

p -adic étale Tate twists and arithmetic duality
(Twists de Tate p -adiques étale et
dualité arithmétique)

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ABSTRACT: In this paper, we define, for arithmetic schemes with semistable reduction, p -adic objects playing the roles of Tate twists in étale topology, and establish their fundamental properties.

RÉSUMÉ: Dans ce papier, nous définissons, pour les schémas arithmétiques à réduction semistable, des objets p -adiques jouant les rôles de twists à la Tate en topologie étale, et nous établissons leurs propriétés fondamentales.

KEYWORDS: étale cohomology theory
 Beilinson-Lichtenbaum axioms for motivic complexes
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 étale sheaves of p -adic vanishing cycles

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1. Introduction

Let k be a finite field of characteristic $p > 0$, and let X be a proper smooth variety over $\text{Spec}(k)$ of dimension d . For a positive integer m prime to p , we have the étale sheaf μ_m on X consisting of m -th roots of unity. The sheaves $\mathbb{Z}/m\mathbb{Z}(n) := \mu_m^{\otimes n}$ ($n \geq 0$), so called Tate twists, satisfy Poincaré duality of the following form: There is a non-degenerate pairing of finite groups for any $i \in \mathbb{Z}$

$$H_{\text{ét}}^i(X, \mathbb{Z}/m\mathbb{Z}(n)) \times H_{\text{ét}}^{2d+1-i}(X, \mathbb{Z}/m\mathbb{Z}(d-n)) \longrightarrow \mathbb{Z}/m\mathbb{Z}.$$

On the other hand, we have the étale subsheaf $W_r \Omega_{X, \log}^n$ ($n \geq 0, r \geq 1$) of the logarithmic part of the Hodge-Witt sheaf $W_r \Omega_X^n$ ([B1], [Il1]). When we put $\mathbb{Z}/p^r\mathbb{Z}(n) := W_r \Omega_{X, \log}^n[-n]$, we have an analogous duality fact due to Milne [Mi1], [Mi2].

In this paper, for a regular scheme X which is flat of finite type over $\text{Spec}(\mathbb{Z})$ and a prime number p , we construct an object $\mathfrak{T}_r(n)_X$ playing the role of ‘ $\mathbb{Z}/p^r\mathbb{Z}(n)$ ’ in $D^b(X_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})$, the derived category of bounded complexes of étale $\mathbb{Z}/p^r\mathbb{Z}$ -sheaves on X . The fundamental idea is due to Schneider [Sch], that is, we will glue $\mu_{p^r}^{\otimes n}$ on $X[1/p]$ and a logarithmic Hodge-Witt sheaf on the fibers of characteristic p to define $\mathfrak{T}_r(n)_X$ (cf. Lemma 1.3.1 below). We will further prove a duality result analogous to the above Poincaré duality. The object $\mathfrak{T}_r(n)_X$ is a p -adic analogue of the Beilinson-Deligne complex $\mathbb{R}(n)_{\mathcal{D}}$ on the complex manifold $(X \otimes_{\mathbb{Z}} \mathbb{C})^{\text{an}}$, while $\mu_{p^r}^{\otimes n}$ on $X[1/p]$ corresponds to $(2\pi\sqrt{-1})^n \cdot \mathbb{R}$ on $(X \otimes_{\mathbb{Z}} \mathbb{C})^{\text{an}}$.

1.1. Existence result. We fix the setting as follows. Let p be a rational prime number. Let A be a Dedekind ring whose fraction field has characteristic zero and which has a residue field of characteristic p . We assume that

every residue field of A of characteristic p is perfect.

Let X be a noetherian regular scheme of pure-dimension which is flat of finite type over $B := \text{Spec}(A)$ and satisfies the following condition:

X is a smooth or semistable family around any fiber of X/B of characteristic p .

Let j be the open immersion $X[1/p] \hookrightarrow X$. The first main result of this paper is the following:

Theorem 1.1.1. *For each $n \geq 0$ and $r \geq 1$, there exists an object $\mathfrak{T}_r(n)_X \in D^b(X_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})$, which we call a p -adic étale Tate twist, satisfying the following properties:*

T1 (Trivialization, cf. 4.2.4). *There is an isomorphism $t : j^* \mathfrak{T}_r(n)_X \simeq \mu_{p^r}^{\otimes n}$.*

T2 (Acyclicity, cf. 4.2.4). *$\mathfrak{T}_r(n)_X$ is concentrated in $[0, n]$, i.e., the q -th cohomology sheaf is zero unless $0 \leq q \leq n$.*

T3 (Purity, cf. 4.4.7). *For a locally closed regular subscheme $i : Z \hookrightarrow X$ of characteristic p and of codimension $c (\geq 1)$, there is a Gysin isomorphism*

$$W_r \Omega_{Z, \log}^{n-c}[-n-c] \xrightarrow{\simeq} \tau_{\leq n+c} Ri^! \mathfrak{T}_r(n)_X \quad \text{in } D^b(Z_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z}).$$

T4 (Compatibility, cf. 6.1.1). *Let $i_y : y \hookrightarrow X$ and $i_x : x \hookrightarrow X$ be points on X with $\text{ch}(x) = p$, $x \in \overline{\{y\}}$ and $\text{codim}_X(x) = \text{codim}_X(y) + 1$. Put $c := \text{codim}_X(x)$. Then the connecting homomorphism*

$$R^{n+c-1} i_{y*} (Ri_y^! \mathfrak{T}_r(n)_X) \longrightarrow R^{n+c} i_{x*} (Ri_x^! \mathfrak{T}_r(n)_X)$$

in localization theory (cf. (1.9.3) below) agrees with the (sheafified) boundary map of Galois cohomology groups due to Kato (cf. §1.8 below)

$$\left\{ \begin{array}{ll} R^{n-c+1}i_{y*}\mu_{p^r}^{\otimes n-c+1} & (\text{ch}(y) = 0) \\ i_{y*}W_r\Omega_{y,\log}^{n-c+1} & (\text{ch}(y) = p) \end{array} \right\} \longrightarrow i_{x*}W_r\Omega_{x,\log}^{n-c}$$

up to a sign depending only on $(\text{ch}(y), c)$, via Gysin isomorphisms. Here the Gysin map for i_y with $\text{ch}(y) = 0$ is defined by the isomorphism t in **T1** and Deligne's cycle class in $R^{2c-2}i_y!\mu_{p^r}^{\otimes c-1}$.

T5 (Product structure, cf. 4.2.6). *There is a unique morphism*

$$\mathfrak{T}_r(m)_X \otimes^{\mathbb{L}} \mathfrak{T}_r(n)_X \longrightarrow \mathfrak{T}_r(m+n)_X \quad \text{in } D^-(X_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})$$

that extends the natural isomorphism $\mu_{p^r}^{\otimes m} \otimes \mu_{p^r}^{\otimes n} \simeq \mu_{p^r}^{\otimes m+n}$ on $X[1/p]$.

If X is smooth over B , the object $\mathfrak{T}_r(n)_X$ is already considered by Schneider [Sch], §7 (see also 7.3.4 below). The properties **T1–T3** and **T5** are $\mathbb{Z}/p^r\mathbb{Z}$ -coefficient variants of the Beilinson-Lichtenbaum axioms on the conjectural étale motivic complex $\Gamma(n)_X^{\text{ét}}$ [Be], [Li2], [Li3]. More precisely, **T1** (resp. **T2**) corresponds to the axiom of Kummer theory for $\Gamma(n)_V^{\text{ét}}$ (resp. the acyclicity axiom for $\Gamma(n)_X^{\text{ét}}$), and **T3** is suggested by the purity axiom and the axiom of Kummer theory for $\Gamma(n-c)_Z^{\text{ét}}$. Although **T4** is not among the Beilinson-Lichtenbaum axioms, it is a natural property to be satisfied. We deal with this rather technical property for two reasons. One is that the pair $(\mathfrak{T}_r(n)_X, t)$ (t is that in **T1**) is characterized by the properties **T2**, **T3** and **T4** (see 1.3.5 below). The other is that we need **T4** to prove the property **T7** in the following functoriality result.

Theorem 1.1.2. *Let X be as in 1.1.1, and let Z be another scheme which is flat of finite type over B and for which the objects $\mathfrak{T}_r(n)_Z$ ($n \geq 0, r \geq 1$) are defined. Let $f : Z \rightarrow X$ be a morphism of schemes and let $\psi : Z[1/p] \rightarrow X[1/p]$ be the induced morphism. Then:*

T6 (Contravariant functoriality, cf. 4.2.8). *There is a unique morphism*

$$f^*\mathfrak{T}_r(n)_X \longrightarrow \mathfrak{T}_r(n)_Z \quad \text{in } D^b(Z_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})$$

that extends the natural isomorphism $\psi^*\mu_{p^r}^{\otimes n} \simeq \mu_{p^r}^{\otimes n}$ on $Z[1/p]$.

T7 (Covariant functoriality, cf. 7.1.1). *Assume that f is proper, and put $c := \dim(X) - \dim(Z)$. Then there is a unique morphism*

$$Rf_*\mathfrak{T}_r(n-c)_Z[-2c] \longrightarrow \mathfrak{T}_r(n)_X \quad \text{in } D^b(X_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})$$

that extends the trace morphism $R\psi_*\mu_{p^r}^{\otimes n-c}[-2c] \rightarrow \mu_{p^r}^{\otimes n}$ on $X[1/p]$.

Furthermore, these morphisms satisfy a projection formula (cf. 7.2.4 below).

We will explain how we find $\mathfrak{T}_r(n)_X$ in §1.3 below.

1.2. Arithmetic duality. We explain the second main result of this paper, the arithmetic duality for p -adic étale Tate twists. We assume that A is an algebraic integer ring, and that X is proper over B . Put $V := X[1/p]$ and $d := \dim(X)$. For a scheme Z which is separated of finite type over B , let $H_c^*(Z, \bullet)$ be the étale cohomology with compact support (cf. §10.2 below). There is a well-known pairing

$$H_c^q(V, \mu_{p^r}^{\otimes n}) \times H_{\text{ét}}^{2d+1-q}(V, \mu_{p^r}^{\otimes d-n}) \longrightarrow \mathbb{Z}/p^r\mathbb{Z},$$

and it is a non-degenerate pairing of finite groups by the Artin-Verdier duality ([AV], [Ma], [Mi3], [De], [Sp]) and the relative Poincaré duality for regular schemes ([SGA4], XVIII, [Th], [FG]). We extend this duality to a twisted duality for X with coefficients in the p -adic étale Tate twists. A key ingredient is a global trace map

$$H_c^{2d+1}(X, \mathfrak{F}_r(d)_X) \longrightarrow \mathbb{Z}/p^r\mathbb{Z},$$

which is obtained from the trace morphism in **T7** for the structural morphism $X \rightarrow B$ and the classical global class field theory. See §10.2 below for details. The product structure **T5** and the global trace map give rise to a pairing

$$H_c^q(X, \mathfrak{F}_r(n)_X) \times H_{\text{ét}}^{2d+1-q}(X, \mathfrak{F}_r(d-n)_X) \longrightarrow \mathbb{Z}/p^r\mathbb{Z}. \quad (1.2.1)$$

The second main result of this paper is the following:

Theorem 1.2.2 (10.1.3). *The pairing (1.2.1) is a non-degenerate pairing of finite groups for any q and n with $0 \leq n \leq d$.*

A crucial point of this duality result is the non-degeneracy of a pairing

$$H_{\text{ét}}^q(X_\Sigma, \mathfrak{F}_r(n)_{X_\Sigma}) \times H_Y^{2d+1-q}(X, \mathfrak{F}_r(d-n)_X) \longrightarrow \mathbb{Z}/p^r\mathbb{Z},$$

which is an extension of a duality result of Niziol [Ni] for crystalline local systems. Here Σ denotes the set of the closed points on B of characteristic p , X_Σ denotes $\coprod_{s \in \Sigma} X \times_B B_s$ with B_s the henselization of B at s , and Y denotes $X \times_B \Sigma$. To calculate this pairing, we will provide an explicit formula (cf. 8.3.8 below) for a pairing of étale sheaves of p -adic vanishing cycles.

We state a consequence of Theorem 1.2.2. For an abelian group M , let $M_{p\text{-tors}}$ be the subgroup of p -primary torsion elements and let $M_{p\text{-cotors}}$ be the quotient of $M_{p\text{-tors}}$ by its maximal p -divisible subgroup. The following corollary is originally due to Cassels and Tate ([Ca], [Ta1], 3.2) in the case that the structural morphism $X \rightarrow B$ has a section, and due to Saito [Sa2] in the general case.

Corollary 1.2.3. *Assume $d = 2$ and either $p \geq 3$ or A has no real places. Then $\text{Br}(X)_{p\text{-cotors}}$ is finite and carries a non-degenerate skew-symmetric bilinear form with values in $\mathbb{Q}_p/\mathbb{Z}_p$. In particular, if $p \geq 3$ then it is alternating and the order of $\text{Br}(X)_{p\text{-cotors}}$ is a square number.*

Indeed, by a Bockstein triangle (cf. §4.3) and a standard limit argument, Theorem 1.2.2 yields a non-degenerate pairing of cofinitely and finitely generated \mathbb{Z}_p -modules

$$H^q(X, \mathfrak{F}_{\mathbb{Q}_p/\mathbb{Z}_p}(1)) \times H^{5-q}(X, \mathfrak{F}_{\mathbb{Z}_p}(1)) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p,$$

where $H^*(X, \mathfrak{F}_{\mathbb{Q}_p/\mathbb{Z}_p}(1)) := \varinjlim_{r \geq 1} H_{\text{ét}}^*(X, \mathfrak{F}_r(1)_X)$ and $H^*(X, \mathfrak{F}_{\mathbb{Z}_p}(1)) := \varprojlim_{r \geq 1} H_{\text{ét}}^*(X, \mathfrak{F}_r(1)_X)$. By the Kummer theory for \mathbb{G}_m (cf. 4.5.1 below), one can easily check that

$$\text{Br}(X)_{p\text{-cotors}} \simeq H_{\text{ét}}^2(X, \mathfrak{F}_\infty(1)_X)_{p\text{-cotors}} \simeq H_{\text{ét}}^3(X, \mathfrak{F}_{\mathbb{Z}_p}(1)_X)_{p\text{-tors}}.$$

Hence the corollary follows from the same argument as for [Ur], 1.5 (cf. [Ta3]) and the fact that the bigraded algebra $\bigoplus_{q, n \geq 0} H_{\text{ét}}^q(X, \mathfrak{F}_r(n)_X)$ with respect to the cup product is anti-commutative in q .

1.3. Construction of $\mathfrak{T}_r(n)_X$. We explain how to find $\mathfrak{T}_r(n)_X$ satisfying the properties in Theorem 1.1.1. Let $Y \subset X$ be the divisor on X defined by the radical ideal of $(p) \subset \mathcal{O}_X$ and let V be the complement $X \setminus Y = X[1/p]$. Let ι and j be as follows:

$$V \xrightarrow{j} X \xleftarrow{\iota} Y.$$

We start with necessary conditions for $\mathfrak{T}_r(n)_X$ to exist.

Lemma 1.3.1. *Assume that there exists an object $\mathfrak{T}_r(n)_X \in D^b(X_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})$ satisfying **T1–T4**. For a point $x \in X$, let i_x be the natural map $x \hookrightarrow X$. Then:*

(1) *There is an exact sequence of sheaves on $X_{\text{ét}}$*

$$R^n j_* \mu_{p^r}^{\otimes n} \longrightarrow \bigoplus_{y \in Y^0} i_{y*} W_r \Omega_{y, \log}^{n-1} \longrightarrow \bigoplus_{x \in Y^1} i_{x*} W_r \Omega_{x, \log}^{n-2}, \quad (1.3.2)$$

where each arrow arises from the boundary maps of Galois cohomology groups.

(2) *There is a distinguished triangle in $D^b(X_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})$ of the form*

$$\iota_* \nu_{Y,r}^{n-1}[-n-1] \xrightarrow{g} \mathfrak{T}_r(n)_X \xrightarrow{t'} \tau_{\leq n} Rj_* \mu_{p^r}^{\otimes n} \xrightarrow{\sigma_{X,r}(n)} \iota_* \nu_{Y,r}^{n-1}[-n]. \quad (1.3.3)$$

*Here t' is induced by t in **T1** and the acyclicity property in **T2**, and $\tau_{\leq n}$ denotes the truncation at degree n . The object $\nu_{Y,r}^{n-1}$ is an étale sheaf on Y defined as the kernel of the second arrow in (1.3.2) (restricted onto Y), and the arrow $\sigma_{X,r}(n)$ is induced by the exact sequence (1.3.2).*

The sheaf $\nu_{Y,r}^{n-1}$ agrees with $W_r \Omega_{Y, \log}^{n-1}$ if Y is smooth. See §2.2 below for fundamental properties of $\nu_{Y,r}^{n-1}$. Because this lemma is quite simple, we include a proof here.

Proof. There is a localization distinguished triangle (cf. (1.9.2) below)

$$\mathfrak{T}_r(n)_X \xrightarrow{j^*} Rj_* j^* \mathfrak{T}_r(n)_X \xrightarrow{\delta_{U,Z}^{\text{loc}}} R\iota_* R\iota^! \mathfrak{T}_r(n)_X[1] \xrightarrow{\iota_*} \mathfrak{T}_r(n)_X[1]. \quad (1.3.4)$$

By **T1**, we have $j^* \mathfrak{T}_r(n)_X \simeq j^* \mu_{p^r}^{\otimes n}$ via t . On the other hand, one can easily check

$$\tau_{\leq n}(R\iota_* R\iota^! \mathfrak{T}_r(n)_X[1]) \simeq \iota_* \nu_{Y,r}^{n-1}[-n]$$

by **T3** and **T4** (cf. (1.9.4) below). Because the map $R^n j_* \mu_{p^r}^{\otimes n} \rightarrow \iota_* \nu_{Y,r}^{n-1}$ of cohomology sheaves induced by $\delta_{U,Z}^{\text{loc}}$ is compatible with Kato's boundary maps up to a sign (again by **T4**), the sequence (1.3.2) must be a complex and we obtain the morphism $\sigma_{X,r}(n)$. Finally by **T2**, we obtain the triangle (1.3.3) by truncating and shifting the triangle (1.3.4) suitably. The exactness of (1.3.2) also follows from **T2**. Thus we obtain the lemma. \square

We will prove the exactness of the sequence (1.3.2), independently of this lemma, in 3.2.4 and 3.4.2 below. By this exactness, we are provided with the morphism $\sigma_{X,r}(n)$ in (1.3.3), and it turns out that any object $\mathfrak{T}_r(n)_X \in D^b(X_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})$ fitting into a distinguished triangle of the form (1.3.3) is concentrated in $[0, n]$. Because $D^b(X_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})$ is a triangulated category, there is at least one such $\mathfrak{T}_r(n)_X$. Moreover, an elementary homological algebra argument (cf. 2.1.2 (3) below) shows that a triple $(\mathfrak{T}_r(n)_X, t', g)$ fitting into (1.3.3) is unique up to a unique isomorphism (and that g is determined by $(\mathfrak{T}_r(n)_X, t')$). Thus there is a unique pair $(\mathfrak{T}_r(n)_X, t')$ fitting into (1.3.3). Our task is to prove that this pair satisfies the listed properties, which will be carried out in §§4–7 below. As a consequence of Theorem 1.1.1 and Lemma 1.3.1, we obtain

Theorem 1.3.5. *The pair $(\mathfrak{T}_r(n)_X, t)$ in 1.1.1 is the only pair that satisfies **T2–T4**, up to a unique isomorphism in $D^b(X_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})$.*

1.4. Comparison with known objects. We mention relations between $\mathfrak{T}_r(n)_X$ and other cohomology theories (or coefficients). Assume that A is local with residue field k and $p \geq n+2$. If X is smooth over B , then $\iota^*\mathfrak{T}_r(n)_X$ is isomorphic to $R\epsilon_*S_r(n)$, where $S_r(n)$ denotes the syntomic complex of Fontaine-Messing [FM] on the crystalline site $(X_r/W_r)_{\text{cris}}$ with $X_r := X \otimes_A A/p^r A$ and $W_r := W_r(k)$, and ϵ denotes the natural continuous map $(X_r/W_r)_{\text{cris}} \rightarrow (X_r)_{\text{ét}}$ of sites. This fact follows from a result of Kurihara [Ku] (cf. [Ka2]) and Lemma 1.3.1 (see also 2.1.2 (3) below). The isomorphism t in **T1** corresponds to the Fontaine-Messing morphism $R\epsilon_*S_r(n) \rightarrow \tau_{\leq n}\iota^*Rj_*\mu_{p^r}^{\otimes n}$. On the other hand, $\iota^*\mathfrak{T}_r(n)_X$ is *not* a log syntomic complex of Kato and Tsuji ([Ka5], [Ts1]) unless $n > \dim(X)$, because the latter object is isomorphic to $\tau_{\leq n}\iota^*Rj_*\mu_{p^r}^{\otimes n}$ by a result of Tsuji [Ts2]. Therefore $\mathfrak{T}_r(n)_X$ is a new object particularly on semistable families.

We turn to the setting in §1.1, and mention what can be hoped for $\mathfrak{T}_r(n)_X$ in comparison with the étale sheafification $\mathbb{Z}(n)_X^{\text{ét}}$ and the Zariski sheafification $\mathbb{Z}(n)_X^{\text{Zar}}$ of Bloch's cycle complex ([Bl2], [Le1]). By works of Levine ([Le1], [Le2]), these two objects are strong candidates for the motivic complexes $\Gamma(n)_X^{\text{ét}}$ and $\Gamma(n)_X^{\text{Zar}}$, respectively. So Theorem 1.3.5 leads us to the following:

Conjecture 1.4.1. (1) *There is an isomorphism in $D^b(X_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})$*

$$\mathbb{Z}(n)_X^{\text{ét}} \otimes^{\mathbb{L}} \mathbb{Z}/p^r\mathbb{Z} \xrightarrow{\cong} \mathfrak{T}_r(n)_X.$$

(2) *Let ε be the natural continuous map $X_{\text{ét}} \rightarrow X_{\text{Zar}}$ of sites. Then the isomorphism in (1) induces an isomorphism in $D^b(X_{\text{Zar}}, \mathbb{Z}/p^r\mathbb{Z})$*

$$\mathbb{Z}(n)_X^{\text{Zar}} \otimes^{\mathbb{L}} \mathbb{Z}/p^r\mathbb{Z} \xrightarrow{\cong} \tau_{\leq n}R\varepsilon_*\mathfrak{T}_r(n)_X.$$

The case $n = 0$ is obvious, because $\mathfrak{T}_r(0)_X = \mathbb{Z}/p^r\mathbb{Z}$ (by definition). The case $n = 1$ holds by the Kummer theory for \mathbb{G}_m (cf. 4.5.1 below) and the isomorphisms

$$\begin{aligned} \mathbb{Z}(1)_X^{\text{ét}} &\simeq \mathbb{G}_m[-1], & \mathbb{Z}(1)_X^{\text{Zar}} &\simeq \varepsilon_*\mathbb{G}_m[-1] \quad (\text{Levine, [Le2], 11.2}), \\ R^1\varepsilon_*\mathbb{G}_m &= 0 \quad (\text{Hilbert's theorem 90}). \end{aligned}$$

As for $n \geq 2$, by results of Geisser ([Ge], 1.2 (2), (4), 1.3), Conjecture 1.4.1 holds if X/B is smooth, under the Bloch-Kato conjecture on Galois symbol maps [BK], §5. A key step in his proof is to show that $\mathbb{Z}(n)_X^{\text{Zar}} \otimes^{\mathbb{L}} \mathbb{Z}/p^r\mathbb{Z}$ is concentrated in degrees $\leq n$. We have nothing to say about this problem for the general case in this paper.

1.5. Guide for the readers. This paper is organized as follows. In §2, we will review some preliminary facts from homological algebra and results in [Sat], which will be used frequently in this paper. In §3, which is the technical heart of this paper, we will provide preliminary results on étale sheaves of p -adic vanishing cycles (cf. Theorem 3.4.2, Corollary 3.5.2) using the Bloch-Kato-Hyodo theorem (Theorem 3.3.7). In §4, we will define p -adic étale Tate twists in a slightly more general situation and prove fundamental properties including the product structure **T5**, the contravariant functoriality **T6**, the purity property **T3** and the Kummer theory for \mathbb{G}_m . In §§5–6, we are concerned with the compatibility property **T4**. Using this property, we will prove the covariant functoriality **T7** and a projection formula in §7. In §§8–10, we will study pairings of p -adic vanishing cycles and prove Theorem 1.2.2. The appendix A due to Kei Hagihara includes a proof of a semi-purity of the étale sheaves of p -adic vanishing cycles (cf. Theorem A.2.6 below), which plays an important role in this paper. He applies his semi-purity result to the coniveau filtration on étale cohomology groups of varieties over p -adic fields (cf. Theorems A.1.4, A.1.5 and Corollary A.1.9 below).

Notation and conventions

1.6. For an abelian group M and a positive integer n , ${}_nM$ and M/n denote the kernel and the cokernel of the map $M \xrightarrow{\times n} M$, respectively. For a field k , \bar{k} denotes a fixed separable closure, and G_k denotes the absolute Galois group $\text{Gal}(\bar{k}/k)$. For a discrete G_k -module M , $H^*(k, M)$ denote the Galois cohomology groups $H_{\text{Gal}}^*(G_k, M)$, which are the same as the étale cohomology groups of $\text{Spec}(k)$ with coefficients in the étale sheaf associated with M .

1.7. Unless indicated otherwise, all cohomology groups of schemes are taken over the étale topology. We fix some general notation for a scheme X . For a point $x \in X$, $\kappa(x)$ denotes its residue field and \bar{x} denotes $\text{Spec}(\overline{\kappa(x)})$. If X has pure dimension, then for a non-negative integer q , X^q denotes the set of all points on X of codimension q . For an étale sheaf Λ of commutative rings on X , we write $D(X_{\text{ét}}, \Lambda)$ for the derived category of étale Λ -modules on X (cf. [Ha1], I, [BBD], §1). We write $D^+(X_{\text{ét}}, \Lambda)$ for the full subcategory of $D(X_{\text{ét}}, \Lambda)$ consisting of objects coming from complexes of étale Λ -modules bounded below. For $x \in X$ and the natural map $i_x : x \hookrightarrow X$, we define the functor $Ri_x^! : D^+(X_{\text{ét}}, \Lambda) \rightarrow D^+(x_{\text{ét}}, \Lambda)$ as $\xi^* R\varphi^!$, where φ denotes the closed immersion $\bar{\{x\}} \hookrightarrow X$ and ξ denotes the natural map $x \hookrightarrow \bar{\{x\}}$. If ξ is of finite type, then $Ri_x^!$ is right adjoint to Ri_{x*} , but otherwise it is not. If x is a generic point of X , $Ri_x^!$ agrees with i_x^* . For $\mathcal{F} \in D^+(X_{\text{ét}}, \Lambda)$, we often write $H_x^*(X, \mathcal{F})$ for $H_x^*(\text{Spec}(\mathcal{O}_{X,x}), \mathcal{F})$.

1.8. We fix some notation of arithmetic objects defined for a scheme X . For a positive integer m invertible on X , μ_m denotes the étale sheaf of m -th roots of unity. If X is a smooth variety over a perfect field of positive characteristic $p > 0$, then for integers $r \geq 1$ and $q \geq 0$, $W_r \Omega_{X, \log}^q$ denotes the étale subsheaf of the logarithmic part of the Hodge-Witt sheaf $W_r \Omega_X^q$ ([B11], [II1]). For $q < 0$, we define $W_r \Omega_{X, \log}^q$ as the zero sheaf. For a noetherian excellent scheme X (all schemes in this paper are of this kind), we will use the following notation. Let y and x be points on X such that x has codimension 1 in the closure $\bar{\{y\}} \subset X$. Let p be a prime number, and let i and n be non-negative integers. In [KCT], §1, Kato defined the boundary maps

$$\begin{aligned} H^{i+1}(y, \mu_{p^r}^{\otimes n+1}) &\longrightarrow H^i(x, \mu_{p^r}^{\otimes n}) && (\text{if } \text{ch}(x) \neq p), \\ H^0(y, W_r \Omega_{y, \log}^{n+1}) &\longrightarrow H^0(x, W_r \Omega_{x, \log}^n) && (\text{if } \text{ch}(y) = \text{ch}(x) = p), \\ H^{n+1}(y, \mu_{p^r}^{\otimes n+1}) &\longrightarrow H^0(x, W_r \Omega_{x, \log}^n) && (\text{if } \text{ch}(y) = 0 \text{ and } \text{ch}(x) = p). \end{aligned}$$

We write $\partial_{y,x}^{\text{val}}$ for these maps. See (3.2.3) for the construction of the last map.

1.9. Let X be a scheme, let $i : Z \hookrightarrow X$ be a closed immersion, and let $j : U \hookrightarrow X$ be the open complement $X \setminus Z$. Let m be a non-negative integer. For $\mathcal{K} \in D^+(X_{\text{ét}}, \mathbb{Z}/m\mathbb{Z})$, we define the morphism

$$\delta_{U,Z}^{\text{loc}}(\mathcal{K}) : Rj_* j^* \mathcal{K} \longrightarrow Ri_* Ri^! \mathcal{K}[1] \quad \text{in } D^+(X_{\text{ét}}, \mathbb{Z}/m\mathbb{Z})$$

as the connecting morphism associated with the semi-splitting short exact sequence of complexes $0 \rightarrow i_* i^! I^\bullet \rightarrow I^\bullet \rightarrow j_* j^* I^\bullet \rightarrow 0$ ([SGA4 $\frac{1}{2}$], Catégories Dérivées, I.1.2.4), where I^\bullet is an injective resolution of \mathcal{K} . The morphism $\delta_{X,U}^{\text{loc}}(\mathcal{K})$ is functorial in \mathcal{K} , and

$$\delta_{U,Z}^{\text{loc}}(\mathcal{K})[q] = (-1)^q \cdot \delta_{U,Z}^{\text{loc}}(\mathcal{K}[q]) \tag{1.9.1}$$

for an integer q . Note also that the triangle

$$\mathcal{K} \xrightarrow{j^*} Rj_* j^* \mathcal{K} \xrightarrow{\delta_{U,Z}^{\text{loc}}(\mathcal{K})} Ri_* Ri^! \mathcal{K}[1] \xrightarrow{i_*} \mathcal{K}[1] \tag{1.9.2}$$

is distinguished in $D^+(X_{\acute{e}t}, \mathbb{Z}/m\mathbb{Z})$, where the arrow i_* (resp. j^*) denotes the adjunction map $Ri_* Ri^! \rightarrow \text{id}$ (resp. $\text{id} \rightarrow Rj_* j^*$). We generalize the above connecting morphism to the following situation. Let y and x be points on X such that x has codimension 1 in the closure $\overline{\{y\}} \subset X$. Put $T := \overline{\{y\}} \subset X$ and $S := \text{Spec}(\mathcal{O}_{T,x})$, and let i_T (resp. i_y, i_x, ψ) be the natural map $T \hookrightarrow X$ (resp. $y \hookrightarrow X, x \hookrightarrow X, S \hookrightarrow T$). Then for $\mathcal{K} \in D^+(X_{\acute{e}t}, \mathbb{Z}/m\mathbb{Z})$, we define

$$\delta_{y,x}^{\text{loc}}(\mathcal{K}) := Ri_{T*} R\psi_* \{ \delta_{y,x}^{\text{loc}}(\psi^* Ri_T^! \mathcal{K}) \} : Ri_{y*} Ri_y^! \mathcal{K} \longrightarrow Ri_{x*} Ri_x^! \mathcal{K}[1] \quad (1.9.3)$$

($Ri_y^!$ and $Ri_x^!$ were defined in §1.6), which is a morphism in $D^+(X_{\acute{e}t}, \mathbb{Z}/m\mathbb{Z})$. These connecting morphisms for all points on X give rise to a local-global spectral sequence of sheaves on $X_{\acute{e}t}$

$$E_1^{u,v} = \bigoplus_{x \in X^u} R^{u+v} i_{x*} (Ri_x^! \mathcal{K}) \implies \mathcal{H}^{u+v}(\mathcal{K}).$$

For a closed immersion $i : Z \hookrightarrow X$, there is a localized variant

$$E_1^{u,v} = \bigoplus_{x \in X^u \cap Z} R^{u+v} i_{x*} (Ri_x^! \mathcal{K}) \implies i_* R^{u+v} i^! \mathcal{K}. \quad (1.9.4)$$

1.10. Let k be a field, and let X be a pure-dimensional scheme which is of finite type over $\text{Spec}(k)$. We call X a *normal crossing scheme* over $\text{Spec}(k)$, if it is everywhere étale locally isomorphic to

$$\text{Spec}(k[T_0, T_1, \dots, T_N]/(T_0 T_1 \cdots T_a))$$

for some integer a with $0 \leq a \leq N = \dim(X)$. This condition is equivalent to the assumption that X is everywhere étale locally embedded into a smooth variety over $\text{Spec}(k)$ as a normal crossing divisor.

1.11. Let A be a discrete valuation ring, and let K (resp. k) be the fraction field (resp. residue field) of A . Let X be a pure-dimensional scheme which is flat of finite type over $\text{Spec}(A)$. We call X a *regular semistable family* over $\text{Spec}(A)$, if it is regular and everywhere étale locally isomorphic to

$$\text{Spec}(A[T_0, T_1, \dots, T_N]/(T_0 T_1 \cdots T_a - \pi))$$

for some integer a with $0 \leq a \leq N = \dim(X/A)$, where π denotes a prime element of A . This condition is equivalent to the assumption that X is regular, $X \otimes_A K$ is smooth over $\text{Spec}(K)$, and $X \otimes_A k$ is reduced and a normal crossing divisor on X . If X is a regular semistable family over $\text{Spec}(A)$, then the closed fiber $X \otimes_A k$ is a normal crossing scheme over $\text{Spec}(k)$.

2. Preliminaries

In this section we review some fundamental facts on homological algebra and results of the author in [Sat], which will be used frequently in this paper.

2.1. Elementary facts from homological algebra. Let \mathcal{A} be an abelian category with enough injective objects, and let $D(\mathcal{A})$ be the derived category of complexes of objects of \mathcal{A} .

Lemma 2.1.1. *Let m and q be integers. Let \mathcal{K} be an object of $D(\mathcal{A})$ concentrated in degrees $\leq m$ and let \mathcal{K}' be an object of $D(\mathcal{A})$ concentrated in degrees ≥ 0 . Then we have*

$$\text{Hom}_{D(\mathcal{A})}(\mathcal{K}, \mathcal{K}'[-q]) = \begin{cases} \text{Hom}_{\mathcal{A}}(\mathcal{H}^m(\mathcal{K}), \mathcal{H}^0(\mathcal{K}')) & (\text{if } q = m), \\ 0 & (\text{if } q > m), \end{cases}$$

where for $n \in \mathbb{Z}$ and $\mathcal{L} \in D(\mathcal{A})$, $\mathcal{H}^n(\mathcal{L})$ denotes the n -th cohomology object of \mathcal{L} .

Proof. By the assumption that \mathcal{A} has enough injectives, the left hand side is written as the group of morphisms in the homotopy category of complexes of objects of \mathcal{A} ([SGA4 $\frac{1}{2}$], Catégories Dérivées, II, 2.3 (4)). The assertion follows from this fact. \square

Lemma 2.1.2. *Let $\mathcal{N}_1 \xrightarrow{f} \mathcal{N}_2 \xrightarrow{g} \mathcal{N}_3 \xrightarrow{h} \mathcal{N}_1[1]$ be a distinguished triangle in $D(\mathcal{A})$.*

- (1) *Let $i : \mathcal{K} \rightarrow \mathcal{N}_2$ be a morphism with $g \circ i = 0$ and suppose that $\mathrm{Hom}_{D(\mathcal{A})}(\mathcal{K}, \mathcal{N}_3[-1]) = 0$. Then there exists a unique morphism $i' : \mathcal{K} \rightarrow \mathcal{N}_1$ that i factors through.*
- (2) *Let $p : \mathcal{N}_2 \rightarrow \mathcal{K}$ be a morphism with $p \circ f = 0$ and suppose that $\mathrm{Hom}_{D(\mathcal{A})}(\mathcal{N}_1[1], \mathcal{K}) = 0$. Then there exists a unique morphism $p' : \mathcal{N}_3 \rightarrow \mathcal{K}$ that p factors through.*
- (3) *Suppose that $\mathrm{Hom}_{D(\mathcal{A})}(\mathcal{N}_2, \mathcal{N}_1) = 0$ (resp. $\mathrm{Hom}_{D(\mathcal{A})}(\mathcal{N}_3, \mathcal{N}_2) = 0$). Then relatively to a fixed triple $(\mathcal{N}_1, \mathcal{N}_3, h)$, the other triple (\mathcal{N}_2, f, g) is unique up to a unique isomorphism, and f (resp. g) is determined by the pair (\mathcal{N}_2, g) (resp. (\mathcal{N}_2, f)).*

Proof. These claims follow from the same arguments as in [BBD], 1.1.9. The details are straightforward and left to the reader. \square

2.2. Logarithmic Hodge-Witt sheaves. Throughout this subsection, n denotes a non-negative integer and r denotes a positive integer. Let k be a perfect field of positive characteristic p . Let X be a pure-dimensional scheme of finite type over $\mathrm{Spec}(k)$. For a point $x \in X$, let i_x be the canonical map $x \hookrightarrow X$. We define the étale sheaves $\nu_{X,r}^n$ and $\lambda_{X,r}^n$ on X as

$$\begin{aligned} \nu_{X,r}^n &:= \mathrm{Ker} \left(\bigoplus_{x \in X^0} i_{x*} W_r \Omega_{x,\log}^n \xrightarrow{\partial^{\mathrm{val}}} \bigoplus_{x \in X^1} i_{x*} W_r \Omega_{x,\log}^{n-1} \right), \\ \lambda_{X,r}^n &:= \mathrm{Im} \left((\mathbb{G}_{\mathrm{m},X})^{\otimes n} \xrightarrow{d\log} \bigoplus_{x \in X^0} i_{x*} W_r \Omega_{x,\log}^n \right), \end{aligned}$$

where ∂^{val} denotes the sum of $\partial_{y,x}^{\mathrm{val}}$'s with $y \in X^0$ and $x \in X^1$ (cf. §1.8). By definition, $\lambda_{X,r}^n$ is a subsheaf of $\nu_{X,r}^n$. If X is smooth, then both $\nu_{X,r}^n$ and $\lambda_{X,r}^n$ agree with the sheaf $W_r \Omega_{X,\log}^n$. See also Remark 3.3.8 (4) below.

We define the Gysin morphism for logarithmic Hodge-Witt sheaves as follows. We define the complex of sheaves $C_r^\bullet(X, n)$ on $X_{\mathrm{ét}}$ to be

$$\begin{array}{ccccccc} \bigoplus_{x \in X^0} i_{x*} W_r \Omega_{x,\log}^n & \xrightarrow{(-1)^{n-1} \cdot \partial} & \bigoplus_{x \in X^1} i_{x*} W_r \Omega_{x,\log}^{n-1} & \xrightarrow{(-1)^{n-1} \cdot \partial} & \dots & & \\ & \xrightarrow{(-1)^{n-1} \cdot \partial} & \bigoplus_{x \in X^q} i_{x*} W_r \Omega_{x,\log}^{n-q} & \xrightarrow{(-1)^{n-1} \cdot \partial} & \dots & & \end{array}$$

Here the first term is placed in degree 0 and ∂ denotes the sum of sheafified variants of $\partial_{y,x}^{\mathrm{val}}$'s with $y \in X^q$ and $x \in X^{q+1}$ (cf. §1.8). The fact $\partial \circ \partial = 0$ is due to Kato ([KCT], 1.7). If X is a normal crossing scheme, this complex is quasi-isomorphic to the sheaf $\nu_{X,r}^n$ by [Sat], 2.2.5 (1).

