PARAMETRIZED HIGHER CATEGORY THEORY

JAY SHAH

Abstract. We develop foundations for the category theory of ∞-categories parametrized by a base ∞-category. Our main contribution is a theory of parametrized homotopy limits and colimits, which recovers and extends the Dotto–Moi theory of G-colimits for G a finite group when the base is chosen to be the orbit category of G. We apply this theory to show that the G-∞-category of G-spaces is freely generated under G-colimits by the contractible G-space, thereby affirming a conjecture of Mike Hill.

Contents

1. Introduction

This thesis lays foundations for a theory of ∞-categories parametrized by a base ∞-category S. Our interest in this project originated in an attempt to locate the core homotopy theories of interest in equivariant homotopy theory - those of G-spaces and G-spectra - within the appropriate ∞-categorical framework. To explain, let G be a finite group and let us review the definitions of the ∞-categories of G-spaces and G-spectra, with a view towards endowing them with universal properties.

Consider a category Top_G of (nice) topological spaces equipped with G-action, with morphisms given by G-equivariant continuous maps. There are various homotopy theories that derive from this category, depending on the class of weak equivalences that one chooses to invert. At one end, we can invert the class $\mathscr{W}_1$ of G-equivariant maps which induce a weak homotopy equivalence of underlying topological spaces, forgetting the G-action. If we let $\mathbf{Top}$ denote the ∞-category of spaces, then we have the identification

$$\mathrm{Top}_G[\mathscr{W}_1^{-1}] \simeq \mathrm{Fun}(BG, \mathbf{Top});$$

inverting $\mathscr{W}_1$ obtains the *∞-category of spaces with G-action*. For many purposes, $\mathrm{Fun}(BG, \mathbf{Top})$ is the homotopy theory that one wishes to contemplate, but here we instead highlight its main deficiency. Namely, passing to this homotopy theory blurs the distinction between homotopy and actual fixed points, in that the functor $\mathrm{Top}_G \longrightarrow \mathrm{Fun}(BG, \mathbf{Top})$ forgets the homotopy types of the various spaces X^H for H a nontrivial subgroup of G. Because many arguments in equivariant homotopy theory involve comparing X^H with the homotopy fixed points X^{hH}, we want to retain this data. To this end, we can instead let $\mathscr{W}$ be the class of G-equivariant maps which induce an equivalence on H-fixed points for every subgroup H of G. Let $\mathbf{Top}_G = \mathrm{Top}_G[\mathscr{W}^{-1}]$; this is the *$\infty$-category of G-spaces.*

Like with $\mathrm{Top}_G[\mathscr{W}_1^{-1}]$, we would like a description of $\mathbf{Top}_G$ which eliminates any reference to topological spaces with G-action, for the purpose of comprehending its universal property. Elmendorf's theorem grants such a description: we have

$$\mathrm{Top}_G[\mathscr{W}^{-1}] \simeq \mathrm{Fun}(\mathbf{O}_G^{op}, \mathbf{Top}),$$

where $\mathbf{O}_G$ is the category of orbits of the group G. Thus, as an ∞-category, $\mathbf{Top}_G$ is the *free cocompletion* of $\mathbf{O}_G$.

It is a more subtle matter to define the homotopy theory of G-spectra. There are at least three possibilities:

(1) The ∞-category of *Borel G-spectra*, i.e. spectra with G-action: This is $\mathrm{Fun}(BG, \mathbf{Sp})$, which is the stabilization of $\mathrm{Fun}(BG, \mathbf{Top})$.
(2) The ∞-category of *'naive' G-spectra*, i.e. spectral presheaves on $\mathbf{O}_G$: This is $\mathrm{Fun}(\mathbf{O}_G^{op}, \mathbf{Sp})$, which is the stabilization of $\mathbf{Top}_G$.
(3) The ∞-category of *'genuine' G-spectra*: Let $A^{\mathit{eff}}(\mathbf{F}_G)$ be the effective Burnside $(2,1)$-category of G, given by taking as objects finite G-sets, as morphisms spans of finite G-sets, and as 2-morphisms isomorphisms between spans. Let $\mathbf{Sp}^G = \mathrm{Fun}^{\oplus}(A^{\mathit{eff}}(\mathbf{F}_G), \mathbf{Sp})$ be the ∞-category of direct-sum preserving functors from $A^{\mathit{eff}}(\mathbf{F}_G)$ to $\mathbf{Sp}$, i.e. that of *spectral Mackey functors*.[1]

The third possibility incorporates essential examples of cohomology theories for G-spaces, such as equivariant K-theory, because G-spectra in this sense possess transfers along maps of finite G-sets, encoded by the covariant maps in $A^{\mathit{eff}}(\mathbf{F}_G)$. It is thus what homotopy theorists customarily mean by G-spectra. However, from a categorical perspective it is a more mysterious object than the ∞-category of naive G-spectra, since it is *not* the stabilization of G-spaces. We are led to ask:

Question: What is the universal property of $\mathbf{Sp}^G$? More precisely, we have an adjunction

$$\Sigma^\infty_+ : \mathbf{Top}_G \rightleftarrows \mathbf{Sp}^G : \Omega^\infty$$

with Ω^∞ given by restriction along the evident map $\mathbf{O}_G^{op} \longrightarrow A^{\mathit{eff}}(\mathbf{F}_G)$, and we would like a universal property for Σ^∞_+ or Ω^∞.

Put another way, what is the categorical procedure which manufactures $\mathbf{Sp}^G$ from $\mathbf{Top}_G$?

The key idea is that for this procedure of 'G-stabilization' one needs to enforce 'G-additivity' over and above the usual additivity satisfied by a stable ∞-category[2]: that is, one wants the coincidence of coproducts and products indexed by finite sets with G-action. Reflecting upon the possible homotopical meaning of such a G-(co)product, we see that for a transitive G-set G/H, $\coprod_{G/H}$ and $\prod_{G/H}$ should be as functors the left and right adjoints to the restriction functor $\mathbf{Sp}^G \longrightarrow \mathbf{Sp}^H$, i.e. the induction and coinduction functors, and G-additivity then becomes the Wirthmüller isomorphism. In particular, we see that G-additivity is not a property that $\mathbf{Sp}^G$ can be said to enjoy in isolation, but rather one satisfied by the *presheaf* $\underline{\mathbf{Sp}}^G$ of ∞-categories indexed by $\mathbf{O}_G$, where $\underline{\mathbf{Sp}}^G(G/-) = \mathbf{Sp}^{(-)}$. Correspondingly, we must rephrase our question so as to inquire after the universal property of the *morphism of $\mathbf{O}_G$-presheaves* $\underline{\Sigma}^\infty_+ : \underline{\mathbf{Top}}_G \longrightarrow \underline{\mathbf{Sp}}^G$, where $\underline{\mathbf{Top}}_G(G/-) = \mathbf{Top}_{(-)}$ and $\underline{\Sigma}^\infty_+$ is objectwise given by suspension.

We now pause to observe that for the purpose of this analysis the group G is of secondary importance as compared to its associated category of orbits $\mathbf{O}_G$. Indeed, we focused on G-additivity as the distinguishing feature of genuine vs. naive G-spectra, as opposed to the invertibility of representation spheres, in order to evade representation theoretic aspects of equivariant stable homotopy theory. In order to frame our situation in its proper generality, let us now dispense with the group G and replace $\mathbf{O}_G$ by an arbitrary ∞-category T. Call a presheaf of ∞-categories on T a *T-category.* The

[1] This is not the definition which first appeared in the literature for G-spectra, but it is equivalent to e.g. the homotopy theory of orthogonal G-spectra by the work of Guillou-May.

[2] We first learned of this perspective on G-spectra from Mike Hopkins.

T-category of T-spaces $\underline{\mathbf{Top}}_T$ is given by the functor $T^{op} \longrightarrow \mathbf{Cat}_\infty$, $t \mapsto \mathrm{Fun}((T_{/t})^{op}, \mathbf{Top})$. Note that this specializes to $\underline{\mathbf{Top}}_G$ when $T = \mathbf{O}_G$ because $\mathbf{O}_H \simeq (\mathbf{O}_G)_{/(G/H)}$; slice categories stand in for subgroups in our theory. With the theory of T-colimits advanced in this thesis, we can then supply a universal property for $\underline{\mathbf{Top}}_T$ *as a T-category*:

1.1. **Theorem.** *Suppose T is any ∞-category. Then $\underline{\mathbf{Top}}_T$ is T-cocomplete, and for any T-category E which is T-cocomplete, the functor of evaluation at the T-final object*

$$\mathrm{Fun}^L(\underline{\mathbf{Top}}_T, E) \longrightarrow E$$

induces an equivalence from the ∞-category of T-functors $\underline{\mathbf{Top}}_T \longrightarrow E$ which strongly preserve T-colimits to E. In other words, $\underline{\mathbf{Top}}_T$ is freely generated under T-colimits by the final T-category.

When $T = \mathbf{O}_G$, this result was originally conjectured by Mike Hill.

To go further and define T-spectra, we need a condition on T so that it supports a theory of spectral Mackey functors. We say that T is *orbital* if T admits multipullbacks, by which we mean that its finite coproduct completion $\mathbf{F}_T$ admits pullbacks. The purpose of the orbitality assumption is to ensure that the effective Burnside category $A^{eff}(\mathbf{F}_T)$ is well-defined. Note that the slice categories $T_{/t}$ are orbital if T is. We define the *T-category of T-spectra* $\underline{\mathbf{Sp}}^T$ to be the functor $T^{op} \longrightarrow \mathbf{Cat}_\infty$ given by $t \mapsto \mathrm{Fun}^{\oplus}(A^{eff}(\mathbf{F}_{T_{/t}}), \mathbf{Sp})$. We then have the following theorem of Denis Nardin concerning $\underline{\mathbf{Sp}}^T$ from [13], which resolves our question:

1.2. **Theorem** (Nardin)**.** *Suppose T is an atomic[3] orbital ∞-category. Then $\underline{\mathbf{Sp}}^T$ is T-stable, and for any pointed T-category C which has all finite T-colimits, the functor of postcomposition by Ω^∞*

$$(\Omega^\infty)_* : \mathrm{Fun}_T^{T-rex}(C, \underline{\mathbf{Sp}}^T) \longrightarrow \mathrm{Lin}^T(C, \underline{\mathbf{Top}}_T)$$

induces an equivalence from the ∞-category of T-functors $C \longrightarrow \underline{\mathbf{Sp}}^T$ which preserve finite T-colimits to the ∞-category of T-linear functors $C \longrightarrow \underline{\mathbf{Top}}_T$, i.e. those T-functors which are fiberwise linear and send finite T-coproducts to T-products.

We hope that the two aforementioned theorems will serve to impress upon the reader the utility of the purely ∞-categorical work that we undertake in this thesis.

Warning. In contrast to this introduction thus far and the conventions adopted elsewhere (e.g. in [13]), we will henceforth speak of S-categories, S-colimits, etc. for $S = T^{op}$.

What is parametrized ∞-category theory? Roughly speaking, parametrized ∞-category theory is an interpretation of the familiar notions of ordinary or 'absolute' ∞-category theory within the $(\infty, 2)$-category of functors $\mathrm{Fun}(S, \mathbf{Cat}_\infty)$, done relative to a fixed 'base' ∞-category S. By 'interpretation', we mean something along the lines of the program of Emily Riehl and Dominic Verity, which axiomatizes the essential properties of an $(\infty, 2)$-category that one needs to do formal category theory into the notion of an *∞-cosmos*, of which $\mathrm{Fun}(S, \mathbf{Cat}_\infty)$ is an example. In an ∞-cosmos, one can write down in a formal way notions of limits and colimits, adjunctions, Kan extensions, and so forth. Working out what this means in the example of $\mathbf{Cat}_\infty$-valued functors is the goal of this thesis. For example, we will see that the Dotto–Moi theory of G-colimits ([5]) coincides with that of $\mathbf{O}_G^{op}$-colimits in the sense of parametrized ∞-category theory.

In contrast to Riehl–Verity, we will work within the model of quasi-categories and not hesitate to use special aspects of our model (e.g. combinatorial arguments involving simplicial sets). We are motivated in this respect by the existence of a highly developed theory of *cocartesian fibrations* due to Jacob Lurie, which we review in §1. Cocartesian fibrations are our preferred way to model $\mathbf{Cat}_\infty$-valued functors, for two reasons:

(1) The data of a functor $F : S \longrightarrow \mathbf{Cat}_\infty$ is overdetermined vs. that of a cocartesian fibration over S, in the sense that to define F one must prescribe an infinite hierarchy of coherence data, which under the functor-fibration correspondence amounts to prescribing an infinite sequence of compatible horn fillings.[4] Because of this, specifying any given cocartesian fibration (which

[3] This is an additional technical hypothesis which we do not explain here. It will not concern us in the body of the paper.

[4] It is for this reason that one speaks of *straightening* a cocartesian fibration to a functor.

one ultimately needs to do in order to connect our theory to applications) is typically an easier task than specifying the corresponding functor.

(2) The Grothendieck construction on a functor $S \longrightarrow \mathbf{Cat}_\infty$ is made visible in the cocartesian fibration setup, as the total category of the cocartesian fibration. Many of our arguments involve direct manipulation of the Grothendieck construction, in order to relate or reduce notions of parametrized ∞-category theory to absolute ∞-category theory.

We have therefore tailored our exposition to the reader familiar with the first five chapters of [9]; the only additional major prerequisite is the part of [11, App. B] dealing with variants of the cocartesian model structure of [9, §3] and functoriality in the base. Let us now give a select summary of the contents of this thesis. Because our work is of a foundational nature, most of our results concern novel *constructions* that we introduce in parametrized ∞-category theory, which parallel simpler constructions in absolute ∞-category theory. These include:

- Functor S-categories (§3) which model the internal hom in $\mathrm{Fun}(S, \mathbf{Cat}_\infty)$ at the level of cocartesian fibrations;
- Join and slice S-categories (§4), which permit us to define S-limits and S-colimits (§5);
- A bestiary of fibrations defined relative to S (§7);
- S-adjunctions (§8);
- S-colimits parametrized by a base S-category (§9), and subsequently S-Kan extensions (§10);
- S-categories of presheaves (§11).

Our main *theorems* concerning these new constructions are:

- A relation between S-slice categories and ordinary slice categories (Thm. 6.6), which permits us to establish the S-cofinality theory (Thm. 6.7);
- Existence and uniqueness of S-Kan extensions (Thm. 10.3);
- The universal property of S-presheaves (Thm. 11.5), which specializes to Thm. 1.1;
- Bousfield–Kan style decomposition results for S-colimits (Thm. 12.6 and Thm. 12.13), which imply in the case where S^{op} is orbital that, in a sense, S-(co)products are the only innovation of our theory of S-(co)limits (Cor. 12.15).

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2. Cocartesian fibrations and model categories of marked simplicial sets

In this section, we give a rapid review of the theory of cocartesian fibrations and the surrounding apparatus of marked simplicial sets. This primarily serves to fix some of our notation and conventions for the remainder of the paper; for a more detailed exposition of these concepts, we refer the reader to [4]. In particular, the reader should be aware of our special notation (Ntn. 2.28) for the S-fibers of a S-functor.

Cocartesian fibrations. We begin with the basic definitions:

2.1. **Definition.** Let $\pi : X \longrightarrow S$ be a map of simplicial sets. Then π is a *cocartesian fibration* if

(1) π is an *inner fibration*: for every $n > 1$, $0 < k < n$ and commutative square

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & S, \end{tikzcd}$$

the dotted lift exists.

(2) For every edge $\alpha : s_0 \to s_1$ in S and $x_0 \in X$ with $\pi(x_0) = s_0$, there exists an edge $e : x_0 \to x_1$ in X with $\pi(e) = \alpha$, such that e is *π-cocartesian*: for every $n > 1$ and commutative square

$$\begin{tikzcd} \Lambda^n_0 \arrow[r, "f"] \arrow[d] & X \arrow[d, "\pi"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

with $f|_{\Delta^{\{0,1\}}} = e$, the dotted lift exists.

Dually, π is a *cartesian fibration* if π^{op} is a cocartesian fibration.

A cocartesian resp. cartesian fibration $\pi : X \longrightarrow S$ is said to be a *left* resp. *right* fibration if for every object $s \in S$ the fiber X_s is a Kan complex.

2.2. **Definition.** Suppose $\pi : X \longrightarrow S$ and $\rho : Y \longrightarrow S$ are (co)cartesian fibrations. Then a *map of (co)cartesian fibrations* $f : X \longrightarrow Y$ is a map of simplicial sets such that $\rho \circ f = \pi$ and f carries π-(co)cartesian edges to ρ-(co)cartesian edges.

2.3. **Definition.** In the case where S is an ∞-category, we introduce alternative terminology for cocartesian fibrations and left fibrations over S:

- An *S-category* resp. *S-space* C is a cocartesian resp. left fibration $\pi : C \longrightarrow S$.
- An *S-functor* $F : C \longrightarrow D$ between S-categories C and D is a map of cocartesian fibrations over S.

We now suppose that S is an ∞-category for the remainder of this section (and indeed, for this paper).

2.4. **Example** (Arrow ∞-categories)**.** The arrow ∞-category $\mathscr{O}(S)$ of S is cocartesian over S via the target morphism ev_1, and cartesian over S via the source morphism ev_0. An edge

$$e : [s_0 \to t_0] \longrightarrow [s_1 \to t_1]$$

in $\mathscr{O}(S)$ is ev_1-cocartesian resp. ev_0-cartesian if and only if $\mathrm{ev}_0(e)$ resp. $\mathrm{ev}_1(e)$ is an equivalence in S.

The fiber of $\mathrm{ev}_0 : \mathscr{O}(S) \longrightarrow S$ over s is isomorphic to Lurie's alternative slice category $S^{s/}$. Using our knowledge of the ev_1-cocartesian edges, we see that ev_1 restricts to a left fibration $S^{s/} \longrightarrow S$. In the terminology of [9, 4.4.4.5], this is a *corepresentable* left fibration. We will refer to the corepresentable left fibrations as *S-points*. Further emphasizing this viewpoint, we will often let $\underline{s}$ denote $S^{s/}$.

To a beginner, the lifting conditions of Dfn. 2.1 can seem opaque. Under the assumption that S is an ∞-category, we have a reformulation of the definition of cocartesian edge, and hence that of cocartesian fibration, which serves to illuminate its homotopical meaning.

2.5. **Proposition.** *Let $\pi : X \longrightarrow S$ be an inner fibration (so X is an ∞-category). Then an edge $e : x_0 \to x_1$ in X is π-cocartesian if and only if for every $x_2 \in X$, the commutative square of mapping spaces*

$$\begin{tikzcd} \mathrm{Map}_X(x_1, x_2) \arrow[r, "e^*"] \arrow[d, "\pi"] & \mathrm{Map}_X(x_0, x_2) \arrow[d, "\pi"] \\ \mathrm{Map}_S(\pi(x_1), \pi(x_2)) \arrow[r, "\pi(e)^*"] & \mathrm{Map}_S(\pi(x_0), \pi(x_2)) \end{tikzcd}$$

is homotopy cartesian.

With some work, Prp. 2.5 can be used to supply an alternative, model-independent definition of a cocartesian fibration: we refer to Mazel-Gee's paper [12] for an exposition along these lines. In any case, the collection of cocartesian fibrations over S and maps thereof organize into a subcategory $\mathbf{Cat}^{cocart}_{\infty/S}$ of the overcategory $\mathbf{Cat}_{\infty/S}$.

2.6. **Example.** Let $\mathbf{Cat}_\infty$ denote the (large) ∞-category of (small) ∞-categories. Then there exists a *universal cocartesian fibration* $\mathscr{U} \longrightarrow \mathbf{Cat}_\infty$, which is characterized up to contractible choice by the requirement that any cocartesian fibration $\pi : X \longrightarrow S$ (with S a (small) simplicial set) fits into a homotopy pullback square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\pi"] & \mathscr{U} \arrow[d] \\ S \arrow[r, "F_\pi"] & \mathbf{Cat}_\infty. \end{tikzcd}$$

Concretely, one can take $\mathscr{U}$ to be the subcategory of the arrow category $\mathscr{O}(\mathbf{Cat}_\infty)$ spanned by the representable right fibrations and morphisms thereof.

As suggested by Exm. 2.6, the functor

$$\mathrm{Fun}(S, \mathbf{Cat}_\infty) \longrightarrow \mathbf{Cat}_{\infty/S}^{cocart}$$

given by pulling back along $\mathscr{U} \longrightarrow \mathbf{Cat}_\infty$ is an equivalence. The composition

$$\mathrm{Gr} : \mathrm{Fun}(S, \mathbf{Cat}_\infty) \xrightarrow{\simeq} \mathbf{Cat}_{\infty/S}^{cocart} \subset \mathbf{Cat}_{\infty/S}$$

is the *Grothendieck construction* functor. Since equivalences in $\mathrm{Fun}(S, \mathbf{Cat}_\infty)$ are detected objectwise, Gr is conservative. Moreover, one can check that Gr preserves limit and colimits, so by the adjoint functor theorem Gr admits both a left and a right adjoint

$$\mathrm{Fr} \dashv \mathrm{Gr} \dashv H.$$

We call Fr the *free cocartesian fibration* functor: concretely, $\mathrm{Fr}(X \longrightarrow S) = X \times_S \mathscr{O}(S) \xrightarrow{ev_1} S$, or as a functor $s \mapsto X \times_S S_{/s}$ with functoriality obtained from $S_{/(-)}$. The functor H can also be concretely described using its universal mapping property: since $\mathrm{Fr}(\{s\} \subset S) = S_{s/}$, the fiber $H(X)_s$ is given by $\mathrm{Fun}_{/S}(S_{s/}, X)$, and the functoriality in S is obtained from that of $S_{(-)/}$.

A model structure for cocartesian fibrations. We want a model structure which has as its fibrant objects the cocartesian fibrations over a fixed simplicial set. However, it is clear that to define it we need some way to remember the data of the cocartesian edges. This leads us to introduce *marked simplicial sets.*

2.7. **Definition.** A marked simplicial set $(X, \mathcal{E})$ is the data of a simplicial set X and a subset $\mathcal{E} \subset X_1$ of the edges of X, such that $\mathcal{E}$ contains all of the degenerate edges. We call $\mathcal{E}$ the set of *marked edges* of X. A map of marked simplicial sets $f : (X, \mathcal{E}) \longrightarrow (Y, \mathcal{F})$ is a map of simplicial sets $f : X \longrightarrow Y$ such that $f(\mathcal{E}) \subset \mathcal{F}$.

2.8. **Notation.** We introduce notation for certain classes of marked simplicial sets. Let X be a simplicial set.

- ▶ $X^\flat$ is X with only the degenerate edges marked.
- ▶ $X^\sharp$ is X with all of its edges marked.
- ▶ Suppose that X is an ∞-category. Then $X^\sim$ is X with its equivalences marked.
- ▶ Suppose that $\pi : X \longrightarrow S$ is an inner fibration. Then ${}_\natural X$ is X with its π-cocartesian edges marked, and $X^\natural$ is X with its π-cartesian edges marked.
- ▶ Let $n > 0$. Let ${}_\natural\Delta^n$ resp. ${}_\natural\Lambda^n_0$ denote Δ^n resp. Λ^n_0 with the edge $\{0,1\}$ marked (if it exists) along with the degenerate edges. Dually, let $\Delta^{n\natural}$ resp. $\Lambda^{n\natural}_n$ denote Δ^n resp. Λ^n_n with the edge $\{n-1, n\}$ marked.

Beware also that we will frequently not indicate the marking in the notation for a marked simplicial set, leaving it either implicit or to be deduced from context.

For the rest of this section, fix a marked simplicial set $(Z, \mathcal{E})$ where Z is an ∞-category and $\mathcal{E}$ contains all of the equivalences in Z; in our applications, Z will generally be some type of fibration over S. Let $s\mathbf{Set}^+_{/(Z,\mathcal{E})}$ be the category of marked simplicial sets over $(Z, \mathcal{E})$. Also denote $s\mathbf{Set}^+_{/Z^\sharp}$ by $s\mathbf{Set}^+_{/Z}$. We will frequently abuse notation by referring to objects $\pi : (X, \mathcal{F}) \longrightarrow (Z, \mathcal{E})$ of $s\mathbf{Set}^+_{/(Z,\mathcal{E})}$ by their domain $(X, \mathcal{F})$ or X.

2.9. **Definition.** An object $(X, \mathcal{F})$ in $s\mathbf{Set}^+_{/(Z,\mathcal{E})}$ is $(Z, \mathcal{E})$-*fibered*[5] if

[5]This differs from the definition in [11, B.0.19].

(1) $\pi : X \longrightarrow Z$ is an inner fibration.

(2) For every $n > 0$ and commutative square

$$\begin{tikzcd} {}_\natural\Lambda^n_0 \arrow[r] \arrow[d] & (X, \mathcal{F}) \arrow[d] \\ {}_\natural\Delta^n \arrow[r] \arrow[ur, dotted] & (Z, \mathcal{E}), \end{tikzcd}$$

a dotted lift exists. In other words, letting $n > 1$, marked edges in X are π-cocartesian, and letting $n = 1$, π-cocartesian lifts exist over marked edges in Z. (Note that condition 2 already guarantees that $X \longrightarrow Z$ is a cocartesian fibration if $\mathscr{E} = Z_1$; however, it may happen that not all of the π-cocartesian edges were marked in X.)

(3) For every commutative square

$$\begin{tikzcd} (\Lambda^2_1)^\sharp \cup_{(\Lambda^2_1)^\flat} (\Delta^2)^\flat \arrow[r] \arrow[d] & (X, \mathcal{F}) \arrow[d] \\ (\Delta^2)^\sharp \arrow[r] \arrow[ur, dotted] & (Z, \mathcal{E}), \end{tikzcd}$$

a dotted lift exists. In other words, marked edges are closed under composition.

(4) Let $Q = \Delta^0 \coprod_{\Delta^{\{0,2\}}} \Delta^3 \coprod_{\Delta^{\{1,3\}}} \Delta^0$. For every commutative square

$$\begin{tikzcd} Q^\flat \arrow[r] \arrow[d] & (X, \mathcal{F}) \arrow[d] \\ Q^\sharp \arrow[r] \arrow[ur, dotted] & (Z, \mathcal{E}), \end{tikzcd}$$

a dotted lift exists. We remark that this lifting property implies that marked edges in X are stable under equivalences in the fiber of the target.

2.10. **Example.** Let $\pi : X \longrightarrow Z$ be an inner fibration. Comparing with Dfn. 2.1, it is clear that $(X, \mathcal{F})$ is $Z^\sharp$-fibered if and only if π is a cocartesian fibration and $(X, \mathcal{F}) = {}_\natural X$. At the other extreme, $(X, \mathcal{F})$ is $Z^\sim$-fibered if and only if π is a categorical fibration and $(X, \mathcal{F}) = X^\sim$.

Recall that a model structure, if it exists, is determined by its cofibrations and fibrant objects. We will define a model structure on $s\mathbf{Set}^+_{/(Z,\mathcal{E})}$ with cofibrations the monomorphisms and fibrant objects given by the $(Z, \mathcal{E})$-fibered objects.

2.11. **Definition.** Define functors

$$\mathrm{Map}_Z(-,-) : s\mathbf{Set}^{+\,op}_{/(Z,\mathcal{E})} \times s\mathbf{Set}^+_{/(Z,\mathcal{E})} \longrightarrow s\mathbf{Set}$$
$$\mathrm{Fun}_Z(-,-) : s\mathbf{Set}^{+\,op}_{/(Z,\mathcal{E})} \times s\mathbf{Set}^+_{/(Z,\mathcal{E})} \longrightarrow s\mathbf{Set}$$

by $\mathrm{Hom}(A, \mathrm{Map}_Z(X, Y)) = \mathrm{Hom}_{/(Z,\mathcal{E})}(A^\sharp \times X, Y)$ and $\mathrm{Hom}(A, \mathrm{Fun}_Z(X, Y)) = \mathrm{Hom}_{/(Z,\mathcal{E})}(A^\flat \times X, Y)$.[6]

2.12. **Definition.** A map $f : A \longrightarrow B$ in $s\mathbf{Set}^+_{/(Z,\mathcal{E})}$ is a *cocartesian equivalence* (with respect to $(Z, \mathcal{E})$) if the following equivalent conditions obtain:

(1) For all $(Z, \mathcal{E})$-fibered X, $f^* : \mathrm{Map}_Z(B, X) \longrightarrow \mathrm{Map}_Z(A, X)$ is an equivalence of Kan complexes.

(2) For all $(Z, \mathcal{E})$-fibered X, $f^* : \mathrm{Fun}_Z(B, X) \longrightarrow \mathrm{Fun}_Z(A, X)$ is an equivalence of ∞-categories.

2.13. **Theorem.** *There exists a left proper combinatorial model structure on the category $s\mathbf{Set}^+_{/(Z,\mathcal{E})}$, which we call the* ***cocartesian model structure****, such that:*

(1) The cofibrations are the monomorphisms.

(2) The weak equivalences are the cocartesian equivalences.

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[6] In [11, App. B], these functors are denoted as $\mathrm{Map}^\sharp_Z$ and $\mathrm{Map}^\flat_Z$ respectively.

(3) The fibrant objects are the $(Z,\mathcal{E})$-fibered objects.

Dually, we define the **cartesian model structure** *on $s\mathbf{Set}^+_{/(Z,\mathcal{E})}$ to be the cocartesian model structure on $s\mathbf{Set}^+_{/(Z,\mathcal{E})^{op}}$ under the isomorphism given by taking opposites.*

We have the following characterization of the cocartesian equivalences between fibrant objects (which is unsurprising, in light of the equivalence $\mathbf{Cat}^{cocart}_{\infty/Z} \simeq \mathrm{Fun}(Z, \mathbf{Cat}_\infty)$).

2.14. **Proposition.** *Let X and Y be fibrant objects in $s\mathbf{Set}^+_{/(Z,\mathcal{E})}$ equipped with the cocartesian model structure, and let $f : X \longrightarrow Y$ be a map in $s\mathbf{Set}^+_{/(Z,\mathcal{E})}$. Then the following are equivalent:*

(1) f is a cocartesian equivalence.
(2) f is a homotopy equivalence, i.e. f admits a homotopy inverse: there exists a map $g : Y \longrightarrow X$ and homotopies $h : (\Delta^1)^\sharp \times X \longrightarrow X$, $h' : (\Delta^1)^\sharp \times Y \longrightarrow Y$ in $s\mathbf{Set}^+_{/(Z,\mathcal{E})}$ connecting $g \circ f$ to id_X and $f \circ g$ to id_Y, respectively.
(3) f is a categorical equivalence.
(4) For every (not necessarily marked) edge $\alpha : \Delta^1 \longrightarrow Z$, $f_\alpha : \Delta^1 \times_Z X \longrightarrow \Delta^1 \times_Z Y$ is a categorical equivalence.

If every edge of Z is marked, then (4) can be replaced by the following apparently weaker condition:

(4') For every object $z \in Z$, $f_z : X_z \longrightarrow Y_z$ is a categorical equivalence.

We also have the following characterization of the fibrations between fibrant objects.

2.15. **Proposition.** *Let $Y = (Y, \mathcal{F})$ be a fibrant object in $s\mathbf{Set}^+_{/(Z,\mathcal{E})}$ equipped with the cocartesian model structure, and let $f : X \longrightarrow Y$ be a map in $s\mathbf{Set}^+_{/(Z,\mathcal{E})}$. Then the following are equivalent:*

(1) f is a fibration.
(2) X is fibrant, and f is a categorical fibration.
(3) f is fibrant in $s\mathbf{Set}^+_{/(Y,\mathcal{F})}$.

2.16. **Corollary.** *Suppose $Z \longrightarrow S$ is a cocartesian fibration. Then the cocartesian model structure $s\mathbf{Set}^+_{/\natural Z}$ coincides with the 'slice' model structure on $(s\mathbf{Set}^+_{/S})_{/\natural Z}$ created by the forgetful functor to $s\mathbf{Set}^+_{/S}$ equipped with its cocartesian model structure.*

2.17. **Example.** Suppose that Z is a Kan complex. Then the cocartesian and cartesian model structures on $s\mathbf{Set}^+_{/Z}$ coincide. In particular, taking $Z = \Delta^0$, we will also refer to the cocartesian model structure on $s\mathbf{Set}^+$ as the *marked model structure*. Since this model structure on $s\mathbf{Set}^+$ is unambiguous, we will always regard $s\mathbf{Set}^+$ as equipped with it. Then the fibrant objects of $s\mathbf{Set}^+$ are precisely the ∞-categories with their equivalences marked.

2.18. **Example.** Suppose that $(Z,\mathcal{E}) = Z^\sim$. Then the cocartesian and cartesian model structures on $s\mathbf{Set}^+_{/Z^\sim}$ coincide. Moreover, we have a Quillen equivalence

$$(-)^\flat : (s\mathbf{Set}_{\mathrm{Joyal}})_{/Z} \rightleftarrows s\mathbf{Set}^+_{/Z^\sim} : U$$

where the functor U forgets the marking.

2.19. **Example.** The inclusion functor $\mathbf{Top} \subset \mathbf{Cat}_\infty$ admits left and right adjoints B and ι, where B is the classifying space functor that inverts all edges and ι is the 'core' functor that takes the maximal sub-∞-groupoid. These two adjunctions are modeled by the two Quillen adjunctions

$$U : s\mathbf{Set}^+ \rightleftarrows s\mathbf{Set}_{\mathrm{Quillen}} : (-)^\sharp,$$

$$(-)^\sharp : s\mathbf{Set}_{\mathrm{Quillen}} \rightleftarrows s\mathbf{Set}^+ : M.$$

Here $M(X, \mathscr{E})$ is the maximal sub-simplicial set of X such that all of its edges are marked.

In particular, we have that $(-)^\flat$ and $(-)^\sharp$ send categorical equivalences resp. weak homotopy equivalences to marked equivalences.

2.20. **Proposition.** *The bifunctor*

$$- \times - : s\mathbf{Set}^+_{/(Z_1,\mathcal{E}_1)} \times s\mathbf{Set}^+_{/(Z_2,\mathcal{E}_2)} \longrightarrow s\mathbf{Set}^+_{/(Z_1\times Z_2,\mathcal{E}_1\times\mathcal{E}_2)}.$$

is left Quillen. Consequently, the bifunctors

$$\begin{aligned} \mathrm{Map}_Z(-,-) :& s\mathbf{Set}^{+\ op}_{/(Z,\mathcal{E})} \times s\mathbf{Set}^+_{/(Z,\mathcal{E})} \longrightarrow s\mathbf{Set}_{Quillen} \\ \mathrm{Fun}_Z(-,-) :& s\mathbf{Set}^{+\ op}_{/(Z,\mathcal{E})} \times s\mathbf{Set}^+_{/(Z,\mathcal{E})} \longrightarrow s\mathbf{Set}_{Joyal} \end{aligned}$$

are right Quillen, so $s\mathbf{Set}^+_{/(Z,E)}$ *is both a simplicial i.e.* $s\mathbf{Set}_{Quillen}$*-enriched model category (with respect to* Map_Z*) and* $s\mathbf{Set}_{Joyal}$*-enriched model category (with respect to* Fun_Z*).*

2.21. **Remark.** By Prp. 2.20, $s\mathbf{Set}^+_{/(Z,\mathcal{E})}$ is an example of an ∞-cosmos in the sense of Riehl-Verity.

Finally, we explain how the formalism of marked simplicial sets can be used to extract the pushforward functors implicitly defined by a cocartesian fibration. First, we need a lemma.

2.22. **Lemma.** *For* $n > 0$*, the inclusion* $i_n : \Delta^{n-1} \cong \Delta^{\{0\}} \star \Delta^{\{2,\ldots,n\}} \longrightarrow {}^\natural\Delta^n$ *is left marked anodyne. Consequently, for a cocartesian fibration* $C \longrightarrow S$*, the map*

$$\mathrm{Fun}({}^\natural\Delta^n, {}^\natural C) \longrightarrow \mathrm{Fun}(\Delta^{n-1}, C) \times_{\mathrm{Fun}(\Delta^{n-1},C)} \mathrm{Fun}(\Delta^n, S)$$

induced by i_n *is a trivial Kan fibration.*

Proof. We proceed by induction on n, the base case $n = 1$ being the left marked anodyne map $\Delta^{\{0\}} \longrightarrow (\Delta^1)^\sharp$. Consider the commutative diagram

$$\begin{tikzcd} \Delta^{\{0\}} \star \partial\Delta^{n-2} \arrow[r] \arrow[d, "\cup i_{n-1}"] & \Delta^{\{0\}} \star \Delta^{\{2,\ldots,n\}} \arrow[d] \arrow[dd, bend left, "i_n"] \\ (\Delta^{\{0\}} \star \Lambda^{n-1}_0, \mathscr{E}) \arrow[r] & {}^\natural\Lambda^n_0 \arrow[d] \\ & {}^\natural\Delta^n \end{tikzcd}$$

where $\mathscr{E}$ is the collection of edges $\{0, i\}$, $0 < i \leq n$ (and the degenerate edges). The square is a pushout, and by the inductive hypothesis, the lefthand vertical map is left marked anodyne. We deduce that i_n is left marked anodyne. The second statement now follows because the lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathrm{Fun}({}^\natural\Delta^n, {}^\natural C) \arrow[d] \\ B \arrow[r] \arrow[ur, dotted] & \mathrm{Fun}(\Delta^{n-1}, C) \times_{\mathrm{Fun}(\Delta^{n-1},C)} \mathrm{Fun}(\Delta^n, S) \end{tikzcd}$$

transposes to

$$\begin{tikzcd} A \times {}^\natural\Delta^n \bigcup_{A\times\Delta^{n-1}} B \times \Delta^{n-1} \arrow[r] \arrow[d] & {}^\natural C \arrow[d] \\ B \times {}^\natural\Delta^n \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

and the lefthand vertical map is left marked anodyne for any cofibration $A \longrightarrow B$ by [9, 3.1.2.3]. $\square$

The main case of interest in Lm. 2.22 is when $n = 1$, which shows that $\mathscr{O}^{cocart}(C) \longrightarrow C \times_S \mathscr{O}(S)$ is a trivial Kan fibration. Let $P : C \times_S \mathscr{O}(S) \longrightarrow \mathscr{O}^{cocart}(C)$ be a section that fixes the inclusion $C \subset \mathscr{O}^{cocart}(C)$. Then we say that P or the further composite $P' = \mathrm{ev}_1 \circ P$ is a *cocartesian pushforward* for $C \longrightarrow S$. Given an edge α of S, $P'_\alpha : C_s \longrightarrow C_t$ is the pushforward functor $\alpha_!$ determined under the equivalence $\mathbf{Cat}^{cocart}_{\infty/S} \simeq \mathrm{Fun}(S, \mathbf{Cat}_\infty)$.

Functoriality in the model structure. Let $\pi : X \longrightarrow Z$ be a map of simplicial sets. Then the pullback functor $\pi^* : s\mathbf{Set}_{/Y} \longrightarrow s\mathbf{Set}_{/X}$ admits a left adjoint $\pi_!$, given by postcomposing with π. In addition, since $s\mathbf{Set}$ is a topos, π^* also admits a right adjoint π_*, which may be thought of as the functor of relative sections because $\mathrm{Hom}_{/X}(A, f_*(B)) \cong \mathrm{Hom}_{/Y}(A \times_X Y, B)$.

Now supposing that π is a map of marked simplicial sets, π^*, $\pi_!$, and π_* extend to functors of marked simplicial sets over X or Z in an evident manner. We then seek conditions under which the adjunctions $\pi_! \dashv \pi^*$ and $\pi^* \dashv \pi_*$ are Quillen with respect to the cocartesian model structures. To this end, we have the following theorem of Lurie:

2.23. **Theorem.** *Let*

$$(Z, \mathcal{E}) \xleftarrow{\pi} (X, \mathcal{F}) \xrightarrow{\rho} (X', \mathcal{F}')$$

be a span of marked simplicial sets such that Z, X, X' are ∞-categories and the collections of markings contain all the equivalences. Then the adjunction

$$\rho_! \colon s\mathbf{Set}^+_{/(X,\mathcal{F})} \rightleftarrows s\mathbf{Set}^+_{/(X',\mathcal{F}')} : \rho^*$$

is Quillen with respect to the cocartesian model structures. Moreover, suppose that

(1) *For every object $x \in X$ and marked edge $f : z \to \pi(x)$ in Z, there exists a locally π-cartesian edge $x_0 \to x$ in X lifting f.*
(2) *π is a flat categorical fibration.*
(3) *$\mathcal{E}$ and $\mathcal{F}$ are closed under composition.*
(4) *Suppose given a commutative diagram*

$$\begin{tikzcd} & x_1 \arrow[dr, "g"] & \\ x_0 \arrow[ur, "f"] \arrow[rr, "h"] & & x_2 \end{tikzcd}$$

in X where g is locally π-cartesian, $\pi(g)$ is marked, and $\pi(f)$ is an equivalence. Then f is marked if and only if h is marked. (Note in particular that, taking f to be an identity morphism, every locally π-cartesian edge lying over a marked edge is itself marked.)

Then the adjunction

$$\pi^* \colon s\mathbf{Set}^+_{/(X,\mathcal{F})} \rightleftarrows s\mathbf{Set}^+_{/(Z,\mathcal{E})} : \pi_*$$

is Quillen with respect to the cocartesian model structures.

We formulated Thm. 2.23 as a theorem concerning a span $X \xleftarrow{\phi} Z \xrightarrow{\rho} X'$ because in applications we will typically be interested in the composite Quillen adjunction

$$\rho_! \pi^* \colon s\mathbf{Set}^+_{/(X,\mathcal{F})} \rightleftarrows s\mathbf{Set}^+_{/(X',\mathcal{F}')} : \pi_* \rho^*.$$

Here are two examples.

2.24. **Example** (Pairing cartesian and cocartesian fibrations)**.** Let $\pi : X \longrightarrow Z$ be a cartesian fibration. Then the span

$$Z^\sharp \xleftarrow{\pi} X^\natural \xrightarrow{\pi} Z^\sharp$$

satisfies the hypotheses of Thm. 2.23. Now given a cocartesian fibration $Y \longrightarrow Z$, define

$$\widetilde{\mathrm{Fun}}_Z(X, Y) = (\pi_* \pi^*)({}_\natural Y \to Z^\sharp).$$

Then the fiber of $\widetilde{\mathrm{Fun}}_Z(X, Y)$ over an object $z \in Z$ is $\mathrm{Fun}(X_z, Y_z)$, and given a morphism $\alpha : z_0 \longrightarrow z_1$, the pushforward functor $\alpha_! : \mathrm{Fun}(X_{z_0}, Y_{z_0}) \longrightarrow \mathrm{Fun}(X_{z_1}, Y_{z_1})$ is given by precomposition in the source and postcomposition in the target.

2.25. **Example** (Right Kan extension)**.** Let $f : Y \longrightarrow Z$ be a functor. We can apply Thm. 2.23 to perform the operation of right Kan extension at the level of cocartesian fibrations. Consider the span

$$Z^\sharp \xleftarrow{\mathrm{ev}_0} (\mathscr{O}(Z) \times_{Z,f} Y)^\sharp \xrightarrow{\mathrm{pr}_Y} Y^\sharp.$$

Then the conditions of Thm. 2.23 are satisfied, so we obtain a Quillen adjunction

$$(\mathrm{pr}_Y)_! (\mathrm{ev}_0)^* \colon s\mathbf{Set}^+_{/Z} \rightleftarrows s\mathbf{Set}^+_{/Y} : (\mathrm{ev}_0)_* (\mathrm{pr}_Y)^*.$$

In addition, the map $C \times_Z Y^\sharp \longrightarrow C \times_Z \mathscr{O}(Z)^\sharp \times_Z Y^\sharp$ induced by the identity section $\iota : Z \longrightarrow \mathscr{O}(Z)$ is a cocartesian equivalence in $s\mathbf{Set}^+_{/Y}$ for $C \longrightarrow Z$ *fibrant* in $s\mathbf{Set}^+_{/Z}$, by [2, 9.8]. Consequently, the induced adjunction of ∞-categories

$$(\mathrm{pr}_Y)_!(\mathrm{ev}_0)^* \colon \mathbf{Cat}^{cocart}_{\infty/Z} \rightleftarrows \mathbf{Cat}^{cocart}_{\infty/Y} : (\mathrm{ev}_0)_*(\mathrm{pr}_Y)^*$$

is equivalent to

$$f^* \colon \mathrm{Fun}(Z, \mathbf{Cat}_\infty) \rightleftarrows \mathrm{Fun}(Y, \mathbf{Cat}_\infty) : f_*$$

under the straightening/unstraightening equivalence (which is natural with respect to pullback).

Note that as a special case, if $Z = \Delta^0$ we recover the formula $\mathrm{Fun}_Y(Y^\sharp, {}_\natural C) \simeq \varprojlim F_C$ of [9, 3.3.3.2] (where $C \longrightarrow Y$ is a cocartesian fibration and $F_C : Y \longrightarrow \mathbf{Cat}_\infty$ the corresponding functor). Indeed, this construction of the right Kan extension of a cocartesian fibration is suggested by that result and the pointwise formula for a right Kan extension.

Finally, we will use the following two observations concerning the interaction of Thm. 2.23 with compositions and homotopy equivalences of spans (which we also recorded in [4]).

2.26. **Lemma.** *Suppose we have spans of marked simplicial sets*

$$X_0 \xrightarrow{\pi_0} Z_0 \xrightarrow{\rho_0} X_1$$

and

$$X_1 \xrightarrow{\pi_1} Z_1 \xrightarrow{\rho_1} X_2$$

which each satisfy the hypotheses of Thm. 2.23. Then the span

$$Z_0 \xrightarrow{\mathrm{pr}_0} Z_0 \times_{X_1} Z_1 \xrightarrow{\mathrm{pr}_1} Z_1$$

also satisfies the hypothesis of Thm. 2.23. Consequently, we obtain a Quillen adjunction

$$(\rho_1 \circ \mathrm{pr}_1)_!(\pi_0 \circ \mathrm{pr}_0)^* \colon s\mathbf{Set}^+_{/X_0} \rightleftarrows s\mathbf{Set}^+_{X_2} : (\pi_0 \circ \mathrm{pr}_0)_*(\rho_1 \circ \mathrm{pr}_1)^*,$$

which is the composite of the Quillen adjunction from $s\mathbf{Set}^+_{/X_0}$ to $s\mathbf{Set}^+_{/X_1}$ with the one from $s\mathbf{Set}^+_{/X_1}$ to $s\mathbf{Set}^+_{/X_2}$.

Proof. The proof is by inspection. However, one should beware that the "long" span

$$X_0 \longleftarrow Z_0 \times_{X_1} Z_1 \longrightarrow X_2$$

may fail to satisfy the hypotheses of Thm. 2.23, because the composition of locally cartesian fibrations may fail to again be locally cartesian; this explains the roundabout formulation of the statement. Finally, observe that if we employ the base-change isomorphism $\rho_0^*\pi_{1,*} \cong \mathrm{pr}_{0,*} \circ \mathrm{pr}_1^*$, then we obtain our Quillen adjunction as the composite of the two given Quillen adjunctions. ☐

2.27. **Lemma.** *Suppose a morphism of spans of marked simplicial sets*

$$\begin{tikzcd} & Z \arrow[dl, "\pi"'] \arrow[d, "f"] \arrow[dr, "\rho"] & \\ X & Z' \arrow[l, "\pi'"] \arrow[r, "\rho'"'] & X' \end{tikzcd}$$

where $\rho_!\pi^*$ and $(\rho')_!(\pi')^*$ are left Quillen with respect to the cocartesian model structures on X and X'. Suppose moreover that f is a homotopy equivalence in $s\mathbf{Set}^+_{/X'}$, so that there exists a homotopy inverse g and homotopies

$$h \colon id \simeq g \circ f \quad \text{and} \quad k \colon id \simeq f \circ g.$$

Then the natural transformation $\rho_!\pi^* \longrightarrow (\rho')_!(\pi')^*$ induced by f is a cocartesian equivalence on all objects, and, consequently, the adjoint natural transformation $(\pi')_*(\rho')^* \longrightarrow \pi_*\rho^*$ is a cocartesian equivalence on all fibrant objects.

Proof. The homotopies h and k pull back to show that for all $X \longrightarrow C$, the map

$$id_X \times_C f \colon X \times_C K \longrightarrow X \times_C L$$

is a homotopy equivalence with inverse $id_X \times_C g$. The last statement now follows from [7, 1.4.4(b)]. ☐

Parametrized fibers. In this brief subsection, we record notation for the S-fibers of an S-functor.

2.28. **Notation.** Given an S-category $\pi : D \longrightarrow S$ and an object $x \in D$, define

$$\mathscr{O}_{x\to}(D) = \{x\} \times_D \mathscr{O}(D).$$

For the full subcategory of cocartesian edges $\mathscr{O}^{cocart}(D) \subset \mathscr{O}(D)$, also define

$$\underline{x} = \mathscr{O}^{cocart}_{x\to}(D).$$

Given an S-functor $\phi : C \longrightarrow D$, define

$$C_{\underline{x}} = \underline{x} \times_{D,\phi} C.$$

By Lm. 12.10, $\underline{x} \longrightarrow S^{\pi x/}$ is a trivial fibration. We will therefore also regard $C_{\underline{x}}$ as a $S^{\pi x/}$-category (and we will sometimes be cavalier about the distinction between $\underline{x}$ and $S^{\pi x/}$). Note however, that the functor $\underline{x} \longrightarrow D$ is canonical in our setup, whereas we need to make a choice of cocartesian pushforward to choose a S-functor $S^{\pi x/} \longrightarrow D$ that selects $x \in D$.

