CLASSIFYING SPACES AND SPECTRAL SEQUENCES

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The following work makes no great claim to originality. The first three sections are devoted to a very general discussion of the representation of categories by topological spaces, and all the ideas are implicit in the work of Grothendieck. But I think the essential simplicity of the situation has never been made quite explicit, and I hope the present popularization will be of some interest.

Apart from this my purpose is to obtain for a generalized cohomology theory k^* a spectral sequence connecting $k^*(X)$ with the ordinary cohomology of X. This has been done in the past [1], when X is a CW-complex, by considering the filtration of X by its skeletons. I give a construction which makes no such assumption on X: the interest of this is that it works also in the case of an equivariant cohomology theory defined on a category of G-spaces, where G is a fixed topological group. But I have not discussed that application here, and I refer the reader to [13]. On the other hand I have explained in detail the context into which the construction fits, and its relation to other spectral sequences obtained in [8] and [12] connected with the bar-construction.

§ 1. SEMI-SIMPLICIAL OBJECTS

A semi-simplicial set is a sequence of sets $A_0, A_1, A_2, \ldots$ together with boundary- and degeneracy-maps which satisfy certain well-known conditions [5]. But it is better regarded as a contravariant functor A from the category *Ord* of finite totally ordered sets to the category of sets. Thus, if $\mathbf{n}$ denotes the ordered set $\{0, 1, \ldots, n\}$, we have $A(\mathbf{n}) = A_n$. The two boundary-maps $A_1 \to A_0$ are induced by the two maps $\mathbf{0} \to \mathbf{1}$, and so on.

More generally, if C is any category, a semi-simplicial object of C is a sequence of objects $A_0, A_1, \ldots$ of C together with various maps; alternatively, it is a contravariant functor $A : Ord \to C$.

A semi-simplicial set A has a realization $\Delta(A)$ as a topological space [9]. If, for a finite set S, $\Delta(S)$ denotes the standard simplex with S as set of vertices, then $\Delta(A)$ is obtained from the topological sum of all $\Delta(S) \times A(S)$, for all finite ordered sets S, by identifying $(x, \theta^* a) \in \Delta(S) \times A(S)$ with $(\theta_* x, a) \in \Delta(T) \times A(T)$ for all $x \in \Delta(S)$, $a \in A(T)$, and $\theta : S \to T$ in *Ord*. ($S \mapsto \Delta(S)$, $S \mapsto A(S)$ are covariant and contravariant functors, respectively. I have written $\theta_* = \Delta(\theta)$ and $\theta^* = A(\theta)$.)

The product of two semi-simplicial sets A and B is defined by

$$(A\times B)(S)=A(S)\times B(S).$$

The natural map $\Delta(A\times B)\to\Delta(A)\times\Delta(B)$ is a bijection, and is a homeomorphism if the product on the right is formed in the category of *compactly generated spaces* or *k-spaces* ([7], p. 230; [14], p. 47).

Now the realization process makes sense also when applied to semi-simplicial spaces instead of sets; in particular it takes semi-simplicial k-spaces to k-spaces, and commutes with products in the latter category, as it is not difficult to verify.

§ 2. CATEGORIES AND CLASSIFYING SPACES

To a category C one can associate a semi-simplicial set NC, which one might call the *nerve* of C, by taking the objects of C as vertices, the morphisms as 1-simplexes, the triangular commutative diagrams as 2-simplexes, and so on. More formally, the definition is as follows. An ordered set S can be regarded as a category with S as set of objects and with just one morphism from x to y whenever $x\leqslant y$. Then define $NC(S)=\mathrm{Funct}(S;C)$, the set of functors $S\to C$.

The semi-simplicial set NC obviously determines C; Grothendieck has pointed out [6] that a category can be defined as a semi-simplicial set A with the property that the natural map $A(S_1\amalg_{S_0}S_2)\to A(S_1)\times_{A(S_0)}A(S_2)$ is an isomorphism whenever the amalgamated sum on the left exists.

I shall write BC for the realization of NC, and shall call it the *classifying space* of C.

More generally, let me define *a topological category* as a category in which the set of objects and the set of morphisms have topologies for which the four structural maps are continuous. If C is a topological category then NC is a semi-simplicial space, and I define BC as its realization, just as before.

The functor $C\mapsto NC$ obviously commutes with products; if one replaces the category of topological spaces by the category of k-spaces throughout, as I shall do tacitly from now on, then B too commutes with products. This has the following interesting consequence.

