Some computations.

Cohomology.

X compact Hausdorff $\qquad M$ ab. group

$$H^i(X, M) \underset{X\ CW}{=} H^i_{sing}(X, M).$$

sheaf cohom

$\shortparallel$

Čech cohom

Embed CHaus $\hookrightarrow$ CondSet. How to recover this within CondSet?

In any topos $\mathcal{T}$ if $X \in \mathcal{T}$ any $M \in Ab(\mathcal{T})$

can define $$H^i_{\mathcal{T}}(X, M) = H^i(\underbrace{\mathcal{T}_{/X}}_{\text{topos}}, \underbrace{M|_X}_{\text{abelian sheaf in } \mathcal{T}_{/X}.})$$

$$= \mathrm{Ext}^i_{Ab(\mathcal{T})}(\mathbb{Z}[X], M).$$

This also works for $\mathcal{T}$ = CondSet.

Thm ('70s) Dyckhoff $\quad X$ compact Hausdorff, M discrete ab. group.

Then $$\underset{\substack{\text{sheaf cohom} \\ \shortparallel \\ \text{Čech}}}{H^i(X, M)} \cong H^i_{Cond}(X, M).$$

Proof. Step 0. X extremally disconnected.

Then X projective object in CondSet, so for **any** $M \in Ab(\mathrm{CondSet}_{/X})$, $H^i_{cond}(X, M) = \begin{cases} M(X) & i=0 \\ 0 & i>0. \end{cases}$

So in our case

$$H^i_{cond}(X, M) = \begin{cases} \mathrm{Cont}(X,M) = \mathrm{LocConst}(X,M) & i=0 \\ 0 & i>0. \end{cases}$$

$$= H^i_{sheaf}(X, M).$$

$X = \varprojlim_i \underbrace{X_i}_{finite}$ $\quad H^j_{sheaf}(X, M) = \varinjlim_i H^j_{sheaf}(X_i, M)$

$$= \begin{cases} \mathrm{LocConst}(X,M) & j=0 \\ 0 & j>0 \end{cases}$$

Consider presheaf

$$X \in \mathrm{CHaus} \longmapsto R\Gamma_{sheaf}(X,M) \in D(\mathbb{Z}).$$

Then Step 0 implies that its sheafification is

$$R\Gamma_{cond}(X,M) \longleftarrow R\Gamma_{sheaf}(X,M).$$

<u>Step 1</u>. X profinite.

To compute $R\Gamma_{cond}(X,M)$, pick a simplicial hypercover

$$\underbrace{\cdots X_2 \substack{\rightarrow\\\rightarrow\\\rightarrow} X_1 \rightrightarrows X_0}_{X_\bullet} \longrightarrow X$$

with all X_i extremally disconnected.

Then $R\Gamma_{\mathrm{cond}}(X, M)$ is computed by

$$0 \to M(X_0) \to M(X_1) \to M(X_2) \to \dots .$$

need to see that

$$(*) \quad 0 \to M(X) \to M(X_0) \to M(X_1) \to \dots \quad \text{is exact.}$$

In fact, this holds for any hypercover of a profinite set X by profinite sets X_i.

One can write

$$\dots X_1 \rightrightarrows X_0 \to X$$

as $\varprojlim_j$ cofilt. limit of hypercovers of fin. sets by fin. sets.

$$\dots X_{2,j} \rightrightarrows X_{1,j} \rightrightarrows X_{0,j} \to X_j .$$

Then
$$(*) = \varinjlim_j (*)_j .$$

enough: $(*)_j$ exact.

Any hypercover of fin. sets by fin. sets splits.

This gives contracting homotopy.

<u>Step 2</u>. General $X \in \mathrm{CHaus}$.

Pick hypercover $X_\bullet \longrightarrow X$ by extr. disc.

$$(\kappa\text{-})\mathrm{CondSet}_{/X} \xrightarrow{f} \mathrm{Sh}(X) \quad \text{map of topoi}$$

$$(f_*\mathcal{F})(U) = \mathcal{F}(U).$$

pullback map $f^* : H^i_{sheaf}(X, M) \longrightarrow H^i_{cond}(X, M)$.

Leray spectral sequence

$$H^i_{sheaf}(X, R^j f_* M) \Longrightarrow H^{i+j}_{cond}(X, M).$$

$$R\Gamma_{sheaf}(X, Rf_*M) = R\Gamma_{cond}(X, M)$$

<u>enough:</u> $(R^j f_* M)_x = 0$ for $j > 0$, $x \in X$.

