# Condensed Mathematics.

Monday: 10-12:15 Topoi

1:30-... Condensed Sets.

Tuesday: morning: cohomology of cond. sets,
locally compact ab. groups.

afternoon: solid abelian groups

Wednesday "alg. topology": morning: condensed anima,
solid spectra, $K(\mathbb{C})$, $K(\mathbb{Q}_p)$

afternoon: Morava E-theory... .

Thursday "functional analysis": morning. p-adic functional analysis

afternoon: real functional analysis.

Friday "analytic geometry": morning: more real functional analysis

afternoon: compact Riemann surfaces

$$
\begin{tikzcd}[arrows=-]
\underline{\text{Condensed Sets}} \arrow[dr] \arrow[rr] & & \underline{\text{Topoi}} \arrow[dl] \\
& \text{Topological Spaces.} \arrow[d] & \\
& \text{Homotopy types ("anima")} &
\end{tikzcd}
$$

## Sites and Topoi.

abstraction of the notion of <u>sheaves</u> on a topological space.

<u>Recall</u>. Let $X$ topological space.

<u>Def'n</u>. 1). A <u>presheaf</u> on $X$ is a functor.

$$
\begin{aligned}
\mathcal{F} : Op(X)^{op} = \{\text{open subsets } U \subset X\}^{op} &\longrightarrow \text{Sets} \\
U &\longmapsto \mathcal{F}(U).
\end{aligned}
$$

2) A <u>sheaf on X</u> is a presheaf $\mathcal{F}$

s.th. for any $U \subset X$ open, $\bigcup_i U_i = U$ open cover of $U$,

$$\mathcal{F}(U) \xrightarrow{\sim} \mathrm{eq}\left(\prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j)\right).$$

$$\mathcal{F}(U) \xrightarrow{\simeq} \left\{ (s_i)_i,\ s_i \in \mathcal{F}(U_i),\ s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \in \mathcal{F}(U_i \cap U_j) \right\}$$

$$s \longmapsto (s|_{U_i})_i.$$

<u>Example</u>. If $Y$ is any other top. space,

$$U \longmapsto \mathrm{Cont}(U, Y) \quad \text{is a sheaf.}$$

If $Y$ discrete, these are the locally constant functions, and this is "the constant sheaf with value $Y$".

This is also the <u>sheafification</u> of the presheaf

$$U \longmapsto \mathcal{F}_0(U) = Y.$$

<u>Sheafification</u>.

<u>Prop</u>. The fully faithful inclusion

$$\mathrm{Sh}(X) \hookrightarrow \mathrm{PSh}(X)$$

admits a left adjoint

$$\mathcal{F} \longmapsto \mathcal{F}^{\#} \qquad \text{"sheafification".}$$

<u>Proof.</u> Can construct $\mathcal{F}^{\#}$ explicitly.

Let $\mathcal{F}^{\natural}$ be defined by

$$\mathcal{F}^{\natural}(U) = \operatorname*{colim}_{(U_i)_i \text{ covers of } U} \operatorname{eq}\Big(\prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j)\Big)$$

$$\uparrow$$

$$\mathcal{F}$$

$$\text{Then } \mathcal{F}^{\#} = (\mathcal{F}^{\natural})^{\natural}.$$

<u>Remark.</u> A presheaf $\mathcal{F}$ is <u>separated</u> if

$$\mathcal{F}(U) \hookrightarrow \prod_i \mathcal{F}(U_i)$$

for any cover $U = \bigcup_i U_i$.

Then $\mathcal{F}^{\natural}$ separated, and if $\mathcal{F}$ is already separated, then $\mathcal{F}^{\natural}$ is a sheaf. $\square$.

<u>Stalks</u> For any $x \in X$, let

$$\mathcal{F}_x = \operatorname*{colim}_{U \ni x} \mathcal{F}(U) \qquad \text{"stalk of } \mathcal{F}\text{".}$$

**Prop.** 1). Sheafification does not change stalks: $\mathcal{F}_x \xrightarrow{\sim} \mathcal{F}^{\#}_x$.

If $\mathcal{F}$ is a sheaf,

2). $\forall s_1, s_2 \in \mathcal{F}(U)$ then $s_1 = s_2$ iff $\forall x \in X$ $s_{1,x} = s_{2,x}$. $\square$.

