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The Adams-Novikov E_2-term for $\pi_*(L_2S^0)$ at the prime 2

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Abstract. In this paper we determine the E_2-term of the Adams-Novikov spectral sequence converging to the homotopy groups $\pi_*(L_2S^0)$ of L_2-localized sphere spectrum at the prime 2. The structure of the E_2-term indicates that the homotopy group $\pi_i(L_2S^0)$ is finite except for $i = 0, -4$ and -5.

1 Introduction

Let $\mathcal{S}_p$ denote the stable homotopy category of p-local spectra, and $\mathcal{L}_n$ the stable homotopy subcategory of $E(n)$-local spectra for the n-th Johnson-Wilson spectrum $E(n)$ (*cf.* [1],[6],[7]). Miller, Ravenel and Wilson [4] introduced the chromatic method to understand the homotopy category $\mathcal{S}_p$ through $\mathcal{L}_n$. Bousfield localization provides a retraction $L_n\colon \mathcal{S}_p \to \mathcal{L}_n$. The $E(n)$-localization L_nS^0 of the p-local sphere spectrum S^0 plays as important a role in $\mathcal{L}_n$ as the sphere spectrum itself does in $\mathcal{S}_p$. Thus the determination of the homotopy groups $\pi_*(L_nS^0)$ is one of crucial problems for understanding the stable homotopy category $\mathcal{L}_n$. We obtain some information about the homotopy groups $\pi_*(S^0)$ of spheres from $\pi_*(L_nS^0)$. For example, some relations among α-elements and β-elements are obtained by studying $\pi_*(L_1S^0)$ and $\pi_*(L_2S^0)$ ([4], [10], [11], [14]).

Turn to the homotopy groups $\pi_*(L_nS^0)$. For $n = 0$, $\pi_*(L_0S^0) = \boldsymbol{Q}$ by Serre [9]. $\pi_*(L_1S^0)$ is determined in [4] for $p > 2$, and in [6] for $p = 2$.

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If $n = 2$ and $p > 3$, then $\pi_*(L_2S^0)$ is determined in [17]. Hopkins arrived at the chromatic splitting conjecture from this result [3]. One statement of the conjecture says that the fiber F_n of $L_nS_p^0 \to L_{K(n)}S^0$ decomposes into $2^n - 1$ nontrivial summands, where S_p^0 denotes the p-completion of S^0. This is based on the fact that $F_2 = \Sigma^{-2}L_1S_p^0 \vee \Sigma^{-4}L_0S_p^0 \vee \Sigma^{-5}L_0S_p^0$ for $n = 2$ and $p > 3$ [3], which is predicted from the fact that the homotopy groups $\pi_*(L_2S^0)$ have three summands $\boldsymbol{Q}/\boldsymbol{Z}_{(p)}$. At the prime 3, we determined $\pi_*(L_2S^0)$ in [16] and found only one summand $\boldsymbol{Q}/\boldsymbol{Z}_{(p)}$ in it. We also showed that $F_2 = \Sigma^{-2}L_1S_3^0$, which is a counter example of the conjecture at the prime 3.

The homotopy groups $\pi_*(L_2S^0)$ are computed by the Adams-Novikov spectral sequence $E_2^* = H^*E(2)_* \Rightarrow \pi_*(L_2S^0)$, where H^*- denotes the functor $\mathrm{Ext}^*_{E(2)_*(E(2))}(E(2)_*, -)$. In this paper, we determine the E_2-term of the Adams-Novikov spectral sequence at the prime 2 (Theorem 2.4). Since summands $\boldsymbol{Q}/\boldsymbol{Z}_{(2)}$ of the E_2-term are infinitely 2-divisible, they survive to the E_∞-term, and we have three summands $\boldsymbol{Q}/\boldsymbol{Z}_{(2)}$ in the homotopy groups $\pi_*(L_2S^0)$ (Corollary 2.5). This indicates a possibility that the fiber F_2 decomposes into 3 summands as in the case $p > 3$. The computation of the differentials of this spectral sequence seems much more difficult than it is in the case $p = 3$, just as the issue of the existence of elements of Kervaire invariant one is harder at $p = 2$ than it is at $p = 3$.

We begin by computing the E_2-term by the chromatic spectral sequence $E_1^* = H^*M_0^i \Rightarrow H^*E(2)_*$ $(i = 0, 1, 2)$ introduced by Miller, Ravenel and Wilson [4]. Here $E(2)_* = \boldsymbol{Z}_{(2)}[v_1, v_2^{\pm 1}]$ with $|v_1| = 2(p-1)$ and $|v_2| = 2(p^2-1)$, and $E(2)_*(E(2))$-comodules M_0^i are defined as follows: Put first $N_0^0 = E(2)_*$ and $M_0^0 = p^{-1}E(2)_*$. Then define N_0^1 by the short exact sequence

$$0 \longrightarrow N_0^0 \overset{\subset}{\longrightarrow} M_0^0 \longrightarrow N_0^1 \longrightarrow 0 \tag{1.1}$$

and put $M_0^1 = v_1^{-1}N_0^1$. M_0^2 is defined by the short exact sequence

$$0 \longrightarrow N_0^1 \overset{\subset}{\longrightarrow} M_0^1 \xrightarrow{f} M_0^2 \longrightarrow 0. \tag{1.2}$$

Applying the functor H^*- to these short exact sequences yields the chromatic spectral sequence. The E_1-term $H^*M_0^0$ of the chromatic spectral sequence is $\boldsymbol{Q}$ concentrated at dimension 0, and $H^*M_0^1$ is given in [4] (see Sect. 4). Now the determination of the Adams-Novikov E_2-term $H^*E(2)_*$ results in the determination of the chromatic E_1-term $H^*M_0^2$. Indeed, after determining the E_1-term, we observe the long exact sequence associated to (1.2) to see that there is only one non-trivial extension in the spectral sequence (see Sect. 18), and obtain $H^*N_0^1$ (Proposition 18.2) and then $H^*E(2)_*$ from the long exact sequence associated to (1.1).

Miller, Ravenel and Wilson introduced in [4] the v_1-Bockstein spectral sequence $H^*M_2^0 \Rightarrow H^*M_1^1$ and the mod p Bockstein spectral sequence $H^*M_1^1 \Rightarrow H^*M_0^2$ associated to the short exact sequences

$$
\begin{aligned}
&0 \longrightarrow M_2^0 \xrightarrow{1/v_1} M_1^1 \xrightarrow{v_1} M_1^1 \longrightarrow 0,\\
&0 \longrightarrow M_1^1 \xrightarrow{1/2} M_0^2 \xrightarrow{2} M_0^2 \longrightarrow 0.
\end{aligned} \tag{1.3}
$$

Here $M_2^0 = E(2)_*/(p, v_1)$, which is also denoted by $K(2)_*$, and $M_1^1 = E(2)_*/(p, v_1^\infty)$. The v-Bockstein spectral sequence $H^*M' \Rightarrow H^*M$ for $(v, M', M) = (v_1, M_2^0, M_1^1)$ or $(2, M_1^1, M_0^2)$ is computed by defining submodules B^s of H^sM fitting into the exact sequence $H^{s-1}M \xrightarrow{\delta} H^sM' \xrightarrow{1/v} B^s \xrightarrow{v} B^s \xrightarrow{\delta} H^*M'$. Indeed, such a module B^s is shown to equal H^sM itself. Generators of B^s are obtained as follows: For each generator ξ of $k(v)$-module H^sM', we have $\xi/v \in H^sM$ and pull it back by v as many times as possible to obtain an element $\xi/v^{a(\xi)} \in H^sM$ for an integer $a(\xi)$ such that $\delta(\xi/v^{a(\xi)}) \neq 0$, which is seen to be a generator of B^s ($k(v) = \boldsymbol{Z}/p$ if $v = v_1$, and $= k(1)_* = \boldsymbol{Z}/p[v_1]$ if $v = 2$). We start the computation of the Bockstein spectral sequences from Ravenel's result:

Theorem (Ravenel [5]). *As a $K(2)_*$-module,*

$$H^*M_2^0 = \Lambda(\zeta_2)\bigotimes K(2)_*\{1, h_{10}, h_{11}, g_0, g_1, g_0h_{11}\} \qquad \textit{if } p > 3,$$
$$H^*M_2^0 = \Lambda(\rho_2)\bigotimes (\boldsymbol{Z}/2[g]\bigotimes\Lambda(\beta)\bigotimes M\bigoplus\Lambda(\zeta)\bigotimes\zeta K(2)_*[h_0]) \qquad \textit{if } p = 2,$$

where $M = K(2)_*[h_{10}, h_{11}]/(h_{10}h_{11}, v_2h_{10}^3 - h_{11}^3)$, and the elements ρ_2, g, β, ζ, h_0, h_{10} and h_{11} for $p = 2$ have bidegree $(1,0)$, $(4,0)$, $(3,0)$, $(1,0)$, $(1,0)$, $(1,2)$ *and* $(1,4)$, *respectively.*

Note that $H^*M_2^0$ is infinite dimensional if $p = 2$, since g and h_0 are the polynomial generators of positive dimensions. In [5], the elements ζh_0^i are denoted by ζ_2 for $i = 0$, α_0 for $i = 1$, $\widetilde{\zeta_2}$ for $i = 2$ and $\widetilde{\alpha_0}$ for $i = 3$. The structure of $H^*M_2^0$ at the prime 3 is given by Henn [2] (*cf.* [12]), which is also infinite dimensional. In order to find the integer $a(v_2^{p^i})$ for the generator $v_2^{p^i}$ of $H^0M_2^0$, Miller, Ravenel and Wilson introduced in [4] the elements x_i (see Sect. 4). Then by these Bockstein spectral sequences, they determined the E_1-term $H^sM_0^2$ for $s = 0$ and $p > 2$. In the same manner, the first author determined $H^0M_0^2$ at $p = 2$ in [10]. For the case $s > 0$, $H^sM_0^2$ for $p \geq 3$ is determined in [17] and [16]. For the prime 2, $H^*M_1^1$ is determined in [14]. Here we determine $H^*M_0^2$ at the prime 2 (Theorem 2.3), whose structure is much more complicated than if $p > 3$ in that it is infinite dimensional.

The computation is similar but very complicated. If we have a cochain x of $\Omega^*E(2)_*$ such that $d(x) \equiv 0 \bmod (2^{n+1}, v_1^{2^n})$ for any $n > 0$, then the cohomology class ξ represented by x acts on $H^*M_0^2$. We have such cochains $\widetilde{z}$ and $\widetilde{G}$ in Sect. 2, r in Sect. 4, and R, B and G in Sect. 6, which represent $\widetilde{\zeta} = v_1v_2\zeta_2$, $v_1^4 g$, $\widetilde{\zeta}/v_1^4$, ρ_2, β and g, respectively. The action of these elements helps computation. For example, $H^*M_0^2 = \widetilde{W}^*\bigotimes\Lambda(\rho_2)$ for a submodule $\widetilde{W}^* = \sum_s \widetilde{W}^s$ in the exact sequence $\widetilde{W}^{s-1} \to W^s \to \widetilde{W}^s \to \widetilde{W}^s \xrightarrow{\delta} W^{s+1}$, where $H^*M_1^1 = W^*\bigotimes\Lambda(\rho_2)$. To make the verification easier, we decompose W^* into ten submodules $M_i = MC_i\bigoplus MI_i$, and find submodules $\widetilde{M}_i$ of $H^*M_0^2$ such that the sequence $0 \to MC_i \xrightarrow{1/2} \widetilde{M}_i \xrightarrow{2} \widetilde{M}_i \xrightarrow{\delta} MI_i \to 0$ is exact. Then we see that $\widetilde{W}^* = \sum_{i=1}^{10} \widetilde{M}_i$ (Lemma 18.1). We rename M_i the i-th letter of D, E, F, J, K, L, M, N, P, Q and define $\widetilde{M}_i$ one by one.

These modules are too complicated to write here, and we give hints of these modules. The explicit definitions of these modules are found in the next section. If we put $RM_i = \{\xi \in H^*M_2^0 \mid \xi/2^a v_1^b \in \widetilde{M}_i$ for some a, $b > 0\}$, then we have

$$
\begin{aligned}
RD &= \overline{K(2)}_*^{(4)}\{v_2, v_2\zeta\}\bigoplus\overline{K(2)}_*^{(2)}\{v_2\zeta^2\}\\
RE &= \overline{K(2)}_*^{(4)}\{1\}\bigoplus\overline{K(2)}_*^{(8)}\{v_2^{-1}\zeta\}\\
RF &= \left(\overline{K(2)}_*^{(4)}\{v_2^{-1}h_{11}^2, v_2^{-1}h_{11}\beta, h_{10}^3\}\bigoplus\overline{K(2)}_*^{(2)}\{v_2\zeta^2 g\}\right)\bigotimes\boldsymbol{Z}/2[g]\\
RJ &= \zeta\boldsymbol{Z}/2[h_0]\\
RK &= \boldsymbol{Z}/2\{v_2, v_2\zeta^2, h_{10}^3, h_{10}^3\beta, v_2^{-1}h_{11}^2, v_2^{-1}h_{11}^2\beta\}\bigotimes\boldsymbol{Z}/2[g]\\
RL &= \left(\boldsymbol{Z}/2[h_{10}]/(h_{10}^4)\bigotimes\Lambda(\beta)\bigoplus\overline{K(2)}_*^{(4)}[h_{10}]/(h_{10}^4)\right)\bigotimes\boldsymbol{Z}/2[g]\\
&\qquad \bigoplus\left(\overline{K(2)}_*^{(2)}\{\zeta, v_2\zeta, v_2\zeta^2\}\right)\bigotimes\boldsymbol{Z}/2[h_0]\\
RM &= \boldsymbol{Z}/2\{v_2^2, v_2^3, v_2^2h_{10}, v_2^3h_{10}, v_2^3h_{11}\}\bigotimes\Lambda(\beta)\bigotimes K(2)_*^{(4)}[g]\\
RN &= \overline{K(2)}_*^{(8)}\{1, v_2\zeta\}\bigoplus\boldsymbol{Z}/2\{v_2^{2^n s}\zeta \mid n > 0, s \equiv -1\ (4)\}\\
RP &= \boldsymbol{Z}/2\{1, v_2\zeta^2\}\\
RQ &= \boldsymbol{Z}/2\{v_2\zeta, v_2\zeta^2\}
\end{aligned}
$$

Here $K(2)_*^{(n)} = \boldsymbol{Z}/2[v_2^{\pm n}]$ and $\overline{K(2)}_*^{(n)} = K(2)_*^{(n)} \setminus \boldsymbol{Z}/2$. The module $H^*M_0^2$ is infinite dimensional, since so are the summands $\widetilde{F}$, $\widetilde{J}$, $\widetilde{K}$, $\widetilde{L}$ and $\widetilde{M}$. The only indecomposable summands of $H^*M_0^2$ which are not finite in every degree are the four summands of $\widetilde{Q}\bigotimes\Lambda(\rho_2)$, each of which contains a $\boldsymbol{Q}/\boldsymbol{Z}_{(2)}$.

We divide the paper into eighteen sections:

1. Introduction
2. Statement of results
3. Cocycles of the cobar complex Ω^*A
4. The Miller-Ravenel-Wilson elements x_i
5. $H^*M_0^1$ revisited
6. $H^*M_1^1$ revisited
7. The cocycles R and B
8. The connecting homomorphism on J
9. The connecting homomorphism on K
10. The connecting homomorphism on E
11. The connecting homomorphism on F
12. The connecting homomorphism on D
13. The connecting homomorphism on L
14. The connecting homomorphism on M
15. The connecting homomorphism on N
16. The connecting homomorphism on P
17. The connecting homomorphism on Q
18. The Adams-Novikov E_2-terms

In the next section, we state the structure of the chromatic E_1-term $H^*M_0^2$ and the Adams-Novikov E_2-term $H^*E(2)_*$ for $\pi_*(L_2S^0)$ by using explicit generators. The Sects. 3 and 4 are devoted to make some preliminary computation in the cobar complexes for studying the mod 2 Bockstein spectral sequence $H^*M_1^1 \Rightarrow H^*M_0^2$. In particular, we construct the cocycles $\widetilde{\widetilde{z}}$ and $\widetilde{G}$ of the cobar complex Ω^*A of $A = E(2)_*$ over $E(2)_*(E(2))$ as well as the cochain r_1 that is obtained from the generator ρ_1 of $H^1M_1^1$. In Sect. 5, we define one of $\boldsymbol{Q}/\boldsymbol{Z}_{(2)}$ summands in $H^1M_0^2$ originating from $H^1M_0^1$. We decompose the E_1-term $H^*M_1^1$ into the summands in Sect. 6. In the next section, we study some cochains arisen from the structure of $H^*M_1^1$ including the cocycles R on $E(2)_*/(2^{n+1}, v_1^{2^n})$ and B on $E(2)_*/(2^{n+5}, v_1^{2^{n+5}})$ for $n \geq 0$, which represent the generators ρ_2 and β, respectively. In Sects. 8 to 17, we study the behavior of the connecting homomorphism $\delta\colon H^*M_0^2 \to H^{*+1}M_1^1$ on each summand of $H^*M_1^1$. Note that each element used to state the results is expressed by its leading term as we did in the previous papers. In the last section, we compute the chromatic spectral sequence, or observe the exact sequence associated to (1.1) and (1.2), to obtain the module $H^*N_0^1$ and prove the theorems on $H^*M_0^2$ and $H^*E(2)_*$.

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2 Statement of results

Ravenel showed the structure of $H^*M_2^0 = H^*K(2)_*$ (*cf.* [7]):

(2.1)

$$H^*M_2^0 = \big(K(2)_*[h_{10}, h_{11}]/(h_{10}h_{11}, v_2h_{10}^3 - h_{11}^3)\bigotimes \Lambda(\beta)\bigotimes \boldsymbol{Z}/2[g] \\ \bigoplus \zeta K(2)_*[h_0]\bigotimes \Lambda(\zeta)\big) \bigotimes \Lambda(\rho_2).$$

From this, $H^*M_1^1$ is determined in [13]:

$$H^*M_1^1 = \left(\sum_{i=0}^{6} MB_i\right) \bigotimes \Lambda(\rho_2), \tag{2.2}$$

where

$$\begin{aligned}
MB_0 &= K(1)_*/k(1)_*\{h_{10}^i, h_{10}^i\beta, \zeta h_0^i, \zeta^2 h_0^i, \widetilde{\zeta} h_0^i, \zeta\widetilde{\zeta} h_0^i \mid i = 0,1,2,3\} \\
&\qquad \bigotimes \boldsymbol{Z}[g] \\
MB_1 &= (v_2/v_1)\boldsymbol{Z}/2[v_2^{\pm 2}]\bigotimes \boldsymbol{Z}/2[h_{11}]/(h_{11}^3)\bigotimes \Lambda(\beta)\bigotimes \boldsymbol{Z}/2[g] \\
MB_2 &= (v_2^2/v_1^2)\boldsymbol{Z}/2[v_1, v_2^{\pm 4}]\bigotimes \Lambda(h_{10}, v_2h_{10}, \beta)\bigotimes \boldsymbol{Z}/2[g] \\
MB_3 &= \sum_{n>0}(v_2^{2^n}\widetilde{\zeta}/v_1^{3\cdot 2^{n-1}+1})\boldsymbol{Z}/2[v_1, v_2^{\pm 2^{n+1}}]\bigotimes \boldsymbol{Z}/2[h_0] \\
MB_4 &= \sum_{n>1}(v_2^{2^n}\widetilde{\zeta}^{(n-1)}/v_1^{2^{n+1}})\boldsymbol{Z}/2[v_1, v_2^{\pm 2^{n+1}}]\bigotimes \boldsymbol{Z}/2[h_0] \\
MB_5 &= \sum_{n>1}(v_2^{2^n}/v_1^{3\cdot 2^{n-1}+3})\boldsymbol{Z}/2[v_1, v_2^{\pm 2^{n+1}}] \\
&\qquad \bigotimes \boldsymbol{Z}/2\{v_1^3, v_1^2h_{10}, v_1h_{10}^2, h_{10}^3\}\bigotimes \boldsymbol{Z}/2[g] \\
MB_6 &= \sum_{n>1}(v_2^{2^n}\widetilde{\zeta}^{(n-1)}/v_1^{2^{n+1}+4})\boldsymbol{Z}/2[v_1, v_2^{\pm 2^{n+1}}] \\
&\qquad \bigotimes \boldsymbol{Z}/2\{\widetilde{\zeta}, v_1\widetilde{\zeta}h_0, v_1^2\widetilde{\zeta}h_0^2, v_1^3\widetilde{\zeta}h_0^3\}\bigotimes \boldsymbol{Z}/2[g].
\end{aligned}$$

Here $\widetilde{\zeta}/v_1^j$ and $\widetilde{\zeta}^{(n-1)}/v_1^j$ denote the elements represented by cocycles $\widetilde{z}/v_1^j$ and $\widetilde{z}^{2^{n-1}}/v_1^j \in \Omega^1 M_1^1$, respectively, where the cocycle $\widetilde{z}$ is defined in the next section. Note that $\widetilde{\zeta}/v_1 = 0$ and $\widetilde{\zeta}^{(n-1)}/v_1^{2^{n-1}} = v_2^{2^{n-1}}\zeta^{2^{n-1}} = 0$.