$(f_*M)_x = M.$

$$\| $$

$$\varinjlim_{U \ni x} H^j_{cond}(U, M) = \varinjlim_{\bar{U} \ni x} \underbrace{H^j_{cond}(\bar{U}, M)}$$

computed by complex

$$0 \longrightarrow M(X_0 \times_X \bar{U}) \longrightarrow M(X_1 \times_X \bar{U}) \longrightarrow \cdots$$

$$= H^j\big(0 \to M(X_0 \times_X x) \to M(X_1 \times_X x) \to \cdots\big).$$

New $\quad \cdots X_1 \times_X x \rightrightarrows X_0 \times_X x \longrightarrow x$

hypercover of x by profinite sets, so got

$$= \begin{cases} M & j=0 \\ 0 & j>0 \end{cases} . \qquad \square$$

<u>Thm.</u> $X \in \mathrm{CHaus}$. Consider $\mathbb{R} \in \mathrm{CondAb}$.

Then $H^i_{\mathrm{cond}}(X,\mathbb{R}) = \begin{cases} \mathrm{Cont}(X,\mathbb{R}) & i=0 \\ 0 & i>0 . \end{cases}$

More precisely, for any hypercover of X by profin. sets X_i, the complex

$$(**)\ 0 \to C(X,\mathbb{R}) \to C(X_0,\mathbb{R}) \to C(X_1,\mathbb{R}) \to \cdots$$

Banach spaces for sup norm.

is exact, and

for any $f_i \in C(X_i,\mathbb{R})$ with $df_i = 0$, $\varepsilon > 0$,

$\exists\ g_{i-1} \in C(X_{i-1},\mathbb{R})$ with $f_i = dg_{i-1}$ $\quad (X_{-1} := X)$

<u>and</u> $\|g_{i-1}\| \le (1+\varepsilon)\|f_i\|$.

(../scholze/Condensed.pdf)

<u>Sketch of proof.</u>

<u>Step 1.</u> Say X and all X_i finite.

Then hypercover splits, so get contracting homotopy

$h_i : C(X_i, \mathbb{R}) \longrightarrow C(X_{i-1}, \mathbb{R})$

induced by pullback along some maps $s_i : X_{i-1} \longrightarrow X_i$.

Then $\|\underbrace{h_i(f_i)}_{= g_{i-1}}\| \leq \|f_i\|$.

<u>Step 2</u>. X (and all X_i) are profinite.

As above, write hypercover as cofilt. limit of hypercovers of fin. sets $\underbrace{X_i \text{ by fin. sets } X_{ij}}$.

Then $(**) = \varinjlim_j (**)_j$

(complete for induced norm).

This stays exact, with desired bounds.

<u>Step 3</u>. X general compact Hausdorff.

<u>Idea</u>. reduce to statement on fibres.

Pick $f_i \in C(X_i, \mathbb{R})$ with $df_i = 0$.

For any $x \in X$, get

$f_{i,x} \in C(X_i \times_X x, \mathbb{R})$ $df_{i,x} = 0$.

$X_{\bullet, x} \longrightarrow x$ hypercover

Step 2 $\exists\, g_{i-1,x} \in C(X_{i-1} \underset{X}{\times} x, \mathbb{R})$ $d g_{i-1,x} = f_{i,x}$

$$\|g_{i-1,x}\| \le (1+\varepsilon)\|f_{i,x}\| .$$

Spread $g_{i-1,x}$: $\exists\, U \ni x$, $g_{i-1,U} \in C(X_{i-1} \underset{X}{\times} U, \mathbb{R})$

s.th. (Urysohn's lemma) $g_{i-1,U}\big|_{X_{i-1} \underset{X}{\times} x} = g_{i-1,x}$,

$$\|g_{i-1,U}\| \le \|g_{i-1,x}\|,$$

$$\|f_{i,U} - d\, g_{i-1,U}\| \le \varepsilon \|f_i\|.$$ (make U small enough).

Pick fin. many such U_j covering X, and a partition of unity

$$1_X = \sum_j \psi_j \in C(X, \mathbb{R})$$

$\operatorname{supp} \psi_j \subseteq U_j$, ψ_j takes values in $[0,1]$.

Let $g_{i-1}' = \sum_j \psi_j\, g_{i-1,U_j} \in C(X, \mathbb{R})$

$$\|g_{i-1}'\| \le (1+\varepsilon)\|f_i\|$$ ($\mathbb{R}$ satisfies triangle ineq.)

$$\|\underbrace{d g_{i-1}' - f_i}\| \le \varepsilon \|f_i\|.$$

Take this as new f_i, continue: Process converges. $\square$.

↯ Need that ℓ^1-averages of small functions are small.
Argument works for Banach spaces V in place of $\mathbb{R}$, but not for ℓ^p $0<p<1$.

