The Davis–Mahowald spectral sequence

John Rognes

University of Oslo, Norway

Nils Baas 70, NTNU, December 1st 2016

Outline

The Adams spectral sequence

The Davis–Mahowald spectral sequence

Calculation of *Ext* over $A(2)$

Outline

The Adams spectral sequence

The Davis–Mahowald spectral sequence

Calculation of *Ext* over $A(2)$

Context

- Joint work with Robert R. Bruner on $THH(tmf)$ and its circle action.
- Aim to study v_3-periodic homotopy detected by

$$S \to K(tmf) \to THH(tmf)^{tS^1} .$$

- Prime $p = 2$, $H = H\mathbb{F}_2$.
- Steenrod algebra $A = H^*(H) = \langle Sq^k \mid k \geq 1 \rangle$.
- Adams spectral sequence for X

$$E_2^{s,t} = Ext_A^{s,t}(H^*(X), \mathbb{F}_2) \Longrightarrow_s \pi_{t-s}(X_2^\wedge) .$$

Baas–Madsen (1972)

- Nils Baas and Ib Madsen constructed spectra X with prescribed cohomology modules
$$H^*(X) = A/A\{Q_i, \dots, Q_j\}\,,$$
where $Q_n \in A$ is the Milnor primitive in degree $2p^n - 1 = 2^{n+1} - 1$.
- Example:
$$H^*(k(n)) = A/A\{Q_n\} = A \otimes_{E[Q_n]} \mathbb{F}_2 = A/\!/E[Q_n]$$
where $k(n) \to K(n)$ is the connective cover of the n-th Morava K-theory, and $E[Q_n] \subset A$ is the exterior algebra.
- Adams SS
$$E_2 = Ext_A(A/\!/E[Q_n], \mathbb{F}_2) \cong Ext_{E[Q_n]}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2[v_n] \Longrightarrow \pi_* k(n)$$
collapses, so $\pi_* k(n) = \mathbb{F}_2[v_n]$ with $|v_n| = 2p^n - 2 = 2^{n+1} - 2$.

Adams SS for $k(2)$

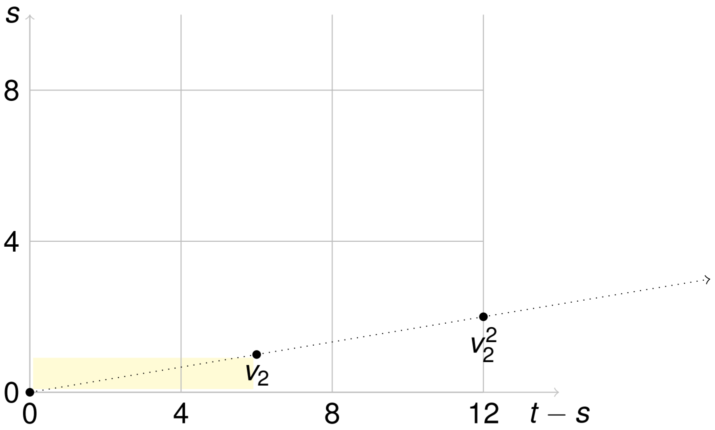
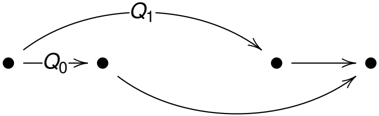


Figure: $Ext^{s,t}_{E[Q_2]}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow_s \pi_{t-s} k(2)$

Complex K-theory

- $H^*(ku) = A//E(1)$ where $ku \to KU$ is the connective cover of complex K-theory.
- $E(1) = E[Q_0, Q_1]$ with $Q_0 = Sq^1$ and $Q_1 = [Sq^1, Sq^2]$.



- Adams SS

$$E_2 = Ext_A(A//E(1), \mathbb{F}_2) = Ext_{E(1)}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2[h_0, v_1] \Longrightarrow \pi_* ku$$

collapses, so $\pi_* ku = \mathbb{Z}[v_1]$ with $|v_1| = 2$.

Adams SS for ku

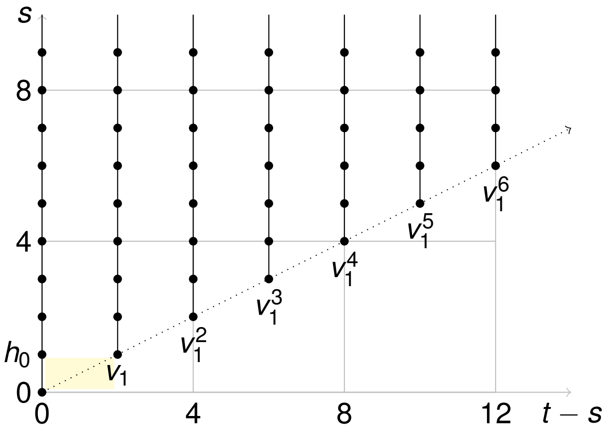
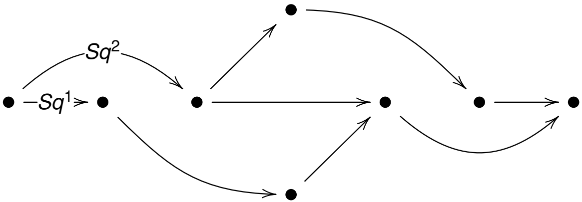


Figure: $Ext^{s,t}_{E(1)}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow_s \pi_{t-s} ku$

Real K-theory

- $H^*(ko) = A//A(1)$ where $ko \to KO$ is the connective cover of real K-theory.
- $A(1) = \langle Sq^1, Sq^2 \rangle \subset A$.



- Adams SS $E_2 = Ext_A(A//A(1), \mathbb{F}_2) = Ext_{A(1)}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2[h_0, h_1, v, w_1]/(h_0 h_1, h_1^3, h_1 v, v^2 = h_0^2 w_1) \Longrightarrow \pi_* ko$ collapses, so $\pi_* ko = \mathbb{Z}[\eta, \alpha, \beta]/(2\eta, \eta^3, \eta\alpha, \alpha^2 = 4\beta)$.

Adams SS for *ko*

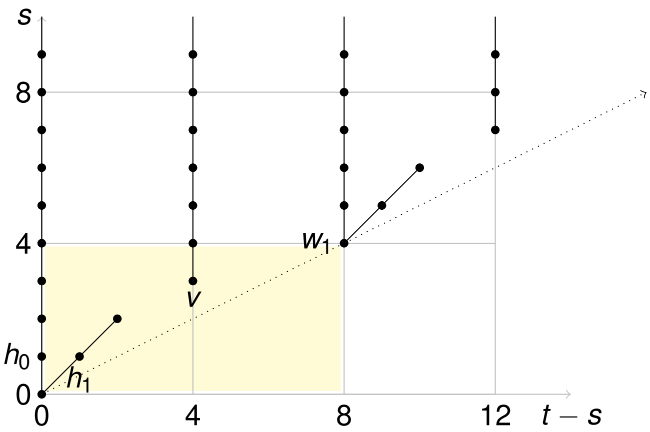


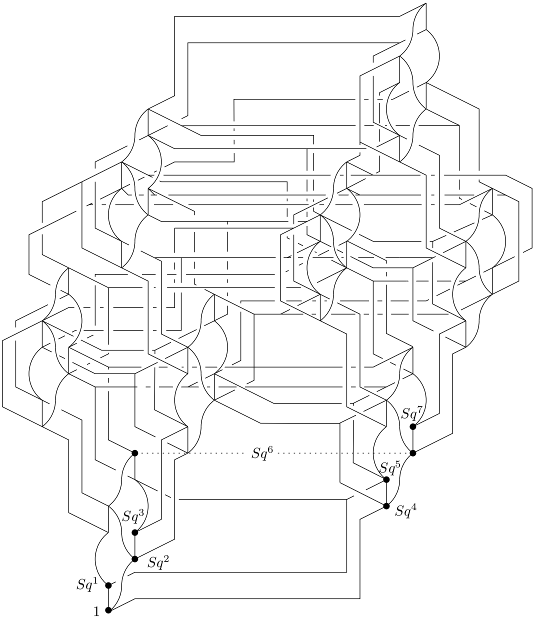
Figure: $Ext_{A(1)}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow_s \pi_{t-s} ko$

Hopkins–Miller / Hopkins–Mahowald (1994)

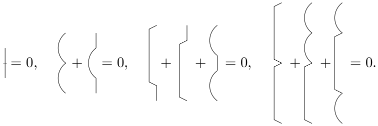
- Mike Hopkins and Mark Mahowald constructed a topological modular forms spectrum *tmf* with $H^*(tmf) = A//A(2)$, where $A(2) = \langle Sq^1, Sq^2, Sq^4 \rangle \subset A$.
- $\dim A(2) = 64$.
- André Henriques (2004) created the picture of $A(2)$ on the next page:

Andre Henriques, december 2004.

The subalgebra $A(2)$ of the Steenrod algebra



with generators Sq^1, Sq^2, Sq^4 and relations given by



Adams SS for *tmf*

- Adams SS

$$E_2 = Ext_A(A//A(2), \mathbb{F}_2) = Ext_{A(2)}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow \pi_* tmf \, .$$

- Theorem (Shimada-Iwai (1967)):

$$Ext_{A(2)}(\mathbb{F}_2, \mathbb{F}_2) \cong \frac{\mathbb{F}_2[h_0, h_1, h_2, c_0, \alpha, d_0, \beta, e_0, \gamma, \delta, g, w_1, w_2]}{\text{(54 relations)}} \, .$$

- Bruner's `ext` can reproduce such calculations in a finite range.
- How to prove that the apparent patterns persist?