Definition 2.2.1 (cf. loc. cit., 2.4.1). *Let X be a normal crossing scheme over $\mathrm{Spec}(k)$ and let $i : Z \hookrightarrow X$ be a closed immersion of pure codimension $c \geq 0$. We define the Gysin morphism*

$$\mathrm{Gys}_i^n : \nu_{Z,r}^{n-c}[-c] \longrightarrow \mathrm{R}i^! \nu_{X,r}^n \quad \text{in } D^b(Z_{\mathrm{ét}}, \mathbb{Z}/p^r \mathbb{Z})$$

as the adjoint morphism of the composite morphism in $D^b(X_{\mathrm{ét}}, \mathbb{Z}/p^r \mathbb{Z})$

$$i_* \nu_{Z,r}^{n-c}[-c] \longrightarrow i_* C_r^\bullet(Z, n-c)[-c] \longrightarrow C_r^\bullet(X, n) \xleftarrow{\cong} \nu_{X,r}^n,$$

where the second arrow is the natural inclusion of complexes. See also Remark 2.2.6 below.

Theorem 2.2.2 (Purity, loc. cit., 2.4.2). *For $i : Z \hookrightarrow X$ as in Definition 2.2.1, Gys_i^n induces an isomorphism $\tau_{\leq c}(\mathrm{Gys}_i^n) : \nu_{Z,r}^{n-c}[-c] \xrightarrow{\cong} \tau_{\leq c} \mathrm{R}i^! \nu_{X,r}^n$.*

We next state the duality result in loc. cit. For integers $m, n \geq 0$, there is a natural biadditive pairing of sheaves

$$\nu_{X,r}^n \times \lambda_{X,r}^m \longrightarrow \nu_{X,r}^{m+n} \quad (2.2.3)$$

induced by the corresponding pairing on the generic points of X (cf. loc. cit., 3.1.1).

Theorem 2.2.4 (Duality, loc. cit., 1.2.2). *Let k be a finite field, and let X be a normal crossing scheme of dimension N which is proper over $\mathrm{Spec}(k)$. Then:*

- (1) *There is a trace map $\mathrm{tr}_X : H^{N+1}(X, \nu_{X,r}^N) \rightarrow \mathbb{Z}/p^r\mathbb{Z}$ such that for an arbitrary closed point $x \in X$ the composite map*

$$H^1(x, \mathbb{Z}/p^r\mathbb{Z}) \xrightarrow{\mathrm{Gys}_{i_x}^N} H^{N+1}(X, \nu_{X,r}^N) \xrightarrow{\mathrm{tr}_X} \mathbb{Z}/p^r\mathbb{Z}$$

coincides with the trace map of x , i.e., the map that sends a continuous character of $G_{\kappa(x)}$ to its value at the Frobenius substitution. Furthermore tr_X is bijective if X is connected.

- (2) *For integers q and n with $0 \leq n \leq N$, the natural pairing*

$$H^q(X, \nu_{X,r}^n) \times H^{N+1-q}(X, \lambda_{X,r}^{N-n}) \xrightarrow{(2.2.3)} H^{N+1}(X, \nu_{X,r}^N) \xrightarrow{\mathrm{tr}_X} \mathbb{Z}/p^r\mathbb{Z} \quad (2.2.5)$$

is a non-degenerate pairing of finite $\mathbb{Z}/p^r\mathbb{Z}$ -modules.

We will give the definition of tr_X in Remark 2.2.6 (4) below.

Remark 2.2.6. *We summarize the properties of the Gysin morphisms and the trace morphisms, which will be used in this paper.*

- (1) *The Gysin morphisms defined in Definition 2.2.1 satisfy the transitivity property.*
(2) *For $i : Z \hookrightarrow X$ as in Definition 2.2.1, Gys_i^n agrees with the Gysin morphism considered in [Sat], 2.4.1, up to a sign of $(-1)^c$. In particular if X and Z are smooth, then Gys_i^n agrees with the Gysin morphism $W_r \Omega_{Z,\log}^{n-c}[-c] \rightarrow R^i W_r \Omega_{X,\log}^n$ of Gros ([Gs], II.1) up to the sign $(-1)^c$ by [Sat], 2.3.1. This fact will be used in Lemma 6.4.1 below.*
(3) *Let X and Z be normal crossing schemes over $\mathrm{Spec}(k)$ of dimension N and d , respectively, and let $f : Z \rightarrow X$ be a separated morphism of schemes. We define the morphism*

$$\mathrm{tr}_f : Rf_! \nu_{Z,r}^d[d] \longrightarrow \nu_{X,r}^N[N] \quad \text{in } D^b(X_{\acute{e}t}, \mathbb{Z}/p^r\mathbb{Z})$$

by applying the same arguments as for [JSS], Theorem 2.9 to the complexes $C_r^\bullet(Z, d)[d]$ and $C_r^\bullet(X, N)[N]$. Then tr_f agrees with that in loc. cit., Theorem 2.9 up to the sign $(-1)^{N-d}$. In particular if X and Z are smooth and f is proper, then tr_f agrees with the Gysin morphism $Rf_ W_r \Omega_{Z,\log}^d[d] \rightarrow W_r \Omega_{X,\log}^N[N]$ due to Gros ([Gs], II.1) up to the sign $(-1)^{N-d}$.*

- (4) *We define the trace map tr_X in Theorem 2.2.4 (1) as the map induced by tr_f for $f : X \rightarrow \mathrm{Spec}(k)$ and the trace map of $\mathrm{Spec}(k)$. The map tr_X agrees with $(-1)^N$ -times of the trace morphism constructed in loc. cit., §3.4.*

3. Boundary maps on the sheaves of p -adic vanishing cycles

This section is devoted to technical preparations on the étale sheaves of p -adic vanishing cycles. The main results of this section are Theorem 3.4.2 and Corollary 3.5.2 below. Throughout this section, n and r denote integers with $n \geq 0$ and $r \geq 1$.

3.1. Milnor K -groups and boundary maps. We prepare some notation on Milnor K -groups. Let R be a commutative ring with unity. We define the 0-th Milnor K -group $K_0^M(R)$ as \mathbb{Z} . For $n \geq 1$, we define the n -th Milnor K -group $K_n^M(R)$ as $(R^\times)^{\otimes n}/J$, where J denotes the subgroup of $(R^\times)^{\otimes n}$ generated by elements of the form $x_1 \otimes \cdots \otimes x_n$ with $x_i + x_j = 0$ or 1 for some $1 \leq i < j \leq n$. An element $x_1 \otimes \cdots \otimes x_n \pmod J$ will be denoted by $\{x_1, \dots, x_n\}$. Now let L be a field endowed with a discrete valuation v . Let O_v be the valuation ring with respect to v , and let F_v be its residue field. Fix a prime element π_v of O_v . We define the homomorphism

$$\begin{aligned} \partial_{\pi_v}^M : K_n^M(L) &\longrightarrow K_{n-1}^M(F_v) \\ (\text{resp. } \text{sp}_{\pi_v} : K_n^M(L) &\longrightarrow K_n^M(F_v)) \end{aligned}$$

by the assignment

$$\begin{aligned} \{\pi_v, x_1, \dots, x_{n-1}\} &\mapsto \{\overline{x_1}, \dots, \overline{x_{n-1}}\} \quad (\text{resp. } 0) \\ \{x_1, \dots, x_n\} &\mapsto 0 \quad (\text{resp. } \{\overline{x_1}, \dots, \overline{x_n}\}) \end{aligned}$$

with each $x_i \in O_v^\times$ (cf. [BT], I.4.3). Here for $x \in O_v^\times$, \overline{x} denotes its residue class in F_v^\times . The map $\partial_{\pi_v}^M$ is called the *boundary map of Milnor K -groups*, and depends only on the valuation ideal $\mathfrak{p}_v \subset O_v$. We will denote $\partial_{\pi_v}^M$ by $\partial_{\mathfrak{p}_v}^M$. On the other hand, *the specialization map* sp_{π_v} depends on the choice of π_v , and its restriction to $\text{Ker}(\partial_{\mathfrak{p}_v}^M) \subset K_n^M(L)$ depends only on \mathfrak{p}_v . Indeed, $\text{Ker}(\partial_{\pi_v}^M)$ is generated by the image of $(O_v^\times)^{\otimes n}$ and symbols of the form $\{1 + a, x_1, \dots, x_{n-1}\}$ with $a \in \mathfrak{p}_v$ and each $x_i \in L^\times$.

3.2. Boundary map in a geometric setting. Let p be a prime number. Let K be a henselian discrete valuation field of characteristic 0 whose residue field k has characteristic p . Let O_K be the integer ring of K . Let X be a regular semistable family over $\text{Spec}(O_K)$ of pure dimension (cf. §1.11), or more generally, a scheme over $\text{Spec}(O_K)$ satisfying the following condition:

Condition 3.2.1. *There exist a discrete valuation subring $O' \subset O_K$ with O_K/O' finite and a pure-dimensional regular semistable family X' over $\text{Spec}(O')$ with $X \simeq X' \otimes_{O'} O_K$.*

Later in §3.4 and §3.5 below, the extension O_K/O' will be assumed to be unramified or tamely ramified. Let Y be the reduced divisor on X defined by a prime element $\pi \in O_K$, and let ι and j be as follows:

$$X_K \xrightarrow{j} X \xleftarrow{\iota} Y.$$

In this section, we are concerned with the étale sheaf

$$M_r^n := \iota^* R^n j_* \mu_{p^r}^{\otimes n}$$

on Y and the composite map of étale sheaves

$$\partial_{X,r}^n : M_r^n \longrightarrow \bigoplus_{y \in Y^0} i_{y*} i_y^* M_r^n \xrightarrow{\partial^{\text{val}}} \bigoplus_{y \in Y^0} i_{y*} W_r \Omega_{y,\log}^{n-1}. \quad (3.2.2)$$

Here for a point $y \in Y$, i_y denotes the canonical map $y \hookrightarrow Y$. For each $y \in Y^0$ the second arrow ∂^{val} is defined as follows:

$$(i_y^* M_r^n)_{\overline{y}} \simeq K_n^M(\mathcal{O}_{X,\overline{y}}^{\text{sh}}[1/p])/p^r \xrightarrow{\partial_{\mathfrak{p}_y}^M} K_{n-1}^M(\kappa(\overline{y}))/p^r \xrightarrow{d\log} W_r \Omega_{\overline{y},\log}^{n-1}, \quad (3.2.3)$$

where \mathfrak{p}_y denotes the maximal ideal of the discrete valuation ring $\mathcal{O}_{X,\overline{y}}^{\text{sh}}$. The first isomorphism is due to Bloch-Kato [BK], (5.12), and the arrow $\partial_{\mathfrak{p}_y}^M$ denotes the boundary map of Milnor K -groups. We first show the following fundamental fact:

Lemma 3.2.4. *The image of $\partial_{X,r}^n$ is contained in $\nu_{Y,r}^{n-1}$. See §2.2 for the definition of $\nu_{Y,r}^{n-1}$.*

Proof. For $x \in X_K$, let ι_x be the natural map $x \hookrightarrow X$. Consider a diagram on $Y_{\text{ét}}$

$$\begin{array}{ccccccc} M_r^n & \xrightarrow{a} & \bigoplus_{x \in (X_K)^0} \iota_x^* R^n \iota_{x*} \mu_{p^r}^{\otimes n} & \xrightarrow{\partial_2} & \bigoplus_{x \in (X_K)^1} \iota_x^* R^{n-1} \iota_{x*} \mu_{p^r}^{\otimes n-1} & & \\ & & \downarrow \partial_1 & & \downarrow \partial_3 & & \\ 0 & \longrightarrow & \nu_{Y,r}^{n-1} & \longrightarrow & \bigoplus_{y \in Y^0} i_{y*} W_r \Omega_{y,\log}^{n-1} & \xrightarrow{\partial_4} & \bigoplus_{y \in Y^1} i_{y*} W_r \Omega_{y,\log}^{n-2}. \end{array}$$

Here a denotes the canonical adjunction map and each ∂_i ($i = 1, \dots, 4$) is the sum of sheafified variants of boundary maps in §1.8. The right square is anti-commutative by a result of Kato [KCT], 1.7. The upper row is a complex by the smoothness of X_K . The lower row is exact by the definition of $\nu_{Y,r}^{n-1}$. Hence we have $\text{Im}(\partial_{X,r}^n) = \text{Im}(\partial_1 \circ a) \subset \nu_{Y,r}^{n-1}$. \square

By this lemma, $\partial_{X,r}^n$ induces a map

$$\sigma_{X,r}^n : M_r^n \longrightarrow \nu_{Y,r}^{n-1}, \quad (3.2.5)$$

which is a geometric version of the boundary map of Milnor K -groups (modulo p^r).

3.3. Bloch-Kato-Hyodo theorem. We give a brief review of the Bloch-Kato-Hyodo theorem on the structure of M_r^n , which will be useful in this and later sections. See also Remark 3.3.8 below. We define the étale sheaf $\mathcal{K}_{n,X_K/Y}^M$ on Y as $(\iota^* j_* \mathcal{O}_{X_K}^\times)^{\otimes n} / J$, where J denotes the subsheaf generated by local sections of the form $x_1 \otimes \cdots \otimes x_n$ ($x_i \in \iota^* j_* \mathcal{O}_{X_K}^\times$) with $x_i + x_j = 0$ or 1 for some $1 \leq i < j \leq n$. There is a natural map due to Bloch and Kato [BK], (1.2)

$$\mathcal{K}_{n,X_K/Y}^M \longrightarrow M_r^n, \quad (3.3.1)$$

which is a geometric version of Tate's Galois symbol map. We define the filtrations U^\bullet and V^\bullet on M_r^n using this map, as follows.

Definition 3.3.2. (1) *Let π be a prime element of \mathcal{O}_K . Let $U_{X_K}^0$ be the full sheaf $\iota^* j_* \mathcal{O}_{X_K}^\times$. For $q \geq 1$, let $U_{X_K}^q$ be the étale subsheaf of $\iota^* j_* \mathcal{O}_{X_K}^\times$ generated by local sections of the form $1 + \pi^q \cdot a$ with $a \in \iota^* \mathcal{O}_X$. We define the subsheaf $U^q \mathcal{K}_{n,X_K/Y}^M$ ($q \geq 0$) of $\mathcal{K}_{n,X_K/Y}^M$ as the part generated by $U_{X_K}^q \otimes \{\iota^* j_* \mathcal{O}_{X_K}^\times\}^{\otimes n-1}$.*

(2) *We define the subsheaf $U^q M_r^n$ ($q \geq 0$) of M_r^n as the image of $U^q \mathcal{K}_{n,X_K/Y}^M$ under (3.3.1). We define the subsheaf $V^q M_r^n$ ($q \geq 0$) of M_r^n as the part generated by $U^{q+1} M_r^n$ and the image of $U^q \mathcal{K}_{n-1,X_K/Y}^M \otimes \langle \pi \rangle$ under (3.3.1).*

Remark 3.3.3. (1) $U^\bullet \mathcal{K}_{n,X_K/Y}^M$ and $U^\bullet M_r^n$ are independent of the choice of $\pi \in \mathcal{O}_K$ by definition.

(2) $V^0 M_r^n$ and $V^\bullet M_1^n$ are independent of the choice of $\pi \in \mathcal{O}_K$ by Theorem 3.3.7 below.

To describe the graded pieces $\text{gr}_{U/V}^q M_r^n := U^q M_r^n / V^q M_r^n$ and $\text{gr}_{V/U}^q M_r^n := V^q M_r^n / U^{q+1} M_r^n$ (especially in the case where Y is not smooth), we introduce some notation from log geometry in étale topology See [Ka3] for the general framework of log schemes in the Zariski topology. See also e.g., [KF], §2 and §3 for the corresponding framework in the étale topology. For a regular scheme Z and a normal crossing divisor D on Z , we define the étale sheaf $\mathcal{L}_Z(D)$ of pointed sets on Z as

$$\mathcal{L}_Z(D) := \{f \in \mathcal{O}_Z; f \text{ is invertible outside of } D\} \subset \mathcal{O}_Z.$$

We regard this sheaf as a sheaf of monoids by the multiplication of functions. The natural inclusion $\mathcal{L}_Z(D) \hookrightarrow \mathcal{O}_Z$ gives a log structure on Z , and the associated sheaf $\mathcal{L}_Z(D)^{\text{gp}}$ of abelian groups is étale locally generated by \mathcal{O}_Z^\times and primes of \mathcal{O}_Z defining irreducible components of D . Now we return to our situation. Put $B := \text{Spec}(O_K)$ and $s := \text{Spec}(k)$, and let \mathcal{L}_s be the inverse image of $\mathcal{L}_B := \mathcal{L}_B(s)$ onto $s_{\text{ét}}$ in the sense of log structures. We define the log structure \mathcal{L}_Y on $Y_{\text{ét}}$ as follows. By 3.2.1, there exist a discrete valuation subring $O' \subset O_K$ and a regular semistable family over $B' := \text{Spec}(O')$ such that O_K/O' is finite totally ramified and such that $X' \otimes_{O'} O_K \simeq X$. Note that Y is a normal crossing divisor on X' . We fix such a pair (O', X') and define the log structure \mathcal{L}_X on $X_{\text{ét}}$ as that obtained from $\mathcal{L}_{X'}(Y)$ by base-change (in the category of log schemes):

$$(X, \mathcal{L}_X) := (X', \mathcal{L}_{X'}(Y)) \times_{(B', \mathcal{L}_{B'}(s))} (B, \mathcal{L}_B).$$

Finally we define \mathcal{L}_Y as the inverse image of \mathcal{L}_X onto $Y_{\text{ét}}$ in the sense of log structures. Let us recall the following fundamental facts:

- The log scheme $(X', \mathcal{L}_{X'}(Y))$ (resp. (X, \mathcal{L}_X) , (Y, \mathcal{L}_Y)) is smooth over the log scheme $(B', \mathcal{L}_{B'}(s))$ (resp. (B, \mathcal{L}_B) , (s, \mathcal{L}_s)) with respect to the natural map induced by the structure map $X' \rightarrow B'$.
- The relative differential modules $\omega_{(Y, \mathcal{L}_Y)/(s, \mathcal{L}_s)}^*$ on $Y_{\text{ét}}$ are locally free \mathcal{O}_Y -modules of finite rank and coincide with the modified differential modules ω_Y^* defined in [Hy1].
- There is a natural surjective homomorphism

$$\iota^* j_* \mathcal{O}_{X_K}^\times \simeq \iota^*(\mathcal{L}_X^{\text{gp}}) \longrightarrow \mathcal{L}_Y^{\text{gp}} \quad (3.3.4)$$

of sheaves of abelian groups on $Y_{\text{ét}}$ (see [Ts1], (3.2.1) for the first isomorphism).

Let us recall further some facts relating log structures and differential modules.

- By the definition of $\omega_{(Y, \mathcal{L}_Y)/(s, \mathcal{L}_s)}^1 = \omega_Y^1$, there is a natural map taking the logarithmic differentials of local sections of $\mathcal{L}_Y^{\text{gp}}$:

$$d\log : \mathcal{L}_Y^{\text{gp}} \longrightarrow \omega_Y^1. \quad (3.3.5)$$

- There is an analogous map for each $n \geq 0$ and $r > 0$

$$d\log : (\mathcal{L}_Y^{\text{gp}})^{\otimes n} \longrightarrow \bigoplus_{y \in Y^0} i_{y*} W_r \Omega_{y, \log}^n. \quad (3.3.6)$$

The modified logarithmic Hodge-Witt sheaf $W_r \omega_{Y, \log}^n$ defined by Hyodo ([Hy1], (1.5)) agrees with the image of this map. See also Remark 3.3.8 (4) below.

Now we state the theorems of Bloch-Kato [BK], (1.4) and Hyodo [Hy1], (1.6). For local sections $x_i \in \iota^* j_* \mathcal{O}_{X_K}^\times$ ($1 \leq i \leq n$), we will denote the image of $\{x_1, x_2, \dots, x_n\} \in \mathcal{K}_{n, X_K/Y}^M$ under the symbol map (3.3.1) again by $\{x_1, x_2, \dots, x_n\}$, for simplicity.

Theorem 3.3.7 (Bloch-Kato/Hyodo). (1) *The symbol map (3.3.1) is surjective, that is, the subsheaf $U^0 M_r^n$ is the full sheaf M_r^n for any $n \geq 0$ and $r > 0$.*

(2) *There are isomorphisms*

$$\text{gr}_{U/V}^0 M_r^n \simeq W_r \omega_{Y, \log}^n; \quad \{x_1, x_2, \dots, x_n\} \bmod V^0 M_r^n \mapsto d\log(\overline{x_1} \otimes \overline{x_2} \otimes \cdots \otimes \overline{x_n}),$$

$$\text{gr}_{V/U}^0 M_r^n \simeq W_r \omega_{Y, \log}^{n-1}; \quad \{x_1, \dots, x_{n-1}, \pi\} \bmod U^1 M_r^n \mapsto d\log(\overline{x_1} \otimes \cdots \otimes \overline{x_{n-1}}),$$

where for $x \in \iota^* j_* \mathcal{O}_{X_K}^\times$, \overline{x} denotes its image into $\mathcal{L}_Y^{\text{gp}}$ via (3.3.4).

- (3) Let e be the absolute ramification index of K , and let $r = 1$. Then for q with $1 \leq q < e' := pe/(p-1)$, there are isomorphisms

$$\begin{aligned} \mathrm{gr}_{U/V}^q M_1^n &\simeq \begin{cases} \omega_Y^{n-1}/\mathcal{B}_Y^{n-1} & (p \nmid q), \\ \omega_Y^{n-1}/\mathcal{Z}_Y^{n-1} & (p \mid q), \end{cases} \\ \mathrm{gr}_{V/U}^q M_1^n &\simeq \omega_Y^{n-2}/\mathcal{Z}_Y^{n-2}, \end{aligned}$$

given by the following, respectively:

$$\begin{aligned} \{1 + \pi^q a, x_1, \dots, x_{n-1}\} \bmod V^q M_1^n &\mapsto \begin{cases} \bar{a} \cdot d\log(\bar{x}_1) \wedge \cdots \wedge d\log(\bar{x}_{n-1}) \bmod \mathcal{B}_Y^{n-1} & (p \nmid q) \\ \bar{a} \cdot d\log(\bar{x}_1) \wedge \cdots \wedge d\log(\bar{x}_{n-1}) \bmod \mathcal{Z}_Y^{n-1} & (p \mid q) \end{cases} \\ \{1 + \pi^q a, x_1, \dots, x_{n-2}, \pi\} \bmod U^{q+1} M_1^n &\mapsto \bar{a} \cdot d\log(\bar{x}_1) \wedge \cdots \wedge d\log(\bar{x}_{n-2}) \bmod \mathcal{Z}_Y^{n-2}, \end{aligned}$$

where \mathcal{B}_Y^m (resp. \mathcal{Z}_Y^m) denotes the image of $d : \omega_Y^{m-1} \rightarrow \omega_Y^m$ (resp. the kernel of $d : \omega_Y^m \rightarrow \omega_Y^{m+1}$), a denotes a local section of \mathcal{O}_X and \bar{a} denotes its residue class in \mathcal{O}_Y .

- (4) We have $U^q M_1^n = V^q M_1^n = 0$ for any $q \geq e'$.

Remark 3.3.8. (1) By Theorem 3.3.7 (1) and (2), the natural adjunction map

$$M_r^n / U^1 M_r^n \longrightarrow \bigoplus_{y \in Y^0} i_{y*} i_y^*(M_r^n / U^1 M_r^n) \quad (3.3.9)$$

is injective. We will use this injectivity to calculate the kernel of the map $\sigma_{X,r}^n$ defined in (3.2.5). See the proof of Theorem 3.4.2 below.

- (2) If Y is smooth over $s = \mathrm{Spec}(k)$, then we have $W_r \omega_{Y,\log}^m = W_r \Omega_{Y,\log}^m$ and $\omega_Y^m = \Omega_Y^m = \Omega_{Y/k}^m$, and the isomorphisms in Theorem 3.3.7 (2) yield the direct decomposition

$$M_r^n / U^1 M_r^n \simeq W_r \Omega_{Y,\log}^n \oplus W_r \Omega_{Y,\log}^{n-1} \quad (3.3.10)$$

(cf. [BK], (1.4.1.i)). By this decomposition, it is easy to see that the kernel of $\sigma_{X,r}^n$ is generated by $U^1 M_r^n$ and the image of $(i^* \mathcal{O}_X^\times)^{\otimes n}$ under (3.3.1). In the next subsection, we will extend the last fact to the regular semistable case, although the decomposition (3.3.10) does not hold any longer in that case.

- (3) Theorem 3.3.7 (3) and (4) will be used in later sections.
(4) There are inclusions of étale sheaves (cf. [Sat], 4.2.1)

$$\lambda_{Y,r}^n \subset W_r \omega_{Y,\log}^n \subset \nu_{Y,r}^n.$$

These inclusions are not equalities, in general (cf. loc. cit., 4.2.3). If $n = \dim(Y)$, then we have $W_r \omega_{Y,\log}^n = \nu_{Y,r}^n$ by loc. cit., 1.3.2.

3.4. Structure of $\mathrm{Ker}(\sigma_{X,r}^n)$. We define the étale subsheaf FM_r^n of M_r^n as the part generated by $U^1 M_r^n$ and the image of $(i^* \mathcal{O}_X^\times)^{\otimes n}$ under (3.3.1). In the rest of this section, we are concerned with the map $\sigma_{X,r}^n$ in (3.2.5) and the filtration

$$0 \subset U^1 M_r^n \subset FM_r^n \subset M_r^n.$$

Remark 3.4.1. Clearly, FM_r^n is contained in the kernel of $\sigma_{X,r}^n$.

The main result of this section is the following theorem, which plays an important role in later sections (see also Corollary 3.5.2 below):

Theorem 3.4.2. *Suppose that X is a regular semistable family over $\mathrm{Spec}(O_K)$. Then $\sigma_{X,r}^n$ induces an isomorphism*

$$M_r^n / FM_r^n \xrightarrow{\cong} \nu_{Y,r}^{n-1}, \quad (3.4.3)$$

that is, $\sigma_{X,r}^n$ is surjective and $FM_r^n = \mathrm{Ker}(\sigma_{X,r}^n)$. Furthermore there is an isomorphism

$$FM_r^n / U^1 M_r^n \xrightarrow{\cong} \lambda_{Y,r}^n \quad (3.4.4)$$

sending the symbol $\{x_1, x_2, \dots, x_n\}$ ($x_i \in \iota^* \mathcal{O}_X^\times$) to $d\log(\overline{x}_1 \otimes \overline{x}_2 \otimes \dots \otimes \overline{x}_n)$. Here for a section $x \in \iota^* \mathcal{O}_X^\times$, \overline{x} denotes its residue class in \mathcal{O}_Y^\times . See §2.2 for the definition of $\lambda_{Y,r}^n$.

Remark 3.4.5. *Theorem 3.4.2 is not included in Theorem 3.3.7 unless X is smooth over O_K . See also Remark 3.3.8 (2). In fact, $V^0 M_r^n$ is not related to FM_r^n directly. However, Theorem 3.3.7 (1) and (2) play a key role in the proof of Theorem 3.4.2 as the injectivity of (3.3.9).*

We first prove the following lemma, which is an essential step in the proof of Theorem 3.4.2:

Lemma 3.4.6. *Let*

$$\tau : \mathrm{Ker}(\sigma_{X,r}^n) / U^1 M_r^n \longrightarrow \bigoplus_{y \in Y^0} i_{y*} W_r \Omega_{y,\log}^n$$

be the natural map induced by the first map in (3.2.2) and an exact sequence

$$0 \longrightarrow \bigoplus_{y \in Y^0} i_{y*} W_r \Omega_{y,\log}^n \longrightarrow \bigoplus_{y \in Y^0} i_{y*} i_y^*(M_r^n / U^1 M_r^n) \xrightarrow{\partial^{\mathrm{val}}} \bigoplus_{y \in Y^0} i_{y*} W_r \Omega_{y,\log}^{n-1}$$

(cf. Remark 3.3.8 (2), see (3.2.2) for ∂^{val}). Then τ is injective, and $\mathrm{Im}(\tau)$ is contained in $\lambda_{Y,r}^n$.

Proof. The injectivity of τ immediately follows from that of (3.3.9). We prove that $\mathrm{Im}(\tau)$ is contained in $\lambda_{Y,r}^n$. Since the problem is étale local on Y , we may assume that Y has simple normal crossings on X . For $y \in Y^0$, let Y_y be the irreducible component of Y whose generic point is y . For $x \in Y$, let i_x be the canonical map $x \hookrightarrow Y$. Let $Y^{(1)}$ (resp. $Y^{(2)}$) be the disjoint union of irreducible components of Y (resp. the disjoint union of intersections of two distinct irreducible components of Y), and let $a_i : Y^{(i)} \rightarrow Y$ ($i = 1, 2$) be the natural map. Fix an ordering on the set Y^0 . There is a Čech restriction map $\check{r} : a_{1*} W_r \Omega_{Y^{(1)},\log}^n \rightarrow a_{2*} W_r \Omega_{Y^{(2)},\log}^n$, and its kernel agrees with $\lambda_{Y,r}^n$ by [Sat], 3.2.1. Our task is to prove the following two claims:

(1) *For arbitrary points $y \in Y^0$ and $x \in (Y_y)^1$, the composite map*

$$\alpha_{y,x} : \mathrm{Ker}(\sigma_{X,r}^n) \xrightarrow{\tau_y} i_{y*} W_r \Omega_{y,\log}^n \xrightarrow{\partial_{y,x}^{\mathrm{val}}} i_{x*} W_r \Omega_{x,\log}^{n-1}$$

is zero, where τ_y denotes the natural map induced by τ . Consequently, τ induces a map $\tau' : \mathrm{Ker}(\sigma_{X,r}^n) \rightarrow a_{1*} W_r \Omega_{Y^{(1)},\log}^n$.

(2) *The following composite map is zero:*

$$\beta : \mathrm{Ker}(\sigma_{X,r}^n) \xrightarrow{\tau'} a_{1*} W_r \Omega_{Y^{(1)},\log}^n \xrightarrow{\check{r}} a_{2*} W_r \Omega_{Y^{(2)},\log}^n.$$

Proof of Claim (1). It suffices to show that the stalk $(\alpha_{y,x})_{\overline{x}}$ is the zero map. Let Y_{sing} be the singular locus of Y . The case $x \notin (Y_{\mathrm{sing}})^0$ immediately follows from the direct decomposition (3.3.10). To show the case $x \in (Y_{\mathrm{sing}})^0$, we fix some notation. Put $R := \mathcal{O}_{X,\overline{x}}^{\mathrm{sh}}$, which is a strict henselian regular local ring of dimension 2. Let T_1 and T_2 be the irreducible components of $\mathrm{Spec}(\mathcal{O}_{Y,\overline{x}}^{\mathrm{sh}})$. We suppose that T_1 lies above Y_y . Fix a prime element $t_i \in R$ ($i = 1, 2$) defining T_i . Put $w_1 := t_1 \bmod (t_2) \in R/(t_2)$ and $w_2 := t_2 \bmod (t_1) \in R/(t_1)$. Because the divisor $T_1 \cup T_2 \subset \mathrm{Spec}(R)$ has simple normal crossings, $R/(t_1)$ and $R/(t_2)$ are discrete valuation rings

and w_1 (resp. w_2) is a prime element in $R/(t_2)$ (resp. $R/(t_1)$). Let η_i ($i = 1, 2$) be the generic point of T_i . There is a commutative diagram with exact rows

$$\begin{array}{ccccc} \text{Ker}(\partial) & \longrightarrow & K_n^M(R[1/p])/p^r & \xrightarrow{\partial} & K_{n-1}^M(\kappa(\eta_1))/p^r \oplus K_{n-1}^M(\kappa(\eta_2))/p^r \\ \downarrow & & \text{(3.3.1)} \downarrow & & d\log \downarrow \simeq \\ \text{Ker}(\sigma_{X,r}^n)_{\bar{x}} & \xrightarrow{\subset} & (M_r^n)_{\bar{x}} & \xrightarrow{(\sigma_{X,r}^n)_{\bar{x}}} & (i_{y^*} W_r \Omega_{y,\log}^{n-1})_{\bar{x}} \oplus (i_{y'^*} W_r \Omega_{y',\log}^{n-1})_{\bar{x}}. \end{array} \quad (3.4.7)$$

Here ∂ is the direct sum of the boundary maps of Milnor K -groups modulo p^r , and y' denotes the generic point of Y corresponding to T_2 . In this diagram, the right vertical arrow is bijective by a theorem of Bloch-Gabber-Kato [BK], (2.1), and the central vertical map is surjective by Theorem 3.3.7 (1). Hence the left vertical map, resulting from the right square, is surjective. On the other hand, there is a composite map

$$\text{sp}_{t_1, R[1/p]} : K_n^M(R[1/p])/p^r \longrightarrow K_n^M(\text{Frac}(R))/p^r \xrightarrow{\text{sp}_{t_1}} K_n^M(\kappa(\eta_1))/p^r. \quad (3.4.8)$$

See §3.1 for sp_{t_1} . The restriction of this map to $\text{Ker}(\partial)$ fits into a commutative diagram

$$\begin{array}{ccccc} \text{Ker}(\partial) & \xrightarrow{\text{sp}_{t_1, R[1/p]}|_{\text{Ker}(\partial)}} & K_n^M(\kappa(\eta_1))/p^r & \xrightarrow{\partial_{(w_2)}^M} & K_{n-1}^M(\kappa(\bar{x}))/p^r \\ \text{surj.} \downarrow & & d\log \downarrow & & d\log \downarrow \\ \text{Ker}(\sigma_{X,r}^n)_{\bar{x}} & \xrightarrow{(\tau_y)_{\bar{x}}} & (i_{y^*} W_r \Omega_{y,\log}^n)_{\bar{x}} & \xrightarrow{(\partial_{y,x}^{\text{val}})_{\bar{x}}} & W_r \Omega_{\bar{x},\log}^{n-1}, \end{array} \quad (3.4.9)$$

where the composite of the lower row gives $(\alpha_{y,x})_{\bar{x}}$. The composite of the upper row is the zero map by a commutative diagram of Milnor K -groups modulo p^r

$$\begin{array}{ccc} K_n^M(R[1/p])/p^r & \xrightarrow{\partial_{(t_2)}^M} & K_{n-1}^M(\kappa(\eta_2))/p^r \\ \text{sp}_{t_1, R[1/p]} \downarrow & & \downarrow \text{sp}_{w_1} \\ K_n^M(\kappa(\eta_1))/p^r & \xrightarrow{\partial_{(w_2)}^M} & K_{n-1}^M(\kappa(\bar{x}))/p^r, \end{array} \quad (3.4.10)$$

whose commutativity is shown explicitly by the direct decomposition $R[1/p]^\times \simeq R^\times \times \langle t_1 \rangle \times \langle t_2 \rangle$. Hence $(\alpha_{y,x})_{\bar{x}}$ is the zero map by the diagram (3.4.9), and we obtain the claim (1).

Proof of Claim (2). Let Z be a connected component of $Y^{(2)}$. Let Y_1 and Y_2 be the irreducible components of Y such that $a_2(Z) \subset Y_1 \cap Y_2$. Our task is to show that the composite map

$$\beta_Z : \text{Ker}(\sigma_{X,r}^n) \xrightarrow{\tau'} a_{1^*}(W_r \Omega_{Y_1,\log}^n \oplus W_r \Omega_{Y_2,\log}^n) \xrightarrow{\check{r}} a_{2^*} W_r \Omega_{Z,\log}^n$$

is zero, where the last map sends (ω_1, ω_2) ($\omega_i \in a_{1^*} W_r \Omega_{Y_i,\log}^n$) to $\omega_1|_Z - \omega_2|_Z$. Let x be the generic point of $a_2(Z)$. Since the canonical map $a_{2^*} W_r \Omega_{Z,\log}^n \rightarrow i_{x^*} W_r \Omega_{x,\log}^n$ is injective, we have only to show that the stalk $(\beta_Z)_{\bar{x}}$ is zero. Put $R := \mathcal{O}_{X,\bar{x}}^{\text{sh}}$, and let the notation be as in the proof of Claim (1). Suppose that T_i ($i = 1, 2$) is the irreducible component of $\text{Spec}(\mathcal{O}_{Y,\bar{x}}^{\text{sh}})$ lying above Y_i . Let N_i ($i = 1, 2$) be the kernel of the boundary map $K_n^M(\kappa(\eta_i))/p^r \rightarrow K_{n-1}^M(\kappa(\bar{x}))/p^r$. By the commutative diagram (3.4.10), $\text{sp}_{t_i, R[1/p]}$ in (3.4.8) induces a map

$$f_i : \text{Ker}(\partial) \longrightarrow N_i.$$

See (3.4.7) for ∂ . This map fits into a commutative diagram

$$\begin{array}{ccccc}
 \text{Ker}(\partial) & \xrightarrow{(f_1, f_2)} & N_1 \oplus N_2 & \xrightarrow{f_3} & K_n^M(\kappa(\bar{x}))/p^r \\
 \text{surj.} \downarrow & & \text{dlog} \downarrow & & \text{dlog} \downarrow \\
 \text{Ker}(\sigma_{X,r}^n)_{\bar{x}} & \xrightarrow{\tau'} & (W_r \Omega_{T_1, \log}^n \oplus W_r \Omega_{T_2, \log}^n)_{\bar{x}} & \xrightarrow{\tilde{r}} & W_r \Omega_{\bar{x}, \log}^n,
 \end{array} \tag{3.4.11}$$

where f_3 sends (u_1, u_2) ($u_i \in N_i$) to $\text{sp}_{w_2}(u_1) - \text{sp}_{w_1}(u_2)$. See (3.4.7) for the left vertical map. The composite of the lower row gives $(\beta_Z)_{\bar{x}}$. The composite of the upper row is zero by a commutative diagram

$$\begin{array}{ccc}
 K_n^M(R[1/p])/p^r & \xrightarrow{\text{sp}_{t_2, R[1/p]}} & K_n^M(\kappa(\eta_2))/p^r \\
 \text{sp}_{t_1, R[1/p]} \downarrow & & \downarrow \text{sp}_{w_1} \\
 K_n^M(\kappa(\eta_1))/p^r & \xrightarrow{\text{sp}_{w_2}} & K_n^M(\kappa(\bar{x}))/p^r,
 \end{array}$$

whose commutativity is checked in the same way as for (3.4.10). Thus $(\beta_Z)_{\bar{x}}$ is the zero map, and we obtain the claim (2) and Lemma 3.4.6. \square

Proof of Theorem 3.4.2. The surjectivity of (3.4.3) follows from the same argument as for [Sat], 2.4.6 (see the surjectivity of the map (2.4.9) in loc. cit.). We prove the injectivity of (3.4.3) and construct the bijection (3.4.4). There are injective maps

$$FM_r^n/U^1M_r^n \hookrightarrow \text{Ker}(\sigma_{X,r}^n)/U^1M_r^n \xrightarrow{\tau} \lambda_{Y,r}^n$$

(see Lemma 3.4.6 for τ). These two arrows are both bijective, because the sheaves $FM_r^n/U^1M_r^n$ and $\lambda_{Y,r}^n$ are generated by symbols from $(\iota^* \mathcal{O}_X^\times)^{\otimes n}$ and $(\mathcal{O}_Y^\times)^{\otimes n}$, respectively. Therefore we have $FM_r^n = \text{Ker}(\sigma_{X,r}^n)$ as subsheaves of M_r^n and the composite of the above two maps gives the desired bijective map (3.4.4). This completes the proof of Theorem 3.4.2.

