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Dieudonné Module Structures for Ungraded and Periodically Graded Hopf Rings

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Abstract Dieudonné theory provides functors between categories of Hopf algebras and categories of modules over rings. These functors define equivalences of such theories, and thus allow us to represent some Hopf algebras by modules over specific rings, which in some cases can ease calculations one needs to perform in the context of Hopf algebras. The theory can be extended to accommodate Hopf rings, which are Hopf algebras with additional structure. This paper reviews the construction of such equivalences of categories in the case of graded connected Hopf algebras (following Ravenel (1975) and Schoeller (Manuscripta Math **3**, 133–155, 1970)) and Hopf rings (following Goerss (1999)); of ungraded connected Hopf algebras (from Bousfield (Math Z **223**, 483–519, 1996)); and of periodically graded connected Hopf algebras (from Sadofsky and Wilson (1998)). We also present an alternative proof of Goerss's equivalence in the graded connected Hopf ring case (Goerss 1999), and complete the picture by constructing the functors for ungraded and periodically graded Hopf rings. Finally, we analyze the unconnected case, focusing on Hopf algebras and rings that are either group-like in degree zero or geometric-like.

Keywords Hopf algebras · Hopf rings · Dieudonné modules · Homotopy theory

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1 Introduction

Let $\mathcal{H}A_*$ be the category of graded, connected, bicommutative Hopf algebras over $\mathbb{F}_p$. Given one such Hopf algebra H, we define its Dieudonné module by

$$D_*H = \{D_nH\}_{n\geq 1} = \{\mathrm{Hom}_{\mathcal{H}A_*}(H(n), H)\}_{n\geq 1}$$

Here, $H(n)$ is the Hopf algebra $\mathbb{F}_p[x_0, x_1, \cdots, x_k]$ (with $n = p^k m$, for m and a prime p, both fixed from the start and such that $(p, m) = 1$) whose coproduct makes the Witt polynomials primitive.

The Frobenius and the Verschiebung on the $H(n)$ induce maps F and V on D_*H that satisfy $FV = VF = p$ and, for any $x \in D_*H$, $V^n x = 0$ for some $n \geq 0$.

We can thus consider the category of Dieudonné modules $\mathcal{D}M_*$ consisting of graded modules over the ring $R = \mathbb{F}_p[F, V]/(FV - p)$.

Ravenel [11] and Schoeller [15] have proved that the functor $D_* : \mathcal{H}A_* \to \mathcal{D}M_*$ above provides an equivalence of categories. Goerss [6] proved that, if we consider the category $\mathcal{H}R_*$ of Hopf rings $H = \{H_n\}_{n\in\mathbb{Z}}$ (where each H_n is in $\mathcal{H}A_*$), we can also get an equivalence to a category of Dieudonné rings (which are Dieudonné modules with additional structure).

A Hopf ring in $\mathcal{H}R_*$ has by definition products $\circ_{ij} : H_i \otimes H_j \to H_{i+j}$. These induce products $\circ'_{ij} : D_*H_i \otimes D_*H_j \to D_*H_{i+j}$ in the corresponding Dieudonné modules. These are defined based on the existence of tensor products $H(m) \otimes H(n) \to H(m) \boxtimes H(n)$. The induced products satisfy precise relations with F and V.

We also have that any product $\circ_{ij} : M_i \otimes M_j \to M_{i+j}$ in Dieudonné modules induces a product $\circ'_{ij} : U_*(H_i) \otimes U_*(H_j) \to U_*(H_{i+j})$ in Hopf algebras (Here, U_* is the inverse functor to D_*).

Given a Hopf ring $H = \{H_n\}_{n\in\mathbb{Z}}$, we prove that $U_*D_*H = \{U_*D_*H_n\}_{n\in\mathbb{Z}}$ (whose products are the composition of the two induced above) is a Hopf ring equivalent to H. A similar reasoning is used to prove that, given a Dieudonné ring $M = \{M_n\}_{n\in\mathbb{Z}}$, the Dieudonné ring $D_*U_*H = \{D_*U_*H_n\}_{n\in\mathbb{Z}}$ (again,with composition of induced products as products) is equivalent to M.

We then give proof of equivalence of categories in other, different settings; in each case, a suitable category of Dieudonné rings is presented for the particular category of Hopf rings at hand (This includes the definition of products in Dieudonné modules and also their relations to V and F). Namely, we deal with $\mathcal{H}R^0_*$, the category of graded Hopf rings that are group-like in degree zero; with $\mathcal{H}R$, the category of ungraded connected Hopf rings (we refer to the work on ungraded connected Hopf algebras in [2]), and $\mathcal{H}R^0$, the category of ungraded geometric-like Hopf rings (i. e., with “degree-zero” part a group ring); with $\mathcal{H}R_m$, the category of $2m$-graded connected Hopf rings ([13] dealt with the Hopf algebra case), and $\mathcal{H}R^0_m$, the same with geometric-like Hopf rings. In each case, the induced products are constructed and shown to produce equivalence on any Hopf ring and Dieudonné ring.

2 The Basics on Hopf Rings

This paper deals with different categories of Hopf algebras. An example of Hopf algebras is $\mathbb{Z}_p[x_0, x_1, \cdots]$, the free commutative algebra over the p-adic integers on the indeterminates $x_0, x_1, \cdots$ (for a fixed prime p), together with a suitable co-product. It is usual to define such a co-product in order to make the Witt polynomials $\omega_n(x) = x_0^{p^n} + px_1^{p^{n-1}} + \cdots + p^n x_n$ primitive (Such structure is unique—see [6]). This is the structure we shall be considering in this work.

We can also consider just the algebra $\mathbb{Z}_p[x_0, x_1, \cdots, x_k]$, restricting the above co-product to this finitely generated algebra and thus obtaining the *Hopf algebra* $CW(k) = \mathbb{Z}_p[x_0, x_1, \cdots, x_k]$.

In the graded case, give each x_i degree $p^i m$ for some fixed m and define $CW_m(k)$ to be the graded Hopf algebra corresponding to $CW(k)$.

Next we want to consider Hopf algebras over $\mathbb{F}_p$ with characteristic p. Define Hopf algebras $H(k) = \mathbb{F}_p \otimes CW(k) = \mathbb{F}_p[x_0, x_1, \cdots, x_k]$. In the graded case, write $H(n) = \mathbb{F}_p[x_0, x_1, \cdots, x_k]$, where $n = p^k m$ for $(p, m) = 1$ and each x_i has degree $p^i m$. We will make no other distinctions in the notation for the graded and ungraded case and it will be clear from the context whether we mean $H(k)$ to have an associated grading or not. We define a morphism $v : H(n) = \mathbb{F}_p[x_0, x_1, \cdots, x_k] \to H(pn) = \mathbb{F}_p[x_0, x_1, \cdots, x_k, x_{k+1}]$ by inclusion and also a map $f : H(pn) \to H(n)$ satisfying $fv = [p]$ and $vf = [p]$, where $[p]$ is p-times the identity map (On $H(n) = \mathbb{F}_p[x_0, x_1, \cdots, x_k]$ for the first and on $H(pn)$ for the second. This map f does $f(x_i) = x_{i-1}^p$ (One has $[p](x_i) \equiv x_{i-1}^p \pmod p$ [6]).

Two automorphisms of a co-commutative Hopf algebra H over $\mathbb{F}_p$ are of special interest, namely the *Frobenius* $F : H \to H$, taking an element x to the element x^p, and the *Verschiebung* $V : H \to H$, which is the dual to the Frobenius in the dual algebra. This last one can be described as follows: If an element $x \in H$ has p-fold co-product

$$\Psi^p(x) = \sum x' \otimes x' \otimes \cdots \otimes x' + \sum_{\text{not all y equal}} c \left(y' \otimes y'' \otimes \cdots \otimes y^{p+1} \right)$$

(where we consider that the second summand is written in terms of a basis) then the Verschiebung on x is $V(x) = \sum x'$.

Hopf rings are abelian graded ring objects with unit in the category of co-algebras $\mathcal{CA}_k$; that is, collections $\{X_k\}_{k\in\mathbb{Z}}$ of Hopf algebras together with a multiplication $\circ = \circ_{ij} \in \mathrm{Hom}_{\mathcal{CA}_k}(X_i \otimes_k X_j, X_{i+j})$ and a multiplicative unit $e \in \mathrm{Hom}_{\mathcal{CA}_k}(k, X_0)$ that satisfy corresponding commutative diagrams (These diagrams declare properties of associativity and commutativity of the new operation, plus distributivity, multiplication by the unit and by zero). Hopf rings appear frequently in applications, and can arise for example as homology modules of certain spaces for certain generalized homology theories (see [12]).

3 Bilinear Maps and Tensor Products

Let $\mathcal{C}$ be a category with finite products and $\mathcal{A} \subseteq \mathcal{C}$ a subcategory of abelian objects. Since any $A \in \mathcal{A}$ is an abelian object, $F_A = \mathrm{Hom}_{\mathcal{C}}(\cdot, A)$ is a functor to abelian groups.

If A, B and C are objects in $\mathcal{A}$, a morphism $\varphi : A \times B \to C$ in $\mathcal{C}$ is a *bilinear map* if for all $X \in \mathcal{C}$ the induced map

$$F_A(X) \times F_B(X) \to F_C(X)$$

is a natural bilinear map of abelian groups. An initial bilinear map $\epsilon : A \times B \to A \boxtimes B$ in $\mathcal{A}$ is called a *tensor product* of A and B.

The following theorem gives conditions for the existence of tensor products. Note that if $A \boxtimes B$ exists, it is unique up to isomorphism in $\mathcal{A}$.