3. Functor categories

Let S be an ∞-category. Then $\mathrm{Fun}(S, \mathbf{Cat}_\infty)$ is cartesian closed, so it possesses an internal hom. As a basic application of the existence of f_* under suitable hypotheses, we will define this internal hom at the level of cocartesian fibrations over S.

3.1. **Proposition.** *Let $C \longrightarrow S$ be a cocartesian fibration. Let $\mathrm{ev}_0, \mathrm{ev}_1 : \mathscr{O}(S) \times_S C \longrightarrow S$ denote the source and target maps. Then the functor*

$$(\mathrm{ev}_1)_!(\mathrm{ev}_0)^* : s\mathbf{Set}^+_{/S} \longrightarrow s\mathbf{Set}^+_{/\mathscr{O}(S)^\sharp \times_S {}_\natural C} \longrightarrow s\mathbf{Set}^+_{/S}$$

is left Quillen with respect to the cocartesian model structures.

Proof. We verify the hypotheses of Thm. 2.23 as applied to the span $S^\sharp \xleftarrow{\mathrm{ev}_0} \mathscr{O}(S)^\sharp \times_S {}_\natural C \xrightarrow{\mathrm{ev}_1} S^\sharp$. By [9, 2.4.7.12], ev_0 is a cartesian fibration and an edge e in $\mathscr{O}(S) \times_S C$ is ev_0-cartesian if and only if its projection to C is an equivalence. (1) thus holds. (2) holds since cartesian fibrations are flat categorical fibrations. (3) is obvious. (4) follows from the stability of cocartesian edges under equivalence. □

We will denote the right adjoint $(\mathrm{ev}_0)_*(\mathrm{ev}_1)^*$ by $\underline{\mathrm{Fun}}_S(C, -)$ or $\underline{\mathrm{Fun}}_S({}_\natural C, -)$. Prp. 3.1 implies that if $D \longrightarrow S$ is a cocartesian fibration, $\underline{\mathrm{Fun}}_S(C, D) \longrightarrow S$ is a cocartesian fibration. Unwinding the definitions, we see that an object of $\underline{\mathrm{Fun}}_S(C, D)$ over $s \in S$ is a $S^{s/}$-functor $S^{s/} \times_S C \longrightarrow S^{s/} \times_S D$, and a cocartesian edge of $\underline{\mathrm{Fun}}_S(C, D)$ over an edge $e : \Delta^1 \longrightarrow S$ is a $\Delta^1 \times_S \mathscr{O}(S)$-functor $\Delta^1 \times_S \mathscr{O}(S) \times_S C \longrightarrow \Delta^1 \times_S \mathscr{O}(S) \times_S D$.

3.2. **Lemma.** *Let $\iota : S \longrightarrow \mathscr{O}(S)$ be the identity section and regard $\mathscr{O}(S)^\sharp$ as a marked simplicial set over S via the target map. Then*

(1) *For every marked simplicial set $X \longrightarrow S$ and cartesian fibration $C \longrightarrow S$,*

$$id_X \times \iota \times id_C : X \times_S C^\natural \longrightarrow X \times_S \mathscr{O}(S)^\sharp \times_S C^\natural$$

is a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$.

(1') *For every marked simplicial set $X \longrightarrow S$ and cartesian fibration $C \longrightarrow S$,*

$$\iota \times id_C : X \times_S C^\natural \longrightarrow \mathrm{Fun}((\Delta^1)^\sharp, X) \times_S C^\natural$$

is a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$, where the marked edges in $\mathrm{Fun}((\Delta^1)^\sharp, X)$ are the marked squares in X.

(2) *For every marked simplicial set $X \longrightarrow S$ and cocartesian fibration $C \longrightarrow S$,*

$$id_C \times \iota \times id_X : {}_\natural C \times_S X \longrightarrow {}_\natural C \times_S \mathscr{O}(S)^\sharp \times_S X$$

is a homotopy equivalence in $s\mathbf{Set}^+_{/S}$.

Proof. (1): Because $-\times_S C^\natural$ preserves cocartesian equivalences, we reduce to the case where $C = S$. By definition, $X \longrightarrow X \times_S \mathscr{O}(S)^\sharp$ is a cocartesian equivalence if and only if for every cocartesian fibration $Z \longrightarrow S$, $\mathrm{Map}_S^\sharp(X \times_S \mathscr{O}(S)^\sharp, {}_\natural Z) \longrightarrow \mathrm{Map}_S^\sharp(X, {}_\natural Z)$ is a trivial Kan fibration. In other words, for every monomorphism of simplicial sets $A \longrightarrow B$ and cocartesian fibration $Z \longrightarrow S$, we need to provide a lift in the following commutative square

$$\begin{tikzcd} B^\sharp \times X \coprod_{A^\sharp\times X} (A^\sharp \times X) \times_S \mathscr{O}(S)^\sharp \arrow[r, "\phi"] \arrow[d] & {}_\natural Z \arrow[d] \\ (B^\sharp \times X) \times_S \mathscr{O}(S)^\sharp \arrow[r] \arrow[ur, dotted] & S^\sharp \end{tikzcd}$$

Define $h_0 : \mathscr{O}(S)^\sharp \times (\Delta^1)^\sharp \longrightarrow \mathscr{O}(S)^\sharp$ to be the adjoint to the map $\mathscr{O}(S)^\sharp \longrightarrow \mathscr{O}(\mathscr{O}(S))^\sharp$ obtained by precomposing by the map of posets $\Delta^1 \times \Delta^1 \longrightarrow \Delta^1$ which sends $(1,1)$ to 1 and the other vertices to 0. Precomposing ϕ by $id_{A^\sharp\times X} \times h_0$, define a homotopy

$$h : (A^\sharp \times X) \times_S \mathscr{O}(S)^\sharp \times (\Delta^1)^\sharp \longrightarrow {}_\natural Z$$

from $\phi|_{A^\sharp\times X} \circ \mathrm{pr}_{A^\sharp\times X}$ to $\phi|_{(A^\sharp\times X)\times_S \mathscr{O}(S)^\sharp}$. Using h and $\phi|_{B^\sharp\times X}$, define a map

$$\psi : B^\sharp \times X \coprod_{A^\sharp\times X} (A^\sharp \times X) \times_S \mathscr{O}(S)^\sharp \longrightarrow \mathrm{Fun}((\Delta^1)^\sharp, {}_\natural Z)$$

such that $\psi|_{B^\sharp\times X}$ is adjoint to $\phi|_{B^\sharp\times X} \circ \mathrm{pr}_{B^\sharp\times X}$ and $\psi|_{(A^\sharp\times X)\times_S\mathscr{O}(S)^\sharp}$ is adjoint to h. Then we may factor the above square through the trivial fibration $\mathrm{Fun}((\Delta^1)^\sharp, {}_\natural Z) \longrightarrow {}_\natural Z \times_S \mathscr{O}(S)^\sharp$ to obtain the commutative rectangle

$$\begin{tikzcd} B^\sharp \times X \coprod_{A^\sharp\times X} (A^\sharp \times X) \times_S \mathscr{O}(S)^\sharp \arrow[r, "\psi"] \arrow[d] & \mathrm{Fun}((\Delta^1)^\sharp, {}_\natural Z) \arrow[r, "e_1"] \arrow[d, two heads, "\simeq"] & {}_\natural Z \arrow[d] \\ (B^\sharp \times X) \times_S \mathscr{O}(S)^\sharp \arrow[r, "\phi|_{B^\sharp\times X}\times id"'] \arrow[ur, dotted, "\widetilde{\psi}"] & {}_\natural Z \times_S \mathscr{O}(S)^\sharp \arrow[r, "e_1"] & S^\sharp. \end{tikzcd}$$

The dotted lift $\widetilde{\psi}$ exists, and $e_1 \circ \widetilde{\psi}$ is our desired lift.

(1'): Repeat the argument of (1) with $\mathrm{Fun}((\Delta^1)^\sharp, X)$ in place of $\mathscr{O}(S)^\sharp$.

(2): Let $p : C \longrightarrow S$ denote the structure map and let P be a lift in the commutative square

$$\begin{tikzcd} {}_\natural C \arrow[r, "\iota_C"] \arrow[d] & \mathrm{Fun}((\Delta^1)^\sharp, {}_\natural C) \arrow[d, two heads, "\simeq"', "{(e_0,\mathscr{O}(p))}"] \\ {}_\natural C \times_S \mathscr{O}(S)^\sharp \arrow[r, "="] \arrow[ur, dotted, "P"] & {}_\natural C \times_S \mathscr{O}(S)^\sharp. \end{tikzcd}$$

Let

$$g = (e_1 \times id_X) \circ (P \times id_X) : {}_\natural C \times_S \mathscr{O}(S)^\sharp \times_S X \longrightarrow {}_\natural C \times_S X.$$

and note that g is map over S. We claim that g is a marked homotopy inverse of $f = id_C \times \iota \times id_X$. By construction, $g \circ f = id$. For the other direction, define $h_0 : \mathrm{Fun}((\Delta^1)^\sharp, {}_\natural C) \times (\Delta^1)^\sharp \longrightarrow \mathrm{Fun}((\Delta^1)^\sharp, {}_\natural C)$ as the adjoint of the map $\mathrm{Fun}((\Delta^1)^\sharp, {}_\natural C) \longrightarrow \mathrm{Fun}((\Delta^1 \times \Delta^1)^\sharp, {}_\natural C)$ obtained by precomposing by the map of posets $\Delta^1 \times \Delta^1 \longrightarrow \Delta^1$ which sends $(0,0)$ to 0 and the other vertices to 1. Define

$$h : {}_\natural C \times_S \mathscr{O}(S)^\sharp \times_S X \times (\Delta^1)^\sharp \longrightarrow {}_\natural C \times_S \mathscr{O}(S)^\sharp \times_S X$$

as the composite $((e_0, \mathscr{O}(p)) \times X) \circ (h_0 \times X) \circ (P \times id_{X\times(\Delta^1)^\sharp})$. Then h is a homotopy over S from id to $f \circ g$. $\square$

3.3. **Proposition.** *Let $C, C', D \longrightarrow S$ be cocartesian fibrations and let $F : C \longrightarrow C'$ be a monomorphism. For all marked simplicial sets Y over S, the map*

$$\mathrm{Fun}_S({}_\natural D, \underline{\mathrm{Fun}}_S({}_\natural C', Y)) \longrightarrow \mathrm{Fun}_S({}_\natural D \times_S {}_\natural C', Y) \times_{\mathrm{Fun}_S({}_\natural D \times_S {}_\natural C, Y)} \mathrm{Fun}_S({}_\natural D, \underline{\mathrm{Fun}}_S({}_\natural C, Y))$$

which precomposes by F is a trivial Kan fibration.

Proof. From the defining adjunction, for all $X, Y \in s\mathbf{Set}^+_{/S}$ we have a natural isomorphism

$$\mathrm{Fun}_S(X, \underline{\mathrm{Fun}}_S({}_\natural C, Y)) \cong \mathrm{Fun}_S(X \times_S \mathscr{O}(S)^\sharp \times_S {}_\natural C, Y)$$

of simplicial sets. Since $\mathrm{Fun}_S(-,-)$ is a right Quillen bifunctor, the assertion reduces to showing that

$${}_\natural D \times_S {}_\natural C' \coprod_{{}_\natural D \times_S {}_\natural C} {}_\natural D \times_S \mathscr{O}(S)^\sharp \times_S {}_\natural C \longrightarrow {}_\natural D \times_S \mathscr{O}(S)^\sharp \times_S {}_\natural C'$$

is a trivial cofibration in $s\mathbf{Set}^+_{/S}$, which follows from 3.2 (2). $\square$

In Prp. 3.3, letting $C = \emptyset$ and $Y = {}_\natural E$ for another cocartesian fibration $E \longrightarrow S$, we deduce that $\underline{\mathrm{Fun}}_S(C, -)$ is right adjoint to $C \times_S -$ as a endofunctor of $\mathrm{Fun}(S, \mathbf{Cat}_\infty)$. Further setting $D = S$, we deduce that the category of cocartesian sections of $\underline{\mathrm{Fun}}_S({}_\natural C, {}_\natural E)$ is equivalent to $\mathrm{Fun}_S({}_\natural C, {}_\natural E)$.

3.4. **Notation.** Given a map $f : {}_\natural C \longrightarrow {}_\natural E$, let σ_f denote the cocartesian section $S^\sharp \longrightarrow \underline{\mathrm{Fun}}_S({}_\natural C, {}_\natural E)$ given by adjointing the map $\mathscr{O}(S)^\sharp \times_S {}_\natural C \xrightarrow{\mathrm{pr}_C} {}_\natural C \xrightarrow{f} {}_\natural E$.

3.5. **Lemma.** *Let $C \longrightarrow D$ be a fibration of marked simplicial sets over S.*

(1) *Let $K \longrightarrow S$ be a cocartesian fibration. Then*

$$\underline{\mathrm{Fun}}_S({}_\natural K, C) \longrightarrow \underline{\mathrm{Fun}}_S({}_\natural K, D) \times_D C$$

is a fibration in $s\mathbf{Set}^+_{/S}$.

(2) *The map*

$$\underline{\mathrm{Fun}}_S(S^\sharp, C) \longrightarrow \underline{\mathrm{Fun}}_S(S^\sharp, D) \times_D C$$

is a trivial fibration in $s\mathbf{Set}^+_{/S}$.

Proof. Let $i : A \longrightarrow B$ be a map of marked simplicial sets. For (1), we use that if i is a trivial cofibration, then

$$B \coprod_A A \times_S \mathscr{O}(S)^\sharp \times_S {}_\natural K \longrightarrow B \times_S \mathscr{O}(S) \times_S {}_\natural K$$

is a trivial cofibration, which follows from Prp. 3.1. For (2), we use that if i is a cofibration, then

$$B \coprod_A A \times_S \mathscr{O}(S)^\sharp \longrightarrow B \times_S \mathscr{O}(S)$$

is a trivial cofibration, which follows from Lm. 3.2 (1). $\square$

3.6. **Proposition.** *The Quillen adjunction*

$$- \times_S \mathscr{O}(S)^\sharp \colon s\mathbf{Set}^+_{/S} \rightleftarrows s\mathbf{Set}^+_{/S} : \underline{\mathrm{Fun}}_S(S^\sharp, -)$$

is a Quillen equivalence.

Proof. We first check that for every cocartesian fibration $C \longrightarrow S$, the counit map

$$\underline{\mathrm{Fun}}_S(S^\sharp, {}_\natural C) \times_S \mathscr{O}(S)^\sharp \longrightarrow {}_\natural C$$

is a cocartesian equivalence. By Lm. 3.2(1), it suffices to show that

$$\underline{\mathrm{Fun}}_S(S^\sharp, {}_\natural C) \longrightarrow {}_\natural C$$

is a trivial marked fibration, which follows from Lm. 3.5(2) (taking $D = S$). We now complete the proof by checking that $- \times_S \mathscr{O}(S)^\sharp$ reflects cocartesian equivalences: i.e., given the commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ A \times_S \mathscr{O}(S)^\sharp \arrow[r] & B \times_S \mathscr{O}(S)^\sharp. \end{tikzcd}$$

if the lower horizontal map is a cocartesian equivalence over S (with respect to the target map) then the upper horizontal map is a cocartesian equivalence over S. But the vertical maps are cocartesian equivalences by Lm. 3.2(1). $\square$

The construction $\underline{\mathrm{Fun}}_S(-,-)$ does not make homotopical sense when the first variable is not fibrant, so it does not yield a Quillen bifunctor. Nevertheless, we can say the following about varying the first variable.

3.7. **Proposition.** *Let K, L, and C be fibrant marked simplicial sets over S, let $f : K \longrightarrow L$ be a map and let*

$$f^* : \underline{\mathrm{Fun}}_S(L, C) \longrightarrow \underline{\mathrm{Fun}}_S(K, C)$$

denote the induced map.

(1) Suppose that $K \longrightarrow L$ is a cocartesian equivalence over S. Then f^* is a cocartesian equivalence over S.
(2) Suppose that $K \longrightarrow L$ is a cofibration. Then f^* is a fibration in $s\mathbf{Set}^+_{/S}$.

Proof. (1): It suffices to check that for all $s \in S$, f^* induces a categorical equivalence between the fibers over s, i.e. that

$$\mathrm{Fun}_S((S^{s/})^\sharp \times_S L, C) \longrightarrow \mathrm{Fun}_S((S^{s/})^\sharp \times_S K, C)$$

is a categorical equivalence. Our assumption implies that $(S^{s/})^\sharp \times_S K \longrightarrow (S^{s/})^\sharp \times_S L$ is a cocartesian equivalence over S, so this holds.

(2): For any trivial cofibration $A \longrightarrow B$ in $s\mathbf{Set}^+_S$, we need to check that

$$A \times_S \mathscr{O}(S) \times_S L \coprod_{A \times_S \mathscr{O}(S) \times_S K} B \times_S \mathscr{O}(S) \times_S K \longrightarrow B \times_S \mathscr{O}(S) \times_S L$$

is a trivial cofibration in $s\mathbf{Set}^+_{/S}$. By Prp. 3.1, $- \times_S \mathscr{O}(S) \times_S K$ preserves trivial cofibrations and ditto for L. The result then follows. □

A final word on notation: since $\underline{\mathrm{Fun}}_S(-,-)$ is only well-defined and fibrant when both variables are fibrant, we will henceforth cease to denote the markings on the variables.

S-categories of S-objects. For the convenience of the reader, we briefly review the construction and basic properties of the S-category of S-objects in an ∞-category C. This material is originally due to D. Nardin in [2, §7].

3.8. **Construction.** The span

$$S^\sharp \xleftarrow{\mathrm{ev}_0} \mathscr{O}(S)^\natural \longrightarrow \Delta^0$$

defines a right Quillen functor $s\mathbf{Set}^+ \longrightarrow s\mathbf{Set}^+_{/S}$, which sends an ∞-category E to $\widetilde{\mathrm{Fun}}_S(\mathscr{O}(S), E \times S)$. This is the *$S$-category of objects in E*, which we will denote by $\underline{E}_S$.

The next proposition shows that the functor $E \mapsto \underline{E}_S$ implements the right adjoint to $\mathbf{Cat}^{cocart}_{\infty/S} \longrightarrow \mathbf{Cat}_\infty$ at the level of cocartesian fibrations.

3.9. **Proposition.** *Suppose C a S-category and E an ∞-category. Then we have an equivalence*

$$\psi : \mathrm{Fun}_S(C, \underline{E}_S) \xrightarrow{\simeq} \mathrm{Fun}(C, E)$$

Proof. Consider the commutative diagram

$$\begin{tikzcd} C^\sim \arrow[r] \arrow[d] & \mathscr{O}(S)^\natural \arrow[r] \arrow[d] & \Delta^0 \\ {}_\natural C \arrow[r] \arrow[d] & S^\sharp & \\ \Delta^0 & & \end{tikzcd}$$

Given an ∞-category E, travelling along the outer span yields $\mathrm{Fun}(C, E)$, travelling along the two inner spans yields $\mathrm{Fun}_S(C, \underline{E}_S)$, and the comparison functor ψ is induced by the map $\iota : C^\sim \longrightarrow {}_\natural C \times_S \mathscr{O}(S)^\natural$. By [2, 6.2], ι is a homotopy equivalence in $s\mathbf{Set}^+_{/S}$. Therefore, combining Lm. 2.26 and Lm. 2.27, we deduce the claim. □

3.10. **Example.** Let $E = \mathbf{Top}$ or $\mathbf{Cat}_\infty$. Then $\underline{\mathbf{Top}}_S$ resp. $\underline{\mathbf{Cat}}_{\infty,S}$ is the S-category of S-spaces resp. S-categories. In particular, suppose $E = \mathbf{Top}$ and $S = \mathbf{O}_G^{op}$. Then we also call $\underline{\mathbf{Top}}_{\mathbf{O}_G^{op}}$ the G-∞-category of G-spaces. Note that the fiber of this cocartesian fibration over a transitive G-set G/H is equivalent to the ∞-category of H-spaces $\mathrm{Fun}(\mathbf{O}_H^{op}, \mathbf{Top})$, and the pushforward functors are given by restriction along a subgroup and conjugation.

4. Join and slice

The join and slice constructions are at the heart of the ∞-categorical approach to limits and colimits. In this section, we introduce relative join and slice constructions and explore their homotopical properties.

The S-join.

4.1. **Definition.** Let $\iota : S \times \partial\Delta^1 \longrightarrow S \times \Delta^1$ be the inclusion. Define the *S-join* to be the functor

$$(- \star_S -) = \iota_* : s\mathbf{Set}_{/S \times \partial\Delta^1} \longrightarrow s\mathbf{Set}_{/S\times\Delta^1}.$$

Define the *marked S-join* to be the functor

$$(- \star_S -) = \iota_* : s\mathbf{Set}^+_{/S^\sharp \times (\partial\Delta^1)^\flat} \longrightarrow s\mathbf{Set}^+_{/S^\sharp\times(\Delta^1)^\flat}.$$

4.2. **Notation.** Given X, Y marked simplicial sets over S, we will usually refer to the structure maps to S by $\pi_1 : X \longrightarrow S$, $\pi_2 : Y \longrightarrow S$, and $\pi : X \star_S Y \longrightarrow S$. Explicitly, a $(i+j+1)$-simplex λ of $X \star_S Y$ is the data of simplices $\sigma : \Delta^i \longrightarrow X$, $\tau : \Delta^j \longrightarrow Y$, and $\pi \circ \lambda : \Delta^i \star \Delta^j \longrightarrow S$ such that the diagram

$$\begin{tikzcd} \Delta^i \arrow[r] \arrow[d, "\sigma"] & \Delta^i \star \Delta^j \arrow[d, "\pi\lambda"] & \Delta^j \arrow[l] \arrow[d, "\tau"] \\ X \arrow[r, "\pi_1"] & S & Y \arrow[l, "\pi_2"'] \end{tikzcd}$$

commutes. We will sometimes write $\lambda = (\sigma, \tau)$ so as to remember the data of the i-simplex of X and the j-simplex of Y in the notation. If given an n-simplex of $X \star_S Y$, we will indicate the decomposition of Δ^n given by the structure map to Δ^1 as $\Delta^{n_0} \star \Delta^{n_1}$ (with either side possibly empty).

4.3. **Proposition.** *Let $\iota : S \times \partial\Delta^1 \longrightarrow S \times \Delta^1$ be the inclusion. Then*

(a) $\iota_* : s\mathbf{Set}_{/S\times\partial\Delta^1} \longrightarrow s\mathbf{Set}_{/S\times\Delta^1}$ is a right Quillen functor.
(b) $\iota_* : s\mathbf{Set}^+_{/S^\sharp\times(\partial\Delta^1)^\flat} \longrightarrow s\mathbf{Set}^+_{/S^\sharp\times(\Delta^1)^\flat}$ is a right Quillen functor.

Consequently, if X and Y are categorical resp. cocartesian fibrations over S, then $X \star_S Y$ is a categorical resp. cocartesian fibration over S, with the cocartesian edges given by those in X and Y.

Proof. For (b), we verify the hypotheses of Thm. 2.23. All of the requirements are immediate except for (1) and (2).

(1): Let (s, i) be a vertex of $S^\sharp \times (\partial\Delta^1)^\flat$, $i = 0$ or 1. Let $f : (s', i') \longrightarrow (s, i)$ be a marked edge in $S^\sharp \times (\Delta^1)^\flat$. Then $i' = i$ and f viewed as an edge in $S^\sharp \times (\partial\Delta^1)^\flat$ is locally ι-cartesian.

(2): It is obvious that $\partial\Delta^1 \longrightarrow \Delta^1$ is a flat categorical fibration, so by stability of flat categorical fibrations under base change, $S \times \partial\Delta^1 \longrightarrow S \times \Delta^1$ is a flat categorical fibration.

(a) also follows from (2) by [11, B.4.5]. By (a), if X and Y are categorical fibrations over S, $X \star_S Y$ is a categorical fibration over $S \times \Delta^1$. The projection map $S \times \Delta^1 \longrightarrow S$ is a categorical fibration, so $X \star_S Y$ is also a categorical fibration over S. By (b), if X and Y are cocartesian fibrations over S, ${}_\natural X \star_S {}_\natural Y$ is fibrant in $s\mathbf{Set}^+_{/S^\sharp\times(\Delta^1)^\flat}$. Since $S^\sharp \times (\Delta^1)^\flat$ is marked as a cocartesian fibration over S, ${}_\natural X \star_S {}_\natural Y$ is marked as a cocartesian fibration over S. $\square$

We have the compatibility of the relative join with base change.

4.4. **Lemma.** *Let $f : T \longrightarrow S$ be a functor and let X and Y be (marked) simplicial sets over S. Then we have a canonical isomorphism*

$$(X \star_S Y) \times_S T \cong (X \times_S T) \star_T (Y \times_S T).$$

Proof. From the pullback square

$$\begin{tikzcd} T \times \partial\Delta^1 \arrow[r, "\iota_T"] \arrow[d, "f\times id"] & T \times \Delta^1 \arrow[d, "f\times id"] \\ S \times \partial\Delta^1 \arrow[r, "\iota_S"] & S \times \Delta^1 \end{tikzcd}$$

we obtain the base-change isomorphism $f^*(\iota_S)_* \cong (\iota_T)_* f^*$. □

In [9, §4.2.2], Lurie introduces the relative 'diamond' join operation $\diamond_S$, which we now recall. Given X and Y marked simplicial sets over S, define

$$X \diamond_S Y = X \sqcup_{X \times_S Y \times \{0\}} X \times_S Y \times (\Delta^1)^\flat \sqcup_{X\times_S Y \times \{1\}} Y.$$

There is a comparison map $\psi_{(X,Y)} : X \diamond_S Y \longrightarrow X \star_S Y = \iota_*(X, Y)$, adjoint to the isomorphism $\iota^*(X \star_S Y) \cong (X, Y)$.

4.5. **Lemma.** *$\psi_{(X,S)} : X \diamond_S S^\sharp \longrightarrow X \star_S S^\sharp$ is a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$. Dually, if X is fibrant, then $\psi_{(S,X)}$ is a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$.*

Proof. We first address the map $\psi_{(X,S)}$. By left properness of the cocartesian model structure, the defining pushout for $X \diamond_S S$ is a homotopy pushout. By Thm. 4.16, $- \star_S S$ preserves cocartesian equivalences. Therefore, choosing a fibrant replacement for X and using naturality of the comparison map $\psi_{(X,S)}$, we may reduce to the case that X is fibrant. Then we have to check that

$$\begin{tikzcd} X \times \{1\} \arrow[r] \arrow[d] & X \times (\Delta^1)^\flat \arrow[d] \\ S^\sharp \arrow[r] & X \star_S S^\sharp \end{tikzcd}$$

is a homotopy pushout square. Since this is a square of fibrant objects, this assertion can be checked fiberwise, in which case it reduces to the equivalence $X_s \diamond \Delta^0 \xrightarrow{\simeq} X^\rhd$ of [9, 4.2.1.2].

The second statement concerning $\psi_{(S,X)}$ follows by the same type of argument. □

4.6. **Warning.** In general, $\psi_{(X,Y)}$ is not a cocartesian equivalence. As a counterexample, consider $S = \Delta^1$, $X = \{0\}$, and $Y = \{1\}$. Then $\psi_{(X,Y)}$ is the inclusion of $X \diamond_S Y \cong \Delta^{\{0\}} \sqcup \Delta^{\{1\}}$ into $X \star_S Y \cong \Delta^1$, which is not a cocartesian equivalence over Δ^1.

We will later need the following strengthening of the conclusion of Prp. 4.3.

4.7. **Proposition.**
1. *Let $C, C', D \longrightarrow S$ be inner fibrations and let $C, C' \longrightarrow D$ be functors. Then $C \star_D C' \longrightarrow S$ is an inner fibration.*
2. *Let $C, C', D \longrightarrow S$ be S-categories and let $C, C' \longrightarrow D$ be S-functors. Then $C \star_D C' \longrightarrow S$ is a S-category with cocartesian edges given by those in C or C', and $C \star_D C' \longrightarrow D$ is a S-functor.*

Proof. (1) Let $0 < k < n$. We need to solve the lifting problem

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\lambda_0"] \arrow[d] & C \star_D C' \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted, "\lambda"] & S. \end{tikzcd}$$

If λ_0 lands entirely in C or C', then we are done by assumption, so suppose not. Let $\overline{\lambda} : \Delta^n \longrightarrow D$ be a lift in the commutative square

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & D \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, "\overline{\lambda}"] & S. \end{tikzcd}$$

Define λ using the data $(\lambda_0|_{\Delta^{n_0}}, \lambda_0|_{\Delta^{n_1}}, \overline{\lambda})$. Then λ is a valid lift.

(2) Consider $C \star_D C'$ as a marked simplicial set with marked edges those in ${}_\natural C$ or in ${}_\natural C'$. We need to solve the lifting problem

$$\begin{tikzcd} {}_\natural\Lambda^n_0 \arrow[r, "\lambda_0"] \arrow[d] & C \star_D C' \arrow[d] \\ {}_\natural\Delta^n \arrow[r] \arrow[ur, dotted, "\lambda"] & S. \end{tikzcd}$$

Again, if λ_0 lands entirely in C or C', then we are done by assumption, so suppose not (so that $n \geq 2$ and the marked edge lies in C). Let $\overline{\lambda} : \Delta^n \longrightarrow D$ be a lift in the commutative square

$$\begin{tikzcd} {}_\natural\Lambda^n_0 \arrow[r] \arrow[d] & {}_\natural D \arrow[d] \\ {}_\natural\Delta^n \arrow[r] \arrow[ur, "\overline{\lambda}"] & S. \end{tikzcd}$$

Define λ using the data $(\lambda_0|_{\Delta^{n_0}}, \lambda_0|_{\Delta^{n_1}}, \overline{\lambda})$. Then λ is a valid lift. Finally, note that we may obviously lift against classes (3) and (4) of [9, 3.1.1.1]. We conclude that $C \star_D C' \longrightarrow S$ is fibrant in $s\mathbf{Set}^+_{/S}$, hence an S-category with cocartesian edges as marked.

□

Since the S-join is defined as a right Kan extension, it is simple to map into. In the other direction, we can offer the following lemma.

4.8. **Lemma.** *Let C, C', D, and E be S-categories and let $C, C' \longrightarrow D$ be S-functors. Then*

$$\mathrm{Fun}_S(C \star_D C', E) \longrightarrow \mathrm{Fun}_S(C, E) \times \mathrm{Fun}_S(C', E)$$

is a bifibration (Dfn. [9, 2.4.7.2]*). Consequently,*

$$\mathrm{Fun}_S(C \star_D C', E) \longrightarrow \mathrm{Fun}_S(C, E)$$

is a cartesian fibration with cartesian edges those sent to equivalences in $\mathrm{Fun}_S(C', E)$*, and*

$$\mathrm{Fun}_S(C \star_D C', E) \longrightarrow \mathrm{Fun}_S(C', E)$$

is a cocartesian fibration with cocartesian edges those sent to equivalences in $\mathrm{Fun}_S(C', E)$*.*

Proof. By inspection, the span

$$(\Delta^1)^\flat \xleftarrow{\pi} {}_\natural(C \star_D C') \xrightarrow{\pi'} S^\sharp$$

satisfies the hypotheses of Thm. 2.23. Therefore, $\pi_* \pi'^* ({}_\natural E \longrightarrow S)$ is a categorical fibration over Δ^1. The claim now follows from [9, 2.4.7.10], and the consequence from [9, 2.4.7.5] and its opposite. □

The Quillen adjunction between S-join and S-slice. Our next goal is to obtain a relative join and slice Quillen adjunction. To this end, we need a good understanding of the combinatorics of the relative join (Prp. 4.11). We prepare for the proof of that proposition with a few lemmas.

4.9. **Lemma.** *Let $i, l \geq -1$ and $j, k \geq 0$. Then*

$$\Delta^i \star \Delta^j \star \partial\Delta^k \star \Delta^l \coprod_{\Delta^j \star \partial\Delta^k \star \Delta^l} \Delta^{j+k+l+2} \longrightarrow \Delta^{i+j+k+l+3}$$

is inner anodyne.

Proof. Let $f : \Delta^{j-1} \longrightarrow \Delta^i \star \Delta^{j-1}$ and $g : \Lambda^{k+1}_0 \longrightarrow \Delta^{k+1}$. The map in question is $f \star g \star \Delta^l$, so is inner anodyne by [9, 2.1.2.3]. □

By [9, 2.1.2.4], the join of a left anodyne map and an inclusion is left anodyne. We need a slight refinement of this result:

4.10. **Lemma.** *Let $f : A_0 \longrightarrow A$ be a cofibration of simplicial sets.*

(1) Let $g : B_0 \longrightarrow B$ be a right marked anodyne map between marked simplicial sets. Then

$$f^\flat \star g : A_0^\flat \star B \coprod_{A_0^\flat \star B_0} A^\flat \star B_0 \longrightarrow A^\flat \star B$$

is a right marked anodyne map.

(2) Let $g : B_0 \longrightarrow B$ be a left marked anodyne map between marked simplicial sets. Then

$$g \star f^\flat : B \star A_0^\flat \coprod_{B_0 \star A_0^\flat} B_0 \star A^\flat \longrightarrow B \star A^\flat$$

is a left marked anodyne map.

Proof. We prove (1); the dual assertion (2) is proven by a similar argument. f lies in the weakly saturated closure of the inclusions $i_m : \partial\Delta^m \longrightarrow \Delta^m$, so it suffices to check that $i_m^\flat \star g$ is right marked anodyne for the four classes of morphisms enumerated in [9, 3.1.1.1]. For $g : (\Lambda_i^n)^\flat \longrightarrow (\Delta^n)^\flat$, $0 < i < n$, $i_m^\flat \star g$ obtained from an inner anodyne map by marking common edges, so is marked right anodyne. For $g : \Lambda_n^{n\natural} \longrightarrow \Delta^{n\natural}$, $i_m^\flat \star g$ is $\Lambda_{n+m+1}^{n+m+1\natural} \longrightarrow \Delta^{n+m+1\natural}$, so $i_m^\flat \star g$ is marked right anodyne. For the remaining two classes, $i_m^\flat \star g$ is the identity because no markings are introduced when joining two marked simplicial sets. $\square$

The following proposition reveals a basic asymmetry of the relative join, which is related to our choice of *cocartesian* fibrations to model functors.

4.11. **Proposition.** *Let K be a marked simplicial set over S.*

(1) *For every ${}_\natural\Lambda_0^n \longrightarrow {}_\natural\Delta^n$ a map of marked simplicial sets over S,*

$$K \star_S ({}_\natural\Lambda_0^n \times_S \mathscr{O}(S)^\natural) \longrightarrow K \star_S ({}_\natural\Delta^n \times_S \mathscr{O}(S)^\natural)$$

is left marked anodyne, where the pullbacks ${}_\natural\Lambda_0^n \times_S \mathscr{O}(S)^\natural$ and ${}_\natural\Delta^n \times_S \mathscr{O}(S)^\natural$ are formed with respect to the source map e_0 and are regarded as marked simplicial sets over S via the target map e_1.

(1') *For every $\Lambda_0^n \longrightarrow \Delta^n$ a map of simplicial sets over S,*

$$\Delta^n \times_S \mathscr{O}(S) \coprod_{\Lambda_0^n \times_S \mathscr{O}(S)} K \star_S (\Lambda_0^n \times_S \mathscr{O}(S)) \longrightarrow K \star_S (\Delta^n \times_S \mathscr{O}(S))$$

is an inner anodyne map.

(2) *Let $e_0 : C \longrightarrow S$ be a cartesian fibration over S and let $e_1 : C \longrightarrow S$ be any map of simplicial sets. For every $\Lambda_k^n \longrightarrow \Delta^n$, $0 < k < n$ a map of simplicial sets over S,*

$$K \star_S (\Lambda_k^n \times_S C) \longrightarrow K \star_S (\Delta^n \times_S C)$$

is inner anodyne, where the pullbacks $\Lambda_k^n \times_S C$ and $\Delta^n \times_S C$ are formed with respect to e_0 and are regarded as simplicial sets over S via e_1.

(3) *For every $\Lambda_n^{n\natural} \longrightarrow \Delta^{n\natural}$ a map of simplicial sets over S,*

$$K \star_S \Lambda_n^{n\natural} \longrightarrow K \star_S \Delta^{n\natural}$$

is right marked anodyne.

Proof. Let I be the set of simplices of K endowed with a total order such that $\sigma < \sigma'$ if the dimension of σ is less than that of σ', where we view the empty set as a simplex of dimension -1. Let J be the set of epimorphisms $\chi : \Delta^j \longrightarrow \Delta^{n-1}$ endowed with a total order such that $\chi < \chi'$ if the dimension of χ is less than that of χ'. Order $I \times J$ by $(\sigma, \chi) < (\sigma', \chi')$ if $\sigma < \sigma'$ or $\sigma = \sigma'$ and $\chi < \chi'$. For any simplex $\tau : \Delta^j \longrightarrow \Delta^n$, we let $r_k(\tau)$ be the pullback

$$\begin{tikzcd}
\Delta^{r_k(\tau)_0} \arrow[r, "r_k(\tau)"] \arrow[d] & \Delta^{n-1} \arrow[d, "d_k"] \\
\Delta^j \arrow[r, "\tau"] & \Delta^n
\end{tikzcd}$$

We will let ι denote the map under consideration. We first prove (1). Given $\sigma \in I$ and $\chi \in J$, let $X_{\sigma,\chi}$ be the sub-marked simplicial set of $K \star_S ({}_\natural\Delta^n \times_S \mathscr{O}(S)^\natural)$ on $K \star_S ({}_\natural\Lambda^n_0 \times_S \mathscr{O}(S)^\natural)$ and simplices $(\sigma', \tau') : \Delta^i \star \Delta^j \longrightarrow K \star_S (\Delta^n \times_S \mathscr{O}(S))$ not in $K \star_S (\Lambda^n_0 \times_S \mathscr{O}(S))$ with $(\sigma', r_0(e_0 \circ \tau')) \leq (\sigma, \chi)$. If $(\sigma, \chi) < (\sigma', \chi')$, then we have an obvious inclusion $X_{\sigma,\chi} \longrightarrow X_{\sigma',\chi'}$, and we let

$$X_{<(\sigma,\chi)} = ({}_\natural\Lambda^n_0 \times_S \mathscr{O}(S)^\natural) \bigcup (\cup_{(\sigma',\chi')<(\sigma,\chi)} X_{\sigma,\chi}).$$

Since $K \star_S ({}_\natural\Delta^n \times_S \mathscr{O}(S)^\natural) = \operatorname{colim}_{(\sigma,\chi)\in I\times J} X_{\sigma,\chi}$, in order to show that ι is left marked anodyne it suffices to show that $X_{<(\sigma,\chi)} \longrightarrow X_{\sigma,\chi}$ is left marked anodyne for all $(\sigma, \chi) \in I \times J$. We will say that a simplex of $X_{\sigma,\chi}$ is *new* if it does not belong to $X_{<(\sigma,\chi)}$.

Let $\sigma : \Delta^i \longrightarrow K$ be an element of I and $\chi : \Delta^j \longrightarrow \Delta^{n-1}$ an element of J. Let $\lambda = (\sigma, \tau) : \Delta^i \star \Delta^j \longrightarrow K \star_S (\Delta^n \times_S \mathscr{O}(S))$ be any nondegenerate new simplex of $X_{\sigma,\chi}$, so $r_0(e_0 \circ \tau) = \chi$. Let $\bar{\chi} : \Delta^{j+1} \longrightarrow \Delta^n$ be the unique epimorphism with $r_0(\bar{\chi}) = \chi$ and let $e : \Delta^1 \longrightarrow \Delta^n \times_S \mathscr{O}(S)$ be a cartesian edge over $\{0, 1\}$ with $e(1) = \tau(0)$. The inclusion $(\Delta^1)^\sharp \coprod_{\Delta^0} \Delta^j \longrightarrow {}_\natural\Delta^{j+1}$ is right marked anodyne, so we have a lift $\bar{\tau}$ in the following diagram

$$\begin{tikzcd}
\Delta^1 \coprod_{\Delta^0} \Delta^j \arrow[r, "\tau \cup e"] \arrow[d] & \Delta^n \times_S \mathscr{O}(S) \arrow[d] \\
\Delta^{j+1} \arrow[r, "\bar{\chi}"] \arrow[ur, dotted, "\bar{\tau}"] & \Delta^n
\end{tikzcd}$$

By Lm. 4.10,

$$\Delta^i \star \Delta^j \coprod_{\Delta^j} {}_\natural\Delta^{j+1} \longrightarrow \Delta^i \star {}_\natural\Delta^{j+1}$$

is right marked anodyne. Using that $(e_1 \circ \bar{\tau})(e)$ is an equivalence, we obtain a lift

$$\begin{tikzcd}
\Delta^i \star \Delta^j \coprod_{\Delta^j} {}_\natural\Delta^{j+1} \arrow[r, "\pi\lambda \cup e_1\bar{\tau}"] \arrow[d] & S^\sim \\
\Delta^i \star {}_\natural\Delta^{j+1} \arrow[ur, dotted] &
\end{tikzcd}$$

which allows us to define $\bar{\lambda} : \Delta^i \star \Delta^{j+1} \longrightarrow K \star_S (\Delta^n \times_S \mathscr{O}(S))$ extending λ and $\bar{\tau}$. Then $\bar{\lambda}$ is nondegenerate and every face of $\bar{\lambda}$ except for $\lambda = d_{i+1}(\bar{\lambda})$ lies in $X_{<(\sigma,\chi)}$. We may thus form the pushout

$$\begin{tikzcd}
\coprod_\lambda (\Lambda^{i+j+2}_{i+1}, \{i+1, i+2\}) \arrow[r] \arrow[d] & X_{<(\sigma,\chi)} \arrow[d] \\
\coprod_\lambda (\Delta^{i+j+2}, \{i+1, i+2\}) \arrow[r] & X_{<(\sigma,\chi),1}
\end{tikzcd}$$

which factors the inclusion $X_{<(\sigma,\chi)} \longrightarrow X_{(\sigma,\chi)}$ as the composition of a left marked anodyne map and an inclusion (there is one further complication involving markings: in the special case $n = 1$, $\sigma = \emptyset$, $j = 1$, we may have that $\lambda = \tau$ is a marked edge, i.e. an equivalence over 1. Then the edges of $\bar{\tau}$ are all marked, so we should form the pushout via maps $(\Lambda^2_0)^\sharp \longrightarrow (\Delta^2)^\sharp$, which are left marked anodyne by [9, 3.1.1.7]).

Now for the inductive step suppose that we have defined a sequence of left marked anodyne maps

$$X_{<(\sigma,\chi)} \longrightarrow \ldots \longrightarrow X_{<(\sigma,\chi),m} \subset X_{(\sigma,\chi)}$$

such that for all $0 < l \leq m$ all new nondegenerate simplices in $X_{(\sigma,\chi)}$ of dimension $i + l + j$ lie in $X_{<(\sigma,\chi),l}$ and admit an extension to a $i + l + j + 1$-simplex with the edge $\{i + l, i + l + 1\}$ marked in $X_{<(\sigma,\chi),l}$, and no new nondegenerate simplices of dimension $> i + j + l$ lie in $X_{<(\sigma,\chi),l}$. Let $\lambda = (\sigma, \tau)$ be any new nondegenerate $i + m + j + 1$-simplex not in $X_{<(\sigma,\chi),m}$. For $0 \leq l < m$ let $\lambda_l = (\sigma, \tau_l)$ be a nondegenerate $i+m+j+1$-simplex in $X_{<(\sigma,\chi),m}$ with $d_{i+m}(\lambda_l) = d_{i+l+1}(\lambda)$ and edge $\{i+m, i+m+1\}$ marked. τ and $\tau_0, ..., \tau_{m-1}$ together define a map

$$\tau' : \Lambda^{m+1}_{m+1} \star \Delta^{j-1} \longrightarrow \Delta^n \times_S \mathscr{O}(S)$$

where the domain of τ is the subset $\{0,...,m-1,m+1,...,m+j+1\}$ and the domain of τ_l is the subset $\{0,...,\widehat{l},...,m+j+1\}$. Observe that the map $\Lambda^{m+1\natural}_{m+1} \star \Delta^{j-1} \longrightarrow \Delta^{m+1\natural} \star \Delta^{j-1}$ is right marked anodyne, since it factors as

$$\Lambda^{m+1\natural}_{m+1} \star \Delta^{j-1} \longrightarrow \Delta^{m+1\natural} \coprod_{\Lambda^{m+1\natural}_{m+1}} \Lambda^{m+1\natural}_{m+1} \star \Delta^{j-1} \longrightarrow \Delta^{m+1\natural} \star \Delta^{j-1}$$

where the first map is obtained as the pushout of the right marked anodyne map $\Lambda^{m+1\natural}_{m+1} \longrightarrow \Delta^{m+1\natural}$ along the inclusion $\Lambda^{m+1\natural}_{m+1} \longrightarrow \Lambda^{m+1\natural}_{m+1} \star \Delta^{j-1}$ and the second map is obtained by marking a common edge of an inner anodyne map. Let $\bar{\chi} : \Delta^{m+j+1} \longrightarrow \Delta^n$ be the unique epimorphism with $r_0(\bar{\chi}) = \chi$. Then we have a lift $\bar{\tau}$ in the following commutative diagram

$$\begin{tikzcd}
\Lambda^{m+1}_{m+1} \star \Delta^{j-1} \arrow[r, "\tau'"] \arrow[d] & \Delta^n \times_S \mathscr{O}(S) \arrow[d] \\
\Delta^{m+1} \star \Delta^{j-1} \arrow[r, "\bar{\chi}"] \arrow[ur, dotted, "\bar{\tau}"] & \Delta^n
\end{tikzcd}$$

By Lm. 4.10, the map

$$\Delta^i \star \Lambda^{m+1\natural}_{m+1} \star \Delta^{j-1} \coprod_{\Lambda^{m+1\natural}_{m+1} \star \Delta^{j-1}} \Delta^{m+1\natural} \star \Delta^{j-1} \longrightarrow \Delta^i \star \Delta^{m+1\natural} \star \Delta^{j-1}$$

is right marked anodyne. Since $(e_1 \circ \bar{\tau})(\{m,m+1\})$ is an equivalence, we may extend $(\cup_l \pi\lambda_l) \cup \pi\lambda \cup e_1\bar{\tau}$ to a map $\Delta^{i+m+j+2} \longrightarrow S$, which defines a nondegenerate $(i+m+j+2)$-simplex $\bar{\lambda}$ with λ as its $(i+m+1)$th face and which extends $\bar{\tau}$. By construction every other face of $\bar{\lambda}$ lies in $X_{<(\sigma,\chi),m}$. Thus we may form the pushout

$$\begin{tikzcd}
\coprod_\lambda (\Lambda^{i+m+j+2}_{i+m+1}, \{i+m+1, i+m+2\}) \arrow[r] \arrow[d] & X_{<(\sigma,\chi),m} \arrow[d] \\
\coprod_\lambda (\Delta^{i+m+j+2}, \{i+m+1, i+m+2\}) \arrow[r] & X_{<(\sigma,\chi),m+1}
\end{tikzcd}$$

and complete the inductive step (again, there is one further complication involving markings: in the special case $i=-1$, $n=1$, $j=0$, $m=1$, we may have that λ is marked. Then every edge of $\bar{\lambda}$ is marked since $(\Lambda^2_2)^\sharp \longrightarrow (\Delta^2)^\sharp$ is right marked anodyne, and we form the pushout along maps $(\Lambda^2_1)^\sharp \longrightarrow (\Delta^2)^\sharp$). Passing to the colimit, we deduce that $X_{<(\sigma,\chi)} \longrightarrow X_{\sigma,\chi}$ is marked left anodyne, which completes the proof.

For (1'), simply observe that if $i > -1$ we are attaching along inner horns.

We now modify the above proof to prove (2). Let $X_{\sigma,\chi}$ be the sub-simplicial set of $K \star_S (\Delta^n \times_S C)$ on $K \star_S (\Lambda^n_k \times_S C)$ and simplices $(\sigma',\tau') : \Delta^i \star \Delta^j \longrightarrow K \star_S (\Delta^n \times_S C)$ not in $K \star_S (\Lambda^n_k \times_S C)$ with $(\sigma', r_k(e_0 \circ \tau')) \leq (\sigma,\chi)$. Let $X_{<(\sigma,\chi)} = (K \star (\Lambda^n_k \times_S C)) \bigcup (\cup_{(\sigma',\chi')<(\sigma,\chi)} X_{\sigma,\chi})$. We will show that $X_{<(\sigma,\chi)} \longrightarrow X_{\sigma,\chi}$ is inner anodyne for all $(\sigma,\chi) \in I \times J$.