3.5. Tamely ramified case. Assume that X satisfies the following condition over O_K :

Condition 3.5.1. *There exist a discrete valuation subring $O' \subset O_K$ with O_K/O' finite tamely ramified and a regular semistable family X' over O' with $X \simeq X' \otimes_{O'} O_K$.*

Let Y and M_r^n (resp. $U^1M_r^n$, FM_r^n) be as we defined in §3.2 (resp. §3.3, §3.4). By Theorems 3.3.7 and 3.4.2, we obtain

Corollary 3.5.2. *The map $\sigma_{X,r}^n$ induces an isomorphism $M_r^n/FM_r^n \simeq \nu_{Y,r}^{n-1}$, and there is an isomorphism $FM_r^n/U^1M_r^n \simeq \lambda_{Y,r}^n$ described in the same way as (3.4.4).*

Proof. The second assertion is an immediate consequence of Remark 3.3.8 (1) for X and the definitions of FM_r^n and $\lambda_{Y,r}^n$ (cf. Lemma 3.4.6). We prove the first assertion. Since the problem is étale local on Y , we may assume that O_K/O' is totally tamely ramified. Then the divisor on X' defined by a prime element $\pi' \in O'$ agrees with Y . Let e_1 be the ramification index of O_K/O' and let ι' and j' be as follows:

$$X'_{K'} \xrightarrow{j'} X' \xleftarrow{\iota'} Y,$$

where K' denotes $\text{Frac}(O')$. Let $M_{r,X'}^n$ be the étale sheaf $\iota'^* R^n j'_* \mu_{p^r}^{\otimes n}$. There is a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & FM_{r,X'}^n/U^1 M_{r,X'}^n & \longrightarrow & M_{r,X'}^n/U^1 M_{r,X'}^n & \xrightarrow{\sigma_{X',r}^n \bmod U^1 M_{r,X'}^n} & \nu_{Y,r} \longrightarrow 0 \\ & & \downarrow & & \text{res} \downarrow & & \times e_1 \downarrow \\ 0 & \longrightarrow & \text{Ker}(\sigma_{X,r}^n)/U^1 M_r^n & \longrightarrow & M_r^n/U^1 M_r^n & \xrightarrow{\sigma_{X,r}^n \bmod U^1 M_r^n} & \nu_{Y,r}, \end{array}$$

where the exactness of the upper row follows from Theorem 3.4.2. Because $(e_1, p) = 1$ by assumption, the central vertical arrow is bijective by Theorem 3.3.7 (1), (2) (for X' and X), and the right vertical arrow is bijective as well. Hence $\sigma_{X,r}^n$ is surjective and the left vertical map is bijective, which implies the equality $FM_r^n = \text{Ker}(\sigma_{X,r}^n)$. \square

4. p -adic étale Tate twists

In this section, we define the objects $\mathfrak{T}_r(n)_X$ ($n \geq 0$, $r \geq 1$) stated in Theorem 1.1.1 and discuss their fundamental properties including **T1**, **T2**, **T3**, **T5** and **T6**.

4.1. Setting. Let A be a Dedekind ring whose fraction field has characteristic zero and which has a maximal ideal of positive characteristic. Let p be a prime number which is not invertible in A , and we assume that the residue fields of A at maximal ideals of characteristic p are *perfect*. Put $B := \text{Spec}(A)$ and write Σ for the set of the closed points on B of characteristic p . For a point s on B , let B_s be the henselization of B at s . Let X be a pure-dimensional scheme which is flat of finite type over B . We assume that X satisfies the following condition, unless mentioned otherwise:

Condition 4.1.1. $X[1/p]$ is regular. For any $s \in \Sigma$, each connected component X' of $X \times_B B_s$ satisfies the condition 3.5.1 over the integral closure of B_s in $\Gamma(X', \mathcal{O}_{X'})$.

We will often work under the following stronger assumption:

Condition 4.1.2. X is regular. For any $s \in \Sigma$, each connected component X' of $X \times_B B_s$ is a regular semistable family over the integral closure of B_s in $\Gamma(X', \mathcal{O}_{X'})$.

Let X be a pure-dimensional flat of finite type B -scheme satisfying 4.1.1. Let $Y \subset X$ be the divisor defined by the radical of $(p) \subset \mathcal{O}_X$. We always assume that Y is non-empty. Let $\nu_{Y,r}^n$ be as in §2.2. Put $V := X \setminus Y = X[1/p]$. Let ι and j be as follows:

$$V \xrightarrow{j} X \xleftarrow{\iota} Y.$$

Define the étale sheaf M_r^n on Y to be $\iota^* R^n j_* \mu_{p^r}^{\otimes n}$.

4.2. Definition of $\mathfrak{T}_r(n)_X$. Let X and p be as before. We define $\mathfrak{T}_r(0)_X := \mathbb{Z}/p^r \mathbb{Z}_X$. For $n \geq 1$, let

$$\sigma_{X,r}(n) : \tau_{\leq n} Rj_* \mu_{p^r}^{\otimes n} \longrightarrow \iota_* \nu_{Y,r}^{n-1}[-n] \quad \text{in } D^b(X_{\text{ét}}, \mathbb{Z}/p^r \mathbb{Z}) \quad (4.2.1)$$

be the morphism induced by the map $\iota_*(\sigma_{X,r}^n) : R^n j_* \mu_{p^r}^{\otimes n} = \iota_* M_r^n \rightarrow \iota_* \nu_{Y,r}^{n-1}$ of sheaves on $X_{\text{ét}}$ (cf. Lemma 2.1.1). See (3.2.5) for $\sigma_{X,r}^n$.

Lemma 4.2.2. *Suppose $n \geq 1$, and let*

$$\iota_* \nu_{Y,r}^{n-1}[-n-1] \xrightarrow{g} \mathcal{K} \xrightarrow{t} \tau_{\leq n} Rj_* \mu_{p^r}^{\otimes n} \xrightarrow{\sigma_{X,r}^{(n)}} \iota_* \nu_{Y,r}^{n-1}[-n] \quad (4.2.3)$$

be a distinguished triangle in $D^b(X_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})$. Then \mathcal{K} is concentrated in $[0, n]$, the triple (\mathcal{K}, t, g) is unique up to a unique isomorphism and g is determined by the pair (\mathcal{K}, t) .

Proof. The map $\iota_*(\sigma_{X,r}^n)$ is surjective by Theorem 3.4.2 and Corollary 3.5.2. Hence \mathcal{K} is acyclic outside of $[0, n]$ and there is no non-zero morphism from \mathcal{K} to $\iota_* \nu_{Y,r}^{n-1}[-n-1]$ by Lemma 2.1.1. The uniqueness assertion follows from this fact and Lemma 2.1.2 (3). \square

Definition 4.2.4. *For $n \geq 1$, we fix a pair (\mathcal{K}, t) fitting into a distinguished triangle of the form (4.2.3), and define $\mathfrak{T}_r(n)_X := \mathcal{K}$. The morphism t determines an isomorphism $j^*\mathcal{K} \simeq \mu_{p^r}^{\otimes n}$, and $\mathfrak{T}_r(n)_X$ is concentrated in $[0, n]$, that is, $\mathfrak{T}_r(n)_X$ satisfies **T1** and **T2** in 1.1.1. Moreover, t induces isomorphisms*

$$\mathcal{H}^q(\mathfrak{T}_r(n)_X) \simeq \begin{cases} R^q j_* \mu_{p^r}^{\otimes n} & (0 \leq q < n), \\ \iota_* FM_r^n & (q = n), \end{cases} \quad (4.2.5)$$

where we have used Theorem 3.4.2 and Corollary 3.5.2 for $q = n$.

We prove here the existence of a natural product structure (**T5** in 1.1.1).

Proposition 4.2.6 (Product structure). *For $m, n \geq 0$, there is a unique morphism*

$$\mathfrak{T}_r(m)_X \otimes^{\mathbb{L}} \mathfrak{T}_r(n)_X \longrightarrow \mathfrak{T}_r(m+n)_X \quad \text{in } D^-(X_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z}) \quad (4.2.7)$$

that extends the natural map $\mu_{p^r}^{\otimes m} \otimes \mu_{p^r}^{\otimes n} \rightarrow \mu_{p^r}^{\otimes m+n}$ on $V_{\text{ét}}$.

Proof. If $m = 0$ or $n = 0$, then the assertion is obvious. Assume $m, n \geq 1$, and put $\mathfrak{L} := \mathfrak{T}_r(m)_X \otimes^{\mathbb{L}} \mathfrak{T}_r(n)_X$. By the definition of $\mathfrak{T}_r(m+n)_X$ and Lemma 2.1.2 (1), it suffices to show that the following composite morphism is zero in $D^-(X_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})$:

$$\mathfrak{L} \longrightarrow \tau_{\leq m} Rj_* \mu_{p^r}^{\otimes m} \otimes^{\mathbb{L}} \tau_{\leq n} Rj_* \mu_{p^r}^{\otimes n} \longrightarrow \tau_{\leq m+n} Rj_* \mu_{p^r}^{\otimes m+n} \xrightarrow{\sigma_{X,r}^{(m+n)}} \iota_* \nu_{Y,r}^{m+n-1}[-m-n],$$

where the second arrow is induced by the natural map $\mu_{p^r}^{\otimes m} \otimes \mu_{p^r}^{\otimes n} \rightarrow \mu_{p^r}^{\otimes m+n}$ on V . We prove this triviality. Because \mathfrak{L} is concentrated in degrees $\leq m+n$, this composite morphism is determined by the composite map of the $(m+n)$ -th cohomology sheaves (cf. Lemma 2.1.1)

$$\mathcal{H}^{m+n}(\mathfrak{L}) \longrightarrow \iota_* M_r^m \otimes \iota_* M_r^n \longrightarrow \iota_* M_r^{m+n} \xrightarrow{\iota_*(\sigma_{X,r}^{m+n})} \iota_* \nu_{Y,r}^{m+n-1}.$$

The image of $\mathcal{H}^{m+n}(\mathfrak{L}) (\simeq \iota_* FM_r^m \otimes \iota_* FM_r^n)$ into $\iota_* M_r^{m+n}$ is contained in $\iota_* FM_r^{m+n}$. Hence this composite map is zero and we obtain Proposition 4.2.6. \square

The following proposition (**T6** in 1.1.2) follows from a similar argument as for Proposition 4.2.6.

Proposition 4.2.8 (Contravariant functoriality). *Let X and Z be flat B -schemes satisfying 4.1.1. Let $f : Z \rightarrow X$ be a morphism of schemes, and let $\psi : Z[1/p] \rightarrow X[1/p]$ be the induced morphism. Then there is a unique morphism*

$$f^* : f^* \mathfrak{T}_r(n)_X \longrightarrow \mathfrak{T}_r(n)_Z \quad \text{in } D^b(Z_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})$$

that extends the natural isomorphism $\psi^* \mu_{p^r}^{\otimes n} \simeq \mu_{p^r}^{\otimes n}$ on $(Z[1/p])_{\text{ét}}$. Consequently, these pull-back morphisms satisfy the transitivity property.

4.3. **Bockstein triangle.** We prove the following proposition:

Proposition 4.3.1. *For $r, s \geq 1$, the following holds:*

- (1) *There is a unique morphism $\underline{p} : \mathfrak{T}_r(n)_X \rightarrow \mathfrak{T}_{r+1}(n)_X$ in $D^b(X_{\text{ét}}, \mathbb{Z}/p^{r+1}\mathbb{Z})$ that extends the natural inclusion $\mu_{p^r}^{\otimes n} \hookrightarrow \mu_{p^{r+1}}^{\otimes n}$ on $V_{\text{ét}}$.*
- (2) *There is a unique morphism $\mathcal{R} : \mathfrak{T}_{r+1}(n)_X \rightarrow \mathfrak{T}_r(n)_X$ in $D^b(X_{\text{ét}}, \mathbb{Z}/p^{r+1}\mathbb{Z})$ that extends the natural projection $\mu_{p^{r+1}}^{\otimes n} \rightarrow \mu_{p^r}^{\otimes n}$ on $V_{\text{ét}}$.*
- (3) *There is a canonical Bockstein morphism $\delta_{s,r} : \mathfrak{T}_s(n)_X \rightarrow \mathfrak{T}_r(n)_X[1]$ in $D^b(X_{\text{ét}})$ satisfying*
 - (3-1) *$\delta_{s,r}$ extends the Bockstein morphism $\mu_{p^s}^{\otimes n} \rightarrow \mu_{p^r}^{\otimes n}[1]$ in $D^b(V_{\text{ét}})$ associated with the short exact sequence $0 \rightarrow \mu_{p^r}^{\otimes n} \rightarrow \mu_{p^{r+s}}^{\otimes n} \rightarrow \mu_{p^s}^{\otimes n} \rightarrow 0$.*
 - (3-2) *$\delta_{s,r}$ fits into a distinguished triangle*

$$\mathfrak{T}_{r+s}(n)_X \xrightarrow{\mathcal{R}^r} \mathfrak{T}_s(n)_X \xrightarrow{\delta_{s,r}} \mathfrak{T}_r(n)_X[1] \xrightarrow{\underline{p}^s[1]} \mathfrak{T}_{r+s}(n)_X[1].$$

Proof. The claims (1) and (2) follow from the fact that $\mathfrak{T}_r(n)_X$ concentrated in $[0, n]$ and Lemma 2.1.2 (1). The details are straight-forward and left to the reader. We prove (3). For two complexes $M^\bullet = (\{M^u\}_{u \in \mathbb{Z}}, \{d_M^u : M^u \rightarrow M^{u+1}\}_{u \in \mathbb{Z}})$, $N^\bullet = (\{N^v\}_{v \in \mathbb{Z}}, \{d_N^v\}_{v \in \mathbb{Z}})$ and a map $h^\bullet : M^\bullet \rightarrow N^\bullet$ of complexes, let $\text{Cone}(h)^\bullet$ be the mapping cone (cf. [SGA4], XVII)

$$\text{Cone}(h)^q := M^{q+1} \oplus N^q, \quad d_{\text{Cone}(h)}^q := (-d_M^{q+1}, h^{q+1} + d_N^q).$$

We construct a morphism $\delta_{s,r}$ satisfying (3-1) and (3-2) in a canonical way. Take injective resolutions $\mu_{p^v}^{\otimes n} \rightarrow I_v^\bullet$ ($v = r, r+s$) and an injective resolution $\mu_{p^s}^{\otimes n} \rightarrow J_s^\bullet$ in the category of sheaves on $V_{\text{ét}}$ for which there is a short exact sequence of complexes of the form

$$0 \longrightarrow I_r^\bullet \longrightarrow I_{r+s}^\bullet \longrightarrow J_s^\bullet \longrightarrow 0.$$

Let $a_v : \tau_{\leq n} j_* I_v^\bullet \rightarrow \iota_* \nu_{Y,v}^{n-1}[-n]$ ($v = r, r+s$) and $b_s : \tau_{\leq n} j_* J_s^\bullet \rightarrow \iota_* \nu_{Y,s}^{n-1}[-n]$ be the natural maps of complexes that represent $\sigma_{X,v}(n) : \tau_{\leq n} Rj_* \mu_{p^v}^{\otimes n} \rightarrow \iota_* \nu_{Y,v}^{n-1}[-n]$ with $v = r, r+s$ and s , respectively (cf. §4.2). The complexes $\text{Cone}^\bullet(a_v)$ ($v = r, r+s$) and $\text{Cone}^\bullet(b_s)$ represent $\mathfrak{T}_v(n)_X$ with $v = r, r+s$ and s , respectively. We show that the sequence of complexes

$$0 \longrightarrow \text{Cone}^\bullet(a_r) \longrightarrow \text{Cone}^\bullet(a_{r+s}) \xrightarrow{f} \text{Cone}^\bullet(b_s) \longrightarrow 0 \quad (4.3.2)$$

is exact. Indeed, this exactness follows from that of the sequence $0 \rightarrow \nu_{Y,r}^{n-1} \rightarrow \nu_{Y,r+s}^{n-1} \rightarrow \nu_{Y,s}^{n-1} \rightarrow 0$ ([Sat], 2.2.5 (2)) and that of the sequence of complexes

$$0 \longrightarrow \tau_{\leq n} j_* I_r^\bullet \longrightarrow \tau_{\leq n} j_* I_{r+s}^\bullet \longrightarrow \tau_{\leq n} j_* J_s^\bullet \longrightarrow 0 \quad (4.3.3)$$

(cf. Theorem 3.3.7 (1)). Finally, we define $\delta_{s,r}$ as the composite $\text{Cone}^\bullet(b_s) \rightarrow \text{Cone}^\bullet(f) \simeq \text{Cone}^\bullet(a_r)[1]$ in $D^b(X_{\text{ét}})$, i.e., connecting morphism associated with (4.3.2). By definition, $\delta_{s,r}$ is canonical and satisfies the properties (3-1) and (3-2). This completes the proof. \square

Remark 4.3.4. *One can construct a map $\delta'_{s,r} : \mathfrak{T}_s(n)_X \rightarrow \mathfrak{T}_r(n)_X[1]$ in $D^b(X_{\text{ét}}, \mathbb{Z}/p^{s+r}\mathbb{Z})$ satisfying (3-1) and (3-2) in the same way as above. Clearly, $\delta'_{s,r} = \delta_{s,r}$ in $D^b(X_{\text{ét}})$.*

4.4. Gysin morphism and purity. We define Gysin morphisms for closed subschemes of X contained in Y and prove **T3** in 1.1.1. See §6 below for a purity result for horizontal subschemes.

Lemma 4.4.1. (1) *There is a unique morphism*

$$g' : \nu_{Y,r}^{n-1}[-n-1] \longrightarrow R\iota^! \mathfrak{T}_r(n)_X \quad \text{in } D^b(Y_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})$$

fitting into a commutative diagram with distinguished rows

$$\begin{array}{ccccccc} \mathfrak{T}_r(n)_X & \xrightarrow{t} & \tau_{\leq n} Rj_* \mu_{p^r}^{\otimes n} & \xrightarrow{-\sigma_{X,r}(n)} & \iota_* \nu_{Y,r}^{n-1}[-n] & \xrightarrow{g[1]} & \mathfrak{T}_r(n)_X[1] \\ \parallel & & \downarrow & & R\iota_*(g')[1] \downarrow & & \parallel \\ \mathfrak{T}_r(n)_X & \xrightarrow{j^*} & Rj_* \mu_{p^r}^{\otimes n} & \xrightarrow{\delta_{V,Y}^{\text{loc}}(\mathbb{Z}/p^r\mathbb{Z}(n)_X)} & R\iota_* R\iota^! \mathfrak{T}_r(n)_X[1] & \xrightarrow{\iota_*} & \mathfrak{T}_r(n)_X[1]. \end{array} \quad (4.4.2)$$

Here t and g denote the same morphisms as in Lemma 4.2.2, and the lower row is the localization distinguished triangle (1.9.2).

(2) g' induces an isomorphism

$$\tau_{\leq n+1}(g') : \nu_{Y,r}^{n-1}[-n-1] \xrightarrow{\cong} \tau_{\leq n+1} R\iota^! \mathfrak{T}_r(n)_X \quad \text{in } D^b(Y_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z}).$$

Proof. We first calculate the cohomology sheaves of $R\iota^! \mathfrak{T}_r(n)_X$. In the lower row of (4.4.2), the map of the q -th cohomology sheaves of $\alpha \circ t$ is bijective (resp. injective) if $q < n$ (resp. $q = n$), by (4.2.5). Hence by **T2**, we obtain

$$R^q \iota^! \mathfrak{T}_r(n)_X \simeq \begin{cases} 0 & (q < n+1), \\ \iota^* R^{q-1} j_* \mu_{p^r}^{\otimes n} & (q > n+1), \end{cases} \quad (4.4.3)$$

and a short exact sequence

$$0 \longrightarrow FM_r^n \longrightarrow M_r^n \xrightarrow{\iota^* \mathcal{H}^n(\delta_{V,Y}^{\text{loc}}(\mathfrak{T}_r(n)_X))} R^{n+1} \iota^! \mathfrak{T}_r(n)_X \longrightarrow 0. \quad (4.4.4)$$

By Lemma 2.1.1, (4.4.3) and **T2**, we have

$$\text{Hom}_{D^b(Y_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})}(\mathfrak{T}_r(n)_X[1], R\iota_* R\iota^! \mathfrak{T}_r(n)_X[1]) = 0.$$

Hence the first assertion of the lemma follows from Lemma 2.1.2 (2). The second assertion follows from (4.4.4). \square

Definition 4.4.5. *Let $\phi : Z \hookrightarrow Y$ be a closed immersion of pure codimension. Put $c := \text{codim}_X(Z)$, and let i be the composite map $Z \hookrightarrow Y \hookrightarrow X$. We define the morphism*

$$\text{Gys}_i^n : \nu_{Z,r}^{n-c}[-n-c] \longrightarrow Ri^! \mathfrak{T}_r(n)_X \quad \text{in } D^b(Z_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z}) \quad (4.4.6)$$

as follows, where $\nu_{Z,r}^{n-c}$ means the zero sheaf if $n < c$. If $Z = Y$ (hence $c = 1$ and $i = \iota$), then we define Gys_ι^n as the morphism g' in Lemma 4.4.1. This morphism agrees with the adjoint of g in Lemma 4.2.2 by the commutativity of the right square of (4.4.2). For a general Z , we define Gys_i^n as the composite

$$\nu_{Z,r}^{n-c}[-n-c] \xrightarrow{\text{Gys}_\phi^{n-1}[-n-1]} R\phi^! \nu_{Y,r}^{n-1}[-n-1] \xrightarrow{R\phi^!(\text{Gys}_\iota^n)} R\phi^! R\iota^! \mathfrak{T}_r(n)_X = Ri^! \mathfrak{T}_r(n)_X.$$

See Definition 2.2.1 for Gys_ϕ^{n-1} .

Theorem 4.4.7 (Purity). *The morphism*

$$\tau_{\leq n+c}(\mathrm{Gys}_i^n) : \nu_{Z,r}^{n-c}[-n-c] \longrightarrow \tau_{\leq n+c}Ri^!\mathfrak{I}_r(n)_X$$

is an isomorphism.

Proof. By the definition of Gys_i^n , the morphism $\tau_{\leq n+c}(\mathrm{Gys}_i^n)$ is decomposed as follows:

$$\begin{aligned} \nu_{Z,r}^{n-c}[-n-c] &\xrightarrow{\tau_{\leq n+c}(\mathrm{Gys}_\phi^{n-1}[-n-1])} \tau_{\leq n+c}(R\phi^!\nu_{Y,r}^{n-1}[-n-1]) \\ &\xrightarrow{\tau_{\leq n+c}R\phi^!\{\tau_{\leq n+1}(\mathrm{Gys}_i^n)\}} \tau_{\leq n+c}\{R\phi^!(\tau_{\leq n+1}Ri^!\mathfrak{I}_r(n)_X)\} \\ &\xrightarrow{\text{canonical}} \tau_{\leq n+c}Ri^!\mathfrak{I}_r(n)_X. \end{aligned}$$

The first two arrows are isomorphisms by Theorem 2.2.2 and Lemma 4.4.1. We show that the last arrow is an isomorphism as well. There is a distinguished triangle of the form

$$\tau_{\leq n+1}Ri^!\mathfrak{I}_r(n)_X \longrightarrow Ri^!\mathfrak{I}_r(n)_X \longrightarrow \tau_{\geq n+2}Ri^!\mathfrak{I}_r(n)_X \longrightarrow (\tau_{\leq n+1}Ri^!\mathfrak{I}_r(n)_X)[1]$$

and we have $\tau_{\geq n+2}Ri^!\mathfrak{I}_r(n)_X \simeq (\tau_{\geq n+1}l^*Rj_*\mu_p^{\otimes n})[-1]$ (cf. (4.4.3)). Hence it suffices to show

$$\tau_{\leq n+c-1}R\phi^!(\tau_{\geq n+1}l^*Rj_*\mu_p^{\otimes n}) = 0. \quad (4.4.8)$$

By the exactness of (4.3.3), there is a distinguished triangle of the form

$$\tau_{\geq n+1}Rj_*\mu_p^{\otimes n} \longrightarrow \tau_{\geq n+1}Rj_*\mu_p^{\otimes n} \longrightarrow \tau_{\geq n+1}Rj_*\mu_p^{\otimes n} \longrightarrow (\tau_{\geq n+1}Rj_*\mu_p^{\otimes n})[1]$$

Hence (4.4.8) is reduced to the case $r = 1$ and then to the following semi-purity due to Hagihara (cf. Theorem A.2.6 below):

$$R^q\phi^!(l^*R^m j_*\mu_p^{\otimes n}) = 0 \quad \text{for any } m, q \text{ with } q \leq c-2,$$

where one must note $c = \mathrm{codim}_Y(Z) + 1$. This completes the proof. \square

Corollary 4.4.9. *Let $i : Z \hookrightarrow X$ be a closed immersion of codimension $\geq n + 1$. Then we have $R^q i^!\mathfrak{I}_r(n)_X = 0$ for any $q \leq 2n + 1$.*

Proof. If $Z[1/p]$ is empty, then we have $R^q i^!\mathfrak{I}_r(n)_X = 0$ for $q \leq 2n + 1$ by Theorem 4.4.7. We next prove the case that $Z[1/p]$ is non-empty. Put $U := Z[1/p]$ and $T := Z \setminus U$. Let $\alpha : T \hookrightarrow Z$, $\beta : U \hookrightarrow Z$ and $\gamma : T \hookrightarrow X$ be the natural immersions. There is a long exact sequence of sheaves on $Z_{\acute{e}t}$

$$\cdots \longrightarrow \alpha_* R^q \gamma^! \mathfrak{I}_r(n)_X \longrightarrow R^q i^! \mathfrak{I}_r(n)_X \longrightarrow R^q \beta_* \beta^* Ri^! \mathfrak{I}_r(n)_X \longrightarrow \cdots,$$

where $\alpha_* R^q \gamma^! \mathfrak{I}_r(n)_X$ is zero for $q \leq 2n + 1$ by the previous case. We show that $R^q \beta_* \beta^* Ri^! \mathfrak{I}_r(n)_X$ is zero for $q \leq 2n + 1$. Indeed, we have $\beta^* Ri^! \mathfrak{I}_r(n)_X = R\psi^! \mu_p^{\otimes n}$ with ψ the closed immersion $U \hookrightarrow V$, and it is concentrated in degrees $\geq 2n + 2$ by the absolute purity of Thomason and Gabber [Th], [FG] and by the assumption that $\mathrm{codim}_X(Z) \geq n + 1$. \square

We next prove a projection formula, which will be used later in §5 and §6.

Proposition 4.4.10 (Projection formula). *Let $i : Z \hookrightarrow X$ be as in 4.4.5. We define the morphism $i^* : \mathfrak{I}_r(n)_X \rightarrow i_* \lambda_{Z,r}^n[-n]$ in $D^b(X_{\acute{e}t}, \mathbb{Z}/p^r\mathbb{Z})$ by the natural pull-back of symbols on*

the n -th cohomology sheaves (cf. (4.2.5)). Then the square

$$\begin{array}{ccc} i_*\nu_{Z,r}^{m-c}[-m-c] \otimes^{\mathbb{L}} \mathfrak{F}_r(n)_X & \xrightarrow{\text{Gys}_i^m \otimes^{\mathbb{L}} \text{id}} & \mathfrak{F}_r(m)_X \otimes^{\mathbb{L}} \mathfrak{F}_r(n)_X \\ \text{\scriptsize (\#)} \downarrow & & \downarrow \text{\scriptsize (4.2.7)} \\ i_*\nu_{Z,r}^{m+n-c}[-m-n-c] & \xrightarrow{\text{Gys}_i^{m+n}} & \mathfrak{F}_r(m+n)_X \end{array}$$

commutes in $D^-(X_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})$. Here the left vertical arrow (\#) is the composite map

$$i_*\nu_{Z,r}^{m-c}[-m-c] \otimes^{\mathbb{L}} \mathfrak{F}_r(n)_X \xrightarrow{\text{id} \otimes^{\mathbb{L}} i^*} i_*\nu_{Z,r}^{m-c}[-m-c] \otimes^{\mathbb{L}} i_*\lambda_{Z,r}^n[-n] \longrightarrow i_*\nu_{Z,r}^{m+n-c}[-m-n-c],$$

and the last arrow is induced by the pairing (2.2.3) on the $(m+n+c)$ th cohomology sheaves.

Proof. One can easily check the case $Z = Y$ by the commutativity of the central square in (4.4.2). The general case is, by the previous case, reduced to the commutativity of a diagram

$$\begin{array}{ccc} \phi_*\nu_{Z,r}^{m-c}[-m-c] \otimes^{\mathbb{L}} \lambda_{Y,r}^n[-n] & \xrightarrow{\text{Gys}_\phi^{m-1}[-m-1] \otimes^{\mathbb{L}} \text{id}[-n]} & \nu_{Y,r}^{m-1}[-m-1] \otimes^{\mathbb{L}} \lambda_{Y,r}^n[-n] \\ \downarrow & & \downarrow \\ \phi_*\nu_{Z,r}^{m+n-c}[-m-n-c] & \xrightarrow{\text{Gys}_\phi^{m+n-1}[-m-n-1]} & \nu_{Y,r}^{m+n-1}[-m-n-1] \end{array}$$

in $D^-(Y_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})$ with $\phi : Z \hookrightarrow Y$. Here the vertical arrows are defined in a similar way as for (\#). We prove the commutativity of this square. For two complexes $M^\bullet = (\{M^u\}_{u \in \mathbb{Z}}, \{d_M^u : M^u \rightarrow M^{u+1}\}_{u \in \mathbb{Z}})$ and $N^\bullet = (\{N^v\}_{v \in \mathbb{Z}}, \{d_N^v\}_{v \in \mathbb{Z}})$, we define the double complex $M^\bullet \otimes N^\bullet$ as

$$(M^\bullet \otimes N^\bullet)^{u,v} := M^u \otimes N^v, \quad \partial_1^{u,v} := d_M^u \otimes \text{id}_{N^v}, \quad \partial_2^{u,v} := (-1)^u \text{id}_{M^u} \otimes d_N^v.$$

We write $(M^\bullet \otimes N^\bullet)^t$ for the associated total complex, whose image into the derived category gives $M^\bullet \otimes^{\mathbb{L}} N^\bullet$ if either M^\bullet or N^\bullet is bounded above and consists of flat objects. Now for $T \in \{Y, Z\}$ and $a \geq 0$, let $C_r^\bullet(T, a)$ be the complex of sheaves defined in §2.2. Because $\lambda_{Y,r}^n$ is flat over $\mathbb{Z}/p^r\mathbb{Z}$ by [Sat], 3.2.3, the commutativity in question follows from that of a diagram of complexes on $Y_{\text{ét}}$

$$\begin{array}{ccc} (\phi_*C_r^\bullet(Z, m-c)[-m-c] \otimes \lambda_{Y,r}^n[-n])^t & \hookrightarrow & (C_r^\bullet(Y, m-1)[-m-1] \otimes \lambda_{Y,r}^n[-n])^t \\ \text{product} \downarrow & & \downarrow \text{product} \\ \phi_*C_r^\bullet(Z, m+n-c)[-m-n-c] & \hookrightarrow & C_r^\bullet(Y, m+n-1)[-m-n-1], \end{array}$$

where the vertical arrows are induced by the pairings (2.2.3) and the horizontal arrows are natural inclusions of complexes. This completes the proof. \square

4.5. Kummer sequence for \mathbb{G}_m and purity of Brauer groups. We study the case $n = 1$.

Proposition 4.5.1. *Put $\mathbb{G}_m := \mathcal{O}_X^\times$. Then there is a unique morphism*

$$\mathbb{G}_m \otimes^{\mathbb{L}} \mathbb{Z}/p^r\mathbb{Z}[-1] \longrightarrow \mathfrak{F}_r(1)_X \quad \text{in } D^b(X_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})$$

that extends the canonical isomorphism $j^*(\mathbb{G}_m \otimes^{\mathbb{L}} \mathbb{Z}/p^r\mathbb{Z}[-1]) \simeq \mu_{p^r}$. Moreover it is an isomorphism, if X satisfies 4.1.2.

Proof. Put $\mathcal{M} := \mathbb{G}_m \otimes^{\mathbb{L}} \mathbb{Z}/p^r\mathbb{Z}[-1]$. By definition, (i) \mathcal{M} is concentrated in $[0, 1]$, and (ii) there are natural isomorphisms

$$\mathcal{H}^0(\mathcal{M}) \simeq \text{Ker}(\mathbb{G}_m \xrightarrow{\times p^r} \mathbb{G}_m) \quad \text{and} \quad \mathcal{H}^1(\mathcal{M}) \simeq \mathbb{G}_m/p^r.$$

Because $j^*\mathcal{M} \simeq \mu_{p^r}$ canonically in $D^b(V_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})$, there is a natural morphism $\mathcal{M} \rightarrow \tau_{\leq 1}Rj_*\mu_{p^r}$ in $D^b(X_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})$ by (i). The composite morphism

$$\mathcal{M} \longrightarrow \tau_{\leq 1}Rj_*\mu_{p^r} \xrightarrow{\sigma_{X,r}(1)} \iota_*\nu_{Y,r}^0[-1]$$

is zero by (ii) and Lemma 2.1.1. Hence by Lemma 2.1.2 (1), we obtain a unique morphism $\mathcal{M} \rightarrow \mathfrak{T}_r(1)_X$ that extends the isomorphism $j^*\mathcal{M} \simeq \mu_{p^r}$. Next we prove that this morphism is bijective on cohomology sheaves, assuming that X satisfies 4.1.2. By the standard purity for \mathbb{G}_m ([Gt], (6.3)–(6.5)), there is an exact sequence

$$0 \longrightarrow \mathbb{G}_m \longrightarrow j_*j^*\mathbb{G}_m \longrightarrow \bigoplus_{y \in Y^0} \iota_{y*}\mathbb{Z} \longrightarrow 0,$$

where for $x \in X$, ι_x denotes the canonical map $x \hookrightarrow X$. Since $\bigoplus_{y \in Y^0} \iota_{y*}\mathbb{Z}$ is torsion-free, we have $\mathcal{H}^0(\mathcal{M}) \simeq j_*\mu_{p^r}$ and there is an exact sequence

$$0 \longrightarrow \mathcal{H}^1(\mathcal{M}) \longrightarrow R^1j_*\mu_{p^r} \longrightarrow \bigoplus_{y \in Y^0} \iota_{y*}\mathbb{Z}/p^r\mathbb{Z} (= \iota_*\nu_{Y,r}^0)$$

by (ii) and the snake lemma. Here we have used the isomorphism $(j_*j^*\mathbb{G}_m)/p^r \simeq R^1j_*\mu_{p^r}$ obtained from Hilbert's theorem 90: $R^1j_*j^*\mathbb{G}_m = 0$. Now the assertion follows from (4.2.5). \square

As an application of Corollary 4.4.9 and Proposition 4.5.1, we prove the p -primary part of the purity of Brauer groups (cf. [Gt], §6).

Corollary 4.5.2 (Purity of Brauer groups). *Assume that X satisfies 4.1.2. Let $i : Z \hookrightarrow X$ be a closed immersion with $\text{codim}_X(Z) \geq 2$. Then the p -primary torsion part of $R^3i^!\mathbb{G}_m$ is zero.*

If $\dim(X) \leq 3$, then the full sheaf $R^3i^!\mathbb{G}_m$ is zero by a theorem of Gabber [Ga].

Proof. By Proposition 4.5.1, there is a distinguished triangle

$$\mathbb{G}_m[-1] \longrightarrow \mathfrak{T}_r(1)_X \longrightarrow \mathbb{G}_m \xrightarrow{\times p^r} \mathbb{G}_m \quad \text{in } D^b(X_{\text{ét}}), \quad (4.5.3)$$

which yields an exact sequence $R^3i^!\mathfrak{T}_r(1)_X \rightarrow R^3i^!\mathbb{G}_m \xrightarrow{\times p^r} R^3i^!\mathbb{G}_m$. Hence the corollary follows from the vanishing result in Corollary 4.4.9. \square

5. Cycle class and intersection property

Throughout this section, we work with the setting in §4.1 and assume that X satisfies the condition 4.1.2. In this section we define the cycle class $\text{cl}_X(Z) \in H_Z^{2n}(X, \mathfrak{T}_r(n))$ for an integral closed subscheme $Z \subset X$ of codimension $n \geq 0$, and prove an ‘intersection formula’

$$\text{cl}_X(Z) \cup \text{cl}_X(Z') = \text{cl}_X(Z \cap Z') \quad \text{in } H_Z^{2(m+n)}(X, \mathfrak{T}_r(m+n)),$$

assuming that Z of codimension m and Z' of codimension n are regular and meet transversally. In §6, we will prove **T4** in Theorem 1.1.1 using this result.

5.1. **Cycle class.** We first note a standard consequence of Corollary 4.4.9.

Lemma 5.1.1. *Let Z be a closed subscheme of X of pure codimension $n \geq 0$. Let Z' be a dense open subset of Z , and let T be the complement $Z \setminus Z'$. Then the natural map*

$$H_Z^{2n}(X, \mathfrak{F}_r(n)_X) \longrightarrow H_{Z'}^{2n}(X \setminus T, \mathfrak{F}_r(n)_{X \setminus T})$$

is bijective.

Proof. There is a long exact sequence of cohomology groups with supports

$$\cdots \rightarrow H_T^{2n}(X, \mathfrak{F}_r(n)_X) \rightarrow H_Z^{2n}(X, \mathfrak{F}_r(n)_X) \rightarrow H_{Z'}^{2n}(X \setminus T, \mathfrak{F}_r(n)_X) \rightarrow H_T^{2n+1}(X, \mathfrak{F}_r(n)_X) \rightarrow \cdots .$$

Since $\text{codim}_X(T) \geq n+1$, we have $H_T^{2n}(X, \mathfrak{F}_r(n)_X) = H_T^{2n+1}(X, \mathfrak{F}_r(n)_X) = 0$ by Corollary 4.4.9, which shows the lemma. \square

Definition 5.1.2. *For an integral closed subscheme $Z \subset X$ of codimension $n \geq 0$, we define the cycle class $\text{cl}_X(Z) \in H_Z^{2n}(X, \mathfrak{F}_r(n)_X)$ as follows.*

- (1) *If Z is regular and contained in Y , then we define $\text{cl}_X(Z)$ to be the image of $1 \in \mathbb{Z}/p^r\mathbb{Z}$ under the Gysin map*

$$\text{Gys}_i^n : \mathbb{Z}/p^r\mathbb{Z} \longrightarrow H_Z^{2n}(X, \mathfrak{F}_r(n)_X)$$

induced by the Gysin morphism defined in Definition 4.4.5.