Theorem 3.1 [8] Assume that the categories $\mathcal{A}$ and $\mathcal{C}$ satisfy:

(a) both $\mathcal{C}$ and $\mathcal{A}$ have all limits and colimits;
(b) the forgetful functor $\mathcal{A} \to \mathcal{C}$ has a left adjoint S.

Then any two objects $A, B \in \mathcal{A}$ have a tensor product $A \boxtimes B$ in $\mathcal{A}$.

If $\mathcal{C}$ is a category of coalgebras over a commutative ring k and $\mathcal{A} \subseteq \mathcal{C}$ is a category of bicommutative Hopf algebras, a bilinear map becomes a morphism of coalgebras $\varphi : H_1 \otimes_k H_2 \to K$ between *Hopf algebras* H_1, H_2 and K. This morphism has to reduce to a bilinear map of abelian groups once we apply the functor $\mathrm{Hom}_{\mathcal{C}}(\cdot, X)$ for any *coalgebra* X. In these cases, we have tensor products $H_1 \otimes H_2 \to H_1 \boxtimes H_2$ [6].

4 Dieudonné Theory for Graded Hopf Algebras and Rings

It is important to obtain efficient computable ways of dealing with the Hopf ring structures that appear in applications. One such way is to devise categories equivalent to those of Hopf rings, where such calculations can be more easily performed.

We start with the case of graded Hopf algebras, and we will consider only bicommutative ones, that is, those for which the product is commutative and the co-product co-commutative. They are said to be *connected* if $H_0 = \mathbb{F}_p$. Call $\mathcal{H}A_*$ the category of graded, connected, bicommutative Hopf algebras over $\mathbb{F}_p$.

The previous morphisms v and f on $\mathbb{F}_p[x_0, x_1, \cdots, x_k]$ allow us to define *Dieudonné Modules* for each of these Hopf algebras.

Given a $H \in \mathcal{H}A_*$, its *Dieudonné module* is the graded $\mathbb{Z}_p$-module

$$\{D_n H\}_{n\geq 1} = \{\mathrm{Hom}_{\mathcal{H}A_*}(H(n), H)\}_{n\geq 1}$$

together with homomorphisms $F : D_n H \to D_{pn} H$ and $V : D_{pn} H \to D_n H$ which come from the previous maps f and v by composition on the left:

$$H(pn) \xrightarrow{f} H(n) \xrightarrow{\varphi} H, \qquad F(\varphi) : H(pn) \to H$$

$$H(n) \xrightarrow{v} H(pn) \xrightarrow{\varphi} H, \qquad V(\varphi) : H(n) \to H$$

These homomorphisms reflect, in Dieudonné modules, the Verschiebung and the Frobenius defined on Hopf algebras.

We have $VF = FV = p$, where p stands for p-times the identity map. Also, if $n = p^k m$ with $(p, m) = 1$, then the order of the identity map in $\mathrm{Hom}_{\mathcal{H}A_*}(H(n), H(n))$ is p^{k+1}, and so $p^{k+1} D_n H = 0$. This suggests the definition of a category of Dieudonné modules, namely $\mathcal{D}M_*$, of graded modules M together with maps V, dividing degree by p, and F, multiplying degree by p, such that $VF = FV = p$. Given any x in one of those modules we have $V^n x = 0$ for some $n \geq 0$. This category could also have been defined as the category of graded modules over the ring $R = \mathbb{F}_p[F, V]/(FV - p)$, where we put $\deg(F) = 1$ and $\deg(V) = -1$ and define $\deg(ax) = p^{\deg(a)}\deg(x)$ for $a \in R$ and $x \in M$ (or zero, if this calculation provides a fraction).

We have defined a functor $D_* : \mathcal{HA}_* \to \mathcal{DM}_*$, which provides the following equivalence of categories:

Theorem 4.1 [11, 15] The above functor D_* has a right adjoint $U_* : \mathcal{DM}_* \to \mathcal{HA}_*$, and the pair (D_*, U_*) forms an equivalence of categories.

This theorem is an instance of a more general result: an abelian category with a set of small projective generators is equivalent to a category of modules over some ring [3, 4].

Next we want to deal with Hopf *rings*. It will be shown that we can also get in this case an equivalent category of modules over some ring, which will be the previous Dieudonné modules with some additional structure.

Let then $H = \{H_n\}_{n\in\mathbb{Z}}$ be a Hopf ring in this case; that is, a graded ring object in the category $\mathcal{CA}_*$. We know from the previous result that there is a corresponding Dieudonné module for each H_n, and thus we get an algebraic object $\{D_*H_n\}_{n\in\mathbb{Z}}$. We are interested in determining how the ring structure in H carries over to $\{D_*H_n\}_{n\in\mathbb{Z}}$ and also how this structure relates to the various homomorphisms V and F in the Dieudonné modules.

Recall that $CW_m(k) = \mathbb{Z}_p[x_0, \ldots, x_k]$, where $n = p^k m$ with $(p, m) = 1$, and that $H(n) = \mathbb{F}_p \otimes CW_m(k) = \mathbb{F}_p[x_0, \ldots, x_k]$. Define $G(n) = QCW_m(k)$, the indecomposables of $CW_m(k)$. We can view $G(n) \otimes G(n')$ as a $\mathbb{Z}_p[V]$-module for each n and n'. We have the following result.

Proposition 4.2 [6] There exists an isomorphism

$$R \otimes_{\mathbb{Z}_p[V]} (G(n) \otimes G(n')) \longrightarrow D_*(H(n) \boxtimes H(n'))$$

where $R = \mathbb{F}_p[F, V]/(FV - p)$.

This isomorphism takes the element $1 \otimes (x_k \otimes x_{k'}) \in R \otimes_{\mathbb{Z}_p[V]} (G(n) \otimes G(n'))$ into an element $\iota_n \otimes \iota_{n'}$ in $D_{n+n'}(H(n) \boxtimes H(n')) = \mathrm{Hom}_{\mathcal{HA}_*}(H(n+n'), H(n) \boxtimes H(n'))$. This element gives to $x_{k+k'}$ the value $\varphi(x_k \otimes x_{k'})$, where $\varphi : H(n) \otimes H(n') \to H(n) \boxtimes H(n')$ is the tensor product map, and commutes with Verschiebungs (details are in [6] - In particular, if $\mathcal{M}_V$ is the category of positively graded $\mathbb{Z}_p[V]$ modules, there is a general homomorphism $(QH)_n \to \mathrm{Hom}_{\mathcal{M}_V}(H(n), QH)$ for any H that is a graded connected torsion-free Hopf algebra over $\mathbb{Z}_p$ that has a lift of the Verschiebung. This homomorphism takes x_k to a map φ with $\varphi(x_k) = x_k$ that commutes with V. Also, the universal map $H \otimes K \to H \boxtimes K$ induces an isomorphism $QH \otimes QK \to Q(H \boxtimes K)$ in $\mathcal{M}_V$). The $\iota_n \otimes \iota_{n'}$, for various values of n and n', will allow us to construct ring structures in Dieudonné modules.

Consider then a bilinear pairing $\circ_{ij} : H_i \otimes H_j \to H_{i+j}$ that is given by the ring structure of H. We can get an induced pairing $D_m H_i \times D_n H_j \to D_{m+n} H_{i+j}$ as follows. Given $x \in D_m H_i = \mathrm{Hom}_{\mathcal{HA}_*}(H(m), H_i)$ and $y \in D_n H_j = \mathrm{Hom}_{\mathcal{HA}_*}(H(n), H_j)$, consider the obvious map $x \otimes y : H(m) \otimes H(n) \to H_i \otimes H_j$. The composition

$$H(m) \otimes H(n) \xrightarrow{x\otimes y} H_i \otimes H_j \longrightarrow H_{i+j}$$

is a bilinear map, so by definition of tensor product we can get a unique map of Hopf algebras $g : H(m) \boxtimes H(n) \to H_{i+j}$ that makes the following diagram commute as coalgebras:

$$\begin{array}{ccc} H(m) \otimes H(n) & \xrightarrow{\varphi} & H(m) \boxtimes H(n) \\ {\scriptstyle x\otimes y}\downarrow & & \downarrow{\scriptstyle g} \\ H_i \otimes H_j & \xrightarrow{\circ_{ij}} & H_{i+j} \end{array}$$

(here the top map is the one given in the definition of tensor product $\boxtimes$).

Note that this homomorphism depends on the original $x \in D_m H_i$ and $y \in D_n H_j$.

Using $g : H(m) \boxtimes H(n) \to H_{i+j}$ we can produce an element in $D_{m+n} H_{i+j}$. Apply the functor D_* to g and get

$$D_* g : D_*(H(m) \boxtimes H(n)) \to D_* H_{i+j}$$

D_* is given simply by composition on the left.

Using the previous proposition, we can obtain an element $D_* g(\iota_m \otimes \iota_n)$ in $D_{m+n} H_{i+j}$, given by the composition $H(m+n) \to H(m) \boxtimes H(n) \to H_{i+j}$, which takes $x_{k+k'}$ to $g(\varphi(x_k \otimes x_{k'}))$.

We obtained pairings

$$D_m H_i \times D_n H_j \to D_{m+n} H_{i+j}$$

and these pairings finally produce a pairing

$$\circ' : D_* H_i \otimes D_* H_j \to D_* H_{i+j}$$

of Dieudonné modules.

Next we see how the homomorphisms V and F defined on each $D_* H_i$ relate to the new ring structure put on them. We have the following result.

