1}$ by the element represented by the cocycle $c(3^ns/k+1)$. Since $d(v_2^3) \equiv 3v_1V \bmod (9, v_1^3)$ by definition, we observe that

$$d(v_2^{3^ns}) \equiv -3^{i+1}sv_1^{3^{n-i-1}}v_2^{3^{n-i}(3^is-1)}V^{3^{n-i-1}} \bmod (3^{i+2}, v_1^{3^{n-i}}).$$

We compute in $\Omega^2 M_0^2$:

$$\begin{aligned} d(c(3^n s/k+1,2)) &= d(\eta_R(v_2^{3^n s})\sigma_{k+1,2}) \\ &= kt_1 \otimes \eta_R(v_2^{3^n s})\tilde{z}/3v_1^{k+1} - \sigma_{k+1,2} \otimes d(v_2^{3^n s}) \\ &= kv_2^{3^n s} t_1 \otimes \tilde{z}/3v_1^{k+1} + sv_2^{3^n(s-1)}\sigma \otimes V^{3^{n-1}}/3v_1^{k+1-3^{n-1}} \end{aligned}$$

whose second term is homologous to $-v_2^{3^n s-3^{n-1}}x^{3^{n-1}}/3v_1^{k-2\times 3^{n-1}}$ by Lemma 5.2. Since ξ_n is represented by $(-1)^n v_2^{-3^n}x^{3^n}$, this represents $(-1)^n v_2^{3^n s}\xi_{n-1}/3v_1^{k-2\times 3^{n-1}}$ as desired. If $k \geqslant 3^n$, then the case $i=0$ follows from the formula $v_1^{3n+3}\delta(v_2^{3^n s}h_{10}/3v_1^{3^i k+1}) = \delta(v_2^{3^n s}h_{10}/3v_1^{3^i k-3^n-2})$.

Suppose the case for i. Then $\delta(v_2^{3^n s}h_{10}/3^{i+1}v_1^{3^{i+1}k+1}) = 0$ if $3^{i+1}k \leqslant 2\times 3^{n-i-1}$. Since we compute

$$d(\eta_R(v_2^{3^n s})\sigma_{3^{i+1}k+1,i+3}) = kv_2^{3^n s}t_1 \otimes z/3v_1^{3^{i+1}k+1} - v_2^{3^{n-i-1}(3^{i+1}s-1)}\sigma \otimes V^{3^{n-i-2}}/3v_1^{3^{i+1}k+1-3^{n-i-2}}$$

in $\Omega^2 M_0^2$, which shows the case for $i+1$, we obtain inductively the first part by Lemma 5.2. Thus, if we denote a cocycle that represents $v_2^a h_{10}/3^c v_1^b$ by $c(a/b,c)$, then

$$d(c(3^n s/3^i k+1,i+2)) = kv_2^{3^n s}t_1 \otimes z/3v_1^{3^i k+1} - (-1)^{n-i}v_2^{3^n s}x(n-i-1)/3v_1^{3^i k-2\times 3^{n-i-1}}, \tag{5.1}$$

where $x(n) = (-1)^n v_2^{-3^n}x^{3^n}$ and so $\sigma \otimes V^{3^n} = (-1)^{n+1}v_1^{3^n+1}v_2^{3^{n+1}}x(n)$ up to homology.

Consider the case $s = 9t-1$. The proof of [10, Lemma 7.7] works also at prime 3 and we obtain

5.5. *The element $v_2^{3^n(9t-1)}h_{10}/3v_1^{3^i k+1}$ of $H^1 M_0^2$ is represented by a cochain* $c(3^n s/3^i k+1,1) = d(x_{n+2}^t)/9tv_1^{4\times 3^n+3^i k} - c(3^{n+1}(3t-1)/3^i k - 8\times 3^n+1,2)$.

