Algebraic Topology: Condensed Anima

?

Homotopy Type.

Recall simplex category

$\Delta = \{$ nonempty finite totally ordered sets, nondecreasing maps $\}$

$\cup$

$[n] = \{0, \dots, n\} \quad n \geq 0.$

$[n] \cong$ category $0 \to 1 \to 2 \to \dots \to n.$

$\mathcal{C}$ category $\rightsquigarrow$ nerve

$N(\mathcal{C}) : \Delta^{op} \to \mathrm{Sets} : [n] \longmapsto \mathrm{Fun}([n], \mathcal{C})$

simplicial set. (simplicial objects in $\mathcal{D}$ $= \mathrm{Fun}(\Delta^{op}, \mathcal{D})$.)

$\rightsquigarrow$ fully faithful functor from the 1-category of categories to simplicial sets, with essential image those simpl. sets $S_\bullet$ s.th.

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & S_\bullet \\ \Delta^n \arrow[ur, dashed, "\exists!"] \end{tikzcd} \qquad 0 < i < n.$$

$\Lambda^n_i \subseteq \Delta^n$ simpl. set. repr. by $[n]$.

$\Lambda^n_i = \Delta^n \setminus$ interior, face opposite to i.

$$\begin{tikzcd} 0 \arrow[dr] & & 2 \\ & 1 \arrow[ur] & \end{tikzcd} \subseteq \begin{tikzcd} 0 \arrow[rr] \arrow[dr] & & 2 \\ & 1 \arrow[ur] & \end{tikzcd}$$

$$\Lambda^2_1 \subseteq \Delta^2.$$

<u>Definition</u>. An <u>∞-category</u> ("$(\infty,1)$-cat.") is a simplicial set $\mathcal{C}$ s.th. for all $0<i<n$,

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & \mathcal{C} \\ \Delta^n \arrow[ur, dotted, "\exists"] & \end{tikzcd}$$

Equivalently:

$$\begin{tikzcd} S' \arrow[r] & \mathrm{Map}(\Delta^2, \mathcal{C}) \arrow[d] \\ S \arrow[u, phantom, "\cap|"] \arrow[ur, dotted, "\exists"] \arrow[r] & \mathrm{Map}(\Lambda^2_1, \mathcal{C}) \end{tikzcd}$$

any inclusion of simpl. sets

"trivial Kan fibration"

<u>Idea</u>. $\mathcal{C}_0$ = "objects of $\mathcal{C}$"

$\mathcal{C}_1$ = "pairs of objects + map $X \to Y$".

$\mathcal{C}_2$ = "comm. diag. $\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[dl] \\ Z & \end{tikzcd}$".

...

<u>Def'n</u>. 1) An <u>equivalence</u> in an ∞-category $\mathcal{C}$ is a map $f: X \to Y$ that fits into a diagram

$$\begin{tikzcd}
 & X \arrow[rr, "\mathrm{id}"] \arrow[dr, "f"] \arrow[r, phantom, "\simeq"] & & X \\
Y \arrow[ur, "g'"] \arrow[rr, "\mathrm{id}_Y"'] \arrow[r, phantom, "\simeq"] & & Y \arrow[ur, "g"'] &
\end{tikzcd}$$

2) An ∞-groupoid (= anima) is an ∞-category $\mathcal{C}$ s.th. all maps are equivalences.

<u>equiv</u>: $\mathcal{C}$ <u>Kan complex</u>.

$$\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d, hook] & \mathcal{C} \\
\Delta^n \arrow[ur, dashed, "\exists"] &
\end{tikzcd} \qquad 0 \le i \le n.$$

If $\mathcal{C}, \mathcal{D}$ ∞-cat's, also

$$\mathrm{Fun}(\mathcal{C}, \mathcal{D}) = \underline{\mathrm{Hom}}(\mathcal{C}, \mathcal{D}) \text{ as an } \infty\text{-cat.}$$

$$\mathrm{Fun}(\mathcal{C}, \mathcal{D})_n = \mathrm{Hom}(\mathcal{C} \times \Delta^n, \mathcal{D}).$$

<u>Def'n</u>. A functor $f: \mathcal{C} \to \mathcal{D}$ is an equiv.

if $\exists\, g: \mathcal{D} \to \mathcal{C}$ s.th. fg equiv. to $\mathrm{id}_{\mathcal{D}}$ in $\mathrm{Fun}(\mathcal{D}, \mathcal{D})$,
gf equiv. to $\mathrm{id}_{\mathcal{C}}$ in $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$.