- (2) *If Z is regular and not contained in Y , then we have Gabber's refined cycle class $\text{cl}_V(U) \in H_U^{2n}(V, \mu_{p^r}^{\otimes n})$ (cf. [FG]), where we put $U := Z[1/p]$ and $V := X[1/p]$. We define $\text{cl}_X(Z)$ as the inverse image of $\text{cl}_V(U)$ under the natural map*

$$H_Z^{2n}(X, \mathfrak{F}_r(n)_X) \longrightarrow H_U^{2n}(V, \mu_{p^r}^{\otimes n}).$$

This map is bijective by Lemma 5.1.1 and excision, and hence $\text{cl}_X(Z)$ is well-defined. Note that Gabber's refined cycle class agrees with Deligne's cycle class ([SGA4], Cycle) in any situation where the latter is defined (cf. [FG], 1.1.5).

- (3) *For a general Z , we take a dense open regular subset $Z' \subset Z$ and define $\text{cl}_X(Z)$ to be the inverse image of $\text{cl}_{X'}(Z') \in H_{Z'}^{2n}(X', \mathfrak{F}_r(n)_{X'})$ ($X' := X \setminus (Z \setminus Z')$) under the natural map*

$$H_Z^{2n}(X, \mathfrak{F}_r(n)_X) \longrightarrow H_{Z'}^{2n}(X', \mathfrak{F}_r(n)_{X'}),$$

which is bijective by Lemma 5.1.1 and $\text{cl}_X(Z)$ is well-defined.

We prove the following result:

Proposition 5.1.3 (Intersection property). *Let Z and Z' be integral regular closed subschemes of X of codimension a and b , respectively. Assume that Z and Z' meet transversally on X . Then we have*

$$\text{cl}_X(Z) \cup \text{cl}_X(Z') = \text{cl}_X(Z \cap Z') \quad \text{in } H_{Z \cap Z'}^{2(a+b)}(X, \mathfrak{F}_r(a+b)_X).$$

Here, if $Z \cap Z'$ is not connected, then $\text{cl}_X(Z \cap Z')$ means the sum of the cycle classes of the connected components.

5.2. Proof of Proposition 5.1.3. Without loss of generality, we may assume that $Z \cap Z'$ is connected (hence integral and regular). If $Z \cap Z'$ is not contained in Y , the assertion follows from Lemma 5.1.1 and the corresponding property of Gabber's refined cycle classes [FG], 1.1.4. We treat the case that $Z \cap Z' \subset Y$. Let x be the generic point of $Z \cap Z'$. By Lemma 5.1.1, we may replace X by $\text{Spec}(\mathcal{O}_{X,x})$. Because Z and Z' are regular and meet transversally at x , there is a normal crossing divisor $D = \cup_{i=1}^{a+b} D_i$ with each D_i integral regular such that $\cap_{i=1}^a D_i = Z$ and $\cap_{i=a+1}^{a+b} D_i = Z'$. Therefore we are reduced to the following local assertion:

Lemma 5.2.1. *Suppose that X is local with closed point x of characteristic p . Put $n := \text{codim}_X(x) \geq 1$. Let $D = \cup_{i=1}^n D_i$ be a normal crossing divisor on X with each D_i integral regular such that $\cap_{i=1}^n D_i = x$. Then the cohomology class*

$$\text{cl}_X(x; D) := \text{cl}_X(D_1) \cup \text{cl}_X(D_2) \cup \cdots \cup \text{cl}_X(D_n) \in H_x^{2n}(X, \mathfrak{F}_r(n)_X)$$

depends only on the flag: $D_1 \supset D_1 \cap D_2 \supset \cdots \supset D_1 \cap \cdots \cap D_{n-1} \supset x$, and agrees with $\text{cl}_X(x)$.

We prove this lemma by induction on $n \geq 1$. The case $n = 1$ is clear. Suppose that $n \geq 2$ and put $S := \cap_{i=1}^{n-1} D_i$. Let ψ (resp. i_x) be the closed immersion $S \hookrightarrow X$ (resp. $x \rightarrow S$). Note that S is regular, local and of dimension 1.

We first show the case that $S \subset Y$. By the induction hypothesis and Lemma 5.1.1, we have $\text{cl}_X(D_1) \cup \cdots \cup \text{cl}_X(D_{n-1}) = \text{cl}_X(S)$, and hence

$$\text{cl}_X(x; D) = \text{cl}_X(S) \cup \text{cl}_X(D_n) = \text{Gys}_\psi^n(\text{Gys}_{i_x}^{n-c}(1)) = \text{cl}_X(x).$$

Here the second equality follows from Proposition 4.4.10 for ψ and the last equality follows from Remark 2.2.6 (1). In particular, $\text{cl}(x; D)$ depends only on the flag of D .

We next show the case that $S \not\subset Y$. Let y be the generic point of S . Since $\text{ch}(y) = 0$, we have $\text{cl}_X(D_1) \cup \cdots \cup \text{cl}_X(D_{n-1}) = \text{cl}_X(S)$ by Lemma 5.1.1 and [FG], 1.1.4. We have to show

Sublemma 5.2.2. *Let E and E' be regular connected divisors on X each of which meets S transversally at x . Then we have $\text{cl}_X(S) \cap \text{cl}_X(E) = \text{cl}_X(S) \cap \text{cl}_X(E')$.*

We first finish the proof of the lemma, admitting this sublemma. It implies that $\text{cl}_X(x; D)$ depends only on the flag of D , and moreover that $\text{cl}_X(x; D)$ is independent of D by [SGA4 $\frac{1}{2}$], Cycle, 2.2.3. Hence we obtain $\text{cl}_X(x; D) = \text{cl}_X(x)$ by the computation in the previous case.

Proof of Sublemma 5.2.2. Let E be a regular divisor on X as in the sublemma. The map

$$H_E^2(X, \mathfrak{F}_r(1)_X) \longrightarrow H_x^{2n}(X, \mathfrak{F}_r(n)_X), \quad \alpha \mapsto \text{cl}_X(S) \cup \alpha$$

factors through a natural pull-back map

$$\psi^* : H_E^2(X, \mathfrak{F}_r(1)_X) \longrightarrow H_x^2(S, \psi^* \mathfrak{F}_r(1)_X).$$

We compute $\psi^*(\text{cl}_X(E))$ as follows. Since $\text{ch}(y) = 0$, we have $(\psi^* \mathfrak{F}_r(1)_X)|_y \simeq \mu_{p^r}$ on $y_{\text{ét}}$ and there is a commutative diagram with exact rows

$$\begin{array}{ccccc} H^1(X, \mathfrak{F}_r(1)_X) & \longrightarrow & H^1(X \setminus E, \mathfrak{F}_r(1)_{X \setminus E}) & \xrightarrow{\delta_1} & H_E^2(X, \mathfrak{F}_r(1)_X) \\ \psi^* \downarrow & & \psi^* \downarrow & & \psi^* \downarrow \\ H^1(S, \psi^* \mathfrak{F}_r(1)_X) & \longrightarrow & H^1(y, \mu_{p^r}) & \xrightarrow{\delta_2} & H_x^2(S, \psi^* \mathfrak{F}_r(1)_X), \end{array}$$

where δ_1 denotes $\delta_{X \setminus E, E}^{\text{loc}}(\psi^* \mathfrak{T}_r(1)_X)$ and δ_2 denotes $\delta_{y, x}^{\text{loc}}(\psi^* \mathfrak{T}_r(1)_X)$ (cf. (1.9.2)). Take a prime element $\pi_E \in \Gamma(X, \mathcal{O}_X)$ which defines E , and let $\{\pi_E\} \in H^1(X \setminus E, \mathfrak{T}_r(1)_{X \setminus E})$ be the image of π_E under the boundary map of Kummer theory (cf. (4.5.3))

$$\Gamma(X \setminus E, \mathcal{O}_{X \setminus E}^\times) \longrightarrow H^1(X \setminus E, \mathfrak{T}_r(1)_{X \setminus E}).$$

We have $\text{cl}_X(E) = -\delta_1(\{\pi_E^{-1}\})$ by [SGA4 $\frac{1}{2}$], Cycle, 2.1.3 (cf. (1.9.1)). By the diagram,

$$\psi^*(\text{cl}_X(E)) = -\delta_2(\psi^*\{\pi_E\}) = -\delta_2(\{\overline{\pi_E}\}).$$

Here $\overline{\pi_E}$ denotes the residue class of π_E in $\mathcal{O}_{S, x}$ and it is a prime element by the assumption that E meets S transversally at x . Moreover we have $\delta_2(\{u\}) = 0$ for any unit $u \in \mathcal{O}_{S, x}^\times$, because every $u \in \mathcal{O}_{S, x}^\times$ lifts to $\mathcal{O}_{X, x}^\times$. Hence for a fixed prime $\pi_x \in \mathcal{O}_{S, x}$ we have $\psi^*(\text{cl}_X(E)) = -\delta_2(\{\pi_x\})$, which shows the sublemma. \square

This completes the proof of Lemma 5.2.1 and Proposition 5.1.3.

6. Compatibility and purity for horizontal subschemes

In this section, we prove **T4** in Theorem 1.1.1. This result is rather technical, but we will need its consequence, Theorem 6.1.3, to prove the covariant functoriality **T7** in §7.

6.1. Gysin maps. We work with the setting in §4.1, and assume that X satisfies 4.1.1. Let b and n be integers with $n \geq b \geq 0$. For $x \in X^b$, we define the complex $\mathbb{Z}/p^r \mathbb{Z}(n)_x$ on $x_{\text{ét}}$ as

$$\mathbb{Z}/p^r \mathbb{Z}(n)_x := \begin{cases} \mu_{p^r}^{\otimes n} & (\text{if } \text{ch}(x) \neq p), \\ W_r \Omega_{x, \log}^n[-n] & (\text{if } \text{ch}(x) = p). \end{cases}$$

We define the Gysin map

$$\text{Gys}_{i_x}^n : H^{n-b}(x, \mathbb{Z}/p^r \mathbb{Z}(n-b)_x) \longrightarrow H_x^{n+b}(X, \mathfrak{T}_r(n)_X) := H_x^{n+b}(\text{Spec}(\mathcal{O}_{X, x}), \mathfrak{T}_r(n)_X)$$

as the map induced by the Gysin morphism for $i_x : x \hookrightarrow X$, if $\text{ch}(x) = p$ (cf. Definition 4.4.5). If $\text{ch}(x) \neq p$, we define $\text{Gys}_{i_x}^n$ by sending $\alpha \in H^{n-b}(x, \mu_{p^r}^{\otimes n-b})$ to $\text{cl}_V(x) \cup \alpha$, where $\text{cl}_V(x) \in H_x^{2b}(V, \mu_{p^r}^{\otimes b})$ denotes Gabber's refined cycle class we mentioned in Definition 5.1.2 (2). The aim of this section is to prove the following two theorems:

Theorem 6.1.1 (Compatibility). *Let x and y be points with $x \in \overline{\{y\}} \cap Y \cap X^b$ and $y \in X^{b-1}$ (hence $\text{ch}(y) = 0$ or p). Then the diagram*

$$\begin{array}{ccc} H^{n-b+1}(y, \mathbb{Z}/p^r \mathbb{Z}(n-b+1)_y) & \xrightarrow{-\partial_{y, x}^{\text{val}}} & H^{n-b}(x, \mathbb{Z}/p^r \mathbb{Z}(n-b)_x) \\ \text{Gys}_{i_y}^n \downarrow & & \downarrow \text{Gys}_{i_x}^n \\ H_y^{n+b-1}(X, \mathfrak{T}_r(n)_X) & \xrightarrow{\delta_{y, x}^{\text{loc}}(\mathfrak{T}_r(n)_X)} & H_x^{n+b}(X, \mathfrak{T}_r(n)_X) \end{array} \quad (6.1.2)$$

is commutative (see (1.9.3) for the definition of the bottom arrow).

Sheafifying this commutative diagram, we obtain **T4** in Theorem 1.1.1. As for the case $x \notin Y$, the corresponding commutativity is proved in [JSS], §1.

Theorem 6.1.3 (Purity). *Let Z be an integral locally closed subscheme of X which is flat over B and satisfies 4.1.1. Put $c := \text{codim}_X(Z)$ and $U := Z[1/p]$. Let i and ψ be the locally closed immersions $Z \hookrightarrow X$ and $U \hookrightarrow V$, respectively. Then for $n \geq c$, there is a unique morphism*

$$\text{Gys}_i^n : \mathfrak{T}_r(n-c)_Z[-2c] \longrightarrow \text{Ri}^1 \mathfrak{T}_r(n)_X \quad \text{in } D^b(Z_{\text{ét}}, \mathbb{Z}/p^r \mathbb{Z})$$

that extends the purity isomorphism (cf. [Th], [FG])

$$\text{Gys}_\psi^n : \mu_{p^r}^{\otimes n-c}[-2c] \xrightarrow{\cong} \text{R}\psi^1 \mu_{p^r}^{\otimes n} \quad \text{in } D^b(U_{\text{ét}}, \mathbb{Z}/p^r \mathbb{Z}).$$

Moreover, Gys_i^n induces an isomorphism

$$\tau_{\leq n+c}(\text{Gys}_i^n : \mathfrak{T}_r(n-c)_Z[-2c]) \xrightarrow{\cong} \tau_{\leq n+c} \text{Ri}^1 \mathfrak{T}_r(n)_X.$$

This result extends Theorem 4.4.7 to horizontal situations. Before starting the proof of these theorems, we state a consequence of Theorem 6.1.1. For a point $x \in X^n$ and a closed subscheme $S \subset X$ containing x there is a natural map

$$\text{H}_x^{2n}(X, \mathfrak{T}_r(n)_X) \longrightarrow \text{H}_S^{2n}(X, \mathfrak{T}_r(n)_X)$$

by Lemma 5.1.1. By Theorem 6.1.1 and [JSS], Theorem 1.1, we obtain

Corollary 6.1.4 (Reciprocity law). *Let y be a point with $y \in X^{n-1}$, and put $S := \overline{\{y\}} \subset X$. Then for any $\alpha \in \text{H}^1(y, \mathbb{Z}/p^r \mathbb{Z}(1)_y)$, we have*

$$\sum_{x \in X^n \cap S} \text{Gys}_{i_x}^n(\partial_{y,x}^{\text{val}}(\alpha)) = 0 \quad \text{in } \text{H}_S^{2n}(X, \mathfrak{T}_r(n)_X).$$

Consequently, the sum of Gysin maps

$$\sum_{x \in X^n} \text{Gys}_{i_x}^n : \bigoplus_{x \in X^n} \mathbb{Z}/p^r \mathbb{Z} \longrightarrow \text{H}^{2n}(X, \mathfrak{T}_r(n)_X)$$

factors through the Chow group of algebraic cycles modulo rational equivalence:

$$\text{cl}_{X,r}^n : \text{CH}^n(X)/p^r \longrightarrow \text{H}^{2n}(X, \mathfrak{T}_r(n)_X).$$

Remark 6.1.5. (1) *The case $\text{ch}(y) = p$ of Theorem 6.1.1 follows from the definition of Gysin maps (cf. 2.2.1, 4.4.5) and a similar arguments as for [Sat], 2.3.1 (see also (1.9.1)). On the other hand, the case $\text{ch}(y) = 0$ of Theorem 6.1.1 is closely related to Theorem 6.1.3.*

(2) *Corollary 6.1.4 is not a new result if X is smooth over B . In fact, by an argument of Geisser [Ge], §6, Proof of 1.3, one can construct a canonical map from higher Chow groups of X to $\text{H}^*(X, \mathfrak{T}_r(n)_X)$. A key ingredient in his argument is the localization exact sequences for higher Chow groups due to Levine [Le2]. In this paper, we give a more elementary proof of Theorem 6.1.1 without using Levine's localization sequences.*

In what follows, we refer the case $\text{ch}(y) = 0$ of Theorem 6.1.1 as Case (M). We will proceed the proof of Theorems 6.1.1 and 6.1.3 in three steps. In §6.2, we will prove Case (M) of Theorem 6.1.1 assuming that X satisfies 4.1.2 and that $S := \overline{\{y\}}$ is normal at x . In §6.3, we will prove Theorem 6.1.3 assuming that X satisfies 4.1.2 and then reduce Case (M) of Theorem 6.1.1 to the case where X is smooth over B (and S is arbitrary). The last case will be proved in §6.4, which will complete the proof of Theorems 6.1.1 and 6.1.3.

6.2. Proof of the theorems, Step 1. In this step, we prove Case (M) of Theorem 6.1.1 assuming that X and $S(=\overline{\{y\}})$ are regular at x . Replacing X by $\text{Spec}(\mathcal{O}_{X,x}^h)$ and replacing y by the point on $\text{Spec}(\mathcal{O}_{X,x}^h)$ lying above y , we suppose that X is regular henselian local with closed point x . Note that it suffices to show the desired compatibility in this situation, and that $\mathcal{O}_{S,x}$ is a henselian discrete valuation ring. By the Bloch-Kato theorem [BK], (5.12), $H^{n-b+1}(y, \mu_{p^r}^{\otimes n-b+1})$ is generated by symbols of the forms

$$(i) \quad \{\beta_1, \dots, \beta_{n-b+1}\} \quad \text{and} \quad (ii) \quad \{\pi_x, \beta_1, \dots, \beta_{n-b}\},$$

where each β_λ belongs to $\mathcal{O}_{S,x}^\times$, and π_x denotes a prime element of $\mathcal{O}_{S,x}$. We show that the diagram (6.1.2) commutes for these two kinds of symbols. Recall that $\text{Gys}_{i_y}^n$ is given by the cup product with the cycle class $\text{cl}_V(y) \in H_y^{2b-2}(V, \mu_{p^r}^{\otimes b-1})$, and that this cycle class extends to the cycle class $\text{cl}_X(S) \in H_S^{2b-2}(X, \mathfrak{T}_r(b-1)_X)$ (cf. Definition 5.1.2). We first show that the diagram (6.1.2) commutes for symbols of the form (i). Because a symbol ω of this form lifts to $\tilde{\omega} \in H^{n-b+1}(X, \mathfrak{T}_r(n-b+1)_X)$, its image $\text{Gys}_{i_y}^n(\omega)$ lifts to $\text{cl}_X(S) \cup \tilde{\omega} \in H_S^{n+b-1}(X, \mathfrak{T}_r(n)_X)$. Hence we have $\delta_{y,x}^{\text{loc}} \circ \text{Gys}_{i_y}^n(\omega) = 0$, which implies the assertion. We next consider symbols of the form (ii). The map $\delta_{y,x}^{\text{loc}}(\mathfrak{T}_r(n)_X) \circ \text{Gys}_{i_y}^n$ sends a symbol $\{\pi_x, \beta_1, \dots, \beta_{n-b}\}$ to

$$\delta(\text{cl}_V(y) \cup \{\pi_x\}) \cup \omega \in H_x^{n+b}(X, \mathfrak{T}_r(n)_X). \quad (6.2.1)$$

Here ω denotes a lift of $\{\beta_1, \dots, \beta_{n-b}\}$ to $H^{n-b}(X, \mathfrak{T}_r(n-b)_X)$, and δ denotes the connecting map $\delta_{y,x}^{\text{loc}}(\mathfrak{T}_r(b)_X)$. Since $\text{cl}_V(y)$ extends to $\text{cl}_X(S)$, we have $\delta(\text{cl}_V(y) \cup \{\pi_x\}) = -\text{cl}_X(x)$ by Proposition 5.1.3 (see also the proof of Sublemma 5.2.2). Hence we have

$$(6.2.1) = -\text{cl}_X(x) \cup \omega = -\text{Gys}_{i_x}^n(\bar{\omega}) = -\text{Gys}_{i_x}^n \circ \partial_{y,x}^{\text{val}}(\{\pi_x, \beta_1, \dots, \beta_{n-b}\}),$$

where $\bar{\omega}$ denotes the residue class of ω in $H^0(x, W_r \Omega_{x,\log}^{n-b})$ and the second equality follows from Proposition 4.4.10 for i_x . Thus we obtain the desired commutativity. This completes Step 1.

6.3. Proof of the theorems, Step 2. In this step we prove Theorem 6.1.3 assuming that X satisfies 4.1.2 (see also Remark 6.3.4 below). Let $i : Z \hookrightarrow X$ and $\psi : U \hookrightarrow V$ be as in Theorem 6.1.3. Let T be the divisor on Z defined by the radical of $(p) \subset \mathcal{O}_Z$. We obtain a commutative diagram of schemes

$$\begin{array}{ccccc} U & \xrightarrow{\beta} & Z & \xleftarrow{\alpha} & T \\ \psi \downarrow & \square & i \downarrow & & \downarrow \\ V & \xrightarrow{j} & X & \xleftarrow{\iota} & Y. \end{array}$$

Put $\phi := i \circ \alpha : T \hookrightarrow X$, $\mathfrak{L} := \mathfrak{T}_r(n-c)_Z[-2c]$ and $\mathfrak{M} := Ri^! \mathfrak{T}_r(n)_X$. By Theorem 4.4.7 for ϕ , there is an isomorphism

$$\nu_{T,r}^{n-c-1}[-n-c-1] \simeq \tau_{\leq n+c+1} R\phi^! \mathfrak{T}_r(n)_X = \tau_{\leq n+c+1} R\alpha^! \mathfrak{M}. \quad (6.3.1)$$

Consider a diagram with distinguished rows in $D^+(Z_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})$

$$\begin{array}{ccccccc} \mathfrak{L} & \xrightarrow{t[-2c]} & (\tau_{\leq n-c} R\beta_* \mu_{p^r}^{\otimes n-c})[-2c] & \xrightarrow{-\sigma[-2c]} & \alpha_* \nu_{T,r}^{n-c-1}[-n-c] & \xrightarrow{g[-2c]} & \mathfrak{L}[1] \\ & & R\beta_*(\text{Gys}_\psi^n) \downarrow & & \downarrow R\alpha_*(\text{Gys}_\phi^n)[1] & & \\ \mathfrak{M} & \xrightarrow{\beta^*} & R\beta_* \beta^* \mathfrak{M} & \xrightarrow{\delta_{U,T}^{\text{loc}}(\mathfrak{M})} & R\alpha_* R\alpha^! \mathfrak{M}[1] & \xrightarrow{\alpha_*} & \mathfrak{M}[1]. \end{array} \quad (6.3.2)$$

Here the upper low is the distinguished triangle defining $\mathfrak{T}_r(n-c)_Z$ shifted by degree $-2c$, and we wrote σ for $\sigma_{Z,r}(n-c)$. The lower row is the localization distinguished triangle for \mathfrak{M} (cf. (1.9.2)). We show that the square of (6.3.2) commutes. Indeed, by (6.3.1) and Lemma 2.1.1, it is enough to show that the induced diagram of the $(n+c)$ -th cohomology sheaves is commutative at the generic points of T , which was shown in Step 1. Hence the square commutes, and there is a unique morphism $\text{Gys}_i^n : \mathcal{L} \rightarrow \mathfrak{M}$ that extends Gys_i^n by Lemma 2.1.2 (1), because

$$\text{Hom}_{D^+(Z_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})}(\mathcal{L}, R\alpha_* R\alpha^! \mathfrak{M}) = 0$$

by (6.3.1) and Lemma 2.1.1.

We next show that $\tau_{\leq n+c}(\text{Gys}_i^n)$ is an isomorphism. By the commutativity of the square of (6.3.2), the morphism $\delta_{U,T}^{\text{loc}}(\mathfrak{M})$ is surjective on the $(n+c)$ -th cohomology sheaves, and there is a distinguished triangle

$$\tau_{\leq n+c} \mathfrak{M} \xrightarrow{\beta^*} \tau_{\leq n+c} R\beta_* \beta^* \mathfrak{M} \xrightarrow{\delta_{U,T}^{\text{loc}}(\mathfrak{M})} (\tau_{\leq n+c+1} R\alpha_* R\alpha^! \mathfrak{M})[1] \xrightarrow{(\alpha_*)'} (\tau_{\leq n+c} \mathfrak{M})[1],$$

where the arrow $(\alpha_*)'$ is obtained by decomposing $\alpha_* : R\alpha_* R\alpha^! \mathfrak{M}[1] \rightarrow \mathfrak{M}[1]$. Replacing the lower row of (6.3.2) with this distinguished triangle, we see that $\tau_{\leq n+c}(\text{Gys}_i^n)$ is an isomorphism. This completes Step 2.

Corollary 6.3.3. *Let $i : Z \hookrightarrow X$ be as in Theorem 6.1.3, and assume further that X satisfies 4.1.2. Let $h : Z' \hookrightarrow Z$ be a closed immersion of pure codimension with $\text{ch}(Z') = p$. Put $g := i \circ h$ and $c' := \text{codim}_X(Z')$. Then we have $\text{Gys}_g^n = Rh^!(\text{Gys}_i^n) \circ (\text{Gys}_h^{n-c}[-2c])$ as morphisms*

$$\nu_{Z'}^{n-c'}[-n-c'] \longrightarrow Rg^! \mathfrak{T}_r(n)_X \quad \text{in } D^b(Z'_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z}).$$

Proof. Because $\tau_{\leq n+c'-1} Rg^! \mathfrak{T}_r(n)_X = 0$ by Theorem 4.4.7, a morphism $\nu_{Z'}^{n-c'}[-n-c'] \rightarrow Rg^! \mathfrak{T}_r(n)_X$ is determined by a map on the $(n+c')$ -th cohomology sheaves (cf. Lemma 2.1.1). Because $R^{n+c'} g^! \mathfrak{T}_r(n)_X$ is isomorphic to the sheaf $\nu_{Z'}^{n-c'}$ by Theorem 4.4.7, we are reduced to the case that X and Z are local with closed point Z' , and moreover, to the case that Z' is a generic point of $Z \otimes_{\mathbb{Z}} \mathbb{F}_p$ (that is, $c' = c + 1$). This last case follows from the commutativity of (6.1.2) proved in Step 1. \square

Remark 6.3.4. (1) *By the results in this step and the bijectivity of $\text{Gys}_{i_x}^n$ in (6.1.2) (cf. Theorem 4.4.7), Case (M) of Theorem 6.1.1 (with X and $\overline{\{y\}}$ arbitrary) is reduced to the case where X is smooth over B . We will prove this case in the next step.*
(2) *Once we finish the proof of Theorem 6.1.1, we will obtain Theorem 6.1.3 by repeating the same arguments as for Step 2.*

6.4. Proof of the theorems, Step 3. Assume that X is smooth over B . In this step, we prove Case (M) of Theorem 6.1.1 for this X , which will complete the proof of Theorems 6.1.1 and 6.1.3 (cf. Remark 6.3.4). We first show Lemma 6.4.1 below. Let m be a positive integer, and put $\mathbb{P} := \mathbb{P}_X^m$. Consider cartesian squares of schemes

$$\begin{array}{ccccc} \mathbb{P}_V & \xrightarrow{\beta} & \mathbb{P} & \xleftarrow{\alpha} & \mathbb{P}_Y \\ \psi \downarrow & & \square & f \downarrow & \square & g \downarrow \\ V & \xrightarrow{j} & X & \xleftarrow{\iota} & Y. \end{array}$$

For this diagram, we prove

Lemma 6.4.1. (1) *There is a unique morphism*

$$\mathrm{tr}_f^n : Rf_* \mathfrak{T}_r(n+m)_{\mathbb{P}}[2m] \longrightarrow \mathfrak{T}_r(n)_X \quad \text{in } D^b(X_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})$$

that extends the trace morphism for ψ ([SGA4], XVIII.2.9, XII.5.3)

$$\mathrm{tr}_\psi^n : R\psi_* \mu_{p^r}^{\otimes n+m}[2m] \longrightarrow \mu_{p^r}^{\otimes n} \quad \text{in } D^b(V_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z}).$$

(2) tr_f^n fits into a commutative diagram

$$\begin{array}{ccc} R\iota_* Rg_* W_r \Omega_{\mathbb{P}_Y, \log}^{n+m-1}[m-n-1] & \xrightarrow{(\natural)} & Rf_* \mathfrak{T}_r(n+m)_{\mathbb{P}}[2m] \\ R\iota_*(\mathrm{tr}_g^{n-1})[-n-1] \downarrow & & \mathrm{tr}_f^n \downarrow \\ \iota_* W_r \Omega_{Y, \log}^{n-1}[-n-1] & \xrightarrow{\mathrm{Gys}_\alpha^n} & \mathfrak{T}_r(n)_X. \end{array}$$

Here tr_g^{n-1} denotes $(-1)^m$ -times of the Gysin morphism of Gros ([Gs], II.1.2.7). The arrow (\natural) is induced by the isomorphism $R\iota_* Rg_* = Rf_* R\alpha_*$ and the Gysin morphism $\mathrm{Gys}_\alpha^{m+n}$.

Proof. Because $Rf_* \mathfrak{T}_r(n+m)_{\mathbb{P}}[2m]$ is concentrated in degrees $\leq n$ ([SGA4], XII.5.2, X.5.2), it is enough to show that the square

$$\begin{array}{ccc} Rf_*(\tau_{\leq n+m} R\beta_* \mu_{p^r}^{\otimes n+m})[2m] & \xrightarrow{Rf_*(\sigma_{\mathbb{P}, r}(n+m))[2m]} & Rf_*(\alpha_* W_r \Omega_{\mathbb{P}_Y, \log}^{n+m-1})[m-n] \\ \downarrow & & \downarrow R\iota_*(\mathrm{tr}_g)[-n] \\ \tau_{\leq n} Rj_* \mu_{p^r}^{\otimes n} & \xrightarrow{\sigma_{X, r}(n)} & \iota_* W_r \Omega_{Y, \log}^{n-1}[-n] \end{array}$$

is commutative in $D^b(X_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})$ (cf. Lemma 2.1.2 (1)). Here the left vertical arrow is defined as the composite of the natural morphism

$$Rf_*(\tau_{\leq n+m} R\beta_* \mu_{p^r}^{\otimes n+m})[2m] \longrightarrow \tau_{\leq n}(Rf_* R\beta_* \mu_{p^r}^{\otimes n+m}[2m]) = \tau_{\leq n}(Rj_* R\psi_* \mu_{p^r}^{\otimes n+m}[2m])$$

and $\tau_{\leq n} Rj_*(\mathrm{tr}_\psi)$. The vertices of this diagram are concentrated in degrees $\leq n$. Hence we are reduced to the commutativity of the diagram of the n -th cohomology sheaves, which one can check by taking a section $s : X \hookrightarrow \mathbb{P}_X$ of f and using the compatibility proved in Step 1 (see also Remark 2.2.6 (2)). More precisely, using (4.4.8) one can construct a Gysin map

$$\mathrm{Gys}_s : R^n j_* \mu_{p^r}^{\otimes n} \longrightarrow R^m f_*(R^{m+n} \beta_* \mu_{p^r}^{\otimes n+m}) = \mathcal{H}^n(Rf_*(\tau_{\leq n+m} R\beta_* \mu_{p^r}^{\otimes n+m})[2m]),$$

induced by that for $s_V : V \hookrightarrow \mathbb{P}_V$. One can further check that it is surjective by Theorem 3.3.7 with $r = 1$ and [Gs], I.2.1.5, I.2.2.3. The details are straight-forward and left to the reader. \square

Now we turn to the proof of Theorem 6.1.1. Replacing X by $\mathrm{Spec}(\mathcal{O}_{X, x})$, we assume that X is local with closed point x . Suppose that $S(= \overline{\{y\}})$ is not normal, and let $\mathbf{n} : T \rightarrow S$ be the normalization of S . Since \mathbf{n} is finite, the composite $T \rightarrow S \hookrightarrow X$ is projective, i.e., factors as $T \xrightarrow{i} \mathbb{P}_X^m =: \mathbb{P} \xrightarrow{f} X$ with i closed, for some $m \geq 1$. Let $\psi : \mathbb{P}_V^m \rightarrow V$ be the morphism induced by f . Let T_x be the fiber $\mathbf{n}^{-1}(x) \subset T$ with reduced structure, and let $h : T_x \rightarrow x$ be the natural map. Consider the diagram in Figure 1 below, where we wrote Gys for Gysin maps for simplicity. In this diagram, the outer large square commutes by the definition of $\delta_{y, x}^{\mathrm{val}}$ and [JSS], Lemma A.1. The square (1) commutes by Step 1. The square (2) commutes by [SGA4 $\frac{1}{2}$], Cycle, 2.3.8 (i). The square (3) commutes by the property $j^*(\mathrm{tr}_f^n) = \mathrm{tr}_\psi^n$ of tr_f^n . The square (4)

$$\begin{array}{ccc}
\mathrm{H}^{n-b+1}(y, \mu_{p^r}^{\otimes n-b+1}) & \xrightarrow{-\partial_{y, T_x}^{\mathrm{val}}} & \mathrm{H}^0(T_x, W_r \Omega_{T_x, \log}^{n-b}) \\
\downarrow \parallel & \begin{array}{c} \searrow \text{Gys} \\ \xrightarrow{(1)} \\ \downarrow \text{tr}_\psi^n \\ \xrightarrow{(2)} \\ \swarrow \text{Gys} \end{array} & \begin{array}{c} \swarrow \text{Gys} \\ \xrightarrow{(3)} \\ \downarrow \text{tr}_f^n \\ \xrightarrow{(4)} \\ \swarrow \text{Gys} \end{array} \\
\mathrm{H}_y^{n+2m+b-1}(\mathbb{P}_V, \mu_{p^r}^{\otimes m+n}) & \xrightarrow{\delta_{y, T_x}^{\mathrm{loc}}(\mathfrak{T}_r(n+m)_\mathbb{F})} & \mathrm{H}_{T_x}^{n+2m+b}(\mathbb{P}, \mathfrak{T}_r(n+m)_\mathbb{F}) \\
\downarrow \text{tr}_\psi^n & \xrightarrow{(3)} & \downarrow \text{tr}_f^n \\
\mathrm{H}_y^{n+b-1}(V, \mu_{p^r}^{\otimes n}) & \xrightarrow{\delta_{y, x}^{\mathrm{loc}}(\mathfrak{T}_r(n)_X)} & \mathrm{H}_x^{n+b}(X, \mathfrak{T}_r(n)_X) \\
\downarrow \text{Gys} & \xrightarrow{(5)} & \downarrow \text{Gys} \\
\mathrm{H}^{n-b+1}(y, \mu_{p^r}^{\otimes n-b+1}) & \xrightarrow{-\partial_{y, x}^{\mathrm{val}}} & \mathrm{H}^0(x, W_r \Omega_{x, \log}^{n-b}) \\
& & \downarrow \text{tr}_h^{n-b}
\end{array}$$

FIGURE 1. diagram for Step 3

commutes by Lemma 6.4.1 (2) and Remark 2.2.6 (1). Hence the square (5) commutes, which is the commutativity of (6.1.2). This completes the proof of Theorems 6.1.1 and 6.1.3. \square

7. Covariant functoriality and relative duality

In this section, we prove the covariant functoriality **T7** in Theorem 1.1.2 and prove a relative duality result (see Theorem 7.3.1 below). Throughout this section, we work with the setting in §4.1. Let X and Z be integral schemes which are flat of finite type over B and satisfy 4.1.1, and let $f : Z \rightarrow X$ be a separated morphism of finite type. Put $c := \dim(X) - \dim(Z)$, and let $\psi : Z[1/p] \rightarrow X[1/p] = V$ be the morphism induced by f . By the absolute purity [Th], [FG], there is a trace morphism

$$\mathrm{tr}_\psi^n : R\psi_! \mu_{p^r}^{\otimes n-c}[-2c] \longrightarrow \mu_{p^r}^{\otimes n} \quad \text{in } D^b(V_{\acute{\mathrm{e}}\mathrm{t}}, \mathbb{Z}/p^r\mathbb{Z}),$$

which extends the trace morphisms for flat morphisms due to Deligne [SGA4], XVIII.2.9 and satisfies the transitivity property.

7.1. Covariant functoriality. The first result of this section is the following:

Theorem 7.1.1 (Covariant functoriality). *For $f : Z \rightarrow X$ as before, there is a unique morphism*

$$\mathrm{tr}_f^n : Rf_! \mathfrak{T}_r(n-c)_Z[-2c] \longrightarrow \mathfrak{T}_r(n)_X \quad \text{in } D^b(X_{\acute{\mathrm{e}}\mathrm{t}}, \mathbb{Z}/p^r\mathbb{Z})$$

that extends tr_ψ^n . Consequently, these trace morphisms satisfy the transitivity property.

This theorem will be proved in the next subsection. In this subsection, we prove the following:

Lemma 7.1.2. *Let k be a perfect field of characteristic $p > 0$, and let Y be a normal crossing variety over $\mathrm{Spec}(k)$. Let $g : T \rightarrow Y$ be a separated morphism of finite type of schemes, and assume that T has dimension $\leq a$. Put $c := \dim(Y) - a$. Assume that $\mathfrak{L} \in D^b(W_{\acute{\mathrm{e}}\mathrm{t}}, \mathbb{Z}/p^r\mathbb{Z})$ and $\mathfrak{M} \in D^-(Y_{\acute{\mathrm{e}}\mathrm{t}}, \mathbb{Z}/p^r\mathbb{Z})$ are concentrated in degrees $\leq \ell$ and $\leq m$, respectively. Let \mathcal{F} be a locally free $(\mathcal{O}_Y)^p$ -module of finite rank. Then for an integer $q < c - \ell - m$, we have*

$$\begin{aligned}
\mathrm{Hom}_{D^-(Y_{\acute{\mathrm{e}}\mathrm{t}}, \mathbb{Z}/p^r\mathbb{Z})}((Rg_! \mathfrak{L}) \otimes^{\mathbb{L}} \mathfrak{M}, \nu_{Y, r}^n[q]) &= 0 \quad \text{and} \\
\mathrm{Hom}_{D^-(Y_{\acute{\mathrm{e}}\mathrm{t}}, \mathbb{Z}/p^r\mathbb{Z})}((Rg_! \mathfrak{L}) \otimes^{\mathbb{L}} \mathfrak{M}, \mathcal{F}[q]) &= 0.
\end{aligned}$$

Proof. We prove the assertion only for $\nu_{Y,r}^n$. One can check the assertion for \mathcal{F} by repeating the same arguments as for $\nu_{Y,r}^n$, using Lemma A.2.8 below.