Remark. The names of the generators are different from those of [13]. The generators $\zeta h_0^i/v_1^j$, $\widetilde{\zeta} h_0^i/v_1^j$ and $\widetilde{\zeta}^{(n)} h_0^i/v_1^j$ correspond to $\zeta_a g^b/v_1^j$, $v_2\zeta_a g^b/v_1^{j-1}$, and $v_2^{2^n}\zeta_a g^b/v_1^{j-2^n}$, respectively, where a and b are integers such that $a = 1, 2, 3, 4$ and $i = 4b + a - 1$.

In order to state the structure of $H^*M_0^2$, we introduce submodules of it, in which s and t in the summation run through $\boldsymbol{Z}$.

$$\begin{aligned}
\widetilde{D} &= \sum_{n\ge 2, 2\nmid s}\left(v_2^{2^ns}\boldsymbol{Z}/2\{v_2/2v_1\}\bigoplus v_2^{2^ns}\boldsymbol{Z}/16\{v_2^{2^{n-1}}\zeta\widetilde{\zeta}/16v_1^2\}\right)\\
&\quad\bigoplus\sum_{n\ge 2, 4|(s-1)}\left(v_2^{2^ns}\boldsymbol{Z}/2^{n+2}\{\widetilde{\zeta}/2^{n+2}v_1^4\}\bigoplus v_2^{2^ns}\boldsymbol{Z}/4\{\widetilde{\zeta}^{(n-1)}\widetilde{\zeta}/4v_1^{2^{n+1}+2}\}\right)\\
&\qquad\bigoplus\sum_{n\ge 2, 4|(s+1)} v_2^{2^ns}\boldsymbol{Z}/2^{n+1}\{\widetilde{\zeta}/2^{n+1}v_1^4\},\\
\widetilde{E} &= \sum_t v_2^{8t}(\boldsymbol{Z}/8\{v_2^4/8v_1^2\}\bigoplus\boldsymbol{Z}/4\{v_2^6\widetilde{\zeta}/4v_1^2, v_2^6\widetilde{\zeta}/4v_1^4\}\\
&\qquad\qquad\bigoplus\boldsymbol{Z}/2\{v_2^4/2v_1^4, v_2^4/2v_1^6\})\\
\widetilde{F} &= \textstyle\sum_{n\ge 2, 2\nmid s} v_2^{2^ns}\left(\boldsymbol{Z}/8\{v_2^{-1}h_{11}^2\beta/8v_1, h_{10}^3/8v_1\}\right.\\
&\qquad\left.\bigoplus\boldsymbol{Z}/4\{v_2^{-1}h_{11}^2/4v_1, \widetilde{\zeta}^{(n-1)}\widetilde{\zeta}g/4v_1^{2^{n+1}+2}\}\right)\bigotimes\boldsymbol{Z}[g]\\
\widetilde{J} &= \boldsymbol{Z}/2\{\zeta/2v_1^j \mid j>0\}\bigotimes\boldsymbol{Z}/2[h_0]\\
\widetilde{K} &= \left(\boldsymbol{Z}/8\{\zeta\widetilde{\zeta}/8v_1^2, h_{10}^3/8v_1, h_{10}^3\beta/8v_1\}\right.\\
&\qquad\bigoplus\boldsymbol{Z}/4\{v_2^{-1}h_{11}^2/4v_1, v_2^{-1}h_{11}^2\beta/4v_1\}\\
&\qquad\left.\bigoplus\boldsymbol{Z}/2\{v_2/2v_1\}\right)\bigotimes\boldsymbol{Z}[g].\\
\widetilde{L} &= \textstyle\sum_{i=0}^7\widetilde{LC_i}\\
\widetilde{M} &= \boldsymbol{Z}/2\{v_2^2/2v_1^2, v_2^2/2v_1, v_2^2h_{10}/2v_1, v_2^3/2v_1, v_2^3h_{10}/2v_1, v_2^3h_{11}/2v_1\}\\
&\qquad\bigotimes\Lambda(\beta)\bigotimes\boldsymbol{Z}/2[v_2^{\pm 4}, g],\\
\widetilde{N} &= \textstyle\sum_{(n,i,k)\in T\cup S}\widetilde{A(n,i,k)}\bigoplus\sum_{(n,i,k)\in T'-T^+}(\widetilde{\zeta}/v_1^4)\widetilde{A(n,i,k)}\\
&\qquad\bigoplus\sum_{n\ge 3}\hat{A}\widetilde{(n,n-1,1)}\bigoplus\sum_{(n,i,k)\in T^+} Z\widetilde{(n,i,k)},\\
\widetilde{P} &= \textstyle\sum_{2\nmid s, n>0}\boldsymbol{Z}/2^{n+2}\{1/2^{n+2}v_1^{2^ns}\}\bigoplus\sum_{j>0}\boldsymbol{Z}/2\{\zeta\widetilde{\zeta}/2v_1^{2j+4}\},\\
\widetilde{Q} &= \boldsymbol{Q}/\boldsymbol{Z}_{(2)}\{\widetilde{\zeta}/v_1^4\}\bigotimes\Lambda(\zeta) = \{\widetilde{\zeta}/2^jv_1^4 \mid j>0\}\bigotimes\Lambda(\zeta)
\end{aligned}$$

Here

$$\begin{aligned}
\widetilde{LC}_0 &= \sum_{j\ge 0}\left(\boldsymbol{Z}/2\{1/2v_1^{2j+1}, h_{10}/2v_1^{2j+1}, h_{10}^2/2v_1^{2j+1}, h_{10}^3/2v_1^{2j+5}\}\right.\\
&\qquad\bigotimes\boldsymbol{Z}/2[g]\bigotimes\Lambda(\beta)\\
&\qquad\left.\bigoplus\boldsymbol{Z}/2\{\widetilde{\zeta}/2v_1^{2j+1}, \widetilde{\zeta}h_0/2v_1^{2j+2} \mid j\ge 0\}\bigotimes\boldsymbol{Z}/2[h_0^2]\bigotimes\Lambda(\zeta)\right),
\end{aligned}$$

$$\widetilde{LC}_1 = \sum_{2\not| s} v_2^{2s}\widetilde{\zeta}\boldsymbol{Z}/2\{1/2v_1^3, h_0/2v_1^2, h_0/2v_1^4\}\bigotimes\boldsymbol{Z}/2[h_0^2],$$
$$\widetilde{LC}_2 = \sum_{n\geq 2, 2\not| s} v_2^{2^n s}\widetilde{\zeta}\boldsymbol{Z}/2[v_1^2]\{1/2v_1^{3\cdot 2^{n-1}+1}, h_0/2v_1^{3\cdot 2^{n-1}}\}\bigotimes\boldsymbol{Z}/2[h_0^2],$$
$$\widetilde{LC}_3 = \sum_{n\geq 2, 2\not| s} v_2^{2^n s+2^{n-1}}\zeta\boldsymbol{Z}/2[v_1^2]\{1/2v_1^{3\cdot 2^{n-1}-1}, h_0/2v_1^{3\cdot 2^{n-1}}\}\bigotimes\boldsymbol{Z}/2[h_0^2],$$
$$\widetilde{LC}_4 = \sum_{n\geq 2, 2\not| s} v_2^{2^n s}\boldsymbol{Z}/2[v_1^2]\{1/2v_1^{3\cdot 2^{n-1}-1}, h_0^3/2v_1^{3\cdot 2^{n-1}}\}\bigotimes\boldsymbol{Z}/2[g],$$
$$\widetilde{LC}_5 = \sum_{n\geq 2, 2\not| s} v_2^{2^n s}\boldsymbol{Z}/2[v_1^2]\{h_{10}/2v_1^{3\cdot 2^{n-1}+1}, h_{10}^2/2v_1^{3\cdot 2^{n-1}+1}\}\bigotimes\boldsymbol{Z}/2[g],$$
$$\widetilde{LC}_6 = \sum_{n\geq 2, 2\not| s} v_2^{2^n s+2^{n-1}}\zeta\widetilde{\zeta}\boldsymbol{Z}/2[v_1^2]\{h_0^2/2v_1^{3\cdot 2^{n-1}+1}, h_0^3/2v_1^{3\cdot 2^{n-1}}\}\bigotimes\boldsymbol{Z}/2[g],$$
$$\widetilde{LC}_7 = \sum_{n\geq 2, 2\not| s} v_2^{2^n s+2^{n-1}}\zeta\widetilde{\zeta}\boldsymbol{Z}/2[v_1^2]\{1/2v_1^{3\cdot 2^{n-1}+3}, h_0/2v_1^{3\cdot 2^{n-1}+2}\}\bigotimes\boldsymbol{Z}/2[g],$$

$$\widetilde{A(n,i,k)} = \begin{cases} \boldsymbol{Z}/2^{n-i}\{v_2^{2^n s}/2^{n-i}v_1^{2^k m} \mid \\ \quad 2\not| sm, \quad 3\cdot 2^{i-1} < 2^k m \leq 3\cdot 2^i\},\ n-i<k+1 \\ \boldsymbol{Z}/2^{k+2}\{v_2^{2^n s}/2^{k+2}v_1^{2^k m} \mid \\ \quad 2\not| sm, \quad 3\cdot 2^{i-1} < 2^k m \leq 3\cdot 2^i\}, \quad n-i>k \end{cases}$$

$$\hat{A}\widetilde{(n,n-1,1)} = \boldsymbol{Z}/4\{v_2^{2^n s}/4v_1^{2m} \mid 2\not| sm,\ 3\cdot 2^{n-2} < 2m \leq 3\cdot 2^{n-1}\}$$
$$\widetilde{Z(n,i,k)} = \boldsymbol{Z}/2^{k+2}\{v_2^{2^n s-2^i}\widetilde{\zeta}^{(i-1)}/2^{k+2}v_1^{2^k m} \mid$$
$$2\not| sm, \quad 2^{i-1} < 2^k m \leq 2^{i+1}\}$$
$$\widetilde{Z(n,i,i)} = \boldsymbol{Z}/2^{i+2}\{v_2^{2^n s-2^i}\widetilde{\zeta}^{(i-1)}/2^{i+2}v_1^{2^{i+1}} \mid 2\not| s\},$$
$$Z\widetilde{(n,i,i+1)} = \boldsymbol{Z}/2^{i+3}\{v_2^{2^n s-2^i}\widetilde{\zeta}^{(i-1)}/2^{i+3}v_1^{2^i} \mid 2\not| s\}.$$

and subsets of triple integers

$$\begin{aligned} T &= \{(n,i,k)\in\boldsymbol{Z}^3 \mid n\geq 3, 2\leq i\leq n-1, 1\leq k\leq i+1\} \\ T' &= \{(n,i,k)\in T \mid (i,k)\neq(n-1,1),(n-1,n-1)\}, \\ S &= \{(n,i,k)\in\boldsymbol{Z}^3 \mid n\geq 3, (i,k)=(0,1),(1,1),(1,2)\} \\ T^+ &= \{(n,i,k)\in T_0 \mid n>i+k+1\} \end{aligned}$$

Put a module

$$EM = \widetilde{D}\bigoplus\widetilde{E}\bigoplus\widetilde{F}\bigoplus\widetilde{J}\bigoplus\widetilde{K}\bigoplus\widetilde{L}\bigoplus\widetilde{M}\bigoplus\widetilde{N}\bigoplus\widetilde{P}\bigoplus\widetilde{Q}.$$

Then we have the E_1-term of the chromatic spectral sequence.

Theorem 2.3 *The module $H^*M_0^2$ is the tensor product of the exterior algebra $\Lambda(\rho_2)$ and EM.*

For the Adams-Novikov E_2-term $H^*E(2)_*$, we further introduce the modules:

$$\begin{aligned}
\widetilde{A}^+ &= \sum_{i,2\nmid s>0} \boldsymbol{Z}\{v_1^{2^i s}/2^{i+2}\}\\
\widetilde{C}^+ &= v_1\boldsymbol{Z}/2[v_1^2]\{1,h_{10},h_{10}^2,h_{10}^3\}\bigotimes\boldsymbol{Z}[g]\bigoplus v_1^3\rho_1\boldsymbol{Z}/2[v_1^2,h_0^2]\bigotimes\Lambda(h_{10})\\
\widetilde{K}' &= \Big(\boldsymbol{Z}/16\{\delta_1(\zeta\widetilde{\zeta}/16v_1^2)\}\bigoplus\boldsymbol{Z}/8\{\delta_1(h_{10}^3\beta/8v_1),v_2^{-1}h_{10}^3/8\}\\
&\quad\bigoplus\boldsymbol{Z}/4\{\delta_1(h_{10}^3/8v_1),\delta_1(v_2^{-1}h_{11}^2\beta/4v_1)\}\bigoplus\boldsymbol{Z}/2\{\delta_1(v_2/2v_1)\}\Big)\bigotimes\boldsymbol{Z}[g],\\
\widetilde{L}' &= \widetilde{LC}_0'\bigoplus\sum_{i=1}^{8}\delta_1(\widetilde{LC}_i),\\
\widetilde{LC}_0' &= \sum_{j>0}(\boldsymbol{Z}/2\{\delta_1(\beta/2v_1^{2j+1}),\delta_1(h_{10}\beta/2v_1^{2j+1}),\delta_1(h_{10}^2\beta/2v_1^{2j+1}),\\
&\qquad\qquad \delta_1(h_{10}^3\beta/2v_1^{2j+5})\}\bigotimes\boldsymbol{Z}/2[g]\\
&\qquad \bigoplus\boldsymbol{Z}/2\{\widetilde{\zeta}\zeta/2v_1^{2j+1},\widetilde{\zeta}\zeta h_0/2v_1^{2j+1}\}\bigotimes\boldsymbol{Z}/2[h_0^2]),\\
\widetilde{P}' &= \sum_{j>0}\boldsymbol{Z}/2\{\delta_1(\zeta\widetilde{\zeta}/2v_1^{2j+4})\},\quad \text{and}\\
\widetilde{Q}' &= \boldsymbol{Q}/\boldsymbol{Z}_{(2)}\{\delta_1(\zeta\widetilde{\zeta}/v_1^4)\}.
\end{aligned}$$

Let δ_0 (resp. δ_1) denote the connecting homomorphism associated to the short exact sequence (1.1) (resp. (1.2)).

Theorem 2.4 *The E_2-term of the Adams-Novikov spectral sequence converging to the homotopy groups $\pi_*(L_2S^0)$ is isomorphic to the direct sum of $\boldsymbol{Z}_{(2)}$, $\delta_0(\widetilde{A}^+)$, $\delta_0(\widetilde{C}^+)$, $\rho_2\delta_0\delta_1(EM)$, $\delta_0\delta_1(\overline{EM})$, $\delta_0(\widetilde{K}')$, $\delta_0(\widetilde{L}')$, $\delta_0(\widetilde{P}')$ and $\delta_0(\widetilde{Q}')$.*

Note that the α-elements are in $\widetilde{A}^+$ and the β-elements are in $\delta_0\delta_1(\widetilde{N})$. Furthermore, $\boldsymbol{Q}/\boldsymbol{Z}_{(2)}$ summands are $\delta_0\delta_1(\rho_2\widetilde{Q})\bigoplus\delta_0(\widetilde{Q}')$.

Corollary 2.5 *The homotopy groups $\pi_*(L_2S^0)$ contain two $\boldsymbol{Q}/\boldsymbol{Z}_{(2)}$ summands in dimension -4 and one in dimension -5. Furthermore, the group $\pi_i(L_2S^0)$ is finite except for $i=0,-4$, and -5.*

Proof. The differentials on $\boldsymbol{Q}/\boldsymbol{Z}_{(2)}$ vanish since it has no finite index subgroups. We see that $E_2^{1,-2}$, $E_2^{2,-2}$ and $E_2^{0,-4}$ are all zero, and so nothing kills the summands. Since the spectrum $E(2)$ is smashing [8], the Adams-Novikov spectral sequence has a horizontal vanishing line. Therefore, the other part follows from the theorem. □

3 Cocycles of the cobar complex $\Omega^* A$

Let (A, Γ) denote the Hopf algebroid $(E(2)_*, E(2)_*E(2)) = (\boldsymbol{Z}_{(2)}[v_1, v_2^{\pm 1}], E(2)_*[t_1, t_2, \dots]\bigotimes_{BP_*} E(2)_*)$ associated to the Johnson-Wilson spectrum $E(2)$. Then the cobar complex $\Omega^* M$ for a right comodule M with structure map $\psi\colon M \to M\bigotimes_A \Gamma$ is a differential graded module $\Omega^s M = M\bigotimes_A \Gamma^{\otimes s}$ with the differential $d\colon \Omega^s M \to \Omega^{s+1} M$ given by $d(m) = \psi(m) - m$, $d(x) = 1 \otimes x - \Delta(x) + x \otimes 1$ and $d(m \otimes x \otimes y) = d(m) \otimes x \otimes y + m \otimes d(x) \otimes y - m \otimes x \otimes d(y)$ for $m \in M$, $x \in \Omega^1 A$ and $y \in \Omega^s A$. Here, $\Delta\colon \Gamma \to \Gamma\bigotimes_A \Gamma$ denotes the diagonal map of Γ. Note that A is a Γ-comodule with the structure map η_R. In this paper, we consider comodules M induced from A and denote the structure map ψ by η_R. We denote the cohomology of the cobar complex by $H^* M$. Then $H^* A$ is the E_2-term of the Adams-Novikov spectral sequence based on $E(2)$ converging to $\pi_*(L_2 S^0)$.