Adams E_2-term for *tmf*, $t - s \leq 12$

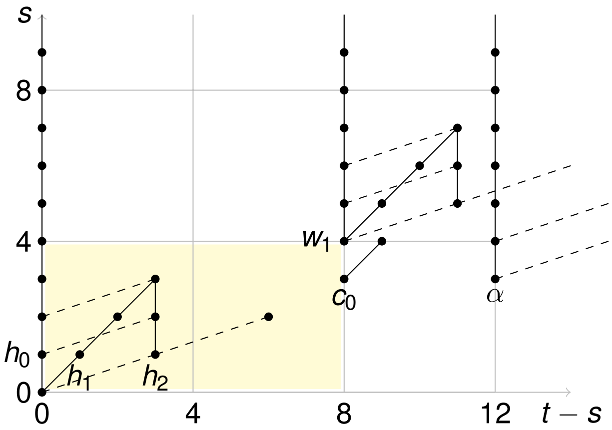


Figure: $Ext^{s,t}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow_s \pi_{t-s} tmf$

Adams E_2-term for *tmf*, $t - s \leq 24$

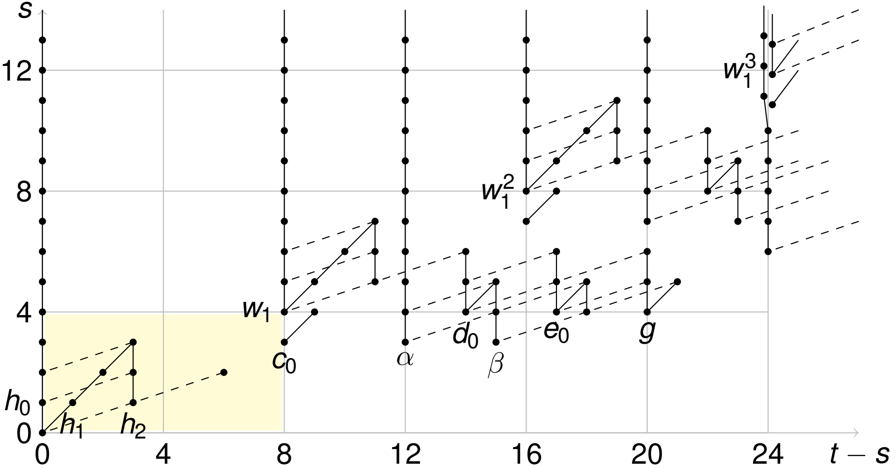


Figure: $Ext_{A(2)}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow_s \pi_{t-s} tmf$

Adams E_2-term for *tmf*, $t - s \leq 48$

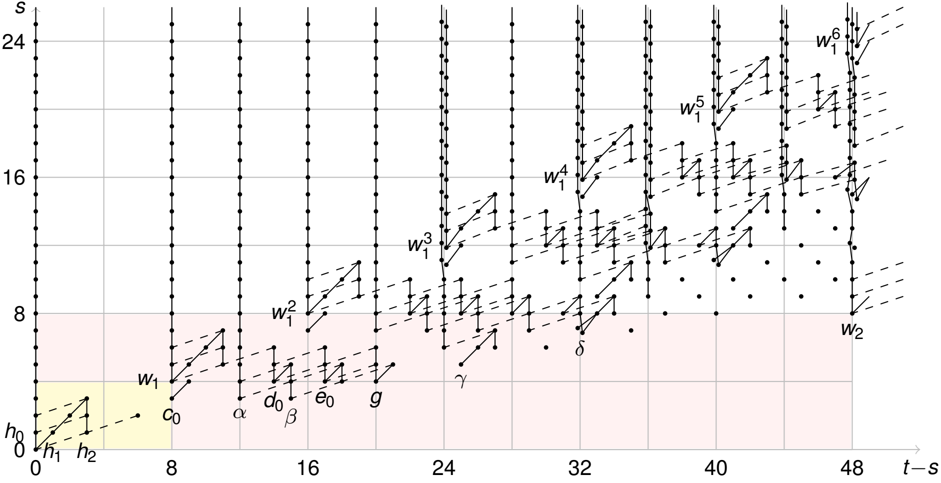


Figure: $Ext_{A(2)}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow_s \pi_{t-s} tmf$

Adams E_2-term for *tmf*, $t - s \leq 96$

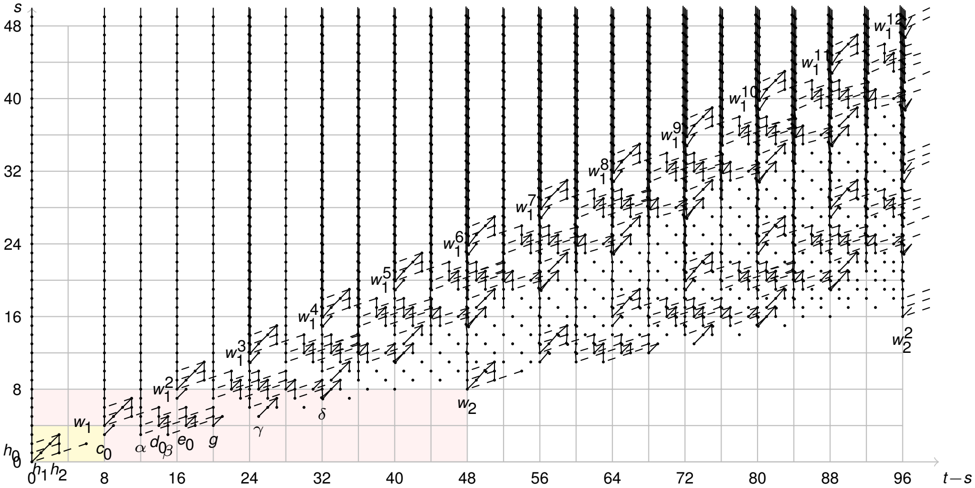


Figure: $Ext^{s,t}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow_s \pi_{t-s} tmf$

Remaining steps to determine $\pi_* tmf$

- Determine Adams d_2, d_3, d_4 and $E_5 = E_\infty$ [Hopkins–Mahowald].
- Examine additive and multiplicative extensions.
- Similar strategy applies for *tmf*-modules like $THH(tmf)$.
- Alternative: Use Adams–Novikov / elliptic SS [Hopkins–Mahowald, Bauer (2008)].

Outline

Davis–Mahowald (1982)

- Don Davis and Mark Mahowald calculated $Ext_{A(2)}(M, \mathbb{F}_2)$ for a number of $A(2)$-modules M related to $\mathbb{F}_2[x] = H^*\mathbb{R}P^\infty$.
- Spectral sequence

$$E_1^{\sigma,s,t} = Ext_{A(1)}^{s,t}(M \otimes N_\sigma, \mathbb{F}_2) \Longrightarrow_\sigma Ext_{A(2)}^{s+\sigma,t}(M, \mathbb{F}_2)$$

  for specific $A(1)$-modules N_σ, $\sigma \geq 0$.
- Multiplicative structure only observed in hindsight.

Module coalgebra

- Steenrod algebra $A = \langle Sq^k \mid k \geq 1 \rangle$.
- Coproduct $\psi(Sq^k) = \sum_{i+j=k} Sq^i \otimes Sq^j$.
- Cocommutative sub Hopf algebras $A(1) \subset A(2) \subset A$.
- $A(1)$ is not normal as a subalgebra of $A(2)$.
-
$$A(2) \twoheadrightarrow A(2)//A(1) = A(2) \otimes_{A(1)} \mathbb{F}_2$$

  is a quotient $A(2)$-module coalgebra, but not a quotient algebra.

Comodule algebra

- Dual Steenrod algebra $A_* = \mathbb{F}_2[\bar{\xi}_k \mid k \geq 1]$ with $|\bar{\xi}_k| = 2^k - 1$.
- Coproduct $\psi(\bar{\xi}_k) = \sum_{i+j=k} \bar{\xi}_i \otimes \bar{\xi}_j^{2^i}$.
- Commutative quotient Hopf algebras $A_* \twoheadrightarrow A(2)_* \twoheadrightarrow A(1)_*$.
- $A(1)_* = \mathbb{F}_2[\xi_1, \bar{\xi}_2]/(\xi_1^4, \bar{\xi}_2^2)$.
- $A(2)_* = \mathbb{F}_2[\xi_1, \bar{\xi}_2, \bar{\xi}_3]/(\xi_1^8, \bar{\xi}_2^4, \bar{\xi}_3^2)$.
- $$A(2)_* \,\Box_{A(1)_*}\, \mathbb{F}_2 = E[\xi_1^4, \bar{\xi}_2^2, \bar{\xi}_3] \rightarrowtail A(2)_*$$

  is a sub $A(2)_*$-comodule algebra, but not a sub coalgebra.

General context (à la Cartan–Eilenberg/Adams)

- Γ Hopf algebra over a field k.
- Λ sub Hopf algebra of Γ.
- $\Omega = \Gamma/\!/\Lambda = \Gamma \otimes_\Lambda k$ quotient Γ-module coalgebra of Γ.

$$\Lambda \rightarrowtail \Gamma \twoheadrightarrow \Omega$$

- Aim: Calculate algebra $Ext^*_\Gamma(k, k)$ in terms of $Ext^*_\Lambda(N, k)$ for suitable Λ-modules N.

Dual context (à la Eilenberg–Moore/Milnor–Moore)

- Γ_* Hopf algebra over a field k.
- Λ_* quotient Hopf algebra of Γ_*.
- $\Omega_* = \Gamma_* \square_{\Lambda_*} k$ sub Γ_*-comodule algebra of Γ_*.

$$\Omega_* \rightarrowtail \Gamma_* \twoheadrightarrow \Lambda_*$$

- Aim: Calculate algebra $Ext^*_{\Gamma_*}(k, k)$ in terms of $Ext^*_{\Lambda_*}(k, R)$ for suitable Λ_*-comodules R.

Cartan–Eilenberg (1956)

- Suppose (temporarily) that Λ is normal in Γ.
- $\Omega = \Gamma//\Lambda$ is a quotient Hopf algebra of Γ.
- Cartan–Eilenberg spectral sequence (CESS)

$$E_2^{\sigma,s} = Ext_{\Omega}^{\sigma}(k, Ext_{\Lambda}^{s}(k,k)) \Longrightarrow_{\sigma} Ext_{\Gamma}^{s+\sigma}(k,k)$$

- Equivalently

$$E_2^{\sigma,s} = Ext_{\Omega_*}^{\sigma}(Ext_{\Lambda_*}^{s}(k,k),k) \Longrightarrow_{\sigma} Ext_{\Gamma_*}^{s+\sigma}(k,k)$$

CESS for $A(1)$

- Example: $\Gamma = A(1)$, $\Lambda = E[Q_1]$, $\Omega = E[Sq^1, Sq^2]$.
- $Ext_{\Lambda}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2[v_1]$ trivial Ω-module, $Ext_{\Omega}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2[h_0, h_1]$.

$$E_2^{*,*,*} = \mathbb{F}_2[h_0, h_1] \otimes \mathbb{F}_2[v_1] \, .$$

- $d_2(v_1) = h_0 h_1$.

$$E_3^{*,*,*} = \mathbb{F}_2[h_0, h_1]/(h_0 h_1) \otimes \mathbb{F}_2[v_1^2] \, .$$

- $d_3(v_1^2) = h_1^3$, $E_4 = E_\infty$.