Proposition 4.3 [6] Given bilinear pairings $\circ : H_i \otimes H_j \to H_{i+j}$, the induced bilinear pairings $\circ' : D_* H_i \otimes D_* H_j \to D_* H_{i+j}$ satisfy:

(a) $V(x \circ' y) = Vx \circ' Vy$;
(b) $Fx \circ' y = F(x \circ' Vy)$;
(c) $x \circ' Fy = F(Vx \circ' y)$.

With this information, it becomes natural to define a category of Dieudonné rings.

Definition 4.4 A *graded connected Dieudonné ring* over $\mathbb{F}_p$ is a sequence $\{M_{*i}\}_{i\in\mathbb{Z}}$ of graded connected Dieudonné modules together with bilinear maps $\circ' : M_{*i} \otimes M_{*j} \to M_{*i+j}$ satisfying the conditions in the previous proposition.

We want to finish the proof on the equivalence between the categories of graded connected Hopf rings and graded connected Dieudonné rings. One way of doing this is, like Goerss [6], to construct universal bilinear pairings on graded connected Dieudonné *modules* and then to show that these are exactly those that carry on, functorally, from those in the category of Hopf algebras. We will present another, more direct proof of the equivalence.

In the following results we will extend somehow the definition of Dieudonné module for a graded connected Hopf algebra: they will not be positively graded modules as before, but non-negatively graded ones, since we will define $D_0 H = \mathrm{Hom}_{\mathcal{H}A_*}(H(0), H)$. (Here, as before, $H(0) = \mathbb{F}_p[\mathbb{Z}]$). Thus, for each graded connected Hopf algebra H in $\mathcal{H}A_*$, $D_0 H \cong \mathbb{F}_p$ as a $\mathbb{Z}_p$-module. With this new convention, the writing of the following results becomes cleaner.

The following Lemma offers a result symmetric to the one before Proposition 4.3.

Lemma 4.5 *Any bilinear pairing* $\circ_{ij} : D_* H_i \otimes D_* H_j \to D_* H_{i+j}$ *induces a bilinear pairing* $\circ'_{ij} : H_i \otimes H_j \to H_{i+j}$.

Proof Suppose first that the characteristic of the base field is zero.

By Saramago (unpublished manuscript), to define uniquely the map $\circ' : H_i \otimes H_j \to H_{i+j}$ it is enough to fix its value on the primitives of $H_i \otimes H_j$ (since this is a connected Hopf algebra). Suppose $x \otimes 1$, with x a primitive of H_i, is such an element (the other only possibility, a $1 \otimes y$ with y a primitive of H_j, can be dealt with similarly). If the degree of x is $n = p^k m$, define the homomorphism $\hat{x} \in D_n H_i$ by $\hat{x}(1) = 1$, $\hat{x}(\omega_k) = x$ and $\hat{x}(\omega_i) = 0$ for $i \neq k$. Define also $\hat{1} \in D_0 H_j$ by $\hat{1}(1) = 1$. Then $\hat{x} \circ \hat{1}$ is in $D_n H_{i+j}$, and we define $x \circ' 1$ as $[\hat{x} \circ \hat{1}](\omega_k)$. (If the degree of x is not of the form $n = p^k m$, define $x \circ' 1 = 0$).

If the characteristic of the base field is a prime, we can run into additional problems, as the reference above indicates. In this case, we have to work from the condition of connectedness. If x in H_i has zero Verschiebung, then we can still define $x \circ' 1$ as in the reference above. If $V(x)$ is non-zero, by connectedness there exists an $r > 0$ such that the repeated Verschiebung $V^r(x)$ is zero but $V^{r-1}(x) = b$ is non-zero. If the degree of b is $n = p^k m$, define $\hat{x} \in D_n H_i$ by $\hat{x}(1) = 1$, $\hat{x}(\omega_k) = b$ and $\hat{x}(\omega_i) = 0$ for $i \neq k$. Then $x \circ' 1$ is defined as $[\hat{x} \circ \hat{1}](\omega_k^r)$. (If the degree of b is not of the form $n = p^k m$, define $x \circ' 1 = 0$). We can similarly define $1 \circ' y$ for $y \in H_j$. If either x or y are primitives, this will coincide with what was done before. Finally, just define $x \circ' y = (x \circ' 1)(1 \circ' y)$.

This definition works for the general case of $p \geq 2$. □

Notice that in the proof above no reference was made to the special properties bilinear maps satisfy on Dieudonné modules; that is, the result is valid for any bilinear map on Dieudonné modules.

As before, we denote by $\mathcal{H}R_*$ the category of Hopf rings for graded connected Hopf algebras and by $\mathcal{D}R_*$ the category of graded connected Dieudonné rings.

Consider the functor $D_*^R : \mathcal{H}R_* \to \mathcal{D}R_*$ that takes each sequence of Hopf algebras $\{H_i\}_{i\in\mathbb{Z}}$ to the sequence of Dieudonné modules $\{D_*(H_i)\}_{i\in\mathbb{Z}}$, where D_* is the previous functor $D_* : \mathcal{H}A_* \to \mathcal{D}M_*$, and such that the products $\circ_{ij} : H_i \otimes H_j \to H_{i+j}$ carry over to the products $\circ'_{ij} : D_* H_i \otimes D_* H_j \to D_* H_{i+j}$ as given before Proposition 4.3.

We want to define a functor $U_*^R : \mathcal{D}R_* \to \mathcal{H}R_*$ in a way that makes the pair (D_*^R, U_*^R) an equivalence of categories. We will use the previous functor U_* from Dieudonné *modules* to Hopf *algebras*. First some notation. We know from before that (D_*, U_*) forms an equivalence of categories between graded connected Hopf algebras and Dieudonné modules. Thus, for each Hopf algebra $H \in \mathcal{H}A_*$, we have $H \cong U_* D_*(H)$. Call φ_H, or simply φ, this isomorphism $\varphi : H \to U_* D_*(H)$. Also, for each Dieudonné module $M \in \mathcal{D}M_*$, there is an isomorphism $\psi : M \to D_* U_*(M)$.

Remark If M is of the form $M = D_* H$ for some Hopf algebra H, then $\psi : D_* H \to D_* U_* D_* H$, and the following diagram

$$\begin{array}{ccc} H(n) & \xrightarrow{\;x\;} & H \\ & \searrow^{\psi(x)} & \downarrow{\scriptstyle \varphi} \\ & & U_* D_* H \end{array}$$

defines, for each n, the isomorphism φ in terms of ψ and vice-versa. That is, we can pick φ and ψ in such a way that $\psi(x)(a) = \varphi(x(a))$ for $x \in M = D_* H$ and $a \in H(n)$ (for all n).

Proposition 4.6 *Any bilinear pairing* $\circ_{ij} : M_i \otimes M_j \to M_{i+j}$ *induces a bilinear pairing* $\circ'_{ij} : U_*(M_i) \otimes U_*(M_j) \to U_*(M_{i+j})$.

Proof Given the pairing $M_i \otimes M_j \xrightarrow{\;\circ\;} M_{i+j}$ we can get a natural pairing

$$D_* U_*(M_i) \otimes D_* U_*(M_j) \xrightarrow{\;\circ\;} D_* U_*(M_{i+j})$$

by $\psi(x) \circ \psi(y) = \psi(x \circ y)$. This defines also a pairing $U_*(M_i) \otimes U_*(M_j) \longrightarrow U_*(M_{i+j})$ as in Lemma 4.5, and so we define the pairing $\circ'$ as this last pairing. □

We can now define the functor $U_*^R : \mathcal{D}R_* \to \mathcal{H}R_*$ as follows.

For each sequence $\{M_i\}_{i\in\mathbb{Z}}$ we get the corresponding sequence $\{U_*(M_i)\}_{i\in\mathbb{Z}}$ of Hopf algebras, where $U_* : \mathcal{D}M_* \to \mathcal{H}A_*$ is the inverse functor to $D_* : \mathcal{H}A_* \to \mathcal{D}M_*$ as given in Theorem 4.1. Then, given a product $\circ_{ij} : M_i \otimes M_j \to M_{i+j}$, we define the product $\circ' : U_*(M_i) \otimes U_*(M_j) \to U_*(M_{i+j})$ as in Proposition 4.6. (Note that, since U_* and D_* are inverse functors, for each $M \in \mathcal{D}M_*$ we have $M \simeq_{\mathcal{D}M_*} \{\mathrm{Hom}_{\mathcal{H}A_*}(H(n), U_*(M))\}_{n>0}$. Thus each $M \in \mathcal{D}M_*$ is in fact isomorphic to $D_*(H)$ for some Hopf algebra H).

Corollary 4.7 *The functor* $D^{\mathcal{R}} : \mathcal{H}R_* \to \mathcal{D}R_*$ *has a right adjoint* $U^{\mathcal{R}} : \mathcal{D}R_* \to \mathcal{H}R_*$ *and the pair* $(D_*^{\mathcal{R}}, U_*^{\mathcal{R}})$ *forms an equivalence of categories.*

Proof The proof comes from the uniqueness of the induced product from Lemma 4.5. □

We now extend our definitions and results to a more general setting. We will say that a graded Hopf algebra H is *group-like in degree zero* if H_0 is a group ring over k. It can thus be written as $H \cong H_0 \otimes_k H_c$, where H_c is a graded connected Hopf algebra over k. Call $\mathcal{HA}^0_*$ the category of bicommutative graded Hopf algebras over $\mathbb{F}_p$ that are group-like in degree zero.

Recall that $H(0) = \mathbb{F}_p[\mathbb{Z}]$.