Let $\sigma : \Delta^i \longrightarrow K$ be an element of I, $\chi : \Delta^j \longrightarrow \Delta^{n-1}$ an element of J, and let k' be the first vertex of χ with $\chi(k') = k$. Let $\lambda = (\sigma,\tau) : \Delta^i \star \Delta^j \longrightarrow K \star_S (\Delta^n \times_S C)$ be any nondegenerate new simplex of $X_{\sigma,\chi}$, so $r_k(e_0 \circ \tau) = \chi$. Let $\bar{\chi} : \Delta^{j+1} \longrightarrow \Delta^n$ be the unique epimorphism with $r_k(\bar{\chi}) = \chi$. Combining [9, 2.1.2.3] and Lm. 4.10, we see that the inclusion

$$d_{k'} : \Delta^j = \Delta^{k'-1} \star \Delta^{j-k'} \longrightarrow \Delta^{k'-1} \star_\natural \Delta^{j-k'+1}$$

is right marked anodyne, so we have a lift $\bar{\tau}$ in the following diagram

$$\begin{tikzcd}
\Delta^j \arrow[r, "\tau"] \arrow[d] & \Delta^n \times_S C \arrow[d] \\
\Delta^{j+1} \arrow[r, "\bar{\chi}"] \arrow[ur, dotted, "\bar{\tau}"] & \Delta^n
\end{tikzcd}$$

where $\bar{\tau}(\{k', k'+1\})$ is a cartesian edge. By Lm. 4.9, $\Delta^i \star \Delta^j \coprod_{\Delta^j} \Delta^{j+1} \longrightarrow \Delta^i \star \Delta^{j+1}$ is inner anodyne. We thus obtain an extension

$$\begin{tikzcd} \Delta^i \star \Delta^j \coprod_{\Delta^j} \Delta^{j+1} \arrow[r, "\pi\lambda \cup e_1\bar{\tau}"] \arrow[d] & S \\ \Delta^i \star \Delta^{j+1} \arrow[ur, dotted] & \end{tikzcd}$$

which allows us to define $\bar{\lambda} : \Delta^i \star \Delta^{j+1} \longrightarrow K \star_S (\Delta^n \times_S C)$ extending λ and $\bar{\tau}$. Then $\bar{\lambda}$ is nondegenerate and every face of $\bar{\lambda}$ except for $\lambda = d_{i+k'+1}(\bar{\lambda})$ lies in $X_{<(\sigma,\chi)}$. We may thus form the pushout

$$\begin{tikzcd} \coprod_\lambda \Lambda^{i+j+2}_{i+k'+1} \arrow[r] \arrow[d] & X_{<(\sigma,\chi)} \arrow[d] \\ \coprod_\lambda \Delta^{i+j+2} \arrow[r] & X_{<(\sigma,\chi),1} \end{tikzcd}$$

which factors the inclusion $X_{<(\sigma,\chi)} \longrightarrow X_{(\sigma,\chi)}$ as the composition of an inner anodyne map and an inclusion.

Now for the inductive step suppose that we have defined a sequence of inner anodyne maps

$$X_{<(\sigma,\chi)} \longrightarrow ... \longrightarrow X_{<(\sigma,\chi),m} \subset X_{(\sigma,\chi)}$$

such that for all $0 < l \leq m$ all new nondegenerate simplices in $X_{(\sigma,\chi)}$ of dimension $i+l+j$ lie in $X_{<(\sigma,\chi),l}$ and admit an extension to a $i+l+j+1$-simplex such that the edge $\{i+k'+l, i+k'+l+1\}$ is sent to a cartesian edge of $\Delta^n \times_S C$, and no new nondegenerate simplices of dimension $> i+j+l$ lie in $X_{<(\sigma,\chi),l}$. Let $\lambda = (\sigma, \tau)$ be any new nondegenerate $i+m+j+1$-simplex not in $X_{<(\sigma,\chi),m}$. For $0 \leq l < m$ let $\lambda_l = (\sigma, \tau_l)$ be a nondegenerate $i+m+j+1$-simplex in $X_{<(\sigma,\chi),m}$ with $d_{i+m+k'}(\lambda_l) = d_{i+l+k'+1}(\lambda)$. τ and $\tau_0, ..., \tau_{m-1}$ together define a map

$$\tau' : \Delta^{k'-1} \star \Lambda^{m+1}_{m+1} \star \Delta^{j-k'-1} \longrightarrow \Delta^n \times_S C$$

where the domain of τ is the subset $\{0, ..., k'+m-1, k'+m+1, ..., m+j+1\}$ and the domain of τ_l is the subset $\{0, ..., \widehat{k'+l}, ..., m+j+1\}$. The map

$$\Delta^{k'-1} \star \Lambda^{m+1 \natural}_{m+1} \star \Delta^{j-k'-1} \longrightarrow \Delta^{k'-1} \star \Delta^{m+1 \natural} \star \Delta^{j-k'-1}$$

is $\Delta^{k'-1}$ joined with a right marked anodyne map, so is right marked anodyne by Lm. 4.10. Let $\bar{\chi} : \Delta^{m+j+1} \longrightarrow \Delta^n$ be the unique epimorphism with $r_k(\bar{\chi}) = \chi$. Then we have a lift $\bar{\tau}$ in the following commutative diagram

$$\begin{tikzcd} \Delta^{k'-1} \star \Lambda^{m+1}_{m+1} \star \Delta^{j-k'-1} \arrow[r, "\tau'"] \arrow[d] & \Delta^n \times_S C \arrow[d] \\ \Delta^{m+j+1} \arrow[r, "\bar{\chi}"] \arrow[ur, dotted, "\bar{\tau}"] & \Delta^n \end{tikzcd}$$

such that $\bar{\tau}(\{k'+m, k'+m+1\})$ is a cartesian edge. By Lemma 4.9, the map

$$\Delta^i \star \Delta^{k'-1} \star \partial\Delta^m \star \Delta^{j-k'} \coprod_{\Delta^{k'-1} \star \partial\Delta^m \star \Delta^{j-k'}} \Delta^{m+j+1} \longrightarrow \Delta^{i+m+j+2}$$

is inner anodyne. Therefore, we may extend $(\cup_l \pi\lambda_l) \cup \pi\lambda \cup e_1\bar{\tau}$ to a map $\Delta^{i+m+j+2} \longrightarrow S$, which defines a nondegenerate $(i+m+j+2)$-simplex $\bar{\lambda}$ with λ as its $(i+k'+m+1)$th face and which extends $\bar{\tau}$. By construction every other face of $\bar{\lambda}$ lies in $X_{<(\sigma,\chi),m}$. Thus we may form the pushout

$$\begin{tikzcd} \coprod_\lambda \Lambda^{i+m+j+2}_{i+k'+m+1} \arrow[r] \arrow[d] & X_{<(\sigma,\chi),m} \arrow[d] \\ \coprod_\lambda \Delta^{i+m+j+2} \arrow[r] & X_{<(\sigma,\chi),m+1} \end{tikzcd}$$

and complete the inductive step. Passing to the colimit, we deduce that $X_{<(\sigma,\chi)} \longrightarrow X_{\sigma,\chi}$ is inner anodyne, which completes the proof.

We finally modify the above proof to prove (3). Given $\sigma \in I$ and $\chi \in J$, let $X_{\sigma,\chi}$ be the sub-marked simplicial set of $K \star_S \Delta^{n\natural}$ on $K \star_S \Lambda^{n\natural}_n$ and simplices $(\sigma',\tau') : \Delta^i \star \Delta^j \longrightarrow K \star_S \Delta^{n\natural}$ not in $K \star_S \Lambda^{n\natural}_n$ with $(\sigma', r_n(\tau')) \leq (\sigma,\chi)$. Let $X_{<(\sigma,\chi)} = (K \star_S \Lambda^{n\natural}_n) \bigcup (\cup_{(\sigma',\chi')<(\sigma,\chi)} X_{\sigma,\chi})$. We will show that $X_{<(\sigma,\chi)} \longrightarrow X_{\sigma,\chi}$ is right marked anodyne for all $(\sigma,\chi) \in I \times J$.

Let $\sigma : \Delta^i \longrightarrow K$ be an element of I and $\chi : \Delta^j \longrightarrow \Delta^{n-1}$ an element of J. Let $\lambda = (\sigma,\tau) : \Delta^i \star \Delta^j \longrightarrow K \star_S \Delta^{n\natural}$ be any nondegenerate new simplex of $X_{\sigma,\chi}$, so $\tau = r_n(\tau) = \chi$. Let $\bar\chi : \Delta^{j+1} \longrightarrow \Delta^n$ be the unique epimorphism with $r_n(\bar\chi) = \chi$. By Lm. 4.9, the inclusion

$$\Delta^i \star \Delta^j \coprod_{\Delta^j} \Delta^{j+1} \longrightarrow \Delta^i \star \Delta^{j+1}$$

is inner anodyne, so we have an extension in the following diagram

$$\begin{tikzcd} \Delta^i \star \Delta^j \coprod_{\Delta^j} \Delta^{j+1} \arrow[r, "\pi\lambda \cup \pi_2\bar\chi"] \arrow[d] & S \\ \Delta^i \star \Delta^{j+1} \arrow[ur, dotted] & \end{tikzcd}$$

which allows us to define $\bar\lambda : \Delta^i \star \Delta^{j+1} \longrightarrow K \star_S \Delta^{n\natural}$ extending λ and $\bar\chi$. Then $\bar\lambda$ is nondegenerate and every face of $\bar\lambda$ except for $\lambda = d_{i+j+2}(\bar\lambda)$ lies in $X_{<(\sigma,\chi)}$. We may thus form the pushout

$$\begin{tikzcd} \coprod_\lambda \Lambda^{i+j+2\natural}_{i+j+2} \arrow[r] \arrow[d] & X_{<(\sigma,\chi)} \arrow[d] \\ \coprod_\lambda \Delta^{i+j+2\natural} \arrow[r] & X_{<(\sigma,\chi),1} \end{tikzcd}$$

which factors the inclusion $X_{<(\sigma,\chi)} \longrightarrow X_{(\sigma,\chi)}$ as the composition of a right marked anodyne map and an inclusion.

Now for the inductive step suppose that we have defined a sequence of right marked anodyne maps

$$X_{<(\sigma,\chi)} \longrightarrow ... \longrightarrow X_{<(\sigma,\chi),m} \subset X_{(\sigma,\chi)}$$

such that for all $0 < l \leq m$ all new nondegenerate simplices in $X_{(\sigma,\chi)}$ of dimension $i+l+j$ lie in $X_{<(\sigma,\chi),l}$ and admit an extension to a $i+l+j+1$-simplex, and no new nondegenerate simplices of dimension $> i+j+l$ lie in $X_{<(\sigma,\chi),l}$. Let $\lambda = (\sigma,\tau)$ be any new nondegenerate $i+m+j+1$-simplex not in $X_{<(\sigma,\chi),m}$. For $0 < l \leq m$ let $\lambda_l = (\sigma,\tau_l)$ be a nondegenerate $i+m+j+1$-simplex in $X_{<(\sigma,\chi),m}$ with $d_{i+m+j+1}(\lambda_l) = d_{i+j+l+1}(\lambda)$ (note that $\tau_l = \tau$). By Lm. 4.9, the map

$$\Delta^i \star \Delta^j \star \partial\Delta^m \coprod_{\Delta^j \star \partial\Delta^m} \Delta^j \star \Delta^m \longrightarrow \Delta^i \star \Delta^j \star \Delta^m$$

is inner anodyne. Therefore, we may extend $\pi\lambda \cup (\cup_l \pi\lambda_l)$ to a map $\Delta^{i+j+m+2} \longrightarrow S$ and define a $(i+j+m+2)$-simplex $\bar\lambda$ of $K \star \Delta^{n\natural}$ with $d_{i+j+m+2}\bar\lambda = \lambda$ and $d_{i+j+l+1}\bar\lambda = \lambda + l$. By construction every face of $\bar\lambda$ except for λ lies in $X_{<(\sigma,\chi),m}$. Thus we may form the pushout

$$\begin{tikzcd} \coprod_\lambda \Lambda^{i+j+m+2\natural}_{i+j+m+2} \arrow[r] \arrow[d] & X_{<(\sigma,\chi),m} \arrow[d] \\ \coprod_\lambda \Delta^{i+j+m+2\natural} \arrow[r] & X_{<(\sigma,\chi),m+1} \end{tikzcd}$$

and complete the inductive step. Passing to the colimit, we deduce that $X_{<(\sigma,\chi)} \longrightarrow X_{\sigma,\chi}$ is right marked anodyne, which completes the proof.

□

4.12. **Remark.** The proof of Proposition 4.11 can be adapted to show that for any cartesian fibration $C \longrightarrow S$, ${}_\natural\Lambda^n_0 \times_S C^\natural \longrightarrow {}_\natural\Delta^n \times_S C^\natural$ is marked left anodyne (in the $\sigma = \emptyset$ case, we only use that $\mathscr{O}(S) \longrightarrow S$ is a cartesian fibration). As well, letting $K = \emptyset$, part (2) of Proposition 4.11 shows that

$\Lambda^n_k \times_S C \longrightarrow \Delta^n \times_S C$ is inner anodyne. This refines the theorem that marked left anodyne maps resp. inner anodyne maps pullback to cocartesian equivalences resp. categorical equivalences along cartesian fibrations.

For later use, we state a criterion for showing that a functor is left Quillen.

4.13. **Lemma.** *Let $\mathscr{M}$ and $\mathscr{N}$ be model categories and let $F : \mathscr{M} \longrightarrow \mathscr{N}$ be a functor which preserves cofibrations. Let I be a weakly saturated subset of the trivial cofibrations in $\mathscr{M}$ such that for every object $A \in \mathscr{M}$, we have a map $f : A \longrightarrow A'$ where $f \in I$ and A' is fibrant. Then F preserves trivial cofibrations if and only if*

(1) For every $f \in I$, $F(f)$ is a trivial cofibration.
(2) F preserves trivial cofibrations between fibrant objects.

Proof. The 'only if' direction is obvious. For the other direction, let $A \longrightarrow B$ be a trivial cofibration in $\mathscr{M}$. We may form the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \arrow[dr] & \\ A' \arrow[r] & A' \coprod_A B \arrow[r] & (A' \coprod_A B)' \end{tikzcd}$$

where the vertical and lower right horizontal arrows are in I. Then our two assumptions along with the two-out-of-three property of the weak equivalences shows that $F(A) \longrightarrow F(B)$ is a trivial cofibration. □

4.14. **Lemma.** *Let K be a simplicial set over S. Then*

$$K \star_S -, - \star_S K : s\mathbf{Set}_{/S} \longrightarrow s\mathbf{Set}_{K//S}$$

are left adjoints. Similarly, for K a marked simplicial set over S,

$$K \star_S -, - \star_S K : s\mathbf{Set}^+_{/S} \longrightarrow s\mathbf{Set}^+_{K//S}$$

are left adjoints.

Proof. We will prove that $K \star_S -$ is a left adjoint in the unmarked case and leave the other cases to the reader. Let F denote $K \star_S -$ and define a functor $G : s\mathbf{Set}_{K//S} \longrightarrow s\mathbf{Set}_{/S}$ by letting $G(K \longrightarrow C)$ be the simplicial set over S which satisfies

$$\mathrm{Hom}_{/S}(\Delta^n, G(K \longrightarrow C)) = \mathrm{Hom}_{K//S}(K \star_S \Delta^n, C);$$

this is evidently natural in $K \longrightarrow C$. Define a unit map $\eta : id \longrightarrow GF$ on objects X by sending $\sigma : \Delta^n \longrightarrow X$ to $K \star_S \sigma : K \star_S \Delta^n \longrightarrow K \star_S X$, which corresponds to $\Delta^n \longrightarrow G(K \star_S X)$. Define a counit map $\eta : FG \longrightarrow id$ on objects $K \longrightarrow C$ by sending $\lambda = (\sigma, \tau) : \Delta^i \star \Delta^j \longrightarrow K \star_S G(K \longrightarrow C)$ to $\Delta^i \star \Delta^j \xrightarrow{(\sigma, id)} K \star_S \Delta^j \xrightarrow{\tau'} C$, where τ' corresponds to $\tau : \Delta^j \longrightarrow G(K \longrightarrow C)$. Then it is straightforward to verify the triangle identities, so F is adjoint to G. □

For the following pair of results, endow $s\mathbf{Set}^+_{/S}$ with the cocartesian model structure and $s\mathbf{Set}^+_{K//S} = (s\mathbf{Set}^+_{/S})_{K/}$ with the model structure created by the forgetful functor to $s\mathbf{Set}^+_{/S}$.

4.15. **Theorem.** *Let K be a marked simplicial set over S. The functor*

$$K \star_S (- \times_S \mathscr{O}(S)^\sharp) : s\mathbf{Set}^+_{/S} \longrightarrow s\mathbf{Set}^+_{K//S}$$

is left Quillen.

Proof. We will denote the functor in question by F. First observe that F is the composite of the three left adjoints e_0^*, $e_{1!}$, and $K \star_S -$, so F is a left adjoint. F evidently preserve cofibrations, so it only remains to check that F preserves the trivial cofibrations. We first verify that F preserves the left marked anodyne maps. Since F preserves colimits it suffices to check that F preserves a collection of morphisms which generate the left marked anodyne maps as a weakly saturated class. We verify that F preserves the four classes of maps enumerated in [9, 3.1.1.1].

(1): For $\iota : (\Lambda^n_k)^\flat \longrightarrow (\Delta^n)^\flat$, $0 < k < n$, the underlying map of simplicial sets of $F(\iota)$ is inner anodyne by Proposition 4.11. $F(\iota)$ is obtained by marking common edges of an inner anodyne map, so is left marked anodyne.

(2): For $\iota : {}_\natural\Lambda^n_0 \longrightarrow {}_\natural\Delta^n$, we observe that the map

$$K \star_S ({}_\natural\Lambda^n_0 \times_S \mathscr{O}(S)^\sharp) \coprod_{K\star_S({}_\natural\Lambda^n_0\times_S\mathscr{O}(S)^\natural)} K \star_S ({}_\natural\Delta^n \times_S \mathscr{O}(S)^\natural) \longrightarrow K \star_S ({}_\natural\Delta^n \times_S \mathscr{O}(S)^\sharp)$$

in the case $n = 1$ is marked left anodyne, since every marked edge in the codomain factors as a composite of two marked edges in the domain, and is the identity if $n > 1$. It thus suffices to show that $K \star_S ({}_\natural\Lambda^n_0 \times_S \mathscr{O}(S)^\natural) \longrightarrow K \star_S ({}_\natural\Delta^n \times_S \mathscr{O}(S)^\natural)$ is left marked anodyne, which is the content of part 1 of 4.11.

(3) and (4): In both of these cases one has a map of marked simplicial sets $A \longrightarrow B$ whose underlying map is an isomorphism of simplicial sets. Then

$$\begin{tikzcd} A \arrow[r] \arrow[d] & F(A) \arrow[d] \\ B \arrow[r] & F(B) \end{tikzcd}$$

is a pushout square, so $F(A) \longrightarrow F(B)$ is left marked anodyne if $A \longrightarrow B$ is.

Next, let $f : {}_\natural C \longrightarrow {}_\natural D$ be a cocartesian equivalence between cocartesian fibrations over S. Let $g : {}_\natural D \longrightarrow {}_\natural C$ be a homotopy inverse of f, so that there exists a homotopy $h : {}_\natural C \times (\Delta^1)^\sharp \longrightarrow {}_\natural C$ over S from id_C to $g \circ f$. Define a map

$$\phi : (K \star_S ({}_\natural C \times_S \mathscr{O}(S)^\sharp)) \times (\Delta^1)^\sharp \longrightarrow K \star_S (({}_\natural C \times_S \mathscr{O}(S)^\sharp) \times (\Delta^1)^\sharp)$$

by sending a $(i+j+1)$-simplex (λ, α) given by the data $\sigma : \Delta^i \longrightarrow K$, $\tau : \Delta^j \longrightarrow {}_\natural C \times_S \mathscr{O}(S)^\sharp$, $\pi \circ \lambda : \Delta^{i+j+1} \longrightarrow \Delta^1$, $\alpha : \Delta^{i+j+1} \longrightarrow \Delta^1$ to a $i+j+1$-simplex λ' given by the data σ, $(\tau, \alpha \circ \iota)$, $\pi \circ \lambda$ where $\iota : \Delta^j \longrightarrow \Delta^i \star \Delta^j$ is the inclusion. It is easy to see that ϕ restricts to an isomorphism on $(K \star_S ({}_\natural C \times_S \mathscr{O}(S)^\sharp)) \times \partial\Delta^1$. We deduce that $F(h) \circ \phi$ is a homotopy from $F(g \circ f)$ to the identity. A similar argument concerning a chosen homotopy from $f \circ g$ to id_D shows that $F(f)$ is a cocartesian equivalence. □

4.16. **Theorem.** *Let K be a marked simplicial set over S. The functor*

$$- \star_S K : s\mathbf{Set}^+_{/S} \longrightarrow s\mathbf{Set}^+_{K//S}$$

is left Quillen.

Proof. We first verify that $- \star_S K$ preserves the four classes of left marked anodyne maps enumerated in [9, 3.1.1.1]. (1) is handled by the dual of part (2) of Prp. 4.11. (2) is handled by the dual of part (3) of Prp. 4.11. (3) and (4) are handled as in the proof of Thm. 4.15. Finally, the case of $A \longrightarrow B$ a cocartesian equivalence between fibrant objects is also handled as in the proof of Thm. 4.15. □

4.17. **Definition.** Let $K, C \longrightarrow S$ be marked simplicial sets over S and let $p : K \longrightarrow C$ be a map over S. Define the marked simplicial set $C_{(p,S)/} \longrightarrow S$ as the value of the right adjoint to $K \star_S (- \times_S \mathscr{O}(S)^\sharp)$ on $K \longrightarrow C \longrightarrow S$ in $s\mathbf{Set}^+_{K//S}$. By Thm. 4.15, if $C \longrightarrow S$ is a S-category, then $C_{(p,S)/} \longrightarrow S$ is a S-category. We will refer to $C_{(p,S)/}$ as a *S-undercategory* of C.

Dually, define the marked simplicial set $C_{/(p,S)} \longrightarrow S$ as the value of the right adjoint to $- \star_S (K \times_S \mathscr{O}(S)^\sharp)$ on $K \longrightarrow C \longrightarrow S$ in $s\mathbf{Set}^+_{K//S}$. By Thm. 4.16 applied to $K \times_S \mathscr{O}(S)^\sharp$, if $C \longrightarrow S$ is a S-category, then $C_{/(p,S)} \longrightarrow S$ is a S-category. We will refer to $C_{/(p,S)}$ as a *S-overcategory* of C.

In the sequel, we will focus our attention on the S-undercategory and leave proofs of the evident dual assertions to the reader.

Functoriality in the diagram. We now study the functoriality of the S-undercategory with respect to the diagram category. Given maps $f : K \longrightarrow L$ and $p : L \longrightarrow X$ of marked simplicial sets over S, we have an induced map $X_{(p,S)/} \longrightarrow X_{(pf,S)/}$, which in terms of the functors that $X_{(p,S)/}$ and $X_{(pf,S)/}$ represent is given by precomposing $L \star_S (A \times_S \mathscr{O}(S)^\sharp) \longrightarrow X$ by $f \star_S id$.

Recall that for category $\mathscr{M}$ admitting pushouts and a map $f : K \longrightarrow L$, we have an adjunction

$$f_! : \mathscr{M}_{K/} \rightleftarrows \mathscr{M}_{L/} : f^*$$

where $f_!(K \longrightarrow X) = X \coprod_K L$ and $f^*(L \xrightarrow{p} X) = p \circ f$. If $\mathscr{M}$ is a model category and $\mathscr{M}_{K/}$, $\mathscr{M}_{L/}$ are provided with the model structures induced from $\mathscr{M}$, then $(f_!, f^*)$ is a Quillen adjunction. Moreover, if $\mathscr{M}$ is a left proper model category and f is a weak equivalence, then $(f_!, f^*)$ is a Quillen equivalence.

4.18. **Proposition.** *Let $f : K \longrightarrow L$ be a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$. Let C be a S-category and let $p : L \longrightarrow \natural C$ be a map. Then $\natural C_{(p,S)/} \longrightarrow \natural C_{(pf,S)/}$ is a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$.*

Proof. Let $F = f_! \circ (K \star_S (- \times_S \mathscr{O}(S)^\sharp))$ and let $F' = L \star_S (- \times_S \mathscr{O}(S)^\sharp)$. Let G and G' be the right adjoints to F and F', respectively. Let $\alpha : F \longrightarrow F'$ be the evident natural transformation and let $\beta : G' \longrightarrow G$ be the dual natural transformation, defined by $G' \xrightarrow{\eta_{G'}} GFG' \xrightarrow{G\alpha G'} GF'G' \xrightarrow{G\epsilon'} G$. Then $\beta_C : \natural C_{(p,S)/} \longrightarrow \natural C_{(pf,S)/}$ is the map under consideration. By Thm. 4.16, α_X is a cocartesian equivalence for all $X \in s\mathbf{Set}^+_{/S}$. Therefore, by [7, 1.4.4(b)], β_C is a cocartesian equivalence. $\square$

4.19. **Proposition.** *Consider a commutative diagram of marked simplicial sets*

$$\begin{tikzcd} K \arrow[r] \arrow[d, "i"'] & C \arrow[d, "q"] \\ L \arrow[r] \arrow[ur, "p"] & D \end{tikzcd}$$

where i is a cofibration and q is a fibration.

(1) The map

$$C_{(p,S)/} \longrightarrow C_{(pi,S)/} \times_{D_{(qpi,S)/}} D_{(qp,S)/}$$

is a fibration.

(2) Let $K = \emptyset$ and $D = S^\sharp$. Then the map

$$C_{(p,S)/} \longrightarrow C_{(pi,S)/} \cong \underline{\mathrm{Fun}}_S(S^\sharp, C)$$

is a left fibration (of the underlying simplicial sets).

Proof. (1): Given a trivial cofibration $A \longrightarrow B$, we need to solve lifting problems of the form

$$\begin{tikzcd} L \star_S (A \times_S \mathscr{O}(S)^\sharp) \coprod_{K \star_S (A \times_S \mathscr{O}(S)^\sharp)} K \star_S (B \times_S \mathscr{O}(S)^\sharp) \arrow[r] \arrow[d] & C \arrow[d] \\ L \star_S (B \times_S \mathscr{O}(S)^\sharp) \arrow[r] \arrow[ur, dotted] & D \end{tikzcd}$$

But the lefthand map is a trivial cofibration by Thm. 4.15.

(2): We need to solve lifting problems of the form

$$\begin{tikzcd} (\Delta^n)^\flat \times_S \mathscr{O}(S)^\sharp \coprod_{(\Lambda^n_i)^\flat} K \star_S ((\Lambda^n_i)^\flat \times_S \mathscr{O}(S)^\sharp) \arrow[r] \arrow[d] & C \arrow[d] \\ K \star_S ((\Delta^n)^\flat \times_S \mathscr{O}(S)^\sharp) \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

where $0 \leq i < n$. But the lefthand map is a trivial cofibration by Prp. 4.11 (1') and (2). $\square$

Combining (2) of the above proposition with Lm. 3.5 (2) (which supplies a trivial marked fibration $\underline{\mathrm{Fun}}_S(S^\sharp, C) \longrightarrow C$), we obtain a map $C_{(p,S)/} \longrightarrow C$ which is a marked fibration and a left fibration, and such that for any $f : K \longrightarrow L$, the triangle

$$\begin{tikzcd} C_{(p,S)/} \arrow[rr] \arrow[dr] & & C_{(pf,S)/} \arrow[dl] \\ & C & \end{tikzcd}$$

commutes.

The universal mapping property of the S-slice. Because the S-join and slice Quillen adjunction is not simplicial, we do not immediately obtain a universal mapping property characterizing the S-slice. Our goal in this subsection is to supply such a universal mapping property (Prp. 4.25). We first digress in order to recall how to slice Quillen bifunctors. Suppose $\mathscr{V}$ is a closed symmetric monoidal category and $\mathscr{M}$ is enriched, tensored, and cotensored over $\mathscr{V}$. Denote the internal hom by

$$\underline{\mathrm{Hom}}(-,-) : \mathscr{M}^{op} \times \mathscr{M} \longrightarrow \mathscr{V}.$$

Define bifunctors

$$\underline{\mathrm{Hom}}_{x/}(-,-) : \mathscr{M}_{x/}^{op} \times \mathscr{M}_{x/} \longrightarrow \mathscr{V}$$
$$\underline{\mathrm{Hom}}_{/x}(-,-) : \mathscr{M}_{/x}^{op} \times \mathscr{M}_{/x} \longrightarrow \mathscr{V}$$

on objects $f : x \longrightarrow a, g : x \longrightarrow b$ and $f' : a \longrightarrow x, g' : b \longrightarrow x$ to be pullbacks

$$\begin{tikzcd} \underline{\mathrm{Hom}}_{x/}(f,g) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(a,b) \arrow[d, "f^*"] \\ 1 \arrow[r, "g"] & \underline{\mathrm{Hom}}(x,b) \end{tikzcd} \qquad \begin{tikzcd} \underline{\mathrm{Hom}}_{/x}(f',g') \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(a,b) \arrow[d, "g'_*"] \\ 1 \arrow[r, "f'"] & \underline{\mathrm{Hom}}(a,x) \end{tikzcd}$$

and on morphisms in the obvious way (we abusively denote by $g : 1 \longrightarrow \underline{\mathrm{Hom}}(x,b)$ the map corresponding to g under the natural isomorphisms $\underline{\mathrm{Hom}}(1, \mathrm{Fun}(x,b)) \cong \underline{\mathrm{Hom}}(1 \otimes x, b) \cong \underline{\mathrm{Hom}}(x,b)$, and likewise for f'). It is easy to see that $\underline{\mathrm{Hom}}_{x/}$ and $\underline{\mathrm{Hom}}_{/x}$ preserve limits separately in each variable.

4.20. **Lemma.** *In the above situation let $\mathscr{M}$ be a model category and $\mathscr{P}$ be a monoidal model category. If $\underline{\mathrm{Hom}}(-,-)$ is a right Quillen bifunctor, then $\underline{\mathrm{Hom}}_{x/}(-,-)$ and $\underline{\mathrm{Hom}}_{/x}(-,-)$ are right Quillen bifunctors, where we endow $\mathscr{M}_{x/}$ and $\mathscr{M}_{/x}$ with the model structures created by the forgetful functor to $\mathscr{M}$.*

Proof. We prove the assertion for $\underline{\mathrm{Hom}}_{x/}(-,-)$, the proof for $\underline{\mathrm{Hom}}_{/x}(-,-)$ being identical. Let $i : a \longrightarrow b$ and $f : c \longrightarrow d$ be morphisms in $\mathscr{M}_{x/}$ (so they are compatible with the structure maps $\pi_a, ..., \pi_d$). In the commutative diagram

$$\begin{tikzcd} \underline{\mathrm{Hom}}_{x/}(\pi_b,\pi_c) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(b,c) \arrow[d] \\ \underline{\mathrm{Hom}}_{x/}(\pi_a,\pi_c) \times_{\underline{\mathrm{Hom}}_{x/}(\pi_a,\pi_d)} \underline{\mathrm{Hom}}_{x/}(\pi_b,\pi_d) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(a,c) \times_{\underline{\mathrm{Hom}}(a,d)} \underline{\mathrm{Hom}}(b,d) \arrow[d] \\ 1 \arrow[r] & \underline{\mathrm{Hom}}(x,c) \end{tikzcd}$$

it is easy to see that the lower square and the rectangle are pullback squares, so the upper square is a pullback square. It is now clear that if $\underline{\mathrm{Hom}}(-,-)$ is a right Quillen bifunctor, then $\underline{\mathrm{Hom}}_{x/}(-,-)$ is as well. □

We apply 4.20 to the bifunctors

$$\mathrm{Map}_{K//S}(-,-) : s\mathbf{Set}^{+\,op}_{K//S} \times s\mathbf{Set}^+_{K//S} \longrightarrow s\mathbf{Set}_{\mathrm{Quillen}}$$
$$\mathrm{Fun}_{K//S}(-,-) : s\mathbf{Set}^{+\,op}_{K//S} \times s\mathbf{Set}^+_{K//S} \longrightarrow s\mathbf{Set}_{\mathrm{Joyal}}$$

induced by $\mathrm{Map}_S(-,-)$ and $\mathrm{Fun}_S(-,-)$.

4.21. **Lemma.** *Let K, A, and B be simplicial sets and define a map*

$$A \times (K \star B) \longrightarrow K \star (A \times B)$$

by sending the data $(\Delta^n \longrightarrow A, \Delta^k \longrightarrow K, \Delta^{n-k-1} \longrightarrow B)$ of a n-simplex of $A \times (K \star B)$ to the data $(\Delta^k \longrightarrow K, \Delta^{n-k-1} \longrightarrow A \times B)$ of a n-simplex of $K \star (A \times B)$. Then

$$\phi : A \times (K \star B) \coprod_{A \times K} K \longrightarrow K \star (A \times B)$$

is a categorical equivalence.

Proof. Recall ([9, 4.2.1.2]) that there is a map

$$\eta_{X,Y} : X \diamond Y = X \coprod_{X \times Y \times \{0\}} X \times Y \times \Delta^1 \coprod_{X \times Y \times \{1\}} Y \longrightarrow X \star Y$$

natural in X and Y which is always a categorical equivalence. Thus

$$f = (A \times \eta_{K,B}) \sqcup id_K : A \times (K \diamond B) \coprod_{A \times K} K \longrightarrow A \times (K \star B) \coprod_{A \times K} K$$

is a categorical equivalence. The domain is isomorphic to $K \diamond (A \times B)$, and it is easy to check that the map $\eta_{K, A\times B}$ is the composite

$$K \diamond (A \times B) \xrightarrow{f} A \times (K \star B) \coprod_{A \times K} K \xrightarrow{\phi} K \star (A \times B).$$

Using the 2 out of 3 property of the categorical equivalences, we deduce that ϕ is a categorical equivalence. $\square$

4.22. **Lemma.** *For all $L \in s\mathbf{Set}^+_{/S}$, we have a natural equivalence*

$$\phi : \mathrm{Fun}_S(L, {}_\natural C_{(p,S)/}) \xrightarrow{\simeq} \mathrm{Fun}_{K//S}(K \star_S (L \times_S \mathscr{O}(S)^\sharp), {}_\natural C).$$

Proof. Define bisimplicial sets $X, Y : \Delta^{op} \longrightarrow s\mathbf{Set}$ by

$$\begin{aligned} X_n &= \mathrm{Map}_{K//S}(K \star_S ((\Delta^n)^\flat \times L \times_S \mathscr{O}(S)^\sharp), {}_\natural C) \\ Y_n &= \mathrm{Map}(\Delta^n, \mathrm{Fun}_{K//S}(K \star_S (L \times_S \mathscr{O}(S)^\sharp), {}_\natural C)) \\ &\cong \mathrm{Map}_{K//S}((\Delta^n)^\flat \times (K \star_S (L \times_S \mathscr{O}(S)^\sharp) \coprod_{(\Delta^n)^\flat \times K} K, {}_\natural C). \end{aligned}$$

and define a map of bisimplicial sets $\Phi : X \longrightarrow Y$ by precomposing levelwise by the map

$$g_{L,n} : (\Delta^n)^\flat \times (K \star_S (L \times_S \mathscr{O}(S)^\sharp)) \coprod_{(\Delta^n)^\flat \times K} K \longrightarrow K \star_S ((\Delta^n)^\flat \times L \times_S \mathscr{O}(S)^\sharp)$$

adjoint as a map over $S \times \Delta^1$ to the identity over $S \times \partial\Delta^1$. Taking levelwise zero simplices then defines the map ϕ, which is clearly natural in L, K, and C. By Thm. 4.16, taking a fibrant replacement of K we may suppose that K is fibrant. We first check that X and Y are complete Segal spaces. Y is a complete Segal space as it arises from a ∞-category ([8, 4.12]). For X, since $\mathrm{Map}_{K//S}(-,-)$ is a right Quillen bifunctor, we only have to observe that:

- ▶ Every monomorphism $A \longrightarrow B$ of simplicial sets induces a cofibration

  $$K \star_S (A^\flat \times L \times_S \mathscr{O}(S)^\sharp) \longrightarrow K \star_S (B^\flat \times L \times_S \mathscr{O}(S)^\sharp)$$

  so X is Reedy fibrant.
- ▶ The spine inclusion $\iota_n : \mathrm{Sp}(n) \longrightarrow \Delta^n$ induces a trivial cofibration

  $$K \star_S (\mathrm{Sp}(n)^\flat \times L \times_S \mathscr{O}(S)^\sharp) \longrightarrow K \star_S ((\Delta^n)^\flat \times L \times_S \mathscr{O}(S)^\sharp);$$

  ι_n is inner anodyne, so this follows from Thm. 4.15 and [9, 3.1.4.2].

- The map $\pi : E \longrightarrow \Delta^0$ where E is the nerve of the contractible groupoid with two elements induces a cocartesian equivalence
$$K \star_S (E^\flat \times L \times_S \mathscr{O}(S)^\sharp) \longrightarrow K \star_S (L \times_S \mathscr{O}(S)^\sharp);$$
$\pi^\flat$ is a cocartesian equivalence (as the composite of $E^\flat \longrightarrow E^\sharp$ and $E^\sharp \longrightarrow \Delta^0$), so this also follows from Thm. 4.15 and [9, 3.1.4.2].

We next prove that Φ is an equivalence in the complete Segal model structure. For this, we will prove that each map $g_{L,n}$ is a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$. Both sides preserves colimits as a functor of L (valued in $s\mathbf{Set}^+_{K//S}$), so by left properness and the stability of cocartesian equivalences under filtered colimits we reduce to the case L is an m-simplex with some marking. In particular, $(\Delta^m)^\flat \times_S \mathscr{O}(S)^\sharp \longrightarrow S$ is fibrant in $s\mathbf{Set}^+_{/S}$. By [9, 4.2.4.1] we may check that the square of fibrant objects
$$\begin{tikzcd}
(\Delta^n)^\flat \times K \arrow[r] \arrow[d] & K \arrow[d] \\
(\Delta^n)^\flat \times (K \star_S ((\Delta^m)^\flat \star_S \mathscr{O}(S)^\sharp)) \arrow[r] & K \star_S ((\Delta^n)^\flat \times (\Delta^m)^\flat \times_S \mathscr{O}(S)^\sharp)
\end{tikzcd}$$
is a homotopy pushout square in the underlying ∞-category $\mathbf{Cat}^{cocart}_{\infty,S} \simeq \mathrm{Fun}(S, \mathbf{Cat}_\infty)$, where colimits are computed objectwise. In other words, we may check that for every $s \in S$, the fiber of the square over s is a homotopy pushout square in $s\mathbf{Set}$, which holds by Lm. 4.21. Pushing out along the cofibration $(\Delta^m)^\flat \times_S \mathscr{O}(S)^\sharp \longrightarrow L \times_S \mathscr{O}(S)^\sharp$ and using left properness, we deduce that $g_{L,m}$ is a cocartesian equivalence. Finally, we invoke [8, 4.11] to deduce that ϕ is a categorical equivalence. □

4.23. **Lemma.** *Let $L \longrightarrow S$ be a cocartesian fibration. Then $id_K \star \iota_L : K \star_S {}_\natural L \longrightarrow K \star_S ({}_\natural L \times_S \mathscr{O}(S)^\sharp)$ is a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$.*

Proof. By Thm. 4.16, taking a fibrant replacement of K we may suppose that K is fibrant. By 13.4, it suffices to show that for every $s \in S$, $K^\sim_s \star L^\sim_s \longrightarrow K^\sim_s \star ({}_\natural L \times_S (S^{/s})^\sharp)$ is a marked equivalence in $s\mathbf{Set}^+$. Observe that the *cartesian* equivalence $\{s\} \longrightarrow (S^{/s})^\sharp$ pulls back by the cocartesian fibration ${}_\natural L \longrightarrow S^\sharp$ to a marked equivalence $L^\sim_s \longrightarrow {}_\natural L \times_S (S^{/s})^\sharp$. Then by Thm. 4.15 for $S = \Delta^0$, $K^\sim_s \star -$ preserves marked equivalences, which concludes the proof. □

4.24. **Notation.** Let K, C, D be S-categories and let $F : K \longrightarrow C$, $G : K \longrightarrow D$ be S-functors. Define $\underline{\mathrm{Fun}}_{K//S}(C, D)$ to be the pullback
$$\begin{tikzcd}
\underline{\mathrm{Fun}}_{K//S}(C, D) \arrow[r] \arrow[d] & \underline{\mathrm{Fun}}_S(C, D) \arrow[d, "F^*"] \\
S \arrow[r, "\sigma_G"] & \underline{\mathrm{Fun}}_S(K, D).
\end{tikzcd}$$

Note that by Prp. 3.7, the defining pullback square is a homotopy pullback square if F is a monomorphism.

4.25. **Proposition.** *Let K, L, C be S-categories and let $p : K \longrightarrow C$, $q : L \longrightarrow C$ be S-functors.*

(1) We have an equivalence
$$\psi : \underline{\mathrm{Fun}}_S(L, C_{(p,S)/}) \xrightarrow{\simeq} \underline{\mathrm{Fun}}_{K//S}(K \star_S L, C).$$
(2) We have an equivalence
$$\psi' : \underline{\mathrm{Fun}}_S(L, C_{/(q,S)}) \xrightarrow{\simeq} \underline{\mathrm{Fun}}_{L//S}(K \star_S L, C)$$
(3) We have equivalences
$$\underline{\mathrm{Fun}}_{/C}(L, C_{(p,S)/}) \xrightarrow[\simeq]{\psi_p} \underline{\mathrm{Fun}}_{K \sqcup L//S}(K \star_S L, C) \xleftarrow[\simeq]{\psi'_q} \underline{\mathrm{Fun}}_{/C}(K, C_{/(q,S)}).$$

Proof. (1): Define the S-functor ψ as follows: suppose given a marked simplicial set A and a map $A \longrightarrow \underline{\mathrm{Fun}}_S(L, C_{(p,S)/})$ over S. This is equivalently given by the datum of a map

$$f_A : {}_\natural K \star_S ((A \times_S \mathscr{O}(S)^\sharp \times_S {}_\natural L) \times_S \mathscr{O}(S)^\sharp) \longrightarrow {}_\natural C$$

under K and over S. Let

$${}_\natural K \coprod_{A \times_S \mathscr{O}(S)^\sharp \times_S {}_\natural K} (A \times_S \mathscr{O}(S)^\sharp) \times_S ({}_\natural K \star_S ({}_\natural L \times_S \mathscr{O}(S)^\sharp)) \longrightarrow K \star_S (A \times_S \mathscr{O}(S)^\sharp \times_S {}_\natural L \times_S \mathscr{O}(S)^\sharp)$$

be the map over $S \times \Delta^1$ adjoint to the identity over $S \times \partial\Delta^1$. Precomposing f_A by this and $\iota_L : {}_\natural L \longrightarrow {}_\natural L \times_S \mathscr{O}(S)^\sharp$ on that factor defines the desired map $A \longrightarrow \underline{\mathrm{Fun}}_{K//S}(K \star_S L, C)$.

Now to check that ψ is an equivalence, we may work fiberwise and combine Lm. 4.22 and Lm. 4.23.

The proof of (2) is by a parallel argument.

(3): We prove that ψ_q is an equivalence; a parallel argument will work for ψ'_p. $\underline{\mathrm{Fun}}_{K \sqcup L//S}(K \star_S L, C)$ fits into a diagram

$$\begin{tikzcd}
\underline{\mathrm{Fun}}_{K \sqcup L//S}(K \star_S L, C) \arrow[r] \arrow[d] & \underline{\mathrm{Fun}}_{K//S}(K \star_S L, C) \arrow[r] \arrow[d] & \underline{\mathrm{Fun}}_S(K \star_S L, C) \arrow[d] \\
S \arrow[r] \arrow[rr, bend left, "\sigma_{p \sqcup q}"] & \underline{\mathrm{Fun}}_{K//S}(K \coprod L, C) \arrow[r] \arrow[d] & \underline{\mathrm{Fun}}_S(K \coprod L, C) \arrow[d] \\
& S \arrow[r, "\sigma_p"] & \underline{\mathrm{Fun}}_S(K, C)
\end{tikzcd}$$

in which every square is a pullback square. The map ψ_q is then defined to be the pullback of the map of spans

$$\begin{tikzcd}
\underline{\mathrm{Fun}}_S(L, C_{(p,S)/}) \arrow[r] \arrow[d, "\psi"] & \underline{\mathrm{Fun}}_S(L, C) \arrow[d, "p \sqcup -"] & S \arrow[l, "\sigma_q"'] \arrow[d, "="] \\
\underline{\mathrm{Fun}}_{K//S}(K \star_S L, C) \arrow[r] & \underline{\mathrm{Fun}}_{K//S}(K \coprod L, C) & S \arrow[l]
\end{tikzcd}$$

in which the vertical arrows are equivalences. By Prp. 4.19 and $\underline{\mathrm{Fun}}_S(L, -)$ being right Quillen, the top left horizontal arrow is a S-fibration, and by Prp. 3.7, the bottom left horizontal arrow is a S-fibration. It follows that ψ_q is an equivalence. □

In light of Prp. 4.25, we have evident 'alternative' S-slice S-categories, whose definition more closely adheres to the intuition that a slice category is a category of extensions.

4.26. **Definition.** Let $p : K \longrightarrow C$ be a S-functor. We define the *alternative S-undercategory*

$$C^{(p,S)/} = \underline{\mathrm{Fun}}_{K//S}(K \star_S S, C).$$

Similarly, we define the *alternative S-overcategory*

$$C^{/(p,S)} = \underline{\mathrm{Fun}}_{K//S}(S \star_S K, C).$$

4.27. **Corollary.** *Let $p : K \longrightarrow C$ and $q : L \longrightarrow C$ be S-functors.*

(1) We have equivalences $C_{(p,S)/} \xrightarrow{\simeq} C^{(p,S)/}$ and $C_{/(q,S)} \xrightarrow{\simeq} C^{/(q,S)}$.

(2) We have an equivalence $\underline{\mathrm{Fun}}_{/C}(L, C^{(p,S)/}) \simeq \underline{\mathrm{Fun}}_{/C}(K, C^{/(q,S)})$ through a natural zig-zag.

Proof. For (1), let $L = S$ and $K = S$ in Prp. 4.25 (1) and (2), respectively. For (2), combine the preceding (1) and Prp. 4.25 (3). □

4.28. **Warning.** When $S = \Delta^0$, the alternative S-undercategory $C^{(p,S)/} \cong \{p\} \times_{\mathrm{Fun}(K,C)} \mathrm{Fun}(K^\triangleright, C)$ differs from Lurie's alternative undercategory $C^{p/}$. However, we have a comparison functor

$$\{p\} \times_{\mathrm{Fun}(K,C)} \mathrm{Fun}(K^\triangleright, C) \longrightarrow C^{p/}$$

which is a categorical equivalence and which factors through the categorical equivalence $C_{p/} \longrightarrow C^{p/}$ of [9, 4.2.1.5].

Slicing over and under S-points. We give a smaller model for slicing over and under S-points in an S-category C.

4.29. **Notation.** Suppose C an S-category. Let $\mathscr{O}_S(C) = \widetilde{\mathrm{Fun}}_S(S \times \Delta^1, C) \cong S \times_{\mathscr{O}(S)} \mathscr{O}(C)$ denote the fiberwise arrow S-category of C. Given an object $x \in C$, let $C^{/\underline{x}} = \mathscr{O}_S(C) \times_C \underline{x}$ and $C^{\underline{x}/} = \underline{x} \times_C \mathscr{O}_S(C)$.

4.30. **Proposition.** *Let $x \in C$ be an object and denote by $i_x : \underline{x} \longrightarrow C_{\underline{x}}$ the $\underline{x}$-functor defined by x. We have natural equivalences of $\underline{x}$-categories*

$$C_{\underline{x}}{}^{/(\underline{x}, i_x)} \simeq C^{/\underline{x}}$$

$$C_{\underline{x}}{}^{/(i_x, \underline{x})} \simeq C^{\underline{x}/}.$$

Proof. For any functor $S' \longrightarrow S$ and S-category C, $\mathscr{O}_S(C) \times_S S' \cong \mathscr{O}_{S'}(C \times_S S')$. Therefore, $\mathscr{O}_S(C) \times_C \underline{x} \cong \mathscr{O}_{\underline{x}}(C_{\underline{x}}) \times_{C_{\underline{x}}} \underline{x}$ and likewise for $\underline{x} \times_C \mathscr{O}_S(C)$. Changing base to $\underline{x}$, we may suppose $S = \underline{x}$ and $i_x = i : S \longrightarrow C$ is any S-functor. The identity section $S \longrightarrow \mathscr{O}(S)$ induces a morphism of spans

$$\begin{tikzcd} S \arrow[r, "\sigma_i"] \arrow[d, "="] & \underline{\mathrm{Fun}}_S(S, C) \arrow[d] & \underline{\mathrm{Fun}}_S(S \times \Delta^1, C) \arrow[l] \arrow[d] \\ S \arrow[r, "i"] & C & \widetilde{\mathrm{Fun}}_S(S \times \Delta^1, C) \arrow[l] \end{tikzcd}$$

with the vertical maps equivalences. Taking pullbacks now yields the claim (where we use the isomorphism $S \star_S S \cong S \times \Delta^1$ to identify the upper pullback with the S-slice category in question). □

4.31. **Proposition.** *We have a natural equivalence $C^{\underline{x}/} \simeq C^{x/}$ of left fibrations over C.*

Proof. Using the marked left anodyne map ${}_\natural\Lambda^2_1 \longrightarrow {}_\natural\Delta^2$ and the map of Lm. 2.22 for $n = 2$, we obtain a span

$$\begin{tikzcd}[column sep=tiny] & \mathrm{Fun}({}_\natural\Delta^2, {}_\natural C) \arrow[dl, two heads, "\simeq"'] \arrow[dr, two heads, "\simeq"] & \\ \mathrm{Fun}((\Delta^{\{0,1\}})^\sharp, {}_\natural C) \times_{C^{\{1\}}} \mathrm{Fun}(\Delta^{\{1,2\}}, C) & & \mathrm{Fun}(\Delta^{\{0,2\}}, C) \times_{S^{\{0,2\}}} \mathrm{Fun}(\Delta^2, S). \end{tikzcd}$$

Pulling back via $\{x\} \times_{C^{\{0\}}} -$ on the left and $- \times_{S^{\{1,2\}}} S$ on the right, and using that the inclusion $\Delta^{\{0,2\}} \longrightarrow \Delta^2 \cup_{\Delta^{\{1,2\}}} \Delta^0$ is a categorical equivalence, we get

$$\begin{tikzcd} & \{x\} \times_{C^{\{0\}}} \mathrm{Fun}({}_\natural\Delta^2, {}_\natural C) \times_{S^{\{1,2\}}} S \arrow[dl, two heads, "\simeq"'] \arrow[dr, two heads, "\simeq"] & \\ C^{\underline{x}/} & & C^{x/}. \end{tikzcd}$$

□

5. Limits and colimits

5.1. **Definition.** Let C be a S-category and $\sigma : S \longrightarrow C$ be a cocartesian section. We say that σ is a *S-initial object* if $\sigma(s)$ is an initial object for all objects $s \in S$.