We first prove the case that \mathcal{L} has constructible cohomology sheaves. If there are closed subschemes $\phi_i : T_i \hookrightarrow T$ ($i = 1, 2$) such that $T = T_1 \cup T_2$ and $\dim(T_1 \cap T_2) \leq a - 1$, then there is a distinguished triangle of the form

$$R\phi_{12*}\phi_{12}^*\mathcal{L}[-1] \longrightarrow \mathcal{L} \longrightarrow R\phi_{1*}\phi_1^*\mathcal{L} \oplus R\phi_{2*}\phi_2^*\mathcal{L} \longrightarrow R\phi_{12*}\phi_{12}^*\mathcal{L}$$

in $D^b(T_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})$, where ϕ_{12} denotes the closed immersion $T_1 \cap T_2 \hookrightarrow T$. Hence by induction on $a \geq 0$, we may assume that T is irreducible. Let b be the dimension of $g(T) \subset Y$. Noting that $Rg_!\mathcal{L}$ has constructible cohomology sheaves by [SGA4], XIV.1.1, we prove

Sublemma 7.1.3. *For $i \in \mathbb{Z}$, the support of $R^i g_!\mathcal{L}$ has dimension $\leq \min\{b, a + \ell - i\}$, i.e., there is a closed subscheme $\phi : W \hookrightarrow Y$ of dimension $\leq \min\{b, a + \ell - i\}$ for which we have $R^i g_!\mathcal{L} = \phi_*\phi^*R^i g_!\mathcal{L}$.*

Proof of Sublemma 7.1.3. Without loss of generality, we may assume that g is proper (hence $Rg_! = Rg_*$). Since $R^i g_*\mathcal{L}$ is zero outside of $g(T)$, the support of $R^i g_*\mathcal{L}$ has dimension $\leq b$. We show that the support of $R^i g_*\mathcal{L}$ is at most $(a + \ell - i)$ -dimensional. Let y be a point on $g(T)$. Put $T_y := T \times_Y y = g^{-1}(y)$. Since $\dim \overline{\{y\}} + \dim T_y$ equals the dimension of the closure of T_y in T ([SGA4], XIV.2.3 (iii)), we have $\dim \overline{\{y\}} + \dim T_y \leq a$. Now suppose that $(R^i g_*\mathcal{L})_{\overline{y}} \neq 0$. Because $(R^q g_*\mathcal{L})_{\overline{y}}$ is zero for $q > \dim T_y + \ell$ (loc. cit., XII.5.2, X.5.2), we have

$$i \leq \dim T_y + \ell \leq a - \dim \overline{\{y\}} + \ell,$$

that is, $\dim \overline{\{y\}} \leq a + \ell - i$. Thus we obtain the sublemma. \square

We now turn to the proof of the lemma and compute a spectral sequence

$$E_2^{u,v} = \text{Ext}_{Y,\mathbb{Z}/p^r\mathbb{Z}}^u((R^{-v} g_!\mathcal{L}) \otimes^{\mathbb{L}} \mathfrak{M}, \nu_{Y,r}^n) \implies \text{Ext}_{Y,\mathbb{Z}/p^r\mathbb{Z}}^{u+v}((Rg_!\mathcal{L}) \otimes^{\mathbb{L}} \mathfrak{M}, \nu_{Y,r}^n).$$

For (u, v) with $u + v < c - \ell - m$ and $b \leq a + \ell + v$, we have $u + m < \dim(Y) - b$ and

$$\begin{aligned} E_2^{u,v} &= \text{Ext}_{Y,\mathbb{Z}/p^r\mathbb{Z}}^u((\phi_*\phi^*R^{-v} g_!\mathcal{L}) \otimes^{\mathbb{L}} \mathfrak{M}, \nu_{Y,r}^n) \\ &= \text{Ext}_{Y,\mathbb{Z}/p^r\mathbb{Z}}^u(R\phi_*\phi^*(R^{-v} g_!\mathcal{L} \otimes^{\mathbb{L}} \mathfrak{M}), \nu_{Y,r}^n) \\ &= \text{Ext}_{W,\mathbb{Z}/p^r\mathbb{Z}}^u(\phi^*(R^{-v} g_!\mathcal{L} \otimes^{\mathbb{L}} \mathfrak{M}), R\phi^!\nu_{Y,r}^n) = 0 \end{aligned}$$

by Theorem 2.2.2 and Lemma 2.1.1. Here W denotes the closure of $g(T)$ and ϕ denotes the closed immersion $W \hookrightarrow Y$. For (u, v) with $u + v < c - \ell - m$ and $b > a + \ell + v$, we have $u + m < \dim(Y) - (a + \ell + v)$. There is a closed subscheme $\phi : W \hookrightarrow Y$ of codimension $\geq \dim(Y) - (a + \ell + v)$ such that $R^{-v} g_!\mathcal{L} = \phi_*\phi^*R^{-v} g_!\mathcal{L}$, by the sublemma. Hence

$$E_2^{u,v} = \text{Ext}_{Y,\mathbb{Z}/p^r\mathbb{Z}}^u(\phi^*(R^{-v} g_!\mathcal{L} \otimes^{\mathbb{L}} \mathfrak{M}), R\phi^!\nu_{Y,r}^n) = 0$$

again by Theorem 2.2.2 and Lemma 2.1.1. Thus we obtain the assertion.

We next prove the case that \mathcal{L} is general. Take a bounded complex of $\mathbb{Z}/p^r\mathbb{Z}$ -sheaves \mathcal{L}^\bullet which is concentrated in degrees $\leq \ell$ and represents \mathcal{L} . Take a filtered inductive system $\{\mathcal{L}_\lambda^\bullet\}_{\lambda \in \Lambda}$ consisting of bounded complexes of constructible $\mathbb{Z}/p^r\mathbb{Z}$ -sheaves which are concentrated in

degrees $\leq \ell$ and whose limit is \mathcal{L}^\bullet . Then for $q < c - \ell - m$, we have

$$\begin{aligned} \mathrm{Ext}_{Y, \mathbb{Z}/p^r\mathbb{Z}}^q((Rg_!\mathcal{L}) \otimes^{\mathbb{L}} \mathfrak{M}, \nu_{Y,r}^n) &= \mathrm{Ext}_{T, \mathbb{Z}/p^r\mathbb{Z}}^q(\mathcal{L}, Rg^!R\mathcal{H}om(\mathfrak{M}, \nu_{Y,r}^n)) \\ &= \varprojlim_{\lambda \in \Lambda} \mathrm{Ext}_{T, \mathbb{Z}/p^r\mathbb{Z}}^q(\mathcal{L}_\lambda, Rg^!R\mathcal{H}om(\mathfrak{M}, \nu_{Y,r}^n)) \\ &= \varprojlim_{\lambda \in \Lambda} \mathrm{Ext}_{Y, \mathbb{Z}/p^r\mathbb{Z}}^q((Rg_!\mathcal{L}_\lambda) \otimes^{\mathbb{L}} \mathfrak{M}, \nu_{Y,r}^n) = 0 \end{aligned}$$

by the previous case. Here $Rg^!$ denotes the twisted inverse image functor of Deligne [SGA4], XVIII, and we have used the adjointness between $Rg^!$ and $Rg_!$. The second equality follows from the vanishing of the groups $\mathrm{Ext}_{Y, \mathbb{Z}/p^r\mathbb{Z}}^{q-1}((Rg_!\mathcal{L}_\lambda) \otimes^{\mathbb{L}} \mathfrak{M}, \nu_{Y,r}^n)$ for all $\lambda \in \Lambda$ and a standard argument which is similar as for (8.4.2) below. This completes the proof of the lemma. \square

As a special case of Lemma 7.1.2, we obtain

Corollary 7.1.4 (Semi-purity). *Under the same setting as in Lemma 7.1.2, we have*

$$\tau_{\leq c-1}Rg^!\nu_{Y,r}^n = \tau_{\leq c-1}Rg^!\mathcal{F} = 0.$$

Proof. For T' étale separated of finite type over T , $\mathcal{G} \in \{\nu_{Y,r}^n, \mathcal{F}\}$ and $q \leq c-1$, we have

$$\mathrm{Hom}_{D^+(T'_{\acute{e}t}, \mathbb{Z}/p^r\mathbb{Z})}(\mathbb{Z}/p^r\mathbb{Z}, Rh^!\mathcal{G}[q]) = \mathrm{Hom}_{D^b(Y'_{\acute{e}t}, \mathbb{Z}/p^r\mathbb{Z})}(Rh_!\mathbb{Z}/p^r\mathbb{Z}, \mathcal{G}[q]) = 0$$

by Lemma 7.1.2, where h denotes the composite map $T' \rightarrow T \rightarrow Y$. Hence $\tau_{\leq c-1}Rg^!\mathcal{G} = 0$. \square

7.2. Proof of Theorem 7.1.1. Let $j : V \hookrightarrow X$ and $\iota : Y \hookrightarrow X$ be as in §4.1. Put $\mathcal{L} := \mathfrak{T}_r(n-c)_Z[-2c]$, for simplicity. We first show

$$\mathrm{Hom}_{D^b(X_{\acute{e}t}, \mathbb{Z}/p^r\mathbb{Z})}(Rf_!\mathcal{L}, R\iota_*R\iota^!\mathfrak{T}_r(n)_X) = 0. \quad (7.2.1)$$

Indeed, for $g : T := Z \times_X Y \rightarrow Y$ induced by f , we have

$$\mathrm{Hom}_{D^b(X_{\acute{e}t}, \mathbb{Z}/p^r\mathbb{Z})}(Rf_!\mathcal{L}, R\iota_*R\iota^!\mathfrak{T}_r(n)_X) = \mathrm{Hom}_{D^b(Y_{\acute{e}t}, \mathbb{Z}/p^r\mathbb{Z})}(Rg_!\alpha^*\mathcal{L}, R\iota^!\mathfrak{T}_r(n)_X)$$

by the adjointness between ι^* and $R\iota_*$ and the proper base-change theorem: $\iota^*Rf_! = Rg_!\alpha^*$, where α denotes the closed immersion $T \hookrightarrow Z$. The latter group is zero by Lemma 7.1.2 and a similar argument as for the vanishing (4.4.8). By (7.2.1) and Lemma 2.1.2 (1), it remains to show that the composite morphism

$$Rf_!\mathcal{L} \xrightarrow{Rj_*(\mathrm{tr}_\psi^n)} Rj_*\mu_{p^r}^{\otimes n} \xrightarrow{\delta_{V,Y}^{\mathrm{loc}}(\mathfrak{T}_r(n)_X)} R\iota_*R\iota^!\mathfrak{T}_r(n)_X[1] \quad (7.2.2)$$

is zero in $D^b(X_{\acute{e}t}, \mathbb{Z}/p^r\mathbb{Z})$. We show the following:

Lemma 7.2.3. (1) *Let $\{Z_\lambda\}_{\lambda \in \Lambda}$ be an open covering of Z with Λ finite, and let $f_\lambda : Z_\lambda \rightarrow X$ be the composite map $Z_\lambda \hookrightarrow Z \rightarrow X$ for each $\lambda \in \Lambda$. Then the adjunction map*

$$\begin{aligned} &\mathrm{Hom}_{D^b(X_{\acute{e}t}, \mathbb{Z}/p^r\mathbb{Z})}(Rf_!\mathcal{L}, R\iota_*R\iota^!\mathfrak{T}_r(n)_X[1]) \\ &\longrightarrow \bigoplus_{\lambda \in \Lambda} \mathrm{Hom}_{D^b(X_{\acute{e}t}, \mathbb{Z}/p^r\mathbb{Z})}(Rf_{\lambda!}(\mathcal{L}|_{Z_\lambda}), R\iota_*R\iota^!\mathfrak{T}_r(n)_X[1]) \end{aligned}$$

is injective.

(2) *Assume that f is flat. Let $Y' \subset Y$ be a closed subscheme of codimension ≥ 1 . Put $U := X \setminus Y'$. Then the following natural restriction map is injective:*

$$\begin{aligned} &\mathrm{Hom}_{D^b(X_{\acute{e}t}, \mathbb{Z}/p^r\mathbb{Z})}(Rf_!\mathcal{L}, R\iota_*R\iota^!\mathfrak{T}_r(n)_X[1]) \\ &\longrightarrow \mathrm{Hom}_{D^b(U_{\acute{e}t}, \mathbb{Z}/p^r\mathbb{Z})}((Rf_!\mathcal{L})|_U, (R\iota_*R\iota^!\mathfrak{T}_r(n)_X[1])|_U). \end{aligned}$$

Proof of Lemma 7.2.3. (1) It suffices to consider the case that $\Lambda = \{1, 2\}$. Put $Z_{12} := Z_1 \cap Z_2$. Let f_{12} be the composite map $Z_{12} \hookrightarrow Z \rightarrow X$. There is a distinguished triangle of the form

$$Rf_{12!}(\mathcal{L}|_{Z_{12}}) \longrightarrow Rf_{1!}(\mathcal{L}|_{Z_1}) \oplus Rf_{2!}(\mathcal{L}|_{Z_2}) \longrightarrow Rf_! \mathcal{L} \longrightarrow Rf_{12!}(\mathcal{L}|_{Z_{12}})[1].$$

Hence the assertion follows from the vanishing (7.2.1) for $f_{12} : Z_{12} \rightarrow X$.

(2) Let ϕ be the closed immersion $Y' \hookrightarrow X$. The kernel of the map in question is a quotient of $\mathrm{Hom}_{D^b(X_{\acute{e}t}, \mathbb{Z}/p^r\mathbb{Z})}(Rf_! \mathcal{L}, R\phi_* R\phi^! \mathfrak{T}_r(n)_X[1])$. One can check that it is zero by a similar argument as for (7.2.1), noting that $Y' \times_X Z$ has codimension ≥ 1 in T by the flatness of f . \square

We show that the composite morphism (7.2.2) is zero. We first assume that $Z = \mathbb{P}_X^m$ ($m \geq 1$) and that f is the natural projection. By Lemma 7.2.3 (2), we are reduced to the case that Y is smooth. In this case, (7.2.2) is zero by Lemma 6.4.1. We next prove the general case. By Lemma 7.2.3 (1), we may assume that f is affine. Take a decomposition $Z \xrightarrow{i} \mathbb{P}_X^m =: \mathbb{P} \xrightarrow{h} X$ of f for some integer $m \geq 0$, where i is a locally closed immersion. We have morphisms

$$Rf_! \mathcal{L} \xrightarrow{Rh_*(\mathrm{Gys}_i^{m+n})[2m]} Rh_* \mathfrak{T}_r(m+n)_{\mathbb{P}}[2m] \xrightarrow{\mathrm{tr}_h^n} \mathfrak{T}_r(n)_X,$$

where tr_h^n is obtained from the vanishing of (7.2.2) for h . See Theorem 6.1.3 for Gys_i^{m+n} . Since this composite morphism extends tr_ψ^n , we see that (7.2.2) is zero. This completes the proof of Theorem 7.1.1. \square

The following corollary is a horizontal variant of Proposition 4.4.10:

Corollary 7.2.4 (Projection formula). *For $f : Z \rightarrow X$ as before, the diagram*

$$\begin{array}{ccccc} Rf_! \mathfrak{T}_r(m-c)_Z[-2c] \otimes^{\mathbb{L}} \mathfrak{T}_r(n)_X & \xrightarrow{\mathrm{tr}_f^m \otimes^{\mathbb{L}} \mathrm{id}} & \mathfrak{T}_r(m)_X \otimes^{\mathbb{L}} \mathfrak{T}_r(n)_X & \xrightarrow{(4.2.7)} & \mathfrak{T}_r(m+n)_X \\ \mathrm{id} \otimes^{\mathbb{L}} f^* \downarrow & & & & \parallel \\ Rf_! \mathfrak{T}_r(m-c)_Z[-2c] \otimes^{\mathbb{L}} Rf_* \mathfrak{T}_r(n)_Z & \xrightarrow{(4.2.7)} & Rf_! \mathfrak{T}_r(m+n-c)_Z[-2c] & \xrightarrow{\mathrm{tr}_f^{m+n}} & \mathfrak{T}_r(m+n)_X \end{array}$$

commutes in $D^-(X_{\acute{e}t}, \mathbb{Z}/p^r\mathbb{Z})$. See Proposition 4.2.8 for $f^ : \mathfrak{T}_r(n)_X \rightarrow Rf_* \mathfrak{T}_r(n)_Z$.*

Proof. Because the diagram in question commutes on $X[1/p]$, the assertion follows from Lemma 2.1.2 (1) and a vanishing

$$\mathrm{Hom}_{D^-(X_{\acute{e}t}, \mathbb{Z}/p^r\mathbb{Z})}(Rf_! \mathfrak{T}_r(m-c)_Z[-2c] \otimes^{\mathbb{L}} \mathfrak{T}_r(n)_X, R\iota_* R\iota^! \mathfrak{T}_r(m+n)_X) = 0,$$

which one can check by Lemma 7.1.2 and a similar argument as for (7.2.1). \square

7.3. Relative duality. Let $f : Z \rightarrow X$ be as before. Let $j : V \hookrightarrow X$ and $\iota : Y \hookrightarrow X$ be as in §4.1. Let T be the divisor on Z defined by the radical of $(p) \subset \mathcal{O}_Z$. There is a commutative diagram of schemes

$$\begin{array}{ccccc} Z[1/p] & \xrightarrow{\beta} & Z & \xleftarrow{\alpha} & T \\ \downarrow & \square & \downarrow f & & \downarrow g \\ V & \xrightarrow{j} & X & \xleftarrow{\iota} & Y. \end{array}$$

Put $d := \dim(X)$, $b := \dim(Z)$ and $c := d - b$. We prove the following result, which was included in the earlier version of [JSS]:

Theorem 7.3.1 (Relative duality). (1) tr_f^d induces an isomorphism

$$\mathrm{tr}^f : \mathfrak{F}_r(b)_Z[-2c] \xrightarrow{\simeq} Rf^! \mathfrak{F}_r(d)_X \quad \text{in } D^b(Z_{\acute{e}t}, \mathbb{Z}/p^r\mathbb{Z}).$$

(2) There is a commutative diagram in $D^b(X_{\acute{e}t}, \mathbb{Z}/p^r\mathbb{Z})$

$$\begin{array}{ccc} R\iota_* Rg_! \nu_{T,r}^{b-1}[b-1] & \xrightarrow{(\natural)} & Rf_! \mathfrak{F}_r(b)_Z[2b] \\ R\iota_*(\mathrm{tr}_g) \downarrow & & \downarrow \mathrm{tr}_f^d[2d] \\ \iota_* \nu_{Y,r}^{d-1}[d-1] & \xrightarrow{\mathrm{Gys}_\alpha^d[2d]} & \mathfrak{F}_r(d)_X[2d], \end{array}$$

where tr_g denotes the trace morphism in Remark 2.2.6 (3), and the arrow (\natural) is induced by the isomorphism $R\iota_* Rg_! = Rf_! R\alpha_*$ and the Gysin morphism Gys_α^b .

To prove this theorem, we first note a standard fact (cf. [JSS], Lemma 3.8).

Lemma 7.3.2. For a torsion sheaf \mathcal{F} on $V_{\acute{e}t}$ and an integer $q > d$, we have $R^q j_* \mathcal{F} = 0$.

As an immediate consequence of Lemma 7.3.2 and (4.4.3), we obtain

Lemma 7.3.3. The natural morphism $\tau_{\leq d+1} R\iota^! \mathfrak{F}_r(d)_X \rightarrow R\iota^! \mathfrak{F}_r(d)_X$ is an isomorphism in $D^b(Y_{\acute{e}t}, \mathbb{Z}/p^r\mathbb{Z})$. Consequently, $\mathrm{Gys}_\alpha^d : \nu_{Y,r}^{d-1}[-d-1] \rightarrow R\iota^! \mathfrak{F}_r(d)_X$ is an isomorphism.

Proof of Theorem 7.3.1. We first show (2). By Lemma 7.1.2 and a similar argument as for the vanishing of (7.2.2), one can reduce the assertion to Lemma 6.4.1 (2) and Corollary 6.3.3 (see also Remark 2.2.6 (3)). The details are straight-forward and left to the reader. We next show (1). Let $\mathrm{tr}^f : \mathfrak{F}_r(b)_Z[-2c] \rightarrow Rf^! \mathfrak{F}_r(d)_X$ be the adjoint morphism of tr_f^d . Because $\beta^*(\mathrm{tr}^f)$ is an isomorphism by the absolute purity ([Th], [FG]), we have only to show that $R\alpha^!(\mathrm{tr}^f)$ is an isomorphism. By (2), there is a commutative diagram in $D^+(T_{\acute{e}t}, \mathbb{Z}/p^r\mathbb{Z})$

$$\begin{array}{ccc} \nu_{T,r}^{b-1}[b-1] & \xrightarrow{\mathrm{Gys}_\alpha^b[2b]} & R\alpha^! \mathfrak{F}_r(b)_Z[2b] \\ \mathrm{tr}^g \downarrow & & \downarrow R\alpha^!(\mathrm{tr}^f)[2d] \\ Rg^! \nu_{Y,r}^{d-1}[d-1] & \xrightarrow{Rg^!(\mathrm{Gys}_\alpha^d)[2d]} & R\alpha^! Rf^! \mathfrak{F}_r(d)_X[2d]. \end{array}$$

The horizontal arrows are isomorphisms by Lemma 7.3.3 for Z and X , respectively. The left vertical arrow, defined as the adjoint morphism of tr_g , is an isomorphism by [JSS], Theorem 2.8. Consequently, $R\alpha^!(\mathrm{tr}^f)$ is an isomorphism. This completes the proof of Theorem 7.3.1. \square

Remark 7.3.4. By Theorem 7.3.1, $\mathfrak{F}_r(d)_X[2d]$ is canonically isomorphic to the object $\mathcal{D}_{X,p^r} \in D^b(X_{\acute{e}t}, \mathbb{Z}/p^r\mathbb{Z})$ considered in [JSS], Theorem 4.4.

8. Explicit formula for p -adic vanishing cycles

In this section we construct a canonical pairing on the sheaves of p -adic vanishing cycles in the derived category, and prove an explicit formula for that pairing, which will be used in §9.

8.1. **Setting.** The setting is the same as in §3.2. Note the condition 3.2.1 assumed there. We further assume that K contains a primitive p -th root of unity and that k is finite. We put

$$\nu_Y^n := \nu_{Y,1}^n, \quad \mu' := \iota^* j_* \mu_p \quad \text{and} \quad \mu := \mu_p(K) \quad (8.1.1)$$

for simplicity. Note that μ' is the constant étale sheaf on Y associated with the abstract group $\mu(\simeq \mathbb{Z}/p\mathbb{Z})$, because the sheaf μ_p on X_K is constant and the sheaf $j_* \mu_p$ on X is also constant by the normality of X (cf. [Tsl], 1.5.1). Now let N be the relative dimension $\dim(X/O_K)$. Let n be a positive integer with $1 \leq n \leq N + 1$. Put $n' := N + 2 - n$, $M^q := M_1^q = \iota^* R^q j_* \mu_p^{\otimes q}$, and let U^\bullet be the filtration on M^q defined in Definition 3.3.2. The aim of this section is to construct a morphism

$$\Theta^n : U^1 M^n \otimes U^1 M^{n'}[-N-2] \longrightarrow \mu' \otimes \nu_Y^N[-N-1] \quad \text{in} \quad D^b(Y_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$$

and to prove an explicit formula for this morphism (cf. Theorem 8.3.8 below).

8.2. **Construction of Θ^n .** Because μ' is (non-canonically) isomorphic to the constant sheaf $\mathbb{Z}/p\mathbb{Z}$, we will write $\mu' \otimes \mathcal{K}$ ($\mathcal{K} \in D^-(Y_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$) for the derived tensor product $\mu' \otimes^{\mathbb{L}} \mathcal{K}$ in $D^-(Y_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$. For q with $1 \leq q \leq N + 1$, fix a distinguished triangle

$$(M^q/U^1 M^q)[-q-1] \xrightarrow{g'} \mathbb{A}(q) \xrightarrow{t'} \tau_{\leq q} \iota^* Rj_* \mu_p^{\otimes q} \longrightarrow (M^q/U^1 M^q)[-q],$$

where the last arrow is defined as the composite $\tau_{\leq q} \iota^* Rj_* \mu_p^{\otimes q} \rightarrow M^q[-q] \rightarrow (M^q/U^1 M^q)[-q]$. Clearly, $\mathbb{A}(q)$ is concentrated in $[0, q]$, and the triple $(\mathbb{A}(q), t', g')$ is unique up to a unique isomorphism (and g' is determined by $(\mathbb{A}(q), t')$) by Lemma 2.1.2 (3). We construct Θ^n by decomposing the morphism

$$\begin{aligned} \mathbb{A}(n) \otimes^{\mathbb{L}} \mathbb{A}(n') &\longrightarrow (\tau_{\leq n} \iota^* Rj_* \mu_p^{\otimes n}) \otimes^{\mathbb{L}} (\tau_{\leq n'} \iota^* Rj_* \mu_p^{\otimes n'}) \\ &\longrightarrow \iota^* Rj_* \mu_p^{\otimes N+2} \end{aligned} \quad (8.2.1)$$

induced by the natural isomorphism $\mu_p^{\otimes n} \otimes \mu_p^{\otimes n'} \simeq \mu_p^{\otimes N+2}$ in characteristic zero. By Lemma 7.3.2 and the assumption that $\zeta_p \in K$, there is a morphism

$$\iota^* Rj_* \mu_p^{\otimes N+2} \simeq \mu' \otimes (\tau_{\leq N+1} \iota^* Rj_* \mu_p^{\otimes N+1}) \xrightarrow{\text{id} \otimes (4.2.1)} \mu' \otimes \nu_Y^N[-N-1], \quad (8.2.2)$$

which, together with (8.2.1), induces a morphism

$$\mathbb{A}(n) \otimes^{\mathbb{L}} \mathbb{A}(n') \longrightarrow \mu' \otimes \nu_Y^N[-N-1]. \quad (8.2.3)$$

Noting that $\mathbb{A}(q)$ is concentrated in $[0, q]$ with $\mathcal{H}^q(\mathbb{A}(q)) \simeq U^1 M^q$, we show the following:

Lemma 8.2.4. *There is a unique morphism*

$$\mathbb{A}(n) \otimes^{\mathbb{L}} (U^1 M^{n'}[-n']) \longrightarrow \mu' \otimes \nu_Y^N[-N-1] \quad \text{in} \quad D^-(Y_{\text{ét}}, \mathbb{Z}/p\mathbb{Z}) \quad (8.2.5)$$

that the morphism (8.2.3) factors through.

Proof. There is a distinguished triangle of the form

$$\mathbb{A}(n) \otimes^{\mathbb{L}} (\tau_{\leq n'-1} \mathbb{A}(n')) \rightarrow \mathbb{A}(n) \otimes^{\mathbb{L}} \mathbb{A}(n') \rightarrow \mathbb{A}(n) \otimes^{\mathbb{L}} (U^1 M^{n'}[-n']) \rightarrow \mathbb{A}(n) \otimes^{\mathbb{L}} (\tau_{\leq n'-1} \mathbb{A}(n'))[1].$$

By Lemma 2.1.2 (2), it suffices to show that (i) the morphism

$$\mathbb{A}(n) \otimes^{\mathbb{L}} (\tau_{\leq n'-1} \mathbb{A}(n')) \longrightarrow \mu' \otimes \nu_Y^N[-N-1]$$

induced by (8.2.3) is zero and that (ii) we have

$$\text{Hom}_{D^-(Y_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})}(\mathbb{A}(n) \otimes^{\mathbb{L}} (\tau_{\leq n'-1} \mathbb{A}(n'))[1], \mu' \otimes \nu_Y^N[-N-1]) = 0.$$

The claim (ii) follows from Lemma 2.1.1. As for the claim (i), because $\mathbb{A}(n) \otimes^{\mathbb{L}} (\tau_{\leq n'-1} \mathbb{A}(n'))$ is concentrated in degrees $\leq N + 1$, the problem is reduced to the triviality of the induced map on the $(N + 1)$ st cohomology sheaves (cf. Lemma 2.1.1). One can check this by a similar argument as for Proposition 4.2.6. \square

Applying a similar argument as for this lemma to the morphism (8.2.5), we obtain a morphism

$$(U^1 M^n[-n]) \otimes^{\mathbb{L}} (U^1 M^{n'}[-n']) \longrightarrow \mu' \otimes \nu_Y^N[-N-1]. \quad (8.2.6)$$

Finally because $\mathbb{Z}/p\mathbb{Z}$ -sheaves are flat over $\mathbb{Z}/p\mathbb{Z}$, there is a natural isomorphism

$$(U^1 M^n[-n]) \otimes^{\mathbb{L}} (U^1 M^{n'}[-n']) \simeq U^1 M^n \otimes U^1 M^{n'}[-N-2] \quad \text{in } D^-(Y_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$$

induced by the identity map on the $(n + n')$ th cohomology sheaves. We thus define Θ^n by composing the inverse of this isomorphism and the morphism (8.2.6).

8.3. Explicit formula for Θ^n . We formulate an explicit formula (see Theorem 8.3.8 below) to calculate the morphism Θ^n . Let

$$\chi : \mu' \otimes (\omega_Y^N / \mathcal{B}_Y^N) \longrightarrow \mu' \otimes \nu_Y^N[1] \quad \text{in } D^b(Y_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$$

be the connecting morphism associated with a short exact sequence ([Hy1], (1.5.1))

$$0 \longrightarrow \mu' \otimes \nu_Y^N \xrightarrow{\text{id} \otimes \text{incl}} \mu' \otimes \omega_Y^N \xrightarrow{\text{id} \otimes (1-C^{-1})} \mu' \otimes (\omega_Y^N / \mathcal{B}_Y^N) \longrightarrow 0.$$

Here \mathcal{B}_Y^N denotes the image of $d : \omega_Y^{N-1} \rightarrow \omega_Y^N$, C^{-1} denotes the inverse Cartier operator defined in loc. cit., (2.5) (cf. (9.3.2) below) and we have used the isomorphism $\omega_{Y, \log}^N \simeq \nu_Y^N$ in Remark 3.3.8 (4). We next construct a key map $f^{q,n}$ (cf. Definition 8.3.6 (2) below). Let e be the absolute ramification index of K and put $e' := pe/(p-1)$. Because K contains primitive p -th roots of unity by assumption, e' is an integer divided by p . Fix a prime element $\pi \in O_K$. Put $s := \text{Spec}(k)$. Let \mathcal{L}_Y (resp. \mathcal{L}_s) be the log structure on Y (resp. on s) defined in §3.3. We use the trivial log structure s^\times on s and a map on $Y_{\text{ét}}$ analogous to (3.3.5)

$$d\log : \mathcal{L}_Y^{\text{gp}} \longrightarrow \omega_{(Y, \mathcal{L}_Y)/(s, s^\times)}^1. \quad (8.3.1)$$

Remark 8.3.2. (1) *The composite of (8.3.1) with the canonical projection $\omega_{(Y, \mathcal{L}_Y)/(s, s^\times)}^1 \rightarrow \omega_Y^1$ agrees with the map $d\log$ in (3.3.5).*

(2) *Let $\bar{\pi}$ be the residue class of π in \mathcal{L}_Y under (3.3.4). Then we have $d\log(\bar{\pi}) = 0$ in ω_Y^1 , but not in $\omega_{(Y, \mathcal{L}_Y)/(s, s^\times)}^1$. Indeed, by the definition of relative differential modules ([Ka3], [KF]), there is a short exact sequence on $Y_{\text{ét}}$*

$$0 \longrightarrow \mathcal{O}_Y \xrightarrow{a \mapsto a \cdot d\log(\bar{\pi})} \omega_{(Y, \mathcal{L}_Y)/(s, s^\times)}^1 \longrightarrow \omega_Y^1 \longrightarrow 0.$$

The isomorphism (8.3.3) below follows from this fact.

Now let n and q satisfy $1 \leq n \leq N + 1$ and $1 \leq q \leq e' - 1$. Put $n' := N + 2 - n$. Let $U_{X_K}^q$ be the étale subsheaf of $\iota^* j_* \mathcal{O}_{X_K}^\times$ defined in Definition 3.3.2, and put

$$\text{Symb}^{q,n} := U_{X_K}^q \otimes (\iota^* j_* \mathcal{O}_{X_K}^\times)^{\otimes n-1} \otimes U_{X_K}^{e'-q} \otimes (\iota^* j_* \mathcal{O}_{X_K}^\times)^{\otimes n'-1}.$$

The sheaf $U^q M^n \otimes U^{e'-q} M^{n'}$ is a quotient of $\text{Symb}^{q,n}$ (cf. Definition 3.3.2):

$$U^q M^n \otimes U^{e'-q} M^{n'} = \text{Im} \left(\text{Symb}^{q,n} \longrightarrow U^1 M^n \otimes U^1 M^{n'} \right).$$

We define the homomorphism of étale sheaves

$$F^{q,n} : \text{Symb}^{q,n} \longrightarrow \omega_Y^N / \mathcal{B}_Y^N$$

by sending a local section $(1 + \pi^q \alpha_1) \otimes (\otimes_{i=1}^{n-1} \beta_i) \otimes (1 + \pi^{e'-q} \alpha_2) \otimes (\otimes_{i=n}^N \beta_i)$ with $\alpha_1, \alpha_2 \in \iota^* \mathcal{O}_X$ and $\beta_1, \dots, \beta_N \in \iota^* j_* \mathcal{O}_{X_K}^\times$, to the following:

$$q \cdot \overline{\alpha_1 \alpha_2} \cdot (\wedge_{i=1}^N d\log \overline{\beta_i}) + g^{-1} \left(\overline{\alpha_2} \cdot d\overline{\alpha_1} \wedge (\wedge_{i=1}^N d\log \overline{\beta_i}) \right) \text{ mod } \mathcal{B}_Y^N,$$

where for $x \in \iota^* \mathcal{O}_X$ (resp. $x \in \iota^* j_* \mathcal{O}_{X_K}^\times$), \overline{x} denotes its residue class in \mathcal{O}_Y (resp. in $\mathcal{L}_Y^{\text{gp}}$ under (3.3.4)) and g denotes the following \mathcal{O}_Y -linear isomorphism (cf. Remark 8.3.2 (2)):

$$g : \omega_Y^N \xrightarrow{\cong} \omega_{(Y, \mathcal{L}_Y)/(s, s^\times)}^{N+1}, \quad \omega \mapsto d\log(\overline{\pi}) \wedge \omega. \quad (8.3.3)$$

Lemma 8.3.4. *Let n and q be as above. Then $F^{q,n}$ factors through $U^q M^n \otimes U^{e'-q} M^{n'}$.*

Proof. Let Y_{sing} be the singular locus of Y , and let j_Y be the open immersion $Y \setminus Y_{\text{sing}} \hookrightarrow Y$. Replacing X by $X \setminus Y_{\text{sing}}$, we may assume that Y is smooth over $s = \text{Spec}(k)$, because $\omega_Y^N / \mathcal{B}_Y^N$ is a locally free $(\mathcal{O}_Y)^p$ -module and the canonical map $\omega_Y^N / \mathcal{B}_Y^N \rightarrow j_{Y*} j_Y^* (\omega_Y^N / \mathcal{B}_Y^N)$ is injective. We show that $F^{q,n}$ factors through $\text{gr}_U^q M^n \otimes \text{gr}_U^{e'-q} M^{n'}$, assuming that Y is smooth. For $m \geq 1$ and ℓ with $1 \leq \ell \leq e' - 1$, let

$$\rho^{\ell,m} : \Omega_Y^{m-2} \oplus \Omega_Y^{m-1} \longrightarrow \text{gr}_U^\ell M^m$$

be the Bloch-Kato map (cf. [BK], (4.3)) defined as

$$\begin{cases} (\alpha \cdot d\log \beta_1 \wedge \cdots \wedge d\log \beta_{m-2}, 0) \mapsto \{1 + \pi^\ell \tilde{\alpha}, \widetilde{\beta_1}, \dots, \widetilde{\beta_{m-2}}, \pi\} \text{ mod } U^{\ell+1} M^m, \\ (0, \alpha \cdot d\log \beta_1 \wedge \cdots \wedge d\log \beta_{m-1}) \mapsto \{1 + \pi^\ell \tilde{\alpha}, \widetilde{\beta_1}, \dots, \widetilde{\beta_{m-1}}\} \text{ mod } U^{\ell+1} M^m \end{cases}$$

for $\alpha \in \mathcal{O}_Y$ and each $\beta_i \in \mathcal{O}_Y^\times$, where $\tilde{\alpha} \in \mathcal{O}_X$ (resp. each $\widetilde{\beta_i} \in \mathcal{O}_X^\times$) denotes a lift of α (resp. β_i). There are short exact sequences

$$\begin{aligned} 0 \longrightarrow \Omega_Y^{m-2} &\xrightarrow{\theta^{\ell,m}} \Omega_Y^{m-2} \oplus \Omega_Y^{m-1} \xrightarrow{\rho^{\ell,m}} \text{gr}_U^\ell M^m \longrightarrow 0 & (\text{if } p \nmid \ell), \\ 0 \longrightarrow \mathcal{Z}_Y^{m-2} \oplus \mathcal{Z}_Y^{m-1} &\xrightarrow{\text{incl.}} \Omega_Y^{m-2} \oplus \Omega_Y^{m-1} \xrightarrow{\rho^{\ell,m}} \text{gr}_U^\ell M^m \longrightarrow 0 & (\text{if } p \mid \ell), \end{aligned} \quad (8.3.5)$$

where $\theta^{\ell,m}$ is given by $\omega \mapsto ((-1)^m \cdot \ell \cdot \omega, d\omega)$ (cf. [BK], Lemma (4.5)). Let

$$h^{\ell,m} : U_{X_K}^\ell \otimes (\iota^* j_* \mathcal{O}_{X_K}^\times)^{\otimes m-1} \longrightarrow \Omega_Y^{m-2} \oplus \Omega_Y^{m-1}$$

be the map that sends $(1 + \pi^\ell \alpha) \otimes (\otimes_{i=1}^{m-1} \beta_i)$ with $\alpha \in \iota^* \mathcal{O}_X$ and $\beta_i \in \iota^* \mathcal{O}_X^\times \cup \{\pi\}$ to

$$\begin{cases} (0, \overline{\alpha} \cdot \wedge_{1 \leq i \leq m-1} d\log \overline{\beta_i}) & (\text{if } \beta_i \in \iota^* \mathcal{O}_X^\times \text{ for all } i), \\ ((-1)^{m-1-i'} \cdot \overline{\alpha} \cdot \wedge_{1 \leq i \leq m-1, i \neq i'} d\log \overline{\beta_i}, 0) & (\text{if } \beta_i = \pi \text{ for exactly one } i =: i'), \\ (0, 0) & (\text{otherwise}). \end{cases}$$

Here for $x \in \iota^* \mathcal{O}_X$ (resp. $x \in \iota^* \mathcal{O}_X^\times$), \overline{x} denotes its residue class in \mathcal{O}_Y (resp. in \mathcal{O}_Y^\times). Now there is a diagram

$$\begin{array}{ccc} \text{Symb}^{q,n} & \xrightarrow{h^{q,n} \otimes h^{e'-q,n'}} & (\Omega_Y^{n-2} \oplus \Omega_Y^{n-1}) \otimes (\Omega_Y^{n'-2} \oplus \Omega_Y^{n'-1}) \xrightarrow{\rho^{q,n} \otimes \rho^{e'-q,n'}} \text{gr}_U^q M^n \otimes \text{gr}_U^{e'-q} M^{n'} \\ & & \varphi^{q,n} \downarrow \\ & & \Omega_Y^N / \mathcal{B}_Y^N, \end{array}$$

where $\varphi^{q,n}$ is defined as

$$(\omega_1, \omega_2) \otimes (\omega_3, \omega_4) \mapsto q \cdot \omega_2 \wedge \omega_4 + (-1)^{n-1} \cdot (d\omega_1) \wedge \omega_4 + (-1)^{n'-1} \cdot \omega_2 \wedge d\omega_3 \quad \text{mod } \mathcal{B}_Y^N.$$

In this diagram, the composite of the top row agrees with the symbol map, and the composite of $h^{q,n} \otimes h^{e'-q,n'}$ and $\varphi^{q,n}$ agrees with $F^{q,n}$ (see also Remark 8.3.2 (2)). Hence to prove the assertion, it suffices to show that the subsheaf

$$\text{Ker}(\rho^{q,n} \otimes \rho^{e'-q,n'}) \subset (\Omega_Y^{n-2} \oplus \Omega_Y^{n-1}) \otimes (\Omega_Y^{n'-2} \oplus \Omega_Y^{n'-1})$$

has trivial image under $\varphi^{q,n}$, which follows from (8.3.5) with $(\ell, m) = (q, n), (e' - q, n')$ (note that $p|q \Leftrightarrow p|(e' - q)$, because $p|e'$). Thus we obtain Lemma 8.3.4. \square

Definition 8.3.6. (1) For $\zeta \in \mu = \mu_p(K)$ with $\zeta \neq 1$, let $v(\zeta) \in k^\times$ be the residue class of $(1 - \zeta)/\pi^{e/(p-1)} \in O_K^\times$. We define $u := \zeta \otimes v(\zeta)^{-p} \in \mu \otimes k$, which is independent of the choice of $\zeta \neq 1$.