If $H \in \mathcal{HA}^0_*$, we have $D_*H = \{D_nH\}_{n\geq 0} = \{\mathrm{Hom}_{\mathcal{H}A^0_*}(H(n), H)\}_{n\geq 0}$. Composing on the right with the Verschiebung $v : H \to H$ and the Frobenius $f : H \to H$ gives maps $V : D_*H \to D_*H$ and $F : D_*H \to D_*H$. As before, if $m > 0$ we have $FVx = VFx = px$ for $x \in \mathrm{Hom}_{\mathcal{H}A^0_*}(H(m), H)$. If $m = 0$ we have $Vx = x$ for $x \in \mathrm{Hom}_{\mathcal{H}A^0_*}(H(0), H)$ since the Verschiebung on group-like elements of H is the identity.

Definition 4.8 The category $\mathcal{DM}^0_*$ has as objects graded $\mathbb{Z}_p$-modules $M = M_0 \oplus M_c$ together with maps $V : M \to M$ and $F : M \to M$ satisfying

(a) M_c is a Dieudonné module in $\mathcal{DM}_*$;
(b) $VF = FV = p$;
(c) V is the identity on M_0.

Clearly, we have:

Theorem 4.9 *The functor $D : \mathcal{HA}^0_* \to \mathcal{DM}^0_*$ gives an equivalence of categories.*

We next want to deal with ring objects.

A bilinear map in $\mathcal{DM}^0_*$ will be a map $\tilde{\varphi} : (M_0 \oplus M_c) \otimes (N_0 \oplus N_c) \to (K_0 \oplus K_c)$, where the restriction $\tilde{\varphi} : M_0 \otimes N_0 \to K_0$ is a product in group rings and the restriction $\tilde{\varphi} : M_c \otimes N_c \to K_c$ is a bilinear map in $\mathcal{DM}_*$ as defined before.

Any pairing $\circ_{ij} : H_i \otimes H_j \to H_{i+j}$ in $\mathcal{HA}^0_*$ will induce a pairing $\circ'_{ij} : DH_i \otimes DH_j \to DH_{i+j}$ in $\mathcal{DM}^0_*$. As before, the diagram

$$\begin{array}{ccc} H(0) \otimes H(0) & \longrightarrow & H(0) \boxtimes H(0) \\ \downarrow{\scriptstyle a\otimes b} & & \downarrow{\scriptstyle g} \\ (H_i)_0 \otimes (H_j)_0 & \xrightarrow{\circ} & (H_{i+j})_0 \end{array}$$

allows us to define products $\circ_{ij} : (D_*H_i)_0 \otimes (D_*H_j)_0 \to (D_*H_{i+j})_0$. Similar diagrams can be drawn for $H(0) \otimes H(n)$ and $H(m) \otimes H(0)$, and putting all together defines the product on Dieudonné modules (The restriction to $M_c \otimes N_c$ gives the previous definition for Dieudonné modules in $\mathcal{DM}_*$.)

We will denote by $\mathcal{HR}^0_*$ the category of graded group-like in degree zero Hopf rings and by $\mathcal{DR}^0_*$ the category of Dieudonné rings in this case.

Consider now the functor $D^R_* : \mathcal{HR}^0_* \to \mathcal{DR}^0_*$ that takes each sequence of Hopf algebras $\{H_i\}_{i\in\mathbb{Z}}$ to the sequence of Dieudonné modules $\{D_*(H_i)\}_{i\in\mathbb{Z}}$, where D_* is the previous functor $D_* : \mathcal{HA}^0_* \to \mathcal{DM}^0_*$, and such that the products $\circ_{ij} : H_i \otimes H_j \to H_{i+j}$ carry over to the products $\circ'_{ij} : D_*H_i \otimes D_*H_j \to D_*H_{i+j}$ as given in the preceding paragraph.

We have to construct an inverse functor $U_*^R : \mathcal{DR}_*^0 \to \mathcal{HR}_*^0$. For this, we generalize the definitions of induced pairings introduced in the connected case.

(It should be remarked here that the Dieudonné module for a group ring is the underlying abelian group, concentrated in degree zero and with identity Verschiebung).

Thus, a pairing $\circ : D_*H_i \otimes D_*H_j \to D_*H_{i+j}$ in $\mathcal{DM}_*^0$ induces a pairing $H_i \otimes H_j \to H_{i+j}$ in $\mathcal{HA}_*^0$. On $(H_i)_c \otimes (H_j)_c$, define $\circ'$ from $\circ$ on $(D_*H_i)_c \otimes (D_*H_j)_c$ as in the connected case; on $(H_i)_0 \otimes (H_j)_0$, $\circ' = \circ$ because of the remark above; plus, on $(H_i)_0 \otimes (H_j)_c$ consider the primitives of $(H_i)_c$ in degree n: if x is such a primitive and $a \in (H_i)_0$, put $a \circ' x = a \circ \hat{x}(\omega_n)$; as in Lemma 4.5., and since $(H_j)_c$ is connected, this defines $\circ'$ everywhere (Similarly for $(H_i)_c \otimes (H_j)_0$).

We have:

Corollary 4.10 *The functor $D_*^{\mathcal{R}} : \mathcal{HR}_*^0 \to \mathcal{DR}_*^0$ has a right adjoint $U_*^{\mathcal{R}} : \mathcal{DR}_*^0 \to \mathcal{HR}_*^0$ and the pair $(D_*^{\mathcal{R}}, U_*^{\mathcal{R}})$ forms an equivalence of categories.*

5 Dieudonné Theory for Ungraded Hopf Algebras and Rings

This section will generalize the results from the previous one to some categories of Hopf algebras over $\mathbb{F}_p$ that now do not necessarily carry a grading. We continue to assume that all Hopf algebras are bicommutative. Suppose A is an ungraded cocommutative, coassociative coalgebra with counit over a ring R, together with a coaugmentation $R \to A$. We construct its *coaugmentation filtration* $\{F_qA\}_{q\geq 0}$ by taking the short exact sequence

$$0 \to R \to A \to J(A) \to 0$$

and, using the iterated coproduct on A, defining $F_qA = \mathrm{k}er(A \to J(A)^{\otimes(q+1)})$ for $q \geq 0$ (In particular, $F_0A \simeq R$ and $F_1A/F_0A \simeq P(A)$, the primitives of A). A coalgebra A as above is *connected* if its coaugmentation filtration exhausts it (that is, if any $x \in A$ is in some F_qA for some $q \geq 0$). An ungraded Hopf algebra will be called *geometric-like* if it can be written as $\mathbb{F}_p[G] \otimes H_c$, where $\mathbb{F}_p[G]$ is a group ring and H_c is a connected Hopf algebra. $\mathcal{CA}$ will denote the category of connected, cocommutative, coassociative coalgebras with counit over $\mathbb{F}_p$ together with a coaugmentation. We will also call $\mathcal{HA}$ the corresponding category of Hopf algebras (abelian group objects of $\mathcal{CA}$). We consider the Hopf algebras $H(n) = \mathbb{F}_p[x_1, \cdots, x_n]$ for $n \geq 0$. We have Hopf algebra maps $\alpha : H(n) \to H(n+1)$ (given by inclusion) and $\bar{V} : H(n+1) \to H(n)$ (defined by $\bar{V}(x_i) = x_{i-1}$ for $i > 0$ and $\bar{V}(x_0) = 0$). $\bar{V}\alpha$ gives the Verschiebung on each $H(n)$. We have thus a sequence

$$\cdots \longrightarrow H(n+1) \xrightarrow{\bar{V}} H(n) \xrightarrow{\bar{V}} H(n-1) \longrightarrow \cdots$$

For each Hopf algebra H in $\mathcal{HA}$, this sequence induces a sequence of $\mathbb{F}_p$-modules

$$\cdots \to \mathrm{Hom}_{\mathcal{HA}}(H(n-1), H) \xrightarrow{\hat{V}} \mathrm{Hom}_{\mathcal{HA}}(H(n), H) \xrightarrow{\hat{V}} \mathrm{Hom}_{\mathcal{HA}}(H(n+1), H) \to \cdots$$

where each $\hat{V}$ is given by composition with $\bar{V}$ on the left.

Consider now the $\mathbb{F}_p$-module $DH = \operatorname{colim}_n \operatorname{Hom}_{\mathcal{HA}}(H(n), H)$. Composing on the right with the Verschiebung $v : H \to H$ and the Frobenius $f : H \to H$ gives maps $V : DH \to DH$ and $F : DH \to DH$. We have $FV = VF = p$. Given a Hopf algebra $H \in \mathcal{HA}$, we define its *Dieudonné module* as the module $DH = \operatorname{colim}_n \operatorname{Hom}_{\mathcal{HA}}(H(n), H)$ together with the homomorphisms $V : DH \to DH$ and $F : DH \to DH$ given above.

Since any $H \in \mathcal{HA}$ is connected, its coaugmentation filtration exhausts it and, moreover, if we write $\psi(x) = 1 \otimes x + x \otimes 1 + \sum x' \otimes x''$ for each $x \in F_q H$, then all the x' and x'' that appear in the expression are in those $F_{q'} H$ that have $q < q'$. Thus, the Verschiebung on such Hopf algebras is eventually zero. This carries over to DH, where we have that for each $x \in DH$ there must exist an $n \geq 0$ such that $V^n x = 0$.

Definition 5.1 The category $\mathcal{DM}$ of *ungraded connected Dieudonné modules* has as objects modules M together with endomorphisms $F : M \to M$ and $V : M \to M$ satisfying $FV = VF = p$ and such that for each $x \in M$ there exists an $n \geq 0$ with $V^n x = 0$.