(E_3, d_3) of CESS for $A(1)$

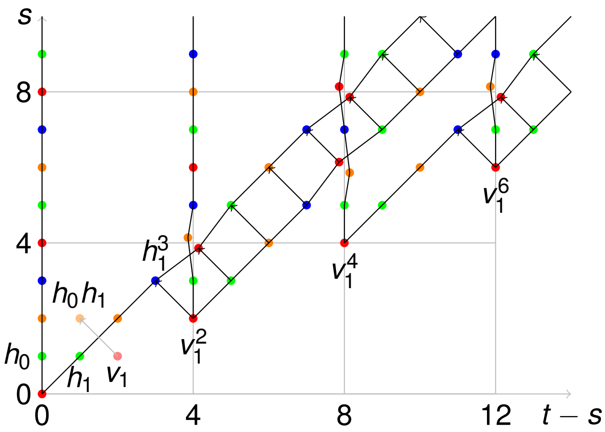


Figure: $E_2^{\sigma,s,*} = Ext^{\sigma,*}_{E[Sq^1,Sq^2]}(\mathbb{F}_2, Ext^{s,*}_{E[Q_1]}(\mathbb{F}_2,\mathbb{F}_2)) \Longrightarrow_\sigma Ext^{s+\sigma,*}_{A(1)}(\mathbb{F}_2,\mathbb{F}_2)$

Davis–Mahowald (1982)

- Allow (from now on) that Λ is not normal in Γ.
- $\Omega = \Gamma//\Lambda$ is a Γ-module coalgebra.
- Require (suitable) Γ-module coalgebra resolution $(\Omega \otimes N_*, d) \to k$.
- Davis–Mahowald spectral sequence (DMSS)

$$E_1^{\sigma,s,*} \Longrightarrow_\sigma Ext_\Gamma^{s+\sigma,*}(k,k)$$

where

$$E_1^{\sigma,s,*} = Ext_\Gamma^{s,*}(\Omega \otimes N_\sigma, k) \cong Ext_\Gamma^{s,*}(\Gamma \otimes_\Lambda N_\sigma, k) \cong Ext_\Lambda^{s,*}(N_\sigma, k), .$$

- Assume Γ is cocommutative to make untwisting $\Omega \otimes N_\sigma \cong \Gamma \otimes_\Lambda N_\sigma$ comultiplicative.

DMSS, dual formulation

- Γ_* commutative Hopf algebra.
- Λ_* quotient Hopf algebra of Γ.
- $\Omega_* = \Gamma_* \,\Box_{\Lambda_*}\, k$ left Γ_*-comodule algebra.
- Require (suitable) Γ_*-comodule algebra resolution $k \to (\Omega_* \otimes R^*, d)$.
- Get multiplicative Davis–Mahowald spectral sequence

$$E_1^{\sigma,s,*} = Ext_{\Lambda_*}^{s,*}(k, R^\sigma) \Longrightarrow_\sigma Ext_{\Gamma_*}^{s+\sigma,*}(k,k)\,.$$

- Untwisting $\Omega_* \otimes R^\sigma \cong \Gamma_* \,\Box_{\Lambda_*}\, R^\sigma$ is multiplicative for commutative Γ_*.

DMSS for $A(1)$

- Example: $\Gamma = A(1)$, $\Lambda = A(0) = E[Sq^1]$, $\Omega_* = E[\xi_1^2, \bar{\xi}_2]$.
- Resolve using $A(1)_*$-comodule algebra $R^* = \mathbb{F}_2[x_2, x_3]$, with coaction $\nu(x_2) = 1 \otimes x_2$, $\nu(x_3) = 1 \otimes x_3 + \xi_1 \otimes x_2$.
- Resolution $\mathbb{F}_2 \to \Omega_* \otimes R^*$ has differential $d(\xi_1^2) = x_2$, $d(\bar{\xi}_2) = x_3$.
- $$R^\sigma = \mathbb{F}_2\{x_2^\sigma, \dots, x_3^\sigma\} = \mathbb{F}_2\{x_2^i x_3^j \mid i + j = \sigma\}\,.$$
- $Ext^{*,*}_{A(0)_*}(\mathbb{F}_2, R^0) = \mathbb{F}_2[h_0]$, $Ext^{*,*}_{A(0)_*}(\mathbb{F}_2, R^1) = \mathbb{F}_2\{x_2\}$.

DMSS (E_1^*, d_1) for $A(1)$

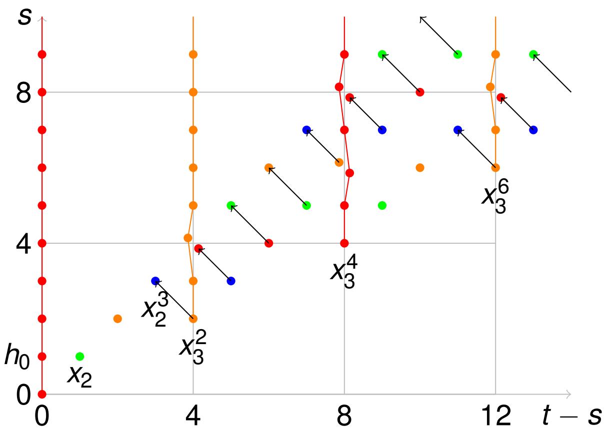


Figure: $E_1^{\sigma,s,*} = Ext_{A(0)_*}^{s,*}(\mathbb{F}_2, R^\sigma) \Longrightarrow_\sigma Ext_{A(1)_*}^{s+\sigma,*}(\mathbb{F}_2, \mathbb{F}_2)$

DMSS $E_\infty^* \Longrightarrow Ext_{A(1)}(\mathbb{F}_2, \mathbb{F}_2)$

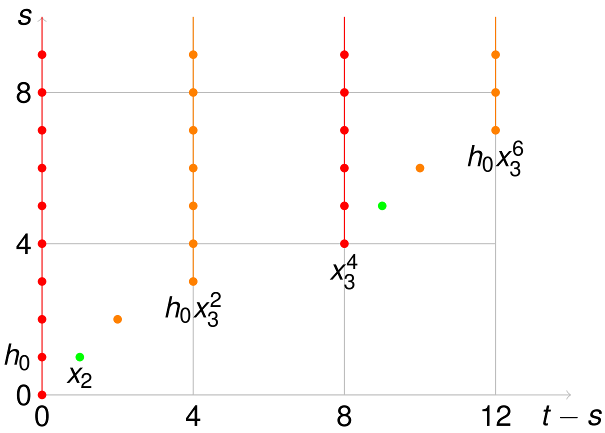


Figure: $h_1 = x_2$, $v = h_0 x_3^2$, $w_1 = x_3^4$

Main construction, 1/4

- $\Gamma_* \twoheadrightarrow \Lambda_*$ and $\Omega_* \subset \Gamma_*$ as above.
- Assume a graded Γ_*-comodule algebra $R^* = \bigoplus_\sigma R^\sigma$ and homomorphisms $d\colon \Omega_* \otimes R^\sigma \to \Omega_* \otimes R^{\sigma+1}$
- such that $(\Omega_* \otimes R^*, d)$ is a differential graded Γ_*-comodule algebra and the unit $k \to (\Omega_* \otimes R^*, d)$ is a quasi-isomorphism.
- Get an exact complex

$$0 \to \mathbb{F}_2 \to \Omega_* \otimes R^0 \xrightarrow{d} \Omega_* \otimes R^1 \xrightarrow{d} \Omega_* \otimes R^2 \xrightarrow{d} \dots .$$

Main construction, 2/4

- Consider Γ_*-comodules M, N.
- Let $(C^*_{\Gamma_*}(k, M), d)$ be the cobar complex with cohomology $Ext^s_{\Gamma_*}(k, M)$ in degree s.
- Here
$$C^s_{\Gamma_*}(k, M) = \Gamma_* \otimes \cdots \otimes \Gamma_* \otimes M$$
with s copies of Γ_*.
- Alexander–Whitney pairing $C^s_{\Gamma_*}(k, M) \otimes C^t_{\Gamma_*}(k, N) \to C^{s+t}_{\Gamma_*}(k, M \otimes N)$ induces cup product in Ext.

Main construction, 3/4

- The unit induces a quasi-isomorphism

$$C^*_{\Gamma_*}(k,k) \to C^*_{\Gamma_*}(k, \Omega_* \otimes R^*)$$

  of differential graded algebras (DGAs).
- The RHS is a filtered DGA, with

$$F^\sigma = C^*_{\Gamma_*}(k, \Omega_* \otimes R^{*\geq\sigma})$$

  and

$$F^\sigma / F^{\sigma+1} = C^*_{\Gamma_*}(k, \Omega_* \otimes R^\sigma)\,.$$

Main construction, 4/4

- Get an algebra spectral sequence

$$E_1^{\sigma,s} \Longrightarrow_\sigma Ext_{\Gamma_*}^{s+\sigma}(k,k)$$

with

$$E_1^{\sigma,s} = Ext_{\Gamma_*}^s(k, \Omega_* \otimes R^\sigma) \cong Ext_{\Gamma_*}^s(k, \Gamma_* \Box_{\Lambda_*} R^\sigma) \cong Ext_{\Lambda_*}^s(k, R^\sigma)\,.$$

- Uses untwisting $\Omega_* \otimes R^\sigma \cong \Gamma_* \Box_{\Lambda_*} R^\sigma$ and change-of-rings along $\Gamma_* \twoheadrightarrow \Lambda_*$.
- Product $E_1^{\sigma,*} \otimes E_1^{\tau,*} \to E_1^{\sigma+\tau,*}$ equals pairing induced by Λ_*-comodule product $R^\sigma \otimes R^\tau \to R^{\sigma+\tau}$.

Outline

The Adams spectral sequence

The Davis–Mahowald spectral sequence

Calculation of *Ext* over $A(2)$

Main example: $A(2)$

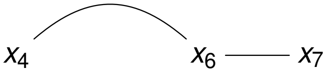
- $\Gamma_* = A(2)_* \twoheadrightarrow A(1)_* = \Lambda_*$ commutative Hopf algebras.
- $$\Omega_* = A(2)_* \,\square_{A(1)_*}\, \mathbb{F}_2 = E[\xi_1^4, \bar{\xi}_2^2, \bar{\xi}_3]$$
  left $A(2)_*$-comodule algebra.
- Resolve by $A(2)_*$-comodule algebra
  $$R^* = \mathbb{F}_2[x_4, x_6, x_7]\,.$$
- Coaction $\nu(x_4) = 1 \otimes x_4$, $\nu(x_6) = 1 \otimes x_6 + \xi_1^2 \otimes x_4$,
  $\nu(x_7) = 1 \otimes x_7 + \xi_1 \otimes x_6 + \bar{\xi}_2 \otimes x_4$.
- Resolution $\mathbb{F}_2 \to \Omega_* \otimes R^*$ has differential $d(\xi_1^4) = x_4$, $d(\bar{\xi}_2^2) = x_6$, $d(\bar{\xi}_3) = x_7$.