(2) Let \underline{k} be the constant sheaf on $Y_{\text{ét}}$ associated with k . We define the homomorphism

$$f^{q,n} : U^q M^n \otimes U^{e'-q} M^{n'} \longrightarrow (\mu' \otimes \underline{k}) \otimes_{\underline{k}} (\omega_Y^N / \mathcal{B}_Y^N) \simeq \mu' \otimes (\omega_Y^N / \mathcal{B}_Y^N)$$

as $u \otimes_k (-1)^{N+n} \overline{F}^{q,n}$. Here we regarded $u \in \mu \otimes k$ as a global section of $\mu' \otimes \underline{k}$, and $\overline{F}^{q,n}$ denotes the map induced by $F^{q,n}$ (cf. Lemma 8.3.4).

Remark 8.3.7. $f^{q,n}$ is independent of the choice of π by the definitions of $F^{q,n}$ and u .

Now we state the main result of this section.

Theorem 8.3.8 (Explicit formula). *Assume that X is proper over B . Then for (q, n) with $1 \leq q \leq e' - 1$ and $1 \leq n \leq N + 1$, the following square commutes in $D^b(Y_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$:*

$$\begin{array}{ccc} U^q M^n \otimes U^{e'-q} M^{n'} & \xrightarrow{f^{q,n}} & \mu' \otimes (\omega_Y^N / \mathcal{B}_Y^N) \\ \text{canonical} \downarrow & & \downarrow \chi \\ U^1 M^n \otimes U^1 M^{n'} & \xrightarrow{\Theta^{n[N+2]}} & \mu' \otimes \nu_Y^N[1]. \end{array}$$

We will prove Theorem 8.3.8 in §§8.4–8.7 below. We will first reduce the problem to an induced diagram of cohomology groups of Y in §8.4, and then to an induced diagram of cohomology groups of higher local fields in §8.5. In §8.6, we will prove a Galois descent of invariant subgroups of Galois modules. We will finish the proof of Theorem 8.3.8 in §8.7 by computing symbols, whose details are standard in higher local class field theory (cf. [Ka1]) but will be included for the convenience of the reader.

8.4. Reduction to cohomology groups. In this step, we reduce Theorem 8.3.8 to the equality (8.4.3) below. We first show the following:

Proposition 8.4.1. *Assume that Y is proper over $\text{Spec}(k)$. Let \mathcal{F} be a $\mathbb{Z}/p\mathbb{Z}$ -sheaf on $Y_{\text{ét}}$. Then for $i \in \mathbb{Z}$, the Yoneda pairing*

$$\text{H}^i(Y, \mathcal{F}) \times \text{Ext}_{Y, \mathbb{Z}/p\mathbb{Z}}^{N-i+1}(\mathcal{F}, \nu_Y^N) \longrightarrow \text{H}^{N+1}(Y, \nu_Y^N) \xrightarrow{\text{tr}_Y} \mathbb{Z}/p\mathbb{Z}$$

(see Theorem 2.2.4 for tr_Y) induces an isomorphism

$$\text{Ext}_{Y, \mathbb{Z}/p\mathbb{Z}}^{N-i+1}(\mathcal{F}, \nu_Y^N) \simeq \text{Hom}(\text{H}^i(Y, \mathcal{F}), \mathbb{Z}/p\mathbb{Z}).$$

Proof. If \mathcal{F} is constructible, then the isomorphism in question is an isomorphism of finite groups by the duality theorem of Moser [Mo] (note that the complex $\tilde{\nu}_{r,Y}^n$ defined in loc. cit. is quasi-isomorphic to the sheaf $\nu_{Y,r}^n$ by [Sat], 2.2.5 (1)). We prove the general case. Write \mathcal{F} as a filtered inductive limit $\varinjlim_{\lambda \in \Lambda} \mathcal{F}_\lambda$, where Λ is a filtered small category and each \mathcal{F}_λ is a constructible $\mathbb{Z}/p\mathbb{Z}$ -sheaf. Replacing $\{\mathcal{F}_\lambda\}_{\lambda \in \Lambda}$ with their images into \mathcal{F} if necessary, we suppose that the transition maps are injective. Since $H^i(Y, \mathcal{F}) \simeq \varinjlim_{\lambda \in \Lambda} H^i(Y, \mathcal{F}_\lambda)$ and

$$\mathrm{Hom}(H^i(Y, \mathcal{F}), \mathbb{Z}/p\mathbb{Z}) \simeq \varprojlim_{\lambda \in \Lambda} \mathrm{Hom}(H^i(Y, \mathcal{F}_\lambda), \mathbb{Z}/p\mathbb{Z}),$$

it is enough to show that

$$\mathrm{Ext}_{Y, \mathbb{Z}/p\mathbb{Z}}^{N-i+1}(\mathcal{F}, \nu_Y^N) \simeq \varprojlim_{\lambda \in \Lambda} \mathrm{Ext}_{Y, \mathbb{Z}/p\mathbb{Z}}^{N-i+1}(\mathcal{F}_\lambda, \nu_Y^N). \quad (8.4.2)$$

Take an injective resolution $\nu_Y^N \rightarrow I^\bullet$ in the category of $\mathbb{Z}/p\mathbb{Z}$ -sheaves on $Y_{\text{ét}}$. The group $\mathrm{Ext}_{Y, \mathbb{Z}/p\mathbb{Z}}^m(\mathcal{F}, \nu_Y^N)$ ($m \in \mathbb{Z}$) is the m -th cohomology group of the complex $\mathrm{Hom}_Y(\mathcal{F}, I^\bullet) \simeq \varinjlim_{\lambda \in \Lambda} \mathrm{Hom}_Y(\mathcal{F}_\lambda, I^\bullet)$. Noting that $\mathrm{Ext}_{Y, \mathbb{Z}/p\mathbb{Z}}^m(\mathcal{F}_\lambda, \nu_Y^N)$ is finite for any $\lambda \in \Lambda$ and that the transition maps $\mathcal{F}_\lambda \rightarrow \mathcal{F}_{\lambda'}$ ($\lambda < \lambda'$) are injective, we are reduced to the following standard fact on projective limits:

Fact. Let Λ be a cofiltered small category, and let $\{C_\lambda^\bullet\}_{\lambda \in \Lambda}$ be a projective system of complexes of abelian groups. For $m \in \mathbb{Z}$ and $\lambda \in \Lambda$, put $H_\lambda^m := H^m(C_\lambda^\bullet)$, the m -th cohomology group of C_λ^\bullet . Now fix $m \in \mathbb{Z}$, and assume that $\{C_\lambda^{m-1}\}_{\lambda \in \Lambda}$, $\{C_\lambda^m\}_{\lambda \in \Lambda}$ and $\{H_\lambda^m\}_{\lambda \in \Lambda}$ satisfy the Mittag-Leffler condition. Then we have $\varprojlim_{\lambda \in \Lambda} H_\lambda^{m+1} \simeq H^{m+1}(\varprojlim_{\lambda \in \Lambda} C_\lambda^\bullet)$.

This completes the proof of Proposition 8.4.1. \square

We now turn to the proof of Theorem 8.3.8. Without loss of generality, we may assume that X is connected. Then by Proposition 8.4.1 for $i = N$, we have

$$\begin{aligned} & \mathrm{Hom}_{D^b(Y_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})}(U^q M^n \otimes U^{e'-q} M^{n'}, \mu' \otimes \nu_Y^N[1]) \\ & \simeq \mathrm{Hom}\left(H^N(Y, U^q M^n \otimes U^{e'-q} M^{n'}), \mu \otimes H^{N+1}(Y, \nu_Y^N)\right). \end{aligned}$$

Hence we are reduced to the equality of induced maps on cohomology groups

$$H^N(Y, \Theta^{q,n}) = H^N(Y, \chi \circ f^{q,n}), \quad (8.4.3)$$

where we wrote $\Theta^{q,n}$ for the composite morphism

$$\Theta^{q,n} : U^q M^n \otimes U^{e'-q} M^{n'} \xrightarrow{\text{canonical}} U^1 M^n \otimes U^1 M^{n'} \xrightarrow{\Theta^n[N+2]} \mu' \otimes \nu_Y^N[1].$$

8.5. Reduction to higher local fields. In this step, (8.4.3) will be reduced to (8.5.3) below. We define a *chain on Y* to be a sequence $(y_0, y_1, y_2, \dots, y_N)$ of points (=spectra of fields) over Y such that y_0 is a closed point on Y and such that for each m with $1 \leq m \leq N$, y_m is a closed point on the scheme

$$\left[\mathrm{Spec} \left(\cdots \left(\left(\mathcal{O}_{Y, y_0}^h \right)_{y_1}^h \right) \cdots \right)_{y_{m-1}}^h \right] \setminus \{y_{m-1}\},$$

where the superscript h means the henselization at the point given on subscript. For a chain (y_0, y_1, \dots, y_N) on Y , each $\kappa(y_m)$ ($0 \leq m \leq N$) is an m -dimensional local field. We write $\mathrm{Ch}(Y)$ for the set of chains on Y . Now for $\mathcal{K} \in D^b(Y_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$ and $\mathfrak{d} = (y_0, y_1, \dots, y_N) \in \mathrm{Ch}(Y)$, there is a composite map

$$H^0(y_N, \mathcal{K}) \rightarrow H_{y_{N-1}}^1(Y_{\mathfrak{d}, N-1}, \mathcal{K}) \rightarrow \cdots \rightarrow H_{y_1}^{N-1}(Y_{\mathfrak{d}, 1}, \mathcal{K}) \rightarrow H_{y_0}^N(Y_{\mathfrak{d}, 0}, \mathcal{K}) \rightarrow H^N(Y, \mathcal{K}).$$

Here $Y_{\mathfrak{d},m}$ ($0 \leq m \leq N$) denotes the henselian local scheme

$$\mathrm{Spec} \left(\cdots \left(\left(\mathcal{O}_{Y,y_0}^{\mathfrak{h}} \right)_{y_1}^{\mathfrak{h}} \right) \cdots \right)_{y_m}^{\mathfrak{h}}$$

and the map $\mathrm{H}_{y_m}^{N-m}(Y_{\mathfrak{d},m}, \mathcal{K}) \rightarrow \mathrm{H}_{y_{m-1}}^{N-m+1}(Y_{\mathfrak{d},m-1}, \mathcal{K})$ ($1 \leq m \leq N$) is defined as the composite

$$\begin{aligned} \mathrm{H}_{y_m}^{N-m}(Y_{\mathfrak{d},m}, \mathcal{K}) = \mathrm{H}_{y_m}^{N-m}(Y_{\mathfrak{d},m-1} \setminus \{y_{m-1}\}, \mathcal{K}) &\longrightarrow \mathrm{H}^{N-m}(Y_{\mathfrak{d},m-1} \setminus \{y_{m-1}\}, \mathcal{K}) \\ &\xrightarrow{\delta^{\mathrm{loc}}(\mathcal{K})} \mathrm{H}_{y_{m-1}}^{N-m+1}(Y_{\mathfrak{d},m-1}, \mathcal{K}). \end{aligned}$$

Taking the direct sum with respect to all chains on Y , we obtain a map

$$\delta_Y(\mathcal{K}) : \bigoplus_{(y_0, y_1, \dots, y_N) \in \mathrm{Ch}(Y)} \mathrm{H}^0(y_N, \mathcal{K}) \longrightarrow \mathrm{H}^N(Y, \mathcal{K}).$$

Lemma 8.5.1. *The map $\delta_Y(U^q M^n \otimes U^{e'-q} M^{n'})$ is surjective.*

Proof. By Theorem 3.3.7, the sheaf $U^q M^n \otimes U^{e'-q} M^{n'}$ is a finitely successive extension of étale sheaves of the form $\mathcal{F} \otimes \mathcal{G}$, where \mathcal{F} and \mathcal{G} are locally free $(\mathcal{O}_Y)^p$ -modules of finite rank and the tensor product is taken over $\mathbb{Z}/p\mathbb{Z}$. We are reduced to the following sublemma.

Sublemma 8.5.2. *Let Z be a noetherian scheme which is of pure-dimension and essentially of finite type over $\mathrm{Spec}(k)$. Put $d := \dim(Z)$. Let \mathcal{F} and \mathcal{G} be locally free $(\mathcal{O}_Z)^p$ -modules of finite rank. Then:*

- (1) *For any $x \in Z$ and $i > \mathrm{codim}_Z(x)$, $\mathrm{H}_x^i(Z, \mathcal{F} \otimes \mathcal{G})$ is zero.*
- (2) *We have $\mathrm{H}^i(Z, \mathcal{F} \otimes \mathcal{G}) = 0$ for $i > d$, and the natural map $\bigoplus_{x \in Z^d} \mathrm{H}_x^d(Z, \mathcal{F} \otimes \mathcal{G}) \rightarrow \mathrm{H}^d(Z, \mathcal{F} \otimes \mathcal{G})$ is surjective.*
- (3) *If Z is henselian local, then $\mathrm{H}^i(Z, \mathcal{F} \otimes \mathcal{G})$ is zero for $i > 0$.*

Proof of Sublemma 8.5.2. Since the absolute Frobenius morphism $F_Z : Z \rightarrow Z$ is finite by assumption, we have $\mathrm{H}^*(Z, \mathcal{F} \otimes \mathcal{G}) \simeq \mathrm{H}^*(Z, F_{Z*}(\mathcal{F} \otimes \mathcal{G}))$ and $\mathrm{H}_x^*(Z, \mathcal{F} \otimes \mathcal{G}) \simeq \mathrm{H}_x^*(Z, F_{Z*}(\mathcal{F} \otimes \mathcal{G}))$ for any $x \in Z$. Hence we are reduced to the case where \mathcal{F} and \mathcal{G} are locally free \mathcal{O}_Z -modules of finite rank.

We first show (3). Let R be the affine ring of Z , which is a henselian local ring by assumption. Let R^{sh} be the strict henselization of R . Without loss of generality, we may assume that $\mathcal{F} = \mathcal{G} = \mathcal{O}_Z$. By the isomorphism $\mathrm{H}^q(Z, \mathcal{O}_Z \otimes \mathcal{O}_Z) \simeq \mathrm{H}_{\mathrm{Gal}}^q(G_R, R^{\mathrm{sh}} \otimes R^{\mathrm{sh}})$ with $G_R := \mathrm{Gal}(R^{\mathrm{sh}}/R)$, our task is to show that the right hand side is zero for $q > 0$. We show that for a finite étale galois extension R'/R with Galois group $G := \mathrm{Gal}(R'/R)$, we have $\mathrm{H}^q(G, R' \otimes R') = 0$ for $q > 0$. Indeed, by taking a normal basis, we have $R' \simeq R[G]$ as left $R[G]$ -modules, and there is an isomorphism of left G -modules

$$R[G] \otimes R[G] \xrightarrow{\simeq} R[G] \otimes (R[G]^\circ), \quad a[g] \otimes b[h] \mapsto a[g] \otimes b[g^{-1}h],$$

where a and b (resp. g and h) are elements of R (resp. of G), and $R[G]^\circ$ denotes the abelian group $R[G]$ with trivial G -action. Hence $R' \otimes R'$ is an induced G -module in the sense of [Se], I.2.5 and we obtain the assertion.

We next prove (1) and (2) by induction on d and a standard local-global argument (cf. [Ra], 1.22). The case $d = 0$ follows from (3). Assume $d \geq 1$ and that (1) and (2) hold true for schemes of dimension $\leq d - 1$. We first show (1). Indeed, the case $\mathrm{codim}_Z(x) = 0$ follows from the case $d = 0$. If $\mathrm{codim}_Z(x) \geq 1$ and $i \geq 1$, then the connecting homomorphism

$$\delta^{\mathrm{loc}}(\mathcal{K}) : \mathrm{H}^{i-1}(\mathrm{Spec}(\mathcal{O}_{Z,x}^{\mathfrak{h}}) \setminus \{x\}, \mathcal{F} \otimes \mathcal{G}) \longrightarrow \mathrm{H}_x^i(\mathrm{Spec}(\mathcal{O}_{Z,x}^{\mathfrak{h}}), \mathcal{F} \otimes \mathcal{G}) = \mathrm{H}_x^i(Z, \mathcal{F} \otimes \mathcal{G})$$

is surjective by (3), and the left hand side is zero for $i > \text{codim}_{\mathbb{Z}}(x)$ by the induction hypothesis. Thus we obtain (1). Finally one can easily check (2) by (1) and a local-global spectral sequence

$$E_1^{u,v} = \bigoplus_{x \in \mathbb{Z}^u} H_x^{u+v}(Z, \mathcal{F} \otimes \mathcal{G}) \implies H^{u+v}(Z, \mathcal{F} \otimes \mathcal{G}).$$

This completes the proof of Sublemma 8.5.2 and Lemma 8.5.1. \square

By Lemma 8.5.1, (8.4.3) is reduced to the formula

$$H^0(y_N, \Theta^{q,n}) = H^0(y_N, \chi \circ f^{q,n}) \quad (8.5.3)$$

for all chains $(y_0, y_1, \dots, y_N) \in \text{Ch}(Y)$, which will be proved in §8.7 below.

8.6. Galois descent by corestriction maps. We prove here the following lemma:

Lemma 8.6.1. *Let F be a field of characteristic $p > 0$. Let V_1 and V_2 be discrete G_F - $\mathbb{Z}/p\mathbb{Z}$ -modules which are finitely successive extensions of finite direct sums of copies of \overline{F} as G_F -modules. Then $(V_1 \otimes V_2)^{G_F}$ agrees with*

$$\bigcup_{F'/F: \text{finite galois}} \text{Im} \left((V_1)^{G_{F'}} \otimes (V_2)^{G_{F'}} \xrightarrow{\subset} (V_1 \otimes V_2)^{G_{F'}} \xrightarrow{\text{cores}_{F'/F}} (V_1 \otimes V_2)^{G_F} \right),$$

where all tensor products are taken over $\mathbb{Z}/p\mathbb{Z}$, and F' runs through all finite galois field extensions of F contained in \overline{F} .

Proof. It suffices to show the case $V_1 = V_2 = \overline{F}$. We prove that the corestriction map

$$\text{cores}_{F'/F} : F' \otimes F' \longrightarrow (F' \otimes F')^G, \quad x \otimes y \mapsto \sum_{g \in G} gx \otimes gy$$

is surjective for a finite galois extension F'/F with $G := \text{Gal}(F'/F)$, which implies the assertion by a limit argument. Since $F' \simeq F[G]$ as $F[G]$ -modules, we have

$$(F' \otimes F')^G \simeq (F \otimes F) \otimes (\mathbb{Z}/p\mathbb{Z}[G] \otimes \mathbb{Z}/p\mathbb{Z}[G])^G$$

by the finiteness of G and the flatness of $\mathbb{Z}/p\mathbb{Z}$ -modules over $\mathbb{Z}/p\mathbb{Z}$. Hence the surjectivity of $\text{cores}_{F'/F}$ follows from that of the map

$$\mathbb{Z}/p\mathbb{Z}[G] \otimes \mathbb{Z}/p\mathbb{Z}[G] \longrightarrow (\mathbb{Z}/p\mathbb{Z}[G] \otimes \mathbb{Z}/p\mathbb{Z}[G])^G, \quad x \otimes y \mapsto \sum_{g \in G} gx \otimes gy.$$

Thus we obtain the lemma. \square

8.7. Proof of (8.5.3). In this step, we finish the proof of Theorem 8.3.8. Fix an arbitrary chain $(y_0, y_1, \dots, y_N) \in \text{Ch}(Y)$. Put $F_N := \kappa(y_N)$ and

$$L_{N+1} := \text{Frac} \left[\left(\dots \left((\mathcal{O}_{X, y_0}^h)_{y_1}^h \right) \dots \right)_{y_N}^h \right],$$

where L_{N+1} is a henselian discrete valuation field (of characteristic 0) with residue field F_N , that is, L_{N+1} is an $(N+1)$ -dimensional local field. Now let F/F_N be a finite separable field extension. Put $y := \text{Spec}(F)$ and

$$A^{q,n}(F) := H^0(y, U^q M^n) \otimes H^0(y, U^{e'-q} M^{n'}) \subset H^0(y, U^q M^n \otimes U^{e'-q} M^{n'}).$$

By Lemma 8.6.1 (for the subfield $(F_N)^p \subset F_N$) and the naturality of corestriction maps, the formula (8.5.3) for y_N is reduced to the formula

$$H^0(y, \Theta^{q,n})|_{A^{q,n}(F)} = H^0(y, \chi \circ f^{q,n})|_{A^{q,n}(F)}. \quad (8.7.1)$$

To prove this equality, we compute the left hand side, i.e., the composite map

$$A^{q,n}(F) \hookrightarrow H^0(y, U^q M^n \otimes U^{e'-q} M^{n'}) \xrightarrow{H^0(y, \Theta^{q,n})} \mu' \otimes H^1(y, \Omega_{y, \log}^N). \quad (8.7.2)$$

Let L/L_{N+1} be the finite unramified extension corresponding to F/F_N . For $i > 0$, put $k_i^M(L) := K_i^M(L)/pK_i^M(L)$. By a similar argument as for [BK], (5.15), we have

$$A^{q,n}(F) = \{U^q k_n^M(L)/U^{e'} k_n^M(L)\} \otimes \{U^{e'-q} k_{n'}^M(L)/U^{e'} k_{n'}^M(L)\}.$$

Let us recall that $1 \leq q \leq e' - 1$ by assumption. In view of the construction of Θ^n (cf. §8.2) and the fact that $U^{e'+1} k_{N+2}^M(L) = 0$ ([BK], (5.1.i)), the map (8.7.2) is written by the product of Milnor K -groups and boundary maps of Galois cohomology groups:

$$\begin{aligned} A^{q,n}(F) &\xrightarrow{\text{product}} U^{e'} k_{N+2}^M(L) \xrightarrow{\text{Galois symbol}} H^{N+2}(L, \mu_p^{\otimes N+2}) \\ &\xrightarrow{\cong} \mu \otimes H^1(F, H^{N+1}(L^{\text{ur}}, \mu_p^{\otimes N+1})) \xrightarrow{\text{id} \otimes (3.2.3)} \mu \otimes H^1(y, \Omega_{y,\log}^N), \end{aligned} \quad (8.7.3)$$

where L^{ur} denotes the maximal unramified extension of L , and the third arrow is obtained from a Hochschild-Serre spectral sequence together with the facts that $\text{cd}_p(F) = 1$ and $\text{cd}_p(L^{\text{ur}}) = N+1$ (cf. Lemma 7.3.2). Here we compute the product of symbols:

Lemma 8.7.4. *For $\alpha_1, \alpha_2 \in O_L \setminus \{0\}$ and $\beta_1, \dots, \beta_N \in L^\times$, we have*

$$\begin{aligned} \{1 + \pi^q \alpha_1, \beta_1, \dots, \beta_{n-1}, 1 + \pi^{e'-q} \alpha_2, \beta_n, \dots, \beta_N\} &= (-1)^{N+n} \cdot q \cdot \{1 + \pi^{e'} \alpha_1 \alpha_2, \beta_1, \dots, \beta_N, \pi\} \\ &\quad + (-1)^n \cdot \{1 + \pi^{e'} \alpha_1 \alpha_2, \alpha_1, \beta_1, \dots, \beta_N\} \end{aligned}$$

in $k_{N+2}^M(L)$. The second term on the right hand side is zero if β_i belongs to O_L^\times for all i .

Proof. We compute the symbol $\{1 + \pi^q \alpha_1, 1 + \pi^{e'-q} \alpha_2\} \in k_2^M(L)$:

$$\begin{aligned} \{1 + \pi^q \alpha_1, 1 + \pi^{e'-q} \alpha_2\} &= \{1 + \pi^q \alpha_1 + \pi^{e'} \alpha_1 \alpha_2, 1 + \pi^{e'-q} \alpha_2\} \\ &\quad - \{(1 + \pi^q \alpha_1 + \pi^{e'} \alpha_1 \alpha_2)(1 + \pi^q \alpha_1)^{-1}, 1 + \pi^{e'-q} \alpha_2\} \\ &\stackrel{(1)}{=} - \{1 + \pi^q \alpha_1 + \pi^{e'} \alpha_1 \alpha_2, -\pi^q \alpha_1\} \\ &\stackrel{(2)}{=} - \{1 + \pi^{e'} \alpha_1 \alpha_2 (1 + \pi^q \alpha_1)^{-1}, -\pi^q \alpha_1\} \\ &\stackrel{(3)}{=} - \{1 + \pi^{e'} \alpha_1 \alpha_2, \pi^q \alpha_1\}. \end{aligned}$$

(1) follows from the equality $\{1 + x_1 x_2, x_1\} = -\{1 + x_1 x_2, -x_2\}$ (applied to the first term) and the fact that the second term is contained in $U^{e'+1} k_2^M(L) = 0$ ([BK], (4.1), (5.1.i)). (2) follows from the equality $\{1 + x, -x\} = 0$, and (3) follows from loc. cit., (4.3). The equality assertion in the lemma follows from this computation. The last assertion follows from loc. cit., (4.3) and the fact that $\Omega_F^{N+1} = 0$. \square

To calculate the last two maps in (8.7.3), we need the following lemma, which is a kind of explicit formula for L (see §8.3 for u and χ):

Lemma 8.7.5. *The following square commutes:*

$$\begin{array}{ccc} \Omega_F^N & \xrightarrow{\omega \mapsto u \otimes_k \omega} & (\mu \otimes k) \otimes_k (\Omega_F^N / \mathcal{B}_F^N) \\ \rho^{e'} \downarrow & & \downarrow H^0(y, \chi) \\ H^{N+2}(L, \mu_p^{\otimes N+2}) & \xrightarrow{\cong} & \mu \otimes H^1(y, \Omega_{y,\log}^N), \end{array} \quad (8.7.6)$$

where the bottom arrow is the composite of the last two maps in (8.7.3) and $\rho^{e'}$ denotes the Bloch-Kato map sending $\alpha \cdot d\log(\beta_1) \wedge \cdots \wedge d\log(\beta_N)$ ($\alpha \in F$, $\beta_i \in F^\times$) to $\{1 + \pi^{e'} \tilde{\alpha}, \tilde{\beta}_1, \dots, \tilde{\beta}_N, \pi\}$ ($\tilde{\alpha}$ and $\tilde{\beta}_i$'s are lifts of α and β_i 's, respectively).

Proof. This commutativity would be well-known to experts (cf. [Ka4] for the case $p > N + 3$, see also [FV], VII.4). However we include here a simple proof using a classical argument originally due to Hasse [Has] to verify the above commutativity including signs. By [Ka1], p. 612, Lemma 2, the bottom horizontal arrow of the diagram (8.7.6) maps

$$\zeta \cup \text{Inf}(x) \cup \{\tilde{\beta}_1, \tilde{\beta}_2, \dots, \tilde{\beta}_N, \pi\} \mapsto \zeta \otimes (-x) \cup (d\log(\beta_1) \wedge \cdots \wedge d\log(\beta_N))$$

for $\zeta \in \mu$, $x \in H^1(F, \mathbb{Z}/p\mathbb{Z})$ and $\beta_i \in F^\times$. Hence it is enough to show the following:

Claim. Fix a primitive p -th root of unity $\zeta_p \in \mu$, and consider the composite map

$$F \longrightarrow H^1(F, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\text{Inf}} H^1(L, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{1 \mapsto \zeta_p} H^1(L, \mu_p) \xrightarrow{\simeq} L^\times / (L^\times)^p,$$

where the first map is the boundary map of Artin-Schreier theory and the last isomorphism is the inverse of the boundary map of Kummer theory. Then this composite map sends $-v(\zeta_p)^{-p}\alpha \in F$ to $1 + \pi^{e'} \tilde{\alpha} \bmod (L^\times)^p$, where $\tilde{\alpha}$ denotes a lift of α to O_L (note that $U^{e'+1}L^\times \subset (L^\times)^p$). See Definition 8.3.6 for the definition of $v(\zeta_p)$.

Proof of Claim. It suffices to show that $\alpha \in F$ maps to $1 - (1 - \zeta_p)^p \tilde{\alpha} \bmod (L^\times)^p$. Consider the following equations in T over F and L , respectively:

$$T^p - T = \alpha, \tag{8.7.7}$$

$$T^p = 1 - (1 - \zeta_p)^p \cdot \tilde{\alpha}. \tag{8.7.8}$$

We show that the Artin-Schreier character $G_F \rightarrow \mathbb{Z}/p\mathbb{Z}$ associated with (8.7.7) induces the Kummer character $G_L \rightarrow \mu$ associated with (8.7.8) by the composition $G_L \rightarrow G_F \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow \mu$. Let $\beta \in \bar{L}$ be a solution to (8.7.8). By the congruity relation

$$(-1)^p \cdot p \equiv (1 - \zeta_p)^{p-1} \bmod \pi^{e'} O_L,$$

one can easily show that β is contained in O_L^{ur} and that

$$\gamma := (1 - \beta)/(1 - \zeta_p) \bmod \pi O_L^{\text{ur}} \in \bar{F}$$

is a solution to (8.7.7). Moreover, $\sigma \in G_L$ satisfies $\sigma(\beta)/\beta = \zeta_p^m \in \mu$ if and only if $\sigma(\gamma) - \gamma = m \in \mathbb{Z}/p\mathbb{Z}$, where G_L acts on \bar{F} via the canonical projection $G_L \rightarrow G_F$. Thus we obtain the claim and Lemma 8.7.5. \square

We now turn to the proof of (8.7.1). Let $\alpha_1, \alpha_2 \in O_L \setminus \{0\}$, and $\beta_i \in O_L^\times \cup \{\pi\}$ ($1 \leq i \leq N$). By Lemmas 8.7.4 and 8.7.5, the value of the symbol

$$\{1 + \pi^q \alpha_1, \beta_1, \dots, \beta_{n-1}\} \otimes \{1 + \pi^{e'-q} \alpha_2, \beta_n, \dots, \beta_N\} \in U^q k_n^M(L) \otimes U^{e'-q} k_n^M(L)$$

under (8.7.2) agrees with the value of the following element of $\mu \otimes (\Omega_F^N / \mathcal{B}_F^N)$ under $H^0(y, \chi)$:

$$\begin{cases} u \otimes_k (-1)^{n+N} [q \cdot \overline{\alpha_1 \alpha_2} \cdot (\wedge_{1 \leq i \leq N} d\log \overline{\beta_i}) \bmod \mathcal{B}_F^N] & \text{(if } \beta_i \in O_L^\times \text{ for all } i), \\ u \otimes_k (-1)^{n+N+i'} [\overline{\alpha_2} \cdot d\overline{\alpha_1} \wedge (\wedge_{1 \leq i \leq N, i \neq i'} d\log \overline{\beta_i}) \bmod \mathcal{B}_F^N] & \text{(if } \beta_i = \pi \text{ for exactly one } i =: i'), \\ 0 & \text{(otherwise),} \end{cases}$$

where for $x \in O_L$ (resp. $x \in O_L^\times$), \bar{x} denotes its residue class in F (resp. in F^\times). Thus comparing this presentation of (8.7.2) with the definition of $f^{q,n}$ (cf. §8.3), we conclude that the equality (8.7.1) holds. This completes the proof of Theorem 8.3.8. \square

9. Duality of p -adic vanishing cycles

In this section we prove Theorem 9.1.1 below, which will be used in §10.

9.1. Statement of the result. Let the notation be as in §8.1. We prove the following:

Theorem 9.1.1. *Let n be $1 \leq n \leq N + 1$ and put $n' := N + 2 - n$. Assume that X is proper over $\text{Spec}(O_K)$. Then for an integer i , the pairing induced by Θ^n and tr_Y (cf. Theorem 2.2.4)*

$$a^i : H^i(Y, U^1 M^n) \times H^{N-i}(Y, U^1 M^{n'}) \xrightarrow{\Theta^n} \mu \otimes H^{N+1}(Y, \nu_Y^N) \xrightarrow{\text{id} \otimes \text{tr}_Y} \mu \quad (9.1.2)$$

is a non-degenerate pairing of finite $\mathbb{Z}/p\mathbb{Z}$ -modules.

To prove this theorem, we first calculate the map $f^{q,n}$ defined in §8.3 (cf. Lemma 9.1.4 below). Let $U^\bullet M^n$ and $V^\bullet M^n$ be as in Definition 3.3.2. We further define the subsheaf $T^q M^n \subset U^q M^n$ ($q \geq 1$) as the part generated by $V^q M^n$ and symbols of the form

$$\{1 + \pi^q \alpha^p, \beta_1, \dots, \beta_{n-1}\}$$

with $\alpha \in \iota^* \mathcal{O}_X$ and each $\beta_i \in \iota^* j_* \mathcal{O}_{X_K}^\times$. By definition we have

$$U^{q+1} M^n \subset V^q M^n \subset T^q M^n \subset U^q M^n.$$

For $q \geq 1$, we put

$$\text{gr}_{U/T}^q M^n := U^q M^n / T^q M^n, \quad \text{gr}_{T/V}^q M^n := T^q M^n / V^q M^n$$

$$\text{and } \text{gr}_{V/U}^q M^n := V^q M^n / U^{q+1} M^n.$$

Let us recall that $e' = pe/(p-1)$ is an integer divided by p (because $\zeta_p \in K$). By Theorem 3.3.7 (3), (4), the sheaf $U^{e'} M^n$ is zero, and for q with $1 \leq q \leq e' - 1$ we have isomorphisms

$$\begin{aligned} \rho_1^{q,n} : \text{gr}_{U/T}^q M^n &\xrightarrow{\simeq} \omega_Y^{n-1} / \mathcal{Z}_Y^{n-1}, \\ \rho_2^{q,n} : \text{gr}_{T/V}^q M^n &\xrightarrow{\simeq} \begin{cases} \mathcal{Z}_Y^{n-1} / \mathcal{B}_Y^{n-1} & (p \nmid q), \\ 0 & (p \mid q), \end{cases} \\ \rho_3^{q,n} : \text{gr}_{V/U}^q M^n &\xrightarrow{\simeq} \omega_Y^{n-2} / \mathcal{Z}_Y^{n-2}, \end{aligned} \quad (9.1.3)$$

given by the following, respectively:

$$\begin{aligned} \rho_1^{q,n} : \{1 + \pi^q \alpha, \beta_1, \dots, \beta_{n-1}\} \bmod T^q M^n &\mapsto \bar{\alpha} \cdot (\wedge_{i=1}^{n-1} d\log \bar{\beta}_i) \bmod \mathcal{Z}_Y^{n-1}, \\ \rho_2^{q,n} : \{1 + \pi^q \alpha^p, \beta_1, \dots, \beta_{n-1}\} \bmod V^q M^n &\mapsto \bar{\alpha}^p \cdot (\wedge_{i=1}^{n-1} d\log \bar{\beta}_i) \bmod \mathcal{B}_Y^{n-1}, \quad (p \nmid q), \\ \rho_3^{q,n} : \{1 + \pi^q \alpha, \beta_1 \cdots, \beta_{n-2}, \pi\} \bmod U^{q+1} M^n &\mapsto \bar{\alpha} \cdot (\wedge_{i=1}^{n-2} d\log \bar{\beta}_i) \bmod \mathcal{Z}_Y^{n-2}. \end{aligned}$$

Here α (resp. each β_i) denotes a local section of $\iota^* \mathcal{O}_X$ (resp. $\iota^* j_* \mathcal{O}_{X_K}^\times$), and $\bar{\alpha}$ (resp. $\bar{\beta}_i$) denotes its residue class in \mathcal{O}_Y (resp. in $\mathcal{L}_Y^{\text{gp}}$ under (3.3.4)). The following lemma follows from straightforward computations on symbols, whose proof is left to the reader (cf. Remark 8.3.2 (2), Definition 8.3.6):

Lemma 9.1.4. *Let n and n' be as in Theorem 9.1.1, and assume $1 \leq q \leq e' - 1$. Then:*

- (1) $f^{q,n}$ annihilates the subsheaf of $U^q M^n \otimes U^{e'-q} M^{n'}$ generated by $U^{q+1} M^n \otimes U^{e'-q} M^{n'}$, $U^q M^n \otimes U^{e'-q+1} M^{n'}$, $V^q M^n \otimes T^{e'-q} M^{n'}$ and $T^q M^n \otimes V^{e'-q} M^{n'}$.
- (2) The composite map

$$\omega_Y^{n-1}/\mathcal{Z}_Y^{n-1} \otimes \omega_Y^{n'-2}/\mathcal{Z}_Y^{n'-2} \xrightarrow{(\rho_1^{q,n} \otimes \rho_3^{e'-q,n'})^{-1}} \mathrm{gr}_{U/T}^q M^n \otimes \mathrm{gr}_{V/U}^{e'-q} M^{n'} \xrightarrow{f^{q,n}} \mu' \otimes (\omega_Y^N/\mathcal{B}_Y^N)$$

sends a local section $x \otimes y$ to $u \otimes_k (-1)^n \cdot (dx) \wedge y$. Similarly, the composite map

$$\omega_Y^{n-2}/\mathcal{Z}_Y^{n-2} \otimes \omega_Y^{n'-1}/\mathcal{Z}_Y^{n'-1} \xrightarrow{(\rho_3^{q,n} \otimes \rho_1^{e'-q,n'})^{-1}} \mathrm{gr}_{V/U}^q M^n \otimes \mathrm{gr}_{U/T}^{e'-q} M^{n'} \xrightarrow{f^{q,n}} \mu' \otimes (\omega_Y^N/\mathcal{B}_Y^N)$$

sends a local section $x \otimes y$ to $u \otimes_k (-1)^N \cdot x \wedge dy$.

- (3) If q is prime to p , then the composite map

$$\mathcal{Z}_Y^{n-1}/\mathcal{B}_Y^{n-1} \otimes \mathcal{Z}_Y^{n'-1}/\mathcal{B}_Y^{n'-1} \xrightarrow{(\rho_2^{q,n} \otimes \rho_2^{e'-q,n'})^{-1}} \mathrm{gr}_{T/V}^q M^n \otimes \mathrm{gr}_{T/V}^{e'-q} M^{n'} \xrightarrow{f^{q,n}} \mu' \otimes (\omega_Y^N/\mathcal{B}_Y^N)$$

sends a local section $x \otimes y$ to $u \otimes_k (-1)^{N+n} q \cdot x \wedge y$.