Proposition **(2.1)**. — *If* C, C' *are topological categories and* $F_0, F_1 : C\to C'$ *are continuous functors, and* $F : F_0\to F_1$ *is a morphism of functors, then the induced maps* $BF_0, BF_1 : BC\to BC'$ *are homotopic.*

Proof. — F can be regarded as a functor $C\times J\to C'$, where J is the ordered set $\{0, 1\}$ regarded as a category. So F induces $BF : B(C\times J)\to BC'$. But $B(C\times J)\cong BC\times BJ$, and BJ is the unit interval I, so BF is a homotopy between BF_0 and BF_1.

Remark. — Because BJ is compact this proposition is true either for topological spaces or for k-spaces.

§ 3. THE CLASSIFYING SPACE OF A TOPOLOGICAL GROUP

Let G be a topological group. It can be identified with a topological category with $\mathrm{ob}(G) = \text{point}$, $\mathrm{mor}(G) = G$. Its semi-simplicial space NG is given by $NG_k = G^k = G \times \ldots \times G$ (k times).

The space BG if often a classifying space for G in the usual sense, as one can see as follows. Consider the category $\overline{G}$ with $\mathrm{ob}(\overline{G}) = G$ and with a unique isomorphism between each pair of elements of G, i.e. $\mathrm{mor}(\overline{G}) = G \times G$. It is equivalent to the trivial category with one object and one morphism, so $B\overline{G}$ is contractible by (2.1). There is a functor $\overline{G} \to G$ which takes the morphism (g_1, g_2) to $g_1^{-1}g_2$, and it induces a map $B\overline{G} \to BG$. Now $N\overline{G}$ is $(G, G \times G, \ldots)$, a semi-simplicial G-space on which G acts freely, so $B\overline{G}$ is a free G-space. We have $B\overline{G}/G \xrightarrow{\cong} BG$, because $N\overline{G}/G \xrightarrow{\cong} NG$ and quotient formations commute among themselves. (If one allows that Δ commutes with fibre products it is immediate that $G \times B\overline{G} \xrightarrow{\cong} B\overline{G} \times_{BG} B\overline{G}$, so G acts freely on $B\overline{G}$ in the strong sense.)

The only thing wrong with the fibration $B\overline{G} \to BG$ is that it may not be locally trivial. If G itself is locally well-behaved (to be precise, if any map of a closed subset F of $\Delta^n \times G \times \ldots \times G$ into G can be extended to a neighbourhood of F in $\Delta^n \times G \times \ldots \times G$; which is true, for example, if G is an absolute neighbourhood retract) one can construct local sections by induction on the " skeletons " of BG, so the fibration is locally trivial; and it can be argued that in the converse case local triviality is not an appropriate concept. But to see the point of the matter one should compare BG with the space $\mathscr{B}G = (G * G * \ldots)/G$ defined by Milnor ([8]; $*$ denotes join). The principal G-bundle on $\mathscr{B}G$ is obviously locally trivial. One can obtain BG from $\mathscr{B}G$ by collapsing degenerate simplexes, i.e. those joining elements $g_1, \ldots, g_k$ of G with two g_i equal; thus it is related to $\mathscr{B}G$ in precisely the way that reduced suspensions are related to suspensions. But $\mathscr{B}G$ fits into my framework, too. If C is a topological category, let $C_{\mathbf{N}}$ be the associated category unravelled over the ordered set $\mathbf{N}$ of natural numbers as follows: $C_{\mathbf{N}}$ is the subcategory of $\mathbf{N} \times C$ obtained by deleting all morphisms of the form $(n, c) \to (n, c')$ except identity-morphisms. Then $\mathscr{B}G = BG_{\mathbf{N}}$, and Milnor's universal space is $B\overline{G}_{\mathbf{N}}$. The contractibility of $B\overline{G}_{\mathbf{N}}$ follows from (2.1), and looking at it from that point of view illuminates the contraction given by Dold [3]. One advantage of BG over $\mathscr{B}G$ is that $B(G \times G') \xrightarrow{\cong} BG \times BG'$, whereas for Milnor's spaces, since $(G \times G')_{\mathbf{N}} \cong (G_{\mathbf{N}}) \times_{\mathbf{N}} (G'_{\mathbf{N}})$, one has $\mathscr{B}(B \times G') \xrightarrow{\cong} \mathscr{B}G \times_{\Delta} \mathscr{B}G'$, where $\Delta = B\mathbf{N}$ is the infinite simplex.