DMSS for $A(2)$

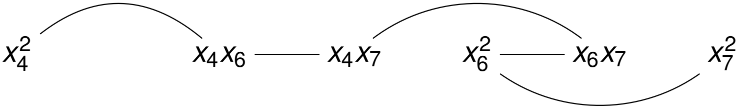
- Graded pieces $R^* = \bigoplus_\sigma R^\sigma$.
- $$R^\sigma = \mathbb{F}_2\{x_4^i x_6^j x_7^k \mid i + j + k = \sigma\}\,.$$
- Algebra spectral sequence
$$E_1^{\sigma,s,*} = Ext_{A(1)_*}^{s,*}(\mathbb{F}_2, R^\sigma) \Longrightarrow_\sigma Ext_{A(2)_*}^{s+\sigma,*}(\mathbb{F}_2, \mathbb{F}_2)$$
- Abutment equals the E_2-term of the Adams SS converging to $\pi_* tmf$.

$A(1)_*$-comodules R^σ

- $R^0 = \mathbb{F}_2$.
- $R^1 = \mathbb{F}_2\{x_4, x_6, x_7\}$:



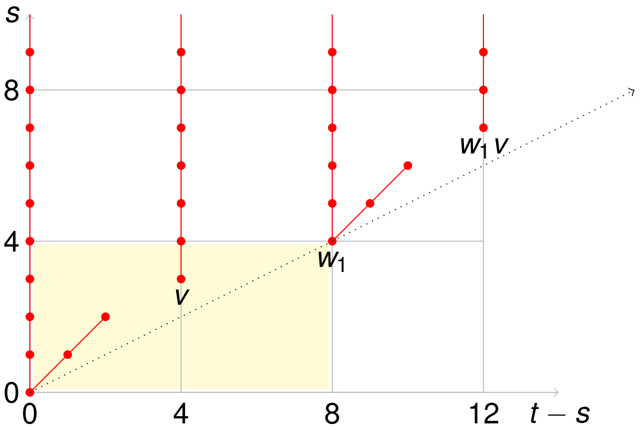
- $R^2 = \mathbb{F}_2\{x_4^2, x_4x_6, x_4x_7, x_6^2, x_6x_7, x_7^2\}$:



- $R^3 = \mathbb{F}_2\{x_4^3, x_4^2x_6, x_4^2x_7, x_4x_6^2, x_4x_6x_7, x_6^3, x_4x_7^2, x_6^2x_7, x_6x_7^2, x_7^3\}$.

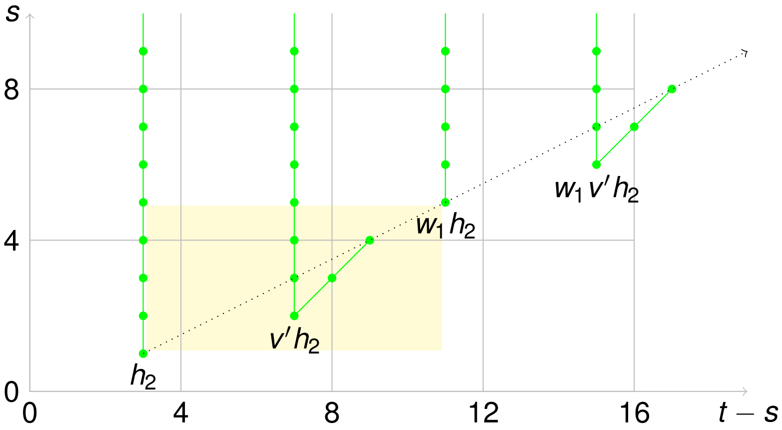
$\sigma = 0$

- $R^0 = \mathbb{F}_2$
- $E_1^{0,*,*} = Ext^{*,*}_{A(1)_*}(\mathbb{F}_2, \mathbb{F}_2) = ko^{*,*}$.



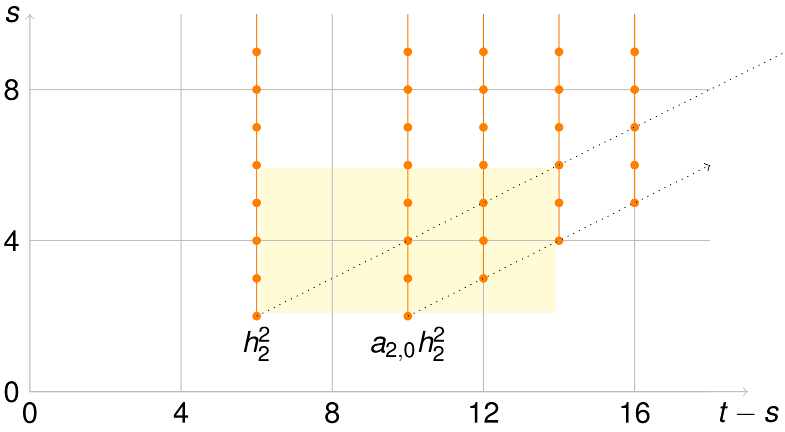
$\sigma = 1$

- $R^1 = \mathbb{F}_2\{x_4, x_6, x_7\} = \Sigma^4 H_*(S \cup_\eta e^2 \cup_2 e^3)$.
- $E_1^{1,*,*} = Ext^{*,*}_{A(1)_*}(\mathbb{F}_2, R^1) = ksp^{*,*}\{h_2\}$.



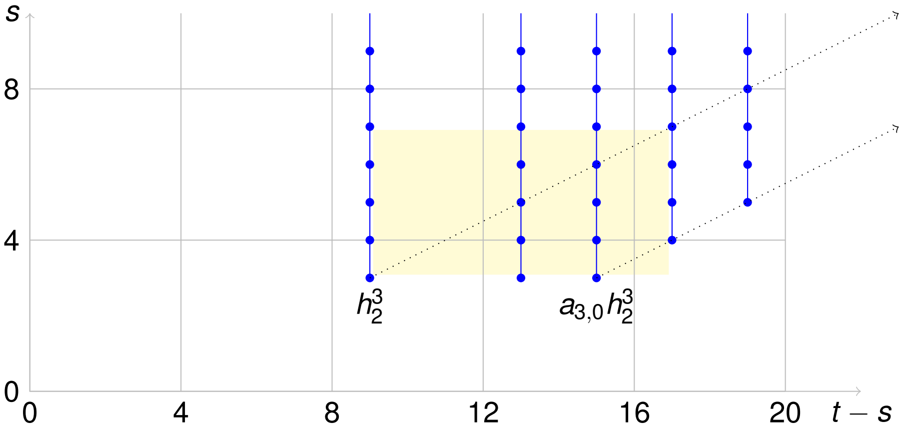
$\sigma = 2$

- $R^2 = \mathbb{F}_2\{x_4^2, x_4x_6, x_4x_7, x_6^2, x_6x_7, x_7^2\}$.
- $E_1^{2,*,*} = Ext_{A(1)_*}^{*,*}(\mathbb{F}_2, R^2) = G_2^{*,*}\{h_2^2\}$.



$\sigma = 3$

- $\dim R^3 = 10$.
- $E_1^{3,*,*} = Ext^{*,*}_{A(1)_*}(\mathbb{F}_2, R^3) = G_3^{*,*}\{h_2^3\}$.



Extensions by $\mathbb{F}_2[x_7^4]$

- $x_7^4 \in R^4$ is $A(1)_*$-comodule primitive.
- $A(1)_*$-comodule algebra extension

$$\mathbb{F}_2[x_7^4] \rightarrowtail R^* \twoheadrightarrow \bar{R}^* \,.$$

- $\bar{R}^* = \bigoplus_\sigma \bar{R}^\sigma$ where $\bar{R}^\sigma = \mathbb{F}_2\{x_4^i x_6^j x_7^k \mid i + j + k = \sigma, 0 \le k \le 3\}$.
- Algebra extension

$$\mathbb{F}_2[x_7^4] \rightarrowtail E_1^{*,*,*} \twoheadrightarrow \bar{E}_1^{*,*,*} \,.$$

The charts $G_\sigma^{*,*}$

- $\bar{E}_1^{\sigma,s,*} = Ext_{A(1)_*}^{s,*}(\mathbb{F}_2, \bar{R}^\sigma)$.
- Starts with $A(1)_*$-comodule primitive x_4^σ in $(t-s,s) = (4\sigma, 0)$.
- Define chart $G_\sigma^{*,*}$ by
$$\Sigma^{4\sigma} G_\sigma^{*,*} = Ext_{A(1)_*}^{*,*}(\mathbb{F}_2, \bar{R}^\sigma)\,.$$
- Starts at origin $(t-s,s) = (0,0)$.
- Contributes $G_\sigma^{s,t}\{h_2^\sigma\}$ towards $Ext_{A(2)_*}^{s+\sigma,t+4\sigma}(\mathbb{F}_2,\mathbb{F}_2)$.
- $G_0^{*,*} = ko^{*,*}$ and $G_1^{*,*} = ksp^{*,*}$.

Adams covers of *ku*

- Can calculate $G_\sigma^{*,*}$ for $\sigma \geq 2$ in terms of Adams covers $ku\langle\sigma\rangle$ of ku.
- Consider minimal Adams tower of $ku = ku\langle 0\rangle$:

$$\begin{array}{ccccccccc}
\dots \longrightarrow & ku\langle\sigma+1\rangle & \longrightarrow & ku\langle\sigma\rangle & \longrightarrow & \dots & \longrightarrow & ku\langle 1\rangle & \longrightarrow & ku \\
 & \nwarrow & & \downarrow & \nwarrow & & & \downarrow & \nwarrow & \downarrow \\
 & & & \bigvee_{i=0}^{\sigma} \Sigma^{2i} H & & & & H \vee \Sigma^2 H & & H
\end{array}$$

- Product $ku \wedge ku \to ku$ lifts to pairings $ku\langle\sigma\rangle \wedge ku\langle\tau\rangle \to ku\langle\sigma+\tau\rangle$.