5.2. **Definition.** Let K and C be S-categories. Let $\overline{p} : K \star_S S \longrightarrow C$ be an extension of a S-functor $p : K \longrightarrow C$. From the commutativity of the diagram

$$\begin{tikzcd} S \arrow[r, "\sigma_{\overline{p}}"] \arrow[d, "="] & \underline{\mathrm{Fun}}_S(K \star_S S, C) \arrow[d] \\ S \arrow[r, "\sigma_p"] & \underline{\mathrm{Fun}}_S(K, C) \end{tikzcd}$$

we see that $\sigma_{\overline{p}}$ defines a cocartesian section of $C^{(p,S)/}$, which we also denote by $\sigma_{\overline{p}}$. We say that $\overline{p}$ is a *S-colimit diagram* if $\sigma_{\overline{p}}$ is a S-initial object. If $\overline{p}$ is a S-colimit diagram, then $\overline{p}|_S : S \longrightarrow C$ is said to be a *S-colimit* of p.

5.3. **Remark.** In view of the comparison result Cor. 4.27, we could also use the S-slice category $C_{(p,S)/}$ to make the definition of a S-colimit diagram. This would yield some additional generality, in that $C_{(p,S)/}$ is defined for an arbitrary marked simplicial set K. However, the construction $C^{(p,S)/}$ is easier to relate to functor categories, which we need to do to show that the left adjoint to the restriction along $K \subset K \star_S S$ computes colimits (a special case of Cor. 9.16).

There are a couple instances where the notion of S-colimit specializes to a notion of ordinary category theory. For example, we have the following pair of propositions computing S-colimits and S-limits in an S-category of objects $\underline{E}_S$ as left or right Kan extensions in E.

5.4. **Proposition.** *Let $\overline{p} : K \star_S S \longrightarrow \underline{C}_S$ be a S-functor extending $p : K \longrightarrow \underline{C}_S$. Suppose further that a left Kan extension of $p^\dagger$ to a functor $K \star_S S \longrightarrow C$ exists. Then the following are equivalent:*

(1) $\overline{p}$ is a S-colimit diagram.
(2) $\overline{p}^\dagger$ is a left Kan extension of $p^\dagger$.
(3) $\overline{p}^\dagger|_{K_s^\rhd}$ is a colimit diagram for all $s \in S$.

Proof. (2) and (3) are equivalent because left Kan extensions along cocartesian fibrations are computed fiberwise. Suppose (3). To prove (1), we want to show that for every $s \in S$, $\overline{p}_{\underline{s}}$ is an initial object in $((\underline{C}_S)^{(p,S)/})_s$. But $((\underline{C}_S)^{(p,S)/})_s$ is equivalent to the fiber of $\mathrm{Fun}(K_{\underline{s}} \star_{\underline{s}} \underline{s}, C) \longrightarrow \mathrm{Fun}(K_{\underline{s}}, C)$ over $p^\dagger|_{K_{\underline{s}}}$, so to prove the claim it suffices to show that the functor $\overline{p}^\dagger|_{K_{\underline{s}}}$ is a left Kan extension of $p|_{K_{\underline{s}}}$. This holds by the equivalence of (2) and (3) for $S^{s/}$.

Conversely, suppose (1). Since we supposed that a left Kan extension of $p^\dagger$ exists, left Kan extensions of $p^\dagger|_{K_s}$ all exist and *any* initial object in the fiber of $\mathrm{Fun}(K_{\underline{s}} \star_{\underline{s}} \underline{s}, C) \longrightarrow \mathrm{Fun}(K_{\underline{s}}, C)$ over $p^\dagger|_{K_{\underline{s}}}$ is a left Kan extension of $p^\dagger|_{K_{\underline{s}}}$, necessarily a fiberwise colimit diagram (we need this hypothesis because Kan extensions as defined in [9, §4.3.2] are always *pointwise* Kan extensions). This implies (3). □

5.5. **Proposition.** *Let $\overline{p} : S \star_S K \longrightarrow \underline{C}_S$ be a S-functor extending $p : K \longrightarrow \underline{C}_S$. Suppose further that a right Kan extension of $p^\dagger$ to a functor $S \star_S K \longrightarrow C$ exists. Then the following are equivalent:*

(1) $\overline{p}$ is a S-limit diagram.
(2) $\overline{p}^\dagger$ is a right Kan extension of $p^\dagger$.
(2') $\overline{p}^\dagger|_{\underline{s} \star_{\underline{s}} K_{\underline{s}}}$ is a right Kan extension of $p^\dagger|_{K_{\underline{s}}}$ for all $s \in S$.
(3) $\overline{p}^\dagger|_{K_{\underline{s}}^\lhd}$ is a limit diagram for all $s \in S$.

Proof. We first observe that because the inclusion $S \longrightarrow S \star_S K$ is left adjoint to the structure map $S \star_S K \longrightarrow S$ of the cocartesian fibration,

$$(S \star_S K)^{s/} \simeq S^{s/} \times_S (S \star_S K) \cong \underline{s} \star_{\underline{s}} K_{\underline{s}}.$$

The equivalence of (2) and (2') now follows from the formula for a right Kan extension. Also, if we view $K_{\underline{s}}^\lhd$ as mapping to $S \star_S K$ via $\{s\} \star K_{\underline{s}} \longrightarrow \underline{s} \star_{\underline{s}} K_{\underline{s}} \longrightarrow S \star_S K$ where the first map is adjoint to $(\{s\} \longrightarrow \underline{s}, id)$, then (2) and (3) are also equivalent by the same argument. Finally, (2') implies (1) by definition, and (1) implies (2') under our additional assumption that a right Kan extension of $p^\dagger$ exists (for the same reason as given in the proof of Prp. 5.4). □

If S is a Kan complex, then the notion of S-colimit reduces to the usual notion of colimit.

5.6. **Proposition.** *Let S be a Kan complex. Then a S-functor $\overline{p} : K \star_S S \longrightarrow C$ is a S-colimit diagram if and only if for every object $s \in S$, $\overline{p}|_s : (K_s)^\rhd \longrightarrow C_s$ is a colimit diagram.*

Proof. If S is a Kan complex, then for every $s \in S$, $S^{s/}$ is a contractible Kan complex. Therefore, for all $s \in S$ we have $(C^{(p,S)/})_s \simeq \{p_s\} \times_{\mathrm{Fun}(K_s, C_s)} \mathrm{Fun}(K_s^\rhd, C_s)$, which proves the claim. □

We say that K is a *constant* S-category if it is equivalent to $S \times L$ for L an ∞-category. We have an isomorphism $L^\rhd \times S \longrightarrow (L \times S) \star_S S$ (defined as a map over $S \times \Delta^1$ to be the adjoint to the identity on $(L \times S, S)$).

5.7. **Proposition.** *A S-functor $\overline{p} : L^\rhd \times S \longrightarrow C$ is a S-colimit diagram if and only if for every object $s \in S$, $\overline{p}_s : L^\rhd \longrightarrow C_s$ is a colimit diagram.*

Proof. Observe that

$$(C^{(p,S)/})_s = \{p_{\underline{s}}\} \times_{\mathrm{Fun}_{S^{s/}}(L\times S^{s/},C_{\underline{s}})} \mathrm{Fun}_{S^{s/}}(L^{\triangleright} \times S^{s/}, C_{\underline{s}}) \simeq \{p_s\} \times_{\mathrm{Fun}(L,C_s)} \mathrm{Fun}(L^{\triangleright}, C_s).$$

Therefore, $\sigma_{\overline{p}} : S \longrightarrow C^{(p,S)/}$ is S-initial if and only if for all $s \in S$, $\{\overline{p}_s\} \in \{p_s\} \times_{\mathrm{Fun}(L,C_s)} \mathrm{Fun}(L^{\triangleright}, C_s)$ is an initial object, which is the claim. □

5.8. **Corollary.** *Suppose C is a S-category such that C_s admits all colimits for every object $s \in S$ and the pushforward functors $\alpha_! : C_s \longrightarrow C_t$ preserve all colimits for every morphism $\alpha : s \to t$ in S. Then C admits all S-colimits indexed by constant diagrams.*

Proof. First suppose that S has an initial object s. Suppose that $p : L \times S \longrightarrow C$ is a S-functor. Let $\overline{p_s} : L^{\triangleright} \longrightarrow C_s$ be a colimit diagram extending p_s. Let $\overline{p} : L^{\triangleright} \times S \longrightarrow C$ be a S-functor corresponding to $\overline{p_s}$ under the equivalence $\mathrm{Fun}_S(L^{\triangleright} \times S, C) \simeq \mathrm{Fun}(L^{\triangleright}, C_s)$, which we may suppose extends p. By Prp. 5.7, $\overline{p}$ is a S-colimit diagram.

The general case now follows from Thm. 9.15, taking $\phi : C \longrightarrow D$ to be $L \times S \longrightarrow S$. □

We now turn to the example of corepresentable fibrations.

5.9. **Definition.** Let $s \in S$ be an object and let K be an $S^{s/}$-category which is equivalent to a coproduct of corepresentable fibrations $\coprod_{i\in I} S^{\alpha_i/} \simeq \coprod_{i\in I} S^{t_i/} \xrightarrow{\coprod \alpha_i^*} S^{s/}$ for $\alpha_i : s \to t_i$ a collection of morphisms in S. Let $p : K \longrightarrow C \times_S S^{s/}$ be a $S^{s/}$-functor, so p selects objects $x_i \in C_{t_i}$. Let $\overline{p} : K \star_{S^{s/}} S^{s/} \longrightarrow C \times_S S^{s/}$ be a $S^{s/}$-colimit diagram extending p, and let $y = \overline{p}(v) \in C_s$ for $v = id_s$ the cone point. Then we say that y is the *S-coproduct* of $\{x_i\}_{i\in I}$ along $\{\alpha_i\}_{i\in I}$, and we adopt the notation $y = \coprod_{\alpha_i} x_i$.

Our choice of terminology is guided by the following result, which shows that a $S^{s/}$-colimit of a $S^{s/}$-functor $p : S^{\alpha/} \simeq S^{t/} \longrightarrow C$ obtains the value of a left adjoint to the pushforward functor $\alpha_!$ on $p(t)$. In the case of $S = \mathbf{O}_G^{op}$, $C = \underline{\mathbf{Top}}_G$ or $\underline{\mathbf{Sp}}^G$, and $K = \mathbf{O}_H^{op}$, this is the induction or indexed coproduct functor from H to G.

5.10. **Proposition.** *Let C be a S-category, let $\alpha : s \to t$ be a morphism in C, and let $\pi : M \longrightarrow \Delta^1$ be a **cartesian** fibration classified by the pushforward functor $\alpha_! : C_s \longrightarrow C_t$. Let $p : S^{t/} \longrightarrow C \times_S S^{s/}$ be a $S^{s/}$-functor and let $x = p(id_t) \in C_t$. Then the data of a $S^{s/}$-colimit diagram extending p yields a π-cocartesian edge e in M with $d_0(e) = x$ and lifting $0 \to 1$.*

Proof. Let $\overline{p} : S^{t/} \star_{S^{s/}} S^{s/} \longrightarrow C \times_S S^{s/}$ be a $S^{s/}$-colimit diagram extending p. Let $y = \overline{p}(id_s)$ and let $f' : \Delta^1 \longrightarrow S^{t/} \star_{S^{s/}} S^{s/}$ be the edge connecting id_t to α. We may suppose that M is given by the relative nerve of $\alpha_!$, so that edges in M over Δ^1 are given by commutative squares

$$\begin{tikzcd} \{1\} \arrow[r] \arrow[d] & C_s \arrow[d, "\alpha_!"] \\ \Delta^1 \arrow[r] & C_t. \end{tikzcd}$$

Then let e be the edge in M determined by y and $f = \overline{p} \circ f' : x \to \alpha_! y$. By definition, $d_0(e) = x$.

We claim that e is π-cocartesian. This holds if and only if for every $y' \in C_s$ the map

$$\mathrm{Map}_{C_s}(y, y') \longrightarrow \mathrm{Map}_{C_t}(x, \alpha_! y')$$

induced by f is an equivalence. But the local variant of the adjunction of Thm. 10.4 implies this (passing to global sections). □

S-coproducts also satisfy a base-change condition. This is awkward to articulate in general, because the pullback of a corepresentable fibration along another need not be corepresentable. However, if we impose the additional hypothesis that $T = S^{op}$ admits multipullbacks, then a pullback of a corepresentable fibration decomposes as a finite coproduct of corepresentable fibrations. In this case, we have the following useful reformulation of the base-change condition. Let $X \subset \mathscr{O}(\mathbf{F}_T)$ be the full subcategory on those arrows whose source lies in T and consider the span

$$(\mathbf{F}_T)^\sharp \xleftarrow{\mathrm{ev}_1} {}_\natural X \xrightarrow{\mathrm{ev}_0} T^\sharp.$$

This satisfies the dual of the hypotheses of Thm. 2.23, so $C^\times := (\mathrm{ev}_0)_*(\mathrm{ev}_1)^*((C^\vee)^\natural)$ is a cartesian fibration over $\mathbf{F}_T$ (with the cartesian edges marked), where $C^\vee \longrightarrow T$ is the dual cartesian fibration of [3]. Unwinding the definitions, given a T-set $U = \coprod_i s_i$, we have that the fiber $(C^\times)_U \simeq \mathrm{Fun}_T(\coprod_i T^{/s_i}, C^\vee) \simeq \prod_i C_{s_i}$, and given a morphism of T-sets $\alpha : U \longrightarrow V$, the pullback functor $\alpha^* : (C^\times)_U \longrightarrow (C^\times)_V$ is induced by restriction.

5.11. **Proposition.** *C admits finite S-coproducts if and only if $\pi : C^\times \longrightarrow \mathbf{F}_T$ is a **Beck-Chevalley fibration**, i.e. π is both cocartesian and cartesian, and for every pullback square*

$$\begin{tikzcd} W \arrow[r, "\alpha'"] \arrow[d, "\beta'"] & V' \arrow[d, "\beta"] \\ U \arrow[r, "\alpha"] & V \end{tikzcd}$$

in $\mathbf{F}_T$, the natural transformation

$$(*) \qquad (\alpha')_!(\beta')^* \longrightarrow \beta^* \alpha_!$$

adjoint to the equivalence $(\beta')^*\alpha^* \simeq (\alpha')^*\beta^*$ is itself an equivalence.

Proof. By Thm. 10.4, C admits finite S-coproducts if and only if for every finite collection of morphisms $\{\alpha_i : s \to t_i\}$, the restriction functor

$$(\coprod \alpha_i)^* : \underline{\mathrm{Fun}}_S(S^{s/}, C) \longrightarrow \underline{\mathrm{Fun}}_S(\coprod_i S^{t_i/}, C)$$

admits a left S-adjoint, in which case that left S-adjoint is computed by the S-coproduct along the α_i. This in turn is immediately equivalent to π being additionally cocartesian and $(*)$ being an equivalence for $\alpha = \coprod \alpha_i : \coprod t_i \to s$ and all morphisms $\beta : s' \to s$ in T. Finally, note that the apparently more general case of $(*)$ being an equivalence for any pullback square is actually determined by this, because any map $\alpha : U = \coprod t_i \to V = \coprod s_j$ is the data of $f : I \to J$ and $\{\alpha_{ij} : s_j \to t_i\}_{i \in f^{-1}(j)}$, whence $\alpha^* = (\alpha_{ij})^* : \prod_j C_{s_j} \longrightarrow \prod_i C_{t_i}$, etc. yields a decomposition of the map $(*)$ in terms of the 'basic' squares that we already handled. □

We conclude this subsection by introducing a bit of useful terminology.

5.12. **Definition.** Let C be a S-category. We say that C is *S-cocomplete* if, for every object $s \in S$ and $S^{s/}$-diagram $p : K \longrightarrow C_{\underline{s}}$ with K small, p admits a $S^{s/}$-colimit.

5.13. **Remark.** Suppose that E is S-cocomplete. Then taking $D = S$ in Thm. 9.15, E admits all (small) S-colimits. However, the converse may fail: if we suppose that E admits all S-colimits, then any $S^{s/}$-diagram $K_{\underline{s}} \longrightarrow E_{\underline{s}}$ pulled back from a S-diagram $K \longrightarrow E$ admits a $S^{s/}$-colimit; however, not every $S^{s/}$-diagram need be of this form.

Vertical opposites. In this subsection we study the vertical opposite construction of [3], with the goal of justifying our intuition that the theory of S-limits can be recovered from that of S-colimits, and vice-versa.

5.14. **Recollection.** Suppose $X \longrightarrow T$ a cocartesian fibration. Then the simplicial set X^{vop} is defined to have n-simplices

$$\begin{tikzcd} {}_\natural\widetilde{\mathscr{O}}(\Delta^n) \arrow[r] \arrow[d, "\mathrm{ev}_1"'] & {}_\natural X \arrow[d] \\ (\Delta^n)^\sharp \arrow[r] & T^\sharp. \end{tikzcd}$$

The forgetful map $X^{vop} \longrightarrow T$ is a cocartesian fibration with cocartesian edges given by $\widetilde{\mathscr{O}}(\Delta^1)^\sharp \longrightarrow {}_\natural X$. For every $t \in T$, we have an equivalence $X^{op}_t \xrightarrow{\simeq} X^{vop}_t$ implemented by the map which precomposes by $\mathrm{ev}_0 : {}_\natural\widetilde{\mathscr{O}}(\Delta^n) \longrightarrow ((\Delta^n)^{op})^\flat$, which is an equivalence in $s\mathbf{Set}^+$.

Dually, suppose $Y \longrightarrow T$ a cartesian fibration. Then the simplicial set Y^{vop} is defined to have n-simplices

$$\begin{tikzcd} (\widetilde{\mathscr{O}}(\Delta^n)^{op})^{\natural} \arrow[r] \arrow[d, "\mathrm{ev}_0^{op}"'] & Y^{\natural} \arrow[d] \\ (\Delta^n)^{\sharp} \arrow[r] & T^{\sharp}. \end{tikzcd}$$

and similarly the forgetful map $Y^{vop} \longrightarrow T$ is a cartesian fibration with fibers $Y_t^{vop} \xleftarrow{\simeq} Y_t^{op}$. As a warning, note that the definition of the underlying simplicial set of $(-)^{vop}$ changes depending on whether the input is a cocartesian or cartesian fibration.

Define a functor $\widetilde{\mathscr{O}}'(-) : s\mathbf{Set}^+_{/S} \longrightarrow s\mathbf{Set}^+_{/S}$ by

$$\widetilde{\mathscr{O}}'(A \xrightarrow{\pi} S) = (\widetilde{\mathscr{O}}(A), \mathscr{E}_A) \xrightarrow{\pi \circ \mathrm{ev}_1} S$$

where an edge e is in $\mathscr{E}_A$ just in case $\mathrm{ev}_0(e)$ is marked in A^{op}. Note that $\widetilde{\mathscr{O}}(-)$ preserves colimits since it is defined as precomposition by $\Delta^{op} \xrightarrow{(rev \star id)^{op}} \Delta^{op}$, and from this it easily follows that $\widetilde{\mathscr{O}}'(-)$ also preserves colimits. By the adjoint functor theorem, $\widetilde{\mathscr{O}}'(-)$ admits a right adjoint, which we label $(-)^{vop}$; this agrees with the previously defined $(-)^{vop}$ for cocartesian fibrations ${}_{\natural}X \longrightarrow S^{\sharp}$.

5.15. **Proposition.** *The adjunction*

$$\widetilde{\mathscr{O}}'(-) \colon s\mathbf{Set}^+_{/S} \rightleftarrows s\mathbf{Set}^+_{/S} : (-)^{vop}$$

is a Quillen equivalence with respect to the cocartesian model structure on $s\mathbf{Set}^+_{/S}$.

Proof. We first prove the adjunction is Quillen by employing the criterion of Lm. 4.13. Consider the four classes of maps which generate the left marked anodyne maps:

(1) $i : \Lambda^n_k \longrightarrow \Delta^n$, $0 < k < n$: By [1, 12.15], $\widetilde{\mathscr{O}}(\Lambda^n_k) \longrightarrow \widetilde{\mathscr{O}}(\Delta^n)$ is inner anodyne, so $\widetilde{\mathscr{O}}'(i)$ is left marked anodyne.
(2) $i : {}_{\natural}\Lambda^n_0 \longrightarrow {}_{\natural}\Delta^n$: We can adapt the proof of [1, 12.16] to show that $\widetilde{\mathscr{O}}'(i)$ is a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$ (even though it fails to be left marked anodyne). The basic fact underlying this is that a *right* marked anodyne map is an equivalence in $s\mathbf{Set}^+$, so in $s\mathbf{Set}^+_{/S}$ if it lies entirely over an object; details are left to the reader.
(3) $i : K^{\flat} \longrightarrow K^{\sharp}$ for K a Kan complex: Because $\widetilde{\mathscr{O}}(K) \longrightarrow K^{op} \times K$ is a left fibration, $\widetilde{\mathscr{O}}(K)$ is then again a Kan complex. It follows that $\widetilde{\mathscr{O}}'(i)$ is left marked anodyne.
(4) $(\Lambda^2_1)^{\sharp} \cup_{\Lambda^2_1} (\Delta^2)^{\flat} \longrightarrow (\Delta^2)^{\sharp}$: Obvious from the definitions.

It remains to show that for a trivial cofibration $f : {}_{\natural}X \longrightarrow {}_{\natural}Y$ between fibrant objects, $\widetilde{\mathscr{O}}'(f)$ is again a trivial cofibration. Since $\widetilde{\mathscr{O}}(X) \longrightarrow \widetilde{\mathscr{O}}(Y)$ is a map of cocartesian fibrations over S and the marking on $\widetilde{\mathscr{O}}'(-)$ contains these cocartesian edges, by Prp. 13.4 it suffices to show that for every object $s \in S$, $\widetilde{\mathscr{O}}'(X)_s \longrightarrow \widetilde{\mathscr{O}}'(Y)_s$ is an equivalence in $s\mathbf{Set}^+$. We have a commutative square

$$\begin{tikzcd} \widetilde{\mathscr{O}}'(X)_s \arrow[r] \arrow[d] & \widetilde{\mathscr{O}}'(Y)_s \arrow[d] \\ X_s^{\sharp} \arrow[r, "f_s"] & Y_s^{\sharp} \end{tikzcd}$$

where the vertical maps are left fibrations and the bottom map is an equivalence in $s\mathbf{Set}^+$. Therefore, the map $X_s^{\sharp} \times_{Y_s^{\sharp}} \widetilde{\mathscr{O}}'(Y)_s \longrightarrow \widetilde{\mathscr{O}}'(Y)_s$ is an equivalence in $s\mathbf{Set}^+$. Applying Prp. 13.4 once more, we reduce to showing that for every object $x_1 \in X$, $\widetilde{\mathscr{O}}'(X)_{x_1} \longrightarrow \widetilde{\mathscr{O}}'(Y)_{f(x_1)}$ is an equivalence in $s\mathbf{Set}^+$.

Now employing the source maps, we have a commutative square

$$\begin{array}{ccc} \widetilde{\mathscr{O}}'(X)_{x_1} & \longrightarrow & \widetilde{\mathscr{O}}'(Y)_{f(x_1)} \\ \big\downarrow & & \big\downarrow \\ X^{op\natural} & \xrightarrow{f^{op}} & Y^{op\natural} \end{array}$$

where the vertical maps are left fibrations and the bottom horizontal map is a *cartesian* equivalence in $s\mathbf{Set}^+_{/S^{op}}$. Therefore, the map $X^{op}\times_{Y^{op}}\widetilde{\mathscr{O}}'(Y)_s \longrightarrow \widetilde{\mathscr{O}}'(Y)_s$ is a cartesian equivalence. By a third application of Prp. 13.4, we reduce to showing that for every object $x_0 \in X$, $\widetilde{\mathscr{O}}'(X)_{(x_0,x_1)} \longrightarrow \widetilde{\mathscr{O}}'(Y)_{(f(x_0),f(x_1))}$ is an equivalence. But now both sides are endowed with the maximal marking and the map is equivalent to $\mathrm{Map}_X(x_0,x_1) \xrightarrow{f_*} \mathrm{Map}_Y(f(x_0),f(x_1))$, which is an equivalence by assumption.

The fact that this Quillen adjunction is an equivalence follows immediately from [3, 1.7]. $\square$

5.16. **Lemma.** *Let $C \longrightarrow S$ be a cocartesian fibration.*

(1) *Let $f : S' \longrightarrow S$ be a functor. Then we have an isomorphism $f^*(C^{vop}) \cong f^*(C)^{vop}$.*
(2) *Let $g : S \longrightarrow T$ be a cartesian fibration and let C be a S-category. Then there is a T-functor $\chi : g_*(C)^{vop} \longrightarrow g_*(C^{vop})$ natural in C which is an equivalence.*

Proof. (1) is obvious from the definitions. For (2), the map χ is defined as follows: an n-simplex of $g_*(C)^{vop}$ over $\sigma \in T_n$ is given by the data of a commutative diagram

$$\begin{array}{ccc} {}_\natural\widetilde{\mathscr{O}}(\Delta^n)\times_{T^\sharp} S^\sharp & \longrightarrow & {}_\natural C \\ \big\downarrow & & \big\downarrow \\ (\Delta^n \times_T S)^\sharp & \xrightarrow{g^*\sigma} & S^\sharp \end{array}$$

and precomposition by the obvious map $\widetilde{\mathscr{O}}(\Delta^n\times_T S) \longrightarrow \widetilde{\mathscr{O}}(\Delta^n)\times_T S$ yields an n-simplex of $g_*(C^{vop})$.

We now show that for all $t \in T$, χ_t is a categorical equivalence. Because χ_t is obtained by taking levelwise 0-simplices of the map of complete Segal spaces

$$\mathrm{Map}_S({}_\natural\widetilde{\mathscr{O}}(\Delta^\bullet)\times S_t^\sharp, {}_\natural C) \longrightarrow \mathrm{Map}_S({}_\natural\widetilde{\mathscr{O}}(\Delta^\bullet)\times \widetilde{\mathscr{O}}(S_t)^\sharp, {}_\natural C),$$

it suffices to show that for all n, ${}_\natural\widetilde{\mathscr{O}}(\Delta^n)\times\widetilde{\mathscr{O}}(S_t)^\sharp \longrightarrow {}_\natural\widetilde{\mathscr{O}}(\Delta^n)\times S_t^\sharp$ is a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$. As a special case of Prp. 6.3, $\widetilde{\mathscr{O}}(S_t)^\sharp \longrightarrow S_t^\sharp$ is a cocartesian equivalence in $s\mathbf{Set}^+_{/S_t}$, so the claim follows. $\square$

5.17. **Lemma.** *The map $\mathrm{ev}^{op} : (\widetilde{\mathscr{O}}(\Delta^n)^{op})^\natural \longrightarrow (\Delta^n)^\sharp \times ((\Delta^n)^{op})^\flat$ is left marked anodyne.*

Proof. For convenience, we will relabel $\widetilde{\mathscr{O}}(\Delta^n)^{op}$ as the nerve of the poset I_n with objects ij, $0 \le i \le j \le n$ and maps $ij \longrightarrow kl$ for $i \le k$ and $j \le l$. Then an edge $ij \to kl$ is marked in I_n just in case $j = l$, and the map ev^{op} becomes the projection $\rho_n : I_n \longrightarrow (\Delta^n)^\sharp \times (\Delta^n)^\flat$, $ij \mapsto (i,j)$. Let $f_n : (\Delta^n)^\flat \longrightarrow I_n$ be the map which sends i to $0i$. Then $\rho_n \circ f_n : \{0\}\times(\Delta^n)^\flat \longrightarrow (\Delta^n)^\sharp\times(\Delta^n)^\flat$ is left marked anodyne, so by the right cancellativity of left marked anodyne maps it suffices to show that i_n is left marked anodyne. For this, we factor f_n as the composition

$$(\Delta^n)^\flat = I_{n,-1} \longrightarrow I_{n,0} \longrightarrow \ \ldots\ \longrightarrow I_{n,n} = I_n$$

where $I_{n,k} \subset I_n$ is the subcategory on objects ij, $i = 0$ or $j \le k$ (and inherits the marking from I_n), and argue that each inclusion $g_k : I_{n,k} \subset I_{n,k+1}$ is left marked anodyne. For this, note that g_k fits into a pushout square

$$\begin{array}{ccc} \{0\}\times(\Delta^{k+1})^\flat \bigcup_{\{0\}\times(\Delta^k)^\flat} (\Delta^{n-k-1})^\sharp\times(\Delta^k)^\flat & \longrightarrow & (\Delta^{n-k-1})^\sharp\times(\Delta^{k+1})^\flat \\ \big\downarrow & & \big\downarrow \\ I_{n,k} & \xrightarrow{g_k} & I_{n,k+1} \end{array}$$

with the upper horizontal map marked left anodyne. $\square$

5.18. **Construction.** Suppose T an ∞-category, $X, Z \longrightarrow T$ cocartesian fibrations, $Y \longrightarrow T$ a cartesian fibration, and a map $\mu : {}_{\natural}X \times_T Y^{\natural} \longrightarrow {}_{\natural}Z$ of marked simplicial sets over T. We define a map $\mu^{vop} : {}_{\natural}X^{vop} \times_T Y^{vop\natural} \longrightarrow {}_{\natural}Z^{vop}$ by the following process:

Let J_n be the nerve of the poset with objects ij, $0 \leq i \leq n$, $-n \leq j \leq n$ and $-j \leq i$ and maps $ij \to kl$ if $i \leq k$, $j \leq l$. Mark edges $ij \longrightarrow kl$ if $j = l$. Let $I_n \subset J_n$ be the subcategory on ij with $j \geq 0$ and $I'_n \subset J_n$ be the subcategory on ij with $j \leq 0$; also give I_n, I'_n the induced markings. We have an inclusion $(\Delta^n)^{\sharp} \longrightarrow J_n$ given by $i \mapsto i0$ which restricts to inclusions $(\Delta^n)^{\sharp} \longrightarrow I_n$, $(\Delta^n)^{\sharp} \longrightarrow I'_n$ and induces a map $\gamma_n : I_n \cup_{(\Delta^n)^{\sharp}} I'_n \subset J_n$.

Define auxiliary (unmarked) simplicial sets $Z' \longrightarrow T$ by $\mathrm{Hom}_{/T}(\Delta^n, Z') = \mathrm{Hom}_{/T}(J_n, {}_{\natural}Z)$ and $Z'' \longrightarrow T$ by $\mathrm{Hom}_{/T}(\Delta^n, Z'') = \mathrm{Hom}_{/T}(I_n \cup_{(\Delta^n)^{\sharp}} I'_n, {}_{\natural}Z)$, where $J_n \longrightarrow \Delta^n$ via $ij \mapsto i$. We have a map $r : Z' \longrightarrow Z''$ given by restriction along the γ_n, which we claim is a trivial fibration. By a standard reduction, for this it suffices to show that γ_n is left marked anodyne. Indeed, this follows from Lm. 5.17 applied to $I_n \longrightarrow (\Delta^n)^{\sharp} \times \Delta^n$ and the observation that the map $\Delta^n \times \Delta^n \cup_{\Delta^n} I'_n \longrightarrow J_n$ is inner anodyne, whose proof we leave to the reader.

Define also a map $Z' \longrightarrow Z^{vop}$ over T by restriction along the map ${}_{\natural}\widetilde{\mathscr{O}}(\Delta^n) \longrightarrow J_n$ which sends ij to jn if $i = 0$ and $j(-i)$ otherwise. Finally, define a map $X^{vop} \times_T Y^{vop} \longrightarrow Z''$ over T as follows: a map $\Delta^n \longrightarrow X^{vop} \times_T Y^{vop}$ is given by the data

$$\begin{tikzcd} {}_{\natural}\widetilde{\mathscr{O}}(\Delta^n) \arrow[r] \arrow[d] & {}_{\natural}X \arrow[d] \\ (\Delta^n)^{\sharp} \arrow[r] & T^{\sharp} \end{tikzcd}, \quad \begin{tikzcd} \left(\widetilde{\mathscr{O}}(\Delta^n)^{op}\right)^{\natural} \arrow[r] \arrow[d] & Y^{\natural} \arrow[d] \\ (\Delta^n)^{\sharp} \arrow[r] & T^{\sharp}. \end{tikzcd}$$

We have isomorphisms ${}_{\natural}\widetilde{\mathscr{O}}(\Delta^n) \cong I'_n$ and $\left(\widetilde{\mathscr{O}}(\Delta^n)^{op}\right)^{\natural} \cong I_n$, and obvious retractions $I_n \cup_{(\Delta^n)^{\sharp}} I'_n \longrightarrow I_n, I'_n$ given by collapsing the complementary part onto Δ^n. Using this, we may define

$$I_n \cup_{(\Delta^n)^{\sharp}} I'_n \longrightarrow {}_{\natural}X \times_T Y^{\natural} \longrightarrow {}_{\natural}Z$$

which is an n-simplex of Z''.

Choosing a section of r, we may compose these maps to define μ^{vop}, which is then easily checked to also preserve the indicated markings. For example, μ^{vop} on edges is given by

$$\left(\begin{tikzcd} & x_{11} \arrow[d] \\ x_{00} \arrow[r] & x_{01}, \\ y_{01} \arrow[r] \arrow[d] & y_{11} \\ y_{00} & \end{tikzcd} \right) \mapsto \left(\begin{tikzcd} & \mu(x_{11}, y_{11}) \arrow[d] \\ \mu(x_{00}, y_{01}) \arrow[r] \arrow[d] & \mu(x_{01}, y_{11}) \arrow[d] \\ \mu(x_{00}, y_{00}) \arrow[r] & \alpha_!\mu(x_{00}, y_{00}) \end{tikzcd} \right) \mapsto \left(\begin{tikzcd} & \mu(x_{11}, y_{11}) \arrow[d] \\ \mu(x_{00}, y_{00}) \arrow[r] & \alpha_!\mu(x_{00}, y_{00}) \end{tikzcd} \right)$$

where $\alpha_!\mu(x_{00}, y_{00})$ is a choice of pushforward for the edge α in T that the diagrams are vertically over.

5.19. **Lemma.** *Let $C \longrightarrow T$ be a cartesian fibration and let $D \longrightarrow T$ be a cocartesian fibration. There exists a T-equivalence $\psi : \widetilde{\mathrm{Fun}}(C, T)^{vop} \longrightarrow \widetilde{\mathrm{Fun}}_T(C^{vop}, D^{vop})$.*

Proof. We have a map $\mu : \widetilde{\mathrm{Fun}}_T(C, D) \times_T C \longrightarrow D$ adjoint to the identity. Employing Cnstr. 5.18 on μ and then adjointing, we obtain our desired T-functor ψ. A chase of the definitions then shows that for all objects $t \in T$, ψ_t is homotopic to the known equivalence $\mathrm{Fun}(C_t, T_t)^{op} \simeq \mathrm{Fun}(C_t^{op}, D_t^{op})$. $\square$

5.20. **Lemma.** *Let K and L be S-categories. Then there exists a S-equivalence*

$$\psi : (K \star_S L)^{vop} \xrightarrow{\simeq} L^{vop} \star_S K^{vop}$$

over $S \times \Delta^1$.

Proof. Note that $(S \times \Delta^1)^{vop} \cong S \times (\Delta^1)^{op}$. View $(K \star_S L)^{vop}$ as lying over $S \times \Delta^1$ via the isomorphism $(\Delta^1)^{op} \cong \Delta^1$. Since $(K \star_S L)^{vop}_0 \cong L^{vop}$ and $(K \star_S L)^{vop}_1 \cong K^{vop}$, we have our S-functor ψ as adjoint to the identity over $S \times \partial\Delta^1$. Fiberwise, ψ_s is homotopic to the known isomorphism $(K_s \star L_s)^{op} \cong L_s^{op} \star K_s^{op}$, so ψ is an equivalence. $\square$

5.21. **Proposition.** *Suppose S-categories K and C.*

(1) The adjoint of the vertical opposite of the evaluation map induces a equivalence

$$\underline{\mathrm{Fun}}_S(K, C)^{vop} \xrightarrow{\simeq} \underline{\mathrm{Fun}}_S(K^{vop}, C^{vop}).$$

(2) Suppose a S-functor $p : K \longrightarrow C$. We have equivalences

$$(C^{(p,S)/})^{vop} \simeq (C^{vop})^{/(p^{vop},S)}, \quad (C^{/(p,S)})^{vop} \simeq (C^{vop})^{(p^{vop},S)/}.$$

Proof. (1): Recall from 6.3.1 the equivalence $\underline{\mathrm{Fun}}_S(K, C) \simeq \pi_* \pi'^* \{K, C\}_S$. By Lm. 5.19 and Lm. 5.16(1), $\{K, C\}_S^{vop} \simeq \{K^{vop}, C^{vop}\}_S$. By Lm. 5.16(1) and (2), $\pi_* \pi'^* \{K, C\}_S^{vop} \simeq (\pi_* \pi'^* \{K, C\}_S)^{vop}$. Combining these equivalences supplies an equivalence $\underline{\mathrm{Fun}}_S(K, C)^{vop} \simeq \underline{\mathrm{Fun}}_S(K^{vop}, C^{vop})$. It is straightforward but tedious to verify that the adjoint of the vertical opposite of the evaluation map $\underline{\mathrm{Fun}}_S(K, C)^{vop} \times_S K^{vop} \longrightarrow C^{vop}$ is homotopic to this equivalence.

(2): Combine (1), Lm. 5.20, Prp. 5.15 (which shows in particular that $(-)^{vop}$ is right Quillen), and the definition of the S-slice category. $\square$

5.22. **Corollary.** *Let $\overline{p} : S \star_S K \longrightarrow C$ be a S-functor. Then $\overline{p}$ is a S-limit diagram if and only if $\overline{p}^{vop} : K^{vop} \star_S S \longrightarrow C^{vop}$ is a S-colimit diagram.*

This allows us to deduce statements about S-limits from statements about S-colimits, and vice-versa. For this reason, we will primarily concentrate our attention on proving statements concerning S-colimits (and eventually, S-left Kan extensions), leaving the formulation of the dual results to the reader.

6. Assembling S-slice categories from ordinary slice categories

Suppose a S-functor $p : K \longrightarrow C$. For every morphism $\alpha : s \to t$ in S, we have a functor $p_\alpha : K_s \longrightarrow C_t$, and we may consider the collection of 'absolute' slice categories $C_{p_\alpha/}$ and examine the functoriality that they satisfy. For this, we have the following basic observation: given a morphism $f : t \to t'$, *covariant* functoriality of slice categories in the target yields a functor $C_{p_\alpha/} \longrightarrow C_{p_{f\alpha}/}$, and given a morphism $g : s' \to s$, *contravariant* functoriality in the source yields a functor $C_{p_\alpha/} \longrightarrow C_{p_{\alpha g}/}$. Elaborating, we will show in this section that there exists a functor $F : \widetilde{\mathscr{O}}(S) \longrightarrow \mathbf{Cat}_\infty$ out of the twisted arrow category $\widetilde{\mathscr{O}}(S)$ such that $F(\alpha) \simeq C_{p_\alpha/}$, which encodes all of this functoriality. Moreover, the right Kan extension of F along the target functor $\widetilde{\mathscr{O}}(S) \longrightarrow S$ is $C_{(p,S)/}$. We will end with some applications of this result to the theory of cofinality and presentability.

We first record a cofinality result which implies that the values of a right Kan extension along $\mathrm{ev}_1 : \widetilde{\mathscr{O}}(S) \longrightarrow S$ are computed as ends.

6.1. **Lemma.** *The functor $\widetilde{\mathscr{O}}(S^{s/}) \longrightarrow \widetilde{\mathscr{O}}(S) \times_S S^{s/}$ is initial.*

Proof. Let $(\alpha : u \longrightarrow t, \beta : s \longrightarrow t)$ be an object of $\widetilde{\mathscr{O}}(S) \times_S S^{s/}$. We will prove that

$$C = \widetilde{\mathscr{O}}(S^{s/}) \times_{\widetilde{\mathscr{O}}(S) \times_S S^{s/}} (\widetilde{\mathscr{O}}(S) \times_S S^{s/})_{/(\alpha,\beta)}$$

is weakly contractible. An object of C is the data of an edge

$$\begin{tikzcd} & s \arrow[dl, "f"'] \arrow[dr, "g"] & \\ x \arrow[rr, "h"] & & y \end{tikzcd}$$

in $S^{s/}$, which we will abbreviate as $f \xrightarrow{h} g$, and an edge

$$\left(\begin{array}{ccc} x & \xrightarrow{h} & y \\ \delta\uparrow & & \downarrow\gamma \\ u & \xrightarrow{\alpha} & t \end{array}, \quad \begin{array}{ccc} s & \xrightarrow{g} & y \\ & \overset{\beta}{\searrow} & \downarrow\gamma \\ & & t \end{array} \right)$$

in $\widetilde{\mathscr{O}}(S) \times_S S^{s/}$, which we will abbreviate as $(h, g) \xrightarrow{(\delta,\gamma)} (\alpha, \beta)$.

Let $C_0 \subset C$ be the full subcategory on objects $c = ((f \xrightarrow{h} g), (h,g) \xrightarrow{(\delta,\gamma)} (\alpha,\beta))$ such that γ is a degenerate edge in $S^{s/}$. We will first show that C_0 is a reflective subcategory of C by verifying the first condition of [9, 5.2.7.8]. Given an object c of C, define c' to be $((f \xrightarrow{\gamma h} \beta), (\gamma h, \beta) \xrightarrow{(\delta, id_t)} (\alpha,\beta))$ and let $e : c \longrightarrow c'$ be the edge given by

$$\left(\begin{tikzcd} f \arrow[r, "h"] & g \\ f \arrow[u, "id_f"] \arrow[r, "\gamma h"] & \beta \arrow[from=1-2, "\gamma"] \end{tikzcd}, \quad \begin{tikzcd} (h,g) \arrow[rr, "(id_x,\gamma)"] \arrow[dr, "(\delta,\gamma))"'] & & (\gamma h,\beta) \arrow[dl, "(\delta, id_t))"] \\ & (\alpha,\beta) & \end{tikzcd} \right).$$

We need to show that for all $d = ((f' \xrightarrow{h'} \beta), (h',\beta) \xrightarrow{(\delta', id)} (\alpha,\beta)) \in C_0$, $\mathrm{Map}_C(c', d) \xrightarrow{e^*} \mathrm{Map}_C(c,d)$ is a homotopy equivalence. The space $\mathrm{Map}_C(c,d)$ lies in a commutative diagram

$$\begin{tikzcd} \mathrm{Map}_C(c,d) \arrow[r] \arrow[d] & \mathrm{Map}_{\widetilde{\mathscr{O}}(S^{s/})}(f \xrightarrow{h} g, f' \xrightarrow{h'} \beta) \arrow[d] \\ \mathrm{Map}_{(\widetilde{\mathscr{O}}(S) \times_S S^{s/})_{/(\alpha,\beta)}}((h,g),(h',\beta)) \arrow[r] \arrow[d] & \mathrm{Map}_{\widetilde{\mathscr{O}}(S) \times_S S^{s/}}((h,g),(h',\beta)) \arrow[d, "(\delta', id)_*"] \\ \Delta^0 \arrow[r, "(\delta,\gamma)"] & \mathrm{Map}_{\widetilde{\mathscr{O}}(S) \times_S S^{s/}}((h,g),(\alpha,\beta)) \end{tikzcd}$$

where the two squares are homotopy pullback squares. We also have the analogous diagram for $\mathrm{Map}_C(c', d)$, and the map e^* is induced by a natural transformation of these diagrams. The assertion then reduces to checking that the upper square in the diagram

$$\begin{tikzcd} \mathrm{Map}_{\widetilde{\mathscr{O}}(S^{s/})}(f \xrightarrow{\gamma h} \beta, f' \xrightarrow{h'} \beta) \arrow[r, "(id_f,\gamma)^*"] \arrow[d] & \mathrm{Map}_{\widetilde{\mathscr{O}}(S^{s/})}(f \xrightarrow{h} g, f' \xrightarrow{h'} \beta) \arrow[d] \\ \mathrm{Map}_{\widetilde{\mathscr{O}}(S) \times_S S^{s/}}((\gamma h, \beta),(\alpha,\beta)) \arrow[r, "(id_x,\gamma))^*"] \arrow[d] & \mathrm{Map}_{\widetilde{\mathscr{O}}(S) \times_S S^{s/}}((h,g),(\alpha,\beta)) \arrow[d] \\ \mathrm{Map}_{S^{s/}}(\beta,\beta) \arrow[r, "\gamma^*"] & \mathrm{Map}_{S^{s/}}(g,\beta) \end{tikzcd}$$

is a homotopy pullback square. Since (id_x, γ) and (id_f, γ) are ev_1-cocartesian edges in $\widetilde{\mathscr{O}}(S)$ and $\widetilde{\mathscr{O}}(S^{s/})$ respectively, the lower and outer squares are homotopy pullback squares (where we implicitly use that the map (δ', id) covers the identity in $S^{s/}$ to identify the long vertical maps with those induced by ev_1), and the claim is proven.

To complete the proof, we will show that $c = (\beta = \beta, (id_t, \beta) \xrightarrow{(\alpha, id_t)} (\alpha,\beta))$ is an initial object in C_0. Let $d \in C_0$ be as above. In the diagram

$$\begin{tikzcd} \Delta^0 \arrow[r, "(h', id_\beta)"] \arrow[d] & \mathrm{Map}_{\widetilde{\mathscr{O}}(S^{s/})}(\beta = \beta, f' \xrightarrow{h'} \beta) \arrow[d] & \\ \Delta^0 \arrow[r, "(\alpha, id_t)"] \arrow[d] & \mathrm{Map}_{\widetilde{\mathscr{O}}(S) \times_S S^{s/}}((id_t, \beta),(\alpha,\beta)) \arrow[r] \arrow[d] & \mathrm{Map}_{\widetilde{\mathscr{O}}(S)}(id_t, \alpha) \arrow[d] \\ \Delta^0 \arrow[r, "id_\beta"] & \mathrm{Map}_{S^{s/}}(\beta,\beta) \arrow[r] & \mathrm{Map}_S(t,t) \end{tikzcd}$$

we need to show that the upper square is a homotopy pullback square in order to prove that $\mathrm{Map}_C(c,d) \simeq *$. The fiber of $\widetilde{\mathscr{O}}(S)$ over $t \in S$ is equivalent to $(S_{/t})^{op}$; in particular, id_t is an initial object in the fiber over t. Therefore, the two outer squares are both homotopy pullbacks. Since the lower right square is a homotopy pullback, this shows that all squares in the diagram are homotopy pullbacks, as desired. $\square$

Suppose K a S-category. Let J_n be the poset with objects ij for $0 \leq i \leq j \leq 2n+1$ which has a unique morphism $ij \longrightarrow kl$ if and only if $k \leq i \leq j \leq l$. Let $I_n \subset J_n$ be the full subcategory on objects ij such that $i \leq n$. In view of the isomorphisms $J_n \cong \widetilde{\mathscr{O}}(\Delta^{2n+1}) \cong \widetilde{\mathscr{O}}((\Delta^n)^{op} \star \Delta^n)$, the I_n and J_n extend to functors $I_\bullet \subset J_\bullet \cong \widetilde{\mathscr{O}}((\Delta^\bullet)^{op} \star \Delta^\bullet) : \Delta \longrightarrow s\mathbf{Set}$. Viewing I_n and J_n as marked simplicial sets where $ij \longrightarrow kl$ is marked just in case $k = i$, we moreover have functors to $s\mathbf{Set}^+$. Define the simplicial set $X : \Delta^{op} \longrightarrow \mathbf{Set}$ to be $\mathrm{Hom}_{s\mathbf{Set}^+}(I_\bullet, {}_\natural K) \times_{\mathrm{Hom}(I_\bullet, S)} \mathrm{Hom}((\Delta^\bullet)^{op} \star \Delta^\bullet, S)$ where $I_\bullet \subset J_\bullet \longrightarrow (\Delta^\bullet)^{op} \star \Delta^\bullet$ is given by the target map. An n-simplex of X is thus the data of a diagram

$$\begin{tikzcd}
 & & & k_{nn} \arrow[r] \arrow[d] & k_{n(n+1)} \arrow[r] \arrow[d] & \dots \arrow[r] & k_{n(2n+1)} \arrow[d] \\
 & & \iddots & \vdots & \vdots & \dots & \vdots \\
 & k_{11} \arrow[r] \arrow[d] & \dots \arrow[r] & k_{1n} \arrow[r] \arrow[d] & k_{1(n+1)} \arrow[r] \arrow[d] & \dots \arrow[r] & k_{1(2n+1)} \arrow[d] \\
k_{00} \arrow[r] & k_{01} \arrow[r] & \dots \arrow[r] & k_{0n} \arrow[r] & k_{0(n+1)} \arrow[r] & \dots \arrow[r] & k_{0(2n+1)}
\end{tikzcd}$$

where the horizontal edges are cocartesian in K and the vertical edges lie over degeneracies in S.