In [4], they introduce the elements $x_i \in E(2)_*$ such that $x_i \equiv v_2^{3^i} \bmod (3,v_1)$ and give the formulas on $d(x_i)$. With a detailed computation, we observe that these elements satisfy $d(x_i) \equiv v_1^{a_i}v_2^{2\times 3^{i-1}}\sigma_{n-1} \bmod (3, v_1^{2\times 3^n-1})$ for $i \geqslant 2$. We then compute with (5.1)

$$\begin{aligned} d(d(x_{n+2}^t)/3^{i+2}tv_1^{4\times 3^n+3^i k}) &= -kt_1 \otimes d(x_{n+2}^t)/3tv_1^{4\times 3^n+3^i k+1} \\ &= -kt_1 \otimes v_2^{3^{n+1}(3t-1)}\sigma/3v_1^{3^i k+1-8\times 3^n} + \cdots, \end{aligned}$$

$$d(-kv_2^{3^{n+1}(3t-1)}t_1^2/3v_1^{3^i k+1-8\times 3^n}) = -kv_2^{3^{n+1}(3t-1)}t_1 \otimes t_1/3v_1^{3^i k+1-8\times 3^n},$$

$$\begin{aligned} &d(c(3^{n+1}(3t-1)/3^i k+1-8\times 3^n, i+2)) \\ &\quad = kv_2^{3^{n+1}(3t-1)}t_1 \otimes z/3v_1^{3^i k+1-8\times 3^n} + (-1)^{n-i}v_2^{3^{n+1}(3t-1)}x(n-i)/3v_1^{3^i k-8\times 3^n-2\times 3^{n-i}}. \end{aligned}$$

They amount to

$$d(c(3^n(9t-1)/3^i k+1,i+1)) = (-1)^{n-i}v_2^{3^{n+1}(3t-1)}x(n-i)/3v_1^{3^i k-8\times 3^n-2\times 3^{n-i}}.$$

We also note the case $n=i=1$ in the same manner, and we obtain part 2.

Parts 3 and 4 follow immediately from 5.5 and computation

$$d(d(x^t_{n+2})/3^{n+3}tv_1^{(4+k)3^n}) = (4+k)t_1 \otimes d(x^t_{n+2})/9tv_1^{(4+k)3^n+1}.$$

In the same way, we obtain part 5 by computing $d(d(x^t_{n+2})/3^{n+4}tv_1^{(4+k)3^n})$ for $k=2,8$ and $d(d(x^t_{n+2})/3^{n+5}tv_1^{3^{n+2}})$ for $k=5$. □

They imply that $\overline{A_1}^1 = \tilde{H} \oplus \widetilde{HI} \oplus \widetilde{X_1}\zeta_2$, and Propositions 5.3 and 5.4 show that the cokernel of $\delta : \overline{A_1}^1 \to A_1^2$ is isomorphic to X_2^*.

Proposition 5.6. *For an element $v_2^{3^{n+l}s}\xi_n/v_1^{3^ik}$ of X_2^*, the connecting homomorphism $\delta : \overline{A_1}^2 \to A_1^3$ acts as follows*:

$$\delta(v_2^{3^{n+l}s}\xi_n/3^{i+1}v_1^{3^ik}) = \pm k(v_2^{3^{n+l}s}\xi_n\zeta_2/v_1^{3^ik} + v_2^{3^{n+l}s-1}\psi_1/v_1^{3^ik} + \cdots).$$

Proof. Let $c \in \Omega^2 M_1^1$ denote a cocycle that represents $v_2^{3^{n+l}s}\xi_n/v_1^{3^jm}$ which is in the image of $\delta : \overline{A_1}^1 \to A_1^2$ with $3^ik \leqslant 3^jm \leqslant 4 \times 3^n$ and $j > i$.

Since the cocycle $c/3 \in \Omega^2 M_0^2$ is bounded, we have a cochain $u \in \Omega^1 M_0^2$ such that $d(u) = c/3$. Then $v_2^{3^{n+l}s}\xi_n/v_1^{3^ik}$ is represented by $c' = v_1^{3^jm-3^ik}c$ and so $c'/3^{i+2} = (v_1^{3^jm-3^ik}/3^{i+1})d(u)$. Therefore, we compute in the cobar complex $\Omega^3 M_0^2$,

$$\begin{aligned} d(c'/3^{i+2}) &= d(1/3^{i+1}v_1^{3^ik-3^jm})d(u) \\ &= -kt_1 \otimes c/3v_1^{3^ik-3^jm+1}, \end{aligned}$$

which represents $\pm k(v_2^{3^{n+l}s}\xi_n\zeta_2/3v_1^{3^ik} + v_2^{3^{n+l}s-1}\psi_1/3v_1^{3^ik} + \cdots)$ by Shimomura [8, Lemma 3.9] as desired. □

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