Comodule syzygies

- $H^*(ku) = A \otimes_{E(1)} \mathbb{F}_2$, so $H_*(ku) = A_* \,\square_{E(1)_*}\, \mathbb{F}_2$.
- Generally,

$$H_*\Sigma^\sigma ku\langle\sigma\rangle \cong A_* \,\square_{E(1)_*}\, \Omega^\sigma_{E(1)_*}(\mathbb{F}_2)$$

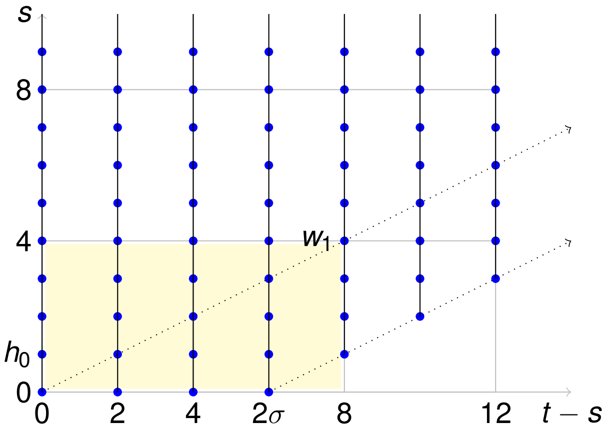
  where $\Omega^\sigma_{E(1)_*}(\mathbb{F}_2)$ is the σ-th $E(1)_*$-comodule syzygy of $\mathbb{F}_2$.
- Adams chart for $\Sigma^\sigma ku\langle\sigma\rangle$ is

$$Ext^{*,*}_{A_*}(\mathbb{F}_2, A_* \,\square_{E(1)_*}\, \Omega^\sigma_{E(1)_*}(\mathbb{F}_2)) \cong Ext^{*,*}_{E(1)_*}(\mathbb{F}_2, \Omega^\sigma_{E(1)_*}(\mathbb{F}_2)) \,.$$

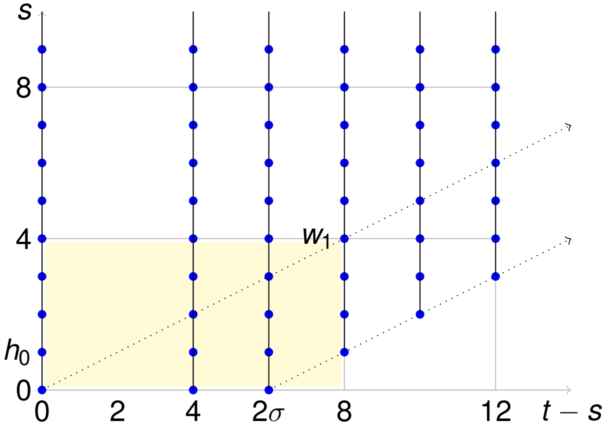
- $E(1)_*$-comodule pairing

$$\Omega^\sigma_{E(1)_*}(\mathbb{F}_2) \otimes \Omega^\tau_{E(1)_*}(\mathbb{F}_2) \to \Omega^{\sigma+\tau}_{E(1)_*}(\mathbb{F}_2) \,.$$

Adams spectral sequence for $ku\langle\sigma\rangle$ (in the case $\sigma = 3$)



The chart $G_\sigma^{*,*}$ (in the case $\sigma = 3$)



A comparison result for $\sigma \geq 2$

- Proposition: $G_\sigma^{*,*}$ is the subchart of the Adams chart for $ku\langle\sigma\rangle$ where the classes with $t - s = 2$ are omitted.
- Additively
$$G_\sigma^{*,*} = \mathbb{F}_2\{a_{n,s} \mid s \geq 0, 0 \leq n \leq s + \sigma, n \neq 1\}$$
with $a_{n,s}$ in Adams bidegree $(t - s, s) = (2n, s)$.
- $ko^{*,*}$-module action is given by $h_0 \cdot a_{n,s} = a_{n,s+1}$, $h_1 \cdot a_{n,s} = 0$, $v \cdot a_{n,s} = a_{n+2,s+3}$ and $w_1 \cdot a_{n,s} = a_{n+4,s+4}$.
- The pairing $G_\sigma^{*,*} \otimes G_\tau^{*,*} \to G_{\sigma+\tau}^{*,*}$ is given by $a_{n,s} \cdot a_{m,t} = a_{n+m,s+t}$.

Proof

- Explicit, injective, $A(1)_*$-comodule algebra homomorphism

$$\phi\colon \bar{R}^* = \mathbb{F}_2[x_4, x_6, x_7]/(x_7^4) \to \bigoplus_\sigma \Sigma^{3\sigma} A(1)_* \,\square_{E(1)_*}\, \Omega^\sigma_{E(1)_*}(\mathbb{F}_2)\,.$$

- For $\sigma \geq 2$, short exact sequence

$$0 \to \bar{R}^\sigma \to \Sigma^{3\sigma} A(1)_* \,\square_{E(1)_*}\, \Omega^\sigma_{E(1)_*}(\mathbb{F}_2) \to \Sigma^{4\sigma+2} A(1)_* \,\square_{A(0)_*}\, \mathbb{F}_2 \to 0$$

induces SES

$$0 \to G_\sigma^{*,*} \to ku\langle\sigma\rangle^{*,*} \to \mathbb{F}_2[h_0]\{a_{1,0}\} \to 0\,.$$

- Graded algebra homomorphism

$$\phi_*\colon \bigoplus_\sigma G_\sigma^{*,*} \to \bigoplus_\sigma ku\langle\sigma\rangle^{*,*}$$

determines product at the LHS.

DMSS (E_1, d_1)-complex

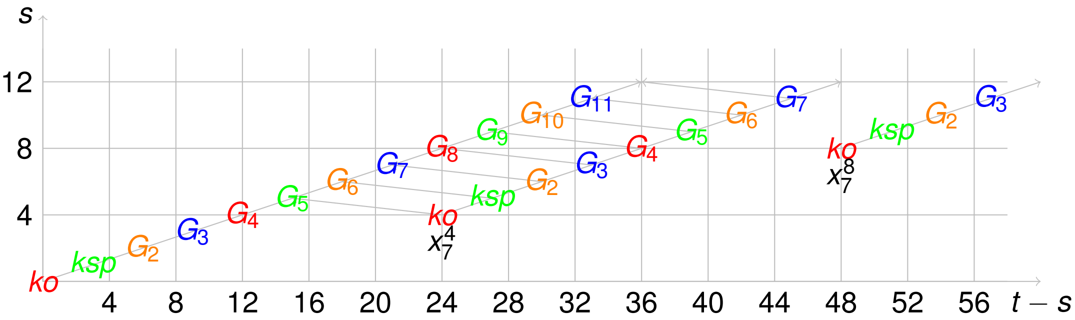


Figure: $\mathbb{F}_2[x_7^4] \to E_1^{*,*,*} \to \bigoplus_\sigma G_\sigma^{*,*}\{h_2^\sigma\}$