Condensed abelian groups

<u>Goal:</u> Compute Ext's between natural examples.

$$\mathbb{Z},\ \mathbb{R},\quad \mathbb{R}/\mathbb{Z},\quad \prod \mathbb{R}/\mathbb{Z},\ \mathbb{Z}_p.$$

$\mathrm{Hom}(-,\mathbb{R}/\mathbb{Z}) =$ Pontrjagin duality. $\mathbb{A}$.

LCA = <u>locally compact abelian groups</u>. A.

$\cup$ discrete, $\supset$ compact, $\supset$ $\mathbb{R}$-vectorspaces.

discrete $\longleftrightarrow$ compact

Cond Ab. $\rightsquigarrow$ Grothendieck abelian category. w/ enough compact projectives

<u>Thm.</u> For $A, B \in \mathrm{LCA}$,

- $\mathrm{Hom}_{\mathrm{cond}}(A,B) =$ usual Hom
- $\mathrm{Ext}^1_{\mathrm{cond}}(A,B) = \mathrm{Ext}^1_{\mathrm{LCA}}(A,B)$
- $\mathrm{Ext}^i_{\mathrm{cond}}(A,B) = 0$ for $i \geq 2$.

For example, if $A = \prod_I \mathbb{R}/\mathbb{Z}$, then

- $R\mathrm{Hom}(A, \mathbb{R}) = 0$
- $R\mathrm{Hom}(A, \underset{\text{discrete}}{M}) = \bigoplus_I M[-1]$.

these imply everything.

What we know so far is

$$R\mathrm{Hom}\left(\mathbb{Z}[X], \begin{Bmatrix} M \\ \mathbb{R} \end{Bmatrix}\right) \quad \text{for } X \in \mathrm{CHaus}.$$
$$\| $$
$$R\Gamma_{\mathrm{cond}}\left(X, \begin{Bmatrix} M \\ \mathbb{R} \end{Bmatrix}\right).$$

need: resolution of A of following form

$$\dots \to \mathbb{Z}[\mathbb{Z}[A]] \to \mathbb{Z}[A] \to A$$

$$\mathbb{Z}[A^3] \oplus \mathbb{Z}[A^2] \longrightarrow \mathbb{Z}[A^2] \longrightarrow \mathbb{Z}[A] \longrightarrow A \longrightarrow 0$$

$(*)$ ......

$$[a] \longmapsto a.$$

$$[(a,b)] \longmapsto [a+b]-[a]-[b]$$

Thm (Breen, Deligne) There <u>is</u> a <u>functorial</u> resolution of an abelian group A of form $(*)$.

$$\dots \to \bigoplus_{j=1}^{n_i} \mathbb{Z}[A^{r_{ij}}] \to \dots \to \mathbb{Z}[A^2] \to \mathbb{Z}[A] \to A \to 0$$

Proof: See notes Condensed.pdf. $\square$.

Functoriality $\Rightarrow$ extends to any topos, and to Cond Ab.

we get resolution as desired.

<u>Proof of Thm on LCA's</u>.

<u>Step 1</u>. $R\mathrm{Hom}(\mathbb{R}, M) = 0$ M discrete.

$\mathbb{Z}[A] \otimes \mathbb{Z}[S] = \mathbb{Z}[A \times S]$

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbb{Z}[\mathbb{R}^2] \arrow[r] & \mathbb{Z}[\mathbb{R}] \arrow[r] & \mathbb{R} \arrow[r] & 0 \\
\cdots \arrow[r] & \mathbb{Z}[0] \arrow[r] \arrow[u, "\alpha"] & \mathbb{Z}[0] \arrow[r] \arrow[u, "\alpha"] & 0 \arrow[r] \arrow[u, "\alpha"] & 0
\end{tikzcd}$$

$$H^i(\mathbb{R}^n \times S, M) = \begin{cases} \mathrm{Cont}(S, M) & i = 0 \\ 0 & i > 0 \end{cases}$$

$$\downarrow \wr$$

$$H^i(0^n \times S, M) = \begin{cases} \mathrm{Cont}(S, M) & i = 0 \\ 0 & i > 0 \end{cases}$$

So $\alpha^*: R\mathrm{Hom}(\mathbb{R}, M) \xrightarrow{\simeq} R\mathrm{Hom}(0, M) = 0$.

<u>Step 2</u>. $R\mathrm{Hom}(\mathbb{R}/\mathbb{Z}, M) = M[-1]$.

Use $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$.

<u>Step 3</u>. $R\mathrm{Hom}\left(\prod_I \mathbb{R}/\mathbb{Z}, M\right) \xleftarrow{\cong} \bigoplus_I M[-1]$.