As a further illustration of Proposition (2.1) I might mention that a conjugation in G induces a map of BG or of $\mathscr{B}G$ which is homotopic to the identity; for as functor the conjugation is equivalent to the identity.

Finally, if C is the category of homogeneous G-spaces, then BC is the classifying space for G-spaces introduced by Palais [11], or, more precisely, it differs from it in the same way that BG differs from $\mathscr{B}$G. (The category G is equivalent to the category of *principal* homogeneous G-spaces.)

§ 4. THE SPACE ASSOCIATED TO A COVERING

Let X be a space, and $U=\{U_\alpha\}_{\alpha\in\Sigma}$ be a covering of X by subsets. If σ is a subset of Σ define $U_\sigma=\bigcap_{\alpha\in\sigma} U_\alpha$. If R_U is the category whose objects are the non-empty U_σ for finite subsets σ of Σ, and whose morphisms are their inclusions, then NR_U is the barycentric subdivision of what is ordinarily called the *nerve* of U. (Observe that a " simplicial complex " does not define a semi-simplicial set until one orders the vertices, but its subdivision has a natural ordering.)

There is also another category X_U associated to U. It is a topological category whose objects are the pairs (x, U_σ) with $x\in U_\sigma$, and whose morphisms $(x, U_\sigma)\to(y, U_\tau)$ are inclusions $i: U_\sigma\to U_\tau$ such that $i(x)=y$. I.e. $\mathrm{ob}(X_U)=\coprod_\sigma U_\sigma$, the sum being over all finite subsets of Σ, and $\mathrm{mor}(X_U)=\coprod_{\sigma\subset\tau} U_\tau$, with the sum over all pairs of finite subsets $\sigma\subset\tau$ of Σ. This category X_U occurs in some places in nature. For example, if G is a topological group, to define a principal G-bundle P on X which is locally trivial with respect to the covering U one can prescribe transition functions which amount in fact to a functor $X_U\to G$, where G is regarded as a category as in § 3. Two functors define the same P if and only if they are equivalent [1]. So to P is associated a map $BX_U\to BG$ determined up to homotopy. Now we shall see in a moment that if the covering U is numerable [3] there is a natural homotopy-equivalence between X and BX_U, so one has a map $X\to BG$ determined up to homotopy. This is a classifying map for P in the usual sense, for if $\overline{U}$ is the covering of P induced by U then a trivialization of P with respect to U is a functor $P_{\overline{U}}\to\overline{G}$ which induces a map $P\to B\overline{G}$ covering $X\to BG$. Furthermore the functors $X_U\to G$, $P_{\overline{U}}\to\overline{G}$ factorize through $G_{\mathbf{N}}$, $\overline{G}_{\mathbf{N}}$; so they induce maps into Milnor's spaces too.

Let us look at the space BX_U more closely. The obvious functor $X_U\to R_U$ induces a map $BX_U\to BR_U$, and the inverse image of a point in the interior of the simplex $[\sigma_0\subset\ldots\subset\sigma_p]$ of BR_U is U_{σ_p}. In fact BX_U can be identified with $\bigcup[\sigma_0\subset\ldots\subset\sigma_p]\times U_{\sigma_p}\subset BR_U\times X$, the last space being the classifying space of the category formed like X_U but with all the U_α replaced by X. (But the topology of BX_U may be finer than that induced from $BR_U\times X$.)

Proposition (4.1). — *The projection* $\mathrm{pr}: BR_U\to X$ *is a homotopy-equivalence if* U *is numerable.*

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(1) Thus the set called $H^1(U; G)$ is the set of equivalence classes of functors $X_U\to G$, just as, if Γ is a group, $H^1(\Gamma; G)$ is the set of conjugacy classes of homomorphisms $\Gamma\to G$.

Proof. — A locally finite partition of unity $\{\varphi_\alpha\}$ subordinate to U defines a map $\varphi : X \to BR_U$. The product $\varphi \times id : X \to BR_U \times X$ factorizes through a map $\psi : X \to BX_U$. For, if $x \in X$, let $\sigma = \{\alpha \in \Sigma : \varphi_\alpha(x) \neq 0\}$. Then $x \in U_\sigma$, and $\varphi(x)$ is in some open simplex $[\sigma_0 \subset \ldots \subset \sigma_p]$ of BR_U with $\sigma_p = \sigma$. To see that $\psi : X \to BX_U$ is continuous it suffices, by localization and functoriality, to consider the case of a finite closed covering, in which case BX_U has the product topology. The map ψ is a homotopy-inverse to $pr : BX_U \to X$, for $pr \circ \psi = id$, while $\psi \circ pr$ can be joined to the identity by a linear homotopy.