9.2. Proof of Theorem 9.1.1. In this subsection, we reduce the theorem to Lemma 9.2.7 below. The finiteness of the groups in the pairing (9.1.2) follows from the finiteness of k , the properness of Y and the fact that the sheaves $U^1 M^n$ and $U^1 M^{n'}$ are finitely successive extensions of coherent $(\mathcal{O}_Y)^p$ -modules (cf. (9.1.3)). To show the non-degeneracy of (9.1.2), we introduce an auxiliary descending filtration $Z^r M^n$ ($r \geq 1$) on $U^1 M^n$ defined as

$$Z^r M^n := \begin{cases} U^q M^n & (\text{if } r \equiv 1 \pmod{3} \text{ and } q = (r+2)/3), \\ T^q M^n & (\text{if } r \equiv 2 \pmod{3} \text{ and } q = (r+1)/3), \\ V^q M^n & (\text{if } r \equiv 0 \pmod{3} \text{ and } q = r/3). \end{cases}$$

Note that $Z^1 M^n = U^1 M^n$ and $Z^r M^n = 0$ for $r \geq 3e' - 2$. We first show

Lemma 9.2.1. *Assume $1 \leq r \leq 3e' - 3$. Then:*

- (1) The composite map

$$\begin{array}{ccc} \mathrm{H}^N(Y, U^1 M^n \otimes Z^{3e'-2-r} M^{n'}) & \longrightarrow & \mathrm{H}^N(Y, U^1 M^n \otimes U^1 M^{n'}) \\ & \xrightarrow{\mathrm{H}^{2N+3}(Y, \Theta^n)} & \mu \otimes \mathrm{H}^{N+1}(Y, \nu_Y^N) \xrightarrow[\simeq]{\mathrm{id} \otimes \mathrm{tr}_Y} \mu \end{array}$$

induces a map

$$\mathrm{H}^N(Y, (U^1 M^n / Z^{r+1} M^n) \otimes Z^{3e'-2-r} M^{n'}) \longrightarrow \mu. \quad (9.2.2)$$

- (2) The composite map

$$\begin{array}{ccc} \mathrm{H}^N(Y, \mathrm{gr}_Z^r M^n \otimes Z^{3e'-2-r} M^{n'}) & \longrightarrow & \mathrm{H}^N(Y, (U^1 M^n / Z^{r+1} M^n) \otimes Z^{3e'-2-r} M^{n'}) \\ & \xrightarrow{(9.2.2)} & \mu \end{array}$$

induces a map

$$\mathrm{H}^N(Y, \mathrm{gr}_Z^r M^n \otimes \mathrm{gr}_Z^{3e'-2-r} M^{n'}) \longrightarrow \mu. \quad (9.2.3)$$

(3) We put

$$\begin{aligned}\mathcal{F}^{r,n} &:= (U^1 M^n / Z^{r+1} M^n) \otimes Z^{3e'-2-r} M^{n'}, \\ \mathcal{G}^{r,n} &:= \mathcal{F}^{r,n} / \left(\text{gr}_Z^r M^n \otimes Z^{3e'-1-r} M^{n'} \right), \\ \mathcal{H}^{r,n} &:= \text{gr}_Z^r M^n \otimes \text{gr}_Z^{3e'-2-r} M^{n'}\end{aligned}$$

(note that $\text{gr}_Z^r M^n \otimes Z^{3e'-1-r} M^{n'}$ is a subsheaf of $\mathcal{F}^{r,n}$, because a $\mathbb{Z}/p\mathbb{Z}$ -sheaf is flat over $\mathbb{Z}/p\mathbb{Z}$). Then the map (9.2.2) induces a map

$$H^N(Y, \mathcal{G}^{r,n}) \longrightarrow \mu. \quad (9.2.4)$$

If $r \geq 2$, then this map makes the following diagram commutative:

$$\begin{array}{ccc} H^N(Y, \mathcal{H}^{r,n}) \oplus H^N(Y, \mathcal{F}^{r-1,n}) & \longrightarrow & H^N(Y, \mathcal{G}^{r,n}) \\ \downarrow (9.2.3) \oplus \downarrow (9.2.2) \text{ for } r-1 & & \downarrow (9.2.4) \\ \mu \oplus \mu & \xrightarrow{\text{product}} & \mu, \end{array} \quad (9.2.5)$$

where the top horizontal arrow is induced by a natural inclusion $\mathcal{H}^{r,n} \oplus \mathcal{F}^{r-1,n} \subset \mathcal{G}^{r,n}$.

Proof of Lemma 9.2.1. We prove only (1). (2) and (3) are similar and left to the reader. We use the notation we fixed in (3). Let q be the maximal integer with $3(q-1) < r$. Noting that $Z^r M^n \otimes Z^{3e'-2-r} M^{n'} \subset U^q M^n \otimes U^{e'-q} M^{n'}$, consider the composite map

$$H^N(Y, Z^r M^n \otimes Z^{3e'-2-r} M^{n'}) \longrightarrow H^N(Y, U^1 M^n \otimes Z^{3e'-2-r} M^{n'}) \xrightarrow{(*)} \mu,$$

where the arrow $(*)$ denotes the first composite map in (1). By Theorem 8.3.8 (cf. (8.4.3)), this composite map agrees with that induced by $\chi \circ f^{q,n}$. By Lemma 9.1.4 (1), $f^{q,n}$ annihilates the subsheaf $Z^{r+1} M^n \otimes Z^{3e'-2-r} M^{n'}$ of $Z^r M^n \otimes Z^{3e'-2-r} M^{n'}$. Hence the arrow $(*)$ induces a map of the form (9.2.2) by the short exact sequence of sheaves

$$0 \longrightarrow Z^{r+1} M^n \otimes Z^{3e'-2-r} M^{n'} \longrightarrow U^1 M^n \otimes Z^{3e'-2-r} M^{n'} \longrightarrow \mathcal{F}^{r,n} \longrightarrow 0$$

and Sublemma 8.5.2 (2) (cf. (9.1.3)). Thus we obtain the lemma. \square

Now we turn to the proof of Theorem 9.1.1. By the trace maps (9.2.2) and (9.2.3), there are induced pairings

$$\begin{aligned} b^{i,r} &: H^i(Y, U^1 M^n / Z^{r+1} M^n) \times H^{N-i}(Y, Z^{3e'-2-r} M^{n'}) \longrightarrow \mu, \\ c^{i,r} &: H^i(Y, \text{gr}_Z^r M^n) \times H^{N-i}(Y, \text{gr}_Z^{3e'-2-r} M^{n'}) \longrightarrow \mu, \end{aligned} \quad (9.2.6)$$

for i and r with $1 \leq r \leq 3e' - 3$. Note that $b^{i,3e'-3} = a^i$ and $b^{i,1} = c^{i,1}$. By the commutative diagram (9.2.5), there is a commutative diagram with exact rows for $r \geq 2$ (after changing the signs of (b) suitably)

$$\begin{array}{ccccccccc} H^{i-1}(Y, U^1 M^n / Z^r M^n) & \rightarrow & H^i(Y, \text{gr}_Z^r M^n) & \rightarrow & H^i(Y, U^1 M^n / Z^{r+1} M^n) & \rightarrow & H^i(Y, U^1 M^n / Z^r M^n) & \rightarrow & H^{i+1}(Y, \text{gr}_Z^r M^n) \\ b^{i-1,r-1} \downarrow & & c^{i,r} \downarrow & & b^{i,r} \downarrow & & b^{i,r-1} \downarrow & & c^{i+1,r} \downarrow \\ H^{\ell+1}(Y, Z^{t+1} M^{n'})^* & \xrightarrow{(b)} & H^\ell(Y, \text{gr}_Z^t M^{n'})^* & \rightarrow & H^\ell(Y, Z^t M^{n'})^* & \rightarrow & H^\ell(Y, Z^{t+1} M^{n'})^* & \xrightarrow{(b)} & H^{\ell-1}(Y, \text{gr}_Z^t M^{n'})^*, \end{array}$$

where we put $\ell := N - i$, $t := 3e' - 2 - r$ and $E^* := \text{Hom}(E, \mu)$ for a $\mathbb{Z}/p\mathbb{Z}$ -module E . Hence Theorem 9.1.1 is reduced to the following lemma by induction on $r \geq 1$ and the five lemma.

Lemma 9.2.7. $c^{i,r}$ in (9.2.6) is non-degenerate for any i and r with $1 \leq r \leq 3e' - 3$.

We prove this lemma in the next subsection.

9.3. Proof of Lemma 9.2.7. We first give a brief review of linear Cartier operators. Let (s, \mathcal{L}_s) and \mathcal{L}_Y be as in §8.3, and let $(Y', \mathcal{L}_{Y'})$ be the log scheme defined by a cartesian diagram

$$\begin{array}{ccc} (Y', \mathcal{L}_{Y'}) & \xrightarrow{\text{pr}_2} & (Y, \mathcal{L}_Y) \\ \text{pr}_1 \downarrow & \square & \downarrow \\ (s, \mathcal{L}_s) & \xrightarrow{\text{F}_{(s, \mathcal{L}_s)}^{\text{abs}}} & (s, \mathcal{L}_s), \end{array} \quad (9.3.1)$$

where $\text{F}_{(s, \mathcal{L}_s)}^{\text{abs}}$ denotes the absolute Frobenius on (s, \mathcal{L}_s) . Let $\underline{\text{pr}}_2 : Y' \rightarrow Y$ be the underlying morphism of schemes of pr_2 , and let $\text{F}_{Y/s} : Y \rightarrow Y'$ be the unique morphism of schemes such that $\underline{\text{pr}}_2 \circ \text{F}_{Y/s}$ agrees with the absolute Frobenius on Y . Note that $\text{F}_{Y/s}$ is a finite morphism of schemes. We put $\omega_{Y'}^N := \omega_{(Y', \mathcal{L}_{Y'})/(s, \mathcal{L}_s)}^N$ for simplicity, where we regarded $(Y', \mathcal{L}_{Y'})$ as a smooth log scheme over (s, \mathcal{L}_s) by pr_1 in (9.3.1). By [KF], 5.3 and the same argument as for [Kz], 7.2, there is a $\mathcal{O}_{Y'}$ -linear isomorphism

$$C_{\text{lin}}^{-1} : \omega_{Y'}^N \xrightarrow{\simeq} \text{F}_{Y/s*}(\omega_Y^N / \mathcal{B}_Y^N). \quad (9.3.2)$$

(The following composite map gives the inverse Cartier operator C^{-1} defined in [Hy1]:

$$\omega_Y^N \xrightarrow{\text{canonical}} \underline{\text{pr}}_{2*} \omega_{Y'}^N \xrightarrow{\underline{\text{pr}}_{2*}(C_{\text{lin}}^{-1})} \underline{\text{pr}}_{2*} \text{F}_{Y/s*}(\omega_Y^N / \mathcal{B}_Y^N) = \omega_Y^N / \mathcal{B}_Y^N.)$$

Now we start the proof of Lemma 9.2.7. Let C_{lin} be the inverse of C_{lin}^{-1} . By [Hy2], 3.2 and the same argument as for [Mil], 1.7, there are $\mathcal{O}_{Y'}$ -bilinear perfect pairings of locally free $\mathcal{O}_{Y'}$ -modules of finite rank

$$\begin{aligned} \text{F}_{Y/s*}(\omega_Y^{n-1} / \mathcal{Z}_Y^{n-1}) \times \text{F}_{Y/s*}(\omega_Y^{n'-2} / \mathcal{Z}_Y^{n'-2}) &\longrightarrow \omega_{Y'}^N, & (x, y) &\mapsto C_{\text{lin}}((dx) \wedge y), \\ \text{F}_{Y/s*}(\mathcal{Z}_Y^{n-1} / \mathcal{B}_Y^{n-1}) \times \text{F}_{Y/s*}(\mathcal{Z}_Y^{n'-1} / \mathcal{B}_Y^{n'-1}) &\longrightarrow \omega_{Y'}^N, & (x, y) &\mapsto C_{\text{lin}}(x \wedge y), \\ \text{F}_{Y/s*}(\omega_Y^{n-2} / \mathcal{Z}_Y^{n-2}) \times \text{F}_{Y/s*}(\omega_Y^{n'-1} / \mathcal{Z}_Y^{n'-1}) &\longrightarrow \omega_{Y'}^N, & (x, y) &\mapsto C_{\text{lin}}(x \wedge dy). \end{aligned}$$

By [Hy2], Theorem 3.1 and the Serre-Hartshorne duality [Ha1], $\omega_{Y'}^N$ is a dualizing sheaf for Y' in the sense of [Ha3], p. 241, Definition. Hence by (9.1.3) and Lemma 9.1.4, the pairing

$$\text{H}^i(Y, \text{gr}_Z^r M^n) \times \text{H}^{N-i}(Y, \text{gr}_Z^{3e'-2-r} M^{n'}) \xrightarrow{f^{q,n}} \mu \otimes \text{H}^N(Y, \omega_Y^N / \mathcal{B}_Y^N) \xrightarrow{\text{id} \otimes \text{tr}'_{Y/s}} \mu \otimes k$$

(q is the maximal integer with $3(q-1) < r$) is a non-degenerate pairing of finite-dimensional k -vector spaces. Here $\text{tr}'_{Y/s}$ denotes the k -linear trace map

$$\text{H}^N(Y, \omega_Y^N / \mathcal{B}_Y^N) = \text{H}^N(Y', \text{F}_{Y/s*}(\omega_Y^N / \mathcal{B}_Y^N)) \xrightarrow{C_{\text{lin}}} \text{H}^N(Y', \omega_{Y'}^N) \longrightarrow k.$$

Finally, $c^{i,r}$ is non-degenerate by commutative squares

$$\begin{array}{ccccc} \text{H}^N(Y, \text{gr}_Z^r M^n \otimes \text{gr}_Z^{3e'-2-r} M^{n'}) & \xrightarrow{f^{q,n}} & \mu \otimes \text{H}^N(Y, \omega_Y^N / \mathcal{B}_Y^N) & \xrightarrow{\text{id} \otimes \text{tr}'_{Y/s}} & \mu \otimes k \\ \text{(9.2.3)} \downarrow & & \downarrow \chi & & \downarrow \text{id} \otimes \text{tr}_{k/\mathbb{F}_p} \\ \mu & \xleftarrow{\text{id} \otimes \text{tr}_Y} & \mu \otimes \text{H}^{N+1}(Y, \nu_Y^N) & \xrightarrow{\text{id} \otimes \text{tr}_Y} & \mu \otimes \mathbb{F}_p, \end{array}$$

where the left square commutes by Theorem 8.3.8 and the right square commutes by a similar argument as for [Sat], 3.4.1. This completes the proof of Lemma 9.2.7 and Theorem 9.1.1.

10. Duality of p -adic étale Tate twists

In this section we prove Theorem 1.2.2 using Theorem 9.1.1.

10.1. Statement of the results. The setting is the same as in §4.1. In this section, we assume that X is *proper* over $B = \text{Spec}(A)$, and that A is either an algebraic integer ring (global case) or a henselian discrete valuation ring whose fraction field has characteristic 0 and whose residue field is finite of characteristic p (local case). Let d be the absolute dimension of X . Throughout this section, n and r denote integers with $0 \leq n \leq d$ and $1 \leq r$. The aim of this section is to prove the following duality results:

Theorem 10.1.1. *Assume that A is local. Then:*

- (1) *There is a canonical trace map $\text{tr}_{(X,Y)} : H_Y^{2d+1}(X, \mathfrak{T}_r(d)_X) \rightarrow \mathbb{Z}/p^r\mathbb{Z}$, which is bijective if X is connected.*
- (2) *For $i \in \mathbb{Z}$, the natural pairing arising from (4.2.7) and $\text{tr}_{(X,Y)}$*

$$H_Y^i(X, \mathfrak{T}_r(n)_X) \times H^{2d+1-i}(X, \mathfrak{T}_r(d-n)_X) \longrightarrow \mathbb{Z}/p^r\mathbb{Z} \quad (10.1.2)$$

is a non-degenerate pairing of finite $\mathbb{Z}/p^r\mathbb{Z}$ -modules.

Theorem 10.1.3 (1.2.2). *Assume that A is global. Then:*

- (1) *There is a canonical trace map $\text{tr}_X : H_c^{2d+1}(X, \mathfrak{T}_r(d)_X) \rightarrow \mathbb{Z}/p^r\mathbb{Z}$, where the subscript c means the étale cohomology with compact support (see §10.2 below). If X is connected, then tr_X is bijective.*
- (2) *For $i \in \mathbb{Z}$, the natural pairing arising from (4.2.7) and tr_X*

$$H_c^i(X, \mathfrak{T}_r(n)_X) \times H^{2d+1-i}(X, \mathfrak{T}_r(d-n)_X) \longrightarrow \mathbb{Z}/p^r\mathbb{Z} \quad (10.1.4)$$

is a non-degenerate pairing of finite $\mathbb{Z}/p^r\mathbb{Z}$ -modules.

In §10.2, we will define the localized trace map $\text{tr}_{(X,Y)}$ and the global trace map tr_X . After showing a compatibility of these trace maps, we will reduce Theorem 10.1.3 (2) to Theorem 10.1.1 (2). We will prove Theorem 10.1.1 (2) in §§10.3–10.5.

Remark 10.1.5. *If A is local, there is a natural pairing of finite $\mathbb{Z}/p^r\mathbb{Z}$ -modules*

$$H^i(V, \mu_{p^r}^{\otimes n}) \times H^{2d-i}(V, \mu_{p^r}^{\otimes d-n}) \longrightarrow H^{2d}(V, \mu_{p^r}^{\otimes d}) \simeq \mathbb{Z}/p^r\mathbb{Z}, \quad (10.1.6)$$

where V denotes X_K with $K := \text{Frac}(A)$. As is well-known, this pairing is non-degenerate by the Tate duality for K and the Poincaré duality for $V_{\overline{K}}$. Theorem 10.1.1 (2) does not follow from these facts, although Theorem 10.1.1 implies the non-degeneracy of (10.1.6). We will deduce Theorem 10.1.1 (2) from Theorems 2.2.4 and 9.1.1.

10.2. Trace maps. We first construct the localized trace map $\text{tr}_{(X,Y)}$, assuming that A is local. Let $\iota : Y \hookrightarrow X$ be the natural closed immersion. By Lemma 7.3.3 and Theorem 2.2.4, $H_Y^i(X, \mathfrak{T}_r(d)_X)$ is zero for any $i \geq 2d + 2$. We define $\text{tr}_{(X,Y)}$ as the composite

$$\text{tr}_{(X,Y)} : H_Y^{2d+1}(X, \mathfrak{T}_r(d)_X) \xrightarrow[\simeq]{(\text{Gys}_i^d)^{-1}} H^d(Y, \nu_{Y,r}^{d-1}) \xrightarrow{\text{tr}_Y} \mathbb{Z}/p^r\mathbb{Z},$$

which is bijective if X is connected (i.e., Y is connected). We next define the global trace map tr_X , assuming that A is global. For a scheme Z which is separated of finite type over B and an object $\mathcal{K} \in D^+(Z_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})$, we define $H_c^*(Z, \mathcal{K})$ as $H_c^*(B, Rf_*\mathcal{K})$, where f denotes the structural morphism $Z \rightarrow B$ and $H_c^*(B, \bullet)$ denote the étale cohomology groups with compact support of

B (cf. [Mi3], II.2). By the Kummer sequence (4.5.3) and the isomorphism $H_c^3(B, \mathbb{G}_m) \simeq \mathbb{Q}/\mathbb{Z}$ (cf. [Mi3], II.2.6), there is a trace map $H_c^3(B, \mathfrak{T}_r(1)_B) \rightarrow \mathbb{Z}/p^r\mathbb{Z}$ (cf. [JSS], Corollary 4.3 (a)). We normalize this map so that for a closed point $i_s : s \hookrightarrow B$, the composite map

$$H^1(s, \mathbb{Z}/p^r\mathbb{Z}) \xrightarrow{\text{Gys}_{i_s}^1} H_c^3(B, \mathfrak{T}_r(1)_B) \longrightarrow \mathbb{Z}/p^r\mathbb{Z}$$

coincides with the trace map of s (defined in 2.2.4 (1)). We define the trace map tr_X as the composite

$$\text{tr}_X : H_c^{2d+1}(X, \mathfrak{T}_r(d)_X) \longrightarrow H_c^3(B, \mathfrak{T}_r(1)_B) \longrightarrow \mathbb{Z}/p^r\mathbb{Z},$$

where the first arrow arises from the trace morphism in Theorem 7.1.1. The bijectivity assertion for tr_X in Theorem 10.1.3 (1) will follow from 10.1.3 (2). We show here the following:

Lemma 10.2.1. *Assume that A is global. Then there is a commutative diagram*

$$\begin{array}{ccc} H_Y^{2d+1}(X, \mathfrak{T}_r(d)_X) & \xrightarrow{\text{tr}_{(X,Y)}} & \mathbb{Z}/p^r\mathbb{Z} \\ \iota_* \downarrow & & \parallel \\ H_c^{2d+1}(X, \mathfrak{T}_r(d)_X) & \xrightarrow{\text{tr}_X} & \mathbb{Z}/p^r\mathbb{Z}, \end{array}$$

where the arrow ι_* denotes the canonical adjunction map and $\text{tr}_{(X,Y)}$ denotes the sum of the localized trace maps for the connected components of Y .

Proof. Let $\{Y_i\}_{i \in I}$ be the connected components of Y . Let x be a closed point on Y with $i_x : x \hookrightarrow X$. Noting that $H_Y^{2d+1}(X, \mathfrak{T}_r(d)_X) \simeq \bigoplus_{i \in I} \mathbb{Z}/p^r\mathbb{Z}$, consider a diagram

$$\begin{array}{ccccc} H^1(x, \mathbb{Z}/p^r\mathbb{Z}) & \xrightarrow{\text{Gys}_{i_x}^d} & H_Y^{2d+1}(X, \mathfrak{T}_r(d)_X) & \xrightarrow{\text{tr}_{(X,Y)}} & \mathbb{Z}/p^r\mathbb{Z} \\ \parallel & & \iota_* \downarrow & & \parallel \\ H^1(x, \mathbb{Z}/p^r\mathbb{Z}) & \xrightarrow{\text{Gys}_{i_x}^d} & H_c^{2d+1}(X, \mathfrak{T}_r(d)_X) & \xrightarrow{\text{tr}_X} & \mathbb{Z}/p^r\mathbb{Z}. \end{array}$$

Since the left square commutes, it suffices to show that the composite of the upper row is bijective and that the outer rectangle is commutative. The composite of the upper row agrees with the trace map for x by Theorem 2.2.4 (1). In particular it is bijective. The composite of the lower row agrees with the trace map for x by Theorem 7.3.1 (2). We are done. \square

We reduce Theorem 10.1.3 (2) to Theorem 10.1.1 (2). Assume that A is global. We use the notation in §4.1. Put $X_\Sigma := \coprod_{s \in \Sigma} X \times_B B_s$. Since $j^*\mathfrak{T}_r(n)_X \simeq \mu_{p^r}^{\otimes n}$, there is a distinguished triangle

$$\mathfrak{T}_r(n)_X \xrightarrow{\iota^*} R\iota_*\iota^*\mathfrak{T}_r(n)_X \longrightarrow j_!\mu_{p^r}^{\otimes n}[1] \xrightarrow{j_!} \mathfrak{T}_r(n)_X[1],$$

where the arrow ι^* (resp. $j_!$) denotes the canonical adjunction morphism $\text{id} \rightarrow R\iota_*\iota^*$ (resp. $Rj_!j^* \rightarrow \text{id}$). By Lemma 10.2.1 and the proper base-change theorem: $H^*(Y, \iota^*\mathfrak{T}_r(n)_X) \simeq H^*(X_\Sigma, \mathfrak{T}_r(n)_{X_\Sigma})$, we obtain a commutative diagram with exact rows (after changing the signs of (b) suitably)

$$\begin{array}{ccccccc} H^{i-1}(X_\Sigma, \mathfrak{T}_r(n)_{X_\Sigma}) & \rightarrow & H_c^i(V, \mu_{p^r}^{\otimes n}) & \rightarrow & H_c^i(X, \mathfrak{T}_r(n)_X) & \rightarrow & H^i(X_\Sigma, \mathfrak{T}_r(n)_{X_\Sigma}) \rightarrow H_c^{i+1}(V, \mu_{p^r}^{\otimes n}) \\ (10.1.2) \downarrow & & a \downarrow & & (10.1.4) \downarrow & & (10.1.2) \downarrow & & a \downarrow \\ H_Y^{\ell+1}(X, \mathfrak{T}_r(m)_X)^* & \xrightarrow{(b)} & H^\ell(V, \mu_{p^r}^{\otimes m})^* & \rightarrow & H^\ell(X, \mathfrak{T}_r(m)_X)^* & \rightarrow & H_Y^\ell(X, \mathfrak{T}_r(m)_X)^* & \xrightarrow{(b)} & H^{\ell-1}(V, \mu_{p^r}^{\otimes m})^*. \end{array}$$

Here the superscript $*$ means the Pontryagin dual and we put $\ell := 2d + 1 - i$ and $m := d - n$. The lower row is the dual of the localization long exact sequence and the vertical arrows arise from duality pairings. The arrows a are isomorphisms of finite groups by the Artin-Verdier duality and the absolute purity [Th], [FG]. Thus Theorem 10.1.3 (2) is reduced to Theorem 10.1.1 (2) by the five lemma.

10.3. Reduction to the case $r = 1$. We start the proof of Theorem 10.1.1 (2), which will be completed in §10.5. By the distinguished triangle in Proposition 4.3.1 (3), the problem is reduced to the case $r = 1$. Furthermore we may assume that $K = \text{Frac}(A)$ contains a primitive p -th root of unity ζ_p . Indeed, otherwise the scalar extension $X_{A'} := X \otimes_A A'$, where A' denotes the integer ring of $K(\zeta_p)$, again satisfies the condition 4.1.1 over $\text{Spec}(A)$. Hence once we show Theorem 10.1.1 (2) for $X_{A'}$, we will obtain Theorem 10.1.1 (2) for X by a standard norm argument and Corollary 7.2.4.

10.4. Descending induction on n . Assume that $\zeta_p \in K$ and $r = 1$. We prove this case of Theorem 10.1.1 (2) by descending induction on $n \leq d$. Let N be the relative dimension $\dim(X/B)$. If $n = N + 1 (= d)$, (10.1.2) is isomorphic to the pairing

$$\mathrm{H}^{i-N-2}(Y, \nu_{Y,1}^N) \times \mathrm{H}^{2N+3-i}(Y, \mathbb{Z}/p\mathbb{Z}) \longrightarrow \mathrm{H}^{N+1}(Y, \nu_{Y,1}^N) \simeq \mathbb{Z}/p\mathbb{Z}$$

by the proper base-change theorem and Lemma 7.3.3. This pairing is a non-degenerate pairing of finite $\mathbb{Z}/p\mathbb{Z}$ -modules by Theorem 2.2.4. To proceed the descending induction on n , we study the inductive structure of $\{\mathfrak{T}_1(n)_X\}_{n \geq 0}$ on n . We fix some notation. Let $\iota : Y \hookrightarrow X$ and $j : V (= X_K) \hookrightarrow X$ be as before. Let ν_Y^n , μ' and μ be as in (8.1.1). See also the remark after (8.1.1). Put $\lambda_Y^n := \lambda_{Y,1}^n$ and $\mathfrak{T}(n)_X := \mathfrak{T}_1(n)_X$. Now for n with $1 \leq n \leq N + 1$, we define the morphism

$$\mathrm{ind}_n : (j_*\mu_p) \otimes \mathfrak{T}(n-1)_X \left(:= (j_*\mu_p) \otimes^{\mathbb{L}} \mathfrak{T}(n-1)_X \right) \longrightarrow \mathfrak{T}(n)_X$$

by restricting the product structure $\mathfrak{T}(1)_X \otimes^{\mathbb{L}} \mathfrak{T}(n-1)_X \rightarrow \mathfrak{T}(n)_X$ to the 0-th cohomology sheaf $j_*\mu_p$ of $\mathfrak{T}(1)_X$.

Lemma 10.4.1. *Let*

$$\mathbb{K}(n)[-1] \xrightarrow{b_n} \mu' \otimes \iota^* \mathfrak{T}(n-1)_X \xrightarrow{\iota^*(\mathrm{ind}_n)} \iota^* \mathfrak{T}(n)_X \xrightarrow{a_n} \mathbb{K}(n) \quad (10.4.2)$$

be a distinguished triangle in $D^b(Y_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$. Then:

- (1) *The triple $(\mathbb{K}(n), a_n, b_n)$ is unique up to a unique isomorphism in $D^b(Y_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$, and b_n is determined by the pair $(\mathbb{K}(n), a_n)$.*
- (2) *$\mathbb{K}(n)$ is concentrated in $[n-1, n]$ and a_n induces isomorphisms*

$$\mathcal{H}^q(\mathbb{K}(n)) \simeq \begin{cases} \mu' \otimes \nu_Y^{n-2} & (q = n-1), \\ FM^n & (q = n), \end{cases}$$

where M^n denotes the étale sheaf $\iota^* R^n j_* \mu_p^{\otimes n}$ on Y , and FM^n denotes the étale subsheaf of M^n defined in §3.4.

- (3) *There is a distinguished triangle in $D^b(Y_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$*

$$\mathbb{K}(n)[-1] \xrightarrow{c_n} \mu' \otimes R\iota^! \mathfrak{T}(n-1)_X \xrightarrow{R\iota^!(\mathrm{ind}_n)} R\iota^! \mathfrak{T}(n)_X \xrightarrow{d_n} \mathbb{K}(n), \quad (10.4.3)$$

where c_n and d_n are morphisms determined by the pair $(\mathbb{K}(n), a_n)$.

(4) *There is an anti-commutative diagram*

$$\begin{array}{ccc}
 \iota^* \mathfrak{T}(n)_X & \xrightarrow{\text{canonical}} & \mu' \otimes \iota^* Rj_* \mu_p^{\otimes n-1} \\
 a_n \downarrow & & \downarrow \text{id}_{\mu'} \otimes \iota^* (\delta_{V,Y}^{\text{loc}}(\mathfrak{T}(n-1)_X)) \\
 \mathbb{K}(n) & \xrightarrow{c_n[1]} & \mu' \otimes R\iota^! \mathfrak{T}(n-1)_X[1].
 \end{array} \tag{10.4.4}$$

Proof. (2) follows from the long exact sequence of cohomology sheaves associated with (10.4.2) and the isomorphism of sheaves $\mu' \otimes \iota^* R^q j_* \mu_p^{\otimes n-1} \simeq \iota^* R^q j_* \mu_p^{\otimes n}$ (cf. (4.2.5)). The details are straight-forward and left to the reader. By (2) and Lemma 2.1.1, there is no non-zero morphism from $\mu' \otimes \iota^* \mathfrak{T}(n-1)_X$ to $\mathbb{K}(n)[-1]$. Lemma 10.4.1 (1) follows from this fact and Lemma 2.1.2 (3). We next prove (3). Let

$$\mathcal{F}[-1] \xrightarrow{b'} (j_* \mu_p) \otimes \mathfrak{T}(n-1)_X \xrightarrow{\text{ind}_n} \mathfrak{T}(n)_X \xrightarrow{a'} \mathcal{F}$$

be a distinguished triangle in $D^b(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$. By a similar argument as for the claim (2), the cohomology sheaves of \mathcal{F} are supported on Y . This implies that $\mathcal{F} = R\iota_* \iota^* \mathcal{F}$. By the uniqueness assertion of the claim (1), the triple $(\iota^* \mathcal{F}, \iota^*(a'), \iota^*(b'))$ is isomorphic to $(\mathbb{K}(n), a_n, b_n)$ by a unique isomorphism. Under this identification, we have $a' \simeq R\iota_*(a_n)$. Moreover, b' is determined by the pair $(\mathcal{F}, a') = (R\iota_* \mathbb{K}(n), R\iota_*(a_n))$ by a similar argument as for the claim (1). Hence applying $R\iota^!$ to the above triangle, we obtain the distinguished triangle (10.4.3) with $c_n = R\iota^!(b')$ and $d_n = R\iota^! R\iota_*(a_n)$. Finally, (4) follows from an elementary computation on connecting morphisms, whose details are straight-forward and left to the reader. \square

In what follows, we fix a pair $(\mathbb{K}(n), a_n)$ fitting into (10.4.2) for each n with $1 \leq n \leq N+1$. By Lemma 10.4.1, the morphisms b_n, c_n and d_n fitting into (10.4.2) and (10.4.3) are determined by $(\mathbb{K}(n), a_n)$. Next we construct a pairing on $\{\mathbb{K}(n)\}_{1 \leq n \leq N+1}$ using $\{a_n\}_{1 \leq n \leq N+1}$. Let us note that for objects $\mathcal{K}_1, \mathcal{K}_2 \in D^-(Y_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$, and $\mathcal{K}_3 \in D^+(Y_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$, we have

$$\text{Hom}_{D(Y_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})}(\mathcal{K}_1 \otimes^{\mathbb{L}} \mathcal{K}_2, \mathcal{K}_3) \simeq \text{Hom}_{D(Y_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})}(\mathcal{K}_1, R\mathcal{H}om_{Y, \mathbb{Z}/p\mathbb{Z}}(\mathcal{K}_2, \mathcal{K}_3)).$$

For $\mathcal{K} \in D^-(Y_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$, we define

$$\mathbb{D}(\mathcal{K}) := R\mathcal{H}om_{Y, \mathbb{Z}/p\mathbb{Z}}(\mathcal{K}, \mu' \otimes \nu_Y^N[-N-2]) \in D^+(Y_{\text{ét}}, \mathbb{Z}/p\mathbb{Z}). \tag{10.4.5}$$

Lemma 10.4.6. *Let n be as before and put $n' := N+2-n$. Then there is a unique morphism*

$$(\mathbb{K}(n)[-1]) \otimes^{\mathbb{L}} \mathbb{K}(n') \longrightarrow \mu' \otimes \nu_Y^N[-N-2] \quad \text{in } D^-(Y_{\text{ét}}, \mathbb{Z}/p\mathbb{Z}) \tag{10.4.7}$$

whose adjoint morphism $\mathbb{K}(n)[-1] \rightarrow \mathbb{D}(\mathbb{K}(n'))$ fits into a commutative diagram with distinguished rows (cf. (10.4.2), (10.4.3))

$$\begin{array}{ccccccc}
 \mathbb{K}(n)[-1] & \xrightarrow{c_n} & \mu' \otimes R\iota^! \mathfrak{T}(n-1)_X & \xrightarrow{R\iota^!(\text{ind}_n)} & R\iota^! \mathfrak{T}(n)_X & \xrightarrow{d_n} & \mathbb{K}(n) \\
 \downarrow & & (\#) \downarrow & & (\#) \downarrow & & \downarrow \\
 \mathbb{D}(\mathbb{K}(n')) & \xrightarrow{\mathbb{D}(a_{n'})} & \mathbb{D}(\iota^* \mathfrak{T}(n')_X) & \xrightarrow{\mathbb{D}(\text{ind}_{n'})} & \mathbb{D}(\mu' \otimes \iota^* \mathfrak{T}(n'-1)_X) & \xrightarrow{-\mathbb{D}(b_{n'})} & \mathbb{D}(\mathbb{K}(n')[-1]).
 \end{array}$$

Here the vertical arrows $(\#)$ come from the product structure of $\{\mathfrak{T}(n)_X\}_{n \geq 0}$, the identity map of μ' and the Gysin isomorphism Gys_ι^{N+1} in Lemma 7.3.3 (the commutativity of the central square is easy and left to the reader).

Proof. The assertion follows from Lemma 2.1.2 (1) and the fact that

$$\begin{aligned} & \mathrm{Hom}_{D^+(Y_{\acute{e}t}, \mathbb{Z}/p\mathbb{Z})}(\mathbb{K}(n), \mathbb{D}(\mu' \otimes \iota^* \mathfrak{T}(n' - 1)_X)) \\ & \simeq \mathrm{Hom}_{D^-(Y_{\acute{e}t}, \mathbb{Z}/p\mathbb{Z})}(\mathbb{K}(n) \otimes^{\mathbb{L}} (\mu' \otimes \iota^* \mathfrak{T}(n' - 1)_X), \mu' \otimes \nu_Y^N[-N - 2]) = 0, \end{aligned}$$

where the last equality follows from Lemma 10.4.1 (2) and Lemma 2.1.1. \square

Now we turn to the proof of Theorem 10.1.1 (2) and claim the following:

Proposition 10.4.8. *Let n and n' be as in Lemma 10.4.6. Then for $i \in \mathbb{Z}$, the pairing*

$$\mathrm{H}^i(Y, \mathbb{K}(n)) \times \mathrm{H}^{2N+2-i}(Y, \mathbb{K}(n')) \longrightarrow \mu \otimes \mathrm{H}^{N+1}(Y, \nu_Y^N) \xrightarrow{\mathrm{id} \otimes \mathrm{tr}_Y} \mu, \quad (10.4.9)$$

induced by (10.4.7), is a non-degenerate pairing of finite $\mathbb{Z}/p\mathbb{Z}$ -modules.