Here we write down some of the formulae on the structure maps of the Hopf algebroid Γ.

$$
\begin{aligned}
\eta_R(v_1) &= v_1 + 2t_1, \\
\eta_R(v_2) &= v_2 - 5v_1 t_1^2 - 3v_1^2 t_1 + 2t_2 - 4t_1^3; \\
\Delta(t_1) &= t_1 \otimes 1 + 1 \otimes t_1, \\
\Delta(t_2) &= t_2 \otimes 1 + t_1 \otimes t_1^2 + 1 \otimes t_2 - v_1 t_1 \otimes t_1.
\end{aligned} \tag{3.1}
$$

Furthermore, the equation $\eta_R(v_3) = 0$ in Γ implies the relation

$$v_2 t_1^4 + t_1 \eta_R(v_2^2) + v_1 t_2^2 + v_1^2 v_2 t_1^2 + v_1^3 v_2 t_1 + v_1^4 (t_2 + t_1^3) \equiv 0 \tag{3.2}$$

mod (2) in Γ (*cf.* [13, (6.10)]). Recall that the generator ζ (which is denoted by ζ_1 in [13] and ζ_2 in [4]) is represented by the cochain $z = v_2^{-1}(t_2 + t_1^3) + v_2^{-2} t_2^2 \in \Omega^1 A$.

Lemma 3.3 *There is a cocycle $\widetilde{z}$ of $\Omega^1 A$ such that $\widetilde{z} \equiv v_1 v_2 z$* mod $(2, v_1^2)$.

We write $\widetilde{\zeta}$ as the homology class of $\widetilde{z}$.

Proof of Lemma 3.3. Put $y = v_1^4 + 8v_1 v_2$ and

$$\widetilde{z} = \eta_R(v_2) t_1 + t_1^4 + v_1 t_2 - v_1^2 t_1^2 - v_1^3 t_1. \tag{3.4}$$

Then we see the congruence $\widetilde{z} \equiv v_1 v_2 z$ mod $(2, v_1^2)$ by (3.2), and the equation

$$d(y) = 16\widetilde{z} \tag{3.5}$$

obtained from the computation in $\Omega^1 A$

$$\begin{aligned}
d(v_1^4) &= (v_1+2t_1)^4 - v_1^4\\
&= 8v_1^3t_1 + 24v_1^2t_1^2 + \underline{32v_1t_1^3}_1 + 16t_1^4\\
d(8v_1v_2) &= 8(v_1+2t_1)(v_2 - 5v_1t_1^2 - 3v_1^2t_1 + 2t_2 - 4t_1^3) - 8v_1v_2\\
&= -40v_1^2t_1^2 - 24v_1^3t_1 + 16v_1t_2 - \underline{32v_1t_1^3}_1 + 16t_1\eta_R(v_2).
\end{aligned}$$

Then $0 = 16d(\widetilde{z})$ by (3.5), and we have $d(\widetilde{z}) = 0$. □

Notation. Hereafter we use underlines on terms to explain the computation. Underlined terms with the same subscript are canceled out each other.

Lemma 3.6 *There are cochains $\widetilde{B}_i \in \Omega^3 A$ for each $i \geq 0$, and $\widetilde{G} \in \Omega^4 A$ such that*

a) $\widetilde{B}_i \equiv v_1^4 B$ *mod* $(2, v_1^5)$ *and* $d(\widetilde{B}_i) \equiv 0$ *mod* $(2^{i+1}, v_1^{2^i})$.
b) $\widetilde{G} \equiv v_1^4 G$ *mod* $(2, v_1^5)$ *and* $d(\widetilde{G}) = 0$.

Here B and G denote representatives of $\beta \in H^3K(2)_*$ and $g \in H^4K(2)_*$, respectively.

Proof. In [13, p.147], it is shown the existence of a cochain W such that $d(W) \equiv \widetilde{z}\otimes z^4 \otimes z^4 + v_1^4 B$ mod $(2, v_1^5)$. The lemma is valid for $\widetilde{B}_i$ defined by the equation

$$d(W) = \widetilde{z} \otimes ZZ_i + \widetilde{B}_i,$$

where ZZ_i is given by $d(z^{2^{i+1}}) \equiv 2ZZ_i$ mod $(v_1^{2^i})$.

In the same way, we define $\widetilde{G}$ by the equation $d(U) = t_1\otimes t_1\otimes t_1\otimes t_1 + \widetilde{G}$, where U is a cochain satisfying $d(U) \equiv t_1\otimes t_1\otimes t_1\otimes t_1 + v_1^4 G$ mod $(2, v_1^5)$ (*cf.* [13, p.152]). □

For the later use, consider a cochain $r_1 \in \Omega^1 v_1^{-1}A$ given by

$$r_1 = \widetilde{z} - t_1^4 + 3v_1^3t_1 + 2v_1(t_1^3 - t_2) - 2t_1^2\eta_R(v_1^{-1}v_2) + 4v_1^2t_1^2. \tag{3.7}$$

Then we compute

$$\begin{aligned}
r_1 &\equiv \widetilde{z} - t_1^4 + 3v_1^3t_1 - 2v_1t_2 + 2v_1t_1^3 + 4v_1^2t_1^2\\
&\qquad -2t_1^2(v_1^{-1} - 2v_1^{-2}t_1)(v_2 - v_1t_1^2 + v_1^2t_1 + 2t_2) \mod (8)\\
&\equiv \widetilde{z} - 3t_1^4 + 3v_1^3t_1 - 2v_1t_2 + 4v_1^2t_1^2 - 2v_1^{-1}v_2t_1^2 + 4v_1^{-1}t_1^2t_2\\
&\qquad +4v_1^{-1}t_1^5 + 4v_1^{-2}v_2t_1^3 \mod (8)
\end{aligned} \tag{3.8}$$

and we see that $v_1^2r_1 \in \Omega^1 A/(8)$.

Lemma 3.9 *In the cobar complex $\Omega^1 A$,*

$$d(v_2^2) \equiv v_1^2(3r_1 - \widetilde{z}) - v_1^3(3v_1t_1^2 + v_1^2t_1) + 4v_2^2 Z \mod (8).$$

for $Z = z + v_1v_2^{-2}(v_2t_1^2 + t_1^5) + v_1^3v_2^{-2}t_2$.

Proof. By (3.8), we have

$$\begin{aligned}3t_1^4 \equiv \widetilde{z} - r_1 + 3v_1^3t_1 - 2v_1^{-1}v_2t_1^2 - 2v_1t_2 + 4v_1^{-2}v_2t_1^3 + 4v_1^{-1}t_1^2t_2\\ +4v_1^{-1}t_1^5 + 4v_1^2t_1^2 \mod (8),\end{aligned}$$

and so

$$\begin{aligned}t_1^4 \equiv 3(\widetilde{z} - r_1) + v_1^3t_1 + 2v_1^{-1}v_2t_1^2 + 2v_1t_2 + 4v_1^{-2}v_2t_1^3 + 4v_1^{-1}t_1^2t_2\\ +4v_1^{-1}t_1^5 + 4v_1^2t_1^2 \mod (8).\end{aligned}$$

We then compute mod (8):

$$\begin{aligned}
d(v_2^2) &\equiv v_1^2t_1^4 + \underline{v_1^4t_1^2} + 4t_2^2 - 2v_1v_2t_1^2 + \underline{2v_1^2v_2t_1} + 4v_2t_2 - \underline{2v_1^3t_1^3}\\
&\quad +4v_1t_1^2t_2 + \underline{4v_1^2t_1t_2}\\
&\equiv v_1^2t_1^4 - v_1^4t_1^2 + 4t_2^2 - 2v_1v_2t_1^2 + 2v_1^2\eta_R(v_2)t_1 + 4v_2t_2 + 4v_1t_1^2t_2\\
&\equiv \underline{v_1^2t_1^4}_1 - \underline{v_1^4t_1^2}_2 + 4t_2^2 - 2v_1v_2t_1^2 + 2v_1^2(\widetilde{z} - \underline{t_1^4}_1 - v_1t_2 + \underline{v_1^2t_1^2}_2\\
&\qquad +v_1^3t_1) + 4v_2t_2 + 4v_1t_1^2t_2\\
&\equiv -v_1^2t_1^4 + v_1^4t_1^2 + 4t_2^2 - 2v_1v_2t_1^2 + 2v_1^2(\widetilde{z} - v_1t_2 + v_1^3t_1)\\
&\qquad +4v_2t_2 + 4v_1t_1^2t_2\\
&\equiv -v_1^2(3(\underline{\widetilde{z}}_1 - r_1) + \underline{v_1^3t_1}_2 + \underline{2v_1^{-1}v_2t_1^2}_3 + \underline{2v_1t_2}_4 + 4v_1^{-2}v_2t_1^3\\
&\quad +\underline{4v_1^{-1}t_1^2t_2}_5 + 4v_1^{-1}t_1^5 + \underline{4v_1^2t_1^2}_6) + \underline{v_1^4t_1^2}_6 + 4t_2^2 - \underline{2v_1v_2t_1^2}_3\\
&\quad +2v_1^2(\underline{\widetilde{z}}_1 - \underline{v_1t_2}_4 + \underline{v_1^3t_1}_2) + 4v_2t_2 + \underline{4v_1t_1^2t_2}_5\\
&\equiv v_1^2(3r_1 - \widetilde{z}) - v_1^5t_1 + 4v_1v_2t_1^2 + 4v_1^3t_2 + 4v_2t_1^3 + 4v_1t_1^5\\
&\quad -3v_1^4t_1^2 + 4t_2^2 + 4v_2t_2\\
&\equiv v_1^2(3r_1 - \widetilde{z}) - v_1^3(3v_1t_1^2 + v_1^2t_1) + 4(v_2^2z + v_1(v_2t_1^2 + t_1^5) + v_1^3t_2)
\end{aligned}$$

□

Lemma 3.10 *In the cobar complex $\Omega^1 v_1^{-1}A$, we have*

$$\begin{aligned}d(v_1v_2) = 2r_1 + 4v_1t_2 - 11v_1^2t_1^2 - 7v_1^3t_1 - 8v_1t_1^3\\ +4t_1^2(v_1^{-1} - 2v_1^{-2}t_1 + 4v_1^{-2}t_1^2\eta_R(v_1^{-1}))\eta_R(v_2)\end{aligned}$$

Proof. Note that

$$\eta_R(v_1^{-1}) = v_1^{-1} - 2v_1^{-2}t_1 + 4v_1^{-2}t_1^2\eta_R(v_1^{-1}). \tag{3.11}$$

The lemma follows from the equations:

$$\begin{aligned} r_1 &= \eta_R(v_2)t_1 - v_1t_2 + 3v_1^2t_1^2 + 2v_1^3t_1 + 2v_1t_1^3 \\ &\qquad -2t_1^2(v_1^{-1} - 2v_1^{-2}t_1 + 4v_1^{-2}t_1^2\eta_R(v_1^{-1}))\eta_R(v_2) \quad \text{and} \\ d(v_1v_2) &= 2t_1\eta_R(v_2) + v_1(-5v_1t_1^2 - 3v_1^2t_1 + 2t_2 - 4t_1^3) \end{aligned}$$

by (3.4), (3.7) and (3.11). □

Lemma 3.12 *In the cobar complex $\Omega^1 A$, we have*

$$d(v_1^{15}v_2) \equiv -2v_1^{14}r_1 + 16v_1^{12}t_1^3\eta_R(v_2) \mod (32, v_1^{16}).$$

Proof. Since $d(v_1^{15}v_2) \equiv v_1^{14}d(v_1v_2) + d(v_1^{14})\eta_R(v_1v_2)$, we obtain $d(v_1^{15}v_2) \equiv 2v_1^{14}r_1 - 4v_1^{14}r_1 + 16v_1^{12}t_1^3\eta_R(v_2)$ by Lemma 3.10 and the definition of r_1. □

Lemma 3.13 *In the cobar complex $\Omega^1 A$, there is a cochain $\widetilde{v_1^6\widetilde{z}^2}$ such that*

$$d(\widetilde{v_1^6\widetilde{z}^2}) \equiv -2v_1^6\widetilde{z} \otimes \widetilde{z} + 8v_1^6X \mod (16, v_1^8)$$

for some X.

Proof. By [13], there are cochains u and u' such that $d(u) \equiv t_1 \otimes z^2$ mod $(2, v_1)$ and $d(v_1u') \equiv t_1 \otimes \widetilde{z}$ mod $(2, v_1^2)$. Put $\widetilde{v_1^6\widetilde{z}^2} = v_1^6\widetilde{z}^2 + 4v_1^7v_2^2u + 4v_1^6u'^2$. By Lemma 3.3, we compute

$$\begin{aligned} d(v_1^6\widetilde{z}^2) &\equiv -4v_1^4(\underline{v_1t_1}_1 + \underline{t_1^2}_2) \otimes \widetilde{z}^2 - 2v_1^6\widetilde{z} \otimes \widetilde{z} \\ d(4v_1^7v_2^2u) &\equiv 8v_1^6t_1 \otimes v_2^2u + 4v_1^7v_2^2(\underline{t_1 \otimes z^2}_1 + 2X_1) \\ d(4v_1^4(v_1u')^2) &\equiv \underline{4v_1^4t_1^2 \otimes \widetilde{z}^2}_2 + 8v_1^6X_2 \end{aligned}$$

mod $(16, v_1^8)$ for some X_1 and X_2. □

4 The Miller-Ravenel-Wilson elements x_i

By the definitions of $\widetilde{z}$ and r_1, we have

$$r_1 \equiv t_1^4 + \widetilde{z} + v_1^3t_1 \equiv v_2t_1 + v_1(t_2 + t_1^3) \mod (2), \tag{4.1}$$

which represents the generator ρ_1 of $H^1M_1^0$ and will be used in this section. Define the element y_1 by

$$y_1 = v_1^2 - 4v_1^{-1}v_2$$

which is $x_{1,1}$ of [4].

Lemma 4.2 *In $v_1^{-1}\Gamma$, we have*

$$d(y_1) = 8v_1^{-2}r_1 \equiv 8v_1^{-2}(t_1^4 + \widetilde{z} + v_1^3 t_1) \mod (16).$$

Proof. Noticing the equation (3.11), we compute

$$\begin{aligned} d(v_1^2) &= 4v_1t_1 + 4t_1^2 \\ d(-4v_1^{-1}v_2) &= -4(v_1^{-1} - 2v_1^{-2}t_1 + 4v_1^{-2}t_1^2\eta_R(v_1^{-1})) \\ &\qquad \times(v_2 - 5v_1t_1^2 - 3v_1^2t_1 + 2t_2 - 4t_1^3) + 4v_1^{-1}v_2 \\ &= 20t_1^2 + 12v_1t_1 - 8v_1^{-1}t_2 + 16v_1^{-1}t_1^3 \\ &\qquad +8v_1^{-2}t_1\eta_R(v_2) - 16v_1^{-2}t_1^2\eta_R(v_1^{-1}v_2). \end{aligned}$$

Therefore, using the relation $\eta_R(v_2)t_1 = \widetilde{z} - t_1^4 - v_1t_2 + v_1^2t_1^2 + v_1^3t_1$ by (3.4), the sum is

$$\begin{aligned} d(y_1) = 32t_1^2 + 24v_1t_1 - 16v_1^{-1}t_2 + 16v_1^{-1}t_1^3 \\ -16v_1^{-2}t_1^2\eta_R(v_1^{-1}v_2) + 8v_1^{-2}(\widetilde{z} - t_1^4). \end{aligned}$$

and we see that the right hand side is $8v_1^{-2}r_1$ by (3.7). □

Define the elements x_i for $i \geq 1$ by

$$\begin{aligned} x_1 &= v_2^2 + v_1^3v_2, \\ x_2 &= v_2^4 + v_1^3v_2^3 + v_1^6v_2^2, \\ x_{2n+1} &= x_{2n}^2 + v_1^{3\cdot 2^{2n+1}-12}x_2 + v_1^{3\cdot 2^{2n+1}-3}v_2 \quad (n > 0), \\ x_{2n+2} &= x_{2n+1}^2 + v_1^{3\cdot 2^{2n+2}-12}x_2 \quad (n > 0). \end{aligned}$$

Then we compute

Lemma 4.3 *In Γ, we have*

$$d(x_1) \equiv v_1^2(r_1 + \widetilde{z}) \mod (2).$$

Furthermore,

$$d(x_n) \equiv v_1^{2^n}\widetilde{z}^{2^{n-1}} + v_1^{3\cdot 2^n-4}\widetilde{z} + \frac{1+(-1)^n}{2}v_1^{3\cdot 2^n-1}t_1 \mod (2)$$

for $n > 1$.

Proof. By (3.1) and Lemma 3.9, we have

$$\begin{aligned} d(v_2^2) &\equiv v_1^2(r_1 + \widetilde{z}) + v_1^3(v_1t_1^2 + v_1^2t_1) \mod (2) \quad \text{and} \\ d(v_1^3v_2) &\equiv v_1^3(v_1t_1^2 + v_1^2t_1) \mod (2). \end{aligned}$$

The sum gives us the first one.

We show the others by induction. By (3.1), (3.2) and (4.1),

$$\begin{aligned}
v_2d(v_2^2)+v_1^2t_1\eta_R(v_2^2) &\equiv v_1^2v_2t_1^4+v_1^2t_1\eta_R(v_2^2)+v_1^4v_2t_1^2\\
&\equiv v_1^3t_2^2+v_1^5v_2t_1+v_1^6(t_2+t_1^3)\\
&\equiv v_1^3t_2^2+v_1^5r_1 \quad \text{mod } (2)
\end{aligned}$$

We also see that $t_1^2\eta_R(v_2^2)\equiv r_1^2+v_1^2t_2^2+v_1^4t_1^4$ mod (2) by (3.1) and (4.1). Since $d(v_2^3)=v_2d(v_2^2)+d(v_2)\eta_R(v_2^2)$, we have

$$\begin{aligned}
d(v_2^3) &\equiv v_1^3t_2^2+v_1^5r_1+v_1r_1^2+v_1^3t_2^2+v_1^5t_1^4 \quad \text{mod } (2)\\
&\equiv v_1r_1^2+v_1^5\widetilde{z}+v_1^8t_1 \quad \text{mod } (2) \quad \text{by (4.1).}
\end{aligned}$$

Then the first step $d(x_2)\equiv v_1^4\widetilde{z}^2+v_1^8\widetilde{z}+v_1^{11}t_1$ mod (2) is obtained as the sum of

$$\begin{aligned}
d(x_1^2) &\equiv v_1^4(r_1^2+\widetilde{z}^2) \quad \text{mod } (2) \quad \text{and}\\
d(v_1^3v_2^3) &\equiv v_1^3(v_1r_1^2+v_1^5\widetilde{z}+v_1^8t_1) \quad \text{mod } (2).
\end{aligned}$$

Inductively suppose that

$$d(x_{2n})\equiv v_1^{2^{2n}}\widetilde{z}^{2^{2n-1}}+v_1^{3\cdot 2^{2n}-4}\widetilde{z}+v_1^{3\cdot 2^{2n}-1}t_1 \quad \text{mod } (2).$$

Then mod (2),

$$\begin{aligned}
d(x_{2n}^2) &\equiv v_1^{2^{2n+1}}\widetilde{z}^{2^{2n}}+\underline{v_1^{3\cdot 2^{2n+1}-8}\widetilde{z}^2}_1+\underline{v_1^{3\cdot 2^{2n+1}-2}t_1^2}_2\\
d(v_1^{3\cdot 2^{2n+1}-12}x_2) &\equiv \underline{v_1^{3\cdot 2^{2n+1}-8}\widetilde{z}^2}_1+v_1^{3\cdot 2^{2n+1}-4}\widetilde{z}+\underline{v_1^{3\cdot 2^{2n+1}-1}t_1}_3\\
d(v_1^{3\cdot 2^{2n+1}-3}v_2) &\equiv \underline{v_1^{3\cdot 2^{2n+1}-2}t_1^2}_2+\underline{v_1^{3\cdot 2^{2n+1}-1}t_1}_3
\end{aligned}$$

and we obtain

$$d(x_{2n+1})\equiv v_1^{2^{2n+1}}\widetilde{z}^{2^{2n}}+v_1^{3\cdot 2^{2n+1}-4}\widetilde{z} \quad \text{mod } (2).$$

Squaring this shows

$$d(x_{2n+1}^2)\equiv v_1^{2^{2n+2}}\widetilde{z}^{2^{2n+1}}+\underline{v_1^{3\cdot 2^{2n+2}-8}\widetilde{z}^2}_a \quad \text{mod } (2).$$

Add

$$d(v_1^{3\cdot 2^{2n+2}-12}x_2)\equiv \underline{v_1^{3\cdot 2^{2n+1}-8}\widetilde{z}^2}_a+v_1^{3\cdot 2^{2n+2}-4}\widetilde{z}+v_1^{3\cdot 2^{2n+2}-1}t_1 \quad \text{mod } (2),$$

and we obtain the case $2n+2$, and the induction completes. □

Remark. In the following, $v_2^{2^n}$ denotes sometimes x_n.