**Cohomology of abelian sheaves.** Can also consider sheaves of groups abelian groups, rings, etc.

**Prop.** $Ab(X) = \{$ abelian sheaves on $X\}$ is an abelian category.

abelian sheaves = sheaves of abelian groups

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \quad \text{exact}$$

$$\iff \forall x \in X \quad 0 \to \mathcal{F}'_x \to \mathcal{F}_x \to \mathcal{F}''_x \to 0.$$

But globally, only get

$$0 \to \mathcal{F}'(X) \to \mathcal{F}(X) \to \mathcal{F}''(X).$$

**Example.** $X = \mathbb{R}/\mathbb{Z} = S^1$.

$\mathcal{F}' = \mathbb{Z}$ constant sheaf

$\mathcal{F} = \mathbb{R} : U \mapsto \mathrm{Cont}(U, \mathbb{R})$.

Then $\mathcal{F}'' = \mathcal{F}/\mathcal{F}' = \mathbb{R}/\mathbb{Z} : U \mapsto \mathrm{Cont}(U, \mathbb{R}/\mathbb{Z})$.

$$\begin{array}{ccc} \mathcal{F}''(X) & = & \mathrm{Cont}(X, \mathbb{R}/\mathbb{Z}) \ni id \\ \uparrow & & \uparrow \\ \mathcal{F}(X) & = & \mathrm{Cont}(X, \mathbb{R}) \end{array}$$

no lift.

<u>Cohomology</u>. $H^i(X, -) : Ab(X) \to Ab$

right derived functor of

$$H^0(X, -) : \mathcal{F} \mapsto \mathcal{F}(X).$$

This is well-defined, as $Ab(X)$ has enough injectives

(If $M$ any divisible group, $x \in X$, then

$$(i_{x,*} M)(U) = \begin{cases} M & x \in U \\ 0 & \text{else} \end{cases}$$ is injective.)

$\rightsquigarrow$ long exact sequences

$$0 \to H^0(X, \mathcal{F}') \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F}'') \to H^1(X, \mathcal{F}') \to H^1(X, \mathcal{F}) \to \dots$$

$\rightsquigarrow$ in the example, we see

$$H^1(\mathbb{R}/\mathbb{Z}, \mathbb{Z}) \neq 0.$$

<u>More generally:</u> If $f : Y \to X$ any map of top. spaces get pullback functor

$$f^* : \underset{Sh}{Ab}(X) \longrightarrow \underset{Sh}{Ab}(Y) \quad \text{with right adjoint}$$

$$f_* : \underset{Sh}{Ab}(Y) \longrightarrow \underset{Sh}{Ab}(X) :$$

$$(f_* \mathcal{F})(U) = \mathcal{F}(f^{-1}(U)).$$

$f^*\mathcal{F}$ sheafification of

$$V \longmapsto \operatorname*{colim}_{U \supset f(V)} \mathcal{F}(U).$$

$$(f^*\mathcal{F})_y = \mathcal{F}_{f(x)} \qquad \text{for any } y \in Y.$$

Then $f^*$ exact, $f_*$ left-exact

$\rightsquigarrow$ right derived functors $R^i f_* : Ab(Y) \to Ab(X)$

$R^i f_* \mathcal{F}$ sheafification of $U \mapsto H^i(f^{-1}(U), \mathcal{F})$.

If $f : X \to *$ projection, then

$f^* : Ab = Ab(*) \to Ab(X)$ "constant sheaves"

$f_* : Ab(X) \to Ab$ "global sections"

Free abelian sheaves.

Prop. The forgetful functor

$$Ab(X) \to Sh(X)$$

admits a left adjoint $\mathcal{F} \mapsto \mathbb{Z}[\mathcal{F}]$,

sheafification of $U \mapsto \mathbb{Z}[\mathcal{F}(U)]$.

If $\mathcal{F}$ is sheaf repr. by some $U \subset X$.

i.e. $\mathcal{F}(V) = \mathrm{Hom}(V, U)$, write $\mathbb{Z}[U]$ for $\mathbb{Z}[\mathcal{F}]$

Then $$H^i(U, \mathcal{G}) \cong \mathrm{Ext}^i_{Ab(X)}(\mathbb{Z}[U], \mathcal{G}).$$

Tensor Products.