<u>equiv:</u> for all $X, Y \in \mathcal{C}$ obj. of ∞-category,
can define an anima $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ of maps
from X to Y.

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(X,Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \Delta^0 \arrow[d] & * \arrow[d, maps to] \\ \mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[r] & \mathcal{C} \times \mathcal{C}. & (X,Y) \end{tikzcd}$$

$$(f: X' \to Y') \longmapsto (X', Y')$$

<u>Thm</u>. $f: \mathcal{C} \to \mathcal{D}$ is an equiv. iff.
- ess surj. : $\forall Y \in \mathcal{D}\ \exists X \in \mathcal{C}, f(X) \simeq Y$.
- fully faithful: $\forall X_1, X_2 \in \mathcal{C}$,

$$\mathrm{Hom}_{\mathcal{C}}(X_1, X_2) \to \mathrm{Hom}_{\mathcal{D}}(f(X_1), f(X_2))$$

equiv. of anima.

"∞-categories are equiv. to categories enriched in anima"

$\rightsquigarrow$ ∞-cat. Cat_∞ of ∞-cat.

objects: ∞ - categories.

morphisms: $\operatorname{Fun}(\mathcal{C}, \mathcal{D})^{\simeq} \subseteq \operatorname{Fun}(\mathcal{C}, \mathcal{D})$

only all those morphisms that are equivalences.

$\mathrm{An} = \mathrm{Grp}_{\infty}$

full ∞ - subcategory:

obj.: ∞ - groupoids = anima.

morphisms: $\operatorname{Fun}(\mathcal{C}, \mathcal{D})$

Anima take the role of <u>sets</u> in higher category theory.

<u>Thm</u> (Yoneda).

$$\mathcal{C} \hookrightarrow \operatorname{Fun}(\mathcal{C}^{op}, \mathrm{An}).$$

fully faithful.

$$X \longmapsto (Y \mapsto \operatorname{Hom}_{\mathcal{C}}(Y, X)).$$

The notion of limits + colimits extends to ∞ - categories.

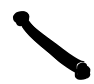
<u>Classical Point of View on Anima</u> / Homotopy Types.

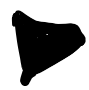
If $S_\bullet$ is a Kan complex.

Then can define geom. realisation of S:

$$\Delta^n \longmapsto |\Delta^n| = \left\{ (t_0, \dots, t_n) \in [0,1] \,/\, \sum t_i = 1 \right\}.$$

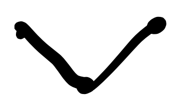
$\Delta^0 \longmapsto$

$\Delta^1 \longmapsto$ 

$\Delta^2 \longmapsto$ 

$\vdots$

There is a unique colimit preserving extension to all simplicial sets.

$\Lambda^1_2 \longmapsto$  ....

defines a functor to CW complexes.

has adjoint, the "singular complex functor", taking X CW complex to

$$\mathrm{Sing}(X)_n = \mathrm{Hom}(|\Delta^n|, X).$$

<u>Thm</u> (Quillen). This defines an equiv. of

"<u>model categories</u>" between simpl. sets & CW complexes.

"presentations of ∞-cat's".

<u>Cor.</u> $(\text{Kan complexes})[\text{equiv}^{-1}] \overset{\mathrm{An}}{\simeq} (\text{CW complexes})[\text{homot. equiv}^{-1}]$

equiv. of ∞-categories

S^1 as top. space 

as anima 

Why "anima"? <u>Beilinson:</u> "animations of identities in K_0".

<u>Sheaves of anima.</u>

(Reference: Lurie's Higher Topos Theory).

<u>Def'n.</u> Let $\mathcal{C}$ some site (w/ finite limits). A <u>presheaf of anima</u> is a functor

$$\mathcal{F} : N(\mathcal{C}^{op}) \longrightarrow \mathrm{An}.$$

A <u>sheaf of anima</u> is a presheaf of anima $\mathcal{F}$

s.th. for all covers $(f_i : X_i \to X)_i$,

$$\mathcal{F}(X) \xrightarrow{\sim} \lim\Big(\prod_i \mathcal{F}(X_i) \rightrightarrows \prod_{i,j} \mathcal{F}(X_i \times_X X_j)$$

$$\substack{\rightarrow\\\rightarrow\\\rightarrow} \prod_{i,j,k} \mathcal{F}(X_i \times_X X_j \times_X X_k) \cdots \Big)$$

is a limit in anima.