Declare an edge e in X to be marked if the corresponding map $I_1 \longrightarrow {}_\natural K$ sends all edges to marked edges. We have a commutative square of marked simplicial sets

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & (K^\vee)^\natural \arrow[d] \\
\widetilde{\mathscr{O}}(S)^\sharp \arrow[r] & (S^{op})^\sharp
\end{tikzcd}$$

where the map $X \longrightarrow K^\vee$ is defined by restricting $I_n \longrightarrow K$ to $I'_n \longrightarrow K$ where I'_n is the full subcategory of I_n on ij with $j \leq n$. Let ψ denote the resulting map from X to the pullback.

6.2. **Lemma.** *$\psi : X \longrightarrow \widetilde{\mathscr{O}}(S)^\sharp \times_{(S^{op})^\sharp} (K^\vee)^\natural$ is a trivial fibration of marked simplicial sets.*

Proof. Since any lift of a marked edge in $\widetilde{\mathscr{O}}(S)^\sharp \times_{(S^{op})^\sharp} (K^\vee)^\natural$ to an edge in X is marked, it suffices to prove that the underlying map of simplicial sets is a trivial fibration.

We first show that $I'_n \subset I_n$ is left marked anodyne. Let $I_{n,k} \subset I_n$ be the full subcategory on objects ij with $i \leq k$ and similarly for $I'_{n,k}$. For $0 \leq k < n$ we have a pushout decomposition

$$\begin{tikzcd}
((\Delta^{n-k})^{op})^\flat \times (\Delta^k)^\sharp \bigcup\limits_{((\Delta^{n-k-1})^{op})^\flat \times (\Delta^k)^\sharp} ((\Delta^{n-k-1})^{op})^\flat \times (\Delta^{n+k+1})^\sharp \arrow[r] \arrow[d] & I'_{n,n-k} \bigcup\limits_{I'_{n,n-k-1}} I_{n,n-k-1} \arrow[d] \\
((\Delta^{n-k})^{op})^\flat \times (\Delta^{n+k+1})^\sharp \arrow[r] & I_{n,n-k},
\end{tikzcd}$$

and the lefthand map is left marked anodyne by [9, 3.1.2.3]. It thus suffices to show that $I'_{n,0} \cong (\Delta^n)^\sharp \longrightarrow I_{n,0} \cong (\Delta^{2n+1})^\sharp$ is left marked anodyne, and this is clear.

We now explain how to solve the lifting problem

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d] & X \arrow[d] \\
\Delta^n \arrow[r] \arrow[ur, dotted] & \widetilde{\mathscr{O}}(S) \times_{S^{op}} K^\vee.
\end{tikzcd}$$

To supply the dotted arrow we must provide a lift in the commutative square

$$\begin{tikzcd}
\partial I_n \cup_{\partial I'_n} I'_n \arrow[r] \arrow[d, "f"] & {}_\natural K \arrow[d] \\
I_n \arrow[r] \arrow[ur, dotted] & S^\sharp.
\end{tikzcd}$$

where $\partial I_n = \bigcup_{[n-1]\subset[n]} I_{n-1}$ as a simplicial subset of I_n and likewise for $\partial I'_n$. Then since $I'_n \longrightarrow \partial I_n \cup_{\partial I'_n} I'_n$ and $I'_n \longrightarrow I_n$ are left marked anodyne, f is a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$, and the lift exists. ☐

For all $s \in S$, we have trivial cofibrations $i_s : K_s \xrightarrow{\simeq} (K^\vee)_s$, and thus commutative squares

$$\begin{tikzcd} K_s \arrow[r, hook] \arrow[d, "id_s"'] & K^\vee \arrow[d] \\ \widetilde{\mathscr{O}}(S) \arrow[r] & S^{op}. \end{tikzcd}$$

From this we obtain a cofibration $\iota : \coprod_{s\in S} K_s \longrightarrow \widetilde{\mathscr{O}}(S) \times_{S^{op}} K^\vee$. We have an explicit lift ι' of ι to X, where $K_s \longrightarrow X$ is given by precomposition by $I_n \longrightarrow \Delta^n$, $ij \mapsto n-i$.

By Lm. 6.2, there exists a lift σ in the commutative square

$$\begin{tikzcd} \coprod_{s\in S} K_s \arrow[r, tail, "\iota'"] \arrow[d, "\iota"] & X \arrow[d, "\psi"] \\ \widetilde{\mathscr{O}}(S) \times_{S^{op}} K^\vee \arrow[r, "="] \arrow[ur, dotted, tail, "\sigma"] & \widetilde{\mathscr{O}}(S) \times_{S^{op}} K^\vee. \end{tikzcd}$$

Let $\chi : X \longrightarrow K$ be the functor induced by $\Delta^n \longrightarrow I_n$, $i \mapsto (n-i)(n+i)$. Define the *twisted pushforward* $\widetilde{P} : \widetilde{\mathscr{O}}(S) \times_{S^{op}} K^\vee \longrightarrow K$ to be the map over S given by the composite $\chi \circ \sigma$. Then for every object $\alpha : s \longrightarrow t$ in $\widetilde{\mathscr{O}}(S)$, $\widetilde{P}_\alpha \circ i_s : K_s \longrightarrow K_t$ is a choice of pushforward functor over α, which is chosen to be the identity if $\alpha = id_s$.

6.3. **Proposition.** *For all $A \in s\mathbf{Set}_{/S}$,*

$$id_A \times_S \widetilde{P} : A^\sharp \times_S (\widetilde{\mathscr{O}}(S)^\sharp \times_{(S^{op})^\sharp} (K^\vee)^\natural) \longrightarrow A^\sharp \times_S {}_\natural K$$

is a cocartesian equivalence in $s\mathbf{Set}^+_{/A}$.

Proof. Let (Z, E) denote the marked simplicial set $\widetilde{\mathscr{O}}(S)^\sharp \times_{(S^{op})^\sharp} (K^\vee)^\natural$. Viewing Z as $\widetilde{\mathscr{O}}(S) \times_{S^{op}\times S} (K^\vee \times S)$, we see that $Z \longrightarrow S$ is a cocartesian fibration with the cocartesian edges a subset of E. Moreover, every edge in E factors as a cocartesian edge followed by an edge in E in the fiber over S. By Prp. 13.4, it suffices to verify that for all $s \in S$, $\widetilde{P}_s$ is a cocartesian equivalence in $s\mathbf{Set}^+$. Since id_s is an initial object in $\widetilde{\mathscr{O}}(S) \times_S \{s\}$, the inclusion of the fiber $(K^\vee)^\sim_s \subset (Z_s, E_s)$ is a cocartesian equivalence in $s\mathbf{Set}^+$ by [9, 3.3.4.1]. We chose $\widetilde{P}$ so as to split the inclusion of K_s in Z, so this completes the proof. ☐

Consider the commutative diagram

$$\begin{tikzcd} \mathscr{O}(S)^\sharp \times_S {}_\natural K \arrow[drrrr, bend left, tail] \arrow[dddr, bend right] & & & & \\ & \mathscr{O}(S)^\sharp \times_S \widetilde{\mathscr{O}}(S)^\sharp \times_{S^{op}} (K^\vee)^\natural \arrow[ul, "id_{\mathscr{O}(S)} \times_S \widetilde{P}"'] \arrow[r] \arrow[d] & \widetilde{\mathscr{O}}(S)^\sharp \times_{S^{op}} (K^\vee)^\natural \arrow[r] \arrow[d] & (K^\vee)^\natural \times S^\sharp \arrow[r, tail, "\mathrm{pr}_S"'] \arrow[d, "q^\vee \times id"'] & S^\sharp \\ & \mathscr{O}(S)^\sharp \times_S \widetilde{\mathscr{O}}(S)^\sharp \arrow[r, "\pi'"] \arrow[d, "\pi"] & \widetilde{\mathscr{O}}(S)^\sharp \arrow[r, "\mathrm{ev}"] & (S^{op})^\sharp \times S^\sharp & \\ & S^\sharp. & & & \end{tikzcd}$$

Since $K^\vee \longrightarrow S^{op}$ is a cartesian fibration, by Thm. 2.23 $(q^\vee \times id)_*$ is right Quillen, and we saw in Exm. 2.25 that π_* is right Quillen. Therefore, given a S-category C, we obtain a $\widetilde{\mathscr{O}}(S)$-category

$$\{K, C\}_S = (\mathrm{ev}^* \circ (q^\vee \times id)_* \circ \mathrm{pr}_S^*)({}_\natural C),$$

and the map $id_{\mathscr{O}(S)} \times_S \widetilde{P}$ induces a S-functor

$$\theta : \underline{\mathrm{Fun}}_S(K, C) \longrightarrow \pi_* \pi'^* \{K, C\}_S, \tag{6.3.1}$$

natural in K and C. By Prp. 6.3 applied to $A = S^{s/}$ for all $s \in S$, θ is an equivalence.

6.4. **Remark.** As a corollary, the global sections of $\{K, C\}_S$ are equivalent to $\mathrm{Fun}_S(K, C)$. If we knew that under the straightening functor St, $\{K, C\}_S$ was equivalent to the composite

$$\widetilde{\mathscr{O}}(S) \longrightarrow S^{op} \times S \xrightarrow{\mathrm{St}_S(K)^{op} \times \mathrm{St}_S(C)} \mathbf{Cat}_\infty^{op} \times \mathbf{Cat}_\infty \xrightarrow{\mathrm{Fun}} \mathbf{Cat}_\infty,$$

then this would yield another proof of the end formula for the ∞-category of natural transformations, as proven in [6, §6]. As we manage to always stay within the environment of cocartesian fibrations, this identification is not necessary for our purposes.

6.5. **Definition.** Given a S-functor $p : K \longrightarrow C$ and a choice of twisted pushforward $\widetilde{P}$ for K, define the cocartesian section $\omega_p : \widetilde{\mathscr{O}}(S) \longrightarrow \{K, C\}_S$ to be the adjoint to

$$p \circ \widetilde{P} : \widetilde{\mathscr{O}}(S)^\sharp \times_{S^{op}} K^{\vee \natural} \longrightarrow {}_\natural K \longrightarrow {}_\natural C.$$

For objects $\alpha : s \to t$ in $\widetilde{\mathscr{O}}(S)$, $\omega_p(\alpha) \in \mathrm{Fun}((K^\vee)_s, C_t)$ is the functor

$$p_t \circ \widetilde{P}_\alpha : (K^\vee)_s \longrightarrow K_t \longrightarrow C_t.$$

Define the *twisted slice category* $C^{\widetilde{p/S}}$ to be $\widetilde{\mathscr{O}}(S) \times_{\{K,C\}_S} \{K \star_S S, C\}_S$ (we omit the dependence on $\widetilde{P}$ from the notation). The fiber of $\widetilde{\mathscr{O}}(S) \times_{\{K,C\}_S} \{K \star_S S, C\}_S$ over an object $\alpha : s \longrightarrow t$ is $C^{p_t \circ \widetilde{P}_\alpha /}$.

We now connect the constructions $C^{\widetilde{p/S}}$ and $C^{p/S}$. A check of the definitions reveals that $\theta \circ \sigma_p = \pi_* \pi'^*(\omega_p)$ for the canonical cocartesian section $\sigma_p : S \longrightarrow \underline{\mathrm{Fun}}_S(K, C)$. We thus have a morphism of spans

$$\begin{tikzcd} S \arrow[r, "\sigma_p"'] \arrow[d, "="] & \underline{\mathrm{Fun}}_S(K, C) \arrow[d, "\simeq"] & \underline{\mathrm{Fun}}_S(K \star_S S, C) \arrow[l] \arrow[d, "\simeq"] \\ S \arrow[r, "\pi_* \pi'^*(\omega_p)"'] & \pi_* \pi'^* \{K, C\}_S & \pi_* \pi'^* \{K \star_S S, C\}_S \arrow[l] \end{tikzcd}$$

with all objects fibrant and the right horizontal maps fibrations by a standard argument. Taking pullbacks, we deduce:

6.6. **Theorem.** *We have an equivalence*

$$\pi_* \pi'^* (C^{\widetilde{p/S}}) \xrightarrow{\simeq} C^{p/S}.$$

In other words, the right Kan extension of $C^{\widetilde{p/S}}$ along the target functor $\mathrm{ev}_1 : \widetilde{\mathscr{O}}(S) \longrightarrow S$ is equivalent to $C^{p/S}$.

Proof. Our interpretation of this equivalence is by Exm. 2.25. □

Relative cofinality. Let us now apply Thm. 6.6. We have the S-analogue of the basic cofinality result [9, 4.1.1.8].

6.7. **Theorem.** *Let $f : K \longrightarrow L$ be a S-functor. The following conditions are equivalent:*

(1) For every object $s \in S$, $f_s : K_s \longrightarrow L_s$ is final.
(2) For every S-functor $p : L \longrightarrow C$, the functor $f^* : C^{p/S} \longrightarrow C^{pf/S}$ is an equivalence.
(3) For every S-colimit diagram $\overline{p} : L \star_S S \longrightarrow C$, $\overline{p} \circ f^\triangleright : K \star_S S \longrightarrow C$ is a S-colimit diagram.

Proof. (1) $\Rightarrow$ (2): Factoring f as the composition of a cofibration and a trivial fibration, we may suppose that f is a cofibration, in which case we may choose compatible twisted pushforward functors $\widetilde{P}_K$ and $\widetilde{P}_L$. Let $p : L \longrightarrow C$ be a S-functor. Precomposition by f yields a $\widetilde{\mathscr{O}}(S)$-functor $\widetilde{f^*}$:

$\widetilde{C^{p/L}} \longrightarrow \widetilde{C^{pf/K}}$. Passing to the fiber over an object $\alpha : s \longrightarrow t$, the compatibility of $\widetilde{P}_K$ and $\widetilde{P}_L$ implies that the diagram

$$\begin{tikzcd}
(K^\vee)_s \arrow[r, "(\widetilde{P}_K)_\alpha"] \arrow[d, "(f^\vee)_s"'] & K_t \arrow[d, "f_t"'] \arrow[dr, "(pf)_t"] & \\
(L^\vee)_s \arrow[r, "(\widetilde{P}_L)_\alpha"] & L_t \arrow[r, "p_t"] & C_t
\end{tikzcd}$$

commutes and that

$$(\widetilde{f^*})_\alpha = (f^\vee)_s^* : C^{p_t \circ (\widetilde{P_L})_\alpha/} \longrightarrow C^{(pf)_t \circ (\widetilde{P_K})_\alpha/}.$$

By [9][4.1.1.10], $(f^\vee)_s$ is final, so by [9][4.1.1.8], $(f^\vee)_s^*$ is an equivalence. Consequently, $\widetilde{f^*}$ is an equivalence. Now by Thm. 6.6, f^* is an equivalence.

(2) $\Rightarrow$ (3): Immediate from the definition.

(3) $\Rightarrow$ (1): Let $s \in S$ be any object and $\overline{p_s} : L_s^\rhd \longrightarrow \mathbf{Top}$ a colimit diagram. Let $\overline{p} : (L \star_S S)_{\underline{s}} \longrightarrow \mathbf{Top}$ be a left Kan extension of $\overline{p_s}$ along the full and faithful inclusion $L_s^\rhd \subset (L \star_S S)_{\underline{s}}$. By transitivity of left Kan extensions, $\overline{p}$ is a left Kan extension of its restriction to $L_{\underline{s}}$. By Prp. 5.4, under the equivalence $\mathrm{Fun}(L, \mathbf{Top}) \simeq \mathrm{Fun}_S(L, \underline{\mathbf{Top}}_S)$, $\overline{p}$ is a $S^{s/}$-colimit diagram. By assumption, $\overline{p} \circ (f^\rhd)_{\underline{s}}$ is a $S^{s/}$-colimit diagram. By Prp. 5.4 again, $\overline{p_s} \circ f_s$ is a colimit diagram, as desired. □

6.8. **Definition.** Let $f : K \longrightarrow L$ be a S-functor. We say that f is S-*final* if it satisfies the equivalent conditions of Thm. 6.7. We say that f is S-*initial* if f^{vop} is S-final.

6.9. **Example.** Let $F : C \rightleftarrows D : G$ be a S-adjunction. Then F is S-initial and G is S-final.

6.10. **Remark.** Any S-functor which is fiberwise a weak homotopy equivalence is a weak homotopy equivalence, by [9, 4.1.2.15], [9, 4.1.2.18], and [9, 3.1.5.7]. In particular, any S-final or S-initial S-functor is a weak homotopy equivalence. However, in general a S-final S-functor is not final.

A remark on presentability. Suppose the functor $S \longrightarrow \mathbf{Cat}_\infty$ classifying the cocartesian fibration $C \longrightarrow S$ factors through $\mathbf{Pr}^R$, i.e. $C \longrightarrow S$ is a *right presentable fibration.* For any X a presentable ∞-category and diagram $f : A \longrightarrow X$, $X^{f/}$ is again presentable and the forgetful functor $X^{f/} \longrightarrow X$ creates limits and filtered colimits. Therefore, the twisted slice category $\widetilde{C^{(p,S)/}}$ is a right presentable fibration. Since the forgetful functor $\mathbf{Cat}_\infty \longrightarrow \mathbf{Pr}^R$ creates limits, by Thm. 6.6 we deduce that $C^{(p,S)/}$ is a right presentable fibration. In particular, in every fiber there exists an initial object. However, these initial objects may fail to be preserved by the pushforward functors. In fact, even if we assume that $C \longrightarrow S$ is both left and right presentable, C may fail to be S-cocomplete.

Another cocartesian fibration over $\widetilde{\mathscr{O}}(S)$. The construction of the slice category $\widetilde{C^{(p,S)/}}$ is in some respects unsatisfactory, because it relies upon an inexplicit choice of twisted pushforward functor. In this subsection, we provide a more explicit construction of an equivalent cocartesian fibration over $\widetilde{\mathscr{O}}(S)$.

6.11. **Definition.** Let K and C be S-categories and let $p : K \longrightarrow C$ be a S-functor. Define the simplicial set $C_{\widetilde{p/S}}$ over $\widetilde{\mathscr{O}}(S)$ by declaring, for a map $\Delta^n \longrightarrow \widetilde{\mathscr{O}}(S)$ and corresponding map $(\sigma, \tau) : \Delta^n \star \Delta^n \longrightarrow S$,

$$\mathrm{Hom}_{/\widetilde{\mathscr{O}}(S)}(\Delta^n, C_{\widetilde{p/S}}) = \mathrm{Hom}_{K|_\sigma//S}(K|_\sigma \star \Delta^n, C)$$

where the set on the right hand side equals the set of dotted arrows making the diagram

$$\begin{tikzcd}
K|_\sigma \arrow[d] \arrow[dr, "p|_\sigma"] & \\
K|_\sigma \star \Delta^n \arrow[r, dotted] \arrow[d] & C \arrow[d] \\
\Delta^n \star \Delta^n \arrow[r, "(\sigma, \tau)"] & S
\end{tikzcd}$$

commute. Endow $C_{\widetilde{p/S}}$ with a marking by declaring an edge to be marked just in case the corresponding map $K|_{\Delta^1} \star \Delta^1 \longrightarrow C$ sends Δ^1 to a cocartesian edge in C.

Observe that we have a forgetful functor $C_{\widetilde{p/S}} \longrightarrow C$ covering the target functor $\widetilde{\mathscr{O}}(S) \longrightarrow S$, given by restricting maps $K|_{\Delta^n} \star \Delta^n \longrightarrow C$ to the copy of Δ^n.

6.12. **Proposition.** *$\pi : C_{\widetilde{p/S}} \longrightarrow \widetilde{\mathscr{O}}(S)$ is fibrant in $s\mathbf{Set}^+_{/\widetilde{\mathscr{O}}(S)}$.*

Proof. We first prove that π is a cocartesian fibration. Let $0 \leq k < n$. We have to solve the lifting problem

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & C_{\widetilde{p/S}} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & \widetilde{\mathscr{O}}(S). \end{tikzcd}$$

with the proviso that if $k = 0$, the edge $\Delta^{\{0,1\}}$ is sent to a marked edge in $C_{\widetilde{p/S}}$. Let $A \subset \Delta^n \star \Delta^n$ be the simplicial subset spanned by the faces $\sigma_0 \star \sigma_1 : \Delta^{n_0} \star \Delta^{n_1} \longrightarrow \Delta^n \star \Delta^n$ such that there exists $l \neq k$, $0 \leq l \leq n$ with $n - l \notin \sigma_0$ and $l \notin \sigma_1$. Then maps $\Lambda^n_k \longrightarrow \widetilde{\mathscr{O}}(S)$ are specified by maps $A \longrightarrow S$. Moreover, we have a factorization of the inclusion as a composite of inclusions

$$A \xrightarrow{\alpha} \Lambda^n_{n-k} \star \Lambda^n_k \xrightarrow{\beta} \Delta^n \cup_{\Lambda^n_{n-k}} \Lambda^n_{n-k} \star \Lambda^n_k \cup_{\Lambda^n_k} \Delta^n \xrightarrow{\gamma} \Delta^n \star \Delta^n.$$

We may thus reformulate our lifting problem as supplying a dotted arrow so as to make the diagram

$$\begin{tikzcd} K|_{\Lambda^n_{n-k}} \arrow[r, "="] \arrow[d] & K|_{\Lambda^n_{n-k}} \arrow[r] \arrow[d] & K|_{\Delta^n} \arrow[r, "="] \arrow[d] & K|_{\Delta^n} \arrow[d] \arrow[dr, "p|_{\Delta^n}"] & \\ A \times_{\Delta^n \star \Delta^n} (K|_{\Delta^n} \star \Delta^n) \arrow[r] \arrow[d] \arrow[rrrr, bend left] & K|_{\Lambda^n_{n-k}} \star \Lambda^n_k \arrow[r] \arrow[d] & K|_{\Delta^n} \underset{K|_{\Lambda^n_{n-k}}}{\cup} K|_{\Lambda^n_{n-k}} \star \Lambda^n_k \underset{\Lambda^n_k}{\cup} \Delta^n \arrow[r] \arrow[d] & K|_{\Delta^n} \star \Delta^n \arrow[r, dotted] \arrow[d] & C \arrow[d] \\ A \arrow[r, "\alpha"] & \Lambda^n_{n-k} \star \Lambda^n_k \arrow[r, "\beta"] & \Delta^n \underset{\Lambda^n_{n-k}}{\cup} \Lambda^n_{n-k} \star \Lambda^n_k \underset{\Lambda^n_k}{\cup} \Delta^n \arrow[r, "\gamma"] & \Delta^n \star \Delta^n \arrow[r] & S \end{tikzcd}$$

commute. To proceed, we make the following observations about the maps α, β, and γ.

- α is inner anodyne: We can slightly modify the proof of [11, 5.2.1.3] to show this. Observe that a face $\sigma = \sigma_0 \star \sigma_1 : \Delta^{n_0} \star \Delta^{n_1} \subset \Lambda^n_{n-k} \star \Lambda^n_k$ does not belong to A just in case the set of vertices $\{i : n - i \in \sigma_0 \text{ or } i \in \sigma_1\}$ contains $\{0, ..., \widehat{k}, ..., n\}$. Partition the collection I of faces in $\Lambda^n_{n-k} \star \Lambda^n_k$ not in A into *primary* and *secondary* faces, where σ is primary if $\{0, ..., n-k-1\} \not\subset \sigma_0$ and secondary otherwise. If σ is primary and $k \notin \sigma_1$, let σ' be the face obtained from σ by adding k to σ_1, and if σ is secondary and $n - k \notin \sigma_0$, let σ' be the face obtained from σ by adding $n - k$ to σ_0. Note that σ' is then still in I, and moreover that $\sigma \mapsto \sigma'$ pairs faces in I uniquely. The rest of the proof is now as in [11, 5.2.1.3].
- If $0 < k < n$, then β is visibly inner anodyne. Otherwise, viewing the edge $\Delta^{\{0,1\}}$ in the second copy of Δ^n as marked, β is the composite of a right anodyne map and a left marked anodyne map.
- γ is inner anodyne: By [9, 2.1.2.3] applied to the right anodyne map $\Lambda^n_{n-k} \longrightarrow \Delta^n$ and $\emptyset \longrightarrow \Lambda^n_k$, the map
$$\Delta^n \cup_{\Lambda^n_{n-k}} \Lambda^n_{n-k} \star \Lambda^n_k \longrightarrow \Delta^n \star \Lambda^n_k$$
is inner anodyne, hence so is the pushout
$$\Delta^n \cup_{\Lambda^n_{n-k}} \Lambda^n_{n-k} \star \Lambda^n_k \cup_{\Lambda^n_k} \Delta^n \longrightarrow \Delta^n \star \Lambda^n_k \cup_{\Lambda^n_k} \Delta^n.$$
Now by [9, 2.1.2.3] applied to $\emptyset \longrightarrow \Delta^n$ and the left anodyne map $\Lambda^n_k \longrightarrow \Delta^n$, the map
$$\Delta^n \star \Lambda^n_k \cup_{\Lambda^n_k} \Delta^n \longrightarrow \Delta^n \star \Delta^n$$
is inner anodyne. Composing these two maps gives the map in question, so we conclude.

To apply these results to our situation in which $K \longrightarrow S$ and $K|_{\Delta^n} \star \Delta^n \longrightarrow \Delta^n \star \Delta^n$ are cocartesian fibrations (the latter by Prp. 4.7), recall that inner anodyne maps pull back along cocartesian fibrations to trivial cofibrations in $s\mathbf{Set}_{\text{Joyal}}$. Thus if $0 < k < n$, we see that

$$i : K|_{\Delta^n} \cup_{K|_{\Lambda^n_{n-k}}} (A \times_{\Delta^n \star \Delta^n} (K|_{\Delta^n} \star \Delta^n)) \longrightarrow K|_{\Delta^n} \star \Delta^n$$

is a trivial cofibration in $s\mathbf{Set}_{\text{Joyal}}$, so the dotted map exists in that case. Now suppose $k = 0$ and consider i as a map of marked simplicial sets with $\Delta^{\{0,1\}}$ in the second copy of Δ^n marked and the cocartesian edges in K marked. We want to show that i is a trivial cofibration in $s\mathbf{Set}^+_{/S}$. Recall that if $A \longrightarrow B$ over S is a trivial cofibration in $s\mathbf{Set}_{\text{Joyal}}$, then $A^\flat \longrightarrow B^\flat$ is a trivial cofibration in $s\mathbf{Set}^+_{/S}$ ([9, 3.1.5.3]), hence $A \longrightarrow B$ with any common marking is also a trivial cofibration in $s\mathbf{Set}^+_{/S}$. Thus, since α and γ are inner anodyne, we are reduced to showing that

$$K|_{\Delta^n} \cup_{K|_{\Lambda^n_n}} K|_{\Lambda^n_n} \star {}_\natural\Lambda^n_0 \longrightarrow K|_{\Delta^n} \cup_{K|_{\Lambda^n_n}} K|_{\Lambda^n_n} \star {}_\natural\Lambda^n_0 \cup_{{}_\natural\Lambda^n_0} {}_\natural\Delta^n$$

is left marked anodyne, which it is, being obtained by pushout from ${}_\natural\Lambda^n_0 \longrightarrow {}_\natural\Delta^n$.

We have shown not only that π is a cocartesian fibration, but also that every marked edge in $C_{\widetilde{p/S}}$ is π-cocartesian and that we may choose our π-cocartesian lifts to be marked edges. To complete the proof, we must verify that every π-cocartesian edge is marked, for which it suffices to show that every equivalence in $C_{\widetilde{p/S}}$ is marked. But this is true by definition, since the forgetful functor $C_{\widetilde{p/S}} \longrightarrow C$ preserves equivalences. □

We now identify the fibers of $C_{\widetilde{p/S}}$ as ordinary slice ∞-categories. Let $\alpha : s \longrightarrow t$ be an edge in S. Choose a lift h in

$$\begin{tikzcd} K_s \times \{0\} \arrow[r, "p|_s"] \arrow[d] & {}_\natural C|_\alpha \arrow[d] \\ K_s \times (\Delta^1)^\sharp \arrow[r] \arrow[ur, dotted, "h"] & (\Delta^1)^\sharp \end{tikzcd}$$

and let $\phi = h|_{K_s \times \{1\}} : K_s \longrightarrow C_t$.

6.13. **Lemma.** *There is a categorical equivalence $(C_{\widetilde{p/S}})_\alpha \simeq (C_t)_{\phi/}$.*

Proof. If $K_s = \emptyset$ then both ∞-categories are isomorphic to C_t and the claim is proved, so suppose not. Consider K_s as a marked simplicial set with the equivalences marked. Define a functor

$$F_1' : \Delta \longrightarrow s\mathbf{Set}^+_{K_s//\Delta^1}$$

by

$$F_1'(n) = (K_s \longrightarrow K_s \sqcup_{K_s \times (\Delta^n)^\sharp} (K_s \times (\Delta^n)^\sharp) \star (\Delta^n)^\flat \longrightarrow \Delta^1)$$

and let F_1 be the unique colimit-preserving extension of F_1' to $s\mathbf{Set}_{\text{Joyal}}$. Define a second colimit-preserving functor

$$F_2 : s\mathbf{Set}_{\text{Joyal}} \longrightarrow s\mathbf{Set}^+_{K_s//\Delta^1}$$

by

$$F_2(A) = (K_s \longrightarrow K_s \star A^\flat \longrightarrow \Delta^1).$$

Let G_1 and G_2 be their right adjoints. Then by definition $G_1(K_s \xrightarrow{p|_s} {}_\natural C|_\alpha \longrightarrow \Delta^1) \cong (C_{\widetilde{p/S}})_\alpha$; define

$$(C|_\alpha)'_{p|_s/} = G_2(K_s \xrightarrow{p|_s} {}_\natural C|_\alpha \longrightarrow \Delta^1).$$

Observe that F_2 is left Quillen and that F_1 preserves cofibrations. Using the commutativity of the square

$$\begin{tikzcd} K_s \times (\Delta^n)^\sharp \arrow[r] \arrow[d] & (K_s \times (\Delta^n)^\sharp) \star (\Delta^n)^\flat \arrow[d] \\ K_s \arrow[r] & K_s \star (\Delta^n)^\flat, \end{tikzcd}$$

define a natural transformation $\theta : F_1 \longrightarrow F_2$, and let $\psi : (C|_\alpha)'_{p|_s/} \longrightarrow (C_{\widetilde{p/S}})_\alpha$ be the adjoint map. We first prove that θ_A is a cocartesian equivalence for all A, which will imply that F_1 is left Quillen and ψ is a categorical equivalence by [7, 1.4.4(b)]. In view of left properness and the stability of cocartesian equivalences under filtered colimits, it suffices to show that θ_{Δ^n} is a cocartesian equivalence for all $n \geq -1$. If $n = -1$, the result is obvious, so suppose $n \geq 0$. Then, since $\Delta^{\{0\}} \subset (\Delta^n)^\sharp$ is left marked anodyne, $(\Delta^n)^\sharp \longrightarrow \Delta^0$ is a cocartesian equivalence in $s\mathbf{Set}^+_{/T}$ for all ∞-categories T and maps $\Delta^0 \longrightarrow T$, so by left properness, [9, 3.1.4.2], and Lm. 4.10, θ_{Δ^n} is a cocartesian equivalence.

Next, we examine the restriction functor

$$\rho : (C|_\alpha)_{h/} \longrightarrow (C|_\alpha)'_{p|_s/}$$

induced by the inclusion $i : K_s \times \{0\} \longrightarrow K_s \times (\Delta^1)^\sharp$. We claim that ρ is a trivial fibration. To show this, for every cofibration $A_0 \longrightarrow A$ we must solve the lifting problem

$$\begin{tikzcd}
(K_s \times \Lambda^1_0) \star A^\flat \underset{(K_s \times \Lambda^1_0) \star A^\flat_0}{\cup} (K_s \times (\Delta^1)^\sharp) \star A^\flat_0 \arrow[r] \arrow[d] & {}_\natural C|_\alpha \arrow[d] \\
(K_s \times (\Delta^1)^\sharp) \star A^\flat \arrow[r] \arrow[ur, dotted] & (\Delta^1)^\sharp,
\end{tikzcd}$$

where the bottom map is defined by

$$(K_s \times \Delta^1) \star A \longrightarrow \Delta^1 \star \Delta^0 \cong \Delta^2 \xrightarrow{s^1} \Delta^1.$$

The left vertical map is left marked anodyne by Lm. 4.10, so the dotted arrow exists. Finally, since the inclusion $j : K_s \times \{1\} \longrightarrow K_s \times \Delta^1$ is right anodyne, by [9, 2.1.2.5] the restriction functor

$$(C|_\alpha)_{h/} \longrightarrow (C_t)_{\phi/}$$

is a trivial fibration. Chaining together these equivalences completes the proof. □

We defer the full comparison between $C_{\widetilde{p/S}}$ and $C^{\widetilde{p/S}}$ to a future work.

7. Types of S-fibrations

In this section we introduce some additional classes of fibrations which are all defined relative to S.

7.1. **Definition.** A S-functor $\phi : C \longrightarrow D$ is a *S-fibration* if it is a categorical fibration. A S-fibration ϕ is an *S-cocartesian* resp. *S-cartesian fibration* if for every object $s \in S$, $\phi_s : C_s \longrightarrow D_s$ is a cocartesian resp. cartesian fibration, and for every $\Delta^1 \times \Delta^1 \longrightarrow C$

$$\begin{tikzcd}
x_s \arrow[r, "h"] \arrow[d, "f"] & x_t \arrow[d, "g"] \\
y_s \arrow[r, "k"] & y_t
\end{tikzcd}$$

with h and k ϕ-cocartesian edges over $\phi(h) = \phi(k)$, if f is a ϕ_s-cocartesian resp. ϕ_s-cartesian edge then g is a ϕ_t-cocartesian resp. ϕ_t-cartesian edge.

Equivalently, $\phi : C \longrightarrow D$ is D-(co)cartesian if it is a categorical fibration, fiberwise (co)cartesian, and for every edge in S, the cocartesian pushforward along that edge preserves (co)cartesian edges in the fibers. (We formulate our definition as above so as to avoid having to make any 'straightening' constructions such as choosing pushforward functors.)

7.2. **Remark.** $\phi : C \longrightarrow D$ is a S-fibration if and only if $\phi : {}_\natural C \longrightarrow {}_\natural D$ is a marked fibration.

7.3. **Remark.** In view of [9, 2.4.2.11], [9, 2.4.2.7], and [9, 2.4.2.8], $\phi : C \longrightarrow D$ is an S-cocartesian fibration if and only if ϕ is a cocartesian fibration. However, there is no corresponding simplification of the definition of an S-cartesian fibration.

7.4. **Lemma.** *Let $\phi : C \longrightarrow D$ be a S-cartesian fibration and let $f : x \longrightarrow y$ be a ϕ_s-cartesian edge in C_s. Then f is a ϕ-cartesian edge.*

Proof. The property of being ϕ-cartesian may be checked after base-change to the 2-simplices of D. Consequently, we may suppose that $S = \Delta^1$ and $s = \{1\}$. We have to verify that for every object $w \in C$ we have a homotopy pullback square

$$\begin{tikzcd} \mathrm{Map}_C(w,x) \arrow[r, "f_*"] \arrow[d, "\phi_*"] & \mathrm{Map}_C(w,y) \arrow[d, "\phi_*"] \\ \mathrm{Map}_D(\phi w, \phi x) \arrow[r, "\phi(f)_*"] & \mathrm{Map}_D(\phi w, \phi y). \end{tikzcd}$$

If $w \in C_0$, for any choice of cocartesian edge $w \longrightarrow w'$ over $0 \longrightarrow 1$, the square is equivalent to

$$\begin{tikzcd} \mathrm{Map}_{C_1}(w',x) \arrow[r, "f_*"] \arrow[d, "\phi_*"] & \mathrm{Map}_{C_1}(w',y) \arrow[d, "\phi_*"] \\ \mathrm{Map}_{D_1}(\phi w', \phi x) \arrow[r, "\phi(f)_*"] & \mathrm{Map}_{D_1}(\phi w', \phi y). \end{tikzcd}$$

Hence we may suppose that $w \in C_1$, in which case the square is a homotopy pullback square since f is a ϕ_1-cartesian edge. □

Recall (Ntn. 4.29) the fiberwise arrow S-category $\mathscr{O}_S(D)$. Fix $\phi : C \longrightarrow D$ a S-functor.

7.5. **Definition.** The *free S-cocartesian* and *free S-cartesian* fibrations on ϕ are the S-functors

$$\mathrm{Fr}^{cocart}(\phi) = \mathrm{ev}_1 \circ \mathrm{pr}_2 : C \times_D \mathscr{O}_S(D) \longrightarrow D,$$
$$\mathrm{Fr}^{cart}(\phi) = \mathrm{ev}_0 \circ \mathrm{pr}_2 : \mathscr{O}_S(D) \times_D C \longrightarrow D$$

.

7.6. **Proposition.** *$\mathrm{Fr}^{cocart}(\phi)$ is a S-cocartesian fibration. Dually, $\mathrm{Fr}^{cart}(\phi)$ is a S-cartesian fibration.*

Proof. We prove the second assertion, the proof of the first being similar but easier. First note that $\mathscr{O}_S(D) \times_D C$ is a subcategory of $\mathscr{O}(D) \times_D C$ stable under equivalences. Therefore, since $\mathrm{ev}_0 : \mathscr{O}(D) \times_D C \longrightarrow D$ is a cartesian fibration, $\mathrm{Fr}^{cart}(\phi)$ is a categorical fibration. Moreover, for every object $s \in S$, $\mathrm{Fr}^{cart}(\phi)_s : \mathscr{O}(D_s) \times_{D_s} C_s$ is the free cartesian fibration on $\phi_s : C_s \longrightarrow D_s$. It remains to show that for every square

$$\begin{tikzcd} (a \to \phi x, x) \arrow[r, "h"] \arrow[d, "f"] & (b \to \phi y, y) \arrow[d, "g"] \\ (a' \to \phi x', x') \arrow[r, "k"] & (b' \to \phi y', y') \end{tikzcd}$$

in $\mathscr{O}_S(D) \times_D C$ with the horizontal edges cocartesian over S and the left vertical edge $\mathrm{Fr}^{cart}(\phi)_s$-cartesian, the right vertical edge is $\mathrm{Fr}^{cart}(\phi)_t$-cartesian. This amounts to verifying that $y \to y'$ is an equivalence in C_t. The above square yields a square

$$\begin{tikzcd} x \arrow[r, "h"] \arrow[d, "f"] & y \arrow[d, "g"] \\ x' \arrow[r, "k"] & y' \end{tikzcd}$$

in C with $x \to x'$ an equivalence and the horizontal edges cocartesian over S, from which the claim follows. □

Define S-functors $\iota_0 : C \longrightarrow C \times_D \mathscr{O}_S(D)$ and $\iota_1 : C \longrightarrow \mathscr{O}_S(D) \times_D C$ via the commutative square

$$\begin{tikzcd} C \arrow[r] \arrow[d, "="] & \mathscr{O}_S(D) \arrow[d, "\mathrm{ev}_i"] \\ C \arrow[r, "\phi"] & D \end{tikzcd}$$

where the upper horizontal map is the composite $C \longrightarrow \mathscr{O}_S(C) \longrightarrow \mathscr{O}_S(D)$.

7.7. **Proposition.** *ι_0 is left S-adjoint to pr_C. Dually, ι_1 is right S-adjoint to pr_C.*

Proof. We prove the first assertion, the proof of the second being similar. To prove that we have a relative S-adjunction $\iota_0 \dashv \mathrm{pr}_C$, we must prove that for each $s \in S$ we have an adjunction $(\iota_0)_s \dashv (\mathrm{pr}_C)_s$. So suppose that $S = \Delta^0$. Since $\mathrm{pr}_C \circ \iota_0 = id$, it suffices by [9, 5.2.2.8] to check that the identity is a unit transformation: that is, for every $x \in C$ and $(y, \phi y \to a) \in C \times_D \mathscr{O}(D)$,

$$\mathrm{pr}_C : \mathrm{Map}_{C \times_D \mathscr{O}(D)}((x, id_{\phi x}), (y, \phi y \to a)) \longrightarrow \mathrm{Map}_C(x, y)$$

is an equivalence. Under the fiber product decomposition

$$\mathrm{Map}_{C \times_D \mathscr{O}(D)}((x, id_{\phi x}), (y, \phi y \to a)) \simeq \mathrm{Map}_C(x, y) \times_{\mathrm{Map}_D(\phi x, \phi y)} \mathrm{Map}_{\mathscr{O}(D)}((id_{\phi x}), (\phi y \to b))$$

the map pr_C is projection onto the first factor. The adjunction $\iota \colon D \rightleftarrows \mathscr{O}(D) : \mathrm{ev}_0$ obtained by exponentiating the adjunction $i_0 \colon \{0\} \rightleftarrows \Delta^1 : p$ implies that

$$\mathrm{Map}_{\mathscr{O}(D)}((id_{\phi x}), (\phi y \to b)) \longrightarrow \mathrm{Map}_D(\phi x, \phi y)$$

is an equivalence, so the claim follows. □

We conclude this section with an observation about the interaction between S-joins and S-cocartesian fibrations which will be used in the sequel.

7.8. **Lemma.** *Let C, C', and D be S-categories and let $\phi, \phi' : C, C' \longrightarrow D$ be S-functors. If ϕ and ϕ' are S-(co)cartesian, then $\phi \star \phi' : C \star_D C' \longrightarrow D$ is S-(co)cartesian.*

Proof. This is an easy corollary of Prp. 4.7. □

7.9. **Definition.** We say that a S-functor $F : C \longrightarrow D \times_S E$ is a S-*bifibration* if for all objects $s \in S$, F_s is a bifibration. Observe it is then automatic that $\mathrm{pr}_D F$ is S-cartesian and $\mathrm{pr}_E F : C \longrightarrow E$ is S-cocartesian.

7.10. **Example.** The S-functor

$$\underline{\mathrm{Fun}}_S(K \star_S L, C) \longrightarrow \underline{\mathrm{Fun}}_S(K, C) \times_S \underline{\mathrm{Fun}}_S(L, C)$$

is a S-bifibration by Lm. 4.8. In particular, for a S-functor $p : K \longrightarrow C$, the S-functors $C^{(p,S)/} \longrightarrow C$ and $C^{/(p,S)} \longrightarrow C$ are S-cocartesian resp. S-cartesian.

8. Relative adjunctions

In [11, 7.3.2], Lurie introduces the notion of a *relative adjunction*.

8.1. **Definition.** Let C and D be S-categories. We call a relative adjunction $F \colon C \rightleftarrows D : G$ (with respect to S) a S-*adjunction* if F and G are S-functors.

We prove some basic results about S-adjunctions in this section. Let us first reformulate the definition of a relative adjunction in terms of a correspondence. Let $F : C \longrightarrow D$ be a S-functor. By the relative nerve construction, F defines a cocartesian fibration $M \longrightarrow \Delta^1$ by prescribing, for every $\Delta^n \cong \Delta^{n_0} \star \Delta^{n_1} \longrightarrow \Delta^1$, the set $\mathrm{Hom}_{\Delta^1}(\Delta^n, M)$ to be the collection of commutative squares

$$\begin{tikzcd} \Delta^{n_0} \arrow[r] \arrow[d] & C \arrow[d, "F"] \\ \Delta^n \arrow[r] & D \end{tikzcd}$$

for $n_1 \geq 0$, and setting $\mathrm{Hom}_{\Delta^1}(\Delta^n, M) = \mathrm{Hom}(\Delta^n, C)$ for $n_1 = -1$. Moreover, the structure maps for C and D to S define a functor $M \longrightarrow S$ by sending $\Delta^n \longrightarrow M$ to $\Delta^n \longrightarrow D \longrightarrow S$ if $n_1 \geq 0$, and $\Delta^n \longrightarrow C \longrightarrow S$ if $n_1 < 0$. Then M is a S-category, $M \longrightarrow S \times \Delta^1$ is a S-cocartesian fibration, and F admits a right S-adjoint if and only if $M \longrightarrow S \times \Delta^1$ is a S-cartesian fibration.

8.2. **Proposition.** *Let $F \colon C \rightleftarrows D : G$ be a S-adjunction and let I be a S-category. Then we have adjunctions*

$$F_* \colon \mathrm{Fun}_S(I, C) \rightleftarrows \mathrm{Fun}_S(I, D) : G_*$$
$$G^* \colon \mathrm{Fun}_S(C, I) \rightleftarrows \mathrm{Fun}_S(D, I) : F^*$$

Proof. Let $M \longrightarrow S \times \Delta^1$ be the S-functor obtained from F. We first produce the adjunction $F_* \dashv G_*$. Invoking Thm. 2.23 on the span

$$(\Delta^1) \xleftarrow{\pi} {}_\natural I \times (\Delta^1)^\sharp \xrightarrow{\pi'} S^\sharp \times (\Delta^1)^\sharp$$

we deduce that $\pi_*\pi'^* : s\mathbf{Set}^+_{/(S^\sharp\times(\Delta^1)^\sharp)} \longrightarrow s\mathbf{Set}^+_{/(\Delta^1)^\sharp}$ is right Quillen. Let $N = \pi_*\pi'^*(M)$. Then $N \longrightarrow \Delta^1$ is a cocartesian fibration classified by the functor

$$F_* : \mathrm{Fun}_S(I, C) \longrightarrow \mathrm{Fun}_S(I, D).$$

Now invoking Thm. 2.23 on the span

$$((\Delta^1)^\sharp)^{op} \xleftarrow{\rho} (I^\sim \times (\Delta^1)^\sharp)^{op} \xrightarrow{\rho'} (S^\sim \times (\Delta^1)^\sharp)^{op}$$

we deduce that with respect to the cartesian model structures $\rho_*\rho'^* : s\mathbf{Set}^+_{/(S^\sim\times(\Delta^1)^\sharp)} \longrightarrow s\mathbf{Set}^+_{/(\Delta^1)^\sharp}$ is right Quillen. Let $N' = \rho_*\rho'^* M$. Since G is right S-adjoint to F, $N' \longrightarrow \Delta^1$ is a cartesian fibration classified by the functor

$$G_* : \mathrm{Fun}_{/S}(I, D) \longrightarrow \mathrm{Fun}_{/S}(I, C)$$

where we view I, C, D as categorical fibrations over S. N is a subcategory of N', and the cartesian edges e in N' with $d_0(e) \in N$ are in N. Hence $N \longrightarrow \Delta^1$ is also a cartesian fibration classified by the functor

$$G_* : \mathrm{Fun}_S(I, D) \longrightarrow \mathrm{Fun}_S(I, C).$$

We now produce the adjunction $G^* \dashv F^*$ by similar methods. Let $\mathscr{E}_0$ be the collection of edges $e : x \longrightarrow y$ in M such that e admits a factorization as a cocartesian edge over S followed by a cartesian edge in the fiber. Note that since $M \longrightarrow S \times \Delta^1$ is a S-cartesian fibration, $\mathscr{E}_0$ is closed under composition of edges. Invoking Thm. 2.23 on the span

$$(\Delta^1)^\sharp \xleftarrow{\mu} (M, \mathscr{E}_0) \xrightarrow{\mu'} S^\sharp \times (\Delta^1)^\sharp$$

we deduce that $\mu_*\mu'^* : s\mathbf{Set}^+_{/(S^\sharp\times(\Delta^1)^\sharp)} \longrightarrow s\mathbf{Set}^+_{/(\Delta^1)^\sharp}$ is right Quillen. Let $P = \mu_*\mu'^*({}_\natural I \times (\Delta^1)^\sharp)$. Then $P \longrightarrow \Delta^1$ is a cocartesian fibration classified by the functor

$$G^* : \mathrm{Fun}_S(C, I) \longrightarrow \mathrm{Fun}_S(D, I).$$

Let $\mathscr{E}_1$ be the collection of edges $e : x \longrightarrow y$ in M such that e is a cocartesian edge over an equivalence in S. Now invoking Thm. 2.23 on the span

$$((\Delta^1)^\sharp)^{op} \xleftarrow{\nu} (M, \mathscr{E}_1)^{op} \xrightarrow{\nu'} (S^\sim \times (\Delta^1)^\sharp)^{op}$$

we deduce that with respect to the cartesian model structures $\nu_*\nu'^* : s\mathbf{Set}^+_{/(S^\sim\times(\Delta^1)^\sharp)} \longrightarrow s\mathbf{Set}^+_{/(\Delta^1)^\sharp}$ is right Quillen. Let $P' = \nu_*\nu'^*(I^\sim \times (\Delta^1)^\sharp)$. $P' \longrightarrow \Delta^1$ is a cartesian fibration with P as a subcategory. One may check that $P \longrightarrow \Delta^1$ inherits the property of being a cartesian fibration, which is classified by the functor $F^* : \mathrm{Fun}_S(D, I) \longrightarrow \mathrm{Fun}_S(C, I)$. ☐

8.3. **Corollary.** *Let $F\colon C \rightleftarrows D : G$ be a S-adjunction and let I be a S-category. Then we have S-adjunctions*

$$F_*\colon \underline{\mathrm{Fun}}_S(I, C) \rightleftarrows \underline{\mathrm{Fun}}_S(I, D) : G_*$$

$$G^*\colon \underline{\mathrm{Fun}}_S(C, I) \rightleftarrows \underline{\mathrm{Fun}}_S(D, I) : F^*$$

Proof. By Prp. 8.2, for every $s \in S$

$$F_*\colon \mathrm{Fun}_{S^{s/}}(I \times_S S^{s/}, C \times_S S^{s/}) \rightleftarrows \mathrm{Fun}_{S^{s/}}(I \times_S S^{s/}, D \times_S S^{s/}) : G_*$$

is an adjunction, and similarly for the contravariant case. ☐

To state the next corollary, it is convenient to introduce a definition.