5 $H^*M_0^1$ revisited

In [4, Th.4.16], it is shown that

$$(5.1)\quad \begin{aligned} H^*M_0^1 = &\left(\textstyle\sum_{i>0,p\nmid s} \boldsymbol{Z}/2^{i+2}\{v_1^{2^is}/2^{i+2}\}\right) \\ &\bigoplus \left(v_1\boldsymbol{Z}/2[v_1^{\pm 2}, h_{10}]\bigotimes \Lambda(\rho_1)\right) \bigoplus \left(\boldsymbol{Q}/\boldsymbol{Z}_{(2)}\bigotimes \Lambda(\rho_1)\right). \end{aligned}$$

In this section we study the image of $\rho_1/2^i$ $(i > 0)$ under the map $H^1M_0^1 \to H^1M_0^2$.

Lemma 5.2 *Consider the formal sum*

$$r = -\frac{1}{16}\sum_{k\geq 1}(-1)^k(16\widetilde{z}/y)^k/k = \widetilde{z}/y - 8(\widetilde{z}/y)^2 + \dots$$

in $\Omega^1 v_1^{-1}A$ *for* $y = v_1^4 + 8v_1v_2$. *Then* $r \equiv \widetilde{z}/v_1^4 \bmod (2)$ *and it defines a cocycle* r *of* $\Omega^1v_1^{-1}A/(2^n)$ *for any* $n > 1$.

Proof. In this proof, we make a computation formally. Let $\log(1 + x) = \sum_{k>0}(-1)^{k-1}x^k/k$, and put $r = d(\log(y))/16$. Note that $\eta_R(y) = y+16\widetilde{z}$ by (3.5). Then $d(r) = 0$ and

$$\begin{aligned} d(\log(y))/16 &= (\eta_R(\log(y)) - \log(y))/16 \\ &= (\log(\eta_R(y)) - \log(y))/16 \\ &= \log(\eta_R(y)/y)/16 \\ &= \log((y + 16\widetilde{z})/y)/16 \end{aligned}$$

as desired. □

Proposition 5.3 *The generator* $\rho_1/2^{n+1}$ *of* $\boldsymbol{Q}/\boldsymbol{Z}_{(2)} \subset H^{1,0}M_0^1 \cong \boldsymbol{Q}/\boldsymbol{Z}_{(2)}\bigoplus\boldsymbol{Z}/2$ *is represented by* $r/2^{n+1}$. *Furthermore, the map* $H^1M_0^1 \to H^1M_0^2$ *sends* $\rho_1/2^{n+2}$ *to* $\widetilde{\zeta}/2^{n+2}v_1^4$.

Proof. By (4.1), $\rho_1/2$ is represented by $r_1/2v_1^4$. Since $\widetilde{z}/2v_1^4$ is homologous to $r_1/2v_1^4$ by Lemma 4.3, we see the proposition by Lemma 5.2. □

6 $H^*M_1^1$ revisited

We give another decomposition of $H^*M_1^1$ to make the book keeping easier. In the following, we decompose each summands MB_i given in (2.2) into smaller ones.

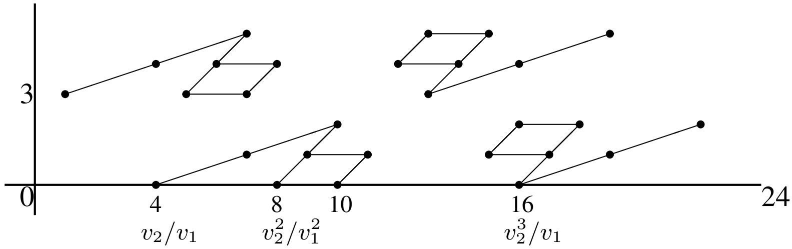
Now we consider the submodules of $K(1)_*/k(1)_*$:

$$\begin{aligned} A_\infty &= \boldsymbol{Z}/2\{1/v_1^{2j} \mid j > 0\} \\ A_n &= \boldsymbol{Z}/2\{1/v_1^{2j} \mid 0 < 2j \le 3 \cdot 2^{n-1}\} \\ A_n^+ &= \boldsymbol{Z}/2\{1/v_1^{2j} \mid 0 < 2j \le 3 \cdot 2^{n-1} + 2\} \\ A_n^- &= \boldsymbol{Z}/2\{1/v_1^{2j} \mid 0 < 2j \le 3 \cdot 2^{n-1} - 4\} \end{aligned}$$

for $n \ge 2$. Then MB_0 is divided into the following seven summands:

$$\begin{aligned} LC_0 &= v_1 A_\infty\{1, h_{10}, h_{10}^2, h_{10}^3/v_1^4\} \bigotimes \boldsymbol{Z}/2[g] \bigotimes \Lambda(\beta) \bigoplus \widetilde{\zeta} A_\infty\{v_1, h_0\} \\ &\qquad \bigotimes \boldsymbol{Z}/2[h_0^2] \bigotimes \Lambda(\zeta), \\ LI_0 &= A_\infty\{h_{10}, h_{10}^2, h_{10}^3, g\} \bigotimes \boldsymbol{Z}/2[g] \bigotimes \Lambda(\beta) \bigoplus \widetilde{\zeta} A_\infty\{v_1 h_0, h_0^2\} \\ &\qquad \bigotimes \boldsymbol{Z}/2[h_0^2] \bigotimes \Lambda(\zeta), \\ JC &= \zeta K(1)_*/k(1)_* \bigotimes \boldsymbol{Z}/2[h_0], \\ JI &= \zeta^2 K(1)_*/k(1)_* \bigotimes \boldsymbol{Z}/2[h_0], \\ Q &= \boldsymbol{Z}/2\{\widetilde{\zeta}/v_1^4\} \bigotimes \Lambda(\zeta) \\ K_0 &= \boldsymbol{Z}/2\{h_{10}^3/v_1^3\} \bigotimes \Lambda(v_1^2, \beta) \bigotimes \boldsymbol{Z}/2[g] \bigoplus \boldsymbol{Z}/2\{\widetilde{\zeta}/v_1^2\} \bigotimes \Lambda(\zeta), \quad \text{and} \\ P &= A_\infty \bigotimes \Lambda(\beta) \bigoplus (\widetilde{\zeta}/v_1^4) A_\infty \bigotimes \Lambda(\zeta). \end{aligned}$$

Note that the direct sum $MB_1 \bigoplus MB_2$ is the tensor product of $\boldsymbol{Z}/2[v_2^{\pm 4}, g]$ and the module displayed as follows:



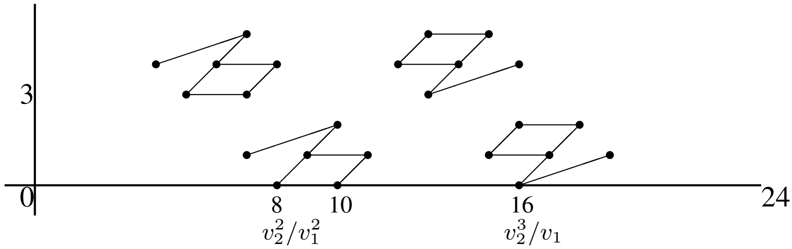
in which the lines of the slope 1 (resp. 1/3, 0) denote the multiplication by h_{10} (resp. h_{11}, v_1). The module $MB_1 \bigoplus MB_2$ is the direct sum of four modules

$$\begin{aligned} K_1 &= (K_1^0 \bigoplus K_1^2) \bigotimes \boldsymbol{Z}/2[g], \\ D_1 &= \sum_{n \ge 2, 2 \nmid s} v_2^{2^n s} K_1^0 \\ F_1 &= \sum_{n \ge 2, 2 \nmid s} v_2^{2^n s} (g K_1^0 \bigoplus K_1^2) \bigotimes \boldsymbol{Z}/2[g] \\ M &= M^0 \bigotimes \boldsymbol{Z}/2[v_2^{\pm 4}, g]. \end{aligned}$$

Here

$$\begin{aligned} K_1^0 &= \boldsymbol{Z}/2\{v_2/v_1\} \bigotimes \Lambda(\beta) \\ K_1^2 &= \boldsymbol{Z}/2\{v_2^{-1}h_{11}^2/v_1\} \bigotimes \Lambda(\beta) \quad \text{and} \\ M^0 &= \big(\boldsymbol{Z}/2\{v_2h_{11}/v_1, v_2^3/v_1\} \bigotimes \Lambda(h_{11}) \\ &\qquad \bigoplus \boldsymbol{Z}/2\{v_2^2/v_1^2\} \bigotimes \Lambda(v_1, h_{10}, v_2h_{10})\big) \bigotimes \Lambda(\beta) \end{aligned}$$

The module M^0 is displayed as follows:



MB_3 is the direct sum of

$$LC_1 = \sum_{2\not| s} \boldsymbol{Z}/2\{v_2^{2s}\widetilde{\zeta}/v_1^3, v_2^{2s}\widetilde{\zeta}h_0/v_1^2, v_2^{2s}\widetilde{\zeta}h_0/v_1^4\} \bigotimes \boldsymbol{Z}/2[h_0^2],$$

$$LI_1 = \sum_{2\not| s} \boldsymbol{Z}/2\{v_2^{2s}\widetilde{\zeta}h_0/v_1^3, v_2^{2s}\widetilde{\zeta}h_0^2/v_1^2, v_2^{2s}\widetilde{\zeta}h_0^2/v_1^4\} \bigotimes \boldsymbol{Z}/2[h_0^2],$$

$$LC_2 = \sum_{n\geq 2, 2\not| s} v_2^{2^n s}\widetilde{\zeta}A_n\{1/v_1, h_0\} \bigotimes \boldsymbol{Z}/2[h_0^2],$$

$$LI_2 = \sum_{n\geq 2, 2\not| s} v_2^{2^n s}\widetilde{\zeta}A_n\{h_0/v_1, h_0^2\} \bigotimes \boldsymbol{Z}/2[h_0^2],$$

$$D_2 = \sum_{n\geq 2, 2\not| s} \boldsymbol{Z}/2\{v_2^{2^n s}\widetilde{\zeta}/v_1^4\} \bigotimes \Lambda(v_1^2)$$

$$E_1 = \sum_{2\not| s} \boldsymbol{Z}/2\{v_2^{2s}\widetilde{\zeta}/v_1^4\} \bigotimes \Lambda(v_1^2)$$

$$N_1 = \sum_{n\geq 3, 2\not| s} v_2^{2^n s}(\widetilde{\zeta}/v_1^4)A_n^- \quad \text{and}$$

$$E_2 = \sum_{2\not| s} \boldsymbol{Z}/2\{v_2^{4s}\widetilde{\zeta}/v_1^6\}.$$

Since $\widetilde{\zeta}^{(n-1)}$ also denotes $v_1^{2^{n-1}}v_2^{2^{n-1}}\zeta$, we use this in the direct summands of MB_4 and MB_6. MB_4 is divided into

$$LC_3 = \sum_{n\geq 2, 2\not| s} v_2^{2^n s+2^{n-1}}\zeta A_n\{v_1, h_0\} \bigotimes \boldsymbol{Z}/2[h_0^2],$$

$$LI_3 = \sum_{n\geq 2, 2\not| s} v_2^{2^n s+2^{n-1}} \zeta A_n\{v_1h_0, h_0^2\} \bigotimes \boldsymbol{Z}/2[h_0^2],$$
$$N_2 = \sum_{n\geq 2, 2\not| s} v_2^{2^n s+2^{n-1}} \zeta A_n.$$

MB_5 is the direct sum of

$$LC_4 = \sum_{n\geq 2, 2\not| s} v_2^{2^n s} A_n\{v_1, h_0^3\} \bigotimes \boldsymbol{Z}/2[g],$$
$$LC_5 = \sum_{n\geq 2, 2\not| s} v_2^{2^n s} A_n^+\{v_1h_{10}, v_1h_{10}^2\} \bigotimes \boldsymbol{Z}/2[g],$$
$$LI_4 = \sum_{n\geq 2, 2\not| s} v_2^{2^n s} A_n\{h_{10}, g\} \bigotimes \boldsymbol{Z}/2[g],$$
$$LI_5 = \sum_{n\geq 2, 2\not| s} v_2^{2^n s} A_n^+\{h_{10}^2, h_{10}^3\} \bigotimes \boldsymbol{Z}/2[g],$$
$$F_2 = \sum_{n\geq 2, 2\not| s} \boldsymbol{Z}/2\{v_2^{2^n s} h_{10}^3/v_1^3\} \bigotimes \Lambda(v_1^2) \bigotimes \boldsymbol{Z}/2[g],$$
$$N_3 = \sum_{n\geq 3, 2\not| s} v_2^{2^n s} A_n \quad \text{and}$$
$$E_3 = \sum_{2\not| s} v_2^{4s} A_2.$$

MB_6 is the direct sum of

$$LC_6 = \sum_{n\geq 2, 2\not| s} v_2^{2^n s+2^{n-1}} \zeta\widetilde{\zeta} A_n\{h_0^2/v_1, h_0^3\} \bigotimes \boldsymbol{Z}/2[g],$$
$$LC_7 = \sum_{n\geq 2, 2\not| s} v_2^{2^n s+2^{n-1}} \zeta\widetilde{\zeta} A_n^+\{1/v_1, h_0\} \bigotimes \boldsymbol{Z}/2[g],$$
$$LI_6 = \sum_{n\geq 2, 2\not| s} v_2^{2^n s+2^{n-1}} \zeta\widetilde{\zeta} A_n\{h_0^3/v_1, g\} \bigotimes \boldsymbol{Z}/2[g],$$
$$LI_7 = \sum_{n\geq 2, 2\not| s} v_2^{2^n s+2^{n-1}} \zeta\widetilde{\zeta} A_n^+\{h_0/v_1, h_0^2\} \bigotimes \boldsymbol{Z}/2[g],$$
$$F_3 = \sum_{n\geq 2, 2\not| s} \boldsymbol{Z}/2\{v_2^{2^n s+2^{n-1}} \zeta\widetilde{\zeta} g/v_1^{3\cdot 2^{n-1}+4}\} \bigotimes \Lambda(v_1^2) \bigotimes \boldsymbol{Z}/2[g]$$
$$D_3 = \sum_{n\geq 2, 4|(s-1)} \boldsymbol{Z}/2\{v_2^{2^n s+2^{n-1}} \zeta\widetilde{\zeta}/v_1^{3\cdot 2^{n-1}+4}\} \bigotimes \Lambda(v_1^2)$$
$$D_4 = \sum_{n\geq 3, 2\not| s} \boldsymbol{Z}/2\{v_2^{2^n s+2^{n-1}} \zeta\widetilde{\zeta}/v_1^4\} \bigotimes \Lambda(v_1^2)$$

$$E_4 = \sum_{2\not| s} \boldsymbol{Z}/2\{v_2^{4s+2}\zeta\widetilde{\zeta}/v_1^4\}\bigotimes\Lambda(v_1^2)$$
$$N_4 = \sum_{n\geq 2, 2\not| s} v_2^{2^ns+2^{n-1}}\zeta(\widetilde{\zeta}/v_1^4)A_n^-$$
$$\bigoplus \sum_{n\geq 2, 4|(s+1)} \boldsymbol{Z}/2\{v_2^{2^ns+2^{n-1}}\zeta\widetilde{\zeta}/v_1^{3\cdot 2^{n-1}+4}\}\bigotimes\Lambda(v_1^2).$$

Now put

$$D = \textstyle\sum_{i=1}^4 D_i, \quad E = \sum_{i=1}^4 E_i, \quad F = \sum_{i=1}^3 F_i, \quad J = JC\bigoplus JI,$$
$$K = K_0\bigoplus K_1, \quad L = \textstyle\sum_{i=0}^7 (LC_i\bigoplus LI_i), \quad \text{and} \quad N = \sum_{i=1}^4 N_i.$$

Then

$$H^*M_1^1 = (D\bigoplus E\bigoplus F\bigoplus J\bigoplus K\bigoplus L\bigoplus M\bigoplus N\bigoplus P\bigoplus Q)\bigotimes\Lambda(\rho_2).$$

7 The cocycles R and B

Consider the $E(2)_*$-module $M(i,j) = E(2)_*/(2^i, v_1^j)$, which is also an $E(2)_*E(2)$-comodule if $2^{i-1}|j$. Then we have the exact sequence

$$(7.1) \qquad H^{s-1,0}M_1^1 \xrightarrow{\delta} H^{s,0}M(1,j) \xrightarrow{1/v_1^j} H^{s,-2j}M_1^1 \xrightarrow{v_1^j} H^{s,0}M_1^1.$$

Lemma 7.2 *Every element of $H^{s,0}M(1,2^n)$ for $s = 2,3,4$ and $n > 4$ is divisible by $v_1^{2^{n-2}}$ except for ζh_0, ζ^2 and $\zeta\rho_2$ if $s = 2$, β, ζh_0^2, $\zeta^2 h_0$, $v_2^{-1}h_{10}^3$, $\zeta h_0\rho_2$ and $\zeta^2\rho_2$ if $s = 3$, and g, ζh_0^3, $\zeta^2 h_0^2$, $\beta\rho_2$, $\zeta h_0^2\rho_2$, $\zeta^2 h_0\rho_2$ and $v_2^{-1}h_{10}^3\rho_2$ if $s = 4$.*

Proof. By (2.2), we see that

$$H^{1,0}M_1^1 = \boldsymbol{Z}/2\{h_{10}/v_1, v_2\zeta/v_1^3\}$$
$$H^{2,0}M_1^1 = \boldsymbol{Z}/2\{h_{10}^2/v_1^2, \widetilde{\zeta}h_0/v_1^4, \zeta\widetilde{\zeta}/v_1^4, v_2^{-1}h_{11}^2/v_1, h_{10}\rho_2/v_1, v_2\zeta\rho_2/v_1^3\}$$
$$H^{3,0}M_1^1 = \boldsymbol{Z}/2\{h_{10}^3/v_1^3, \widetilde{\zeta}h_0^2/v_1^4, \zeta\widetilde{\zeta}h_0/v_1^4, h_{10}^2\rho_2/v_1^2, \widetilde{\zeta}h_0\rho_2/v_1^4,$$
$$\zeta\widetilde{\zeta}\rho_2/v_1^4, v_2^{-1}h_{11}^2\rho_2/v_1\}.$$

Then $\operatorname{Im}\delta = 0$ if $s = 2$, $= \boldsymbol{Z}/2\{v_2^{-1}h_{10}^3\}$ if $s = 3$, and $= \boldsymbol{Z}/2\{v_2^{-1}h_{10}^3\rho_2\}$ if $s = 4$, which gives the exceptional elements.