A <u>hypercomplete sheaf of anima</u> is a sheaf of anima $\mathcal{F}$ s.th. for all hypercovers $X_\bullet \to X$, the map

$$\mathcal{F}(X) \xrightarrow{\sim} \lim_{\Delta} \mathcal{F}(X_\bullet)$$

$$= \lim\Big(\mathcal{F}(X_0) \rightrightarrows \mathcal{F}(X_1) \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots \Big)$$

is an equivalence.

<u>Def'n</u>. The ∞-category of <u>condensed anima</u> is given $\overset{\text{equiv.}}{\text{by}}$ (modulo set-th. issues, resolved as before)

- the ∞-cat. of hypercomplete sheaves of anima on CHaus

- the ∞-cat. of hypercomplete sheaves of anima on ProFin
- the ∞-cat. of sheaves of anima on ExtrDisc.

i.e. of functors

$$\mathrm{ExtrDisc}^{op} \longrightarrow \mathrm{An}$$

taking finite disj. unions to finite products.

<u>Picture:</u>

$$\begin{tikzcd}
 & \mathrm{Cond\,An} & & \\
\text{"top. space" direction} \quad \mathrm{Cond\,Set} \arrow[ur, phantom, "\subset" sloped] & X & \mathrm{An}. \arrow[ul, phantom, "\supset" sloped] & \text{homotopy theoretic direction.} \\
\mathrm{CW\ complexes} \arrow[u, phantom, "\cup" sloped] \arrow[urr, "h"] & & &
\end{tikzcd}$$

$$\mathrm{Set} = \mathrm{CondSet} \cap \mathrm{An} \subset \mathrm{Cond\,An}.$$

<u>Language</u>. Say $X \in \mathrm{Cond\,An}$ is

- <u>discrete</u> if in essential image of An.
- <u>static</u> if in essential image of CondSet. (= 0-truncated)

<u>Prop'n.</u> Let X CW complex. Then there is an initial anima $h(X)$ with a map

$$X \longrightarrow h(X) \quad \text{in CondAn.}$$

In fact, $h(X)$ = usual homotopy type of X.

$$\cong \mathrm{Sing}(X).$$

<u>Proof.</u> <u>Key</u>: If $X = [0,1]^n$, then $h(X) = *$ does the job.

<u>Recall.</u> If $Y \in \mathrm{An}$, have follow. inv:

- $\pi_0 Y$ = "set of conn. components"

for all $y \in Y$

- $\pi_1(Y, y)$ group
- $\pi_n(Y, y)$ ab group if $n \geq 2$.

In fact $Y = \lim_n \tau_{\leq n} Y$ "Postnikov tower"

$$\mathrm{An} \underset{\tau_{\leq n}}{\rightleftarrows} \mathrm{An}_{\leq n} \quad \text{"n-truncated anima"}$$

$(\pi_i = 0 \ \forall i > n.)$ "Eilenberg-Maclane anima".

$\mathrm{An}_{\leq 0} = \mathrm{Set}$. $K(G, n) \hookleftarrow G$

$$\{ Y \in An_{\le n,*} \mid \tau_{\le n-1} Y \simeq * \} \underset{\pi_n}{\cong} \begin{cases} \text{Grp} & n=1 \\ \text{Ab} & n \geq 2. \end{cases}$$

<u>need to see:</u> for all anima Y,

$$\pi_0 Y \xrightarrow{\sim} \pi_0 \operatorname{Hom}_{\text{CondAn}}([0,1]^n, Y).$$

(even true internally).

Use Postnikov tower: reduce to Y m-truncated.

$$Y = K(G,m). \qquad \text{or } Y \text{ a set.}$$

Y a set: follows from $[0,1]^n$ connected.

$Y = K(G,m)$:

$$\pi_0 \operatorname{Hom}(X, K(G,m)) = H^m(X, G).$$

So need:

$$H^1_{\text{cond}}([0,1]^n, G) = * \qquad \text{for any group } G$$

$$H^m_{\text{cond}}([0,1]^n, G) = 0 \qquad \text{for any ab. grp } G. \quad m \geq 2.$$

$$\parallel$$

$$H^m_{\text{sheaf}}([0,1]^n, G) = 0.$$

classifies G-torsors over $[0,1]^n$,
these would be repr. by top. space.
But $[0,1]^n$ is simply connected, so get result. $\square$ (Prop).

Thus, the left adj. to

$$\mathrm{An} \subset \mathrm{Cond}\,\mathrm{An}.$$

exists on $[0,1]^n$ (with value $*$),
thus on everything generated under
colim's by $[0,1]^n$'s.
This includes CW complexes.
Also, $X \mapsto h(X)$: CW compl. $\to$ An.
is the unique (nice) colim.-preserving ext. of

$$[0,1]^n \longmapsto *, \text{ so given}$$

by usual homotopy type. $\square$

Funny condensed anima. $\operatorname{colim}(G \times G \rightrightarrows G \to *)$
$\quad //$

G condensed group $\rightsquigarrow$ $BG = [*/G]$

is condensed anima.