The above considerations give us a functor $D : \mathcal{HA} \to \mathcal{DM}$ that takes a Hopf algebra $H \in \mathcal{HA}$ and produces its Dieudonné module $DH = \operatorname{colim}_n \operatorname{Hom}_{\mathcal{HA}}(H(n), H)$.

Theorem 5.2 [2] The functor $D : \mathcal{HA} \to \mathcal{DM}$ has a left adjoint $U : \mathcal{DM} \to \mathcal{HA}$, and the pair (D, U) forms an equivalence of categories.

From the remarks in Section 3, we know there exist tensor products in this category, which we will denote by $\boxtimes_u$. In order to study Dieudonné ring categories for Hopf rings, we want to be able to induce from products $\circ_{ij} : H_i \otimes H_j \to H_{i+j}$ in Hopf algebras some products $DH_i \otimes DH_j \to DH_{i+j}$ in the corresponding Dieudonné modules. First consider $x \in \operatorname{Hom}_{\mathcal{HA}}(H(m), H_i)$ and $y \in \operatorname{Hom}_{\mathcal{HA}}(H(n), H_j)$. We can get a unique $g : H(m) \boxtimes_u H(n) \to H_{i+j}$ making the following diagram commute

$$\begin{array}{ccc} H(m) \otimes H(n) & \xrightarrow{\varphi} & H(m) \boxtimes_u H(n) \\ {\scriptstyle x\otimes y}\downarrow & & \downarrow{\scriptstyle g} \\ H_i \otimes H_j & \xrightarrow{\circ_{ij}} & H_{i+j} \end{array}$$

As before, we consider the map $\iota_m \otimes \iota_n : H(m+n) \to H(m) \boxtimes_u H(n)$ taking x_{m+n} to $\varphi(x_m \otimes x_n)$ and commuting with Verschiebungs.

The composition $H(m+n) \xrightarrow{\iota_m \otimes \iota_n} H(m) \boxtimes_u H(n) \xrightarrow{g} H_{i+j}$ gives us an element $x \circ y$ in $\operatorname{Hom}_{\mathcal{HA}}(H(m+n), H_{i+j})$.

If $\alpha \in DH_i$ and $\beta \in DH_j$, write $\alpha = \varphi^m(x)$ for $x \in \operatorname{Hom}_{\mathcal{HA}}(H(m), H_i)$ and $\beta = \varphi^n(y)$ for $y \in \operatorname{Hom}_{\mathcal{HA}}(H(n), H_j)$, where φ^k represents, for each k, the map $\operatorname{Hom}_{\mathcal{HA}}(H(k), H) \to DH$ given by the definition of colimit. We get, as above, an element $x \circ y \in \operatorname{Hom}_{\mathcal{HA}}(H(m+n), H_{i+j})$, and also an element $\alpha \circ \beta = \varphi^{m+n}(x \circ y) \in DH_{i+j}$.

Proposition 5.3 *The element $\alpha \circ \beta$ does not depend on the x and y picked. That is, if $\varphi^m(x) = \varphi^{m'}(x')$ and $\varphi^n(y) = \varphi^{n'}(y')$ then $\varphi^{m+n}(x \circ y) = \varphi^{m'+n'}(x' \circ y')$*

Proof The proof is formal and straightforward.

Suppose $\varphi^m(x) = \varphi^{m+1}(x')$, where we have that $x \in \mathrm{Hom}_{\mathcal{HA}}(H(m), H_i)$ and $x' \in \mathrm{Hom}_{\mathcal{HA}}(H(m+1), H_j)$. By definition of colimit we have a commutative diagram

$$\begin{array}{ccc} H(m+1) & & \\ {\scriptstyle \bar V}\downarrow & \searrow^{x'} & \\ H(m) & \xrightarrow{\;x\;} & H_i \end{array}$$

We thus get a diagram

$$\begin{array}{ccc} H(m+1) \otimes H(n) & \xrightarrow{\varphi_2} & H(m+1) \boxtimes_u H(n) \\ {\scriptstyle \bar V \otimes 1}\downarrow & & \downarrow{\scriptstyle f} \\ H(m) \otimes H(n) & \xrightarrow{\varphi_1} & H(m) \boxtimes_u H(n) \\ {\scriptstyle x \otimes y}\downarrow & & \downarrow{\scriptstyle g} \\ H_i \otimes H_j & \xrightarrow{\circ_{ij}} & H_{i+j} \end{array}$$

(Here, f is the unique map given by the definition of tensor product $H(m+1) \otimes H(n) \to H(m+1) \boxtimes_u H(n)$). Then $x \circ y \in \mathrm{Hom}_{\mathcal{HA}}(H(m+n), H_{i+j})$ is the Hopf algebra map completely determined by its value $g\varphi_1(x_m \otimes x_n)$ on x_{m+n} and commuting with Verschiebungs. Also, $x' \circ y \in \mathrm{Hom}_{\mathcal{HA}}(H(m+n+1), H_{i+j})$ is completely determined by its value $gf\varphi_2(x_{m+1} \otimes x_n)$ on x_{m+n+1}. We have:

$$\begin{aligned} x' \circ y(x_{m+n+1}) &= gf\varphi_2(x_{m+1} \otimes x_n) = g\varphi_1(\bar V x_{m+1} \otimes x_n) = g\varphi_1(x_m \otimes x_n) \\ &= x \circ y(x_{m+n}) = x \circ y(\bar V x_{m+n+1}), \end{aligned}$$

and so in this case $\varphi^{m+n}(x \circ y) = \varphi^{m+n+1}(x' \circ y)$. If $\varphi^n(y) = \varphi^{n+1}(y')$, we similarly prove $\varphi^{m+n}(x \circ y) = \varphi^{m+n+1}(x \circ y')$ for any $x \in \mathrm{Hom}_{\mathcal{HA}}(H(m), H_i)$. The general case $\varphi^{m+n}(x \circ y) = \varphi^{m'+n'}(x' \circ y')$ now follows by induction on both $m' - m$ and $n' - n$. □

Corollary 5.4 *Every bilinear pairing $H_i \otimes H_j \to H_{i+j}$ in $\mathcal{HA}$ induces a pairing $DH_i \otimes DH_j \to DH_{i+j}$ in $\mathcal{DM}$.*

Proposition 5.5 *Given bilinear pairings $\circ_{ij} : H_i \otimes H_j \to H_{i+j}$, the induced pairings $\circ' : DH_i \otimes DH_j \to DH_{i+j}$ satisfy:*

(a) $V(\alpha \circ \beta) = V\alpha \circ V\beta$;
(b) $F\alpha \circ \beta = F(\alpha \circ V\beta)$;
(c) $\alpha \circ F\beta = F(V\alpha \circ \beta)$.

Proof We prove the first condition. The others follow similarly. Suppose $\alpha = \varphi^m(x)$ and $\beta = \varphi^n(y)$. Then

$$V(\alpha \circ \beta) = V(\varphi^{m+n}(x \circ y)) = \varphi^{m+n}(V(x \circ y))$$

where $V(x \circ y)$ is given by composition on the right with the Verschiebung on H_{i+j}. We then get:

$$\varphi^{m+n}(Vx \circ Vy)) = \varphi^m(Vx) \circ \varphi^n(Vy) = V\alpha \circ V\beta$$

where we use the fact that $V(x \circ y) = Vx \circ Vy$ for $x \in \mathrm{Hom}_{\mathcal{HA}}(H(m), H_i)$ and $y \in \mathrm{Hom}_{\mathcal{HA}}(H(n), H_j)$ (The proof of this last fact is exactly the same as in [6] once one ignores the definition of degree at each step). □

Definition 5.6 A *bilinear pairing in $\mathcal{DM}$ is a map $f : M_1 \times M_2 \to N$ of abelian groups satisfying $Vf(x, y) = f(Vx, Vy)$, $Ff(x, Vy)$ and $Ff(Vx, y) = f(x, Fy)$.*

Proposition 5.5 above stated that a bilinear pairing $\circ : H_1 \otimes H_2 \to K$ in $\mathcal{HA}$ induces a bilinear pairing $\circ' : DH_1 \otimes DH_2 \to DK$ in $\mathcal{DM}$.

An *ungraded connected Dieudonné ring* over $\mathbb{F}_p$ is a sequence $\{M_i\}_{i\in\mathbb{Z}}$ of ungraded connected Dieudonné modules together with bilinear maps $\circ_{ij} : M_i \otimes M_j \to M_{i+j}$ satisfying the conditions in the previous proposition.

$\mathcal{HR}$ will denote the category of ungraded connected Hopf rings and $\mathcal{DR}$ the category of ungraded connected Dieudonné rings. Define a functor $D^R : \mathcal{HR} \to \mathcal{DR}$ by considering for each sequence of Hopf algebras $\{H_i\}_{i\in\mathbb{Z}}$ in $\mathcal{HR}$ the sequence of Dieudonné modules $\{D(H_i)\}_{i\in\mathbb{Z}}$, where $D : \mathcal{HA} \to \mathcal{DM}$ is the previously defined functor, and such that the products $\circ_{ij} : H_i \otimes H_j \to H_{i+j}$ carry over to the products $\circ'_{ij} : DH_i \otimes DH_j \to DH_{i+j}$ as before. We will construct a right adjoint U^R for D^R. We start with a Lemma that offers a result symmetric to the one on Corollary 5.4.

Lemma 5.7 *Any bilinear pairing $\circ_{ij} : DH_i \otimes DH_j \to DH_{i+j}$ induces a bilinear pairing $\circ'_{ij} : H_i \otimes H_j \to H_{i+j}$ of Hopf algebras.*

Proof We focus on the primitive elements of $H_i \otimes H_j$.