$\bigoplus_I$ of map from Step 2

equiv.,

$$R\mathrm{Hom}\Big(\prod_I \mathbb{R}/\mathbb{Z}, M\Big)$$

$$\Big\uparrow 2$$

$$\operatorname*{colim}_{\substack{J\subset I\\ \text{finite}}} \underbrace{R\mathrm{Hom}\Big(\prod_J \mathbb{R}/\mathbb{Z}, M\Big)}_{\simeq\ \bigoplus_J M[-1]}.$$

Use Breen - Deligne resolution $\quad A = \prod \mathbb{R}/\mathbb{Z}$

$$\dots \to \mathbb{Z}[A^2] \to \mathbb{Z}[A] \to A \to 0$$

Thus, suffices that

$$R\mathrm{Hom}\Big(\mathbb{Z}\Big[\Big(\prod_I \mathbb{R}/\mathbb{Z}\Big)^{r}\times S\Big], M\Big) = R\Gamma_{\mathrm{cond}}\Big(\prod_I(\mathbb{R}/\mathbb{Z})^{r}\times S, M\Big)$$

$$\Big\uparrow 2 \qquad\qquad\qquad\qquad \Big\uparrow 2$$

$$\operatorname*{colim}_{J\subset I} R\mathrm{Hom}\Big(\mathbb{Z}\Big[\Big(\prod_J \mathbb{R}/\mathbb{Z}\Big)^{r}\times S\Big], M\Big) = \operatorname*{colim}_{J\subset I} R\Gamma_{\mathrm{cond}}\Big(\prod_J(\mathbb{R}/\mathbb{Z})^{r}\times S, M\Big)$$

isom. for Čech cohomology (see Monday).

remains: $\quad R\mathrm{Hom}(A, \mathbb{R}) = 0$ for any compact abelian group A.

using Breen - Deligne resolution, this is computed by

$$(*) \quad 0 \to C(A\times S, \mathbb{R}) \to C(A^2\times S, \mathbb{R}) \to C(A^2\times S, \mathbb{R}) \oplus C(A^3\times S, \mathbb{R}) \to \dots$$

$$f \mapsto \big(\tilde{f} : (a,b) \mapsto \dots$$

$f(a+b)-f(a)-f(b)$).

Use further property of Breen–Deligne resolution:

<u>Prop'n.</u> For any choice of a functorial BD resolution,
$C_\bullet(A) \to A$.

the maps

$$2 : C_\bullet(A) \to C_\bullet(A) \quad \text{mult. by } 2$$

and $[2] : C_\bullet(A) \to C_\bullet(A)$ induced by mult. by 2 on A

are homotopic via a functorial homotopy.

<u>ex.</u> $2[a] - [2a] = \pm d([a,a])$.

$\Rightarrow$ identity on $(\cdot)$ homotopic to $\frac{1}{2}\cdot[2]$,
$\overset{\shortparallel}{\frac{1}{2}\cdot 2}$ homotopic to $\frac{1}{4}\cdot[4]$, to $\frac{1}{8}\cdot[8]$, ...,
to $\frac{1}{2^n}[2^n]$.

<u>and</u> the homotopies witnessing this stay of bounded norm.

"Pass to limit": all $[2^n]$ are of norm ≤ 1.
$\frac{1}{2^n}[2^n]$ norm $\leq \frac{1}{2^n}$.

$\leadsto$ homotopic to zero, so complex is acyclic. $\square$.

Cor. $R\mathrm{Hom}(\prod_I \mathbb{Z}, M) = \bigoplus_I M$. <u>M discrete</u>.

Proof. Short exact sequence

$$0 \to \prod_I \mathbb{Z} \to \prod_I \mathbb{R} \to \prod_I \mathbb{R}/\mathbb{Z} \to 0.$$

Infinite Products are exact in CondAb!

$$R\mathrm{Hom}(\prod_I \mathbb{R}/\mathbb{Z}, M) = \bigoplus_I M[-1].$$

enough: $R\mathrm{Hom}(\underbrace{\prod_I \mathbb{R}}_{\text{module over the condensed ring } \mathbb{R}}, M) = 0.$

$$/\!/$$

$$R\mathrm{Hom}_{\mathbb{R}}(\prod_I \mathbb{R}, \underbrace{R\underline{\mathrm{Hom}}(\mathbb{R}, M)}_{=0}) = 0.$$

need: <u>internal</u> Hom $R\underline{\mathrm{Hom}}(\mathbb{R}, M) = 0$.

Thm above on LCA's holds true "internally".

$$R\underline{\mathrm{Hom}}(\prod_I \mathbb{R}/\mathbb{Z}, M) = \bigoplus_I M[-1]$$

$$R\underline{\mathrm{Hom}}(\text{compact}, \mathbb{R}) = 0.$$

$$R\underline{\mathrm{Hom}}(\mathbb{R}, M) = 0.$$

"same" proof.

$$\underline{R\mathrm{Hom}}(A, B)(S) = R\mathrm{Hom}(A \otimes \mathbb{Z}[S], B).$$

S extr. disc.

$\square$.