Only one point $* \in BG$ (up to iso.)

$\mathrm{Aut}_{BG}(*) = G$ as a condensed group.

$//$

$* \times_{BG} *$

$\mathrm{Cond\,An} = \lim_n \mathrm{Cond\,An}_{\leq n}$. "Postnikov tower".

$\pi_0 X$ cond set.

$\pi_1(X,x)$ cond. group.

$\pi_n(X,x)$ cond. ab. group $n \geq 2$.

X conn. CW complex $x \in X$.

$\tilde{X} \longrightarrow X$ universal cover.

$\downarrow \ulcorner \quad \downarrow$ [illegible] Cond An pullback in low π_n.

$$\begin{tikzcd} \arrow[d] & \arrow[d] & \\ h(X)_{\geq 2} \arrow[r] & h(X) \arrow[r] & h(X)_{\leq 1} \arrow[d, phantom, "\parallel"] \\ & & B\pi_1(X,x) \end{tikzcd}$$

2-connected cover.

$$\begin{tikzcd} \tilde{\tilde{X}} \arrow[r] \arrow[d] & X \arrow[d] \\ * \arrow[r] & h(X) \end{tikzcd}$$

$* \longmapsto x^0$

in general, neither static nor discrete.

variant of univ. cover also trivializing all higher homotopy groups.

<u>Exercise</u>. Describe this for $X = S^2$.

<u>Relation to solidification</u>.

<u>Propn</u>. X CW complex

$\rightsquigarrow \quad X \longrightarrow h(X)$.

induces an equivalence

$$\mathbb{Z}[X]^{\blacksquare} \longrightarrow \mathbb{Z}[h(X)]$$

$$\begin{array}{c} T \\ | \\ H_\bullet(X) \end{array}$$

<u>Proof.</u> Both sides commute with colimits, so enough to consider $X = [0,1]^n$.

Then $\mathbb{Z}[[0,1]^n]^\bullet \cong \mathbb{Z} \cong \mathbb{Z}[*]$. $\square$

cf. Dustin

<u>Animation</u>.

Let $\mathcal{C}$ cocomplete category generated under colim's by $\underbrace{\text{compact proj.}}$ objects $\mathcal{C}^{cp} \subset \mathcal{C}$.

$\mathrm{Hom}(X, -)$ comm. w/ 1-sifted colimits.

$\leadsto \mathcal{C} = 1\text{-SiftedInd}(\mathcal{C}^{cp})$

<u>freely</u> generated under 1-sifted colim's by $\mathcal{C}^{cp}$.

$\cong \mathrm{Fun}((\mathcal{C}^{cp})^{op}, \mathrm{Set})$ taking finite disj. unions to finite prod.

<u>Def'n</u>. The <u>animation of $\mathcal{C}$</u> is the ∞-category freely generated under <u>sifted colim's</u> by $\mathcal{C}^{op}$.

$\subset$ filt. colim's + colim$_{\Delta^{op}}$

$$\mathrm{Ani}(\mathcal{C}) = \mathrm{SiftedInd}(\mathcal{C}^{op})$$

$\cong$ $\mathrm{Fun}(\Delta^{op}, \mathcal{C})[eq^{-1}]$.

$$\cong \mathrm{Fun}((\mathcal{C}^{op})^{op}, \mathrm{An})$$

taking finite disj. unions to fin. products.

$$\rightsquigarrow \quad \mathrm{Ani}(\mathcal{C}) \underset{\hookleftarrow}{\xrightarrow{\pi_0}} \mathcal{C}$$

fully faithful.

= Quillen's "nonabelian derived category".

<u>Examples</u> 1). $\mathcal{C} = \mathrm{Sets}$.
$\mathcal{C}^{op} = \mathrm{FinSets}$.

$\mathrm{Ani}(\mathcal{C}) = \mathrm{An}.$

2). $\mathcal{C} = \mathrm{CondSet}$

$\mathcal{C}^{sp} = \mathrm{ExtrDisc}$

$\mathrm{Ani}(\mathcal{C}) = \mathrm{CondAn}.$

3). $\mathcal{C} = \mathrm{Ab}.$

$\mathcal{C}^{sp} = \mathrm{FinFree}_{\mathbb{Z}}$

$\mathrm{Ani}(\mathcal{C}) = \mathcal{D}_{\geq 0}(\mathbb{Z})$.