Prop. The abelian category $Ab(X)$ has a natural symmetric monoidal tensor product

$$- \otimes - : Ab(X) \times Ab(X) \to Ab(X),$$

commuting with colimits in both variables,

$\mathcal{F} \otimes \mathcal{G}$ sheafification of

$$U \mapsto \mathcal{F}(U) \otimes \mathcal{G}(U).$$

It admits a partial right adjoint "internal Hom"

$$\mathcal{H}om : Ab(X)^{op} \times Ab(X) \to Ab(X)$$

$$(\mathcal{F}, \mathcal{G}) \mapsto \left( \mathcal{H}om(\mathcal{F}, \mathcal{G}) : U \mapsto \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U) \right).$$

$$\mathrm{Hom}(\mathcal{F}, \mathcal{H}om(\mathcal{G}, \mathcal{H})) \cong \mathrm{Hom}(\mathcal{F} \otimes \mathcal{G}, \mathcal{H}).$$

Similarly, $Sh(X)$ have symm. monoidal str. given by $\times$ cartesian product, partial right adjoint

$$\mathcal{H}om(\mathcal{F}, \mathcal{G}) : U \mapsto \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U),$$

$$\mathcal{F} \mapsto \mathbb{Z}[\mathcal{F}] : Sh(X) \to Ab(X)$$

is symmetric monoidal. $\square$

## Compact Hausdorff Spaces.

**Prop** Let $X$ compact Hausdorff, $M$ ab. group.

1). If $X$ CW complex, then
$$H^i(X,M) = H^i_{sing}(X,M).$$

2) In general
$$H^i(X,\underline{M}) = \check{H}^i(X,\underline{M})$$
$$\check{H}^i(X,\underline{M}) = \operatorname*{colim}_{(U_j)_j \text{ cover of } X} H^i\Big(\prod_j \mathcal{F}(U_j) \to \prod_{j_1,j_2} \mathcal{F}(U_{j_1}\cap U_{j_2}) \to \prod_{j_1,j_2,j_3} \mathcal{F}(U_{j_1}\cap U_{j_2}\cap U_{j_3}) \to \dots\Big).$$

3) If $X = \varprojlim_j X_j$ (cofiltered), $X_j$ compact Hausdorff, then
$$H^i(X,M) \xleftarrow{\sim} \operatorname*{colim}_j H^i(X_j,M).$$
(e.g. $X = \prod_{\text{infinite}} \mathbb{R}/\mathbb{Z}$).

4). $f: Y \to X$ map of compact Hausdorff spaces then $$(R^i f_* \mathcal{F})_x = H^i(Y_x, \mathcal{F}|_{Y_x})$$

where $Y_x \subset Y$ fibre over $x \in X$. 3)

"Proper Base Change".

$$\operatorname*{colim}_{\substack{U \ni x \\ \text{open nbhd}}} H^i(f^{-1}(U), \mathcal{F}) = \operatorname*{colim}_{\substack{\bar{U} \ni x \\ \text{closed nbhd}}} H^i(f^{-1}(\bar{U}), \mathcal{F})$$

$\square$.

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## Sites.

Abstraction of $Op(X)$ with its notion of covers.

**Def'n.** A site is a category $\mathcal{C}$ together with a collection of covers $\mathrm{Cov}(X) = \{(f_i : X_i \to X)_i\}$ for any $X \in \mathcal{C}$

s.th. — elements are collections $(f_i : X_i \to X)_i$.

1) Pullbacks of covers exist and are covers
2) Composites of covers are covers.
3) Isomorphisms are covers.

Def'n. 1) A presheaf on $\mathcal{C}$ is a functor

$$\mathcal{F} : \mathcal{C}^{op} \longrightarrow \text{Sets}.$$

2) A sheaf on $\mathcal{C}$ is a presheaf $\mathcal{F}$ s.th.

$\forall X \in \mathcal{C}$, $\forall (f_i : X_i \to X)_i$ covers,

$$\mathcal{F}(X) \xrightarrow{\sim} \mathrm{eq}\Big( \prod_i \mathcal{F}(X_i) \rightrightarrows \prod_{i,j} \mathcal{F}(X_i \times_X X_j) \Big)$$

Essentially everything carries over, in particular:

- sheafification
- $Ab(\mathcal{C})$ abelian category with enough injectives.
- free abelian sheaves

  $$\mathcal{F} \longmapsto \mathbb{Z}[\mathcal{F}] : \text{sheafification of } X \mapsto \mathbb{Z}[\mathcal{F}(X)].$$
- tensor product

  $\mathcal{F} \otimes \mathcal{G}$ : sheafification of $X \mapsto \mathcal{F}(X) \otimes \mathcal{G}(X)$.
- internal Hom (on $Sh(\mathcal{C})$, $Ab(\mathcal{C})$)

  $$\mathcal{H}om(\mathcal{F}, \mathcal{G}) : X \longmapsto \mathrm{Hom}(\mathcal{F}|_X, \mathcal{G}|_X).$$

$\mathcal{F}|_X$ : sheaf on $\mathcal{C}_{/X}$ : category $\{Y, f: Y \to X\}$ with induced covers.