§ 5. SPECTRAL SEQUENCES

In this section $k^* = \{k^q\}_{q \in \mathbf{Z}}$ will be a generalized cohomology theory defined on a category of pairs (X, A), where X is a topological space and A is a closed subspace. I shall assume k^* has the following properties:

(i) It is a contravariant ∂-functor in the sense of [4].

(ii) If $f_0 \simeq f_1 : (X, A) \to (Y, B)$, then $k^*(f_0) = k^*(f_1)$.

(iii) If $f : (X, A) \to (Y, B)$ is a relative homeomorphism, in the sense that it induces a homeomorphism $X/A \to Y/B$, then $k^*(f)$ is an isomorphism.

(iv) $k^*(\coprod_\alpha X_\alpha) \xrightarrow{\cong} \prod_\alpha k^*(X_\alpha)$ for any family of spaces $\{X_\alpha\}$, where $\coprod$ denotes the topological sum [10].

If A is a semi-simplicial space its realization ΔA has a natural filtration $\Delta^0 A \subset \Delta^1 A \subset \ldots \subset \Delta A$, where $\Delta^p A$ is the image of $\Delta^p \times A_p$ in A. (In fact $\Delta^p A$ is a quotient space of $\Delta^p \times A_p$.) This filtration leads to a spectral sequence.

Proposition (5.1). — *To a semi-simplicial space* A *is associated naturally a spectral sequence whose termination is* $k^*(\Delta A)$, *with* $E_2^{pq} = H^p(k^q(A))$, *the p-th cohomology group of the semi-simplicial cochain complex* $k^q(A)$.

Proof. — The filtration leads by the method of [2], p. 333 to a spectral sequence with $E_1^{pq} = k^{p+q}(\Delta^p A, \Delta^{p-1} A)$. There is a relative homeomorphism

$$(\Delta^p \times A_p, (\Delta^p \times A_p^d) \cup (\dot{\Delta}^p \times A_p)) \to (\Delta^p A, \Delta^{p-1} A),$$

where $\dot{\Delta}^p$ is the $(p-1)$-skeleton of the simplex Δ^p, and A_p^d is the degenerate part of A_p (the union of the images of all the maps $A_r \to A_p$ with $r < p$). Thus the pair $(\Delta^p A, \Delta^{p-1} A)$ can be identified with the p-fold suspension of (A_p, A_p^d), and accordingly $E_1^{pq} \cong k^q(A_p, A_p^d)$. I shall show that the natural map $E_1^{pq} \to k^q(A_p)$ is compatible with the differential of the cochain complex $k^q(A)$. When that is done it follows that E_2^{pq} can be calculated from $k^q(A)$, for $k^q(A_p, A_p^d)$ is a direct summand in $k^q(A_p)$ complementary to the subgroup of degenerate cochains: indeed $k^q(A(S)) \cong \bigoplus_T k^q(A(T), A^d(T))$, where T runs through the quotients of S.

The compatibility of differentials follows from the commutativity of the diagram

$$\begin{tikzcd}
 & k^*(\Delta^p A, \Delta^{p-1}A) \arrow[r, "d"] \arrow[dl] \arrow[dr] & k^*(\Delta^{p+1}A, \Delta^p A) \arrow[dr] & \\
k^*(\Delta^p \times A_p, \dot{\Delta}^p \times A_p) \arrow[r, "\theta"] & \coprod_p k^*(\Delta^p \times A_{p+1}, \dot{\Delta}^p \times A_{p+1}) & k^*(\dot{\Delta}^{p+1} \times A_{p+1}, \ddot{\Delta}^{p+1} \times A_{p+1}) \arrow[l, "\cong"'] \arrow[r, "d"] & k^*(\Delta^{p+1} \times A_{p+1}, \dot{\Delta}^{p+1} \times A_{p+1}) \\
k^*(A_p) \arrow[u, "E^p"', "\cong"] \arrow[r, "\theta"] & \coprod_p k^*(A_{p+1}) \arrow[u, "E^p"', "\cong"] \arrow[rr, "\Sigma"] & & k^*(A_{p+1}) \arrow[u, "E^{p+1}"', "\cong"]
\end{tikzcd}$$