4) $\mathcal{C}$ ab. cat. gen. by compact proj.

then $\mathrm{Ani}(\mathcal{C}) = \mathcal{D}_{\geq 0}(\mathcal{C}).$

$\rightsquigarrow$ the functor

$$\mathrm{CondSet} \longrightarrow \mathrm{Cond\,Ab} : X \mapsto \mathbb{Z}[X]$$

animates to a functor

$$\mathrm{Cond\,An} \longrightarrow \mathcal{D}_{\geq 0}(\mathrm{Cond\,Ab}) : X \mapsto \mathbb{Z}[X].$$

$K(\mathcal{C})$.

Reminder on alg. K-theory:

R ring.

$$K_0(R) = \left(\{\text{fin. proj. } R\text{-modules}\}/\cong, \oplus\right)^{gp}.$$

Better: $\mathrm{Proj}(R) = \{\text{fin. proj. } R\text{-modules}\}$ as groupoid
$\cap$
An.

Then $(\mathrm{Proj}(R), \oplus)$ forms an "E_∞-monoid"
in An.

$$\mathrm{Mon}_{E_\infty}(\mathrm{An}) \xrightarrow{\text{group completion}} \supseteq \mathrm{Grp}_{E_\infty}(\mathrm{An})$$ "comm. monoid".

<u>Def.</u> $K(R) = (\mathrm{Proj}(R), \oplus)^{gp}$
$\cap$
$\mathrm{Sp}_{\geq 0} \simeq \mathrm{Grp}_{E_\infty}(\mathrm{An})$.

$\cup\!|$ — connective spectra.

$Sp.$ ⟵ stable ∞-cat. of spectra.

If R is a condensed ring, then $K(R)$ is naturally a condensed spectrum.

$$S \in \mathrm{Extr\,Disc} \longmapsto K(R(S))$$

<u>Propn.</u>

$$\begin{array}{lcl} \mathrm{CondSp} & = & \text{condensed spectra.} \\ \cup\!| \ \text{(solid.)} & & \\ \mathrm{SolidSp} & & \text{well-behaved full stable } \infty\text{-subcategory:} \\ \| & & \end{array}$$

$$\{X \in \mathrm{CondSp} \mid \forall i \in \mathbb{Z} \ \ \pi_i X \in \mathrm{Solid}_{\mathbb{Z}} \subseteq \mathrm{CondAb}\}$$

$$\mathbb{S}[S]^{\blacksquare} \xrightarrow{\sim} \lim_i \mathbb{S}[S_i]$$

for $S = \lim_i S_i$ profinite set.

$$\begin{tikzcd}
\Sigma^\infty_+ X \arrow[d, phantom, "\cong" sloped] & & \\
\mathbb{S}[X] & \mathrm{Cond}\,\mathrm{Sp} \arrow[d, shift left] & = \mathrm{Cond}(\mathcal{D}(\mathbb{S})). \\
X \arrow[u, mapsto] & \mathrm{Cond}\,\mathrm{An} \arrow[u, bend left] &
\end{tikzcd}$$

Again, if X CW complex,

$$(\Sigma^\infty_+ X)^\blacksquare = \mathbb{S}[X]^\blacksquare \xrightarrow{\sim} \mathbb{S}[h(X)].$$

<u>Thm.</u> $K(\mathbb{C})^\blacksquare \xrightarrow{\sim} ku \in \mathrm{Sp} \subseteq \mathrm{Cond}\,\mathrm{Sp}.$

top. K-theory.

<u>Proof.</u>

$K(\mathbb{C})$ = group completion of E_∞-monoid in $\mathrm{Cond}\,\mathrm{An}$

$$\coprod_n B\,GL_n(\mathbb{C}) = \mathrm{Proj}(\mathbb{C}).$$

ku = group completion of

$\mathbb{E}_\infty$ - monoid in (Cond) An.

$$\coprod_n B\, h(GL_n(\mathbb{C})).$$

$\rightsquigarrow$ natural map.

$$K(\mathbb{C}) \longrightarrow ku.$$

rest follows from

<u>Lemma.</u> Let $A \longrightarrow B$ be a map in $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathrm{CondAn})$. Assume.

$$\mathbb{S}[A]^{\blacksquare} \xrightarrow{\sim} \mathbb{S}[B]^{\blacksquare}.$$

Then $(A^{gp})^{\blacksquare} \xrightarrow{\sim} (B^{gp})^{\blacksquare}$.