In general, "not enough points" $f: Y \to X$

**Definition.** A continuous morphism of sites $\mathcal{C}' = \varphi(Y) \to \mathcal{C} = \varphi(X)$

$f: \mathcal{C}' \longrightarrow \mathcal{C}$ having finite limits

is a functor $f^{-1}: \mathcal{C} \longrightarrow \mathcal{C}'$ comm. with finite limits, taking covers to covers.

In this situation, get a pullback functor

$$f^*: Sh(\mathcal{C}) \to Sh(\mathcal{C}') \text{ with right adjoint}$$

$$f_*: Sh(\mathcal{C}') \longrightarrow Sh(\mathcal{C}).$$

$$(f_* \mathcal{F})(X) = \mathcal{F}(f^{-1}(X)).$$

$X \in \mathcal{C}$

$\mathcal{F} \in Sh(\mathcal{C}')$

$f^*\mathcal{F}$ : sheafification of $Y \mapsto \operatorname{colim}_{X,\, Y \to f^{-1}(X)} \mathcal{F}(X)$.

$\mathcal{F} \in Sh(\mathcal{C})$, $Y \in \mathcal{C}'$

$f^* : Ab(\mathcal{C}) \to Ab(\mathcal{C}')$ exact

$f_* : Ab(\mathcal{C}') \to Ab(\mathcal{C})$ left-exact

<u>Def'n.</u> $R^i f_* : Ab(\mathcal{C}') \to Ab(\mathcal{C})$ right derived functor of $f_*$.

$R^i f_* \mathcal{F}$ sheafification of $X \mapsto H^i(f^{-1}(X), \mathcal{F})$.

## Examples of sites.

1) $Op(X)$.

2) Étale site of a scheme $X$:

$$\underset{\displaystyle X_{ét}}{\overset{\shortparallel}{\mathcal{C}}} = \{ f : Y \to X \text{ étale} \}.$$

covers: $(f_i : Y_i \to Y)_i$ cover if

$$\begin{tikzcd} Y_i \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & X & \end{tikzcd}$$

$|Y| = \bigcup_i f_i(|Y_i|)$.

If $X = \operatorname{Spec} K$, $K$ field, then

$$X_{ét} = \{ f : \underset{\displaystyle \bigsqcup_i \operatorname{Spec} K_i}{\overset{}{\underset{\shortparallel}{Y}}} \to \operatorname{Spec} K \text{ étale} \}.$$

$K_i / K$ finite separable.

$$\begin{array}{c} Y \\ \downarrow \\ Y(\bar{K}) \end{array}$$

$$\underset{\text{Galois theory}}{\cong} \left\{ \begin{array}{c} \text{sets with a continuous} \\ \text{action of } Gal(\overline{K}/K) \end{array} \right\}$$

3) If $G$ any profinite group (or even $\underset{\text{discrete}}{\underset{\cup}{\text{prodiscrete}}}$ group),

can consider

$$\mathcal{C} = G\text{-sets} = \left\{ \begin{array}{c} \text{sets with cont.} \\ G\text{-action} \end{array} \right\}.$$

covers = covers. $\rightsquigarrow H^i(G,-)$ "cont. group cohom."

4) If $G$ profinite group, can consider

$$\mathcal{C} = G\text{-pfsets} = \left\{ \begin{array}{c} \text{profinite sets} \\ \text{with } \underset{\text{cont.}}{\wedge} G\text{-action} \end{array} \right\}.$$

profinite sets = tot. disc. compact Hausdorff space

Covers: $(f_i : S_i \to S)_{i \in I}$, $G \circlearrowright$, is a <u>cover</u> if

there is a finite subset $J \subseteq I$ s.th.

$$S = \bigcup_{i \in J} f_i(S_i).$$

not allowed: $S \leftarrow \bigsqcup_{x \in S} \{x\}$. if $S$ infinite.

If $G = Gal(\overline{K}/K)$, this is $X_{proet}$. $X = \operatorname{Spec} K$.

This gives site-theoretic interpretation of continuous group cohomology $H^i(G, M)$ for $G$ acting on top. ab grps $M$.