We will prove this proposition in the next subsection. We first finish the proof of Theorem 10.1.1 (2) by descending induction on $n \leq N + 1$, admitting Proposition 10.4.8. See the beginning of this subsection for the case $n = N + 1$. Indeed, we obtain Theorem 10.1.1 (2) from Proposition 10.4.8, applying the following general lemma to the commutative diagram in Lemma 10.4.6:

Lemma 10.4.10. *Let $\mathcal{K}_1 \rightarrow \mathcal{K}_2 \rightarrow \mathcal{K}_3 \rightarrow \mathcal{K}_1[1]$ and $\mathcal{L}_3 \rightarrow \mathcal{L}_2 \rightarrow \mathcal{L}_1 \rightarrow \mathcal{L}_3[1]$ be distinguished triangles in $D^b(Y_{\acute{e}t}, \mathbb{Z}/p\mathbb{Z})$, and suppose that we are given a commutative diagram*

$$\begin{array}{ccccccc} \mathcal{K}_1 & \longrightarrow & \mathcal{K}_2 & \longrightarrow & \mathcal{K}_3 & \longrightarrow & \mathcal{K}_1[1] \\ \alpha_1 \downarrow & & \alpha_2 \downarrow & & \alpha_3 \downarrow & & \alpha_1[1] \downarrow \\ \mathbb{D}(\mathcal{L}_1) & \longrightarrow & \mathbb{D}(\mathcal{L}_2) & \longrightarrow & \mathbb{D}(\mathcal{L}_3) & \longrightarrow & \mathbb{D}(\mathcal{L}_1)[1] \end{array}$$

(with distinguished rows) in $D^+(Y_{\acute{e}t}, \mathbb{Z}/p\mathbb{Z})$. For $m \in \{1, 2, 3\}$ and $i \in \mathbb{Z}$, let

$$\beta_m^i : \mathrm{H}^i(Y, \mathcal{K}_m) \times \mathrm{H}^{2N+3-i}(Y, \mathcal{L}_m) \longrightarrow \mu \otimes \mathrm{H}^{N+1}(Y, \nu_Y^N) \xrightarrow{\mathrm{id} \otimes \mathrm{tr}_Y} \mu$$

be the pairing induced by the adjoint morphism $\mathcal{K}_m \otimes^{\mathbb{L}} \mathcal{L}_m \rightarrow \mu' \otimes \nu_Y^N[-N - 2]$ of α_m . Put $\ell := 2N + 3 - i$. Then there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} \mathrm{H}^{i-1}(Y, \mathcal{K}_3) & \longrightarrow & \mathrm{H}^i(Y, \mathcal{K}_1) & \longrightarrow & \mathrm{H}^i(Y, \mathcal{K}_2) & \longrightarrow & \mathrm{H}^i(Y, \mathcal{K}_3) & \longrightarrow & \mathrm{H}^{i+1}(Y, \mathcal{K}_1) \\ \gamma_3^{i-1} \downarrow & & \gamma_1^i \downarrow & & \gamma_2^i \downarrow & & \gamma_3^i \downarrow & & \gamma_1^{i+1} \downarrow \\ \mathrm{H}^{\ell+1}(Y, \mathcal{L}_3)^* & \xrightarrow{(b)} & \mathrm{H}^{\ell}(Y, \mathcal{L}_1)^* & \longrightarrow & \mathrm{H}^{\ell}(Y, \mathcal{L}_2)^* & \longrightarrow & \mathrm{H}^{\ell}(Y, \mathcal{L}_3)^* & \xrightarrow{(b)} & \mathrm{H}^{\ell-1}(Y, \mathcal{L}_1)^* \end{array}$$

after changing the signs of (b) suitably. Here for a $\mathbb{Z}/p\mathbb{Z}$ -module E , E^ denotes $\mathrm{Hom}_{\mathbb{Z}/p\mathbb{Z}}(E, \mu)$, and γ_m^i denotes the natural map induced by β_m^i . Furthermore, if γ_1^i and γ_3^i are bijective for any i , then γ_2^i is bijective for any i .*

Proof. For each m and i , γ_m^i factors as follows:

$$\mathrm{H}^i(Y, \mathcal{K}_m) \xrightarrow{\mathrm{H}^i(Y, \alpha_m)} \mathrm{Ext}_{Y, \mathbb{Z}/p\mathbb{Z}}^{i-N-2}(\mathcal{L}_m, \mu' \otimes \nu_Y^N) \longrightarrow \mathrm{Hom}_{\mathbb{Z}/p\mathbb{Z}}(\mathrm{H}^{2N+3-i}(Y, \mathcal{L}_m), \mu),$$

where the last map arises from a Yoneda pairing and the trace isomorphism $\mathrm{H}^{N+1}(Y, \mu' \otimes \nu_Y^N) \simeq \mu$. The commutativity of the diagram of cohomology groups in the lemma follows from the functoriality of this decomposition. The last assertion follows from the five lemma. \square

10.5. Proof of Proposition 10.4.8. Let us recall that the canonical pairings

$$\begin{aligned} \mathrm{H}^i(Y, \nu_Y^q) \times \mathrm{H}^{N+1-i}(Y, \lambda_Y^{N-q}) &\xrightarrow{(2.2.5)} \mathbb{Z}/p\mathbb{Z}, \\ \mathrm{H}^i(Y, U^1M^n) \times \mathrm{H}^{N-i}(Y, U^1M^{n'}) &\xrightarrow{(9.1.2)} \mu \end{aligned} \quad (10.5.1)$$

($q = n' - 2$ or $n - 2$) are non-degenerate pairings of finite groups for any i by Theorems 2.2.4 and 9.1.1, respectively. We deduce Proposition 10.4.8 from these results. Let $\mathbb{U}(n)$ be an object of $D^b(Y_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$ fitting into a distinguished triangle

$$\lambda_Y^n[-n-1] \longrightarrow \mathbb{U}(n) \longrightarrow \mathbb{K}(n) \longrightarrow \lambda_Y^n[-n],$$

where the last morphism is defined as the composite $\mathbb{K}(n) \rightarrow \mathcal{H}^n(\mathbb{K}(n))[-n] \simeq FM^n[-n] \rightarrow \lambda_Y^n[-n]$ (cf. Lemma 10.4.1 (2), Theorem 3.4.2, Corollary 3.5.2). By Lemma 10.4.1 (2) and Lemma 2.1.2 (3), $\mathbb{U}(n)$ is concentrated in $[n-1, n]$ and unique up to a unique isomorphism. We have

$$\mathcal{H}^q(\mathbb{U}(n)) \simeq \begin{cases} \mu' \otimes \nu_Y^{n-2} & (q = n-1), \\ U^1M^n & (q = n). \end{cases} \quad (10.5.2)$$

For $\mathcal{K} \in D^b(Y_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$, let $\mathbb{D}(\mathcal{K})$ be as in (10.4.5). In view of Lemma 10.4.10 and the non-degeneracy of the pairings in (10.5.1), we have only to show the following:

Lemma 10.5.3. (1) *There is a unique morphism*

$$f : \mathbb{U}(n)[-1] \longrightarrow \mathbb{D}(FM^{n'}[-n']) \quad \text{in } D^+(Y_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$$

fitting into a commutative diagram with distinguished rows

$$\begin{array}{ccccccc} \mathbb{U}(n)[-1] & \longrightarrow & \mathbb{K}(n)[-1] & \longrightarrow & \lambda_Y^n[-n-1] & \longrightarrow & \mathbb{U}(n) \\ f \downarrow & & (10.4.7) \downarrow & & (-1)^n \cdot f_1 \downarrow & & f[1] \downarrow \end{array} \quad (10.5.4)$$

$$\mathbb{D}(FM^{n'}[-n']) \longrightarrow \mathbb{D}(\mathbb{K}(n')) \longrightarrow \mathbb{D}(\mu' \otimes \nu_Y^{n'-2}[-n'+1]) \longrightarrow \mathbb{D}(FM^{n'}[-n'])[1].$$

Here the lower row arises from a distinguished triangle obtained by truncation

$$\mu' \otimes \nu_Y^{n'-2}[-n'+1] \longrightarrow \mathbb{K}(n') \longrightarrow FM^{n'}[-n'] \longrightarrow \mu' \otimes \nu_Y^{n'-2}[-n'+2]$$

(cf. Lemma 10.4.1 (2)), and we have chosen the signs of the connecting morphisms (=the last arrows) of the both rows suitably. The arrow f_1 is defined as the adjoint morphism of the map $\lambda_Y^n[-n-1] \otimes^{\mathbb{L}} (\mu' \otimes \nu_Y^{n'-2})[-n'+1] \rightarrow \mu' \otimes \nu_Y^N[-N-2]$ induced by the identity map of μ' and the pairing (2.2.3).

(2) *There is a commutative diagram with distinguished rows in $D^+(Y_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$*

$$\begin{array}{ccccccc} \mu' \otimes \nu_Y^{n-2}[-n] & \rightarrow & \mathbb{U}(n)[-1] & \rightarrow & U^1M^n[-n-1] & \rightarrow & \mu' \otimes \nu_Y^{n-2}[-n+1] \\ f_2 \downarrow & & f \downarrow & & f_3 \downarrow & & f_2[1] \downarrow \end{array} \quad (10.5.5)$$

$$\mathbb{D}(\lambda_Y^{n'}[-n']) \rightarrow \mathbb{D}(FM^{n'}[-n']) \rightarrow \mathbb{D}(U^1M^{n'}[-n']) \rightarrow \mathbb{D}(\lambda_Y^{n'}[-n'])[1].$$

Here the upper row is the distinguished triangle obtained by truncation (cf. (10.5.2)), the lower row arises from the short exact sequence $0 \rightarrow U^1M^n \rightarrow FM^n \rightarrow \lambda_Y^n \rightarrow 0$, and we have chosen the signs of the connecting morphisms (=the last arrows) of the both rows suitably. The arrow f_2 is defined in a similar way as for f_1 in (1), and f_3 denotes the morphism induced by $\Theta^n[-1]$. See §8.2 for Θ^n .

To prove Lemma 10.5.3, we first show Lemma 10.5.6 below. Note that for $\mathcal{K} \in D^b(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$, $Rj_*j^*\mathcal{K}$ and $R\iota^!\mathcal{K}$ are both bounded (cf. Lemma 7.3.2). For $\mathcal{K}, \mathcal{L} \in D^b(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$, $\mathcal{K} \otimes^{\mathbb{L}} \mathcal{L}$ is bounded, because a $\mathbb{Z}/p\mathbb{Z}$ -sheaf is flat over $\mathbb{Z}/p\mathbb{Z}$.

Lemma 10.5.6. *For $\mathcal{K}, \mathcal{L} \in D^b(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$, there is a commutative diagram*

$$\begin{array}{ccc} (Rj_*j^*\mathcal{K}) \otimes^{\mathbb{L}} \mathcal{L} & \longrightarrow & Rj_*j^*(\mathcal{K} \otimes^{\mathbb{L}} \mathcal{L}) \\ \delta_{V,Y}^{\text{loc}}(\mathcal{K}) \otimes^{\mathbb{L}} \text{id} \downarrow & & \downarrow \delta_{V,Y}^{\text{loc}}(\mathcal{K} \otimes^{\mathbb{L}} \mathcal{L}) \\ (\iota_*R\iota^!\mathcal{K}[1]) \otimes^{\mathbb{L}} \mathcal{L} & \longrightarrow & \iota_*R\iota^!(\mathcal{K} \otimes^{\mathbb{L}} \mathcal{L})[1], \end{array} \quad (10.5.7)$$

where the horizontal arrows are natural product morphisms.

The commutativity of the induced diagram of cohomology sheaves of (10.5.7) would be well-known. However, we include a proof of the lemma, because we need the commutativity in the derived category to show especially Lemma 10.5.3 (2).

Proof of Lemma 10.5.6. For two complexes M^\bullet and N^\bullet , let $(M^\bullet \otimes N^\bullet)^t$ be as in Proof of Proposition 4.4.10. For a map $h^\bullet : M^\bullet \rightarrow N^\bullet$ of complexes, let $\text{Cone}(h)^\bullet$ be as in Proof of Proposition 4.3.1 and let $u_h : N^\bullet \rightarrow \text{Cone}(h)^\bullet$ be the canonical map. Let $C^b(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$ be the category of bounded complexes of $\mathbb{Z}/p\mathbb{Z}$ -sheaves on $X_{\text{ét}}$. Take an $\iota^!$ -acyclic resolution $K^\bullet \in C^b(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$ of \mathcal{K} (see the remark before Lemma 10.5.6) and a bounded complex $L^\bullet \in C^b(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$ which represents \mathcal{L} . Note that K^\bullet is a j_* -acyclic resolution of \mathcal{K} as well. We further take an injective resolution $J^\bullet \in C^+(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$ of $(K^\bullet \otimes L^\bullet)^t$. Let $f : K^\bullet \rightarrow j_*j^*K^\bullet$ and $g : J^\bullet \rightarrow j_*j^*J^\bullet$ be the canonical maps, and let $f' : (K^\bullet \otimes L^\bullet)^t \rightarrow ((j_*j^*K^\bullet) \otimes L^\bullet)^t$ be the map induced by f . Then in $D^b(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$, the diagram (10.5.7) decomposes as follows:

$$\begin{array}{ccccccc} ((j_*j^*K^\bullet) \otimes L^\bullet)^t & \xlongequal{\quad} & ((j_*j^*K^\bullet) \otimes L^\bullet)^t & \xlongequal{\quad} & ((j_*j^*K^\bullet) \otimes L^\bullet)^t & \xrightarrow{\varphi_1} & j_*j^*J^\bullet & \xlongequal{\quad} & j_*j^*J^\bullet \\ \delta_{V,Y}^{\text{loc}}(\mathcal{K}) \otimes^{\mathbb{L}} \text{id} \downarrow & & (1) \quad (u_f \otimes \text{id})^t \downarrow & & (2) \quad u_{f'} \downarrow & & (3) \quad u_g \downarrow & & (4) \quad \downarrow \delta_{V,Y}^{\text{loc}}(\mathcal{K} \otimes^{\mathbb{L}} \mathcal{L}) \\ ((\iota_*\iota^!K^\bullet[1]) \otimes L^\bullet)^t & \xrightarrow[\cong]{\varphi_2} & (\text{Cone}(f)^\bullet \otimes L^\bullet)^t & \xrightarrow[\cong]{\varphi_3} & \text{Cone}(f')^\bullet & \xrightarrow{\varphi_4} & \text{Cone}(g)^\bullet & \xleftarrow[\cong]{\varphi_5} & \iota_*\iota^!J^\bullet[1], \end{array}$$

where $\varphi_1, \varphi_2, \varphi_4$ and φ_5 are canonical maps of complexes and φ_2 and φ_5 are isomorphisms in $D^b(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$. The arrow φ_3 is defined as the natural identification of complexes, and the composite of the lower row agrees with the bottom arrow in (10.5.7). The squares (2) and (3) commute in the category of complexes, and the squares (1) and (4) commute in $D^b(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$ by the definition of connecting morphisms. Thus the diagram (10.5.7) commutes. \square

Proof of Lemma 10.5.3. There is a commutative diagram in $D^b(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$

$$\begin{array}{ccc} (Rj_*\mu_p^{\otimes n-1}[-1]) \otimes^{\mathbb{L}} \mathfrak{T}(n')_X & \longrightarrow & Rj_*\mu_p^{\otimes N+1}[-1] \xrightarrow{(4.2.1)} \iota_*\nu_Y^N[-N-2] \\ (\delta_1[-1]) \otimes^{\mathbb{L}} \text{id} \downarrow & & \delta_2[-1] \downarrow \quad \parallel \\ \iota_*R\iota^!\mathfrak{T}(n-1)_X \otimes^{\mathbb{L}} \mathfrak{T}(n')_X & \longrightarrow & \iota_*R\iota^!\mathfrak{T}(N+1)_X \xleftarrow[\cong]{-\text{Gys}_\iota^{N+1}} \iota_*\nu_Y^N[-N-2], \end{array}$$

where the left horizontal arrows are product morphisms and we wrote δ_1 and δ_2 for $\delta_{V,Y}^{\text{loc}}(\mathfrak{T}(n-1)_X)$ and $\delta_{V,Y}^{\text{loc}}(\mathfrak{T}(N+1)_X)$, respectively. The left square commutes by Lemma 10.5.6, and the right square commutes by (4.4.2). By this commutative diagram and the anti-commutativity

of (10.4.4), the square

$$\begin{array}{ccc}
 (\iota^* \mathfrak{T}(n)_X[-1]) \otimes^{\mathbb{L}} \iota^* \mathfrak{T}(n')_X & \xrightarrow{\text{product}} & \iota^* Rj_* \mu_p^{\otimes N+2}[-1] \\
 \downarrow (a_n[-1])^{\otimes \mathbb{L}} a_{n'} & & \downarrow (8.2.2) \\
 (\mathbb{K}(n)[-1]) \otimes^{\mathbb{L}} \mathbb{K}(n') & \xrightarrow{(10.4.7)} & \mu' \otimes \nu_Y^N[-N-2]
 \end{array} \tag{10.5.8}$$

commutes in $D^b(Y_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$ (cf. the diagram in Lemma 10.4.6). Now we prove Lemma 10.5.3, using a similar argument as for Lemma 10.4.6. We first show (1). Because there is no non-zero morphism from $\mathbb{U}(n)$ to $\mathbb{D}(\mu' \otimes \nu_Y^{n'-2}[-n'+1])$, it suffices to show the commutativity of the central square in (10.5.4). Our task is to show that the composite morphism

$$(\mathbb{K}(n)[-1]) \otimes^{\mathbb{L}} (\mu' \otimes \nu_Y^{n'-2})[-n'+1] \longrightarrow (\mathbb{K}(n)[-1]) \otimes^{\mathbb{L}} \mathbb{K}(n') \xrightarrow{(10.4.7)} \mu' \otimes \nu_Y^N[-N-2]$$

induces $(-1)^n \cdot f_1$, which follows from the commutativity of (10.5.8) and Lemmas 10.4.1 (2) and 2.1.1. The details are straight-forward and left to the reader. We next show (2). There are no non-zero morphisms from $\mu' \otimes \nu_Y^{n'-2}[-n+1]$ to $\mathbb{D}(U^1 M^{n'}[-n'])$, and the left square in (10.5.5) commutes by a similar argument as for (1). Hence there is a unique morphism $f_4 : U^1 M^n[-n] \rightarrow \mathbb{D}(U^1 M^{n'}[-n'-1])$ fitting into (10.5.5) (cf. Lemma 2.1.2 (2)), which necessarily agrees with f_3 by the commutativity of (10.5.8) and the construction of these maps. Thus we obtain the lemma. \square

This completes the proof of Proposition 10.4.8 and Theorems 10.1.1 and 10.1.3. \square

10.6. Consequences in the local case. We state some consequences of Theorem 10.1.1. Let the notation be as in Theorem 10.1.1 and Remark 10.1.5. Let $H_{\text{ur}}^i(V, \mu_{p^r}^{\otimes n})$ be the image of the canonical map $H^i(X, \mathfrak{T}_r(n)_X) \rightarrow H^i(V, \mu_{p^r}^{\otimes n})$.

Corollary 10.6.1. $H_{\text{ur}}^i(V, \mu_{p^r}^{\otimes n})$ and $H_{\text{ur}}^{2d-i}(V, \mu_{p^r}^{\otimes d-n})$ are exact annihilators of each other under the non-degenerate pairing (10.1.6).

Proof. By Theorem 6.1.1 and a similar argument as for Lemma 10.2.1, one can easily check that the composite map

$$H^{2d}(V, \mu_{p^r}^{\otimes d}) \xrightarrow{\delta_{V,Y}^{\text{loc}}(\mathfrak{T}_r(d)_X)} H_Y^{2d+1}(X, \mathfrak{T}_r(d)_X) \xrightarrow{\text{tr}_{(X,Y)}} \mathbb{Z}/p^r\mathbb{Z}$$

agrees with the trace map tr_V . Hence the diagram with exact rows

$$\begin{array}{ccccc}
 H^i(X, \mathfrak{T}_r(n)_X) & \longrightarrow & H^i(V, \mu_{p^r}^{\otimes n}) & \xrightarrow{\delta^{\text{loc}}} & H_Y^{i+1}(X, \mathfrak{T}_r(n)_X) \\
 (10.1.2) \downarrow \simeq & & (10.1.6) \downarrow \simeq & & (10.1.2) \downarrow \simeq \\
 H_Y^{2d+1-i}(X, \mathfrak{T}_r(d-n)_X)^* & \xrightarrow{(\delta^{\text{loc}})^*} & H^{2d-i}(V, \mu_{p^r}^{\otimes d-n})^* & \longrightarrow & H^{2d-i}(X, \mathfrak{T}_r(d-n)_X)^*
 \end{array}$$

commutes up to signs. Here the superscript $*$ means the Pontryagin dual, and the bijectivity of the left and the right vertical arrows follows from Theorem 10.1.1. Now the assertion follows from a simple diagram chase on this diagram. \square

Corollary 10.6.1 includes some non-trivial duality theorems in the local class field theory. More precisely, we have the following consequence, where $K := \text{Frac}(A)$ and $\text{Br}(C) := H^2(C, \mathbb{G}_m)$:

Corollary 10.6.2. *Let C be a proper smooth curve over K with semistable reduction. Then there is a non-degenerate pairing of finite $\mathbb{Z}/p^r\mathbb{Z}$ -modules*

$$\mathrm{Pic}(C)/p^r \times {}_{p^r}\mathrm{Br}(C) \longrightarrow \mathbb{Z}/p^r\mathbb{Z}.$$

This corollary recovers the p -adic part of the Lichtenbaum duality [Li1] for C and the Tate duality [Ta1] for the Jacobian variety of C (cf. [Sa1], p. 413). However our proof is not new, because we use Artin's proper base-change theorem for Brauer groups.

Proof. Take a proper flat regular model X over B of C with semistable reduction. Let Y be the closed fiber of X/B , and define $\mathrm{Br}(X) := \mathrm{H}^2(X, \mathbb{G}_m)$. There is a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Pic}(X)/p^r & \longrightarrow & \mathrm{H}^2(X, \mathfrak{I}_r(1)_X) & \longrightarrow & {}_{p^r}\mathrm{Br}(X) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Pic}(C)/p^r & \longrightarrow & \mathrm{H}^2(C, \mu_{p^r}) & \longrightarrow & {}_{p^r}\mathrm{Br}(C) \longrightarrow 0 \end{array}$$

See (4.5.3) for the upper row. In view of Corollary 10.6.1, our task is to show $\mathrm{Pic}(C)/p^r = \mathrm{H}_{\mathrm{ur}}^2(C, \mu_{p^r})$. Because the left vertical arrow is surjective, it is enough to show $\mathrm{Br}(X) = 0$. Now by Artin's proper base-change theorem: $\mathrm{Br}(X) \simeq \mathrm{H}^2(Y, \mathbb{G}_m)$ (cf. [Gt], III.3.1), we are reduced to showing $\mathrm{H}^2(Y, \mathbb{G}_m) = 0$, which follows from the classical Hasse principle for the function fields of Y (cf. [Sa1], §3, p. 388). Thus we obtain the corollary. \square

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APPENDIX A. An application of p -adic Hodge theory to the coniveau filtration

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A.1. Every 'suitable' cohomology theory H^* for schemes, for example étale cohomology, is naturally accompanied with an important filtration called *coniveau filtration*, which is defined as follows:

$$\begin{aligned} N^r \mathrm{H}^i(X) &:= \mathrm{Im} \left(\varinjlim_{Z \in X^{\geq r}} \mathrm{H}_Z^i(X) \longrightarrow \mathrm{H}^i(X) \right) \\ &= \mathrm{Ker} \left(\mathrm{H}^i(X) \longrightarrow \varinjlim_{Z \in X^{\geq r}} \mathrm{H}^i(X - Z) \right), \end{aligned}$$

where $\mathrm{H}_Z^*(X)$ denote cohomology groups with support in Z and we put

$$X^{\geq r} = \{Z \subset X \mid \text{closed in } X \text{ and } \mathrm{codim}_X(Z) \geq r\}$$

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for non-negative integer r . This filtration, built into any cohomology group, is intimately related to algebraic cycles and often enables us to control their behavior by various cohomological tools, although the filtration *per se* has not been well-understood yet. The aim of this appendix is to analyze this interesting filtration on étale cohomology groups by means of p -adic Hodge theory. More precisely, we give an upper bound of it, assuming that X is a variety over a ‘ p -adic’ field.

A.1.1. To state our results more precisely, we fix the setting. Let A be a henselian discrete valuation ring whose fraction field K has characteristic 0 and whose residue field k is perfect of characteristic $p > 0$. Consider the following diagram of schemes:

$$\begin{array}{ccccc} Y & \xrightarrow{i} & \mathfrak{X} & \xleftarrow{j} & X \\ \downarrow & & \square & & \downarrow \\ \text{Spec}(k) & \longrightarrow & \text{Spec}(A) & \longleftarrow & \text{Spec}(K), \end{array}$$

where the vertical arrows are proper and flat, and both squares are cartesian. We assume that \mathfrak{X} is a regular semistable family over A , i.e., \mathfrak{X} is regular, X is smooth over K and Y is a reduced divisor on \mathfrak{X} with normal crossings. Fix an algebraic closure \overline{K} of K , let \overline{A} be the integral closure of A in \overline{K} and let \overline{k} be its residue field. We denote $Y \otimes_k \overline{k}$, $\mathfrak{X} \otimes_A \overline{A}$ and $X \otimes_K \overline{K}$ by \overline{Y} , $\overline{\mathfrak{X}}$ and \overline{X} , respectively, and write \overline{i} and \overline{j} for the canonical maps $\overline{Y} \rightarrow \overline{\mathfrak{X}}$ and $\overline{X} \rightarrow \overline{\mathfrak{X}}$, respectively. For simplicity we always suppose that \overline{X} and $\overline{\mathfrak{X}}$ are connected. Throughout this appendix, we use the general notation fixed in §§1.6–1.7 of the main body.

A.1.2. By standard theorems in étale cohomology theory, we have spectral sequences

$$\begin{aligned} E_2^{a,b} &= H^a(Y, i^* R^b j_* \mathbb{Z}/p^n(m)) \implies H^{a+b}(X, \mathbb{Z}/p^n(m)), \\ E_2^{a,b} &= H^a(\overline{Y}, \overline{i}^* R^b \overline{j}_* \mathbb{Z}/p^n) \implies H^{a+b}(\overline{X}, \mathbb{Z}/p^n), \end{aligned}$$

where $\mathbb{Z}/p^n(m)$ denotes the sheaf $\mu_{p^n}^{\otimes m}$ on $X_{\text{ét}}$. We define the filtration $F^\bullet \subset H^q(X, \mathbb{Z}/p^n(m))$ as that induced by the former spectral sequence. Alternatively, one can define

$$F^r H^q(X, \mathbb{Z}/p^n(m)) := \text{Im} (H^q(\mathfrak{X}, \tau_{\leq q-r} Rj_* \mathbb{Z}/p^n(m)) \rightarrow H^q(X, \mathbb{Z}/p^n(m))).$$

Now we have two filtrations N^\bullet and F^\bullet on $H^q(X, \mathbb{Z}/p^n(m))$. One defines the filtrations N^\bullet and F^\bullet on $H^i(\overline{X}, \mathbb{Z}/p^n)$ as well in the same way.

A.1.3. Our results are stated as follows.

Theorem A.1.4. *Let r, s and n be non-negative integers with $0 \leq r \leq s/2$. Then*

$$N^r H^s(X, \mathbb{Z}/p^n(s-r)) \subset F^r H^s(X, \mathbb{Z}/p^n(s-r)).$$

Theorem A.1.5. *Let r, s and n be non-negative integers with $0 \leq r \leq s/2$. Then*

$$N^r H^s(\overline{X}, \mathbb{Z}/p^n) \subset F^r H^s(\overline{X}, \mathbb{Z}/p^n).$$

Remark A.1.6. *If r is outside of this interval, these assertions are straight-forward by coniveau spectral sequences (cf. [BO]).*

A.1.7. If $r = 1$, $s = 3$ and \mathfrak{X} is smooth, then Theorem A.1.4 is originally due to Langer and Saito ([LS], 5.4). Their proof is K -theoretic and reduces the problem to semi-purity of cohomology groups with coefficients in \mathcal{K}_2 -sheaves. On the other hand, our proof is p -adic Hodge theoretic, i.e., we will reduce the problem to semi-purity of cohomology groups with coefficients in étale sheaves of p -adic vanishing cycles.

A.1.8. The filtration F^\bullet is highly non-trivial in the p -adic coefficients case, in contrast with the ' ℓ -adic coefficients' case, where for instance in the good reduction case, the corresponding filtration is trivial. In fact, as an application of Theorem A.1.5 we will prove the following:

Corollary A.1.9. *Let s, r and n be non-negative integers with $0 \leq r \leq s/2$. Assume that \mathfrak{X} is ordinary, i.e., $H^a(\overline{Y}, \mathcal{B}_{\overline{Y}}^b) = 0$ for all a and b (see Theorem 3.3.7 for $\mathcal{B}_{\overline{Y}}^b$). Then we have*

$$\text{length}_{\mathbb{Z}/p^n} N^r H^s(\overline{X}, \mathbb{Z}/p^n) \leq \sum_{r \leq a \leq s} \text{length}_{W_n(\overline{k})} H^a(\overline{Y}, W_n \omega_{\overline{Y}}^{s-a}),$$

where $W_n \omega_{\overline{Y}}^\bullet$ denotes the de Rham-Witt complex defined in [Hy1].

A.1.10. Bloch and Esnault [BE] proved that $\Gamma(Y, \Omega_Y^m) \neq 0 \Rightarrow N^1 H^m(\overline{X}, \mathbb{Z}/p) \neq H^m(\overline{X}, \mathbb{Z}/p)$, assuming that X has ordinary good reduction and that the spectral sequence

$$E_2^{a,b} = H^a(\overline{Y}, \overline{i}^* R^b j_* \mathbb{Z}/p) \Longrightarrow H^{a+b}(\overline{X}, \mathbb{Z}/p)$$

degenerates at E_2 -terms. Corollary A.1.9 recovers and generalizes this fact.

A.1.11. We will prove Theorems A.1.4, A.1.5 and Corollary A.1.9 in §A.2, §A.3 and §A.4 below, respectively.

A.2. Proof of Theorem A.1.4. We first reduce Theorem A.1.4 to Lemma A.2.2 below. For $Z \in X^{\geq r}$, let \mathfrak{Z} be the closure of Z in \mathfrak{X} . There is a commutative diagram

$$\begin{array}{ccc} H_Z^s(X, \mathbb{Z}/p^n(s-r)) & \longrightarrow & H^s(X, \mathbb{Z}/p^n(s-r)) \\ \uparrow & & \uparrow \\ H_{\mathfrak{Z}}^s(\mathfrak{X}, \tau_{\leq s-r} Rj_* \mathbb{Z}/p^n(s-r)) & \longrightarrow & H^s(\mathfrak{X}, \tau_{\leq s-r} Rj_* \mathbb{Z}/p^n(s-r)). \end{array} \quad (\text{A.2.1})$$

Put $\mathcal{A} := Rj_* \mathbb{Z}/p^n(s-r)$. Since $H_{\mathfrak{Z}}^i(\mathfrak{X}, \mathcal{A}) \simeq H_Z^i(X, \mathbb{Z}/p^n(s-r))$, there is a long exact sequence

$$\cdots \rightarrow H_{\mathfrak{Z}}^s(\mathfrak{X}, \tau_{\leq s-r} \mathcal{A}) \rightarrow H_Z^s(X, \mathbb{Z}/p^n(s-r)) \rightarrow H_{\mathfrak{Z}}^s(\mathfrak{X}, \tau_{\geq s-r+1} \mathcal{A}) \rightarrow H_{\mathfrak{Z}}^{s+1}(\mathfrak{X}, \tau_{\leq s-r} \mathcal{A}) \rightarrow \cdots$$

Now Theorem A.1.4 is reduced to

Lemma A.2.2. $H_{\mathfrak{Z}}^s(\mathfrak{X}, \tau_{\geq s-r+1} \mathcal{A}) = 0$ for any $Z \in X^{\geq r}$ and any $r, s \in \mathbb{Z}$ as in the theorem.

Indeed, by this lemma the left vertical arrow in (A.2.1) is surjective, and Theorem A.1.4 follows from a diagram chase on (A.2.1).

A.2.3. The rest of this subsection is devoted to Lemma A.2.2. The following sublemma follows from a simple argument on flatness, whose proof is left to the reader:

Lemma A.2.4. For $Z \in X^{\geq r}$, put $Z_p := \mathfrak{Z} \otimes_A k$ with \mathfrak{Z} the closure of $Z \subset \mathfrak{X}$. Then $Z_p \in Y^{\geq r}$.

A.2.5. Let Z, \mathfrak{Z} and Z_p be as in Lemma A.2.4. For $q \in \mathbb{Z}$, we put $\mathcal{C}_n(q) := i^* Rj_* \mathbb{Z}/p^n(q)$. Since $R^m j_* \mathbb{Z}/p^n(q) \simeq i_* i^* R^m j_* \mathbb{Z}/p^n(q)$ for $m > 0$, we have

$$H_{\mathfrak{Z}}^s(\mathfrak{X}, \tau_{\geq s-r+1} Rj_* \mathbb{Z}/p^n(s-r)) \simeq H_{Z_p}^s(Y, \tau_{\geq s-r+1} \mathcal{C}_n(s-r)).$$

We prove that the right hand side is zero. There are distinguished triangles

$$\begin{aligned} \tau_{\geq s-r+1} \mathcal{C}_{n-1}(s-r) &\longrightarrow \tau_{\geq s-r+1} \mathcal{C}_n(s-r) \longrightarrow \tau_{\geq s-r+1} \mathcal{C}_1(s-r), \\ i^* R^m j_* \mathbb{Z}/p(s-r)[-m] &\longrightarrow \tau_{\geq m} \mathcal{C}_1(s-r) \longrightarrow \tau_{\geq m+1} \mathcal{C}_1(s-r) \end{aligned}$$

in $D^+(Y_{\text{ét}})$, where the former triangle is obtained from the short exact sequence

$$0 \longrightarrow \mathbb{Z}/p^{n-1}(s-r) \longrightarrow \mathbb{Z}/p^n(s-r) \longrightarrow \mathbb{Z}/p(s-r) \longrightarrow 0$$

and, in fact, distinguished because the map $i^*R^{s-r}j_*\mathbb{Z}/p^n(s-r) \rightarrow i^*R^{s-r}j_*\mathbb{Z}/p(s-r)$ is surjective (cf. Theorem 3.3.7 (1) in the main body). By these distinguished triangles and Lemma A.2.4, Lemma A.2.2 is reduced to the following semi-purity result:

Theorem A.2.6 (Semi-purity). *For any $Z_p \in Y^{\geq r}$ and any a, m, q with $a \leq r-1$, we have*

$$H_{Z_p}^a(Y, i^*R^m j_*\mathbb{Z}/p(q)) = 0.$$

A.2.7. By a standard norm argument, Theorem A.2.6 is reduced to the case where K contains primitive p -th roots of unity. In this case, we have $\mathbb{Z}/p(q) \simeq \mathbb{Z}/p(m)$ on X , and it suffices to consider the case $m = q$. Hence Theorem A.2.6 is reduced to the vanishing

$$H_{Z_p}^a(Y, \omega_{Y, \log}^b) = H_{Z_p}^a(Y, \omega_Y^b/\mathcal{B}_Y^b) = H_{Z_p}^a(Y, \mathcal{B}_Y^b) = 0 \text{ for any } a, b \text{ with } a \leq r-1$$

by the Bloch-Kato-Hyodo theorem (cf. Theorem 3.3.7). By a similar argument as for [Mi1], 1.7, the sheaves $\omega_Y^b/\mathcal{B}_Y^b$ and \mathcal{B}_Y^b are locally free $(\mathcal{O}_Y)^p$ -modules of finite rank. By [Hy1], (1.5.1), there is an exact sequence

$$0 \longrightarrow \omega_{Y, \log}^b \longrightarrow \omega_Y^b \xrightarrow{1-C^{-1}} \omega_Y^b/\mathcal{B}_Y^b \longrightarrow 0.$$

Therefore we are further reduced to the following lemma:

Lemma A.2.8. *Let \mathcal{F} be a locally free $(\mathcal{O}_Y)^p$ -module of finite rank. Then $H_{Z_p}^a(Y, \mathcal{F})$ is zero for any $a \leq r-1$.*

A.2.9. Since the absolute Frobenius morphism $F_Y : Y \rightarrow Y$ is finite, $H_{Z_p}^*(Y, \mathcal{F})$ is isomorphic to $H_{Z_p}^*(Y, F_{Y*}(\mathcal{F}))$. Hence we are reduced to the case that \mathcal{F} is a locally free \mathcal{O}_Y -module of finite rank. Take an étale covering $\{U_i\}_{i \in I}$ of Y which trivializes \mathcal{F} . By a local-global spectral sequence ([SGA4], V.6.4 (3)), it suffices to prove that

$$\mathcal{H}_{Z_p \times_Y U_i}^a(U_i, \mathcal{O}_{U_i}) = 0 \text{ for any } a \leq r-1 \text{ and any } i \in I,$$

where $\mathcal{H}_{Z_p}^*(X, \bullet)$ denotes the sheaf of cohomology groups with support ([SGA4], V.6). One can easily check this triviality by the comparison theorem on Zariski and étale cohomology groups for coherent sheaves ([SGA4], VII.4.3) and standard facts on depth (see, e.g., [Ha2], (3.8)), noting that Y and U_i ($i \in I$) are Cohen-Macaulay ([AK], VII.4.8). This completes the proof of Claim, Lemma A.2.2 and Theorem A.1.4. \square

Corollary A.2.11. *Let q, r and s be integers with $0 \leq r \leq s/2$. Then*

$$N^r H^s(X, \mathbb{Z}/p(q)) \subset F^r H^s(X, \mathbb{Z}/p(q)).$$

Proof. Indeed the restriction on Tate twists is unnecessary in this $n = 1$ case by a standard norm argument. \square

A.3. Proof of Theorem A.1.5. Because we do not need to care about Tate twists on \overline{X} , the proof becomes much simpler. As in the proof of Theorem A.1.4, it is enough to show that

$$H_{Z_p}^s(\overline{Y}, \tau_{\geq s-r+1} \overline{i}^* R \overline{j}_* \mathbb{Z}/p^n) = 0$$

for arbitrary $Z_p \in \overline{Y}^{\geq r}$. Since $\mathbb{Z}/p^n(1) \simeq \mathbb{Z}/p^n$ on \overline{X} for any n , we have an exact sequence

$$0 \longrightarrow \overline{i}^* R^q \overline{j}_* \mathbb{Z}/p^{n-1}(q) \longrightarrow \overline{i}^* R^q \overline{j}_* \mathbb{Z}/p^n(q) \longrightarrow \overline{i}^* R^q \overline{j}_* \mathbb{Z}/p(q) \longrightarrow 0$$

(cf. [BK], p. 142, line 9 and [Hy1], (1.11.1)). Hence it suffices to show that

$$H_{Z_p}^a(\overline{Y}, \overline{i}^* R^q \overline{j}_* \mathbb{Z}/p(q)) = 0$$

for any a, q with $0 \leq a \leq r-1$. Take a finite extension k_0/k over which $Z_p \subset \overline{Y}$ is defined, and take a closed subset $Z_{p,0}$ of $Y \otimes_k k_0$ such that $Z_{p,0} \otimes_{k_0} \overline{k} \simeq Z_p$ under the isomorphism $(Y \otimes_k k_0) \otimes_{k_0} \overline{k} \simeq \overline{Y}$. Now let K' be a finite extension of K whose residue field k' contains k_0 , and let A' be the integer ring of K' . By [SGA4], VII.5.8, our task is to show that

$$H_{Z_{p,0} \otimes_{k_0} k'}^a(Y \otimes_k k', i'^* R^q j'_* \mathbb{Z}/p(q)) = 0$$

for any a, q with $0 \leq a \leq r-1$, where i' (resp. j') denotes the morphism $Y \otimes_k k' \rightarrow \mathfrak{X} \otimes_A A'$ (resp. $X \otimes_K K' \rightarrow \mathfrak{X} \otimes_A A'$). This assertion follows from the same argument as in Theorem A.2.6. Thus we obtain Theorem A.1.5. \square

A.4. Proof of Corollary A.1.9. Let $W_n \omega_{\overline{Y}, \log}^*$ be the modified logarithmic Hodge-Witt sheaves (cf. §3.3 of the main body). The ordinarity assumption implies that

$$H^a(\overline{Y}, \overline{i}^* R^b \overline{j}_* \mathbb{Z}/p^n) \simeq H^a(\overline{Y}, W_n \omega_{\overline{Y}, \log}^b)$$

([BK], (9.2), [Hy1], (1.10)) and that

$$H^a(\overline{Y}, W_n \omega_{\overline{Y}, \log}^b) \otimes_{\mathbb{Z}/p^n} W_n(\overline{k}) \simeq H^a(\overline{Y}, W_n \omega_{\overline{Y}}^b)$$

([BK], (7.3), [Il2], (2.3)). Hence Corollary A.1.9 follows from Theorem A.1.5. \square

Remark A.4.1. *The theorem of Bloch-Esnault in [BE], (1.2) is a direct consequence of Corollary A.1.9 with $r = n = 1$ and \mathfrak{X}/A smooth. They derived some interesting results on algebraic cycles from this case. Therefore Corollary A.1.9 would provide us with much more information.*

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