8.4. **Definition.** Suppose $\pi : C \longrightarrow D$ a S-fibration. Define the ∞-category $\mathrm{Sect}_{D/S}(\pi)$ of S-sections of π to be the pullback

$$\begin{tikzcd}
\mathrm{Sect}_{D/S}(\pi) \arrow[r] \arrow[d] & \mathrm{Fun}_S(D, C) \arrow[d, "\pi_*"] \\
\Delta^0 \arrow[r, "id_D"] & \mathrm{Fun}_S(D, D).
\end{tikzcd}$$

Define the S-category $\underline{\mathrm{Sect}}_{D/S}(C)$ to be the pullback

$$\begin{tikzcd}
\underline{\mathrm{Sect}}_{D/S}(\pi) \arrow[r] \arrow[d] & \mathrm{Fun}_S(D, C) \arrow[d, "\pi_*"] \\
S \arrow[r, "\sigma_{id_D}"] & \mathrm{Fun}_S(D, D).
\end{tikzcd}$$

We will often denote $\mathrm{Sect}_{D/S}(\pi)$ by $\mathrm{Sect}_{D/S}(C)$, the S-functor π being left implicit.

Note that for any object $s \in S$, the fiber $\underline{\mathrm{Sect}}_{D/S}(C)_s$ is isomorphic to $\mathrm{Sect}_{D_{\underline{s}}/\underline{s}}(\pi_{\underline{s}})$.

8.5. **Corollary.** *Let $p : C \longrightarrow E$ and $q : D \longrightarrow E$ be S-fibrations. Let $F\colon C \rightleftarrows D : G$ be an adjunction relative to E where F and G are S-functors. Then for any S-category I,*

$$F_*\colon \mathrm{Fun}_S(I, C) \rightleftarrows \mathrm{Fun}_S(I, D) : G_*$$

is an adjunction relative to $\mathrm{Fun}_S(I, E)$. In particular, taking $I = E$ and the fiber over the identity, we deduce that

$$F_*\colon \mathrm{Sect}_{E/S}(p) \rightleftarrows \mathrm{Sect}_{E/S}(q) : G_*$$

is an adjunction, and also that

$$F_*\colon \underline{\mathrm{Sect}}_{E/S}(p) \rightleftarrows \underline{\mathrm{Sect}}_{E/S}(q) : G_*$$

is a S-adjunction.

Proof. The proof of Prp. 8.2 shows that the unit for the adjunction $F_* \dashv G_*$ is sent by p_* to a natural transformation through equivalences. $\square$

8.6. **Lemma.** *Let $F\colon C \rightleftarrows D : G$ be a S-adjunction. For every S-functor $p : K \longrightarrow D$, we have a homotopy pullback square in $s\mathbf{Set}^+_{/S}$*

$$\begin{tikzcd}
C^{/(Gp,S)} \arrow[r] \arrow[d, "\mathrm{ev}_0^C"] & D^{/(p,S)} \arrow[d, "\mathrm{ev}_0^D"] \\
C \arrow[r, "F"] & D
\end{tikzcd}$$

where the upper horizontal map is defined to be the composite $C^{/(Gp,S)} \xrightarrow{F} C^{/(FGp,S)} \xrightarrow{\epsilon(p)_!} D^{/(p,S)}$. Dually, for every S-functor $p : K \longrightarrow D$, we have a homotopy pullback square in $s\mathbf{Set}^+_{/S}$

$$\begin{tikzcd}
D^{(Fp,S))/} \arrow[r] \arrow[d, "\mathrm{ev}_1^D"] & C^{(p,S)/} \arrow[d, "\mathrm{ev}_1^C"] \\
D \arrow[r, "G"] & C.
\end{tikzcd}$$

where the upper horizontal map is defined to be the composite $D^{(Fp,S))/} \xrightarrow{G} C^{(GFp,S)/} \xrightarrow{\eta(p)^*} C^{(p,S)/}$.

Proof. We prove the first assertion; the second then follows by taking vertical opposites. We first explain how to define the map $\epsilon(p)_!$. Choose a counit transformation $\epsilon : D \times \Delta^1 \longrightarrow D$ for $F \dashv G$ such that $\pi_D \circ \epsilon$ is the identity natural transformation from π_D to itself. Then $\epsilon \circ (p \times id)$ is adjoint to a S-functor $\epsilon(p) : S \times \Delta^1 \longrightarrow \underline{\mathrm{Fun}}_S(K, D)$ with $\epsilon(p)_0 = \sigma_{FGp}$ and $\epsilon(p)_0 = \sigma_p$. Because $\underline{\mathrm{Fun}}_S(S \star_S K, D) \longrightarrow D \times_S \underline{\mathrm{Fun}}_S(K, D)$ is an S-bifibration, from $\epsilon(p)$ we obtain a pushforward S-functor $\epsilon(p)_! : D^{/(FGp,S)} \longrightarrow D^{/(p,S)}$ compatible with the source maps to D.

We need to check that for every object $s \in S$, passage to the fiber over s yields a homotopy pullback square of ∞-categories. Because $(D^{/(p,S)})_s \cong (D_{\underline{s}}^{/(p_{\underline{s}},\underline{s})})_s$, we may replace S by $S^{s/}$ and thereby suppose that s is an initial object in S.

Let $r : \{s\} \star S \longrightarrow S$ be a left Kan extension of the identity $S \longrightarrow S$. By the formula for a left Kan extension, $r(s)$ is an initial object in S, which without loss of generality we may suppose to be s. Using $r \circ (id \star \pi_K)$ as the structure map for $\{s\} \star K$ over S, define $\phi' : \{s\} \star {}_\natural K \longrightarrow \{s\} \star_S {}_\natural K$ as adjoint to the identity over $S \times \partial\Delta^1$. It is easy to show that ϕ' is a trivial cofibration in $s\mathbf{Set}^+_{/S}$. Moreover, since the inclusion $\{s\} \longrightarrow S^\sharp$ is a trivial cofibration, $\{s\} \star_S {}_\natural K \longrightarrow S^\sharp \star_S {}_\natural K$ is a trivial cofibration in $s\mathbf{Set}^+_{/S}$ by Thm. 4.16. Let ϕ be the composition of these two maps. Then because $\mathrm{Fun}_S(-,-)$ is a right Quillen bifunctor, $\phi^* : \mathrm{Fun}_S(S^\sharp \star_S {}_\natural K, {}_\natural D) \longrightarrow \mathrm{Fun}_S(\{s\} \star {}_\natural K, {}_\natural D)$ is a trivial Kan fibration.

We further claim that the inclusion $j : \mathrm{Fun}_S(\{s\} \star {}_\natural K, {}_\natural D) \longrightarrow D_s \times_D \mathrm{Fun}(\{s\} \star K, D) \times_{\mathrm{Fun}(K,D)} \mathrm{Fun}_S({}_\natural K, {}_\natural D)$ is an equivalence. Indeed, we have the pullback square

$$\begin{tikzcd}
\mathrm{Fun}_S(\{s\} \star {}_\natural K, {}_\natural D) \arrow[r] \arrow[d] & D_s \times_D \mathrm{Fun}(\{s\} \star K, D) \times_{\mathrm{Fun}(K,D)} \mathrm{Fun}_S({}_\natural K, {}_\natural D) \arrow[d] \\
\Delta^0 \arrow[r, "r \circ (id \star \pi_K)"] & \{s\} \times_S \mathrm{Fun}(\{s\} \star K, S) \times_{\mathrm{Fun}(K,S)} \{\pi_K\}
\end{tikzcd}$$

and the term in the lower right is contractible since it is equivalent to the full subcategory $\mathrm{Fun}'(\{s\} \star K, S) \subset \mathrm{Fun}(\{s\} \star K, S)$ of functors which are left Kan extensions of π_K.

Now taking the pullback of the composition $j \circ \phi^*$ over $\{p\}$, we obtain an equivalence

$$(D^{/(p,S)})_s \longrightarrow D_s \times_D D^{/p}.$$

Similarly, we have an equivalence

$$(C^{/(Gp,S)})_s \longrightarrow C_s \times_C C^{/Gp}.$$

Since $F \dashv G$ is in particular an adjunction, by [9, 5.2.5.5] $C^{/Gp} \longrightarrow C \times_D D^{/p}$ is an equivalence. Taking the fiber over s, we deduce the claim. □

8.7. **Corollary.** *Let $F : C \rightleftarrows D : G$ be a S-adjunction. Then F preserves S-colimits and G preserves S-limits.*

Proof. Let $\overline{p} : K \star_S S \longrightarrow C$ be a S-colimit diagram. To show that $F\overline{p}$ is a S-colimit diagram, it suffices to prove that the restriction map $D^{(F\overline{p},S)/} \longrightarrow D^{(Fp,S)/}$ is an equivalence. We have the commutative square

$$\begin{tikzcd}
D^{(F\overline{p},S)/} \arrow[r] \arrow[d] & C^{(\overline{p},S)/} \times_C D \arrow[d] \\
D^{(Fp,S)/} \arrow[r] & C^{(p,S)/} \times_C D
\end{tikzcd}$$

(here we suppress some details about the naturality of $\epsilon(-)_!$). The righthand vertical map is an equivalence by assumption, and the horizontal maps are equivalences by Lm. 8.6. Thus the lefthand vertical map is an equivalence. □

9. Parametrized colimits

In this section, we first introduce a parametrized generalization of Lurie's pairing construction [9, 3.2.2.13]. We then employ it to study *D-parametrized S-(co)limits.* This material recovers and extends [9, §4.2.2] (in view of Lm. 4.5). It is a precursor to our study of Kan extensions.

An S-pairing construction.

9.1. **Construction.** Let $p : C \longrightarrow S$, $q : D \longrightarrow S$ be S-categories and let $\phi : C \longrightarrow D$ be a S-functor. Let $\pi, \pi' : \mathscr{O}^{cocart}(D) \times_D C \longrightarrow D$ be given by $\pi = \mathrm{ev}_0 \circ \mathrm{pr}_1$, $\pi' = \mathrm{ev}_1 \circ \mathrm{pr}_1$. Let $\mathscr{E}$ denote the collection of edges e in $\mathscr{O}^{cocart}(D) \times_{\mathrm{ev}_1, D, \phi} C$ such that $\pi(e)$ is q-cocartesian and $\mathrm{pr}_2(e)$ is p-cocartesian (so $\pi'(e)$ is q-cocartesian). Then the span

$${}_\natural D \xleftarrow{\pi} (\mathscr{O}^{cocart}(D) \times_D C, \mathscr{E}) \xrightarrow{\pi'} {}_\natural D$$

defines a functor

$$\pi_*\pi'^* : s\mathbf{Set}^+_{/_\natural D} \longrightarrow s\mathbf{Set}^+_{/_\natural D}.$$

For a S-category E and a S-functor $\psi : E \longrightarrow D$, define

$$(\widetilde{\mathrm{Fun}}_{D/S}(C, E) \longrightarrow {}_\natural D) = \pi_*\pi'^*({}_\natural E \longrightarrow {}_\natural D).$$

9.2. **Lemma.** *Let $q : D \longrightarrow S$ be a S-category.*

(1) $\mathrm{ev}_0 : \mathscr{O}^{cocart}(D) \longrightarrow D$ is a cartesian fibration, and an edge e in $\mathscr{O}^{cocart}(D)$ is ev_0-cartesian if and only $(\mathrm{ev}_{S,1} \circ q)(e)$ is an equivalence in S. In particular, if $\mathrm{ev}_0(e)$ is q-cocartesian, then e is ev_0-cartesian if and only if $\mathrm{ev}_1(e)$ is an equivalence in D.

(2) If $f : x \longrightarrow y$ is an edge in D such that $q(f)$ is an equivalence, then there exists a ev_0-cocartesian edge e over f. Moreover, an edge e over f is ev_0-cocartesian if and only if it is ev_0-cartesian.

Proof. $\mathrm{ev}_0 : \mathscr{O}^{cocart}(D) \longrightarrow D$ factors as

$$\mathscr{O}^{cocart}(D) \longrightarrow D \times_S \mathscr{O}(S) \longrightarrow D$$

where the first functor is a trivial fibration and the second is a cartesian fibration, as the pullback of $\mathrm{ev}_{S,0} : \mathscr{O}(S) \longrightarrow S$. Thus ev_0 is a cartesian fibration with cartesian edges as indicated. Moreover, since $\mathrm{ev}_{S,0} : \mathscr{O}(S) \longrightarrow S$ is a categorical fibration, the second claim follows from [11, B.2.9]. □

We have designed our construction so that for any object $x \in D$ and cocartesian section $S^{qx/} \longrightarrow D$, the fiber of $\widetilde{\mathrm{Fun}}_{D/S}(C, E) \longrightarrow D$ over x is equivalent to $\mathrm{Fun}_{S^{qx/}}(C \times_D S^{qx/}, E \times_D S^{qx/})$. For this reason, we think of $\widetilde{\mathrm{Fun}}_{D/S}(-, -)$ as the parametrized generalization of the pairing construction $\widetilde{\mathrm{Fun}}_D(-, -)$, to which it reduces when $S = \Delta^0$.

9.3. **Theorem.** *Notation as in 9.1, $\widetilde{\mathrm{Fun}}_{D/S}(C, E)$ enjoys the following functoriality:*

(1) If ϕ is either a S-cartesian fibration or a S-cocartesian fibration and ψ is a categorical fibration, then $\widetilde{\mathrm{Fun}}_{D/S}(C, E) \longrightarrow S$ is a S-category with cocartesian edges marked as indicated in 9.1, and $\widetilde{\mathrm{Fun}}_{D/S}(C, E) \longrightarrow D$ is a categorical fibration.

(2) If ϕ is a S-cartesian fibration and ψ is a S-cocartesian fibration, then $\widetilde{\mathrm{Fun}}_{D/S}(C, E) \longrightarrow D$ is a S-cocartesian fibration.

(3) If ϕ is a S-cocartesian fibration and ψ is a S-cartesian fibration, then $\widetilde{\mathrm{Fun}}_{D/S}(C, E) \longrightarrow D$ is a S-cartesian fibration.

Proof. (1) It suffices to check that Thm. 2.23 applies to the span

$${}_\natural D \xleftarrow{\pi} (\mathscr{O}^{cocart}(D) \times_D C, \mathscr{E}) \xrightarrow{\pi'} {}_\natural D.$$

In the remainder of this proof we will verify that $\mathscr{O}^{cocart}(D) \times_D C \longrightarrow D$ is a flat categorical fibration. For condition (4) we appeal to Lm. 9.2. The rest of the conditions are easy verifications.

(2) By Lm. 9.2 and 7.4, $\pi : \mathscr{O}^{cocart}(D) \times_D C \longrightarrow D$ is a cartesian fibration (hence flat) with an edge e π-cartesian if and only if $\mathrm{pr}_1(e)$ is ev_0-cartesian and $\mathrm{pr}_2(e)$ is ϕ-cartesian. Let $\mathscr{E}'$ be the collection of edges e in $\mathscr{O}^{cocart}(D) \times_{\mathrm{ev}_1, D} C$ such that for any π-cartesian lift e' of $\pi(e)$, the induced edge $d_1(e) \longrightarrow d_1(e')$ is in $\mathscr{E}$. Note that since ϕ is S-cartesian (and not just fiberwise cartesian), $\mathscr{E}'$ is closed under composition. Invoking Thm. 2.23 on the span

$$D^\sharp \xleftarrow{\pi} (\mathscr{O}^{cocart}(D) \times_D C, \mathscr{E}') \xrightarrow{\pi'} D^\sharp$$

we deduce that $\pi_*\pi'^* : s\mathbf{Set}^+_{/D} \longrightarrow s\mathbf{Set}^+_{/D}$ is right Quillen. Note that there is no conflict of notation with the functor $\pi_*\pi'^* : s\mathbf{Set}^+_{/_\natural D} \longrightarrow s\mathbf{Set}^+_{/_\natural D}$ defined before since $\mathscr{E} \subset \mathscr{E}'$ and the two restrict to the same collections of marked edges in the fibers of π. Since S-cocartesian fibrations are cocartesian fibrations over D (Rm. 7.3), we conclude.

(3) First note that π factors as a cocartesian fibration followed by a cartesian fibration, so is flat. Let $\mathscr{F}$ be the collection of edges f in D such that $q(f)$ is an equivalence. By Lm. 9.2, we have that $\pi : \mathscr{O}^{cocart}(D) \times_{\mathrm{ev}_1, D} C \longrightarrow D$ admits cocartesian lifts of edges in $\mathscr{F}$. Let $\mathscr{E}''$ be the collection of those π-cocartesian edges. Invoking Thm. 2.23 on the span

$$(D, \mathscr{F})^{op} \xleftarrow{\rho} (\mathscr{O}^{cocart}(D) \times_D C, \mathscr{E}'')^{op} \xrightarrow{\rho'} (D, \mathscr{F})^{op}$$

we deduce that with respect to the cartesian model structures $\rho_* \rho'^* : s\mathbf{Set}^+_{/(D,\mathscr{F})} \longrightarrow s\mathbf{Set}^+_{/(D,\mathscr{F})}$ is right Quillen. We have that $\widetilde{\mathrm{Fun}}_{D/S}(C, E)$ is a full subcategory of $\rho_* \rho'^*(\psi)$. Moreover, the compatibility condition in the definition of a S-cartesian fibration ensures that $\widetilde{\mathrm{Fun}}_{D/S}(C, E) \longrightarrow D$ inherits the property of being fibrant in $s\mathbf{Set}^+_{/(D,\mathscr{F})}$. Another routine verification shows that $\widetilde{\mathrm{Fun}}_{D/S}(C, E) \longrightarrow D$ is indeed S-cartesian.

□

9.4. **Lemma.** *Let $C \longrightarrow C'$ be a monomorphism between S-cartesian or S-cocartesian fibrations over D and let $E \longrightarrow D$ be a S-fibration. Then the induced functor*

$$\widetilde{\mathrm{Fun}}_{D/S}(C', E) \longrightarrow \widetilde{\mathrm{Fun}}_{D/S}(C, E)$$

is a categorical fibration.

Proof. Given a trivial cofibration $A \longrightarrow B$ in $s\mathbf{Set}_{\mathrm{Joyal}}$, we need to solve the lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \widetilde{\mathrm{Fun}}_{D/S}(C', E) \arrow[d] \\ B \arrow[r] \arrow[ur, dotted] & \widetilde{\mathrm{Fun}}_{D/S}(C, E). \end{tikzcd}$$

This diagram transposes to

$$\begin{tikzcd} A \times_D \mathscr{O}^{cocart}(D) \times_D C' \bigcup_{A \times_D \mathscr{O}^{cocart}(D) \times_D C} B \times_D \mathscr{O}^{cocart}(D) \times_D C \arrow[r] \arrow[d] & E \arrow[d] \\ B \times_D \mathscr{O}^{cocart}(D) \times_D C' \arrow[r] \arrow[ur, dotted] & D. \end{tikzcd}$$

By the proof of Thm. 9.3, $\mathscr{O}^{cocart}(D) \times_D C \longrightarrow D$ is a flat categorical fibration. Therefore, by [11, B.4.5] the left vertical arrow is a trivial cofibration in $s\mathbf{Set}_{\mathrm{Joyal}}$. □

For later use, we analyze some degenerate instances of the S-pairing construction.

9.5. **Lemma.** *There is a natural equivalence $\widetilde{\mathrm{Fun}}_{D/S}(D, E) \xrightarrow{\simeq} E$ of S-categories over D.*

Proof. The map is induced by the identity section $\iota_D : D \longrightarrow \mathscr{O}^{cocart}(D)$ fitting into a morphism of spans

$$\begin{tikzcd} & {}^\natural D \arrow[dl, "="'] \arrow[d, "\iota_D"] \arrow[dr, "="] & \\ {}^\natural D & (\mathscr{O}^{cocart}(D), \mathscr{E}) \arrow[l] \arrow[r] & {}^\natural D. \end{tikzcd}$$

By Lm. 3.2(1'), ι_D is a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$ via the target map. Since the cocartesian model structure on $s\mathbf{Set}^+_{/{}^\natural D}$ is created by the forgetful functor to $s\mathbf{Set}^+_{/S}$, the assertion follows. □

9.6. **Lemma.** *Let $C' \longrightarrow D'$ be a cartesian fibration of ∞-categories and let E' be a S-category. For all $s \in S$, there is a natural equivalence*

$$\widetilde{\mathrm{Fun}}_{D' \times S/S}(C' \times S, D' \times E')_s \xrightarrow{\simeq} \widetilde{\mathrm{Fun}}_{D'}(C', D' \times E'_s)$$

of cartesian fibrations over D'.

Proof. The lefthand side is defined using the span

$$(D')^\sharp \times \{s\} \longleftarrow ((D')^\sharp \times \{s\}) \times_{D' \times S} (\mathscr{O}^{cocart}(D' \times S) \times_{D'} C', \mathscr{E}') \longrightarrow S^\sharp$$

with $\mathscr{E}'$ as in the proof of Thm. 9.3. Cocartesian edges (over S) in $D' \times S$ are precisely those edges which become equivalences when projected to D', so $\mathscr{O}^{cocart}(D' \times S) \cong \mathrm{Fun}((\Delta^1)^\sharp, (D')^\sim) \times \mathscr{O}(S)$, and the identity section $\iota_{D'} : D' \longrightarrow \mathrm{Fun}((\Delta^1)^\sharp, (D')^\sim)$ is a categorical equivalence. Therefore, the map

$$(D' \times S^{s/})^\sharp \longrightarrow ((D')^\sharp \times \{s\}) \times_{D' \times S} (\mathscr{O}^{cocart}(D' \times S), \mathscr{E})$$

induced by $\iota_{D'}$ is a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$. Since $C' \times S \longrightarrow D' \times S$ is a cartesian fibration, it follows that

$$(C')^\natural \times (S^{s/})^\sharp \longrightarrow ((D')^\sharp \times \{s\}) \times_{D' \times S} (\mathscr{O}^{cocart}(D' \times S) \times_{D'} C', \mathscr{E}')$$

is also a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$. Finally, using the inclusion $C' \times \{s\} \longrightarrow C' \times S^{s/}$, we obtain a morphism from the span

$$(D')^\sharp \longleftarrow (C')^\natural \longrightarrow \{s\} \subset S^\sharp$$

through a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$. This yields the equivalence of the lemma. □

Directly from the definition, we have that for an object $x \in D$, the fiber $\widetilde{\mathrm{Fun}}_{D/S}(C, E)_x$ is isomorphic to $\mathrm{Fun}_{\underline{x}}(C_{\underline{x}}, E_{\underline{x}})$. We now proceed to identify the S-fiber $\widetilde{\mathrm{Fun}}_{D/S}(C, E)_{\underline{x}}$.

9.7. **Proposition.** *There is a $\underline{x}$-functor*

$$\epsilon^* : \widetilde{\mathrm{Fun}}_{D/S}(C, E)_{\underline{x}} \longrightarrow \underline{\mathrm{Fun}}_{\underline{x}}(C_{\underline{x}}, E_{\underline{x}})$$

which is a cocartesian equivalence in $s\mathbf{Set}^+_{/\underline{x}}$.

Proof. We first define the $\underline{x}$-functor ϵ^*. The data of maps of marked simplicial sets

$$A \longrightarrow {}_\natural\widetilde{\mathrm{Fun}}_{D/S}(C, E)_{\underline{x}}$$
$$A \longrightarrow {}_\natural\underline{\mathrm{Fun}}_{\underline{x}}(C_{\underline{x}}, (E \times_S D)_{\underline{x}})$$

over $\underline{x}$ is identical to the data of maps

$$A \times_{\underline{x}} \underline{x}^\sharp \times_D (\mathscr{O}^{cocart}(D), \mathscr{E}) \times_D {}_\natural C \longrightarrow {}_\natural E$$
$$A \times_{\underline{x}} \mathscr{O}(\underline{x})^\sharp \times_{\mathrm{ev}_1 \circ \mathrm{ev}_1, D} {}_\natural C \longrightarrow {}_\natural E$$

over ${}_\natural D$ (where $\mathscr{E}$ is the collection of edges e in $\mathscr{O}^{cocart}(D)$ such that $\mathrm{ev}_0(e)$ and $\mathrm{ev}_1(e)$ are cocartesian). We have a commutative square

$$\begin{tikzcd}
\mathscr{O}(\underline{x})^\sharp \arrow[r, "\mathrm{ev}_0"] \arrow[d, "\mathscr{O}(\mathrm{ev}_1)"] & \underline{x}^\sharp \arrow[d, "\mathrm{ev}_1"] \\
(\mathscr{O}^{cocart}(D), \mathscr{E}) \arrow[r, "\mathrm{ev}_0"] & {}_\natural D
\end{tikzcd}$$

which defines the functor $\epsilon : \mathscr{O}(\underline{x}) \longrightarrow \underline{x} \times_D \mathscr{O}^{cocart}(D)$, and this in turn induces the functor ϵ^*. To show that ϵ^* is a cocartesian equivalence, it will suffice to show that ϵ is a trivial fibration, for then a choice of section σ and homotopy $\sigma \circ \epsilon \simeq id$ will furnish a strong homotopy inverse to ϵ^* in the sense of [9, 3.1.3.5]. Since we have a pullback diagram

$$\begin{tikzcd}
\mathscr{O}(\underline{x}) \arrow[r] \arrow[d, "\epsilon"] & D \times_{\mathrm{Fun}(\Delta^1, D)} \mathrm{Fun}(\Delta^1 \times \Delta^1, D) \arrow[d, "\epsilon'"] \\
\underline{x} \times_D \mathscr{O}^{cocart}(D) \arrow[r] & \mathrm{Fun}(\Lambda^2_1, D)
\end{tikzcd}$$

it will further suffice to show that ϵ' is a trivial Kan fibration. ϵ' factors as the composition

$$D \times_{\mathrm{Fun}(\Delta^1,D)} \mathrm{Fun}(\Delta^1 \times \Delta^1, D) \xrightarrow{\epsilon''} \mathrm{Fun}(\Delta^2, D) \xrightarrow{\epsilon'''} \mathrm{Fun}(\Lambda^2_1, D).$$

where ϵ'' is defined by precomposing by the inclusion $i : \Delta^2 \longrightarrow \Delta^1 \times \Delta^1$ which avoids the degenerate edge for objects in $D \times_{\mathrm{Fun}(\Delta^1,D)} \mathrm{Fun}(\Delta^1 \times \Delta^1, D)$, and ϵ''' is precomposition by $\Lambda^2_1 \longrightarrow \Delta^2$. ϵ''' is a trivial fibration since $\Lambda^2_1 \longrightarrow \Delta^2$ is inner anodyne. To argue that ϵ'' is a trivial fibration, first note that ϵ'' inherits the property of being a categorical fibration from $i^* : \mathrm{Fun}(\Delta^1 \times \Delta^1, D) \longrightarrow \mathrm{Fun}(\Delta^2, D)$. Define an inverse σ'' by precomposing by the unique retraction $r : \Delta^1 \times \Delta^1 \longrightarrow \Delta^2$ chosen so that $r \circ i = id$. Then σ'' is a section of ϵ'' and one can write down an explicit homotopy through equivalences of the identity functor on $D \times_{\mathrm{Fun}(\Delta^1,D)} \mathrm{Fun}(\Delta^1 \times \Delta^1, D)$ to $\sigma'' \circ \epsilon''$, so ϵ'' is a trivial fibration. □

D-parametrized slice. We now study another slice construction defined using the S-pairing construction.

9.8. **Construction.** Let $\phi : C \longrightarrow D$ be a S-cocartesian fibration and let $F : C \longrightarrow E$ be a S-functor over D. Then F defines a cocartesian section

$$\tau_F : D \longrightarrow \widetilde{\mathrm{Fun}}_{D/S}(C, E)$$

as adjoint to the functor $\mathscr{O}^{cocart}(D) \times_{ev_1,D} C \longrightarrow C \xrightarrow{F} E$. Define

$$E^{(\phi,F)/S} = D \times_{\widetilde{\mathrm{Fun}}_{D/S}(C,E)} \widetilde{\mathrm{Fun}}_{D/S}(C \star_D D, E)$$

and let $\pi_{(\phi,F)}$ denote the projection $E^{(\phi,F)/S} \longrightarrow D$.

Given an object $x \in D$, the functor $\tau_F : D \longrightarrow \widetilde{\mathrm{Fun}}_{D/S}(C, E)$ induces via pullback a $\underline{x}$-functor

$$\tau_{F_{\underline{x}}} : \underline{x} \longrightarrow \widetilde{\mathrm{Fun}}_{D/S}(C, E)_{\underline{x}}.$$

We also have the $\underline{x}$-functor

$$\sigma_{F_{\underline{x}}} : \underline{x} \longrightarrow \underline{\mathrm{Fun}}_{\underline{x}}(C_{\underline{x}}, E_{\underline{x}})$$

adjoint to

$$\mathscr{O}(\underline{x}) \times_{\underline{x}} C_{\underline{x}} \xrightarrow{\mathrm{pr}_2} C_{\underline{x}} \xrightarrow{F_{\underline{x}}} E_{\underline{x}}.$$

An inspection of the definition of the comparison functor ϵ^* of 9.7 shows that the triangle

$$\begin{tikzcd} \underline{x} \arrow[r, "\tau_{F_{\underline{x}}}"] \arrow[dr, "\sigma_{F_{\underline{x}}}"'] & \widetilde{\mathrm{Fun}}_{D/S}(C, E)_{\underline{x}} \arrow[d, "\epsilon^*"] \\ & \underline{\mathrm{Fun}}_{\underline{x}}(C_{\underline{x}}, E_{\underline{x}}) \end{tikzcd}$$

commutes. Recalling the definitions

$$(E^{(\phi,F)/S})_{\underline{x}} = \underline{x} \times_{\widetilde{\mathrm{Fun}}_{D/S}(C,E)_{\underline{x}}} \widetilde{\mathrm{Fun}}_{D/S}(C \star_D D, E)_{\underline{x}}$$

$$(E_{\underline{x}})^{F_{\underline{x}}/\underline{x}} = \underline{x} \times_{\underline{\mathrm{Fun}}_{\underline{x}}(C_{\underline{x}},E_{\underline{x}})} \underline{\mathrm{Fun}}_{\underline{x}}(C_{\underline{x}} \star_{\underline{x}} \underline{x}, E_{\underline{x}})$$

we therefore obtain a comparison $\underline{x}$-functor

$$\psi : (E^{(\phi,F)/S})_{\underline{x}} \longrightarrow (E_{\underline{x}})^{F_{\underline{x}}/\underline{x}}.$$

9.9. **Corollary.** *The functor ψ is a cocartesian equivalence in $s\mathbf{Set}^+_{/\underline{x}}$.*

Proof. By [9, 3.3.1.5], we have to verify that ψ induces a categorical equivalence on the fibers. But after passage to the fiber over an object $e = [x \longrightarrow y]$ in $\underline{x}$, by Lm. 4.8 ψ_e is a functor between two pullback squares in which one leg is a cartesian fibration. Therefore, by Prp. 9.7 and [9, 3.3.1.4], ψ_e is a categorical equivalence. □

9.10. **Proposition.** *$\pi_{(\phi,F)} : E^{(\phi,F)/S} \longrightarrow D$ is a S-cartesian fibration.*

Proof. By Lm. 9.4, $\pi_{(\phi,F)}$ is a categorical fibration. By Thm. 9.3, Lm. 9.4, and Lm. 4.8, the functor

$$(\iota_C^*)_s : \widetilde{\mathrm{Fun}}_{D/S}(C \star_D D, E)_s \longrightarrow \widetilde{\mathrm{Fun}}_{D/S}(C, E)_s$$

over D_s satisfies the hypotheses of [9, 2.4.2.11], hence is a locally cartesian fibration. To then show that $(\iota_C^*)_s$ is a cartesian fibration, it suffices to check that for every square

$$\begin{tikzcd} \left[G : C_{\underline{x}} \star_{\underline{x}} \underline{x} \longrightarrow E_{\underline{x}}\right] \arrow[r] \arrow[d] & \left[G' : C_{\underline{y}} \star_{\underline{y}} \underline{y} \longrightarrow E_{\underline{y}}\right] \arrow[d] \\ \left[H : C_{\underline{x}} \star_{\underline{x}} \underline{x} \longrightarrow E_{\underline{x}}\right] \arrow[r] & \left[H' : C_{\underline{y}} \star_{\underline{y}} \underline{y} \longrightarrow E_{\underline{y}}\right] \end{tikzcd}$$

in $\widetilde{\mathrm{Fun}}_{D/S}(C \star_D D, E)_s$ lying over an edge $e : x \longrightarrow y$ in D_s, if the horizontal edges are cartesian lifts over e and the right vertical edge is $(\iota_C^*)_{s,y}$-cartesian, then the left vertical edge is $(\iota_C^*)_{s,x}$-cartesian. In other words, if we let $e_! : C_{\underline{x}} \star_{\underline{x}} \underline{x} \longrightarrow C_{\underline{y}} \star_{\underline{y}} \underline{y}$ and $e^* : E_{\underline{y}} \longrightarrow E_{\underline{x}}$ denote choices of pushforward and pullback functors, then we want to show that given $G \simeq e^* \circ G' \circ e_!$, $H \simeq e^* \circ H' \circ e_!$, and $G'|_{\underline{y}} \simeq H'|_{\underline{y}}$, we have that $G|_{\underline{x}} \simeq H|_{\underline{x}}$; this is clear. We deduce that $(\pi_{(\phi,F)})_s$, being pulled back from $(\iota_C^*)_s$, is a cartesian fibration. For the final verification, let us abbreviate objects

$$(x \in D, \left[G : C_{\underline{x}} \star_{\underline{x}} \underline{x} \longrightarrow E_{\underline{x}}\right] : G|_{C_{\underline{x}}} = F_{\underline{x}}) \in E^{(\phi,F)/S}$$

as $\left[G : C_{\underline{x}} \star_{\underline{x}} \underline{x} \longrightarrow E_{\underline{x}}\right]$, the restriction to $C_{\underline{x}}$ equaling $F_{\underline{x}}$ being left implicit. We must check that given a square

$$\begin{tikzcd} x \arrow[r, "\widetilde{\alpha}_x"] \arrow[d, "e"] & x' \arrow[d, "e'"] \\ y \arrow[r, "\widetilde{\alpha}_y"] & y' \end{tikzcd}$$

in D lying over $\alpha : s \longrightarrow t$ with the vertical edges in the fiber and the horizontal edges cocartesian lifts of α, and given a lift of that square to a square

$$\begin{tikzcd} \left[G : C_{\underline{x}} \star_{\underline{x}} \underline{x} \longrightarrow E_{\underline{x}}\right] \arrow[r] \arrow[d] & \left[G' : C_{\underline{x'}} \star_{\underline{x'}} \underline{x'} \longrightarrow E_{\underline{x'}}\right] \arrow[d] \\ \left[H : C_{\underline{y}} \star_{\underline{y}} \underline{y} \longrightarrow E_{\underline{y}}\right] \arrow[r] & \left[H' : C_{\underline{y'}} \star_{\underline{y'}} \underline{y'} \longrightarrow E_{\underline{y'}}\right] \end{tikzcd}$$

in $E^{(\phi,F)/S}$ with the horizontal edges cocartesian lifts of α and the left vertical edge $(\pi_{(\phi,F)})_s$-cartesian, then the right vertical edge is $(\pi_{(\phi,F)})_t$-cartesian. We will once more translate this compatibility statement into a more obvious looking one so as to conclude. Let $e_!, e^*, e'_!, e'^*$ be defined as above. Let $\alpha^* : \underline{x'} \longrightarrow \underline{x}$, $\alpha^* : \underline{y'} \longrightarrow \underline{y}$ be choices of pullback functors (e.g. the first sends a cocartesian edge $f : x' \longrightarrow z$ to $f \circ \widetilde{\alpha}_x : x \longrightarrow z$), and also label related functors by α^*. Then the cocartesianness of the horizontal edges amounts to the equivalences $G' \simeq G \circ \alpha^*$ and $H' \simeq H \circ \alpha^*$, and the cartesianness of the left vertical edge amounts to the equivalence $G|_{\underline{x}} \simeq (e^* \circ H \circ e_!)|_{\underline{x}}$. Our desired assertion now is implied by the homotopy commutativity of the diagram

$$\begin{tikzcd} \underline{x'} \arrow[r, "\alpha^*"] \arrow[d, "e'_!"] & \underline{x} \arrow[r, "G|_{\underline{x}}"] \arrow[d, "e_!"] & E_{\underline{x}} \\ \underline{y'} \arrow[r, "\alpha^*"] & \underline{y} \arrow[r, "H|_{\underline{y}}"] & E_{\underline{y}} \arrow[u, "e^*"] \end{tikzcd}$$

(the content being in the commutativity of the first square), for this demonstrates that $G'|_{\underline{x'}} \simeq (e'^* \circ H' \circ e'_!)|_{\underline{x'}}$. ∎

9.11. **Lemma.** *Let $p : W \longrightarrow S$, $q : D \longrightarrow S$ be S-categories and let $\pi : W \longrightarrow D$ be a S-fibration such that for every object $s \in S$, π_s is a cartesian fibration.*

(1) Suppose that:

(a) For every object $x \in D$, there exists an initial object in W_x.
(b) For every p-cocartesian edge $w \to w'$ in W, if w is an initial object in $W_{\pi(w)}$, then w' is an initial object in $W_{\pi(w')}$.

Let $W' \subset W$ be the full simplicial subset of W spanned by those objects $w \in W$ which are initial in $W_{\pi(w)}$ and let $\pi' = \pi|_{W'}$. Then W' is a full S-subcategory of W and π' is a trivial fibration.

(2) *Let $\sigma : D \longrightarrow W$ be a S-functor which is a section of π. Then σ is a left adjoint of π relative to D if and only if, for every object $x \in D$, $\sigma(x)$ is an initial object of W_x.*

Proof. (1) Condition (b) ensures that W' is a S-subcategory of W. By [9, 2.4.4.9], for every object $s \in S$, π'_s is a trivial fibration. In particular, π' is S-cocartesian fibration (the compatibility condition being vacuous since all edges in W'_s are π'_s-cocartesian). By Rm. 7.3, π' is a cocartesian fibration. As a cocartesian fibration with contractible fibers, π' is a trivial fibration.

(2) Since relative adjunctions are stable under base change, if σ is a left adjoint of π relative to D, passage to the fiber over $x \in D$ shows that $\sigma(x)$ is an initial object of W_x. Conversely, if for all $x \in D$, $\sigma(x)$ is an initial object of W_x, then by [9, 5.2.4.3], σ_s is left adjoint to π_s for all $s \in S$. Since σ is already given as a S-functor, this implies that σ is S-left adjoint to π; in particular, σ is left adjoint to π. The existence of σ implies the hypotheses of (1), so σ is fully faithful. Now by the definition [11, 7.3.2.1], σ is left adjoint to π relative to D. □

We now connect the construction $\widetilde{\mathrm{Fun}}_{D/S}(-,-)$ with $\underline{\mathrm{Fun}}_S(-,-)$. To this end, consider the commutative diagram

$$\begin{tikzcd}
\mathscr{O}(S)^\sharp \times_S {}_\natural C \arrow[dr, "i"] \arrow[rrrd, bend left] \arrow[dddr] & & & \\
& \mathscr{O}(S)^\sharp \times_S (\mathscr{O}^{cocart}(D) \times_D C, \mathscr{E}) \arrow[r] \arrow[d] & (\mathscr{O}^{cocart}(D) \times_D C, \mathscr{E}) \arrow[r] \arrow[d] & S^\sharp \\
& \mathscr{O}(S)^\sharp \times_S {}_\natural D \arrow[r, "\mathrm{pr}_D"] \arrow[d, "\mathrm{ev}_0"] & {}_\natural D & \\
& S^\sharp, & &
\end{tikzcd}$$

where the map i is induced by the identity section $D \longrightarrow \mathscr{O}^{cocart}(D)$.

9.12. **Lemma.** *i is a homotopy equivalence in $s\mathbf{Set}^+_{/S}$ (considered over S via $p : C \longrightarrow S$).*

Proof. Define a map $h' : \mathscr{O}(S) \times_S \mathscr{O}^{cocart}(D) \longrightarrow \mathrm{Fun}(\Delta^1, \mathscr{O}(S) \times_S \mathscr{O}^{cocart}(D))$ to be the product of the following three maps:

(1) Choose a lift σ

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^{\{0,1\}}, S) \arrow[r, "s_1"] \arrow[d] & \mathrm{Fun}(\Delta^2, S) \arrow[d, "\sim"] \\
\mathrm{Fun}(\Lambda^2_1, S) \arrow[r, "="] \arrow[ur, dotted, "\sigma"] & \mathrm{Fun}(\Lambda^2_1, S)
\end{tikzcd}$$

and let $\Delta^1 \times \Delta^1 \longrightarrow \Delta^2$ be the unique map so that the induced map $\mathrm{Fun}(\Delta^2, S) \longrightarrow \mathrm{Fun}(\Delta^1 \times \Delta^1, S) \cong \mathrm{Fun}(\Delta^1, \mathscr{O}(S))$ sends $(s \longrightarrow t \longrightarrow u)$ to $[s \longrightarrow t] \longrightarrow [s \longrightarrow u]$. Use these two maps to define

$$\mathscr{O}(S) \times_S \mathscr{O}^{cocart}(D) \times_D C \longrightarrow \mathscr{O}(S) \times_S \mathscr{O}(S) \cong \mathrm{Fun}(\Lambda^2_1, S) \longrightarrow \mathrm{Fun}(\Delta^1, \mathscr{O}(S)).$$

(2) Use the unique map $\Delta^1 \times \Delta^1 \longrightarrow \Delta^1$ which sends $(0,0)$ to 0 and all other vertices to 1 to define

$$\mathscr{O}(S) \times_S \mathscr{O}^{cocart}(D) \times_D C \longrightarrow \mathscr{O}^{cocart}(D) \longrightarrow \mathrm{Fun}(\Delta^1, \mathscr{O}^{cocart}(D)).$$

(3) The degeneracy map $s_0 : C \longrightarrow \mathrm{Fun}(\Delta^1, C)$ defines

$$\mathscr{O}(S) \times_S \mathscr{O}^{cocart}(D) \times_D C \longrightarrow C \longrightarrow \mathrm{Fun}(\Delta^1, C).$$

Then h' is adjoint to a map of marked simplicial sets over S

$$h : (\Delta^1)^\sharp \times \mathscr{O}(S)^\sharp \times_S (\mathscr{O}^{cocart}(D) \times_D C, \mathscr{E}) \longrightarrow \mathscr{O}(S)^\sharp \times_S (\mathscr{O}^{cocart}(D) \times_D C, \mathscr{E})$$

such that $h_0 = id$ and h_1 factors as a composition

$$\mathscr{O}(S)^\sharp \times_S (\mathscr{O}^{cocart}(D) \times_D C, \mathscr{E}) \xrightarrow{r} \mathscr{O}(S)^\sharp \times_S {}_\natural C \xrightarrow{i} \mathscr{O}(S)^\sharp \times_S (\mathscr{O}^{cocart}(D) \times_D C, \mathscr{E})$$

where r is defined by

$$\mathscr{O}(S)^\sharp \times_S (\mathscr{O}^{cocart}(D) \times_D C, \mathscr{E}) \longrightarrow \mathrm{Fun}(\Lambda^2_1, S)^\sharp \times_S {}_\natural C \xrightarrow{d_1 \circ \sigma} \mathscr{O}(S)^\sharp \times_S {}_\natural C.$$

Our choice of σ ensures that $r \circ i = id$, completing the proof. □

Note that for any S-fibration $\pi : X \longrightarrow D$, the S-category $\underline{\mathrm{Sect}}_{D/S}(\pi)$ defined in 8.4 may be identified with $(\mathrm{ev}_0)_*(\mathrm{pr}_D)^*({}_\natural X \xrightarrow{\pi} {}_\natural D)$. Combining Lm. 9.12, Lm. 2.26, and Lm. 2.27, we see that if E is a S-category and $C \longrightarrow D$ is S-cocartesian or S-cartesian, then the map induced by i

$$i^* : \underline{\mathrm{Sect}}_{D/S}(\widetilde{\mathrm{Fun}}_{D/S}(C, E \times_S D)) \longrightarrow \underline{\mathrm{Fun}}_S(C, E)$$

is a equivalence of S-categories. Moreover, a chase of the definitions reveals that for every S-functor $F : C \longrightarrow E$, we have an identification

$$i^* \circ \underline{\mathrm{Sect}}_{D/S}(\tau_{F \times \phi}) = \sigma_F : S \longrightarrow \underline{\mathrm{Fun}}_S(C, E).$$

We thus have a morphism of spans

$$\begin{tikzcd} S \arrow[r, "\underline{\mathrm{Sect}}_{D/S}(\tau_{F\times\phi})"] \arrow[d, "="] & \underline{\mathrm{Sect}}_{D/S}(\widetilde{\mathrm{Fun}}_{D/S}(C, E \times_S D)) \arrow[d, "\simeq"] & \underline{\mathrm{Sect}}_{D/S}(\widetilde{\mathrm{Fun}}_{D/S}(C \star_D D, E \times_S D)) \arrow[l] \arrow[d, "\simeq"] \\ S \arrow[r, "\sigma_F"] & \underline{\mathrm{Fun}}_S(C, E) & \underline{\mathrm{Fun}}_S(C \star_D D, E). \arrow[l] \end{tikzcd}$$

The right horizontal maps are S-fibrations by Lm. 9.4 and [2, 9.11(2)], so taking pullbacks yields an equivalence

$$\underline{\mathrm{Sect}}_{D/S}((E \times_S D)^{(\phi, F \times \phi)/S}) \xrightarrow{\simeq} S \times_{\sigma_F, \underline{\mathrm{Fun}}_S(C,E)} \underline{\mathrm{Fun}}_S(C \star_D D, E). \tag{9.12.1}$$

We are now prepared to introduce the main definition of this section.

9.13. **Definition.** Let $\phi : C \longrightarrow D$ be a S-cocartesian fibration. A S-functor $\overline{F} : C \star_D D \longrightarrow E$ is a *D-parametrized S-colimit diagram* if for every object $x \in D$, the $\underline{x}$-functor $\overline{F}|_{C_{\underline{x}} \star_{\underline{x}} \underline{x}} : C_{\underline{x}} \star_{\underline{x}} \underline{x} \longrightarrow E_{\underline{s}}$ is a $\underline{s}$-colimit diagram.

9.14. **Proposition.** *Let $\phi : C \longrightarrow D$ be a S-cocartesian fibration, let $F : C \longrightarrow E$ be a S-functor, and let $\overline{F} : C \star_D D \longrightarrow E$ be a D-parametrized S-colimit diagram extending F. Then the section*

$$id_S \times \sigma_{\overline{F}} : S \longrightarrow S \times_{\sigma_F, \underline{\mathrm{Fun}}_S(C,E)} \underline{\mathrm{Fun}}_S(C \star_D D, E)$$

is a S-initial object.

Proof. Combine Eqn. 9.12.1, Lm. 9.11(2), and Cor. 8.5. □

We have the following existence and uniqueness result for D-parametrized S-colimits.

9.15. **Theorem.** *Let $\phi : C \longrightarrow D$ be a S-cocartesian fibration and let $F : C \longrightarrow E$ be a S-functor. Suppose that for every object $x \in D$, the $\underline{s}$-functor $F|_{C_{\underline{x}}} : C_{\underline{x}} \longrightarrow E_{\underline{s}}$ admits a $\underline{s}$-colimit. Then there exists a D-parametrized S-colimit diagram $\overline{F} : C \star_D D \longrightarrow E$ extending F. Moreover, the full subcategory of $\{F\} \times_{\mathrm{Fun}_S(C,E)} \mathrm{Fun}_S(C \star_D D, E)$ spanned by the D-parametrized S-colimit diagrams coincides with that spanned by the initial objects.*

Proof. By Prp. 9.10 and Cor. 9.9, the functor

$$\pi_{(\phi,F\times\phi)} : (E \times_S D)^{(\phi,F\times\phi)/S} \longrightarrow D$$

is a S-cartesian fibration with $\underline{x}$-fibers equivalent to $(E_{\underline{s}})^{(F|_{C_{\underline{x}}},\underline{s})/}$. Our hypothesis ensures that the conditions of Lm. 9.11(1) are satisfied, so $\pi_{(\phi,F\times\phi)}$ admits a section σ which is a S-functor that selects an initial object in each fiber. The resulting S-functor $D \longrightarrow \widetilde{\mathrm{Fun}}_{D/S}(C \star_D D, E \times_S D)$ covering $\tau_{F\times\phi}$ is adjoint to a S-functor $\overline{F} : C \star_D D \longrightarrow E$ extending F, which is a D-parametrized S-colimit diagram. Having proven existence, the second statement now follows from Prp. 9.14. □

Thm. 9.15 also admits the following 'global' consequence.

9.16. **Corollary.** *Suppose that E is S-cocomplete. Then $U : \underline{\mathrm{Fun}}_S(C \star_D D, E) \longrightarrow \underline{\mathrm{Fun}}_S(C, E)$ admits a left S-adjoint L which is a section of U such that for every object $F : C_{\underline{s}} \longrightarrow E_{\underline{s}}$, $L(F)$ is a $D_{\underline{s}}$-parametrized $S^{s/}$-colimit diagram.*

Proof. The assumption that E is S-cocomplete implies that $E_{\underline{s}}$ is $\underline{s}$-cocomplete for all $s \in S$. By Thm. 9.15 and the stability of parametrized colimit diagrams under base change, the conditions of Lm. 9.11(1) are satisfied. Thus U admits a section L which selects an initial object in each fiber, necessarily a parametrized colimit diagram. By Lm. 9.11(2), L is a left adjoint of U relative to $\underline{\mathrm{Fun}}_S(C, E)$; in particular, L is S-left adjoint to U. □

Application: Functor categories.