$M \rightsquigarrow$ sheaf

$$\underline{M} : \underset{G}{\overset{\circlearrowleft}{S}} \mapsto \mathrm{Cont}^G(S, M).$$

$$H^i_{cts}(G, M) \cong H^i(G\text{-pfsets}, \underline{M})$$

for virtually all $M$.

5) $G = *$. $\rightsquigarrow$

site $\mathcal{C} = \{$ profinite sets $\}$.

covers = finite covers as above

$\rightsquigarrow$ __condensed sets__ $= Sh(\mathcal{C})$.

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## Topoi.

(Grothendieck topoi)

__Def'n.__ A __topos__ is a category $\mathcal{T}$ that is equiv. to $Sh(\mathcal{C})$ for some site $\mathcal{C}$.

__Remark.__ This admits a characterization in terms of certain axioms on $\mathcal{T}$. (Giraud).

<u>Def'n</u>. A functor of topoi $\mathcal{T}' \to \mathcal{T}$ is a pair of adjoint functors

$$f^* : \mathcal{T} \longrightarrow \mathcal{T}',$$
$$f_* : \mathcal{T}' \longrightarrow \mathcal{T}$$

<u>such that</u> $f^*$ commutes with finite limits.

A <u>point of a topos</u> is a map of topoi Sets $\longrightarrow \mathcal{T}$.

$\mathcal{T}$ <u>has enough points</u> if

$$\{ f_i^* : \mathcal{T} \longrightarrow \text{Sets} \mid f_i : \text{Sets} \xrightarrow[\text{point}]{} \mathcal{T} \}$$

conservative family.

This is not always the case.
Even if it is, points may be so inexplicit that they are not helpful.

## <u>More relevant:</u>

<u>Def'n:</u> 1) An object $X \in \mathcal{J}$ is <u>projective</u> if $\mathrm{Hom}(X,-)$ commutes with reflexive coequalizers.

$$\begin{tikzcd} & Y'' \arrow[d, shift left] \arrow[d, shift right] \\ & Y' \arrow[u, bend left] \arrow[d] \\ X \arrow[r] \arrow[ur, dotted] & Y = \mathrm{coeq}(Y'' \rightrightarrows Y'). \end{tikzcd}$$

$$\mathrm{Hom}(X,Y) = \mathrm{coeq}\big(\mathrm{Hom}(X,Y'') \rightrightarrows \mathrm{Hom}(X,Y')\big).$$

2) $X$ <u>compact</u> if $\mathrm{Hom}(X,-)$ commutes with all filtered colimits.

$\rightsquigarrow$ for $X$ <u>compact projective</u>,

$\mathrm{Hom}(X,-)$ commutes with all <u>sifted colimits</u>. (and all limits).

$\rightsquigarrow$ $\mathrm{Hom}(X,-)$ almost as good as a point.

only missing bit: commutation with finite disjoint unions.

We will have lots of compact projective $X$, use $\mathrm{Hom}(X,-)$ in place of stalks at points.

Also, $Ab(\mathcal{T})$ has enough projectives then. $\mathbb{Z}[X]$.

So will use projective resolutions instead of injective resolutions.

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## Quasicompact / Quasiseparatedness.

__Def'n.__ 1) An object $X \in \mathcal{T}$ is <u>quasicompact</u> (qc) if for any $(f_i : X_i \to X)_{i\in I}$ s.th. $\bigsqcup X_i \to X$ surjective, there is some finite subset $J \subseteq I$ s.th.

$$\bigsqcup_{i\in J} X_i \to X \text{ is surjective.}$$

2). A map $f: Y \to X$ is <u>quasicompact</u> if $Y \in \mathcal{T}_{/X}$ quasicompact.

If $\mathcal{T}$ is gen by quasicompact objects then is equiv. to: $\forall Z \in \mathcal{T}$ qc, $Z \to X$,

$Y \times_X Z$ is again qc.

(qs)

3). $X \in \mathcal{T}$ is <u>quasiseparated</u> if

$\forall Y, Z \to X$ , $Y \times_X Z$ is qc.

$Y, Z$ qc

qcqs: qc + qs.

Assume $\mathcal{T}$ is gen. by qcqs objects.

4) $f: Y \to X$ is <u>quasiseparated</u>

if $\forall Z$ qcqs, $Z \to X$, $\Updownarrow$

also $Y \times_Z X$ is qs. $Y \in \mathcal{T}_{/X}$ is qs.

<u>Example</u> $\left([0,1] \sqcup [0,1]\right) \big/ [0,1) \sim [0,1)$.

—: not quasiseparated.