If $x/v_1^j \in H^{s,-2^{n+1}}M_1^1$ is in the image of $1/v_1^{2^n}$, then $v_1^{2^n-j}x \in H^{s,0}M(1,2^n)$. So if the element $v_1^{2^n-j}x$ of $H^{s,0}M(1,2^n)$ is divisible by $v_1^{2^{n-2}}$, then $2^n - j \geq 2^{n-2}$. We will find elements x/v_1^j with $j > 2^n - 2^{n-2}$. There is no such elements in $MB_1\bigoplus MB_2$, since $j \leq 2$. For a generator $v_2^{2^m(2t+1)}\xi_i/v_1^j$ of MB_i, $2^n - 2^{n-2} < j \leq 3\cdot 2^{m-1} + 4$ and

$6\cdot 2^m(2t+1)+|\xi_i|-2j=-2^{n+1}$. Here $|\xi_i|=8$ if $i=3$, $=6\cdot 2^{m-1}$ if $i=4$, $=2s$ if $i=5$ and $s=2,3$, $=0$ if $i=5$ and $s=4$, and $=6\cdot 2^{m-1}+8$ if $i=6$. Then we see that there is no solution if $n\geq 4$. In MB_0, we have

$$h_{10}^2/v_1^{2^n+2}, \zeta h_0/v_1^{2^n}, \zeta^2/v_1^{2^n}, \widetilde{\zeta}h_0/v_1^{2^n+4}, \zeta\widetilde{\zeta}/v_1^{2^n+4},\\ h_{10}\rho_2/v_1^{2^n+1}, \zeta\rho_2/v_1^{2^n}, \quad \text{and} \quad \widetilde{\zeta}\rho_2/v_1^{2^n+4}$$

in $H^{2,-2^{n+1}}M_1^1$,

$$h_{10}^3/v_1^{2^n+3}, \beta/v_1^{2^n}, \zeta h_0^2/v_1^{2^n}, \zeta^2h_0/v_1^{2^n}, \widetilde{\zeta}h_0^2/v_1^{2^n+4}, \zeta\widetilde{\zeta}h_0/v_1^{2^n+4}\\ h_{10}^2\rho_2/v_1^{2^n+2}, \zeta h_0\rho_2/v_1^{2^n}, \zeta^2\rho_2/v_1^{2^n}, \widetilde{\zeta}h_0\rho_2/v_1^{2^n+4}, \quad \text{and} \quad \zeta\widetilde{\zeta}\rho_2/v_1^{2^n+4},$$

in $H^{3,-2^{n+1}}M_1^1$, and

$$g/v_1^{2^n}, h_{10}\beta/v_1^{2^n+1}, \zeta h_0^3/v_1^{2^n}, \zeta^2h_0^2/v_1^{2^n}, \widetilde{\zeta}h_0^3/v_1^{2^n+4}, \zeta\widetilde{\zeta}h_0^2/v_1^{2^n+4},\\ h_{10}^3\rho_2/v_1^{2^n+3}, \beta\rho_2/v_1^{2^n}, \zeta h_0^2\rho_2/v_1^{2^n}, \zeta^2h_0\rho_2/v_1^{2^n},\\ \widetilde{\zeta}h_0^2\rho_2/v_1^{2^n+4}, \quad \text{and} \quad \zeta\widetilde{\zeta}h_0\rho_2/v_1^{2^n+4},$$

in $H^{4,-2^{n+1}}M_1^1$. Since the generators of the form $x/v_1^{2^n}$ are pulled back to x of $H^*M(1,2)$, we obtain the other exceptional elements. □

Lemma 7.3 *Let z_i denote a representative of ζh_0^{i-1} of $H^{i,0}M(1,2^n)$. Then*

$$\begin{aligned} d(z_1) &\equiv 2z_1\otimes z_1 \mod (4, v_1^{2^n})\\ d(z_2) &\equiv 2z_1\otimes z_2+2z_3+2kv_2^{-1}t_1\otimes t_1\otimes t_1 \mod (4, v_1^{2^n})\\ d(z_3) &\equiv 2z_1\otimes z_3+2k'v_2^{-1}t_1\otimes t_1\otimes t_1\otimes R \mod (4, v_1^{2^n})\\ d(z_4) &\equiv 2z_1\otimes z_4+2z_1\otimes G \mod (4, v_1^{2^n}) \end{aligned}$$

for some integers k and k'. Here R represents the generator ρ_2.

Proof. Since $z_1=z^{2^n}$ is primitive mod $(2, v_1^{2^n})$ and z_1 is homologous to z_1^2, the first one is obvious if we replace z_1 by z_1^2.

Put $d(z_2)\equiv 2c_n$ mod $(4, v_1^{2^{n+2}})$. Then

$$\begin{aligned} c_n \equiv\ & k_1B+k_2z_3+k_3z_1\otimes z_2+k_4v_2^{-1}t_1\otimes t_1\otimes t_1\\ &+k_5z_2\otimes R+k_6z_1\otimes z_1\otimes R+d(c_n') \end{aligned} \tag{7.4}$$

mod $(2, v_1^{2^n})$ for some c_n' by Lemma 7.2. Recall [13, Lemma 6.8] that there are elements u_i such that $d(u_1)\equiv z_1\otimes t_1+v_1z_2$ and $d(u_2)\equiv t_1\otimes z_2+v_1z_3$

mod $(2, v_1^{2^n})$. Define elements w to fit in $d(u_1) \equiv z_1 \otimes t_1 + v_1 z_2 + 2w$ mod $(4, v_1^{2^n})$. Then

$$\begin{aligned} 0 &\equiv 2z_1 \otimes z_1 \otimes t_1 + 2t_1 \otimes z_2 + v_1 d(z_2) + 2d(w) \\ &\equiv 2d(z_1 \otimes u_1) + 2v_1 z_1 \otimes z_2 + 2d(u_2) + 2v_1 z_3 \\ &\quad +2v_1 c_n + 2d(w) \quad \text{mod } (4, v_1^{2^n}). \end{aligned}$$

Comparing with (7.4) shows the second.

In the same manner, we see the others. □

Proposition 7.5 *There is a cocycle $R_n \in \Omega^1 M(n+1, 2^n)$, which represents ρ_2 of $H^{1,0}M(1,1) = H^{1,0}K(2)_*$.*

Proof. Suppose first $n > 3$. Raising a representative of ρ_2 of $H^{1,0}K(2)_*$ to 2^{3n} power yields a representative of $\rho_2 \in H^{1,0}M(1, 2^{3n})$. Let R denote a representative of ρ_2. Then R^2 also represents ρ_2, and we compute $d(R^2) \equiv 2R \otimes R$ mod $(4, v_1^{3n})$. The relation $\rho_2^2 = 0$ of $H^*K(2)_*$ shows the existence of an element S such that $d(S) \equiv R \otimes R$ mod $(2, v_1^{3n})$. Then $R_1 = R^2 + 2S$ satisfies the relation $d(R_1) \equiv 0$ mod $(4, v_1^{3n})$.

Suppose inductively that there exists a cocycle R_k such that $d(R_k) \equiv 0$ mod $(2^{k+1}, v_1^{2^{3n+2-2k}})$ for $k \geq 1$. Then $d(R_k) \equiv 2^{k+1}x$ mod $(2^{k+2}, v_1^{2^{3n+2-2k}})$ for some x, and we see that x represents an element of $H^{2,0}M(1, 2^{3n+2-2k})$. Lemma 7.2 shows that $x \equiv k_1 z_2 + k_2 z_1 \otimes z_1 + k_3 z_1 R_k + 2y + d(w)$ mod $(4, v_1^{2^{3n-2k}})$ for some cochains y and w. Since $d(x) \equiv 0$ mod $(4, 2^{3n+2-2k})$, we see that

$$0 = k_1(z_1 \otimes z_2 + z_3 + kv_2^{-1} t_1 \otimes t_1 \otimes t_1) + k_3 z_1 \otimes z_1 \otimes R_k + d(y)$$

mod $(2, v_1^{2^{3n-2k}})$ by Lemma 7.3, which implies a relation $0 = k_1(\zeta^2 h_0 + \zeta h_0^2 + kv_2^{-1} h_{10}^3) + k_3 \zeta^2 \rho_2$ of $H^{3,0}K(2)_*$ and we have $k_1 = 0 = k_3$. Therefore, $d(R_k) \equiv 2^{k+1} k_2 z_1 \otimes z_1 + d(2^{k+1} w)$ mod $(2^{k+2}, v_1^{2^{3n-2k}})$. Put $R_{k+1} = R_k + 2^k k_2 z_1 + 2^{k+1} w$ and we have $d(R_{k+1}) \equiv 0$ mod $(2^{k+2}, v_1^{2^{3n-2k}})$, which completes the induction.

Now take R_n for a representative of $\widetilde{\rho}_2$ and we have the result for $n > 3$.

If $n \leq 3$, just consider the projection $M(5, 2^4) \to M(n+1, 2^n) \to M(1,1)$. □

We write R as R_n as long as no confusion arises.

Corollary 7.6 *In the cobar complex for $M(n+1, j)$ with $2^n \mid j$, there is a cochain S such that*

$$d(S) = R \otimes R.$$

Proof. Since R is homologous to R^2 in the cobar complex $\Omega^1 M(1,j)$ for any j, we have cochains U_j and S_j such that

$$d(U_j) \equiv R^2 - R - 2S_j \mod (v_1^j).$$

Now send this by the differential d, and we have

$$0 \equiv 2R \otimes R - 2d(S_j) \mod (v_1^j)$$

as desired. $\square$

Lemma 7.7 *There is a cochain H such that $H \equiv v_1^3 v_2^{-1} h_{11}^2 \mod (2, v_1^4)$ and*

$$d(H) \equiv 4v_1 t_1 \otimes t_1 \otimes t_1 \mod (8, v_1^4).$$

Proof. Put $H = v_1^2 v_2^{-1} d(v_2 t_1^2) - 2v_1^3 v_2^{-2} t_1^2 t_2 \otimes t_1^2 - 4v_1 v_2^{-1} D$. Here D denotes a cochain such that $d(D) \equiv t_1^4 \otimes t_1 \otimes t_1 + t_1^2 \otimes t_1^2 \otimes t_1^2 \mod (2)$ given in [13, p.149]. Then, we obtain

$$d(H) \equiv 4v_1 t_1 \otimes t_1 \otimes t_1 \mod (8, 4v_1^2, v_1^4).$$

from computation

$$\begin{aligned}
&d(v_1^2 v_2^{-1} d(v_2 t_1^2)) \\
&\equiv v_1^2 d(v_2^{-1}) \otimes d(v_2 t_1^2) + (4v_1 t_1 + 4t_1^2) \otimes v_2^{-1} d(v_2 t_1^2) \\
&\equiv v_1^2 v_2^{-2}(-v_1 t_1^2 + 2t_2) \otimes (-v_1 t_1^2 + 2t_2) \otimes t_1^2 \\
&\qquad -2v_1^2 v_2^{-2}(-v_1 t_1^2 + 2t_2) \otimes v_2 t_1 \otimes t_1 + 4v_1 v_2^{-1} t_1^2 \otimes t_1^2 \otimes t_1^2 \\
&\equiv 2v_1^3 v_2^{-2} t_1^3 \otimes t_1^2 \otimes t_1^2 - 2v_1^3 v_2^{-2} t_1^2 \otimes t_2 \otimes t_1^2 \\
&\qquad -2v_1^3 v_2^{-2} t_2 \otimes t_1^2 \otimes t_1^2 + 2v_1^3 v_2^{-2} t_1^2 \otimes v_2 t_1 \otimes t_1 + 4v_1 v_2^{-1} t_1^2 \otimes t_1^2 \otimes t_1^2 \\
&d(-2v_1^3 v_2^{-2} t_1^2 t_2 \otimes t_1^2) \\
&\equiv 2v_1^3 v_2^{-2}(t_1^3 \otimes t_1^2 + t_1 \otimes t_1^4 + t_1^2 \otimes t_2 + t_2 \otimes t_1^2) \otimes t_1^2 \\
&d(-4v_1 v_2^{-1} D) \\
&\equiv -4v_1 v_2^{-1}(t_1^4 \otimes t_1 \otimes t_1 + t_1^2 \otimes t_1^2 \otimes t_1^2).
\end{aligned}$$

mod $(8, 4v_1^2, v_1^4)$. Here note that $v_2 t_1 \equiv t_1^4 \mod (2, v_1)$ in Γ by (3.2).

Thus we put $d(H) \equiv 4v_1 t_1 \otimes t_1 \otimes t_1 + 4v_1^2 C_1 \mod (8, v_1^4)$ for some C_1. Since $d(C_1) \equiv 0 \mod (2, v_1^2)$, C_1 represents an element of $H^{3,4}M(1,2) = \boldsymbol{Z}/2\{h_{10}^2 \rho_2, v_1 v_2^{-1} h_{11}^2 \rho_2\}$. Since $d(v_1 t_1 \otimes R) = 2t_1 \otimes t_1 \otimes R$, we may assume that C_1 represents a multiple of $v_1 v_2^{-1} h_{11}^2 \rho_2$. Checking the above computation carefully shows that there appears no R and we see that $v_1^2 C_1$ is homologous to zero. After suitable replacement, we may take $C_1 = 0$. $\square$

Lemma 7.8 *There is a cochain C of $\Omega^{3,0}M(4, 2^n)$ for $n > 4$ such that $d(C) \equiv 8G \mod (16, v_1^{2^n})$ and C represents the element $v_2^{-1} h_{10}^3 \in H^{3,0}M(1, 2^n)$. Furthermore, $v_1 C$ is bounded in $\Omega^{3,0}M(3, 2^n)$.*

Proof. We have $d(H) \equiv v_1^4 v_2^{-1} t_1 \otimes t_1 \otimes t_1 + \cdots \bmod (2)$. Therefore, we see that

$$d(H) \equiv 4v_1 t_1 \otimes t_1 \otimes t_1 + y_1^2 C + 8U + 8C_2 \mod (16, v_1^{2^n}) \tag{7.9}$$

for some C_2, $C = v_2^{-1} t_1 \otimes t_1 \otimes t_1 + \dots$ and U in the proof of Lemma 3.6. Note that $d(C) \equiv 0 \bmod (8, v_1^{2^n})$ and $d(C_2) \equiv 0 \bmod (2, v_1^4)$. Then we may put $d(C_2) \equiv v_1^4 C_3 \bmod (2)$ for some C_3. The cochain C_3 does not represents g since g/v_1 is a generator of $H^4 M_1^1$. Since we see that $d(C) \equiv 8G + 8C_3$, we replace the representative G by $G + C_3$ to obtain the lemma. □

Corollary 7.10 *There exists a cocycle G of $\Omega^{4,0} M(i+1, 2^i s)$ for any i, s that represents the polynomial generator g.*

Lemma 7.11 *There is a cochain C' of $\Omega^{3,0} M(4,8)$ for $n > 4$ such that $d(C') \equiv 8v_1^7 v_2 G \bmod (16, v_1^8)$ and C' represents the element $v_1^7 h_{10}^3 \in H^{3,0} M(1,8)$.*

Proof. By Lemmas 3.12 and 7.8, we compute

$$\begin{aligned}
&d(v_1^7 v_2 C) \\
&\equiv -2v_1^6 r_1 \otimes C + 8v_1^7 v_2 G \mod (16, v_1^8) \\
&\equiv -2v_1^2 r_1 \otimes (d(H) - 4v_1 t_1 \otimes t_1 \otimes t_1) + 8v_1^7 v_2 G \bmod (16, v_1^8) \text{ by (7.9)} \\
&d(2v_1^2 r_1 \otimes H) \\
&\equiv 2v_1^2 r_1 \otimes d(H), \quad \text{since } d(v_1^2 r_1) \equiv 0 \bmod (8) \text{ by Lemma 4.2.}
\end{aligned}$$

Since we see that $v_1^3 r_1 \otimes t_1 \otimes t_1 \otimes t_1$ is homologous to $v_1^6 \widetilde{\zeta} h_0^3 \bmod (2, v_1^8)$, we have the desired cocycle. □

Lemma 7.12 *In the cobar complex $\Omega^2 M(n+2, 2^{n+1})$, there exists a cochain $z_{n,s}$ for $n > 3$ and odd s such that $z_{n,s} \equiv v_1^{2^{n+1}-4} v_2^{2^n s} \widetilde{z} \bmod (2, v_1^{2^{n+1}})$ and*

$$d(z_{n,s}) = 2^{n+1} v_1^{2^{n+1}-4} \widetilde{z} \otimes v_2^{2^n s} Z + 2^{n+1} v_1^{2^{n+1}-2} X',$$

for some X'. Here Z is the element given in Lemma 3.9.

Proof. By Lemma 5.2,

$$\begin{aligned}
&d(v_1^{2^{n+1}} \eta_R(v_2^{2^n s}) r) \\
&= v_1^{2^{n+1}} r \otimes d(v_2^{2^n s}) \\
&= \sum_{k \geq 1} \frac{(-16)^{k-1}}{k} v_1^{2^{n+1}} (\widetilde{z}/y)^k \otimes d(v_2^{2^n s}) \\
&= \sum_{k \geq 1} \sum_{i \geq 0} \binom{k+i-1}{i} \frac{(-1)^{k+i-1} 2^{4k-4+3i}}{k} v_1^{2^{n+1}-4k-3i} v_2^i \widetilde{z} \otimes d(v_2^{2^n s})
\end{aligned}$$

in $\Omega^2M(n+2,2^{n+1})$. Here we see that

$$\begin{aligned}&2^{4k-4+3i}v_1^{2^{n+1}-4k-3i}\otimes d(v_2^{2^ns})\\ &=2^{4k-4+3i}v_1^{2^{n+1}-4k-3i}\otimes((v_2^{2^{4k+3i-5}}+v_1^{2^{4k+3i-5}}A^{2^{4k+3i-5}})^{2^{n+2-4k-3i+3}s}\\ &\qquad -v_2^{2^ns})\end{aligned}$$

equals zero if $2^{4k+3i-5}\geq 4k+3i$, which is satisfied when $k\geq 2$ and when $k=1$ and $i\geq 2$. Therefore,

$$\begin{aligned}&d(v_1^{2^{n+1}}\eta_R(v_2^{2^ns})r)\\ &=-v_1^{2^{n+1}-4}\widetilde{z}\otimes d(v_2^{2^ns})+2^3v_1^{2^{n+1}-7}v_2\widetilde{z}\otimes d(v_2^{2^ns})\\ &=-v_1^{2^{n+1}-4}\widetilde{z}\otimes d(v_2^{2^ns})+2^{n+1}sv_1^{2^{n+1}-7}v_2\widetilde{z}\otimes(v_1^4v_2^{2^ns-4}t_1^8)\\ &=-v_1^{2^{n+1}-4}\widetilde{z}\otimes d(v_2^{2^ns})+2^{n+1}sv_1^{2^{n+1}-3}v_2^{2^ns-3}\widetilde{z}\otimes t_1^8\\ &=v_1^{2^{n+1}-4}\widetilde{z}\otimes(2^{n-1}v_2^{2^ns-2}(v_1^2(3r-\widetilde{z})+4v_2^2Z)\\ &\qquad -2^{n+1}v_1^2v_2^{2^ns-2}(3r-\widetilde{z})Z)+2^{n+1}sv_1^{2^{n+1}-3}v_2^{2^ns-3}\widetilde{z}\otimes t_1^8\\ &=2^{n-1}v_1^{2^{n+1}-4}\widetilde{z}\otimes v_2^{2^ns-2}(v_1^2(3r-\widetilde{z})+4v_2^2Z)\\ &\qquad -2^{n+1}v_1^{2^{n+1}-2}\widetilde{z}\otimes v_2^{2^ns-2}(3r-\widetilde{z})Z)\\ &\qquad +2^{n+1}sv_1^{2^{n+1}-3}v_2^{2^ns-3}\widetilde{z}\otimes t_1^8\end{aligned}$$

Next consider $Y=-3\widetilde{z}\eta_R(v_1^{15}v_2)+v_1^8\widetilde{v_1^6\widetilde{z}^2}$ for the cochain $\widetilde{v_1^6\widetilde{z}^2}$ of Lemma 3.13, and we compute

$$d(Y)=2v_1^{14}\widetilde{z}\otimes(3r-\widetilde{z})+8v_1^{14}X,$$

for some X by Lemmas 3.12 and 3.13, which equals

$$d(Y)=2v_1^{12}\widetilde{z}\otimes v_1^2(3r-\widetilde{z})+8v_1^{12}(v_1t_1+t_1^2)\widetilde{z}\otimes(3r-\widetilde{z})+8v_1^{14}X.$$

Put $z_{n,s}=v_1^{2^{n+1}}\eta_R(v_2^{2^ns})r-2^{n-2}v_1^{2^{n+1}-16}v_2^{2^ns-2}Y$ and we obtain the lemma. Indeed, $2^{n+1}sv_1^{2^{n+1}-3}v_2^{2^ns-3}\widetilde{z}\otimes t_1^8$ is of the form $2^{n+1}v_1^{2^{n+1}-2}X$ for some X since $\widetilde{z}\equiv v_1v_2z$ mod $(2,v_1^2)$. □

Lemma 7.13 *Suppose* $n>4$. *Then there is a cocycle* $\widetilde{zz_n}$ *of* $\Omega^2M(n+1,2^n)$ *such that* $\widetilde{zz_n}\equiv v_1^{2^n-4}\widetilde{z}\otimes z^4+v_1^{2^n-2}Y_1+2v_1^{2^n-4}W+2Y_2$ mod $(4,v_1^{2^n})$ *for the cochain* W *in the proof of Lemma 3.6 and cocycles* Y_i $(i=1,2)$ *such that* Y_1 *represents a linear combination of* h_{10}^2, $2v_1h_{10}\rho_2$ *and* $h_{11}\rho_2$.