Given $x \in H_i$ and $1 \in H_j$, pick a positive m and consider $\hat{x} \in \mathrm{Hom}_{\mathcal{HA}}(H(m), H_i)$ (given by $\hat{x}(1) = 1$, $\hat{x}(x_m) = x$ and $\hat{x}(x_i) = 0$ for $i \neq m$). Consider also the homomorphism $\hat{1} \in \mathrm{Hom}_{\mathcal{HA}}(H(0), H_j)$ given by $\hat{1}(1) = 1$. Then we have $\varphi^m(\hat{x}) \circ \varphi^0(\hat{1}) \in DH_{i+j}$, and so $\varphi^m(\hat{x}) \circ \varphi^0(\hat{1}) = \varphi^m(\alpha)$ for some $\alpha \in \mathrm{Hom}_{\mathcal{HA}}(H(m), H_i)$. Call this element $\hat{x} \circ \hat{1}$. Finally, define $x \circ' 1$ as $[\hat{x} \circ \hat{1}](\omega_m)$.

This construction is independent of m: as before, if $\varphi^m(\hat{x}) = \varphi^{m+1}(\widehat{x'})$, then we have $\varphi^m(\hat{x} \circ \hat{1}) = \varphi^{m+1}(\widehat{x'} \circ \hat{1})$, and so $[\hat{x} \circ \hat{1}](\omega_m) = [\widehat{x'} \circ \hat{1}](\omega_{m+n+1})$, proving that the induced pairing is well defined.

By Saramago (unpublished manuscript), this suffices whenever the characteristic of the base field is zero. If not, we can follow the lines in the proof of Lemma 4.5., making the appropriate changes. □

We also have that any pairing $\circ_{ij} : H_i \otimes H_j \to H_{i+j}$ will induce a pairing $\circ_{ij} : UD(H_i) \otimes UD(H_j) \to UD(H_{i+j})$ directly by $UD(x) \circ UD(y) = UD(x \circ y)$, and pairings $\circ_{ij} : M_i \otimes M_j \to M_{i+j}$ induce pairings $\circ_{ij} : DU(M_i) \otimes DU(M_j) \to DU(M_{i+j})$ by $DU(x) \circ DU(y) = DU(x \circ y)$.

Finally, pairings $\circ : M_i \circ M_j \to M_{i+j}$ induce pairings $\circ' : U(M_i) \circ U(M_j) \to U(M_{i+j})$ given by the composition of this last one with the one on Lemma 5.7.

Theorem 5.8 *The functor $D^R : \mathcal{HR} \to \mathcal{DR}$ has a right adjoint $U^R : \mathcal{DR} \to \mathcal{HR}$ and the pair (D^R, U^R) forms an equivalence of categories.*

Proof As in the proof for the graded case, the uniqueness of the induced products gives the result. □

Next we generalize the previous results in order to accommodate the case of ungraded geometric-like Hopf algebras. We will work as in chapter 4.

Suppose H is an ungraded Hopf algebra over a commutative ring k with unit and consider $X(H) = \mathrm{Hom}_{\mathcal{CA}}(k, H)$. Define H_0 as the group ring $k[X(H)]$ (Call the elements in this group ring the *group-like elements of* H). Define also $H_c = k \otimes_{H_0} H$. Then H is an ungraded connected Hopf algebra over k.

Definition 5.9 An ungraded Hopf algebra H over k is *geometric-like if it can be written as $H \cong H_0 \otimes_k H_c$, where H_0 is a group ring over k and H_c is an ungraded connected Hopf algebra over k. Call $\mathcal{HA}^0$ the category of bicommutative Hopf algebras over $\mathbb{F}_p$ that are geometric-like.*

For a Hopf algebra H in $\mathcal{HA}^0$, define $DH = \mathrm{colim}\{D_nH = \mathrm{Hom}_{\mathcal{HA}^0}(H(n), H)\}_{n\geq 0}$. Composing on the right with the Verschiebung $v : H \to H$ and the Frobenius $f : H \to H$ gives maps $V : DH \to DH$ and $F : DH \to DH$. As before, if $m > 0$ we have $FV(\varphi^m(\alpha)) = VF(\varphi^m(\alpha)) = p\varphi^m(\alpha)$ for $\alpha \in \mathrm{Hom}_{\mathcal{HA}}(H(m), H)$. If $m = 0$ we have $V\varphi^0(\alpha) = \varphi^0(\alpha)$ for $\alpha \in \mathrm{Hom}_{\mathcal{HA}}(H(0), H)$ since the Verschiebung on group-like elements of H is the identity.

Definition 5.10 The category $\mathcal{DM}^0$ has as objects modules $M = M_0 \oplus M_c$ together with maps $V : M \to M$ and $F : M \to M$ satisfying: M_c is a Dieudonné module in $\mathcal{DM}$; $VF = FV = p$; and V is the identity on M_0.

As before, we have:

Theorem 5.11 *The functor $D : \mathcal{HA}^0 \to \mathcal{DM}^0$ gives an equivalence of categories.*

We next want to deal with ring objects.

A bilinear map in $\mathcal{DM}^0$ will be a map $\tilde{\varphi} : (M_0 \oplus M_c) \otimes (N_0 \oplus N_c) \to (K_0 \oplus K_c)$ that restricts to a product in group rings $\varphi : M_0 \otimes N_0 \to K_0$ and to a bilinear map in $\mathcal{DM}$ $\varphi : M_c \otimes N_c \to K_c$.

Any pairing $\circ_{ij} : H_i \otimes H_j \to H_{i+j}$ in $\mathcal{HA}^0$ will induce a pairing $\circ'_{ij} : DH_i \otimes DH_j \to DH_{i+j}$ in $\mathcal{DM}^0$.

We will denote by $\mathcal{HR}^0$ the category of ungraded geometric-like Hopf rings and by $\mathcal{DR}^0$ the category of Dieudonné rings in this case.

Consider the functor $D^R : \mathcal{HR}^0 \to \mathcal{DR}^0$ that takes each sequence of Hopf algebras $\{H_i\}_{i\in\mathbb{Z}}$ to the sequence of Dieudonné modules $\{D(H_i)\}_{i\in\mathbb{Z}}$, where D is the previous functor $D : \mathcal{HA}^0 \to \mathcal{DM}^0$, and such that the products $\circ_{ij} : H_i \otimes H_j \to H_{i+j}$

carry over to the products $\circ'_{ij} : DH_i \otimes DH_j \to DH_{i+j}$ as given in the preceding paragraph.

To construct an inverse functor $U^R : \mathcal{DR}^0 \to \mathcal{HR}^0$, we again generalize the definitions of induced pairings introduced in the connected case.

A pairing $\circ : DH_i \otimes DH_j \to DH_{i+j}$ in $\mathcal{DM}^0$ induces a pairing $\circ : H_i \otimes H_j \to H_{i+j}$ in $\mathcal{HA}^0$ (similarly to what was done in the graded case). Also, a pairing $\circ : H_i \otimes H_j \to H_{i+j}$ in $\mathcal{HA}^0$ induces a pairing $\circ : \varphi(H_i) \otimes \varphi(H_j) \to \varphi(H_{i+j})$ in $\mathcal{HA}^0$ by $\varphi(x \circ y) = \varphi(x) \circ \varphi(y)$.

Then a pairing $\circ : M_i \otimes M_j \to M_{i+j}$ in $\mathcal{DM}^0$ induces a pairing $\circ' : H_i \otimes H_j \to H_{i+j}$ in $\mathcal{HA}^0$ by composing the last two.

The proof of the next corollary follows directly from the work we did in the connected case in this section and the methods used in the graded case generalizations of Section 4.

Corollary 5.12 *The functor $D^R : \mathcal{HR}^0 \to \mathcal{DR}^0$ has a right adjoint $U^R : \mathcal{DR}^0 \to \mathcal{HR}^0$ and the pair (D^R, U^R) forms an equivalence of categories.*

6 Dieudonné Theory for Periodically Graded Hopf Algebras and Rings

In this section we consider periodically graded bicommutative Hopf algebras over $\mathbb{F}_p$ and, as before, we study the corresponding Dieudonné modules, plus the corresponding Dieudonné rings for the Hopf rings. We again focus on connected and geometric-like Hopf algebras, with definitions similar to those we presented in the two previous chapters.

We will have to restrict ourselves to a special grading if we want the maps we define to be in the category of graded Hopf algebras. For this, start by fixing any $n > 0$. We will write $m = p^n - 1$ and consider $2m$-graded bicommutative Hopf algebras, that is, those that are graded over $\mathbb{Z}/2(p^n - 1)$, and will focus on those Hopf algebras that are concentrated in even degrees (having elements in odd degrees implies, by commutativity, that the Hopf algebra has elements squaring to zero; in fact, the category of $2m$-graded bicommutative Hopf algebras splits as a direct product of the category of those concentrated in even degrees and a category of primitively generated exterior algebras—see [8]).

A $2m$-graded Hopf algebra concentrated in even degrees will be called *connected* if it has an exhaustive coaugmentation filtration.

We will denote by $\mathcal{HA}_m$ the category whose objects are $2m$-graded connected bicommutative Hopf algebras over $\mathbb{F}_p$ concentrated in even degrees.

The definition of Dieudonné modules from Section 5 carries over to this section and we have $DH = \text{colim}\{\text{Hom}_{\mathcal{H}A}(H(n), H)\}$ for any $H \in \mathcal{HA}_m$. In this case, though, we want a new definition that takes care of the grading, as the maps $H(n) \to H$ in the colimit above are not necessarily in $\mathcal{HA}_m$.