where: $\dot{\Delta}^p$ (resp. $\ddot{\Delta}^p$) means the $(p-1)$-skeleton (resp. $(p-2)$-skeleton) of Δ^p,
the maps θ are induced by the $p+2$ injections $\mathbf{p} \to \mathbf{p+1}$,
E^p denotes p-fold suspension, and
Σ denotes summation with alternating signs, so that the bottom line of the diagram is the differential of the semi-simplicial cochain complex.

Proposition (5.1) applies in particular to the classifying space for a topological category C. (I shall write $B^p C$ for $\Delta^p NC$.) For instance when G is a topological group we have a spectral sequence beginning with the semi-simplicial cochain complex

$$k^*(\text{point}) \to k^*(G) \to k^*(G \times G) \to \ldots,$$

and ending with $k^*(BG)$. This has been used by Milnor, Moore, Steenrod, Rothenberg, etc. [8], [12].

If the category C is discrete, and k^* is ordinary cohomology, the spectral sequence collapses (for $E_1^{pq} = 0$ unless $q = 0$), and the cohomology of BC — which one might well call the cohomology of C — can be calculated from the complex $H^*(NC)$. In the case of a group this is the bar-construction.

The case of the category associated in § 4 to a covering $U = \{U_\alpha\}$ of a space X is interesting and less well-known. Then the E_2-term is the cohomology of the nerve of the covering with coefficients in the system $\sigma \mapsto k^*(U_\sigma)$. The termination is $k^*(BX_U)$. But if U is numerable we have seen that the natural map $k^*(X) \to k^*(BX_U)$ is an isomorphism. The resulting spectral sequence $H^p(U; k^q) \Rightarrow k^*(X)$ is the Leray spectral sequence of the covering U in the theory k^*.

One can prove that $k^*(X) \xrightarrow{\cong} k^*(BX_U)$ in some other cases too, for example when the covering U is finite-dimensional and closed. Let X_k be the part of X contained in at least $k+1$ sets of U, and let $B_k = pr^{-1}X_k$. Then $B_k - B_{k+1} = (X_k - X_{k+1}) \times \Delta^k$, whence

$$k^*(X_k, X_{k+1}) \xrightarrow{\cong} k^*(B_k, B_{k+1}),$$

both being $\prod_{\dim \sigma = k} k^*(U_\sigma, U_\sigma \cap X_{k+1})$; so $k^*(X, X_k) \xrightarrow{\cong} k^*(B, B_k)$ for all k.

In the case of a covering by two sets $X = X_1 \cup X_2$ the Leray spectral sequence reduces to the Mayer-Vietoris sequence. Then $BX_U = (X_1 \times 0) \cup (X_{12} \times I) \cup (X_2 \times 1) \subset X \times I$, and the Mayer-Vietoris sequence is the exact sequence for the pair

$$(BX_U, B^0X_U) = (X_1 \times 0) \cup (X_2 \times 1).$$

This way of obtaining the sequence, unlike the hexagonal argument of Eilenberg and Steenrod, depends on the homotopy axiom for k^*. It would be interesting to generalize the hexagonal argument to obtain the Leray spectral sequence for a finite covering without using the homotopy axiom.

From the spectral sequence for a covering one can obtain the spectral sequence for a map. This reduces, when the map is the identity, to the spectral sequence mentioned in the introduction linking k^* to ordinary cohomology.

Proposition **(5.2)**. — *If* X *is a paracompact space, and* $f: Y \to X$ *is a continuous map, there is a spectral sequence with termination* $k^*(Y)$ *and with* $E_2^{pq} = H^p(X; k^q f)$, *where* $k^q f$ *is the sheaf associated to the presheaf* $U \mapsto k^q(f^{-1}U)$ *on* X.