9.17. **Proposition.** *Let K, I, and C be S-categories.*

(1) *Suppose that for all $s \in S$, $C_{\underline{s}}$ admits all $K_{\underline{s}}$-indexed colimits. $\overline{p} : K \star_S S \longrightarrow \underline{\mathrm{Fun}}_S(I, C)$ is a S-colimit diagram if and only if, for every object $x \in I$ over s,*

$$K_{\underline{s}} \star_{\underline{s}} \underline{s} \xrightarrow{\overline{p}_{\underline{s}}} \underline{\mathrm{Fun}}_{\underline{s}}(I_{\underline{s}}, C_{\underline{s}}) \xrightarrow{\mathrm{ev}_x} C_{\underline{s}}$$

is a $S^{s/}$-colimit diagram.

(2) *A S-functor $p : K \longrightarrow \underline{\mathrm{Fun}}_S(I, C)$ admits an extension to a S-colimit diagram $\overline{p}$ if for all $x \in I$, $\mathrm{ev}_x \circ p_{\underline{s}}$ admits an extension to a $S^{s/}$-colimit diagram.*

Proof. We prove (1), the proof for (2) being similar. Let $\overline{p'} : (K \times_S I) \star_I I \cong (K \star_S S) \times_S I \longrightarrow C$ be a choice of adjoint of p under the equivalence $\mathrm{Fun}_S(K \star_S S, \underline{\mathrm{Fun}}_S(I, C)) \simeq \mathrm{Fun}_S((K \star_S S) \times_S I, C)$. By Thm. 9.15 applied to the S-cocartesian fibration $K \times_S I \longrightarrow I$ and the hypothesis on C, there exists an I-parametrized S-colimit diagram p'' extending $p' = \overline{p'}|_{K\times_S I}$. By Prp. 9.14, p'' defines an S-initial object in

$$S \times_{\underline{\mathrm{Fun}}_S(K\times_S I, C)} \underline{\mathrm{Fun}}_S((K \times_S I) \star_I I, C) \simeq \underline{\mathrm{Fun}}_S(I, C)^{(p,S)/}$$

so *its* adjoint is a S-colimit diagram. For the 'if' direction, supposing that $\overline{p}$ is a S-colimit diagram, then by the uniqueness of S-initial objects, p'' is equivalent to $\overline{p'}$. Then $\mathrm{ev}_x \circ \overline{p}_{\underline{s}}$ is equivalent to $p''_{\underline{x}}$, which is a $S^{s/}$-colimit diagram by definition of I-parametrized S-colimit diagram. For the 'only if' direction, supposing that all the $\mathrm{ev}_x \overline{p}_{\underline{s}}$ are $S^{s/}$-colimit diagrams, we get that $\overline{p'}$ is a I-parametrized S-colimit diagram, so is equivalent to p''. □

9.18. **Corollary.** *Suppose C is S-cocomplete and I is a S-category. Then $\underline{\mathrm{Fun}}_S(I, C)$ is S-cocomplete.*

10. Kan extensions

We now combine the theory of S-colimits parametrized by a base S-category D and that of free S-cocartesian fibrations to establish the theory of left S-Kan extensions.

10.1. **Definition.** Suppose a diagram of S-categories

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\eta\Downarrow"] & E \\ D \arrow[ur, "G"'] & \end{tikzcd}$$

where by the '2-cell' η we mean exactly the datum of a S-functor $\eta : C \times \Delta^1 \longrightarrow E$ restricting to F on 0 and $G \circ \phi$ on 1. Let

$$G' : (C \times_D \mathscr{O}_S(D)) \star_D D \xrightarrow{\pi_D} D \xrightarrow{G} E,$$

let

$$\theta : (C \times_D \mathscr{O}_S(D)) \times \Delta^1 \longrightarrow E$$

be the natural transformation adjoint to $G_* : C \times_D \mathscr{O}_S(D) \longrightarrow \mathscr{O}_S(E)$, let

$$\eta' : (C \times_D \mathscr{O}_S(D)) \times \Delta^1 \longrightarrow C \times \Delta^1 \xrightarrow{\eta} E$$

be the natural transformation obtained from η, and let $\theta' = \theta \circ \eta'$ be a choice of composition in $\mathrm{Fun}_S(C \times_D \mathscr{O}_S(D), E)$. Let

$$r : \mathrm{Fun}_S((C \times_D \mathscr{O}_S(D)) \star_D D, E) \longrightarrow \mathrm{Fun}_S(C \times_D \mathscr{O}_S(D), E)$$

denote the restriction functor. By Lm. 4.8, we may select a r-cartesian edge e in $\mathrm{Fun}_S((C \times_D \mathscr{O}_S(D)) \star_D D, E)$ with $d_0(e) = G'$ covering θ', chosen so that $e|_D$ is degenerate. Let $G'' = d_1(e)$.

We say that G is a *left S-Kan extension* of F along ϕ if G'' is a D-parametrized S-colimit diagram.

10.2. **Remark.** The following are equivalent:

(1) G is a left S-Kan extension of F along ϕ.
(2) For all $s \in S$, $G_{\underline{s}}$ is a left $S^{s/}$-Kan extension of $F_{\underline{s}}$ along $\phi_{\underline{s}}$.
(3) For all $s \in S$ and $x \in D_s$, $G|_{\underline{x}} : \underline{x} \longrightarrow E_{\underline{s}}$ is a left $S^{s/}$-Kan extension of $F|_{C_{\underline{x}}} : C_{\underline{x}} \longrightarrow E_{\underline{s}}$ along $\phi_{\underline{x}} : C_{\underline{x}} \longrightarrow \underline{x}$.

In other words, our notion of S-Kan extension generalizes the concept of *pointwise* Kan extensions.

We can bootstrap Thm. 9.15 to prove existence and uniqueness of left S-Kan extensions.

10.3. **Theorem.** *Let $\phi : C \longrightarrow D$ and $F : C \longrightarrow E$ be S-functors. Suppose that for every object $x \in D$, the $S^{s/}$-functor*

$$C \times_D D^{/\underline{x}} \longrightarrow C_{\underline{s}} \xrightarrow{F_{\underline{s}}} E_{\underline{s}}$$

admits a $S^{s/}$-colimit. Then there exists a left S-Kan extension $G : D \longrightarrow E$ of F along ϕ, uniquely specified up to contractible choice.

Proof. We spell out the details of existence and leave the proof of uniqueness to the reader. By Thm. 9.15, there exists a D-parametrized S-colimit diagram

$$\overline{F} : (C \times_D \mathscr{O}_S(D)) \star_D D \longrightarrow E$$

extending $C \times_D \mathscr{O}_S(D) \longrightarrow C \xrightarrow{F} E$. Let $G = \overline{F}|_D$. Define a map

$$h : C \times \Delta^1 \longrightarrow (C \times_D \mathscr{O}_S(D)) \star_D D$$

over $D \times \Delta^1$ as adjoint to $(C \xrightarrow{(id, \iota\phi)} C \times_D \mathscr{O}_S(D), C \xrightarrow{\phi} D)$ and let $\eta = \overline{F} \circ h$, so that η is a natural transformation from F to $G \circ \phi$.

We claim that η exhibits G as a left Kan extension of F along ϕ. To show this, we will exhibit a r-cartesian edge e from $\overline{F}$ to G' such that the restriction $r(e)$ of e to $C \times_D \mathscr{O}_S(D)$ is a choice of composition $\theta \circ \eta'$. Define

$$e' : (C \times_D \mathscr{O}_S(D)) \star_D D \times \Delta^1 \longrightarrow (C \times_D \mathscr{O}_S(D)) \star_D D$$

over $D \times \Delta^1$ as adjoint to (id, π_D), and let $e = \overline{F} \circ e'$, so that e is an edge from $\overline{F}$ to G'. Since $(\pi_D)|_D = id_D$, $e|_D$ is a degenerate edge in $\mathrm{Fun}_S(D, E)$, so e is r-cartesian.

To finish the proof, we need to introduce a few more maps. Define

$$\alpha = (\mathrm{pr}_C, \alpha') : C \times_D \mathscr{O}_S(D) \times \Delta^1 \longrightarrow C \times_D \mathscr{O}_S(D)$$

where α' is adjoint to

$$C \times_D \mathscr{O}_S(D) \longrightarrow \mathscr{O}_S(D) = \widetilde{\mathrm{Fun}}_S(S \times \Delta^1, D) \xrightarrow{\min^*} \widetilde{\mathrm{Fun}}_S(S \times \Delta^1 \times \Delta^1, D).$$

Here $\min : \Delta^1 \times \Delta^1 \longrightarrow \Delta^1$ is the functor which takes the minimum. Define

$$\beta : C \times_D \mathscr{O}_S(D) \times \Delta^1 \longrightarrow \mathscr{O}_S(D) \times \Delta^1 \xrightarrow{\mathrm{ev}} D.$$

Use α and β to define

$$\gamma : C \times_D \mathscr{O}_S(D) \times \Delta^1 \times \Delta^1 \longrightarrow (C \times_D \mathscr{O}_S(D)) \star_D D$$

so that on objects $(c, \phi c \xrightarrow{f} d)$, γ sends $\Delta^1 \times \Delta^1$ to the square

$$\begin{tikzcd}
(c, \phi c = \phi c) \arrow[r] \arrow[d, "(id,f)"] & \phi c \arrow[d, "f"] \\
(c, \phi c \xrightarrow{f} d) \arrow[r] & d.
\end{tikzcd}$$

Then $\overline{F} \circ \gamma$ defines a square

$$\begin{tikzcd}
F \circ \mathrm{pr}_C \arrow[r, "\eta'"] \arrow[d, "="] & G \circ \phi \circ \mathrm{pr}_C \arrow[d, "\theta"] \\
F \circ \mathrm{pr}_C \arrow[r, "r(e)"] & G'.
\end{tikzcd}$$

in $\mathrm{Fun}_S(C \times_D \mathscr{O}_S(D), E)$, which proves that $r(e) \simeq \theta \circ \eta'$. ☐

We also have the Kan extension counterpart to Cor. 9.16.

10.4. **Theorem.** *Let $\phi : C \longrightarrow D$ be a S-functor and E a S-category. Suppose that E is S-cocomplete. Then the S-functor*

$$\phi^* : \underline{\mathrm{Fun}}_S(D, E) \longrightarrow \underline{\mathrm{Fun}}_S(C, E)$$

given by restriction along ϕ admits a left S-adjoint $\phi_!$ such that for every S-functor $F : C \longrightarrow E$, the unit map $F \longrightarrow \phi^*\phi_! F$ exhibits $\phi_! F$ as a left S-Kan extension of F along ϕ.

Proof. Factor ϕ as the composition

$$C \xrightarrow{\iota_C} C \times_D \mathscr{O}_S(D) \xrightarrow{i} (C \times_D \mathscr{O}_S(D)) \star_D D \xrightarrow{\pi_D} D.$$

Then ϕ^* factors as the composition

$$\underline{\mathrm{Fun}}_S(D, E) \xrightarrow{\pi_D^*} \underline{\mathrm{Fun}}_S((C \times_D \mathscr{O}_S(D)) \star_D D, E) \xrightarrow{i^*} \underline{\mathrm{Fun}}_S(C \times_D \mathscr{O}_S(D), E) \xrightarrow{\iota_C^*} \underline{\mathrm{Fun}}_S(C, E).$$

By Prp. 7.7 and Cor. 8.3, pr_C^* is left S-adjoint to ι_C^*. Since i_D is right S-adjoint to π_D, by Cor. 8.3 again i_D^* is left S-adjoint to π_D^*. By Thm. 9.15, i^* admits a left S-adjoint L which extends functors to D-parametrized S-colimit diagrams. Let $\phi_!$ be the composite of these three functors. The proof of Thm. 10.3 shows that $\phi_!(F)$ is as asserted. ☐

The next proposition permits us to eliminate the datum of the natural transformation η from the definition of a left S-Kan extension when ϕ is fully faithful.

10.5. **Proposition.** *Suppose $\phi : C \longrightarrow D$ is the inclusion of a full S-subcategory. Then for any left S-Kan extension G of $F : C \longrightarrow E$ along ϕ, η is a natural transformation through equivalences. Consequently, G is homotopic to a functor $\overline{F} : D \longrightarrow E$ which is both an extension of F and a left S-Kan extension (with the natural transformation $F \longrightarrow \overline{F} \circ \phi = F$ chosen to be the identity).*

Proof. Let $G'' : (C \times_D \mathscr{O}_S(D)) \star_D D \longrightarrow E$ be as in the definition of a left S-Kan extension. Because D-parametrized S-colimit diagrams are stable under restriction to S-subcategories,

$$(G'')_C : (C \times_D \mathscr{O}_S(D) \times_D C) \star_C C \longrightarrow E$$

is a C-parametrized S-colimit diagram. The additional assumption that C is a full S-subcategory has the consequence that $(C \times_D \mathscr{O}_S(D) \times_D C) \cong \mathscr{O}_S(C)$. Also, for any object $x \in C$, the inclusion $\underline{x}$-functor $i_x : \underline{x} \longrightarrow C^{/\underline{x}}$ is $\underline{x}$-final, using the first criterion of Thm. 6.7. Therefore, $\mathscr{O}_S(C) \star_C C \xrightarrow{\pi_C} C \xrightarrow{F} E$ is a C-parametrized S-colimit diagram extending $\mathscr{O}_S(C) \xrightarrow{\mathrm{ev}_0} C \xrightarrow{F} E$, so $(G'')_C \simeq F \circ \pi_C$.

The map h in the proof of Thm. 10.3 factors as

$$C \times \Delta^1 \xrightarrow{h'} \mathscr{O}_S(C) \star_C C \longrightarrow (C \times_D \mathscr{O}_S(D)) \star_D D.$$

We have the chain of equivalences

$$\eta \simeq G'' \circ h \simeq F \circ \pi_C \circ h' = F \circ \mathrm{pr}_C,$$

proving the first assertion. For the second assertion, use that

$$({}_\natural D \times \{1\}) \cup_{{}_\natural C \times \{1\}} ({}_\natural C \times (\Delta^1)^\sharp) \longrightarrow {}_\natural D \times (\Delta^1)^\sharp$$

is a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$ to extend (G, η) to a homotopy between G and an extension $\overline{F}$, which is then necessarily a left S-Kan extension. □

10.6. **Corollary.** *Suppose $\phi : C \longrightarrow D$ a fully faithful S-functor and E a S-cocomplete S-category. Then the left S-adjoint $\phi_!$ to the restriction S-functor ϕ^* exists and is fully faithful.*

Proof. Combine Thm. 10.4 and Prp. 10.5. □

As expected, S-colimit diagrams are examples of S-left Kan extensions.

10.7. **Proposition.** *Suppose $\phi : C \longrightarrow D$ a S-cocartesian fibration and $\overline{F} : C \star_D D \longrightarrow E$ a S-functor extending $F : C \longrightarrow E$. Then $\overline{F}$ is a D-parametrized S-colimit diagram if and only if $\overline{F}$ is a S-left Kan extension of F.*

Proof. We may check the assertion objectwise on D, so let $x \in D_s$. Consider the commutative diagram

$$\begin{tikzcd} C_{\underline{x}} \arrow[r, hook] \arrow[d, hook, "\theta"'] & C_{\underline{s}} \arrow[d, "F_{\underline{s}}"] \\ C \times_{C \star_D D} (C \star_D D)^{/\underline{x}} \arrow[ur, "\mathrm{pr}_C"] \arrow[r] & E_{\underline{s}}. \end{tikzcd}$$

The value of a D-parametrized colimit of F on x is computed as the $S^{s/}$-colimit of $(F_{\underline{s}})|_{C_{\underline{x}}}$, and that of a S-left Kan extension of F as the $S^{s/}$-colimit of $F_{\underline{s}} \circ \mathrm{pr}_C$. Therefore, it suffices to prove that θ is $\underline{x}$-final. Let $f : x \to y$ be an object in $\underline{x}$, i.e. a cocartesian edge in D, which lies over $s \longrightarrow t$. Then θ_f is equivalent to the inclusion

$$C_y \cong C_y \times_{(C_y)^\rhd} ((C_y)^\rhd)^{/\{\infty\}} \longrightarrow C_t \times_{C_t \star_{D_t} D_t} (C_t \star_{D_t} D_t)^{/y}.$$

Applying Lm. 10.8 to the map $C_t \longrightarrow C_t \star_{D_t} D_t$ of cocartesian fibrations over D_t, we deduce that θ_f is final. □

10.8. **Lemma.** *Let $X \longrightarrow Y$ be a map of cocartesian fibrations over Z and let $y \in Y$ be an object over $z \in S$. Then the inclusion $X_z \times_{Y_z} Y_z^{/y} \longrightarrow X \times_Y Y^{/y}$ is final.*

Proof. By the dual of [11, 3.4.1.10], $X \times_Y Y^{/y} \longrightarrow Z^{/z}$ is a cocartesian fibration. We have a pullback square

$$\begin{tikzcd} X_z \times_{Y_z} Y_z^{/y} \arrow[r] \arrow[d] & X \times_Y Y^{/y} \arrow[d] \\ \{z\} \arrow[r, "id_z"] & Z^{/z}. \end{tikzcd}$$

Since the bottom horizontal map is final and cocartesian fibrations are smooth, the top horizontal map is final. □

As with S-colimits, S-left Kan extensions reduce to the usual notion of left Kan extension when taken in a S-category of objects.

10.9. **Proposition.** *Suppose a diagram of S-categories*

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\eta\Downarrow"] & \underline{E}_S \\ D \arrow[ur, "G"'] & \end{tikzcd}.$$

The following are equivalent:

(1) G is a left S-Kan extension of F along ϕ.
(2) $G^\dagger$ is a left Kan extension of $F^\dagger$ along ϕ.
(3) For all objects $s \in S$, $G^\dagger|_{D_s}$ is a left Kan extension of $F^\dagger|_{C_s}$ along ϕ_s.

Proof. We first prove that (1) and (2) are equivalent. Factor $\phi : C \longrightarrow D$ through the free S-cocartesian fibration on ϕ:

$$\phi : C \xrightarrow{\iota_C} C \times_D \mathscr{O}_S D \xrightarrow{\mathrm{Fr}^{cocart}(\phi)} D.$$

Since ι_C is S-left adjoint to pr_C, it is also left adjoint. Therefore, the S-left Kan extension resp. the left Kan extension of F resp. $F^\dagger$ along ι_C is computed by $F \circ \mathrm{pr}_C$ resp. $F^\dagger \circ \mathrm{pr}_C$. By transitivity of Kan extensions, we thereby reduce to the case that ϕ is S-cocartesian. The claim now follows easily by combining Prp. 5.4 and Prp. 10.7.

We next prove that (2) and (3) are equivalent. For this, it suffices to observe that for all objects $d \in D$ over some $s \in S$, $C_s \times_{D_s} D_s^{/d} \longrightarrow C \times_D D^{/d}$ is final by Lm. 10.8 applied to $C \longrightarrow D$. □

11. Yoneda lemma

By Prp. 5.4, $\underline{\mathbf{Top}}_S$ is S-cocomplete, so by Cor. 9.18, the S-category of presheaves $\mathbf{P}_S(C) = \underline{\mathrm{Fun}}_S(C^{vop}, \underline{\mathbf{Top}}_S)$ is S-cocomplete. The S-Yoneda embedding $j : C \longrightarrow \mathbf{P}_S(C)$ was constructed in [2, §10] via straightening the left fibration $\widetilde{\mathscr{O}}_S(C) \longrightarrow C^{vop} \times_S C$. It was shown there that j is fully faithful. In this section, we prove that $\mathbf{P}_S(C)$ is the free S-cocompletion of C.

11.1. **Lemma** (S-Yoneda lemma). *Let $j : C \longrightarrow \mathbf{P}_S(C)$ denote the S-Yoneda embedding. Then the identity on $\mathbf{P}_S(C)$ is a S-left Kan extension of j along itself.*

Proof. By Prp. 9.17, it suffices to show that for every $s \in S$ and object $x \in C_s$, $\mathrm{ev}_x : \mathbf{P}_{\underline{s}}(C_{\underline{s}}) \longrightarrow \underline{\mathbf{Top}}_{\underline{s}}$ is a $S^{s/}$-left Kan extension of $\mathrm{ev}_x\, j_{\underline{s}}$. To ease notation, let us replace $S^{s/}$ by S and suppose that $s \in S$ is an initial object.

We claim that $(\mathrm{ev}_x\, j)^\dagger : C \longrightarrow \mathbf{Top}$ is homotopic to $\mathrm{Map}_C(x, -)$. By definition of the S-Yoneda embedding, $(\mathrm{ev}_x\, j)^\dagger$ classifies the left fibration $\mathrm{ev}_1 : \widetilde{\mathscr{O}}_S(C)_{x\rightarrow} \longrightarrow C$ pulled back from $\widetilde{\mathscr{O}}_S(C) \longrightarrow C^{vop} \times_S C$ via the cocartesian section $\sigma : S \longrightarrow C^{vop}$ defined by $\sigma(s) = x$. By [9, 4.4.4.5], it suffices to show that id_x is an initial object in $\widetilde{\mathscr{O}}_S(C)_{x\rightarrow}$. For this, because $s \in S$ is an initial object we reduce to checking that for all edges $\alpha : s \to t$, the pushforward of id_x by α is an initial object in the fiber $(\widetilde{\mathscr{O}}_S(C)_{x\rightarrow})_t$. But this fiber is equivalent to $\widetilde{\mathscr{O}}(C_t)_{\alpha_! x\rightarrow} \simeq (C_t)^{\alpha_! x/}$.

Applying Prp. 10.9, we reduce to showing that for all $t \in S$, $(\mathrm{ev}_x)^\dagger|_{\mathbf{P}_S(C)_t}$ is a left Kan extension of $(\mathrm{ev}_x\, j)^\dagger|_{C_t}$. Note that for y any cocartesian pushforward of x over the essentially unique edge $s \to t$, we have both that $(\mathrm{ev}_x\, j)^\dagger|_{C_t}$ is homotopic to $\mathrm{Map}_{C_t}(y, -)$ and $(\mathrm{ev}_x)^\dagger|_{\mathbf{P}_S(C)_t}$ is homotopic to ev_y (regarding y as an object in $C_{\underline{t}}^{vop}$). The inclusion $C_t \longrightarrow \mathbf{P}_S(C)_t \simeq \mathrm{Fun}(C_{\underline{t}}^{vop}, \mathbf{Top})$ factors through $\mathbf{P}(C_t)$ with $\mathbf{P}(C_t) \longrightarrow \mathrm{Fun}(C_{\underline{t}}^{vop}, \mathbf{Top})$ left adjoint to precomposition by the inclusion $i : C_t^{op} \longrightarrow C_{\underline{t}}^{vop}$. By the usual Yoneda lemma for ∞-categories, $\mathrm{ev}_y : \mathbf{P}(C_t) \longrightarrow \mathbf{Top}$ is the left Kan extension of $\mathrm{Map}_{C_t}(y, -)$. The left Kan extension of ev_y to $\mathbf{P}_S(C)_t$ is then given by precomposition by i, so is again ev_y. □

To state the universal property of $\mathbf{P}_S(C)$, we need to introduce a bit of terminology.

11.2. **Definition.** Let $F : C \longrightarrow D$ be a S-functor. We say that F *strongly preserves S-(co)limits* if for all $s \in S$, $F_{\underline{s}}$ preserves $S^{s/}$-(co)limits.

11.3. **Remark.** If F strongly preserves S-colimits then F preserves S-colimits. However, the converse is not necessarily true.

11.4. **Notation.** Suppose that C and D are S-cocomplete categories. Let $\mathrm{Fun}_S^L(C, D)$ denote the full subcategory of $\mathrm{Fun}_S(C, D)$ on the S-functors F which strongly preserve S-colimits. Let $\underline{\mathrm{Fun}}_S^L(C, D)$ denote the full S-subcategory of $\underline{\mathrm{Fun}}_S(C, D)$ with fibers $\mathrm{Fun}_{S^{s/}}^L(C, D)$ over $s \in S$.

11.5. **Theorem.** *Let E be a S-cocomplete category. Then restriction along the S-Yoneda embedding defines equivalences*

$$\mathrm{Fun}_S^L(\mathbf{P}_S(C), E) \xrightarrow{\simeq} \mathrm{Fun}_S(C, E)$$
$$\underline{\mathrm{Fun}}_S^L(\mathbf{P}_S(C), E) \xrightarrow{\simeq} \underline{\mathrm{Fun}}_S(C, E)$$

with the inverse given by S-left Kan extension.

We prepare for the proof of Thm. 11.5 with some necessary results concerning S-mapping spaces. Recall that given an ∞-category C, we have a number of equivalent options for describing mapping spaces in C. The relevant ones to consider for us are:

1. Straightening the left fibration $\widetilde{\mathscr{O}}(C) \longrightarrow C^{op} \times C$, we obtain the mapping space functor $\mathrm{Map}_C(-,-) : C^{op} \times C \longrightarrow \mathbf{Top}$;
2. Fixing an object $x \in C$, straightening the left fibration $C^{x/} \longrightarrow C$ also yields the functor $\mathrm{Map}_C(x,-) : C \longrightarrow \mathbf{Top}$;
3. Fixing objects $x, y \in C$, we have that the space $\mathrm{Map}_C(x,y)$ is given by $\{x\} \times_C \mathscr{O}(C) \times_C \{y\}$.

Likewise, given a S-category C, we have these possibilities:

1. The functor $\underline{\mathrm{Map}}_C(-,-) : C^{vop} \times_S C \longrightarrow \mathbf{Top}$ given by the straightening of $\widetilde{\mathscr{O}}_S(C) \longrightarrow C^{vop} \times_S C$;
2. Fixing an object $x \in C$, the left fibration $C^{\underline{x}/} = \underline{x} \times_C \mathscr{O}_S(C) \longrightarrow C$;
3. Fixing an object $x \in C$, the left fibration $C^{x/} \longrightarrow C$;
4. Fixing objects $x, y \in C$, the left fibration $\underline{x} \times_C \mathscr{O}_S(C) \times_C \underline{y} \longrightarrow \underline{y}$, which when specializing to the case that x and y are in the same fiber, yields $\underline{\mathrm{Map}}_C(x,y)$ as the fiber over $\{(x,y)\}$.

In the proof of Lm. 11.1, we showed that (1) and (3) were equivalent, and by Prp. 4.31, (2) and (3) are equivalent. In keeping with our usual abuse of notation for mapping spaces, we will interchangeably refer to any of these options when we write $\underline{\mathrm{Map}}_C(-,-)$.

Our next goal is to prove that $\underline{\mathrm{Map}}_C(-,-)$ preserves S-limits in the second variable, and dually, takes S-colimits in the first variable to S-limits. For this, we need a few lemmas.

11.6. **Lemma.** *Let $F : X \longrightarrow Y$ be a map of S-cocartesian or S-cartesian fibrations over an S-category C. The following are equivalent:*

(1) F is an equivalence.
(2) For all $s \in S$ and $S^{s/}$-functors $Z \longrightarrow C_{\underline{s}}$, $\underline{\mathrm{Fun}}_{/C_{\underline{s}}}(Z, X_{\underline{s}}) \longrightarrow \underline{\mathrm{Fun}}_{/C_{\underline{s}}}(Z, Y_{\underline{s}})$ is an equivalence.
(3) For all $s \in S$ and $c \in C_s$, $\underline{\mathrm{Fun}}_{/C_{\underline{s}}}(\underline{c}, X_{\underline{s}}) \longrightarrow \underline{\mathrm{Fun}}_{/C_{\underline{s}}}(\underline{c}, Y_{\underline{s}})$ is an equivalence.
(4) For all $c \in C$, $F_c : X_c \longrightarrow Y_c$ is an equivalence.

If X and Y are S-left or S-right fibrations over C, then all instances of $\underline{\mathrm{Fun}}$ can be replaced by $\underline{\mathrm{Map}}$.

Proof. (1) $\Rightarrow$ (2): If F is an equivalence, so is $F_{\underline{s}}$ for all $s \in S$. The map in question is then induced by a map of pullbacks through equivalences in which two matching legs are S-fibrations, so is an equivalence. (2) $\Rightarrow$ (3) is obvious. (3) $\Rightarrow$ (4): Given $c \in C_s$, take fibers over $\{s\} \in \underline{s}$ and note that $\underline{\mathrm{Fun}}_{/C_{\underline{s}}}(\underline{c}, X_{\underline{s}})_s \simeq \mathrm{Fun}_{/C_c}(\{c\}, X_s) \simeq X_c$. (4) $\Rightarrow$ (1): We must check that F_s is an equivalence for all $s \in S$, for which it suffices to check fiberwise over C_s by the hypothesis. $\square$

11.7. **Lemma.** *Let $\overline{q} : S \star_S K \longrightarrow \underline{\mathbf{Top}}_S$ be a S-functor which extends $q : K \longrightarrow \underline{\mathbf{Top}}_S$. Let $\overline{X} \longrightarrow S \star_S K$ be a left fibration which is an unstraightening of $\overline{q}^{\dagger}$, and let $X = \overline{X} \times_{S \star_S K} K$. Then $\overline{q}$ is a S-limit diagram if and only if the restriction S-functor*

$$R : \underline{\mathrm{Map}}_{/S \star_S K}(S \star_S K, \overline{X}) \longrightarrow \underline{\mathrm{Map}}_{/S \star_S K}(K, \overline{X}) \cong \underline{\mathrm{Map}}_{/K}(K, X)$$

is an equivalence.

Proof. In view of [9, 3.3.3.4], R_s is a map from the limit of $\overline{q}^{\dagger}|_{\underline{s} \star_{\underline{s}} K_{\underline{s}}}$ to the limit of $q^{\dagger}|_{K_{\underline{s}}}$ induced by precomposition on the diagram. But by Prp. 5.5, $\overline{q}$ is a S-limit diagram if and only if $\overline{q}^{\dagger}$ is a right Kan extension of $q^{\dagger}$, in which case both of the limits in question are equivalent to $\overline{q}^{\dagger}(s)$. The assertion now follows. $\square$

11.8. **Proposition.** *Let $\overline{p} : S \star_S K \longrightarrow C$ be a S-functor. The following are equivalent:*

(1) $\overline{p}$ *is a* S*-limit diagram.*

(2) For all $s \in S$ *and* $c \in C_s$, $\underline{\mathrm{Map}}_{C_{\underline{s}}}(c, \overline{p}_{\underline{s}}(-)) : \underline{s} \star_{\underline{s}} K_{\underline{s}} \longrightarrow \underline{\mathbf{Top}}_{S^{s/}}$ *is a* $S^{s/}$*-limit diagram.*

(3) For all $s \in S$ *and* $c \in C_s$, $\underline{\mathrm{Map}}_{/C_{\underline{s}}}(\underline{c}, C_{\underline{s}}^{/(\overline{p}_{\underline{s}}, S^{s/})}) \longrightarrow \underline{\mathrm{Map}}_{/C_{\underline{s}}}(\underline{c}, C_{\underline{s}}^{/(p_{\underline{s}}, S^{s/})})$ *is an equivalence.*

Moreover, if the above conditions obtain, then

$$\underline{\mathrm{Map}}_{/C_{\underline{s}}}(\underline{c}, C_{\underline{s}}^{/(p_{\underline{s}}, S^{s/})}) \simeq \underline{\mathrm{Map}}_{C_{\underline{s}}}(c, \overline{p}_{\underline{s}}(v))$$

where v *is the cone point* $\{s\} \in \underline{s} \star_{\underline{s}} K_{\underline{s}}$.

Proof. (2) $\Leftrightarrow$ (3): We will show that the statements match after fixing $c \in C_s$. To ease notation, let us replace $S^{s/}$ by S and suppose that $s \in S$ is an initial object. By Lm. 11.7 and using that $C^{\underline{c}/}$ is the S-unstraightening of $\underline{\mathrm{Map}}_C(c, -)$, $\underline{\mathrm{Map}}_C(c, \overline{p}(-))$ is a S-limit diagram if and only if

$$\underline{\mathrm{Map}}_{/C}(S \star_S K, C^{\underline{c}/}) \longrightarrow \underline{\mathrm{Map}}_{/C}(K, C^{\underline{c}/})$$

is an equivalence. By Cor. 4.27, this map is equivalent by a zig-zag to the map

$$\underline{\mathrm{Map}}_{/C}(\underline{c}, C^{/(\overline{p}, S)}) \longrightarrow \underline{\mathrm{Map}}_{/C}(\underline{c}, C^{/(p, S)}).$$

The assertion now follows. The last assertion also follows in view of the equivalence $C^{/(\overline{p}, S)} \simeq C^{/\underline{\overline{p}(v)}}$ and $\underline{\mathrm{Map}}_{/C}(\underline{c}, C^{/\underline{\overline{p}(v)}}) \simeq \underline{c} \times_C C^{/\underline{\overline{p}(v)}} \simeq \underline{\mathrm{Map}}_C(c, \overline{p}(v))$.

(1) $\Leftrightarrow$ (3): This follows from Lm. 11.6 applied to $C^{/(\overline{p}, S)} \longrightarrow C^{/(p, S)}$, which is a map of S-right fibrations over C. □

11.9. **Corollary.** *Let* $F : C \longrightarrow D$ *be a* S*-functor. Then*

(1) F *strongly preserves* S*-limits if and only if for all* $s \in S$ *and* $d \in D_s$, $\underline{\mathrm{Map}}_{D_{\underline{s}}}(d, F_{\underline{s}}(-)) : C_{\underline{s}} \longrightarrow \underline{\mathbf{Top}}_{S^{s/}}$ *preserves* $S^{s/}$*-limits.*

(2) F *strongly preserves* S*-colimits if and only if for all* $s \in S$ *and* $d \in D_s$, $\underline{\mathrm{Map}}_{D_{\underline{s}}}(F_{\underline{s}}(-), d) = \underline{\mathrm{Map}}_{D_{\underline{s}}^{vop}}(d, F_{\underline{s}}^{vop}(-)) : C_{\underline{s}}^{vop} \longrightarrow \underline{\mathbf{Top}}_{S^{s/}}$ *preserves* $S^{s/}$*-limits.*

11.10. **Corollary.** *Let* C *be a* S*-category. The Yoneda embedding* $j : C \longrightarrow \mathbf{P}_S(C)$ *strongly preserves and detects* S*-limits.*

Proof. Combine Prp. 11.8 and Prp. 9.17. □

Proof of Thm. 11.5. By Thm. 10.4, we have a S-adjunction

$$j_! : \underline{\mathrm{Fun}}_S(C, E) \rightleftarrows \underline{\mathrm{Fun}}_S(\mathbf{P}_S(C), E) : j^*$$

with $j^* j_! \simeq id$ and the essential image of $j_!$ spanned by the left $S^{s/}$-Kan extensions ranging over all $s \in S$. By Prp. 8.2, taking cocartesian sections yields an adjunction

$$j_! : \mathrm{Fun}_S(C, E) \rightleftarrows \mathrm{Fun}_S(\mathbf{P}_S(C), E) : j^*$$

again with $j^* j_! \simeq id$ and the essential image of $j_!$ spanned by the left S-Kan extensions. Both assertions will therefore follow if we prove that for a S-functor $F : \mathbf{P}_S(C) \longrightarrow E$, F strongly preserves S-colimits if and only if F is a left S-Kan extension of its restriction $f = F|_C$.

For the ‘only if’ direction, because $id_{\mathbf{P}_S(C)}$ is a S-left Kan extension of j by the S-Yoneda lemma 11.1, $F = F \circ id_{\mathbf{P}_S(C)}$ is a left S-Kan extension as it is the postcomposition of $id_{\mathbf{P}_S(C)}$ with a strongly S-colimit preserving functor.

For the ‘if’ direction, we use the criterion of Cor. 11.9. Replacing $S^{s/}$ by S and supposing that $s \in S$ is an initial object, we reduce to showing that for all $x \in E_s$, $\underline{\mathrm{Map}}_E(F(-), x) : \mathbf{P}_S(C)^{vop} \longrightarrow \underline{\mathbf{Top}}_S$ preserves S-limits. We first observe that F^{vop} is a S-right Kan extension (of f^{vop}), hence so is $\underline{\mathrm{Map}}_E(F(-), x) = \underline{\mathrm{Map}}_{E^{vop}}(x, -) \circ F^{vop}$ as the postcomposition of a S-right Kan extension with a strongly S-limit preserving functor. However, by the vertical opposite of the S-Yoneda lemma, for any S-functor $G : C^{vop} \longrightarrow \underline{\mathbf{Top}}_S$, the strongly S-limit preserving S-functor $\underline{\mathrm{Map}}_{\mathbf{P}_S(C)}(-, G)$ is a S-right Kan extension of G. Applying this for $G = \underline{\mathrm{Map}}_E(f(-), x)$, we conclude. □

12. Bousfield–Kan formula

In this section, we prove two decomposition formulas for S-colimits which resemble the classical Bousfield–Kan formula for computing homotopy colimits. We first study the situation when $S = \Delta^0$.

12.1. **Notation.** Let K be a simplicial set and let $\Delta_{/K}$ be the nerve of the category of simplices of K. We denote the first vertex map by $\upsilon_K : \Delta^{op}_{/K} \longrightarrow K$ and the last vertex map by $\mu_K : \Delta_{/K} \longrightarrow K$.

By [9, 4.2.3.14], μ_K is final. Unfortunately, this is the wrong direction for the purposes of obtaining a Bousfield–Kan type formula, since $\Delta_{/K}$ is a *cartesian* fibration over Δ. To rectify this state of affairs, we prove that υ_K is in fact final.

12.2. **Proposition.** *Let K be a simplicial set. Then the first vertex map $\upsilon_K : \Delta^{op}_{/K} \longrightarrow K$ is final. Equivalently, the last vertex map $\mu_{K^{op}}$ is initial.*

Proof. Note that υ_K is natural in K and that $\Delta^{op}_{/(-)} : s\mathbf{Set} \longrightarrow s\mathbf{Set}$ preserves colimits. Recall from [9, 4.1.2.5] that a map $f : X \longrightarrow Y$ is final if and only if it is a contravariant equivalence in $s\mathbf{Set}_{/Y}$. It follows that the class of final maps is stable under filtered colimits, so we may suppose that K has finitely many nondegenerate simplices. Using left properness of the contravariant model structure, by induction we reduce to the assertion for $K = \Delta^n$. But in this case υ_K is final by the proof of [9, 4.2.3.15] (which proves the result when K is the nerve of a category).

For the second assertion, we note that the reversal isomorphism $\Delta_{/K^{op}} \cong \Delta_{/K}$ interchanges $\mu_{K^{op}}$ and $(\upsilon_K)^{op}$. ☐

12.3. **Corollary** (Bousfield–Kan formula)**.** *Suppose that C admits (finite) coproducts. Then for a (finite) simplicial set K and a map $p : K \longrightarrow C$, the colimit of p exists if and only if the geometric realization*

$$\left| \textstyle\coprod_{x \in K_0} p(x) \Lleftarrow \coprod_{\alpha \in K_1} p(\alpha(0)) \Lleftarrow\!\!\!\Lleftarrow \coprod_{\sigma \in K_2} p(\sigma(0)) \ \ldots \right|$$

exists, in which case the colimit of p is computed by the geometric realization.

Proof. The fibers of the cocartesian fibration $\pi_K : \Delta^{op}_{/K} \longrightarrow \Delta^{op}$ are the discrete sets K_n. Therefore, the left Kan extension of $p \circ \upsilon_K$ along π_K exists. By Prp. 12.2, $\operatorname{colim} p \simeq \operatorname{colim} p \circ \upsilon_K$, and the latter is computed as the colimit of $(\pi_K)_!(p \circ \upsilon_K)$ by the transitivity of left Kan extensions. ☐

We also have a variant of Cor 12.3 where the coproducts over K_n are replaced by colimits indexed by the spaces $\mathrm{Map}(\Delta^n, K)$. To formulate this, we need to introduce some auxiliary constructions. Let $\xi : W \longrightarrow \Delta^{op}$ be the opposite of the relative nerve of the inclusion $\Delta \longrightarrow s\mathbf{Set}$; this is a cartesian fibration which is an explicit model for the tautological cartesian fibration over Δ^{op} pulled back from the universal cartesian fibration over $\mathbf{Cat}^{op}_\infty$. Let $\lambda : \Delta^{op} \longrightarrow W$ be the 'first vertex' section of ξ which sends an n-simplex $\Delta^{a_0} \leftarrow \ldots \leftarrow \Delta^{a_n}$ to the n-simplex

$$\begin{tikzcd}
\Delta^n \arrow[d, "(\lambda a)_0"] & \ldots \arrow[l] & \Delta^{\{n-1,n\}} \arrow[l] \arrow[d, "(\lambda a)_{n-1}"] & \Delta^{\{n\}} \arrow[l] \arrow[d, "(\lambda a)_n"] \\
\Delta^{a_0} & \ldots \arrow[l] & \Delta^{a_{n-1}} \arrow[l] & \Delta^{a_n} \arrow[l]
\end{tikzcd}$$

of W specified by $(\lambda a)_i(0) = 0$ for all $0 \leq i \leq n$.

For an ∞-category C, let $Z_C = \widetilde{\mathrm{Fun}}_{\Delta^{op}}(W, C \times \Delta^{op})$ and let $Z'_C \subset Z_C$ be the sub-simplicial set on the simplices σ such that every edge of σ is cocartesian (with respect to the structure map to Δ^{op}), so that $Z'_C \longrightarrow \Delta^{op}$ is the maximal sub-left fibration in $Z_C \longrightarrow \Delta^{op}$. Define a Δ^{op}-functor $\Delta^{op}_{/C} \longrightarrow Z_C$ as adjoint to the map $\Delta^{op}_C \times_{\Delta^{op}} W \longrightarrow C$ which sends an n-simplex

$$\begin{tikzcd}
\Delta^n \arrow[d, "(\lambda a)_0"] & \ldots \arrow[l] & \Delta^{\{n-1,n\}} \arrow[l] \arrow[d, "(\lambda a)_{n-1}"] & \Delta^{\{n\}} \arrow[l] \arrow[d, "(\lambda a)_n"] \\
\Delta^{a_0} \arrow[d, "\tau"] & \ldots \arrow[l] & \Delta^{a_{n-1}} \arrow[l] \arrow[dll] & \Delta^{a_n} \arrow[l] \arrow[dlll] \\
C & & &
\end{tikzcd}$$

to $\tau \circ (\lambda a)_0 \in C_n$. Note that since $\Delta^{op}_{/C} \longrightarrow \Delta^{op}$ is a left fibration, this functor factors through Z'_C.

Define a 'first vertex' functor $\Upsilon_C : Z_C \longrightarrow C$ by precomposition with ι (using the isomorphism $\widetilde{\mathrm{Fun}}_{\Delta^{op}}(\Delta^{op}, C \times \Delta^{op}) \cong C \times \Delta^{op}$). We then have a factorization of the first vertex map as

$$\Delta^{op}_{/C} \longrightarrow Z'_C \longrightarrow Z_C \xrightarrow{\Upsilon_C} C.$$

12.4. **Proposition.** *The functors Υ_C and $\Upsilon'_C = (\Upsilon_C)|_{Z'_C}$ are final.*

Proof. We first prove that Υ_C is final by verifying the hypotheses of [9, 4.1.3.1]. Let $c \in C$. The map $Z_C \longrightarrow C$ is functorial in C, so we have a map $Z_{C_{c/}} \longrightarrow Z_C \times_C C_{c/}$. We claim that this map is a trivial Kan fibration. Unwinding the definitions, this amounts to showing that for every cofibration $A \longrightarrow B$ of simplicial sets over Δ^{op}, we can solve the lifting problem

$$\begin{tikzcd} B \cup_A A \times_{\Delta^{op}} W \arrow[r] \arrow[d] & C_{c/} \arrow[d] \\ B \times_{\Delta^{op}} W \arrow[r] \arrow[ur, dotted] & C. \end{tikzcd}$$

Since the class of left anodyne morphisms is right cancellative, we may suppose $A = \varnothing$. It thus suffices to prove that $\lambda_B = B \times_{\Delta^{op}} \lambda : B \longrightarrow B \times_{\Delta^{op}} W$ is left anodyne for any map of simplicial sets $B \longrightarrow \Delta^{op}$. Observe that even though λ is not a cartesian section, it is a left adjoint relative to Δ^{op} to ξ by [11, 7.3.2.6] and the uniqueness of adjoints, since on the fibers it restricts to the adjunction $\{0\} \rightleftarrows \Delta^n$. Consequently, for any ∞-category B and functor $B \longrightarrow \Delta^{op}$, by [11, 7.3.2.5] λ_B is a left adjoint, hence left anodyne. From this, we deduce the general case by using the characterization in [9, 4.1.2.1] of the left anodyne maps $X \longrightarrow Y$ as the trivial cofibrations in $s\mathbf{Set}_{/Y}$ equipped with the covariant model structure. Indeed, arguing as in the proof of Prp. 12.2, by induction on the nondegenerate simplices of B we reduce to the known case $B = \Delta^n$.

We next prove that Z_C is weakly contractible if C is, which will conclude the proof for Υ_C. For this, another application of (the opposite of) [11, 7.3.2.6] shows that the Δ^{op}-functor $C \times \Delta^{op} \longrightarrow Z_C$ defined by precomposition by ξ is a left adjoint relative to Δ^{op} to the functor $(\Upsilon_C, id_{\Delta^{op}})$, because it restricts to the adjunction $\iota\colon C \rightleftarrows \mathrm{Fun}(\Delta^n, C) :\mathrm{ev}_0$ on the fibers. Hence, $|Z_C| \simeq |C \times \Delta^{op}| \simeq |C|$, and the latter is contractible by hypothesis.

We employ the same strategy to show that Υ'_C is final. Since $C_{c/} \longrightarrow C$ is conservative, the trivial Kan fibration above restricts to yield a trivial Kan fibration $Z'_{C_{c/}} \longrightarrow Z'_C \times_C C_{c/}$. Thus it suffices to show that Z'_C is weakly contractible if C is. By (the opposite of) [6, 7.3], the cocartesian fibration $Z'_C \longrightarrow \Delta^{op}$ is classified by the functor $\Delta^{op} \xrightarrow{i^{op}} \mathbf{Cat}_\infty \xrightarrow{\mathrm{Map}(-,C)} \mathbf{Top}$. Let R denote the right adjoint to the colimit-preserving functor $L : \mathrm{Fun}(\Delta^{op}, \mathbf{Top}) \longrightarrow \mathbf{Cat}_\infty$ left Kan extended from the inclusion $i : \Delta \subset \mathbf{Cat}_\infty$; R sends an ∞-category to its corresponding complete Segal space. Then $R(C) \simeq \mathrm{Map}(-, C) \circ i^{op}$. For any $X_\bullet \in \mathrm{Fun}(\Delta^{op}, \mathbf{Top})$, we have $\operatorname{colim} X \simeq |L(X_\bullet)|$, hence $\operatorname{colim} R(C) \simeq |(L \circ R)(C)| \simeq |C|$, where $L \circ R \simeq id$ by [10, 4.3.16]. By [9, 3.3.4.6], $|Z'_C| \simeq \operatorname{colim} R(C)$, so we conclude that $|Z'_C|$ is contractible. ∎

12.5. **Corollary** (Bousfield–Kan formula, 'simplicial' variant)**.** *Suppose that C admits colimits indexed by spaces. Then for any ∞-category K and functor $p : K \longrightarrow C$, the colimit of p exists if and only if the geometric realization*

$$\left| \operatorname*{colim}_{x \in \mathrm{Map}(\Delta^0, K)} p(x) \leftleftarrows \operatorname*{colim}_{\alpha \in \mathrm{Map}(\Delta^1, K)} p(\alpha(0)) \;\substack{\longleftarrow\\[-2pt]\longleftarrow\\[-2pt]\longleftarrow}\; \operatorname*{colim}_{\sigma \in \mathrm{Map}(\Delta^2, K)} p(\sigma(0)) \;\cdots \right|$$

exists, in which case the colimit of p is computed by the geometric realization.

Proof. Using Prp. 12.4, we may repeat the proof of Cor. 12.3, now using the span

$$\Delta^{op} \leftarrow Z'_K \xrightarrow{\Upsilon'_K} K.$$

∎

We now proceed to relativize the above picture, starting with the map Υ_C. Let $C \longrightarrow S$ be a S-category. Define the map

$$\Upsilon_{C,S} : \widetilde{\mathrm{Fun}}_{\Delta^{op}\times S/S}(W \times S, \Delta^{op} \times C) \longrightarrow C$$

to be the composition of the map to $\widetilde{\mathrm{Fun}}_{\Delta^{op}\times S/S}(\Delta^{op} \times S, \Delta^{op} \times C)$ given by precomposition by $\lambda \times id_S$, together with the equivalence of Lm. 9.5 of this to $\Delta^{op} \times C$ and the projection to C. Define $\Upsilon'_{C,S}$ to be the restriction of $\Upsilon_{C,S}$ to the maximal sub-left fibration (with respect to $\Delta^{op} \times S$).