We start by giving each indeterminate x_i degree $2p^i t \pmod{2m}$ for a fixed t satisfying $1 \le t \le p^n - 1$ and define $H(s,t) = \mathbb{F}_p[x_0, x_1, \cdots, x_s]$ for each s, giving it the unique Hopf algebra structure graded over $2m$ that makes the Witt polynomials primitive. The Verschiebung on these $H(s,t)$ is given by $V(x_i) = x_{i-1}$ for $i > 0$.

The Hopf algebra maps $\bar{V} : H(s+1,t) \to H(s,t)$ corresponding to the ones used in the beginning of Section 5 to define the colimit are not necessarily in $\mathcal{HA}_m$, but we can choose some iterate of $\bar{V}$ that does preserve degree: in fact, since the Verschiebung divides degree by p and our grading imposes $p^n = 1$, we have that $\bar{V}^n$ is a map in $\mathcal{HA}_m$ and so we can define Dieudonné modules for $H \in \mathcal{HA}_m$ by

$$DH = \operatorname{colim}\{\operatorname{Hom}_{\mathcal{H}A_m}(H(s,t), H)\}$$

Notice that each $\bar{V}^n$ is a map from a $H(s,t)$ with $s > n$ to $H(s-n,t)$. This means that, by definition of colimit, an element $\alpha \in \operatorname{Hom}_{\mathcal{H}A_m}(H(s,t), H)$ whose image under the colimit map is in DH is completely determined by the restriction of that map to $H(s',t)$, where $0 \le s' < n$ and $s' = s \pmod n$. If we now vary t, we get a collection $\{D_t H\}_{1 \le t \le p^n - 1}$ (One should notice that, since $p^n = 1 \pmod n$ in our grading, the modules $D_t H$ for $t > p^n - 1$ would start repeating the ones already determined, and so $DH = \{D_t H\}_{1 \le t \le p^n - 1}$ is an m-graded module).

Next we define the homomorphisms V and F on DH.

For each s and t, the map $H(s,pt) \to H(s,t)$ that takes x_i to x_i^p preserves degree, and so is a homomorphism in $\mathcal{H}A_m$. The following diagram is commutative because the composition of the inclusion $H(s,t) \to H(s,pt)$ with the above map gives the Frobenius F on $H(s,t)$ and the Frobenius commutes with the Verschiebung.

$$\begin{array}{ccccccccccc}
H(0,t) & \longleftarrow & H(n,t) & \longleftarrow & \cdots & \longleftarrow & H(n(s-1),t) & \longleftarrow & H(ns,t) & \longleftarrow & \cdots \\
\uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \\
H(0,pt) & \longleftarrow & H(n,pt) & \longleftarrow & \cdots & \longleftarrow & H(n(s-1),pt) & \longleftarrow & H(ns,pt) & \longleftarrow & \cdots
\end{array}$$

This diagram allows us to define the homomorphism $F : D_t H \to D_{pt} H$.

Similarly, the composition $H(s,t/p) \to H(s-1,t) \to H(s,t)$ taking x_i to x_{i-1} preserves degree, and so the diagram

$$\begin{array}{ccccccccccc}
H(0,t) & \longleftarrow & H(n,t) & \longleftarrow & \cdots & \longleftarrow & H(n(s-1),t) & \longleftarrow & H(ns,t) & \longleftarrow & \cdots \\
\uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \\
H(0,t/p) & \longleftarrow & H(n,t/p) & \longleftarrow & \cdots & \longleftarrow & H(n(s-1),t/p) & \longleftarrow & H(ns,t/p) & \longleftarrow & \cdots
\end{array}$$

allows us to define a homomorphism $V : D_t H \to D_{p/t} H$.

Definition 6.1 The Dieudonné module for a Hopf algebra $H \in \mathcal{H}A_m$ is the m-graded module

$$DH = \{D_t H = \operatorname{colim}\{\operatorname{Hom}_{\mathcal{H}A_m}(H(s,t), H)\}\}$$

together with the above homomorphisms $F : D_t H \to D_{pt} H$ and $V : D_t H \to D_{p/t} H$.

We call $\mathcal{D}M_m$ the category of m-graded modules M together with maps V and F satisfying $FV = VF = p$ and such that for each $x \in M$ there exists an $r \ge 1$ with $V^r(x) = 0$.

We have the following equivalence between Hopf algebras and Dieudonné modules:

Theorem 6.2 [13] The above map D is a functor $D : \mathcal{H}A_m \to \mathcal{D}M_m$ that has a right adjoint $U : \mathcal{D}M_m \to \mathcal{H}A_m$, and the pair (D, U) forms an equivalence of categories.

Denote by $\mathcal{H}R_m$ the category of Hopf rings in the $2m$-graded connected case, that is, whose objects are collections $\{H_i\}$ of $2m$-graded connected Hopf algebras (concentrated in even degrees) together with maps $H_i \otimes H_j \to H_{i+j}$ of $2m$-graded coalgebras. We want to show this category is equivalent to a category of Dieudonné rings.

Proposition 6.3 *Every bilinear pairing $H_i \otimes H_j \to H_{i+j}$ in $\mathcal{H}A_m$ induces a pairing $DH_i \otimes DH_j \to DH_{i+j}$ in $\mathcal{D}M_m$.*

Proof Suppose that $\circ_{ij} : H_i \otimes H_j \to H_{i+j}$ is a product of $2m$-graded connected Hopf algebras. Consider $x \in D_r H_i$ and $y \in D_s H_j$. As we said above, we can assume x to be an element in $\mathrm{Hom}_{\mathcal{H}A_m}(H(n,r), H_i)$ (It is completely determined by its values on $x_0, \cdots, x_n$). Similarly, $y \in \mathrm{Hom}_{\mathcal{H}A_m}(H(n,s), H_j)$. We can get a map in $\mathrm{Hom}_{\mathcal{H}A_m}(H(n,r+s), H_{i+j})$ using a diagram similar to the ones we used before:

$$\begin{array}{ccc} H(n,r) \otimes H(n,s) & \longrightarrow & H(n,r) \boxtimes_m H(n,s) \\ \downarrow{\scriptstyle x\otimes y} & & \downarrow{\scriptstyle g} \\ H_i \otimes H_j & \xrightarrow{\circ_{ij}} & H_{i+j} \end{array}$$

Here, as before, $\boxtimes_m$ is the tensor product in our case.

We have a map $\iota_r \otimes \iota_s : H(n, r+s) \to H(n,r) \boxtimes_m H(n,s)$ and can form the composition

$$H(n, r+s) \xrightarrow{\iota_r \otimes \iota_s} H(n,r) \boxtimes_m H(n,s) \xrightarrow{g} H_{i+j}$$

This element completely determines a map in $\mathrm{Hom}_{\mathcal{H}A_m}(H(n, r+s), H_{i+j})$, and this map defines an $x \circ y \in D_{r+s} H_{i+j}$. □

As before, we analyze how these pairings relate to V and F defined on the Dieudonné modules.

Proposition 6.4 *Given bilinear pairings $\circ_{ij} : H_i \otimes H_j \to H_{i+j}$, the induced pairings $\circ'_{ij} : DH_i \otimes DH_j \to DH_{i+j}$ satisfy:*

(a) $V(x \circ y) = Vx \circ Vy$;
(b) $Fx \circ y = F(x \circ Vy)$;
(c) $x \circ Fy = F(Vx \circ y)$.

Proof Consider the first condition. The others follow similarly.

Suppose $x \in \mathrm{Hom}_{\mathcal{H}A_m}(H(n,r), H_i)$ and $y \in \mathrm{Hom}_{\mathcal{H}A_m}(H(n,s), H_j)$. Then $x \circ y$ is given by the following diagram.

$$\begin{array}{ccc} & & H(n,r+s) \\ & & \downarrow \\ H(n,r)\otimes H(n,s) & \longrightarrow & H(n,r)\boxtimes_m H(n,s) \\ \downarrow{\scriptstyle x\otimes y} & & \downarrow{\scriptstyle g} \\ H_i\otimes H_j & \xrightarrow{\circ_{ij}} & H_{i+j} \end{array}$$

The element Vx can be seen to be given by the composition $H(n, r/p) \to H(n,r) \to H_i$, where the left map takes x_i to x_{i-1}. This definition agrees with the previous one, since the colimit defining $x \in D_r H_i$ is completely defined by a map $H(n,r) \to H_i$. At the same time, the element Vy is given by the composition $H(n, s/p) \to H(n,s) \to H_j$. There exists a unique map $H(n, r/p) \boxtimes_m H(n, s/p) \to H(n,r) \boxtimes_m H(n,s)$ (because of the definition of tensor products). Putting this map into the following diagram defining $Vx \circ Vy$ proves the result.

$$\begin{array}{ccccc} & & H(n,(r+s)/p) & \longrightarrow & \\ & & \downarrow & & \downarrow \\ H(n,r/p)\otimes H(n,s/p) & \longrightarrow & H(n,r/p)\boxtimes_m H(n,s/p) & & H(n,r+s) \\ \downarrow & & \downarrow & \swarrow & \\ H(n,r)\otimes H(n,s) & \longrightarrow & H(n,r)\boxtimes_m H(n,s) & & \\ \downarrow & & \downarrow & & \\ H_i\otimes H_j & \longrightarrow & H_{i+j} & \longleftarrow & H(n,r+s) \end{array}$$

□

Definition 6.5 A bilinear pairing in $\mathcal{D}M_m$ is a map $f : M_1 \times M_2 \to N$ of modules satisfying $Vf(x, y) = f(Vx, Vy)$, $Ff(x, Vy) = f(Fx, y)$ and $Ff(Vx, y) = f(x, Fy)$.