DMSS $d_1^0 : E_1^{0,*,*} \to E_1^{1,*,*}$

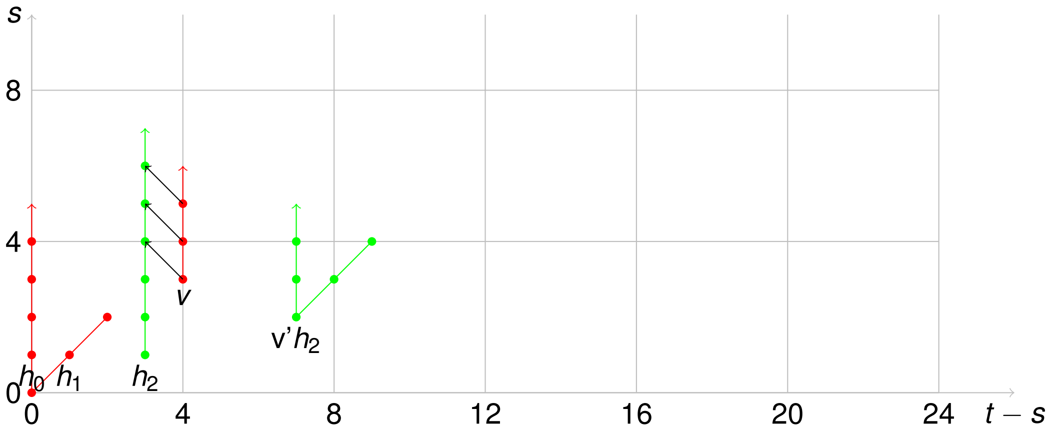


Figure: $d_1^0(v) = h_0^3 h_2$

DMSS $d_1^1 : E_1^{1,*,*} \to E_1^{2,*,*}$

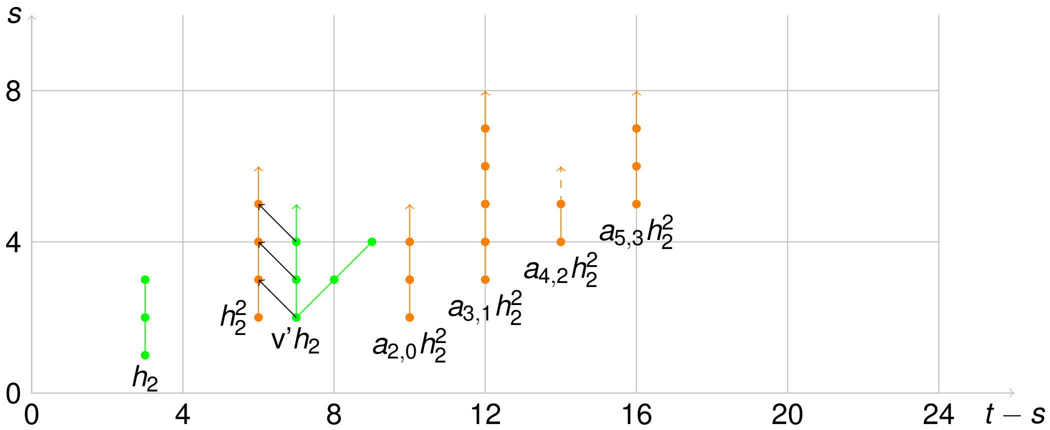


Figure: $d_1^1(v' h_2) = h_0 h_2^2$

DMSS $d_1^2 : E_1^{2,*,*} \to E_1^{3,*,*}$

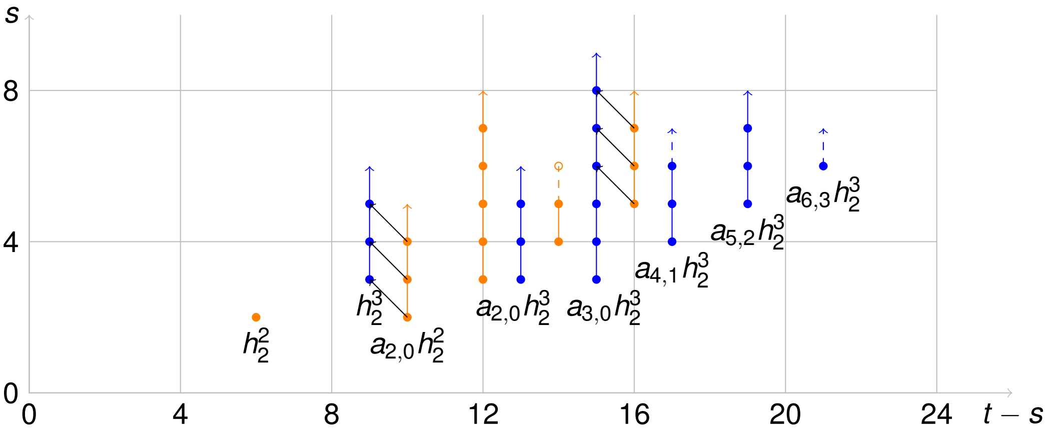


Figure: $d_1^2(a_{2,0}h_2^2) = h_2^3$, $d_1^2(a_{5,3}h_2^2) = a_{3,3}h_2^3$

DMSS $d_1^3 : E_1^{3,*,*} \to \bar{E}_1^{4,*,*}$

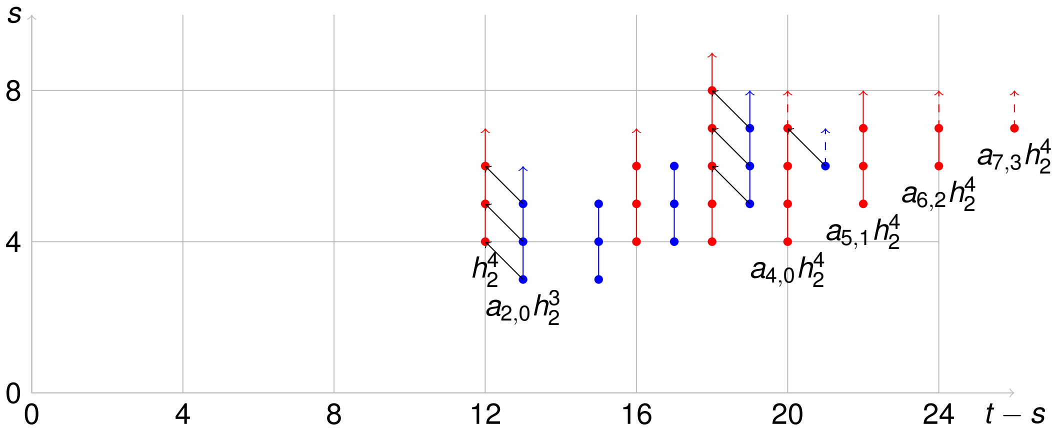


Figure: $d_1^3(a_{2,0}h_2^3) = h_2^4$, $d_1^3(a_{5,2}h_2^3) = a_{3,2}h_2^4$, $d_1^3(a_{6,3}h_2^3) = a_{4,3}h_2^4$

DMSS $d_1^4 : \bar{E}_1^{4,*,*} \to \bar{E}_1^{5,*,*}$

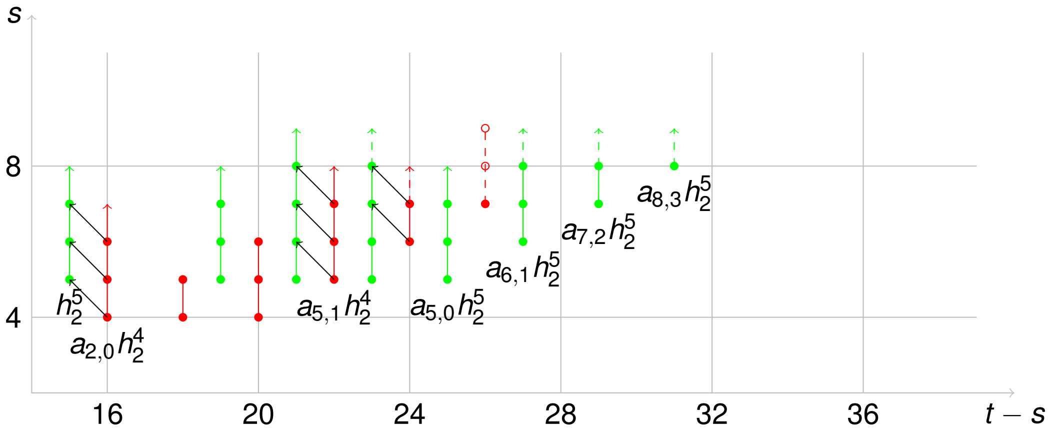


Figure: $d_1^4(a_{2,0}h_2^4) = h_2^5$, $d_1^4(a_{5,1}h_2^4) = a_{3,1}h_2^5$, $d_1^4(a_{6,2}h_2^4) = a_{4,2}h_2^5$

DMSS $d_1^5 \colon \bar{E}_1^{5,*,*} \to \bar{E}_1^{6,*,*}$

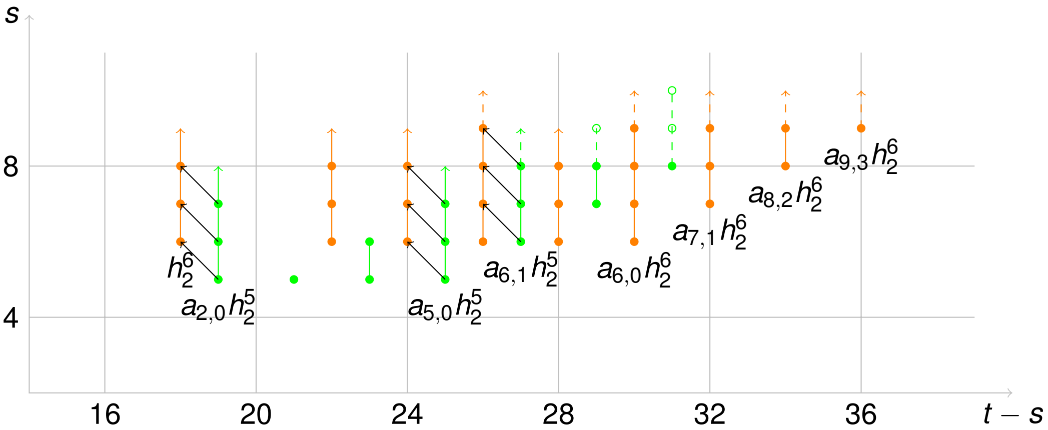


Figure: $d_1^5(a_{2,0}h_2^5) = h_2^6$, $d_1^5(a_{5,0}h_2^5) = a_{3,0}h_2^6$, $d_1^5(a_{6,1}h_2^5) = a_{4,1}h_2^6$

DMSS $d_1^6 \colon \bar{E}_1^{6,*,*} \to \bar{E}_1^{7,*,*}$

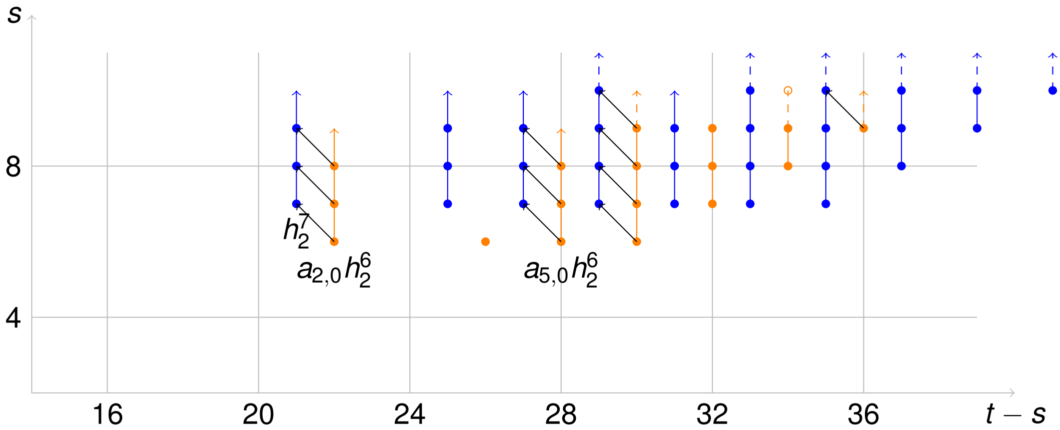


Figure: $d_1^6(a_{2,0}h_2^6) = h_2^7$, $d_1^6(a_{5,0}h_2^6) = a_{3,0}h_2^7$, $d_1^6(a_{6,0}h_2^6) = a_{4,0}h_2^7$, $d_1^6(a_{9,3}h_2^6) = a_{7,3}h_2^7$