Proof. — If U is an open covering of X, form the spectral sequence E(U) for the numerable covering $f^{-1}U$ of Y. This terminates with $k^*(Y)$ and begins with the Čech cohomology of the covering U with coefficients in the presheaf $V \mapsto k^q(f^{-1}V)$. The desired spectral sequence is obtained by taking the direct limit of the family $\{E(U)\}$ indexed by all the open coverings of X. Notice that, if V, W are two coverings of Y, and V refines W, there are evident continuous functors $Y_V \to Y_W$; and if F_0, F_1 are two such functors one can find a third, F, with morphisms $F \to F_0$, $F \to F_1$. So $BF_0 \simeq BF_1 : BY_V \to BY_W$. The homotopy BF_t between them does not preserve the filtration, but $BF_t(B^pY_V) \subset B^{p+1}Y_W$ for all t, so the two morphisms of spectral sequences coincide from the E_2-term onwards ([2], p. 336).

As a final application of the method I shall mention the filtration of $k^*(X)$. I recall [1] that when X is a CW-complex it is customary to define

$$k_p^*(X) = \ker\,(k^*(X) \to k^*(X^{p-1})),$$

where X^{p-1} is the $(p-1)$-skeleton of X. For a general space X I propose to define $\xi \in k_p^*(X)$ if $\xi \in \ker\,(k^*(X) \to k^*(B^{p-1}X_U))$ for some numerable covering U of X. This coincides with the former definition when X is a finite simplicial complex (for if U is the star-covering of X then $B^pX_U \simeq X^p$; but any covering can be refined by the star-covering of a barycentric subdivision). If k^* is a multiplicative theory, i.e. if there is a functorial product $k^*(X, A) \otimes k^*(X, B) \to k^*(X, A \cup B)$, then

Proposition (5.3). — $k^*(X)$ *is a filtered ring*, i.e. $k^*_p(X).k^*_q(X) \subset k^*_{p+q}(X)$.

Proof. — Suppose $\xi \in \ker(k^*(X) \to k^*(B^{p-1}X_U))$ and $\eta \in \ker(k^*(X) \to k^*(B^{q-1}X_V))$. Then I assert $\xi.\eta \in \ker(k^*(X) \to k^*(B^{p+q-1}X_{U\cap V}))$. One can assume $U = V$, and I shall write $B = BX_U$, $B^p = B^pX_U$. Let $\bar\xi$, $\bar\eta$ be the images of ξ, η in $k^*(B)$. Then $\bar\xi$ comes from $k^*(B, B^{p-1})$, and $\bar\eta$ from $k^*(B, B^{q-1})$. Hence $\bar\xi.\bar\eta$ comes by the diagonal map from $k^*(B\times B, (B\times B^{q-1}) \cup (B^{p-1}\times B))$, and it suffices to show that its image in $k^*(B^{p+q-1})$ is zero. That is a consequence of the following lemma.

Lemma (5.4). — *If* A *is a semi-simplicial space let us give* $\Delta A \times \Delta A$ *the product filtration* $(\Delta A \times \Delta A)^n = \bigcup_{p+q=n} (\Delta^p A \times \Delta^q A)$. *Then the diagonal map* $\Delta A \to \Delta A \times \Delta A$ *is homotopic to a filtration-preserving map.*

Proof. — I shall produce two deformations of the identity-map of ΔA. Let us regard an n-simplex $\Delta(S)$ as the subspace $\{t : 0 = t_0 \leqslant t_1 \leqslant \ldots \leqslant t_n \leqslant 1\}$ of $\mathbf{R}^S$. Then define $h_S : \Delta(S) \to \Delta(S)$ by $h_S(t)_i = \inf(2t_i, 1)$. h_S depends functorially on S, so it induces a map $h_A : \Delta A \to \Delta A$ for any semi-simplicial space A. The map h_S, and therefore also h_A, is linearly homotopic to the identity. Similarly, define $h'_S : \Delta(S) \to \Delta(S)$ by $h'_S(t)_i = \sup(0, 2t_i - 1)$. This leads to $h'_A : \Delta A \to \Delta A$. The product

$$h_{\mathbf{n}} \times h'_{\mathbf{n}} : \Delta(\mathbf{n}) \to \Delta(\mathbf{n}) \times \Delta(\mathbf{n})$$

is filtration-preserving, in fact $(h_{\mathbf{n}}(t), h'_{\mathbf{n}}(t)) \in \Delta(\{0, 1, \ldots, p\}) \times \Delta(\{p, p+1, \ldots, n\})$, where $p = \inf\left\{ i : t_i \geqslant \frac{1}{2} \right\}$. Hence $h_A \times h'_A : \Delta A \to \Delta A \times \Delta A$ is filtration-preserving for any A, and is the required deformation of the diagonal.

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