We just saw that a bilinear pairing $\circ : H_1 \otimes H_2 \to K$ in $\mathcal{H}A_m$ induces a bilinear pairing $\circ' : DH_1 \otimes DH_2 \to DK$ in $\mathcal{D}M_m$.

Definition 6.6 An m-graded connected Dieudonné ring is a sequence $\{M_i\}_{i\in\mathbb{Z}}$ of m-graded connected Dieudonné modules together with bilinear maps $\circ_{ij} : M_i \otimes M_j \to M_{i+j}$ satisfying the conditions in the previous proposition.

Denote by $\mathcal{H}R_m$ the category of Hopf rings in the $2m$-graded connected case and by $\mathcal{D}R_m$ the category of m-graded connected Dieudonné rings.

Define a functor $D^R : \mathcal{H}R_m \to \mathcal{D}R_m$ by considering for each sequence of Hopf algebras $\{H_i\}_{i\in\mathbb{Z}}$ in $\mathcal{H}R_m$ the sequence of Dieudonné modules $\{D(H_i)\}_{i\in\mathbb{Z}}$, where $D : \mathcal{H}A_m \to \mathcal{D}M_m$ is the previously defined functor, and such that the products $\circ_{ij} : H_i \otimes H_j \to H_{i+j}$ carry over to the products $\circ'_{ij} : DH_i \otimes DH_j \to DH_{i+j}$ as before.

To construct a right adjoint U^R for D^R, we extend the results in the previous chapters.

Lemma 6.7 *Any bilinear pairing* $\circ_{ij} : DH_i \otimes DH_j \to DH_{i+j}$ *induces a bilinear pairing* $\circ'_{ij} : H_i \otimes H_j \to H_{i+j}$ *of Hopf algebras.*

Proof The proof follows as in the graded and ungraded case.

Given $x \in H_i$ primitive and $1 \in H_j$, consider $\hat{x} \in \mathrm{Hom}_{\mathcal{HA}_m}(H(n, r), H_i)$ (given by $\hat{x}(1) = 1$, $\hat{x}(x_n) = x$ and $\hat{x}(x_i) = 0$ for $i \neq n$). Consider also the homomorphism $\hat{1} \in \mathrm{Hom}_{\mathcal{HA}_m}(H(n, 0), H_j)$ given by $\hat{1}(1) = 1$. Then we define $x\circ'1 = [\hat{x} \circ \hat{1}](\omega_n)$. □

Also, pairings $\circ_{ij} : H_i \otimes H_j \to H_{i+j}$ induce pairings $\circ_{ij} : UD(H_i) \otimes UD(H_j) \to UD(H_{i+j})$ directly by $UD(x) \circ UD(y) = UD(x \circ y)$, and pairings $\circ_{ij} : M_i \otimes M_j \to M_{i+j}$ induce pairings $\circ_{ij} : DU(M_i) \otimes DU(M_j) \to DU(M_{i+j})$ by $DU(x) \circ DU(y) = DU(x \circ y)$.

Finally, pairings $\circ_{ij} : M_i \otimes M_j \to M_{i+j}$ induce pairings $\circ'_{ij} : U(M_i) \otimes U(M_j) \to U(M_{i+j})$ (the composition of the two previous ones).

Now consider the functor $D^R : \mathcal{H}R_m \to \mathcal{D}R_m$ that takes each sequence of Hopf algebras $\{H_i\}_{i\in\mathbb{Z}}$ to the sequence of Dieudonné modules $\{D(H_i)\}_{i\in\mathbb{Z}}$, where D is the previous functor $D : \mathcal{H}A_m \to \mathcal{D}M_m$, and such that the products $\circ_{ij} : H_i \otimes H_j \to H_{i+j}$ carry over to the products $\circ'_{ij} : DH_i \otimes DH_j \to DH_{i+j}$ as given above.

Define a functor $U^R : \mathcal{HR}_m \to \mathcal{DR}_m$ by assigning to each sequence $\{M_i\}_{i\in\mathbb{Z}}$ of Dieudonné modules the corresponding sequence $\{U(M_i)\}_{i\in\mathbb{Z}}$ of Hopf algebras, where $U : \mathcal{DM}_m \to \mathcal{HA}_m$ is the inverse functor to $U : \mathcal{HA}_m \to \mathcal{DM}_m$.

Then, given a product $\circ_{ij} : M_i \otimes M_j \to M_{i+j}$, we define the product $\circ'_{ij} : U(M_i) \otimes U(M_j) \to U(M_{i+j})$ as above.

As in the previous chapters, we have:

Theorem 6.8 *The functor* $D^R : \mathcal{H}R_m \to \mathcal{D}R_m$ *has a right adjoint* $U^R : \mathcal{D}R_m \to \mathcal{H}R_m$ *and the pair* (D^R, U^R) *forms an equivalence of categories.*

Next we generalize the results in the previous sections in order to accommodate the case of $2m$-graded geometric-like Hopf algebras.

Following chapter 5, we will call *geometric-like* those m-graded Hopf algebras that can be written as $H \cong H_0 \otimes_k H_c$, where H_0 is a group ring over k and H_c is a $2m$-graded connected Hopf algebra over k (concentrated in even degrees). Call $\mathcal{HA}_m^0$ the category of bicommutative Hopf algebras over $\mathbb{F}_p$ that are geometric-like.

Now define $H(0, s) = \mathbb{F}_p[\mathbb{Z}]$ for any s and remember that, for $n > 0$, $H(n, s)$ was defined as $H(n, s) = \mathbb{F}_p[x_0, \cdots, x_n]$, with each x_i in degree $2p^i s$.

For a Hopf algebra H in $\mathcal{HA}_m^0$, define $D_0 H = \mathrm{Hom}_{\mathcal{HA}_m^0}(H(0, s), H)$ and $D_t H$ (for $t > 0$) as before. We have that the Verschiebung on $H(0, s)$ is the identity, and so we can extend V on Dieudonné modules by declaring it to be the identity on D_0. Also, define F on D_0 in a way that preserves $FV = VF = p$.

We can thus define the category $\mathcal{DM}_m^0$ whose objects are modules $M = M_0 \oplus M_c$, where M_c is a Dieudonné module in $\mathcal{DM}_m$, furnished with maps $V : M \to M$ and $F : M \to M$ that satisfy $VF = FV = p$ and such that V is the identity on M_0.

Theorem 6.9 *The functor $D : \mathcal{HA}_m^0 \to \mathcal{DM}_m^0$ gives an equivalence of categories.*

Next we deal with ring objects.

A bilinear map in $\mathcal{DM}_m^0$ will be a map $\tilde{\varphi} : (M_0 \oplus M_c) \otimes (N_0 \oplus N_c) \to (K_0 \oplus K_c)$ restricting to a product in group rings $\varphi : M_0 \otimes N_0 \to K_0$ and to a bilinear map in $\mathcal{DM}_m$ $\varphi : M_c \otimes N_c \to K_c$ as defined before.

Any pairing $\circ_{ij} : H_i \otimes H_j \to H_{i+j}$ in $\mathcal{HA}_m^0$ induces a pairing $\circ'_{ij} : DH_i \otimes DH_j \to DH_{i+j}$ in $\mathcal{DM}_m^0$ (same reasoning as for the two previous cases).

We will denote by $\mathcal{HR}_m^0$ the category of $2m$-graded geometric-like Hopf rings (that is, of the ring objects for $\mathcal{HA}_m^0$) and by $\mathcal{DR}_m^0$ the category of Dieudonné rings in this case.

There is a functor $D^R : \mathcal{HR}_m^0 \to \mathcal{DR}_m^0$ taking each sequence of Hopf algebras $\{H_i\}_{i\in\mathbb{Z}}$ to the sequence of Dieudonné modules $\{D(H_i)\}_{i\in\mathbb{Z}}$, where D is the previous functor $D : \mathcal{HA}_m^0 \to \mathcal{DM}_m^0$, and such that the products $\circ_{ij} : H_i \otimes H_j \to H_{i+j}$ carry over to the products $\circ'_{ij} : DH_i \otimes DH_j \to DH_{i+j}$.

To construct an inverse functor $U^R : \mathcal{DR}_m^0 \to \mathcal{HR}_m^0$, we again generalize the definitions of induced pairings introduced in the connected case.

A pairing $\circ_{ij} : DH_i \otimes DH_j \to DH_{i+j}$ in $\mathcal{DM}_m^0$ induces a pairing $\circ'_{ij} : H_i \otimes H_j \to H_{i+j}$ in $\mathcal{HA}_m^0$.

Also, a pairing $\circ_{ij} : H_i \otimes H_j \to H_{i+j}$ in $\mathcal{HA}_m^0$ induces a pairing $\circ_{ij} : \varphi(H_i) \otimes \varphi(H_j) \to \varphi(H_{i+j})$ in $\mathcal{HA}_m^0$ by $\varphi(x) \circ \varphi(y) = \varphi(x \circ y)$.

Then a pairing $\circ_{ij} : M_i \otimes M_j \to M_{i+j}$ in $\mathcal{DM}_m^0$ induces a pairing $\circ'_{ij} : H_i \otimes H_j \to H_{i+j}$ in $\mathcal{HA}_m^0$ by composing the two.

The proof of the next corollary follows directly from the work we did in the connected case in this section and the methods used in the graded case generalizations of Section 4.

Corollary 6.10 *The functor $D^R : \mathcal{HR}_m^0 \to \mathcal{DR}_m^0$ has a right adjoint $U^R : \mathcal{DR}_m^0 \to \mathcal{HR}_m^0$, and the pair (D^R, U^R) forms an equivalence of categories.*

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