DMSS E_2 for $\sigma \leq 0$

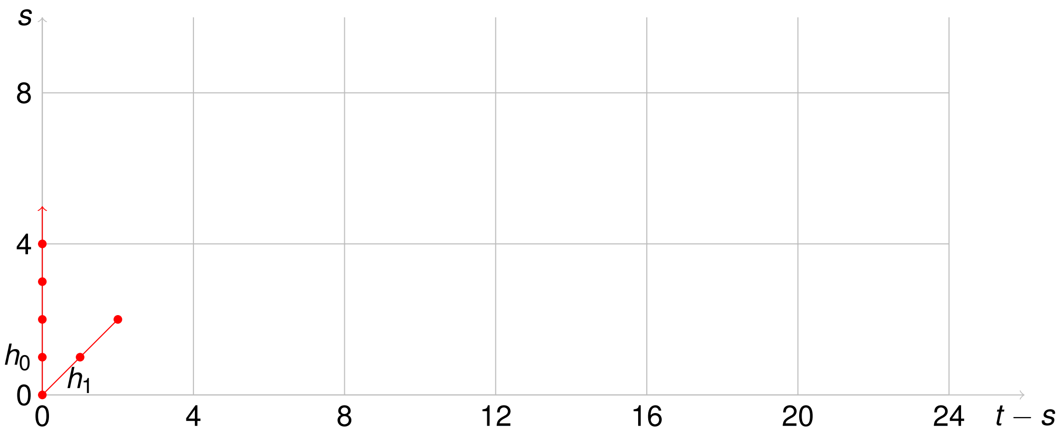


Figure: $\mathbb{F}_2[w_1]$-basis for E_2^0

DMSS E_2 for $\sigma \leq 1$

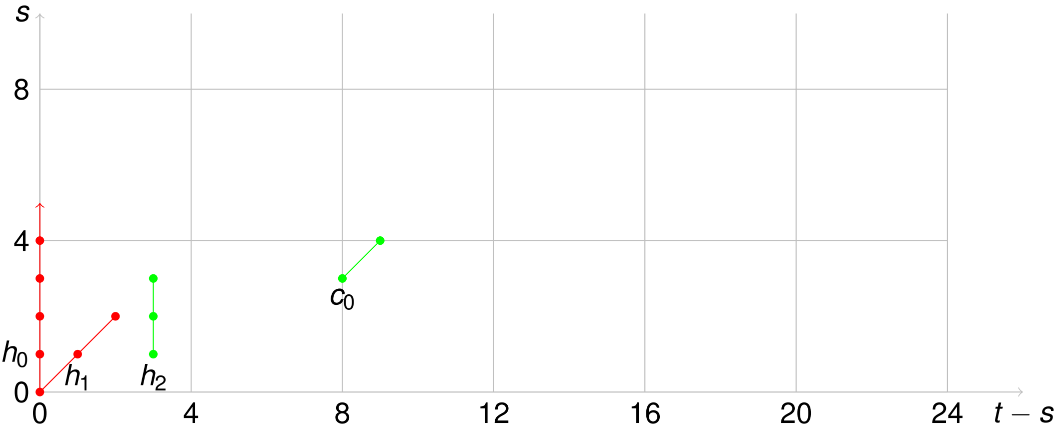


Figure: $c_0 = h_1 v' h_2$

DMSS E_2 for $\sigma \leq 2$

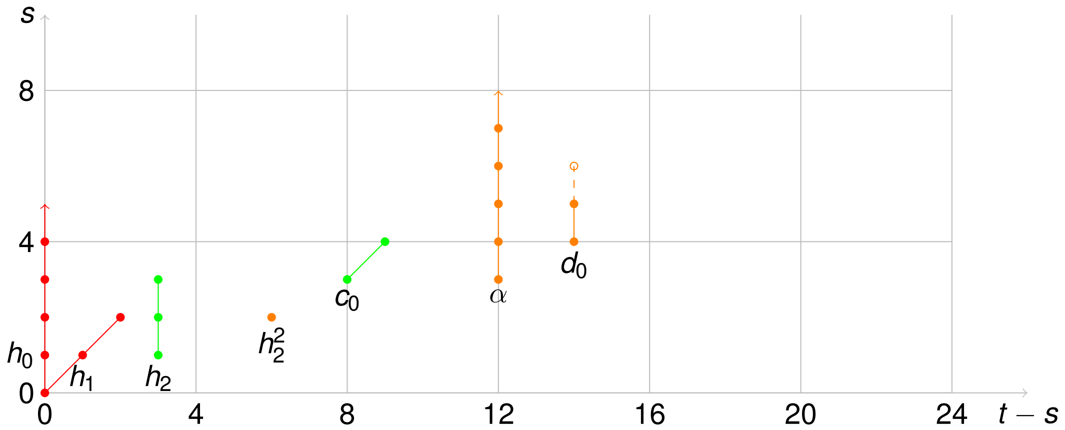


Figure: $\alpha = a_{3,1} h_2^2$, $d_0 = a_{4,2} h_2^2$, $w_1 h_2^2 = h_0^2 d_0$

DMSS E_2 for $\sigma \leq 3$

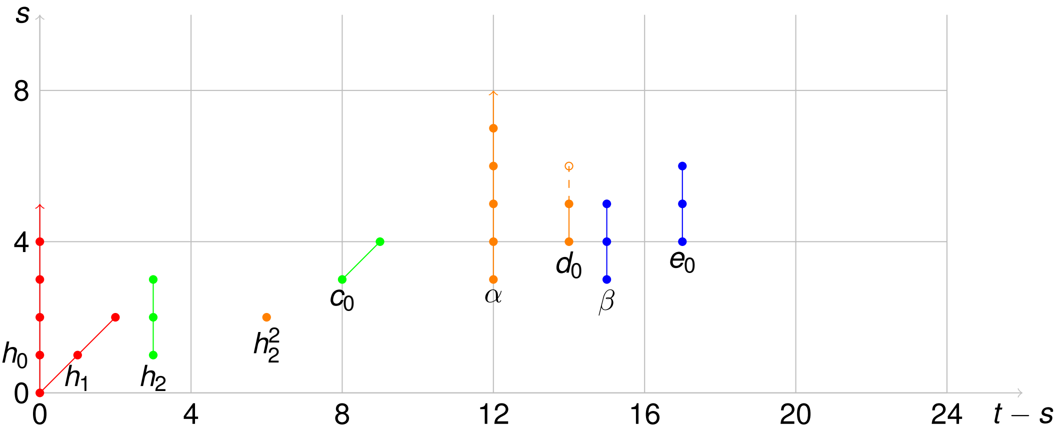


Figure: $\beta = a_{3,0} h_2^3$, $e_0 = a_{4,1} h_2^3$

DMSS $\bar{E}_2$ for $\sigma \leq 4$

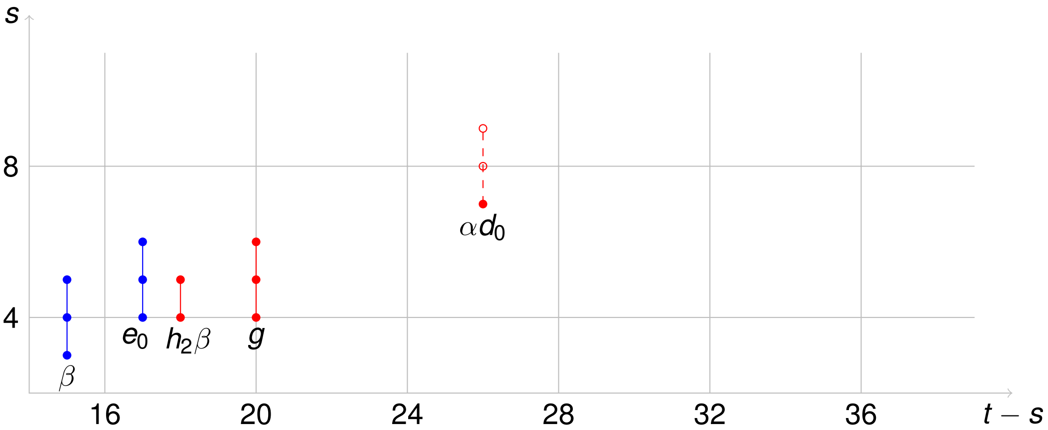


Figure: $h_2\beta = a_{3,0}h_2^4$, $g = a_{4,0}h_2^4$, $\alpha d_0 = a_{7,3}h_2^4$

DMSS $\bar{E}_2$ for $\sigma \leq 5$

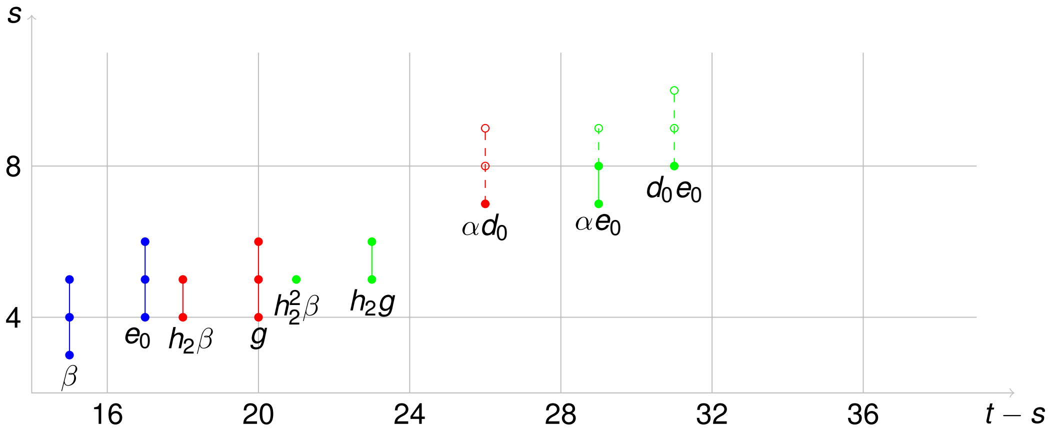


Figure: $h_2^2\beta = a_{3,0}h_2^5$, $h_2 g = a_{4,0}h_2^5$, $\alpha e_0 = a_{7,2}h_2^5$, $d_0 e_0 = a_{8,3}h_2^5$

DMSS $\bar{E}_2$ for $\sigma \leq 6$

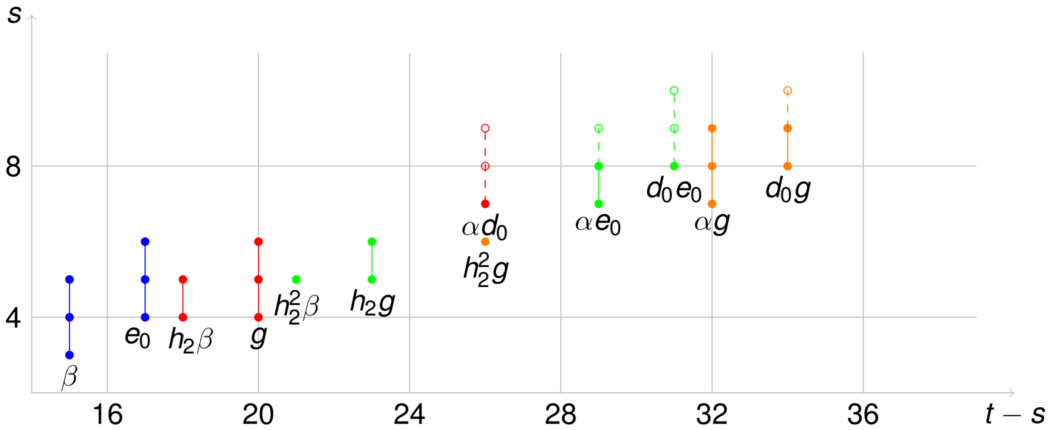


Figure: $h_2^2 g = a_{4,0} h_2^6$, $\alpha g = a_{7,1} h_2^7$, $d_0 g = a_{8,2} h_2^7$

Summary of $\bar{E}_2^*$

- Section $S\colon \bar{R}^* \to R^*$ makes $(\bar{E}_1^*, d_1)$ a subcomplex of (E_1^*, d_1).
- $d_1^\sigma(a_{n,s}h_2^\sigma) = a_{n-2,s}h_2^{\sigma+1}$ for $n \equiv 1, 2 \mod 4$, is zero otherwise.
- Homology $\bar{E}_2^\sigma = H^\sigma(\bar{E}_1^*, d_1)$ repeats with

$$g\colon \bar{E}_2^\sigma \to \bar{E}_2^{\sigma+4}$$

  surjection for $\sigma = 2$, isomorphism for $\sigma \geq 3$.
- For each $\sigma \geq 3$,

$$\bar{E}_2^\sigma = \mathbb{F}_2[w_1]\{a_{n,s}h_2^\sigma\}$$

  is free of rank six, where $0 \leq s \leq 3$, $s + \sigma - 2 \leq n \leq s + \sigma$ and $n \equiv 0, 3 \mod 4$.

Short and long exact sequences

- $x_7^8 = (x_7^4)^2 \in R^8$ is $A(2)_*$-comodule primitive. (Will represent $w_2 \mapsto v_2^8$.)
- Short exact sequence of cochain complexes

$$0 \to \bar{E}_1^* \xrightarrow{S} E_1^*/(x_7^8) \to \bar{E}_1^*\{x_7^4\} \to 0$$

  induces long exact sequence

$$\cdots \to \bar{E}_2^* \xrightarrow{S} E_2^*/(x_7^8) \to \bar{E}_2^*\{x_7^4\} \xrightarrow{\delta} \bar{E}_2^{*+1} \to \dots$$

- Connecting homomorphism $\delta\colon \bar{E}_2^\sigma\{x_7^4\} \to \bar{E}_2^{\sigma+5}$ contains “remainder” of d_1-differential.