Proof. Lemma 7.12 implies that $c=d(z_{2n,1})/2^{2n+1}$ yields a cocycle of $\Omega^2M(n+1,2^{2n+1})$ whose leading term is $v_1^{2^{2n+1}-4}\widetilde{z}\otimes v_2^{2^{2n}}Z$. Consider the exact sequence $M(n+1,2^n)\xrightarrow{v_1^{2^{2n+1}-2^n}}M(n+1,2^{2n+1})\longrightarrow M(n+$

$1, 2^{2n+1} - 2^n)$. Then by the definition of $zz_{2n,1}$ we see that c is pulled back to c' of $M(n+1, 2^n)$ whose leading term is $v_1^{2^n-4}\widetilde{z} \otimes v_2^{2^{2n}} Z$. In fact, we see that $c = 0$ in $\Omega^2 M(3n+3, 2^{2n+1} - 2^n)$ by the computation

$$\begin{aligned}
2^{2n+1} c &= d(v_1^{2^{2n+1}} \eta_R(v_2^{2^{2n}}) r) \\
&= \sum_{k\geq 1} \sum_{i \geq 0} \binom{k+i-1}{i} \\
&\quad \frac{(-1)^{k+i-1} 2^{4k-4+3i}}{k} v_1^{2^{2n+1}-4k-3i} v_2^i \widetilde{z} \otimes d(v_2^{2^{2n}} s) \\
&= 0,
\end{aligned}$$

since $2^n - 4 \geq 3n + 3$ if $n > 4$. Then there is an isomorphism $v_2^{2^{2n}} \colon M(n+1, 2^n) \to M(n+1, 2^n)$ of Γ-comodules and we obtain a cocycle $\widetilde{zz_n}'$ such that

$$\widetilde{zz_n}' \equiv v_1^{2^n-4} \widetilde{z} \otimes Z + v_1^{2^n-2} X'' \mod (2, v_1^{2^n})$$

for some X''. Since $\widetilde{z} \otimes Z$ is homologous to $\widetilde{z} \otimes z^{2^5}$ mod $(2, v_1^2)$, we replace it by $\widetilde{zz_n}$ such that

$$\widetilde{zz_n} \equiv v_1^{2^n-4} \widetilde{z} \otimes z^{2^5} + v_1^{2^n-2} Y_1 + 2 v_1^{2^n-4} W + 2 Y_2 \mod (8, v_1^{2^n})$$

for some Y_1 and Y_2. Then we have $v_1^{2^n-2} d(Y_1) \equiv 2d(Y_2)$ mod $(4, v_1^{2^n})$, and Y_1 represents an element of $H^{2,4}M(1,2) = \boldsymbol{Z}/2\{h_{10}^2, v_1 h_{10} \rho_2, h_{11}\rho_2, v_1 v_2^{-1} h_{11}^2\}$. Therefore, we see that $d(Y_2) \equiv k t_1 \otimes t_1 \otimes R$ for some $k \in \boldsymbol{Z}/2$. Since $h_{10}^2 \rho_2 / v_1$ is not zero in $H^1 M_1^1$, we see that $k = 0$. Thus Y_1 represents $k_1 h_{10}^2 + 2k_2 v_1 h_{10} \rho_2 + k_3 h_{11} \rho_2 + k_4 v_1 v_2^{-1} h_{11}^2$ of $H^{2,4}M(2,2)$ for some $k_i \in \boldsymbol{Z}/2$. Then we have $d(Y_2) \equiv 2k_4 v_1^{2^n-1} t_1 \otimes t_1 \otimes t_1$ by Lemma 7.7. If $k_4 \not\equiv 0$, then $v_1^{2^n-1} t_1 \otimes t_1 \otimes t_1$ represents an element of $H^3 M(3, 2^n)$. Consider the composite

$$H^3 M(3, 2^n) \xrightarrow{1/8v_1^{2^n}} H^3 M_0^2 \xrightarrow{\delta} H^4 M_1^1,$$

where δ is the connecting homomorphism associated to the short exact sequence (1.3), which sends $v_1^{2^n-1} t_1 \otimes t_1 \otimes t_1$ to $v_2 g / v_1$ by Lemma 7.11. On the other hand, $h_{10}^3 / 8v_1 = [t_1 \otimes t_1 \otimes t_1 / 8v_1] = [d(Y_1)/16 v_1^{2^n}] = 0$, which contradict that $v_2 g / v_1$ is a generator. Therefore, $k_4 = 0$. □

Proposition 7.14 *There is an element $\beta \in H^{3,0}M(n, 2^n)$ for each $n > 4$ which goes to $\beta \in H^{3,0}M(1, 2^n)$ by the projection.*

Proof. Consider a commutative diagram with exact sequences

$$\begin{tikzcd}
H^{3,0}M(n,2^n) \arrow[rr, "2"] \arrow[dd, "pr"'] & & H^{3,0}M(n+1,2^n) \arrow[r, "pr"] \arrow[dd, "pr"] & H^{3,0}M(1,2^n) \arrow[dd, equal] \\
& H^{2,2^{n+1}}M(n+1,2^n) \arrow[ur, "\delta"] \arrow[dd, "pr"] & & \\
H^{3,0}M(1,2^n) \arrow[rr, "2"] & & H^{3,0}M(2,2^n) \arrow[r, "pr"] & H^{3,0}M(1,2^n) \\
& H^{2,2^{n+1}}M(2,2^n) \arrow[ur, "\delta"] & &
\end{tikzcd}$$

where δ denotes the connecting homomorphism associated to the short exact sequence $0 \to M(i,2^n) \xrightarrow{v_1^{2^n}} M(i,2^{2n}) \to M(i,2^n) \to 0$. We denote an element represented by $\widetilde{zz_n}$ by $\widetilde{\zeta\zeta} \in H^{2,2^{n+1}}M(n+1,2^n)$. By Lemma 7.13, we see that $pr(\widetilde{\zeta\zeta}) = v_1^{2^n-4}\widetilde{\zeta}\zeta + v_1^{2^n-2}\xi_1 + 2\xi_2$ for some ξ_i such that ξ_1 is a linear combination of h_{10}^2, $2v_1h_{10}\rho_2$ and $h_{11}\rho_2$. Since δ sends ξ_1 to zero, $\delta(pr(\widetilde{\zeta\zeta})) = 2\beta + 2\delta(\xi_2)$ and so $pr\delta(pr(\widetilde{\zeta\zeta})) = 0$. Therefore, we see that $pr(\widetilde{2\beta}) = 0$ for $\widetilde{2\beta} = \delta(\widetilde{\zeta\zeta}) \in H^{3,0}M(n+1,2^n)$ such that $pr(\widetilde{2\beta}) = 2\beta + 2\delta(\xi_2)$. Thus we obtain $\beta \in H^{3,0}M(n,2^n)$ such that $2\beta = \widetilde{2\beta}$ as desired. □

Lemma 7.15 *Let B denote a cochain that represents β of the above lemma. Then there is a cochain W such that $d(W) \equiv \widetilde{z}\otimes ZZ_5 + v_1^4B + 4v_1W_1 + 2v_1^2W_2 \bmod (8, v_1^8)$ for some cochains W_i, where W_1 represents a linear combination of h_{10}^3, $v_2\zeta^2h_0$ and $v_2\zeta h_0\rho_2$.*

Proof. As in the proof of Lemma 3.6, we have a cochain W such that

$$d(W) = \widetilde{z}\otimes ZZ_4 + v_1^4B + 2X_1 + 2v_1^2X_2 \tag{7.16}$$

for some cochains X_1 and X_2. Then we have $0 \equiv 2d(X_1) + 2v_1^2d(X_2) \bmod (8, v_1^{2^n})$ under the differential d, which shows that $X_1 + v_1^2X_2$ represents an element of $H^{3,8}M(2,4)$. We read off that $H^{3,8}M(1,4)$ is the $\boldsymbol{Z}/2$-module generated by $h_{11}^2\rho_2$, $v_1h_{10}^3$, $\widetilde{\zeta}h_0^2$, $\widetilde{\zeta}\zeta h_0$, $\widetilde{\zeta}h_0\rho_2$, $\widetilde{\zeta}\zeta\rho_2$, $v_1^2h_{10}^2\rho_2$, and $v_1^3v_2^{-1}h_{11}^2\rho_2$ from the structure (2.1) of $H^*K(2)_*$. (Here $v_1v_2\beta$ is taken away since $\delta(v_1v_2\beta) = v_1^2h_{11}\beta$.) Therefore, we see that $2X_1$ represents a linear combination of $v_1h_{10}^3$, $4\widetilde{\zeta}\zeta h_0$, $4\widetilde{\zeta}h_0\rho_2$ and $2\widetilde{\zeta}\zeta\rho_2$, and $2v_1^2X_2$ is a multiple of $v_1^3v_2^{-1}h_{11}^2\rho_2$. If we consider the equation (7.16) mod $(8, v_1^8)$, then we see that $2X_1$ represents a linear combination of $4v_1h_{10}^3$ $4\widetilde{\zeta}\zeta h_0$ and $4\widetilde{\zeta}h_0\rho_2$, since δ sends those generators to zero except for $\delta(2\widetilde{\zeta}\zeta\rho_2) = v_1^4\beta\rho_2$. Here δ denotes the connecting homomorphism associated to the short exact sequence $0 \to M(1,8) \xrightarrow{4} M(3,8) \to M(2,8) \to 0$. Now put $v_1W_1 = X_1$ and $W_2 = X_2$, and we see the lemma. □

8 The connecting homomorphism on J

Proposition 8.1 *For the connecting homomorphism $\delta : H^*M_0^2 \to H^{*+1}M_1^1$, $\delta(\zeta h_0^i/2v_1^j) = \zeta^2 h_0^i/v_1^j + (i+j)\zeta h_0^{i+1}/v_1^j$ for each $i \geq 0$ and $j > 0$.*

Proof. Put $t_1^{\otimes i} = t_1 \otimes \cdots \otimes t_1 \in \Omega^i A$. Then $z \otimes t_1^{\otimes i}/v_1^{i+j}$ represents $\zeta h_0^i/v_1^j$ of $H^{i+1}M_1^1$. Take an integer n to be $2^{n-1} > i+j$, and we compute $d(z^{2^n} \otimes t_1^{\otimes i}/4v_1^{i+j}) = z^{2^{n-1}} \otimes z^{2^{n-1}} \otimes t_1^{\otimes i}/2v_1^{i+j} + (i+j)z^{2^n} \otimes t_1^{\otimes(i+1)}/2v_1^{i+j+1}$. □

The above proposition implies immediately the following

Lemma 8.2 *We have the exact sequence*

$$0 \longrightarrow JC \xrightarrow{1/2} \widetilde{J} \xrightarrow{2} \widetilde{J} \xrightarrow{\delta} JI \longrightarrow 0.$$

9 The connecting homomorphism on K

Consider the submodules KC and KI:

$$KC = \boldsymbol{Z}/2\{h_{10}^3/v_1, v_2^{-1}h_{11}^2/v_1\} \bigotimes \Lambda(\beta) \bigotimes \boldsymbol{Z}/2[g] \bigoplus \boldsymbol{Z}/2\{v_2/v_1, \zeta\widetilde{\zeta}/v_1^2\}$$
$$KI = \boldsymbol{Z}/2\{v_2 g/v_1, h_{10}^3/v_1^3\} \bigotimes \Lambda(\beta) \bigotimes \boldsymbol{Z}/2[g] \bigoplus \boldsymbol{Z}/2\{\widetilde{\zeta}/v_1^2, v_2\beta/v_1\}.$$

Then $K = KI \bigoplus KC$.

Proposition 9.1 *For the generators of KC, we have*

1. $\delta(h_{10}^3/8v_1) = v_2 g/v_1$
2. $\delta(v_2^{-1}h_{11}^2/4v_1) = h_{10}^3/v_1^3$
3. $\delta(h_{10}^3\beta/8v_1) = v_2\beta g/v_1$
4. $\delta(v_2^{-1}h_{11}^2\beta/4v_1) = h_{10}^3\beta/v_1^3$
5. $\delta(v_2/2v_1) = v_2\zeta/v_1 = \widetilde{\zeta}/v_1^2$
6. $\delta(\zeta\widetilde{\zeta}/8v_1^2) = v_2\beta/v_1$

Proof. We prove them one by one.

1. Lemma 7.11 shows this.
2. This follows from Lemma 7.7.
3. By Proposition 7.14, $\delta(\xi\beta) = \delta(\xi)\beta$ for any $\xi \in H^*M_0^2$. Therefore, this follows from the first one.
4. This also follows from Proposition 7.14 and the second one.

5. This is shown in [10].
6. Note that $r \equiv v_1^{-4}\widetilde{z}$ mod (8) and $d(r) = 0$ by Lemma 5.2. Then we compute

$$d(y_1 r \otimes z^{2^6}/16) = r_1 \otimes \widetilde{z} \otimes z^{2^6}/2v_1^6 + y_1\widetilde{z} \otimes ZZ_5/8.$$

for y_1 of Lemma 4.2 and ZZ_5 of Lemma 3.6. We see that the first term is bounded by computation:

$$\begin{aligned} d(x_1\widetilde{z} \otimes z^{2^5}/2v_1^8) &= (r_1 + \widetilde{\underline{z}}_1) \otimes \widetilde{z} \otimes z^{2^5}/2v_1^6 \\ d(\widetilde{z}^2 \otimes z^{2^5}/4v_1^6) &= \underline{\widetilde{z} \otimes \widetilde{z} \otimes z^{2^5}/2v_1^6}_1 + \underline{\widetilde{z}^2 \otimes z^{2^4} \otimes z^{2^4}/2v_1^6}_2 \\ d(W^2/2v_1^6) &= \underline{\widetilde{z}^2 \otimes z^{2^4} \otimes z^{2^4}/2v_1^6}_2. \end{aligned}$$

On the other hand, by Lemma 7.15, we see that

$$d(y_1 W/8v_1^4) = y_1\widetilde{z} \otimes ZZ_5/8v_1^4 + v_2 B/2v_1 + W_1/2v_1$$

as desired. □

Note that Corollary 7.10 implies the following

Lemma 9.2 *If $\delta(\xi/2^i v_1^j) = \chi/v_1^k$ for $i \le 3$, then $\delta(\xi g^l/2^i v_1^j) = \chi g^l/v_1^k$ for $l \ge 0$.*

Now these imply immediately the following

Lemma 9.3 *We have the exact sequence*

$$0 \longrightarrow KC \xrightarrow{1/2} \widetilde{K} \xrightarrow{2} \widetilde{K} \longrightarrow KI \longrightarrow 0.$$

10 The connecting homomorphism on E

Consider the submodules EC and EI:

$$\begin{aligned} EC &= \boldsymbol{Z}/2\{v_2^{4s}/v_1^{2j}, v_2^{8t+6}\widetilde{\zeta}/v_1^{2j'} \mid s,t \in \boldsymbol{Z}, 2 \nmid s, j = 1,2,3, j' = 1,2\} \\ EI &= \boldsymbol{Z}/2\{v_2^{4s}\widetilde{\zeta}/v_1^6, v_1^{8t+2}\widetilde{\zeta}/v_1^{2j'}, v_2^{4s+2}\zeta\widetilde{\zeta}/v_1^{2j'} \mid \\ &\qquad\qquad s,t \in \boldsymbol{Z}, 2 \nmid s, j' = 1,2\}. \end{aligned}$$

Then $E = EI \bigoplus EC$, and we have the exact sequence.

Lemma 10.1 *We have the exact sequence*

$$0 \longrightarrow EC \xrightarrow{1/2} \widetilde{E} \xrightarrow{2} \widetilde{E} \xrightarrow{\delta} EI \longrightarrow 0.$$

This follows immediately from the following:

Proposition 10.2 *For the generators of EC, we have*

$$\begin{array}{ll} 1. & \delta(v_2^{4s}/8v_1^2) = v_2^{4s}\widetilde{\zeta}/v_1^6 \\ 2. & \delta(v_2^{4s}/2v_1^{2j}) = v_2^{4s-2}\widetilde{\zeta}/v_1^{2j-2} \qquad (j=2,3) \\ 3. & \delta(v_2^{8t+6}\widetilde{\zeta}/4v_1^{2j}) = v_2^{8t+6}\zeta\widetilde{\zeta}/v_1^{2j} \qquad (j=1,2) \end{array}$$

Proof. The first two equations are shown in [10, Lemma 3.17].