12.6. **Theorem.** *The S-functors $\Upsilon_{C,S}$ and $\Upsilon'_{C,S}$ are S-final.*

Proof. For every object $s \in S$, we have a commutative diagram

$$\begin{tikzcd}
\widetilde{\mathrm{Fun}}_{\Delta^{op}\times S/S}(W \times S, \Delta^{op} \times C)_s \arrow[r, "(\lambda\times id_S)_s^*"] \arrow[d, "\simeq"] \arrow[rr, bend left, "(\Upsilon_{C,S})_s"] & \widetilde{\mathrm{Fun}}_{\Delta^{op}\times S/S}(\Delta^{op} \times S, \Delta^{op} \times C)_s \arrow[r] \arrow[d, "\simeq"] & C_s \arrow[d, "="] \\
\widetilde{\mathrm{Fun}}_{\Delta^{op}}(W, \Delta^{op} \times C_s) \arrow[r, "\lambda^*"] \arrow[rr, bend right, "\Upsilon_{C_s}"'] & \widetilde{\mathrm{Fun}}_{\Delta^{op}}(\Delta^{op}, \Delta^{op} \times C_s) \cong \Delta^{op} \times C_s \arrow[r, "\mathrm{pr}_{C_s}"] & C_s
\end{tikzcd}$$

where the left two vertical maps are given by the natural categorical equivalences of Lm. 9.6; the only point to note is that the equivalences of Lm. 9.5 and Lm. 9.6 coincide when the first variable is trivial. By Prp. 12.4, Υ_{C_s} is final, so $(\Upsilon_{C,S})_s$ is final. By the S-cofinality Thm. 6.7, $\Upsilon_{C,S}$ is S-final. A similar argument shows that $\Upsilon'_{C,S}$ is S-final. □

The process of relativizing v_C is considerably more involved. We begin with some preliminaries on the relative nerve construction. Let J be a category.

12.7. **Lemma.** *The adjunctions*

$$\mathfrak{F}_J : s\mathbf{Set}_{/N(J)} \rightleftarrows \mathrm{Fun}(J, s\mathbf{Set}) : N_J$$
$$\mathfrak{F}_J^+ : s\mathbf{Set}^+_{/N(J)} \rightleftarrows \mathrm{Fun}(J, s\mathbf{Set}^+) : N_J^+$$

of [9, §3.2.5] *are simplicial.*

Proof. Let $\underline{K} : J \longrightarrow s\mathbf{Set}$ denote the constant functor at a simplical set K. We have an obvious map $\chi_K : N(J) \times K \longrightarrow N_J(\underline{K})$ natural in K and hence a map

$$(\eta_X, \chi_K \circ \mathrm{pr}) : X \times K \longrightarrow N_J(\mathfrak{F}_J X \times \underline{K}) \cong N_J \mathfrak{F}_J X \times N_J(\underline{K})$$

natural in X and K. We want to show the adjoint

$$\theta_{X,K} : \mathfrak{F}_J(X \times K) \longrightarrow \mathfrak{F}_J(X) \times \underline{K}$$

is an isomorphism. Both sides preserve colimits separately in each variable, so we may suppose $X = \Delta^n \longrightarrow J$ and $K = \Delta^m$. By [9, 3.2.5.6], $\mathfrak{F}_I(I)(-) \cong N(I_{/-})$, and by [9, 3.2.5.8], for any functor $f : I \longrightarrow J$, the square

$$\begin{tikzcd}
s\mathbf{Set}_{/N(I)} \arrow[r, "f_!"] \arrow[d, "\mathfrak{F}_I"] & s\mathbf{Set}_{/N(J)} \arrow[d, "\mathfrak{F}_J"] \\
\mathrm{Fun}(I, s\mathbf{Set}) \arrow[r, "f_!"] & \mathrm{Fun}(J, s\mathbf{Set})
\end{tikzcd}$$

commutes. Letting $I = \Delta^n \times \Delta^m$ and $f : I \longrightarrow J$ be the structure map, we have

$$\mathfrak{F}_I(\Delta^n \times \Delta^m)(k, l) \cong (\Delta^n)_{/k} \times (\Delta^m)_{/l} \cong \Delta^k \times \Delta^l.$$

Factoring f as $\Delta^n \times \Delta^m \xrightarrow{g} \Delta^n \xrightarrow{h} J$, we then have

$$g_! \mathfrak{F}_I(\Delta^n \times \Delta^m)(k) \cong \Delta^i \times \Delta^m.$$

Let $G = g_!\mathfrak{F}_I(\Delta^n \times \Delta^m)$, so that $\mathfrak{F}_J(\Delta^n \times \Delta^m)(j) \cong (h_!G)(j)$. Then

$$(h_!G)(j) \cong \underset{\Delta^n \times_J J_{/j}}{\operatorname{colim}} \big((k, h(k) \to j) \mapsto \Delta^k\big) \times \Delta^m \cong \mathfrak{F}_J(\Delta^n)(j) \times \Delta^m$$

and one can verify that $\theta_{X,K}$ implements this isomorphism. For the assertion about $\mathfrak{F}_J^+ \dashv N_J^+$, recall that the simplicial tensor $s\mathbf{Set} \times s\mathbf{Set}^+ \longrightarrow s\mathbf{Set}^+$ is given by $(K, X) \mapsto K^\sharp \times X$. Consequently, in the above argument we may simply replace Δ^m by $(\Delta^m)^\sharp$ to conclude. $\square$

Since $N_J^+(\underline{S^\sharp}) = N(J) \times S^\sharp$, the adjunction $\mathfrak{F}_J^+ \dashv N_J^+$ lifts to an adjunction

$$\mathfrak{F}_{J,S}^+ \colon s\mathbf{Set}^+_{/N(J)\times S} \rightleftarrows \operatorname{Fun}(J, s\mathbf{Set}^+_{/S}) : N_{J,S}^+$$

between the overcategories. Moreover, for any functor $f : T \longrightarrow S$, the square

$$\begin{tikzcd} \operatorname{Fun}(J, s\mathbf{Set}^+_{/S}) \arrow[r, "N^+_{J,S}"] \arrow[d, "f^*"] & s\mathbf{Set}^+_{/N(J)\times S} \arrow[d, "(id\times f)^*"] \\ \operatorname{Fun}(J, s\mathbf{Set}^+_{/T}) \arrow[r, "N^+_{J,T}"] & s\mathbf{Set}^+_{/N(J)\times T}, \end{tikzcd}$$

commutes.

12.8. **Proposition.** *Equip* $s\mathbf{Set}^+_{/N(J)\times S}$ *with the cocartesian model structure and* $\operatorname{Fun}(J, s\mathbf{Set}^+_{/S})$ *with the projective model structure, where* $s\mathbf{Set}^+_{/S}$ *has the cocartesian model structure. Then the adjunction*

$$\mathfrak{F}_{J,S}^+ \colon s\mathbf{Set}^+_{/N(J)\times S} \rightleftarrows \operatorname{Fun}(J, s\mathbf{Set}^+_{/S}) : N_{J,S}^+$$

is a Quillen equivalence.

Proof. We first prove that the adjunction is Quillen. Because this is a simplicial adjunction between left proper simplicial model categories, it suffices to show that $\mathfrak{F}^+_{J,S}$ preserves cofibrations and $N^+_{J,S}$ preserves fibrant objects. Observe that the slice model structure on $s\mathbf{Set}^+_{/N(J)\times S} \cong (s\mathbf{Set}^+_{/N(J)})_{/(N(J)\times S)^\sharp}$ is a localization of the cocartesian model structure. Similarly, the slice model structure on $\operatorname{Fun}(J, s\mathbf{Set}^+_{/S}) \cong \operatorname{Fun}(J, s\mathbf{Set}^+)_{/\underline{S^\sharp}}$ is a localization of the projective model structure, since the trivial fibrations for the two model structures coincide and postcomposition by $\pi_! : s\mathbf{Set}^+_{/S} \longrightarrow s\mathbf{Set}^+$ gives a Quillen left adjoint between the projective model structures. Since the lift of a Quillen adjunction $L \colon M \rightleftarrows N : R$ to the adjunction $\widetilde{L} \colon M_{/R(x)} \rightleftarrows N_{/x} : \widetilde{R}$ is Quillen for the slice model structures, we deduce that $\mathfrak{F}^+_{J,S}$ preserves cofibrations.

Now suppose $F : J \longrightarrow s\mathbf{Set}^+_{/S}$ is fibrant. Since S is an ∞-category, $F \to \underline{S}$ is a fibration in $\operatorname{Fun}(J, s\mathbf{Set})$. Hence $N_{J,S}(F) \longrightarrow N(J) \times S$ is a categorical fibration. We verify that it is a cocartesian fibration (with every marked edge cocartesian) by solving the lifting problem ($n \geq 1$)

$$\begin{tikzcd} {}^\natural\Lambda^n_0 \arrow[r] \arrow[d] & N^+_{J,S}(F) \arrow[d] \\ {}^\natural\Delta^n \arrow[r, "(j_\bullet, s_\bullet)"'] \arrow[ur, dotted] & (N(J) \times S)^\sharp. \end{tikzcd}$$

Unwinding the definitions, this amounts to solving the lifting problem

$$\begin{tikzcd} {}^\natural\Lambda^n_0 \arrow[r] \arrow[d] & F(j_n) \arrow[d] \\ {}^\natural\Delta^n \arrow[r, "s_\bullet"'] \arrow[ur, dotted] & S^\sharp, \end{tikzcd}$$

and the dotted lift exists because $F(j_n)$ is cocartesian over S with the cocartesian edges marked. Finally, it is easy to see that marked edges compose and are stable under equivalence. We conclude that $N^+_{J,S}(F)$ is fibrant in $s\mathbf{Set}^+_{/N(J)\times S}$.

To prove that the Quillen adjunction is a Quillen equivalence, we will show that the induced adjunction of ∞-categories

$$\mathfrak{F}'^{+}_{J,S} \colon N((s\mathbf{Set}^{+}_{/N(J)\times S})^{\circ}) \rightleftarrows N(\mathrm{Fun}(J, s\mathbf{Set}^{+}_{/S})^{\circ}) : N'^{+}_{J,S}$$

is an adjoint equivalence, where $N'^{+}_{J,S}$ is the simplicial nerve of $N^{+}_{J,S}$ and $\mathfrak{F}'^{+}_{J,S}$ is any left adjoint to $N'^{+}_{J,S}$. We first check that $N'^{+}_{J,S}$ is conservative. Indeed, for this we may work in the model category: for a natural transformation $\alpha : F \to G$ in $\mathrm{Fun}(J, s\mathbf{Set}^{+}_{/S})$, $N^{+}_{J,S}(F) \longrightarrow N^{+}_{J,S}(G)$ on fibers is given by $F(j)_s \longrightarrow G(j)_s$, hence if F, G are fibrant and $N^{+}_{J,S}(\alpha)$ is an equivalence then α is as well. It now suffices to show that the unit transformation $\eta : id \longrightarrow N'^{+}_{J,S}\mathfrak{F}'^{+}_{J,S}$ is an equivalence. We have the known equivalence $N((s\mathbf{Set}^{+}_{/N(J)\times S})^{\circ}) \simeq \mathrm{Fun}(N(J) \times S, \mathbf{Cat}_{\infty})$ so it further suffices to check that the map

$$(id \times i_s)^* \longrightarrow (id \times i_s)^* N'^{+}_{J,S}\mathfrak{F}'^{+}_{J,S} \simeq N'^{+}_{J} i_s^* \mathfrak{F}'^{+}_{J,S}$$

is an equivalence for all $s \in S$, $i_s : \{s\} \longrightarrow S$ the inclusion. Equivalently, since $\mathfrak{F}^{+}_{J} \dashv N^{+}_{J}$ is a Quillen equivalence ([9, 3.2.5.18]), we must show that the adjoint map

$$\mathfrak{F}'^{+}_{J} i_s^* \longrightarrow (id \times i_s)^* \mathfrak{F}'^{+}_{J,S}$$

is an equivalence. This statement is in turn equivalent to the adjoint map

$$\theta : N'^{+}_{J,S}(i_s)_* \longrightarrow (id \times i_s)_* N'^{+}_{J}$$

being an equivalence. Recall that for a functor $f : T \longrightarrow S$, $f_* : \mathrm{Fun}(T, \mathbf{Cat}_{\infty}) \longrightarrow \mathrm{Fun}(S, \mathbf{Cat}_{\infty})$ is induced by $\pi_* \rho^* : s\mathbf{Set}^{+}_{/T} \longrightarrow s\mathbf{Set}^{+}_{/S}$ for the span

$$S^{\sharp} \xleftarrow{\pi} (\mathscr{O}(S) \times_S T)^{\sharp} \xrightarrow{\rho} T^{\sharp}$$

with π given by evaluation at 0 and ρ projection to T. Moreover, for a functor $id \times f : U \times T \longrightarrow U \times S$, we may elect to use the span

$$(U \times S)^{\sharp} \xleftarrow{id\times\pi} (U \times \mathscr{O}(S) \times_S T)^{\sharp} \xrightarrow{id\times\rho} (U \times T)^{\sharp}$$

to model $(id \times f)_*$. Letting $f = i_s$, we see that θ is induced by the map

$$N^{+}_{J,S}\pi_*\rho^* \longrightarrow (id \times \pi)_* N^{+}_{J,S^{s/}}\rho^* \cong (id \times \pi)_*(id \times \rho)^* N^{+}_{J}.$$

where the first map is adjoint to the isomorphism $(id \times \pi)^* N^{+}_{J,S} \cong N^{+}_{J,S^{s/}}\pi^*$. Direct computation reveals that this map is an equivalence on fibrant $F : J \longrightarrow s\mathbf{Set}^{+}$. □

We now return to the situation of interest. Let C be a S-category with structure map $\pi : C \longrightarrow S$. We first extend our existing notation $\underline{x}$ for objects $x \in C$.

12.9. **Notation.** For an n-simplex σ of C, define

$$\underline{\sigma} = \{\sigma\} \times_{\mathrm{Fun}(\Delta^n \times \{0\}, C)} \mathrm{Fun}((\Delta^n)^{\flat} \times (\Delta^1)^{\sharp}, {}_{\natural}C) \times_{\mathrm{Fun}(\Delta^n \times \{1\}, S)} S.$$

12.10. **Lemma.** *There exists a map $b_\sigma : \underline{\sigma} \longrightarrow \{\pi\sigma(n)\} \times_S \mathscr{O}(S) = S^{\pi\sigma(n)/}$ which is a trivial Kan fibration.*

Proof. First define a map $b'_\sigma : \underline{\sigma} \longrightarrow \underline{\pi\sigma}$ to be the pullback of the map

$$(e_0, \mathscr{O}(\pi))_* : \mathrm{Fun}(\Delta^n, \mathscr{O}^{cocart}(C)) \longrightarrow C^{\Delta^n} \times_{S^{\Delta^n}} \mathrm{Fun}(\Delta^n, \mathscr{O}(S))$$

over $\{\sigma\}$ and S. Since $(e_0, \mathscr{O}(\pi))$ is a trivial Kan fibration, so is b'_σ. Next, let K be the pushout $\Delta^n \times \{0\} \cup_{\{n\}\times\{0\}} \{n\} \times \Delta^1$. We claim that the map $\mathrm{Fun}(\Delta^n, \mathscr{O}(S)) \times_{S^{\Delta^n}} S \longrightarrow \mathrm{Fun}(K, S)$ induced by $K \subset \Delta^n \times \Delta^1$ is a trivial Kan fibration. For a monomorphism $A \longrightarrow B$, we need to solve the lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^n, \mathscr{O}(S)) \times_{S^{\Delta^n}} S \arrow[d] \\ B \arrow[r] \arrow[ur, dotted] & \mathrm{Fun}(K, S). \end{tikzcd}$$

This transposes to

$$\begin{tikzcd}
A \times \Delta^n \bigcup_{A\times\{n\}} B \times \{n\} \arrow[r] \arrow[d] & \mathscr{O}(S) \arrow[d, "\mathrm{ev}_0"] \\
B \times \Delta^n \arrow[r] \arrow[ur, dotted] & S
\end{tikzcd}$$

and the lefthand map is right anodyne by [9, 2.1.2.7], hence the dotted lift exists as ev_0 is a cartesian fibration. Now define b''_σ to be the pullback

$$\underline{\pi\sigma} = \{\pi\sigma\} \times_{S^{\Delta^n}} \mathrm{Fun}(\Delta^n, \mathscr{O}(S)) \times_{S^{\Delta^n}} S \longrightarrow \{\pi\sigma\} \times_{S^{\Delta^n}} \mathrm{Fun}(K, S) \cong S^{\pi\sigma(n)/};$$

this is also a trivial Kan fibration. Finally, let $b_\sigma = b''_\sigma \circ b'_\sigma$. □

We will regard $\underline{\sigma}$ as a $S^{\pi\sigma(n)/}$ or S-category via b_σ. We also have a target map $\underline{\sigma} \longrightarrow C^{\Delta^n}$ induced by $\Delta^n \times \{1\} \subset \Delta^n \times \Delta^1$. This covers the target map $S^{\pi\sigma(n)/} \longrightarrow S$ and is a S-functor.

Define a functor $F_C : \Delta^{op} \longrightarrow s\mathbf{Set}^+_{/S}$ on objects $[n]$ by

$$F_C([n]) = \bigsqcup_{\sigma \in C_n} \underline{\sigma}^\sharp$$

and on morphisms $\alpha : [m] \to [n]$ by the map $\underline{\sigma} \longrightarrow \underline{\sigma\alpha}$ induced by precomposition by $\alpha : \Delta^m \longrightarrow \Delta^n$.

12.11. **Remark.** The map $\underline{\sigma} \longrightarrow \underline{\sigma(n)}$ is compatible with the maps b_σ and $b_{\sigma(n)}$ of Lm. 12.10, hence is a categorical equivalence (in fact, a trivial Kan fibration). Consequently, given a morphism $f : x \to y$ in C, by choosing an inverse to $\underline{f} \xrightarrow{\simeq} \underline{y}$ we obtain a map $f^* : \underline{y} \longrightarrow \underline{x}$, unique up to contractible choice. Moreover, if f lies over an equivalence, then $\underline{f} \longrightarrow \underline{x}$ is a trivial Kan fibration, so we also obtain a map $f_! : \underline{x} \longrightarrow \underline{y}$.

In order to define the S-first vertex map $N^+_{\Delta^{op},S}(F_C) \longrightarrow C$, we need to introduce a few preliminary constructions. Let $A_n \subset \mathscr{O}(\Delta^n)$ be the sub-simplicial set where a k-simplex $x_0y_0 \to ... \to x_ky_k$ is in A_n if and only if $x_k \leq y_0$. For the reader's aid we draw a picture of the inclusion $A_n \subset \mathscr{O}(\Delta^n)$ for $n = 2$, where dashed edges are not in A_2:

$$\begin{tikzcd}
00 \arrow[d] \arrow[dd, bend right] \arrow[dr, dotted] \arrow[ddr, dotted] \arrow[ddrr, dotted, bend left] & & \\
01 \arrow[r] \arrow[d] \arrow[dr] \arrow[drr, dotted] & 11 \arrow[d] \arrow[dr, dotted] & \\
02 \arrow[r] \arrow[rr, bend right] & 12 \arrow[r] & 22.
\end{tikzcd}$$

12.12. **Lemma.** *The inclusion $A_n \longrightarrow \mathscr{O}(\Delta^n)$ is inner anodyne.*

Proof. In this proof we adopt the notation $[x_0y_0, ..., x_ky_k]$ for a k-simplex of $\mathscr{O}(\Delta^n)$. Let E be the collection of edges $[ab, xy]$ in $\mathscr{O}(\Delta^n)$ where $x > b$, and choose a total ordering $\leq$ on E such that if we have a factorization

$$\begin{tikzcd}
ab \arrow[r] \arrow[d] & xy \arrow[d] \\
a'b' \arrow[r] & x'y'
\end{tikzcd}$$

then $[a'b', x'y'] \leq [ab, xy]$. Index edges in E by $I = \{0, ..., N\}$. Define simplicial subsets $A_{n,i}$ of $\mathscr{O}(\Delta^n)$ such that $A_{n,i}$ is obtained by expanding A_n to contain every k-simplex $[x_0y_0, ..., x_ky_k]$ with $[x_0y_0, x_ky_k]$ in $E_{<i}$. We will show that each inclusion $A_{n,i} \longrightarrow A_{n,i+1}$ is inner anodyne. We may divide the nondegenerate k-simplices $[x_0y_0, x_1y_1, ..., x_ky_k]$ in $A_{n,i+1}$ but not in $A_{n,i}$ into six classes:

- $A1$: $x_1y_1 \neq x_0(y_0 + 1)$ and $y_1 > y_0$.
- $A2$: $x_1y_1 = x_0(y_0 + 1)$.
- $B1$: $x_1y_1 = (x_0 + 1)y_0$, $y_2 > y_0$, and $x_2y_2 \neq (x_0 + 1)(y_0 + 1)$.
- $B2$: $x_1y_1 = (x_0 + 1)y_0$ and $x_2y_2 = (x_0 + 1)(y_0 + 1)$.

- $C1$: $x_1y_1 \neq (x_0+1)y_0$ and $y_1 = y_0$.
- $C2$: $x_1y_1 = (x_0+1)y_0$ and $y_2 = y_0$.

We have bijections between classes of form 1 and classes of form 2 given by

- A: $[x_0y_0, x_1y_1, ..., x_ky_k] \mapsto [x_0y_0, x_0(y_0+1), x_1y_1, ..., x_ky_k]$.
- B: $[x_0y_0, x_0+1y_1, x_2y_2, ..., x_ky_k] \mapsto [x_0y_0, (x_0+1)y_0, (x_0+1)(y_0+1), x_2y_2, ..., x_ky_k]$.
- C: $[x_0y_0, x_1y_1, ..., x_ky_k] \mapsto [x_0y_0, (x_0+1)y_0, x_1y_1, ..., x_ky_k]$.

Moreover, this identifies simplices in a class of form 1 as inner faces of simplices in the corresponding class of form 2. Let P be the collection of pairs $\tau \subset \tau'$ of nondegenerate $k-1$ and k-simplices matched by this bijection. Choose a total ordering on P where pairs are ordered first by the dimension of the smaller simplex, and then by $A < B < C$, and then randomly. Let $J = \{0, ..., M\}$ be the indexing set for P. We define a sequence of inner anodyne maps

$$A_{n,i} = A_{n,i,0} \longrightarrow A_{n,i,1} \longrightarrow ... \longrightarrow A_{n,i,M+1} = A_{n,i+1}$$

such that $A_{n,i,j+1}$ is obtained from $A_{n,i,j}$ by attaching the jth pair $\tau \subset \tau'$ along an inner horn. For this to be valid, we need the other faces of τ' to already be in $A_{n,i,j}$. The ordering on E was chosen so that the outer faces of τ' are in $A_{n,i}$. The argument for the inner faces proceeds by cases:

- τ' is in class $A2$: The other inner faces are also in class $A2$ since they contain $x_0(y_0+1)$, hence were added at some earlier stage.
- τ' is in class $B2$: The other inner faces of $[x_0y_0, (x_0+1)y_0, (x_0+1)(y_0+1), x_2y_2, ..., x_ky_k]$ are all in class $B2$, except for $[x_0y_0, (x_0+1)(y_0+1), x_2y_2, ..., x_ky_k]$, which is in class $A1$. Both of these were added at an earlier stage.
- τ' is in class $C2$: The other inner faces are in class $C2$ or $B1$ since they contain $(x_0+1)y_0$, hence were added at some earlier stage.

$\square$

Let $E_n \subset (A_n)_1 \subset \mathscr{O}(\Delta^n)_1$ be the subset of edges $x_0y_0 \to x_1y_1$ where $y_0 = y_1$. Define simplicial sets C' and C'' to be the pullbacks

$$\begin{tikzcd} C'_\bullet \arrow[r] \arrow[d] & \mathrm{Hom}((\mathscr{O}(\Delta^\bullet), E_\bullet), \natural C) \arrow[d] \\ \mathrm{Hom}(\Delta^\bullet, S) \arrow[r, "\mathrm{ev}_0^*"] & \mathrm{Hom}(\mathscr{O}(\Delta^\bullet), S) \end{tikzcd} , \quad \begin{tikzcd} C''_\bullet \arrow[r] \arrow[d] & \mathrm{Hom}((A_\bullet, E_\bullet), \natural C) \arrow[d] \\ \mathrm{Hom}(\Delta^\bullet, S) \arrow[r, "\mathrm{ev}_0^*"] & \mathrm{Hom}(A_\bullet, S). \end{tikzcd}$$

We now show that the map $C' \longrightarrow C''$ induced by precomposition by $A_\bullet \longrightarrow \mathscr{O}(\Delta^\bullet)$ is a trivial Kan fibration. Indeed, in order to solve the lifting problem

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & C' \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & C'' \end{tikzcd}$$

we must supply a lift

$$\begin{tikzcd} A_n \bigcup_{\cup A_{n-1}} (\bigcup \mathscr{O}(\Delta^{n-1})) \arrow[r] \arrow[d] & C \arrow[d] \\ \mathscr{O}(\Delta^n) \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

and the left vertical map is a trivial cofibration by Lm. 12.12. Let $\sigma : C'' \longrightarrow C'$ be any section. Also let $\delta : C' \longrightarrow C$ be the map induced by precomposition by the identity section $\Delta^\bullet \longrightarrow \mathscr{O}(\Delta^\bullet)$.

Define a map $\upsilon_{C,S} : N^+_{\Delta^{op},S}(F_C) \longrightarrow C$ over S as follows: the data of an n-simplex of $N^+_{\Delta^{op},S}(F_C)$ consists of

- an n-simplex $\Delta^{a_0} \leftarrow ... \leftarrow \Delta^{a_n}$ in Δ^{op} (so we have maps $f_{ij} : \Delta^{a_j} \longrightarrow \Delta^{a_i}$ for $i \leq j$);
- an n-simplex $s_\bullet : \Delta^n \longrightarrow S$;
- a choice of a_0-simplex $\sigma_0 \in C_{a_0}$;
- for $0 \leq i \leq n$, a map $\gamma_i : \Delta^i \longrightarrow \underline{\sigma_i}$, where $\sigma_i = \sigma_0 \circ f_{0i}$

such that for all $0 \leq i \leq j \leq n$, the diagram

$$\begin{tikzcd} \Delta^i \arrow[r, "\gamma_i"] \arrow[d, "\{0,\dots,i\}\subset[j]"'] & \underline{\sigma_i} \arrow[d, "f^*_{ij}"] \\ \Delta^j \arrow[r, "\gamma_j"] \arrow[dr, "(s_\bullet)|_{\{0,\dots,j\}}"'] & \underline{\sigma_j} \arrow[d] \\ & S \end{tikzcd}$$

commutes. Let $\overline{\gamma_i} : \Delta^i \times \Delta^{a_i} \times \Delta^1 \longrightarrow C$ denote the adjoint map.

We now define a map $A_n \longrightarrow C$ to be that uniquely specified by sending for all $0 \leq k \leq n$ the rectangle $\Delta^k \times \Delta^{n-k} \subset A_n$ given by $00 \mapsto 0k$ and $k(n-k) \mapsto kn$ to

$$\Delta^k \times \Delta^{n-k} \xrightarrow{id\times(\lambda a)_k} \Delta^k \times \Delta^{a_k} \times \{1\} \xrightarrow{\overline{\gamma_i}|_{\{1\}}} C$$

where the maps $(\lambda a)_k$ are obtained from the first vertex section of $W \longrightarrow \Delta^{op}$ restricted to $a_\bullet$ as before. One may check that the composite $A_n \longrightarrow C \longrightarrow S$ factors as $A_n \longrightarrow \Delta^n \xrightarrow{s_\bullet} S$, so this defines a n-simplex of C''. This procedure is natural in n, so yields a map $N^+_{\Delta^{op},S}(F_C) \longrightarrow C''$. Finally, postcomposition by $\delta \circ \sigma : C'' \longrightarrow C$ define our desired map $\upsilon_{C,S}$. By Prp. 12.8, $N^+_{\Delta^{op},S}(F_C) \xrightarrow{\pi'} S$ is an S-category with an edge π'-cocartesian if and only if it is degenerate when projected to Δ^{op}. These edges are evidently sent to π-cocartesian edges in C, so υ_C is a S-functor.

12.13. **Theorem.** *The S-first vertex map $\upsilon_{C,S} : N^+_{\Delta^{op},S}(F_C) \longrightarrow C$ is fiberwise a weak homotopy equivalence. Moreover, $\upsilon_{C,S}$ is S-final if either $C \longrightarrow S$ is a left fibration, or S is equivalent to the nerve of a 1-category.*

Proof. Let $t \in S$ be an object and $i_t : \{t\} \longrightarrow S$ the inclusion. Then $N^+_{\Delta^{op},S}(F_C)_t \cong N^+_{\Delta^{op}}(i^*_t F_C)$. We have a map $N^+_{\Delta^{op}}(i^*_t F_C) \longrightarrow \Delta^{op}_{/C} \cong N^+_{\Delta^{op}}(C_\bullet)$ of left fibrations over Δ^{op} induced by the natural transformation $i^*_t F_C \longrightarrow C_\bullet$ which collapses each $\underline{\sigma} \times_S \{t\}$ to a point. Moreover, this natural transformation is objectwise a Kan fibration, so the map itself is a left fibration. Also define a map $N^+_{\Delta^{op}}(i^*_t F_C) \longrightarrow (S^{/t})^{op}$ as follows: in the above notation, the γ_0 map in the data of an n-simplex $(a_\bullet, \gamma_i : \Delta^i \longrightarrow \underline{\sigma_i} \times_S \{t\})$ yields a map $\pi\gamma_0 : \Delta^{a_0} \longrightarrow \mathscr{O}(S) \times_S \{t\} = S^{/t}$, and we send the n-simplex to

$$\Delta^n \xrightarrow{(\lambda a^{rev})_0} (\Delta^{a_0})^{op} \xrightarrow{(\pi\gamma_0)^{op}} (S^{/t})^{op}$$

where $a^{rev}_\bullet$ is $(\Delta^{a_0})^{op} \leftarrow \dots \leftarrow (\Delta^{a_n})^{op}$. Using these maps we obtain a commutative square

$$\begin{tikzcd} N^+_{\Delta^{op}}(i^*_t F_C) \arrow[r] \arrow[d] & C^{op} \times_{S^{op}} (S^{/t})^{op} \arrow[d] \\ \Delta^{op}_{/C} \arrow[r, "\mu^{op}_C"] & C^{op}. \end{tikzcd}$$

We claim that the map

$$\theta_{C,t} : N^+_{\Delta^{op}}(i^*_t F_C) \longrightarrow (\Delta^{op}_{/C}) \times_{C^{op}} (C \times_S S^{/t})^{op}$$

is a categorical equivalence. Since $\theta_{C,t}$ is a map of left fibrations over $\Delta^{op}_{/C}$, it suffices to check that for every object $\sigma \in \Delta^{op}_{/C}$, the map on fibers

$$\underline{\sigma} \times_S \{t\} \longrightarrow (S^{op})^{t/} \times_{S^{op}} \{\pi\sigma(n)\} \simeq \{\pi\sigma(n)\} \times_S S^{/t}$$

is a homotopy equivalence. But this is the pullback of the trivial Kan fibration of Lm. 12.10 over $\{t\}$.

We next define a map $N^+_{\Delta^{op}}(i_t^* F_C) \longrightarrow S^{/t}$ by sending $(a_\bullet, \gamma_i)$ to $\pi\gamma_0 \circ (\lambda a)_0$. Then the outer rectangle

$$\begin{tikzcd} N^+_{\Delta^{op}}(i_t^* F_C) \arrow[r, dotted, "\upsilon'_{C,t}"'] \arrow[d] & C \times_S S^{/t} \arrow[r] \arrow[d] & S^{/t} \arrow[d] \\ \Delta^{op}_{/C} \arrow[r, "\upsilon_C"] & C \arrow[r, "\pi"] & S \end{tikzcd}$$

commutes so we obtain the dotted map $\upsilon'_{C,t}$.

Next, we choose a section P of the trivial Kan fibration $\mathscr{O}^{cocart}(C) \longrightarrow C \times_S \mathscr{O}(S)$ which restricts to the identity section on C. P restricts to a map $P_t : C \times_S S^{/t} \longrightarrow \mathscr{O}^{cocart}(C) \times_S \{t\}$, and it is tedious but straightforward to construct a homotopy between the composition $(\mathrm{ev}_1 P_t) \circ \upsilon'_{C,t}$ and $(\upsilon_{C,S})_t$. Finally, we define a map $\upsilon''_{C,t} : \Delta^{op}_{/C \times_S S^{/t}} \longrightarrow N^+_{\Delta^{op}}(i_t^* F_C)$ as follows: given an n-simplex

$$\begin{tikzcd} \Delta^{a_0} \arrow[d, "\tau_0"] & \ldots \arrow[l] & \Delta^{a_n} \arrow[l] \arrow[dll, "\tau_n"] \\ C \times_S S^{/t} & & \end{tikzcd}$$

let $\sigma_i = \mathrm{pr}_C \circ \tau_i$, and define $\gamma_i : \Delta^i \longrightarrow \underline{\sigma_i} \times_S \{t\}$ as the composition of the projection to Δ^0 and the adjoint of the map $P_t \circ \tau_i$. Then $(a_\bullet, \gamma_i)$ assembles to yield an n-simplex of $N^+_{\Delta^{op}}(i_t^* F_C)$.

Unwinding the definitions of the various maps, we identify the composition $\upsilon'_{C,t} \circ \upsilon''_{C,t}$ as given by $\upsilon_{C \times_S S^{/t}}$, and the composition $\theta_{C,t} \circ \upsilon''_{C,t}$ as given by the map $\Delta^{op}_{/\mathrm{pr}_C}$ to the factor $\Delta^{op}_{/C}$ and the map $(\mu_{C \times_S S^{/t}})^{op}$ to the factor $(C \times_S S^{/t})^{op}$. By Prp. 12.2 and the fact that final maps pull back along cocartesian fibrations, we deduce that in

$$\begin{tikzcd} \Delta^{op}_{/C \times_S S^{/t}} \arrow[r] \arrow[rr, bend left] & \Delta^{op}_{/C} \times_{C^{op}} (C \times_S S^{/t})^{op} \arrow[r] & (C \times_S S^{/t})^{op} \end{tikzcd}$$

the long composition and the second map are both final. Consequently, $\theta_{C,t} \circ \upsilon''_{C,t}$ is a weak homotopy equivalence. Moreover, if S is equivalent to the nerve of a 1-category then $\theta_{C,t} \circ \upsilon''_{C,t}$ is a categorical equivalence, as may be verified by checking that the map is a fiberwise equivalence over $\Delta^{op}_{/C}$. Since $\theta_{C,t}$ is a categorical equivalence, $\upsilon''_{C,t}$ is then a weak homotopy equivalence resp. a categorical equivalence. Since $\upsilon_{C \times_S S^{/t}}$ is final, $\upsilon'_{C,t}$ is then a weak homotopy equivalence resp. final.

For the last step, let $j_t : C_t \longrightarrow C \times_S S^{/t}$ denote the inclusion. As the inclusion of the fiber over a final object into a cocartesian fibration, j_t is final. $(\mathrm{ev}_1 P_t) \circ j_t = id_{C_t}$, so by right cancellativity of final maps, $\mathrm{ev}_1 P_t$ is final. We conclude that $(\upsilon_{C,S})_t$ is a weak homotopy equivalence resp. final. In addition, if $C \longrightarrow S$ is a left fibration, $(\upsilon_{C,S})_t$ has target a Kan complex, so is final ([11, 2.3.4.6]). Invoking the S-cofinality Thm. 6.7, we conclude the proof. □

12.14. **Remark.** The above proof that the S-first vertex map $\upsilon_{C,S}$ is final in special cases hinges upon the finality of the map $\theta_{C,t} \circ \upsilon''_{C,t}$. We believe, but are unable to currently prove, that this map is always final.

We conclude this section with our main application to decomposing S-colimits.

12.15. **Corollary.** *Suppose that S^{op} admits multipullbacks. Then C is S-cocomplete if and only C admits all S-coproducts and geometric realizations.*

Proof. We prove the if direction, the only if direction being obvious. Let K be a $S^{s/}$-category and $p : K \longrightarrow C_{\underline{s}}$ a $S^{s/}$-diagram. First suppose that $K \longrightarrow S^{s/}$ is a left fibration. Consider the diagram

$$\begin{tikzcd} N^+_{\Delta^{op}, S^{s/}}(F_K) \arrow[r, "\upsilon_{K,S^{s/}}"] \arrow[d, "\rho"] & K \arrow[r, "p"] & C_{\underline{s}} \\ \Delta^{op} \times S^{s/}. & & \end{tikzcd}$$

By Thm. 12.13, the $S^{s/}$-colimit of p is equivalent to that of $p \circ \upsilon_{K,S^{s/}}$. Since ρ is S-cocartesian, by Thm. 9.15 the $S^{s/}$-left Kan extension of $p \circ \upsilon_{K,S^{s/}}$ along ρ exists provided that for all $n \in \Delta^{op}$ and $f : s \to t$, the $S^{t/}$-colimit exists for $(p \circ \upsilon_{K,S^{s/}})_{\underline{(n,f)}}$. To understand the domain of this map, note that because the pullback of ρ along $f^* : \Delta^{op} \times S^{t/} \longrightarrow \Delta^{op} \times S^{s/}$ is given by $N^+_{\Delta^{op},S^{t/}}(f^* F_K)$, the assumption that S^{op} admits multipullbacks ensures that the $\underline{(n,f)}$-fibers of ρ decompose as coproducts of representable left fibrations. Therefore, these colimits exist since C is assumed to admit S-coproducts. Now by transitivity of left $S^{s/}$-Kan extensions, the $S^{s/}$-colimit of $p \circ \upsilon_{K,S^{s/}}$ is equivalent to that of $\rho_!(p \circ \upsilon_{K,S^{s/}})$, and this exists since C is assumed to admit geometric realizations.

Now suppose that $K \longrightarrow S^{s/}$ is any cocartesian fibration. Consider the diagram

$$\begin{tikzcd}
\iota\widetilde{\mathrm{Fun}}_{\Delta^{op} \times S^{s/}}(W \times S^{s/}, \Delta^{op} \times K) \arrow[r, "\Upsilon'_{K,S^{s/}}"] \arrow[d, "\rho'"] & K \arrow[r, "p"] & C_{\underline{s}} \\
\Delta^{op} \times S^{s/}. & &
\end{tikzcd}$$

By Thm. 12.6, the $S^{s/}$-colimit of p is equivalent to that of $p \circ \Upsilon'_{K,S^{s/}}$. By Prp. 9.7, the $\underline{(n,f)}$-fiber of ρ' is equivalent to $\iota\underline{\mathrm{Fun}}_{S^{t/}}(\Delta^n \times S^{t/}, K \times_{S^{s/}} S^{t/})$, which in any case remains a left fibration. We just showed that for all $t \in S$, $C_{\underline{t}}$ admits $S^{t/}$-colimits indexed by left fibrations. We are thereby able to repeat the above proof in order to show that the $S^{s/}$-colimit of p exists. $\square$

13. Appendix: Fiberwise fibrant replacement

In this appendix, we formulate a result (Prp. 13.4) which will allow us to recognize a map as a cocartesian equivalence if it is a marked equivalence on the fibers. We begin by introducing a marked variant of Lurie's mapping simplex construction.

13.1. **Definition.** Suppose a functor $\phi : [n] \longrightarrow s\mathbf{Set}^+$, $A_0 \longrightarrow \dots \longrightarrow A_n$. Define $M(\phi)$ to be the simplicial set which is the opposite of the mapping simplex construction of [9, §3.2.2], so that a m-simplex of $M(\phi)$ is given by the data of a map $\alpha : \Delta^m \longrightarrow \Delta^n$ together with a map $\beta : \Delta^m \longrightarrow A_{\alpha(0)}$. Endow $M(\phi)$ with a marking by declaring an edge $e = (\alpha, \beta)$ of $M(\phi)$ to be marked if and only if β is a marked edge of $A_{\alpha(0)}$. Note that if each A_i is given the degenerate marking, then the marking on $M(\phi)$ is that of [9, 3.2.2.3].

13.2. **Lemma.** *Suppose $\eta : \phi \longrightarrow \psi$ is a natural transformation between functors $[n] \longrightarrow s\mathbf{Set}^+$ such that for all $0 \le i \le n$, $\eta_i : A_i \longrightarrow B_i$ is a cocartesian equivalence. Then $M(\eta) : M(\phi) \longrightarrow M(\psi)$ is a cocartesian equivalence in $s\mathbf{Set}^+_{/\Delta^n}$.*

Proof. Using the decomposition of $M(\phi)$ as the pushout $M(\phi') \cup_{A_0 \times \Delta^{n-1}} A_0 \times \Delta^n$ for $\phi' : A_1 \longrightarrow \dots \longrightarrow A_n$, this follows by an inductive argument in view of the left properness of $s\mathbf{Set}^+_{/\Delta^n}$. $\square$

13.3. **Construction.** Let $X \longrightarrow \Delta^n$ be a cocartesian fibration, let σ be a section of the trivial Kan fibration $\mathscr{O}^{cocart}(X) \longrightarrow X \times_{\Delta^n} \mathscr{O}(\Delta^n)$ which restricts to the identity section on X, and let $P = \mathrm{ev}_1 \circ \sigma$ be the corresponding choice of pushforward functor. For $0 \le i < n$, define $f_i : X_i \times \Delta^1 \longrightarrow X$ by $P \circ (id_{X_i} \times f'_i)$ where $f'_i : \Delta^1 \longrightarrow \mathscr{O}(\Delta^n)$ is the edge $(i = i) \longrightarrow (i \longrightarrow i+1)$, and let $\phi : X_0^\sim \longrightarrow \dots \longrightarrow X_n^\sim$ be the sequence obtained from the $f_i \times \{1\}$. We will explain how to produce a map $M(\phi) \longrightarrow X$ over Δ^n via an inductive procedure. Begin by defining the map $M(\phi)_n = X_n \longrightarrow X_n$ to be the identity. Proceeding, observe that $M(\phi)$ is the pushout

$$\begin{tikzcd}
X_0 \times \Delta^{\{1,\dots,n\}} \arrow[r] \arrow[d, "\gamma"] & X_0 \times \Delta^n \arrow[d] \\
M(\phi') \arrow[r] & M(\phi)
\end{tikzcd}$$

with ϕ' the composable sequence $X_1 \longrightarrow ... \longrightarrow X_n$ and the map γ given by $X_0 \times \Delta^{n-1} \longrightarrow X_1 \times \Delta^{n-1} \longrightarrow M(\phi')$. Given a map $g' : M(\phi') \longrightarrow X$ over Δ^{n-1}, we have a commutative square

$$\begin{tikzcd} X_0 \times \Delta^1 \cup_{X_0 \times \Delta^{\{1\}}} X_0 \times \Delta^{\{1,\dots,n\}} \arrow[r, "{(f_0, g' \circ \gamma)}"] \arrow[d] & X \arrow[d] \\ X_0 \times \Delta^n \arrow[r] \arrow[ur, dotted] & \Delta^n, \end{tikzcd}$$

and the left vertical map is inner anodyne by [9, 2.1.2.3] and [9, 2.3.2.4]. Thus a dotted lift exists and we may extend g' to $g : M(\phi) \longrightarrow X$.

Note that g_i is the identity for all $0 \leq i \leq n$. Therefore, if we instead take the marking on $M(\phi)$ which arises from the degenerate marking on the X_i, then g is (the opposite of) a quasi-equivalence in the terminology of [9, 3.2.2.6], hence a cocartesian equivalence in $s\mathbf{Set}^+_{/\Delta^n}$ by [9, 3.2.2.14]. Now by Lm. 13.2, g with the given marking is a cocartesian equivalence.

This construction of $M(\phi) \longrightarrow X$ enjoys a convenient functoriality property: given a cofibration $F : X \longrightarrow Y$ between cocartesian fibrations over Δ^n, we may first choose σ_X as above, and then define σ_Y to be a lift in the diagram

$$\begin{tikzcd} (X \times_{\Delta^n} \mathscr{O}(\Delta^n)) \cup_X Y \arrow[r, "{(F \circ \sigma_X, \iota)}"] \arrow[d] & \mathscr{O}^{cocart}(Y) \arrow[d, "\sim"'] \\ Y \times_{\Delta^n} \mathscr{O}(\Delta^n) \arrow[r, "="] \arrow[ur, dotted, "\sigma_Y"] & Y \times_{\Delta^n} \mathscr{O}(\Delta^n). \end{tikzcd}$$

Consequently, we obtain compatible pushforward functors and a natural transformation $\eta : \phi_X \longrightarrow \phi_Y$, which yields, by a similar argument, a commutative square

$$\begin{tikzcd} M(\phi_X) \arrow[r, "M(\eta)"] \arrow[d] & M(\phi_Y) \arrow[d] \\ X \arrow[r, "F"] & Y. \end{tikzcd}$$

where the vertical maps are cocartesian equivalences in $s\mathbf{Set}^+_{/\Delta^n}$.

13.4. **Proposition.** *Let $p : X \longrightarrow S$ and $q : Y \longrightarrow S$ be cocartesian fibrations over S and let $F : X \longrightarrow Y$ be a S-functor. Suppose collections of edges $\mathscr{E}_X$, $\mathscr{E}_Y$ of X, Y such that*

1. *$\mathscr{E}_X$ resp. $\mathscr{E}_Y$ contains the p resp. q-cocartesian edges;*
2. *For $\mathscr{E}^0_X \subset \mathscr{E}_X$ the subset of edges which are either p-cocartesian or lie in a fiber, we have that $(X, \mathscr{E}^0_X) \subset (X, \mathscr{E}_X)$ is a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$, and ditto for Y;*
3. *$F(\mathscr{E}_X) \subset \mathscr{E}_Y$;*
4. *For all $s \in S$, $F_s : (X_s, (\mathscr{E}_X)_s) \longrightarrow (Y_s, (\mathscr{E}_Y)_s)$ is a cocartesian equivalence in $s\mathbf{Set}^+$.*

Let $X' = (X, \mathscr{E}_X)$, $Y' = (Y, \mathscr{E}_Y)$, and $F' : X' \longrightarrow Y'$ be the map given on underlying simplicial sets by F. Then for all simplicial sets U and maps $U \longrightarrow S$, F'_U is a cocartesian equivalence in $s\mathbf{Set}^+_{/U}$.

Proof. Without loss of generality, we may assume that an edge e is in $\mathscr{E}_X$ if and only if either e is p-cocartesian or $p(e)$ is degenerate, and ditto for $\mathscr{E}_Y$. First suppose that F is a trivial fibration in $s\mathbf{Set}^+_{/S}$ and for all $s \in S$, F'_s reflects marked edges. Then F' is again a trivial fibration because F' has the right lifting property against all cofibrations. For the general case, factor F as $X \xrightarrow{G} Z \xrightarrow{H} Y$ where G is a cofibration and H is a trivial fibration, and let $Z' = (Z, \mathscr{E}_Z)$ for $\mathscr{E}_Z$ the collection of edges e where e is in $\mathscr{E}_Z$ if and only if $H(e)$ is in $\mathscr{E}_Y$. Then for all $s \in S$, $Z'_s \longrightarrow Y'_s$ is a trivial fibration in $s\mathbf{Set}^+$, so as we just showed $H' : Z' \longrightarrow Y'$ is a trivial fibration. We thereby reduce to the case that F is a cofibration.

Let $\mathscr{U}$ denote the collection of simplicial sets U such that for every map $U \longrightarrow S$, F'_U is a cocartesian equivalence in $s\mathbf{Set}^+_{/U}$. We need to prove that every simplicial set belongs to $\mathscr{U}$. For this, we will verify the hypotheses of [9, 2.2.3.5]. Conditions (i) and (ii) are obvious, condition (iv) follows from left properness of the cocartesian model structure and [11, B.2.9], and condition (v) follows from the stability of cocartesian equivalences under filtered colimits and [11, B.2.9]. It remains to check that every n-simplex belongs to $\mathscr{U}$, so suppose $S = \Delta^n$. Let

$$\begin{tikzcd} M(\phi_X) \arrow[r, "M(\eta)"] \arrow[d] & M(\phi_Y) \arrow[d] \\ X \arrow[r, "F"] & Y \end{tikzcd}$$

be as in Cnstr. 13.3. Let ϕ'_X be the sequence $X'_0 \longrightarrow ... \longrightarrow X'_n$, where the maps are the same as in ϕ_X, and similarly define ϕ'_Y and η'. Then we have pushout squares

$$\begin{tikzcd} M(\phi_X) \arrow[r] \arrow[d] & M(\phi'_X) \arrow[d] \\ X \arrow[r] & X'' \end{tikzcd}, \quad \begin{tikzcd} M(\phi_Y) \arrow[r] \arrow[d] & M(\phi'_Y) \arrow[d] \\ Y \arrow[r] & Y'' \end{tikzcd}$$

with all four vertical maps cocartesian equivalences in $s\mathbf{Set}^+_{/\Delta^n}$. Here we replace X' by X'', which has the same underlying simplicial set X but more edges marked with $X' \subset X''$ left marked anodyne, so that the vertical maps $M(\phi'_X) \longrightarrow X''$ are defined and the squares are pushout squares (again, ditto for Y''). Note that F defines a map $F'' : X'' \longrightarrow Y''$.

Finally, we have the commutative square

$$\begin{tikzcd} M(\phi'_X) \arrow[r, "M(\eta')"] \arrow[d] & M(\phi'_Y) \arrow[d] \\ X'' \arrow[r, "F''"] & Y''. \end{tikzcd}$$

By assumption, $\eta' : \phi'_X \longrightarrow \phi'_Y$ is a natural transformation through cocartesian equivalences in $s\mathbf{Set}^+$. By Lm. 13.2, $M(\eta')$ is a cocartesian equivalence in $s\mathbf{Set}^+_{/\Delta^n}$. We deduce that F'', hence F', is as well. □

13.5. **Remark.** By a simple modification of the above arguments, we may further prove that for any marked simplicial set $A \longrightarrow S$, F'_A is a cocartesian equivalence in $s\mathbf{Set}^+_{/A}$. We leave the details of this to the reader.

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Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139-4307, USA
E-mail address: jshah@math.mit.edu