$\bar{E}_2^*$ free over $\mathbb{F}_2[w_1]$

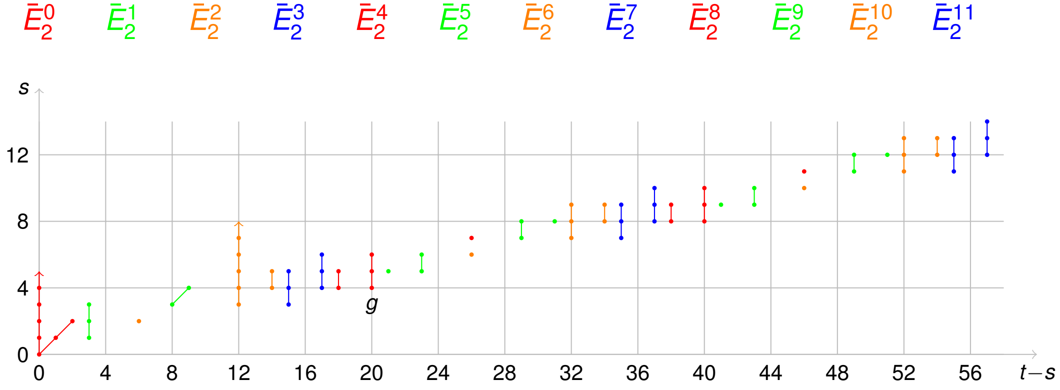


Figure: $g\colon \bar{E}_2^\sigma \to \bar{E}_2^{\sigma+4}$ onto for $\sigma = 2$, iso for $\sigma \geq 3$

$\bar{E}_2^* \oplus \bar{E}_2^*\{x_7^4\}$ free over $\mathbb{F}_2[w_1]$

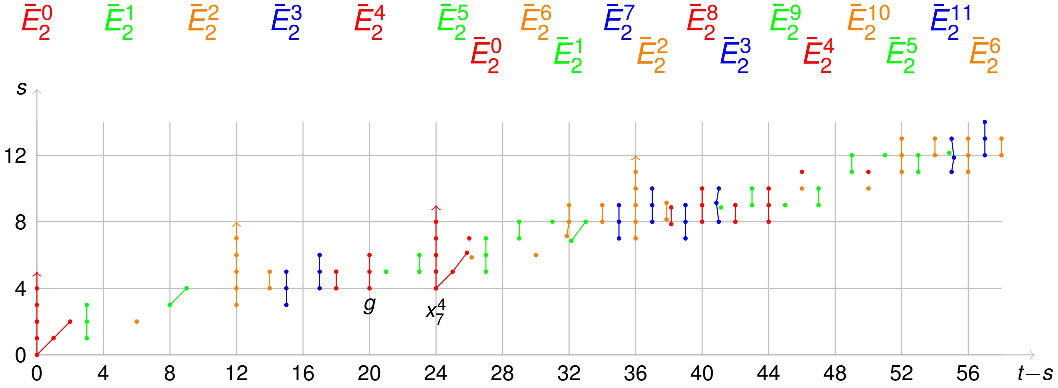


Figure: Add $\bar{E}_2^*\{x_7^4\}$ to $\bar{E}_2^*$

$\delta\colon \bar{E}_2^*\{x_7^4\} \to \bar{E}_2^{*+5}$

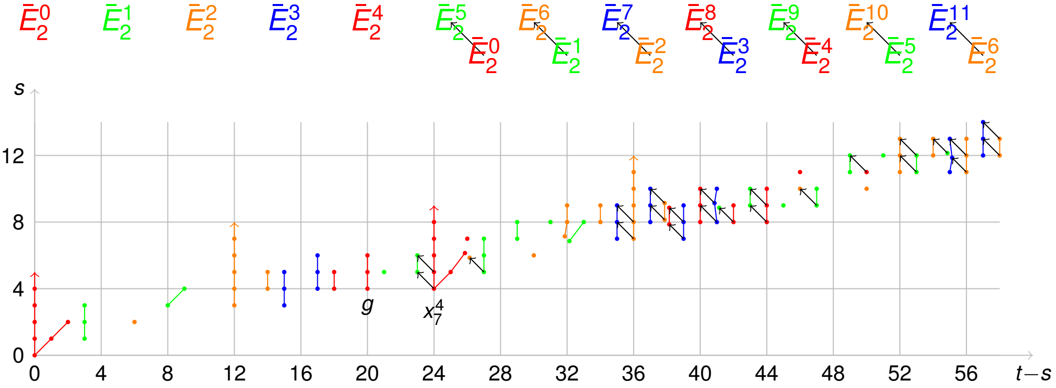


Figure: $\delta(x_7^4) = h_2 g$

DMSS $E_2^*/(x_7^8)$

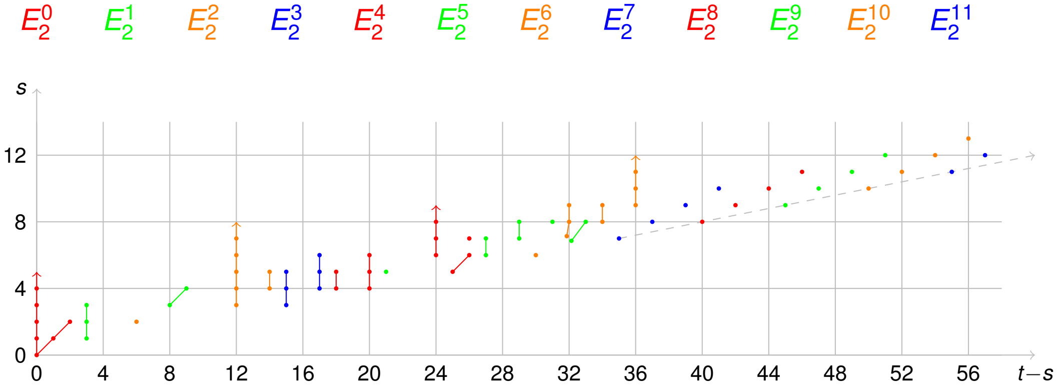


Figure: $\mathbb{F}_2[w_1]$-basis for $E_2^*/(x_7^8)$

DMSS E_2^* free over $\mathbb{F}_2[w_1, w_2]$

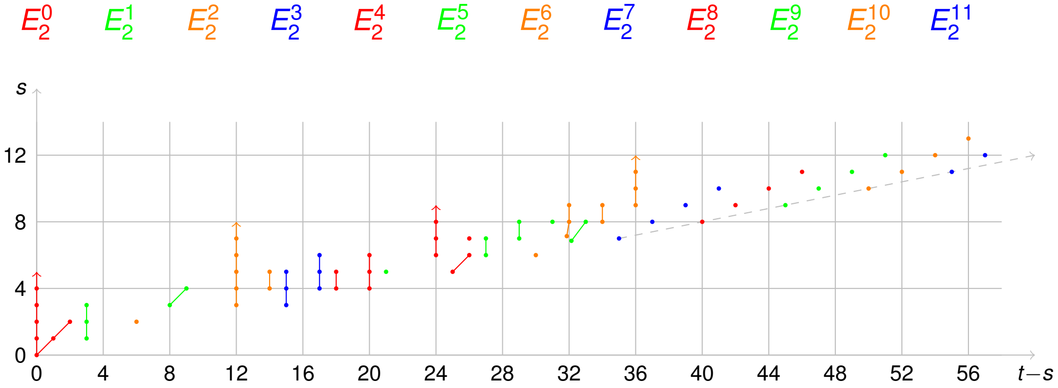


Figure: $\mathbb{F}_2[w_1, w_2]$-basis for E_2^*

Collapse at E_2^*

- No room for further differentials: $E_2^* = E_\infty^*$.
- Algebra generators for E_∞^*:

	h_0	h_1	h_2	c_0	α	d_0	β	e_0	γ	δ	g	w_1	w_2
$t-s$	0	1	3	8	12	14	15	17	25	32	20	8	48
s	1	1	1	3	3	4	3	4	5	7	4	4	8

DMSS $E_\infty^* \Longrightarrow Ext_{A(2)}(\mathbb{F}_2, \mathbb{F}_2)$

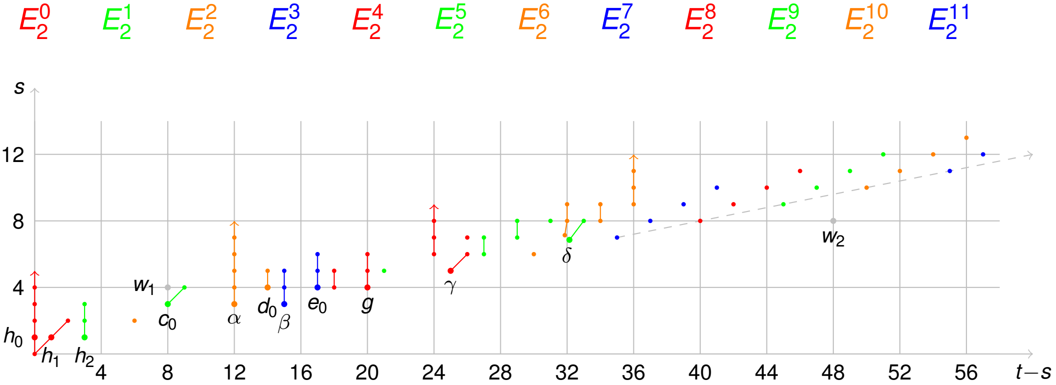


Figure: Algebra generators for $E_\infty^* \Longrightarrow Ext_{A(2)}(\mathbb{F}_2, \mathbb{F}_2)$

The Shimada-Iwai algebra

- Theorem (Shimada-Iwai (1967)):

$$Ext_{A(2)}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2[h_0, h_1, h_2, c_0, \alpha, d_0, \beta, e_0, \gamma, \delta, g, w_1, w_2]/I$$

  (13 generators) where the ideal I is generated by

$$h_0 h_1 \, , \; h_1 h_2 \, , \; h_0^2 h_2 = h_1^3 \, , \; h_0 h_2^2 \, , \; h_2^3 \, , \; \dots \, ,$$
$$\alpha^4 = h_0^4 w_2 + w_1 g^2 \, , \; h_2 \alpha^2 = h_1^2 \gamma \, , \; h_2 \alpha \beta \, , \; h_0 \alpha d_0 = h_2 w_1 \beta \, , \; h_2 \beta^2$$

  (54 relations).
- Verify relations in $Ext_{A(2)}(\mathbb{F}_2, \mathbb{F}_2)$ by machine calculation.
- Use a Gröbner basis to check that the Shimada–Iwai algebra is free as an $\mathbb{F}_2[w_1, w_2]$-module, on an explicit list of generators.
- Observe that the Davis–Mahowald E_∞-term is free over $\mathbb{F}_2[w_1, w_2]$, with the same number of generators in each bidegree.

$Ext_{A(2)}(\mathbb{F}_2, \mathbb{F}_2)$