For the third, we compute by Lemma 3.9,

$$\begin{aligned} d(v_2^{8t+6}\widetilde{z}/8v_1^{2j}) &\equiv d(v_1^{4-2j}v_2^{8t+6}\widetilde{z}/8v_1^4) \\ &\equiv j(t_1^2+v_1t_1)\otimes v_2^{8t+6}\widetilde{z}/2v_1^{2j+2} \\ &\quad -v_2^{8t+4}(r_1+\widetilde{z})\otimes\widetilde{z}/8v_1^{2j-2}+v_2^{8t+6}Z\otimes\widetilde{z}/2v_1^{2j}. \end{aligned} \tag{10.3}$$

If $j=1$, then the first term is $v_2^{8t+5}t_1^4\otimes v_2z/2v_1^2$, since

$$d(v_2^{8t+7}\widetilde{z}/2v_1^5) \equiv (v_1t_1^2+v_1^2t_1)\otimes v_2^{8t+6}\widetilde{z}/2v_1^5+v_2^{8t+5}t_1^4\otimes\widetilde{z}/2v_1^3.$$

Suppose that $j=2$. Then $d(v_2^{8t+4}(v_1^{15}v_2)\widetilde{z}/16v_1^{16}) = v_2^{8t+4}Z\otimes v_1v_2\widetilde{z}/2v_1^2 + v_2^{8t+4}r_1\otimes\widetilde{z}/8v_1^2$ by Lemma 3.12 and $d(v_2^{8t+4}\widetilde{z}^2/16y_1) = r_1\otimes v_2^{8t+4}\widetilde{z}^2/2v_1^6+(v_1^{-1}v_2)(r_1+\widetilde{z})\otimes\widetilde{z}^2/4v_1^2 + v_2^{8t+4}Z\otimes\widetilde{z}^2/2v_1^2 - v_2^{8t+4}\widetilde{z}\otimes\widetilde{z}/8v_1^2+v_2^{8t+5}\widetilde{z}\otimes\widetilde{z}/2v_1^5$. Notice that $\widetilde{z}^2/4v_1^2 = 0$. These imply that the second term of (10.3) is homologous to an element of the form $x/2v_1^3$ with $x\in\Gamma\bigotimes_A\Gamma$. □

11 The connecting homomorphism on F

Consider the submodules FC and FI such that $F=FC\bigoplus FI$:

$$\begin{aligned} FC = \sum_{n\ge 2, 2\nmid s} v_2^{2^ns}\Big(&\boldsymbol{Z}/2\{v_2^{-1}h_{11}^2/v_1\}\bigotimes\Lambda(\beta) \\ &\bigoplus\boldsymbol{Z}/2\{h_{10}^3/v_1, \widetilde{\zeta}^{(n-1)}\widetilde{\zeta}g/v_1^{2^{n+1}+2}\}\Big)\bigotimes\boldsymbol{Z}/2[g] \\ FI = \sum_{n\ge 2, 2\nmid s} v_2^{2^ns}\Big(&\boldsymbol{Z}/2\{v_2g/v_1\}\bigotimes\Lambda(\beta) \\ &\bigoplus\boldsymbol{Z}/2\{h_{10}^3/v_1^3, \widetilde{\zeta}^{(n-1)}\widetilde{\zeta}g/v_1^{2^{n+1}+4}\}\Big)\bigotimes\boldsymbol{Z}/2[g]. \end{aligned}$$

Lemma 11.1 *We have the exact sequence*

$$0\longrightarrow FC\xrightarrow{1/2}\widetilde{F}\xrightarrow{2}\widetilde{F}\xrightarrow{\delta}FI\longrightarrow 0.$$

This follows from Lemma 9.2 and

Proposition 11.2 *The connecting homomorphism δ acts on the generators of $\widetilde{F}$ as follows :*

1. $$\delta(v_2^{2^ns-1}h_{11}^2/4v_1) = v_2^{2^ns}h_{10}^3/v_1^3$$
2. $$\delta(v_2^{2^ns-1}h_{11}^2\beta/8v_1) = v_2^{2^ns}\widetilde{\zeta}^{(n-1)}\widetilde{\zeta}g/v_1^{2^{n+1}+4}$$
3. $$\delta(v_2^{2^ns}h_{10}^3/8v_1) = v_2^{2^ns+1}g/v_1$$
4. $$\delta(v_2^{2^ns}\widetilde{\zeta}^{(n-1)}\widetilde{\zeta}g/4v_1^{2^{n+1}+2}) = v_2^{2^ns+1}\beta g/v_1$$

Proof. We prove them one by one.

1. This follows from the second one of Proposition 9.1.
2. Suppose that $n > 4$. Then $d(v_2^{2^n}) \equiv 0 \bmod (16, y_1^2)$. For the cochain H of Lemma 7.7 and the cocycle B of Lemma 7.14, we have
$$d(v_2^{2^ns}B\otimes H/16y_1^2) = v_2^{2^ns}B\otimes t_1\otimes t_1\otimes t_1/4v_1^3 + v_2^{2^ns}B\otimes H_1/2v_1^4 + v_2^{2^ns+1}B\otimes H_2/2v_1^3.$$
We also have
$$d(v_2^{2^ns}W\otimes t_1\otimes t_1\otimes t_1/4v_1^7) = v_2^{2^ns}(\widetilde{z}\otimes ZZ_4+v_1^4B)\otimes t_1\otimes t_1\otimes t_1/4v_1^7$$
for W of Lemma 7.14. Since $ZZ_4 \equiv z^8\otimes z^8 \bmod (2, v_1^8)$,
(11.3)
$$\begin{aligned}
&d(v_2^{2^ns}\widetilde{z}^{2^{n-1}}\otimes\widetilde{z}\otimes t_1\otimes t_1\otimes t_1/4v_1^{2^{n+1}+7})\\
&= v_1^{2^n}v_2^{2^n(s-1)}\widetilde{z}^{2^{n-1}}\otimes\widetilde{z}^{2^{n-1}}\otimes\widetilde{z}\otimes t_1\otimes t_1\otimes t_1/4v_1^{2^{n+1}+7}\\
&\quad +2t_1\otimes v_2^{2^ns}\widetilde{z}^{2^{n-1}}\otimes\widetilde{z}\otimes t_1\otimes t_1\otimes t_1/4v_1^{2^{n+1}+8}\\
&\quad +2v_1^{2^{n-1}}v_2^{2^{n-1}(2s-1)}\widetilde{z}^{2^{n-2}}\otimes\widehat{z}^{(n-1)}\otimes\widetilde{z}\otimes t_1\otimes t_1\otimes t_1/4v_1^{2^{n+1}+7}\\
&\quad +2v_2^{2^ns}\widetilde{z}^{2^{n-1}}\otimes\widetilde{z}^{2^{n-1}}\otimes\widetilde{z}\otimes t_1\otimes t_1\otimes t_1/4v_1^{2^{n+1}+7}\\
&= v_2^{2^ns}z^{2^{n-1}}\otimes z^{2^{n-1}}\otimes\widetilde{z}\otimes t_1\otimes t_1\otimes t_1/4v_1^7\\
&\quad +t_1\otimes v_2^{2^ns}\widetilde{z}^{2^{n-1}}\otimes\widetilde{z}\otimes t_1\otimes t_1\otimes t_1/2v_1^{2^{n+1}+8}\\
&\quad +v_1^{2^{n-1}}v_2^{2^{n-1}(2s-1)}\widetilde{z}^{2^{n-2}}\otimes\widehat{z}^{(n-1)}\otimes\widetilde{z}\otimes t_1\otimes t_1\otimes t_1/2v_1^{2^{n+1}+7}\\
&\quad +v_2^{2^ns}\widetilde{z}^{2^{n-1}}\otimes\widetilde{z}^{2^{n-1}}\otimes\widetilde{z}\otimes t_1\otimes t_1\otimes t_1/2v_1^{2^{n+1}+7}.
\end{aligned}$$
This shows that $v_2^{2^ns}B\otimes t_1\otimes t_1\otimes t_1/4v_1^3$ is homologous to $v_2^{2^ns}\widetilde{z}^{2^{n-1}}\otimes\widetilde{z}\otimes t_1\otimes t_1\otimes t_1\otimes t_1/2v_1^{2^{n+1}+8}$, as desired.
For $n = 4$,
$$\begin{aligned}
&d(v_2^{16s}B\otimes H/16y_1^2)\\
&= v_2^{16s-2}(r_1+\widetilde{z})\otimes B\otimes H/2v_1^2 + v_2^{16s}B\otimes t_1\otimes t_1\otimes t_1/4v_1^3\\
&\quad +v_2^{16s}B\otimes H_1/2v_1^4 + v_2^{16s+1}B\otimes H_2/2v_1^3
\end{aligned}$$

and we obtain the same result. For $n = 3$,

$$\begin{aligned} &d(v_2^{8s} B \otimes H/16y_1^2) \\ &= v_2^{8s-2}(r_1 + \widetilde{z}) \otimes B \otimes H/4v_1^2 + v_2^{8s-1} d(v_2) \otimes B \otimes H/2v_1^4 \\ &\quad + v_2^{8s} B \otimes t_1 \otimes t_1 \otimes t_1/4v_1^3 + v_2^{8s} B \otimes H_1/2v_1^4 + v_2^{8s+1} B \otimes H_2/2v_1^3 \end{aligned}$$

and $H \equiv 0 \bmod (4, v_1^2)$, which implies the same result as the above one. Similarly, for $n = 2$,

$$\begin{aligned} &d(v_2^{4s} B \otimes H/16y_1^2) \\ &= v_2^{4s-2}(r_1 + \widetilde{z}) \otimes B \otimes H/8v_1^2 + v_2^{4s-1} d(v_2) \otimes B \otimes H/4v_1^4 \\ &\quad + v_2^{4s} B \otimes t_1 \otimes t_1 \otimes t_1/4v_1^3 + v_2^{4s} B \otimes H_1/2v_1^4 + v_2^{4s+1} B \otimes H_2/2v_1^3 \end{aligned}$$

and the congruence $H \equiv 0 \bmod (4, v_1^2)$ also shows the result.

3. This follows from the first one of Proposition 9.1.
4. The cocycle $x/2$ that represents $v_2^{2^n s + 2^{n-1}} \zeta\widetilde{\zeta} g/2v_1^{2^{n+1}+2}$ is homologous to $v_2^{2^n s} t_1 \otimes t_1 \otimes t_1 \otimes B/4v_1$ by (11.3). The equation 3 above indicates the existence of a cochain c such that $d(c) = v_2^{2^n s+1} g/2v_1$ and the leading term of c is $v_2^{2^n s} t_1 \otimes t_1 \otimes t_1/16v_1$. Then $d(x/8) = d(c \otimes B) = v_2^{2^n s+1} g \otimes B/2v_1$ as desired. □

12 The connecting homomorphism on D

Proposition 12.1 *The connecting homomorphism δ behaves as follows* :

1. $\delta(v_2^{2^n s} \widetilde{\zeta}^{(n-1)} \widetilde{\zeta}/4v_1^{2^{n+1}+2}) = v_2^{2^n s+1} \beta/v_1 \quad s = 4t + 1$
2. $\delta(v_2^{2^{n-1}(4t+3)} \zeta\widetilde{\zeta}/16v_1^2) = v_2^{2^{n-1}(4t+3)+1} \beta/v_1 \quad$ *for* $n \geq 3$
3. $\delta(v_2^{2^n s+1}/2v_1) = v_2^{2^n s} \widetilde{\zeta}/v_1^2$
4. $\delta(v_2^{2^n s} \widetilde{\zeta}/2^{n+2} v_1^4) = v_2^{2^n s} \widetilde{\zeta}^{(n-1)} \widetilde{\zeta}/v_1^{2^{n+1}+4} \quad s \equiv 1\ (4)$
   $\delta(v_2^{2^n s} \widetilde{\zeta}/2^{n+1} v_1^4) = v_2^{2^n s} \zeta\widetilde{\zeta}/v_1^4 \quad s \equiv -1\ (4)$

Proof.

1. First we note that $v_2^{2^n(4t+1)} \widetilde{z}^{2^{n-1}} \otimes \widetilde{z}/2v_1^{2^{n+1}+2}$ is homologous to $v_2^{2^{n+2}t+2^n} z^{2^n} \otimes \widetilde{z}/4v_1^2$ by

$$\begin{aligned} &d(v_2^{2^{n+1}(2t+1)} \widetilde{z}/4v_1^{3\cdot 2^n+2}) \\ &= v_2^{2^n(4t+1)} \widetilde{z}^{2^{n-1}} \otimes \widetilde{z}/2v_1^{2^{n+1}+2} + v_2^{2^{n+2}t} \widetilde{z}^{2^n} \otimes \widetilde{z}/4v_1^{2^n+2}. \end{aligned}$$

We compute

$$\begin{aligned}
&d(v_2^{2^n(4t+1)}\widetilde{z}^{2^{n-1}} \otimes \widetilde{z}/8v_1^{2^{n+1}+2})\\
&= d(v_2^{2^{n+2}t+2^n} z^{2^n} \otimes \widetilde{z}/16v_1^2)\\
&= d(v_2^{2^{n+2}t+2^n}) \otimes z^{2^n} \otimes \widetilde{z}/16v_1^2 + v_2^{2^{n+2}t+2^n} d(z^{2^n} \otimes \widetilde{z}/16v_1^2).
\end{aligned}$$

If $n > 3$, $d(v_2^{2^n}) \equiv 0 \bmod (16, v_1^2)$. If $n = 3$, then $d(v_2^8) \equiv 8v_1v_2^7t_1^2 \bmod (16, v_1^2)$. Therefore, this follows from Proposition 9.1.6 if $n \geq 3$. For the case $n = 2$, it also follows from Proposition 9.1.6, since $\widetilde{z} \equiv 2(v_2^{-1}t_3 + t_1t_2) \bmod (4, v_1)$.

2. This is similar to the above one.
3. This is immediate from Proposition 9.1.5.
4. Lemma 7.12 shows

$$d(z_{n,s}/2^{n+2}v_1^{2^{n+1}}) = \widetilde{z} \otimes v_2^{2^n s}Z/2v_1^4 + X'/2v_1^2 \tag{12.2}$$

which implies the case $s \equiv -1 \bmod p$.

Since $H^{2,3\cdot 2^{n+1}}M_1^1 = \boldsymbol{Z}/2\{v_2^{2^n-1}h_{11}^2/v_1,\ v_2^{2^n+2^{n-1}}\zeta\rho_2/v_1^{3\cdot 2^{n-1}}, v_2^{2^n+2^{n-1}}\zeta\widetilde{\zeta}/v_1^{3\cdot 2^{n-1}+4}\}$, the right hand side of (12.2) for $s = 1$ is bounded, and so we have an element $U/2$ such that $r\eta_R(v_2^{2^n})/2^{n+2} - v_2^{2^n-2}Y/16v_1^{16} + U/2$ is a cocycle. Then by (11.3), we see the case where $s = 1$. In other words, there is a cochain $X = r\eta_R(v_2^{2^n})/2^{n+3} - v_2^{2^n-2}Y/32v_1^{16} + U/4 + U'/2$ for some U' and $d(X) = v_2^{2^n+2^{n-1}}z \otimes \widetilde{z}/v_1^{3\cdot 2^{n-1}+4} + \cdots$. Since we compute that $X \otimes d(v_2^{2^{n+2}}) = 0$, we have $d(v_2^{2^{n+2}}X) = v_2^{2^{n+2}}d(X)$, and obtain the case $s \equiv 1 \bmod (4)$. □

Consider the submodules DC and DI such that $D = DC \bigoplus DI$:

$$\begin{aligned}
DC &= \sum_{n\geq 2, 2\nmid s} v_2^{2^n s}\boldsymbol{Z}/2\{v_2/v_1, \widetilde{\zeta}/v_1^4, v_2^{2^{n-1}}\zeta\widetilde{\zeta}/v_1^2\}\\
&\quad \bigoplus \sum_{n\geq 2, 4|(s-1)} v_2^{2^n s}\boldsymbol{Z}/2\{\widetilde{\zeta}^{(n-1)}\widetilde{\zeta}/v_1^{2^{n+1}+2}\}\\
DI &= \sum_{n\geq 2, 2\nmid s} v_2^{2^n s}\boldsymbol{Z}/2\{v_2\beta/v_1, \widetilde{\zeta}/v_1^2, v_2^{2^{n-1}}\zeta\widetilde{\zeta}/v_1^4\}\\
&\quad \bigoplus \sum_{n\geq 2, 4|(s-1)} v_2^{2^n s}\boldsymbol{Z}/2\{\widetilde{\zeta}^{(n-1)}\widetilde{\zeta}/v_1^{2^{n+1}+4}\}
\end{aligned}$$

Then we see the following

Lemma 12.3 *We have the exact sequence*

$$0 \longrightarrow DC \xrightarrow{1} /2\widetilde{D} \xrightarrow{2} \widetilde{D} \xrightarrow{\delta} DI \longrightarrow 0.$$

13 The connecting homomorphism on L

Note that $\widetilde{L} = (1/2)_*(LC)$ for the map $(1/2)_*\colon H^*M_1^1 \longrightarrow H^*M_0^2$. We compute the δ-image of each element of $\widetilde{L}$.

Proposition 13.1 *For each element of $\widetilde{L}$, we have*

$$\delta(\xi) = \xi h_0 + \cdots.$$

Proof. Suppose that ξ is represented by an element $x/2v_1^{2j+1}$. Then we compute $d(x/4v_1^{2j+1}) = t_1/2v_1^{2j+2} \otimes x + x'/2v_1^{2j+1}$ for $d(x) = 2x'$, since $d(v_1) = 2t_1$. The part $\cdots$ is an element represented by x'/v_1^{2j+1}. □

Corollary 13.2 *The connecting homomorphism induces an isomorphism $\delta\colon \widetilde{L} \to LI$.*

Proof. Since $LI_i = h_0 LC_i$ if $i \neq 1$ and $= \widetilde{\zeta} LC_1$ if $i = 1$, the above lemma shows an isomorphism $\widetilde{LC_i} \to LI_i$ for each i. □

Lemma 13.3 *We have the exact sequence*

$$0 \longrightarrow LC \xrightarrow{1/2} \widetilde{L} \xrightarrow{2} \widetilde{L} \xrightarrow{\delta} LI \longrightarrow 0.$$

Proof. Since each generator of LI is not trivial, the isomorphism δ of Corollary 13.2 implies that each generator of $\widetilde{L}$ is not trivial, which shows that the homomorphism $1/2$ is a monomorphism. Since $1/2$ is an epimorphism by definition, we see that $1/2$ is an isomorphism. □

14 The connecting homomorphism on M

We begin with factoring M into the direct sum of MC and MI. Consider the submodules MC and MI:

$$\begin{aligned} MC = \boldsymbol{Z}/2\{&v_2^2/v_1^2, v_2^2/v_1, v_2^2h_{10}/v_1, v_2^3/v_1, v_2^3h_{10}/v_1, v_2^3h_{11}/v_1\} \\ &\bigotimes \Lambda(\beta) \bigotimes \boldsymbol{Z}/2[v_2^{\pm 4}, g] \\ MI = \boldsymbol{Z}/2\{&v_2^2h_{11}/v_1^2, v_2^2h_{10}/v_1^2, v_2h_{11}^2/v_1, v_2^3h_{10}/v_1^2, v_2^3h_{10}^2/v_1^2, \\ &v_2^3h_{10}^2/v_1\} \bigotimes \Lambda(\beta) \bigotimes \boldsymbol{Z}/2[v_2^{\pm 4}, g] \end{aligned}$$

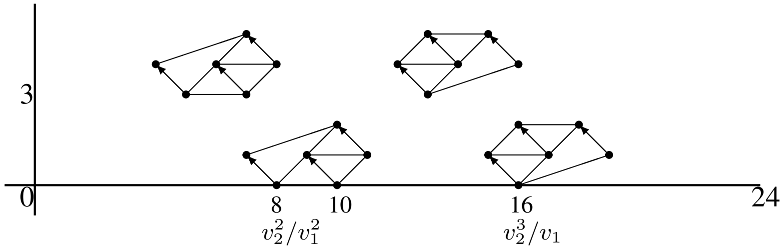
Then we have

Proposition 14.1 *For the generators of MC, we have*

$$\begin{aligned} \delta(v_2^2/2v_1^2) &= v_2h_{11}/v_1 \\ \delta(v_2^2/2v_1) &= v_2^2h_{10}/v_1^2 \\ \delta(v_2^2h_{10}/2v_1) &= v_2^2h_{10}^2/v_1^2 \\ \delta(v_2^3/2v_1) &= v_2^3h_{10}/v_1^2 \\ \delta(v_2^3h_{10}/2v_1) &= v_2^3h_{10}^2/v_1^2 \\ \delta(v_2^3h_{11}/2v_1) &= v_2^3h_{10}^2/v_1. \end{aligned}$$

Proof. The first, the second and the fourth equations are shown in [10, Lemma 3.17]. The third and the fifth are obtained immediately from the second and the fourth ones, since h_{10} is represented by t_1 which is primitive. The last one is also follows from the fourth. In fact, $v_2^3h_{10}/v_1^2$ is represented by $v_2^2r_1/v_1^2$ by (4.1) and h_{11} is represented by $t_1^2+v_1t_1$, and $v_2^2r_1\otimes(t_1^2+v_1t_1)/v_1^2$ is homologous to $v_2^2t_1\otimes t_1^4/v_1$ by $d(v_2^3t_2/v_1^2+v_2^2t_1^2t_2/v_1)$. □

This is displayed as follows:



We see the following lemma by using Lemma 7.14.

Lemma 14.2 *We have the exact sequence*

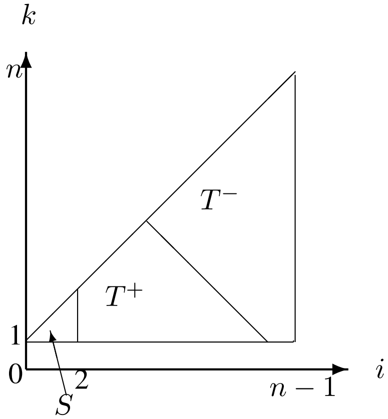
$$0 \longrightarrow MC \xrightarrow{1/2} \widetilde{M} \xrightarrow{2} \widetilde{M} \xrightarrow{\delta} MI \longrightarrow 0.$$

15 The connecting homomorphism on N

We introduce subsets of the set of triple integers:

$$\begin{aligned}
T &= \{(n,i,k)\in \boldsymbol{Z}^3 \mid n\ge 3, 2\le i\le n-1, 1\le k\le i+1\}\\
T' &= \{(n,i,k)\in T \mid (i,k)\ne(n-1,1),(n-1,n-1)\},\\
S &= \{(n,i,k)\in \boldsymbol{Z}^3 \mid n\ge 3, (i,k)=(0,1),(1,1),(1,2)\}\\
T^+ &= \{(n,i,k)\in T_0 \mid n> i+k+1\}\\
T^- &= \{(n,i,k)\in T_0 \mid n\le i+k+1\}
\end{aligned}$$

Here $(T\cup S)\cap(\{n\}\times \boldsymbol{Z}^2)$ for each n is described as follows:



Consider the modules

$$
\begin{aligned}
A(n,i,k) &= \boldsymbol{Z}/2\{v_2^{2^n s}/v_1^{2^k m} \mid 2 \not| \, sm, \quad 3\cdot 2^{i-1} < 2^k m \le 3\cdot 2^i\},\\
\hat{A}(n,n-1,1) &= \boldsymbol{Z}/2\{v_2^{2^n s}/v_1^{2m} \mid \\
&\qquad 2 \not| \, sm, \quad 3\cdot 2^{n-2} < 2m \le 3\cdot 2^{n-1}-4\},\\
Z(n,i,k) &= \boldsymbol{Z}/2\{v_2^{2^i(2^{n-i}s-1)}\widetilde{\zeta}^{(i-1)}/v_1^{2^k m} \mid \\
&\qquad 2 \not| \, sm, \quad 2^{i-1} < 2^k m \le 2^{i+1}\} \quad \text{for } i > k,\\
\hat{Z}(n,n-1,1) &= \boldsymbol{Z}/2\{v_2^{2^{n-1}(2s-1)}\widetilde{\zeta}^{(n-2)}/v_1^{2m} \mid \\
&\qquad 2 \not| \, sm, \quad 2^{n-2} < 2m \le 2^n - 4\}\\
Z(n,i,i) &= \boldsymbol{Z}/2\{v_2^{2^n s - 2^i}\widetilde{\zeta}^{(i-1)}/v_1^{2^{i+1}} \mid 2 \not| \, s\},\\
Z(n,i,i+1) &= \boldsymbol{Z}/2\{v_2^{2^n s - 2^i}\widetilde{\zeta}^{(i-1)}/v_1^{2^i} \mid 2 \not| \, s\}.
\end{aligned}
$$

Then N_i's are rewritten as follows:

$$
\begin{aligned}
N_3 &= \sum_{(n,i,k)\in T\cup S} A(n,i,k)\\
NI_1 &= \sum_{(n,i,k)\in T^+\cup S} (\widetilde{\zeta}/v_1^4)A(n,i,k)\\
NC_1 &= \sum_{(n,i,k)\in T'-T^+} (\widetilde{\zeta}/v_1^4)A(n,i,k) \bigoplus \sum_{n\ge 3} (\widetilde{\zeta}/v_1^4)\hat{A}(n,n-1,1)\\
NI_2 &= \sum_{(n,i,k)\in T^-} Z(n,i,k)\\
NC_2 &= \sum_{(n,i,k)\in T^+} Z(n,i,k)\\
N_4 &= \sum_{(n,i,k)\in T'} (\widetilde{\zeta}/v_1^4)Z(n,i,k) \bigoplus \sum_{n\ge 3} (\widetilde{\zeta}/v_1^4)\hat{Z}(n,n-1,1),
\end{aligned}
$$

in which $N_i = NI_i \bigoplus NC_i$ for $i = 1, 2$. Now N is divided into two direct summands:

$$NI = NI_1 \bigoplus NI_2 \bigoplus N_4$$
$$NC = N_3 \bigoplus NC_1 \bigoplus NC_2.$$

We read off the following from [10, Lemma 3.17].

Proposition 15.1 *The connecting homomorphism δ behaves as follows* :

$$\delta(\widetilde{A(n,i,k)}) = \begin{cases} (\widetilde{\zeta}/v_1^4)A(n,i,k) & n > i+k+1 \\ Z(n,i,k) & n \le i+k+1 \end{cases}$$

$$\delta((\widetilde{\zeta}/v_1^4)\widetilde{A(n,i,k)}) = (\widetilde{\zeta}/v_1^4)Z(n,i,k) \quad \textit{for } n \le i+k+1$$

$$\delta((\widetilde{\zeta}/v_1^4)\hat{A}\widetilde{(n,n-1,1)}) = (\widetilde{\zeta}/v_1^4)\hat{Z}(n,n-1,1)$$

$$\delta(Z\widetilde{(n,i,k)}) = (\widetilde{\zeta}/v_1^4)Z(n,i,k) \quad \textit{for } n > i+k+1.$$

Proof. The first equation is shown in [10, Lemma 3.17], and the second and the third ones are verified by Proposition 5.3.

For the last, note that

$$d(x_2^{2^n s}) \equiv 2^{n-i} s v_1^{2^i} v_2^{2^n s - 2^i} \widetilde{\zeta}^{2^{i-1}} + \cdots \quad \bmod (v_1^{3\cdot 2^i})$$

obtained from Lemma 4.3, where $\cdots$ denotes an element divisible by 2^{n-i+1}. If we write $2^{n-i}sx(n,i,k)$ as the right hand side, then $d(x(n,i,k)) \equiv 0 \bmod (v_1^{3\cdot 2^i})$. If $k < i$, we compute

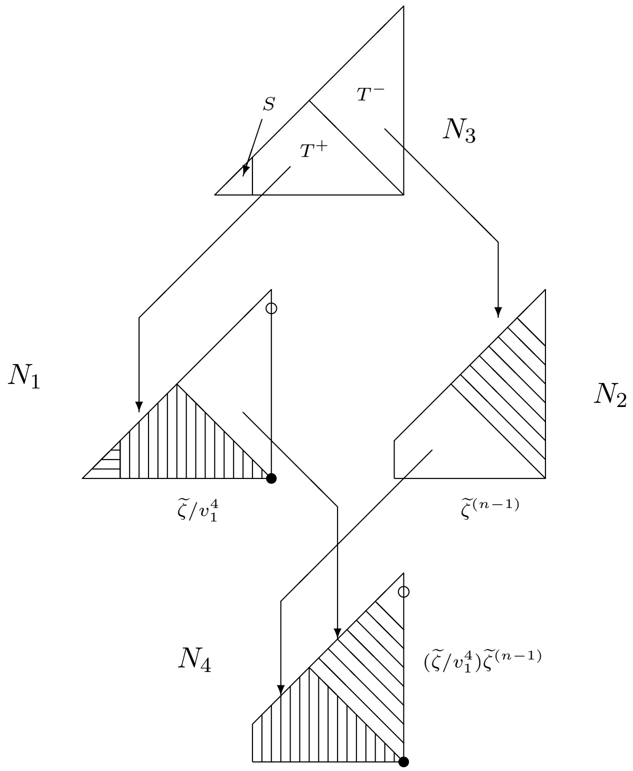
$$\begin{aligned} d(v_2^{2^n s - 2^i}\widetilde{\zeta}^{(i-1)}/2^{k+3}v_1^{2^k m}) &= d(x(n,i,k)/2^{k+3}v_1^{2^k m + 2^i}) \\ &= r \otimes x(n,i,k)/2v_1^{2^k m + 2^i + 4} \end{aligned}$$

which is homologous to $\widetilde{z} \otimes x(n,i,k)/2v_1^{2^k m + 2^i + 4}$. In the same way we see the cases for $k = i$ and $i+1$. In fact, the exponent $2^k m + 2^i$ of v_1 of the denominator is divisible by 2^{i+1} (resp. 2^i) if $k = i$ (resp. $k = i+1$). $\square$

Lemma 15.2 *We have the exact sequence*

$$0 \longrightarrow NC \xrightarrow{1/2} \widetilde{N} \xrightarrow{2} \widetilde{N} \xrightarrow{\delta} NI \longrightarrow 0.$$

This is displayed as follows:



16 The connecting homomorphism on P

Proposition 16.1 *The connecting homomorphism* δ *acts as follows*:

$$\delta(1/2^{n+2}v_1^{2^n}) = \widetilde{\zeta}/v_1^{2^n+4}$$
$$\delta(\zeta\widetilde{\zeta}/2v_1^{2j+4}) = \beta/v_1^{2j}.$$

Proof. By Lemma 4.2, we see that $d(1/2^{n+3}y_1^{2^{n-1}}) = r_1/2v_1^{2^n+4}$. Lemma 4.3 shows that $r_1/2v_1^{2^n+4}$ is homologous to $\widetilde{z}/2v_1^{2^n+4}$, which implies the first equation.

Take $n > 2j + 4$. Then, we obtain the second by the computation

$$d(\widetilde{z} \otimes z^{2^{n+1}}/4v_1^{2j+4}) = \widetilde{z} \otimes z^{2^n} \otimes z^{2^n}/2v_1^{2j+4}$$
$$d(W/2v_1^{2j+4}) = (\widetilde{z} \otimes z^{2^n} \otimes z^{2^n} + v_1^4 B)/2v_1^{2j+4},$$

where W is the one of Lemma 7.14. $\square$

Put $PC = A_\infty \bigoplus (\zeta\widetilde{\zeta}/v_1^4)A_\infty$ and $PI = (\widetilde{\zeta}/v_1^4)A_\infty \bigoplus \beta A_\infty$. Then we have

Lemma 16.2 *We have the exact sequence*

$$0 \longrightarrow PC \xrightarrow{1/2} \widetilde{P} \xrightarrow{2} \widetilde{P} \xrightarrow{\delta} PI \longrightarrow 0.$$

17 The connecting homomorphism on Q

By Lemma 7.13, we have a cocycle $\widetilde{zz}_{n-1}/2^n v_1^{2^{n-1}} \in \Omega^{2,0}M_0^2$, which represents $\widetilde{\zeta}\zeta/2^n v_1^4 \in H^{2,0}M_0^2$.

Proposition 17.1 *The connecting homomorphism $\delta : H^*M_0^2 \longrightarrow H^{*+1}M_1^1$ acts trivially on $\widetilde{\zeta}/2^n v_1^4$ and $\widetilde{\zeta}\zeta/2^n v_1^4$ for each n.*

Proof. By the definition of δ, $\delta(\widetilde{\zeta}/2^n v_1^4) = 0$ follows from Proposition 5.3. The other half follows from Lemma 7.13. $\square$

Lemma 17.2 *There is an exact sequence*

$$0 \longrightarrow Q \xrightarrow{1/2} \widetilde{Q} \xrightarrow{2} \widetilde{Q} \xrightarrow{\delta} 0\ .$$

18 The Adams-Novikov E_2-terms

We begin with restating the lemma given in [4, Remark 3.11].

Lemma 18.1 *Suppose that $H^*M_1^1$ is a direct sum of submodules M_i which is also a direct sum of two modules MI_i and MC_i. If we have a submodule $\widetilde{M_i}$ of $H^*M_0^2$ in an exact sequence*

$$(18.2) \qquad 0 \longrightarrow MC_i \xrightarrow{1/2} \widetilde{M_i} \xrightarrow{2} \widetilde{M_i} \longrightarrow MI_i \longrightarrow 0$$

for each i, where $1/2(x) = x/2$, then $H^*M_0^2$ is the direct sum of $\widetilde{M_i}$.

Proof of Theorem 2.3. Take M_i to be X and $\rho_2 X$ for $X = D$, E, F, J, K, L, M, N, P and Q. Then we have the theorem from Lemmas 12.3, 10.1, 11.1, 8.2, 9.3, 13.3, 14.2, 15.2, 16.2, 17.2 and Proposition 7.5. $\square$

Consider the exact sequences

$$\cdots \longrightarrow H^{*-1}M_0^2 \xrightarrow{\delta_1} H^*N_0^1 \xrightarrow{i_1} H^*M_0^1 \xrightarrow{j_1} H^*M_0^2 \longrightarrow \cdots,$$

associated to the short exact sequence (1.2). Consider the submodules $\widetilde{A}$ and $\widetilde{C}$ of $H^*M_0^1$:

$$\widetilde{A} = \sum_{i,2\not| s} \boldsymbol{Z}\{v_1^{2^i s}/2^{i+2}\}, \quad \widetilde{C} = v_1\boldsymbol{Z}/2[v_1^{\pm 2}, h_{10}]\bigotimes\Lambda(\rho_1).$$

Note that $H^*M_0^1 = \widetilde{A}\bigoplus\widetilde{C}\bigoplus\boldsymbol{Q}/\boldsymbol{Z}_{(2)}\bigotimes\Lambda(\rho_1)$ by (5.1). Furthermore, divide $\widetilde{C}$ into the direct sum of the six submodules $\widetilde{C}_i$ given by

$$\widetilde{C}_1 = v_1\boldsymbol{Z}/2[v_1^2]\{1, h_{10}, h_{10}^2, h_{10}^3\}\bigotimes\boldsymbol{Z}[g], \quad \widetilde{C}_2 = v_1^3\rho_1\boldsymbol{Z}/2[v_1^2, h_0^2],$$
$$\widetilde{C}_3 = v_1^3\rho_1 h_{10}\boldsymbol{Z}/2[v_1^2, h_0^2], \quad \widetilde{C}_4 = v_1 A_\infty\{1, h_{10}, h_{10}^2, h_{10}^3\}\bigotimes\boldsymbol{Z}[g],$$
$$\widetilde{C}_5 = v_1^3\rho_1 A_\infty[h_0^2] \quad \text{and} \quad \widetilde{C}_6 = v_1^3\rho_1 h_{10} A_\infty[h_0^2]$$

Then the module $\widetilde{A}$ yields submodule $\widetilde{A}^+ = \sum_{i,2\not| s>0}\boldsymbol{Z}\{v_1^{2^i s}/2^{i+2}\}$ of $H^0N_0^1$ and kills the first summand of $\widetilde{P}$. Note that j_1 assigns ρ_1 to $\widetilde{\zeta}/v_1^4$ by Proposition 5.3. The direct sum $\widetilde{C}^+ = \sum_{i=1}^3 \widetilde{C}_i$ is pulled back to $H^*N_0^1$ by i_1 and $\sum_{i=4}^6 \widetilde{C}_i$ kills the submodule $(1/2)_*(v_1A_\infty\{1, h_{10}, h_{10}^2, h_{10}^3/v_1^4\}\bigotimes\boldsymbol{Z}[g]\bigoplus\widetilde{\zeta}A_\infty\{v_1, h_0\}\bigotimes\boldsymbol{Z}/2[h_0^2])$ of $(1/2)_*(LC_0)$. Furthermore, $h_{10}^3/2v_1$ in $\widetilde{K}$ is in the image of j_1 and $h_{10}^3/2v_1^3$ yields $\boldsymbol{Z}/8$ summand of $H^3N_0^1$. In fact, the cochain C of $\Omega^3N_0^1$ given in Lemma 7.8 defines an element $v_2^{-1}h_{10}^3/8 \in H^3N_0^1$ such that $\delta_1(v_2^{-1}h_{11}^2/4v_1) = v_2^{-1}h_{10}^3/4$ for $v_2^{-1}h_{11}^2/4v_1 \in \widetilde{K}$ and $i_1(v_2^{-1}h_{10}^3/2) = h_{10}^3/2v_1^3$ by (7.9). The submodule $\boldsymbol{Q}/\boldsymbol{Z}_{(2)}\bigotimes\Lambda(\rho_1)$ of $H^*M_0^1$ also yields $\boldsymbol{Q}/\boldsymbol{Z}_{(2)}$ and kills the first summand of $\widetilde{Q}$. We also put

$$\overline{EM} = \widetilde{D}\bigoplus\widetilde{E}\bigoplus\widetilde{F}\bigoplus\widetilde{J}\bigoplus\widetilde{M}\bigoplus\widetilde{N}.$$

Therefore, we have

Proposition 18.3 *The module $H^*N_0^1$ is the direct sum of $\boldsymbol{Q}/\boldsymbol{Z}_{(2)}$, $\widetilde{A}^+$, $\widetilde{C}^+$, $\rho_2\delta_1(EM)$, $\delta_1(\overline{EM})$, $\widetilde{K}'$, $\widetilde{L}'$, $\widetilde{P}'$ and $\widetilde{Q}'$. Here the modules are given in Sect. 2.*

Proof of Theorem 2.4. Consider the exact sequence

$$\cdots \longrightarrow H^*N_0^1 \xrightarrow{\delta_0} H^{*+1}N_0^0 \longrightarrow H^{*+1}M_0^0 \longrightarrow \cdots.$$

Here $H^*N_0^0 = H^*E(2)_*$ is the Adams-Novikov E_2-term converging to the homotopy groups $\pi_*(L_2S^0)$. Since $H^sM_0^0 = \boldsymbol{Q}$ concentrated at dimension zero, the above sequence splits into the exact sequence $0 \to H^0N_0^0 \to \boldsymbol{Q} \to H^0N_0^1 \xrightarrow{\delta} H^1N_0^0 \to 0$ and the isomorphism $\delta\colon H^sN_0^1 = H^{s+1}N_0^0$ for $s > 0$. The first sequence yields $\boldsymbol{Z}_{(2)}$ and kills $\boldsymbol{Q}/\boldsymbol{Z}_{(2)}$, and we have the E_2-term. $\square$

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