1. Introduction

Consider a commutative ring $R$, with sum and product operations. The category of representations of $R$ inherits a commutative rig structure, given by direct sum and tensor product. In other words, the category $\text{Mod}(R)$ of $R$-modules inherits a bipermutative structure. Continuing, one can consider the categorical representations of $\text{Mod}(R)$, and these in turn form a 2-category $\text{Mod}(\text{Mod}(R))$, with a ring-like structure. Iterating, one can consider an $n$-category of higher representations, for each $n \geq 1$.

All of these constructions can take place within the limiting context of structured ring spectra, or commutative $S$-algebras. From the category of (finite cell) modules over a commutative $S$-algebra $B$ we can distill a new commutative $S$-algebra, the algebraic $K$-theory spectrum $K(B)$. Continuing, one can form $K(K(B))$, etc. When $B = HR$ is the Eilenberg–Mac Lane spectrum of an ordinary ring, the $n$-fold algebraic $K$-theory $K^{(n)}(B)$ is extracted from the $n$-category of higher representations, as above. In this sense, $n$-fold iterated algebraic $K$-theory has something to do with $n$-categories.

From this point of view it is surprising that $n$-fold iterated algebraic $K$-theory also has something to do with formal group laws of height $n$, i.e., one-dimensional commutative formal group laws $F$ in characteristic $p$ where the series expansion $[p]_F(x)$ for the multiplication-by-$p$ map starts with a unit times $x^{p^n}$. This is essentially a statement about the formal coproduct on $K^{(n)}(B)^+ (\mathbb{C}P^\infty)$ that comes from the product on $\mathbb{C}P^\infty$. Hesselholt–Madsen asked about the chromatic filtration of iterated topological cyclic homology in [HM97, p. 61], but could almost as well have asked about the chromatic filtration of iterated algebraic $K$-theory.

In a strong form, this connection implies that the algebraic $K$-theory of a structured ring spectrum related to formal group laws of height $n$ will be related to formal group laws of height $n + 1$. In terms of the periodic families of stable homotopy theory, if the homotopy of $B$ is $v_n$-periodic but not $v_{n+1}$-periodic, then frequently $K(B)$ is $v_{n+1}$-periodic but not $v_{n+2}$-periodic.

Since the (fundamental) period $|v_{n+1}| = 2p^{n+1} - 2$ of $v_{n+1}$-periodicity is longer than the period $|v_n| = 2p^n - 2$ of $v_n$-periodicity, we think of this phenomenon as an increase, or lengthening, of wavelengths. This is what we informally call a “redshift”. In a related fashion, the $v_{n+1}$-periodic phenomena are usually hidden or nested behind the $v_n$-periodic ones, hence more subtle and difficult to detect. Again this corresponds informally to less energetic light, propagating at lower frequencies.

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The height filtration is also related to the sequence of Hopf subalgebras
\[ 0 \subset \cdots \subset \mathcal{E}(n) = E(Q_0, \ldots, Q_n) \subset \cdots \]
of the Steenrod algebra \( \mathcal{A} \), and their annihilating subalgebras
\[ \mathcal{A}_* \supset \cdots \supset (\mathcal{A} / \mathcal{E}(n))^* = P(\xi_k | k \geq 1) \oplus E(\eta_k | k \geq n + 1) \supset \cdots . \]
The latter nested sequence of \( \mathcal{A}_* \)-comodule subalgebras are invariant under the Dyer–Lashof operations that arise from thinking of the dual Steenrod algebra \( \mathcal{A}_* \) as \( H_*(H) \), where \( H = HF_p \) is a commutative structured ring spectrum.

2. Redshift in the \( K \)-theory of rings

We start with examples of chromatic redshift in the algebraic \( K \)-theory of discrete rings.

Let \( k \) be a finite field of characteristic \( p \), with algebraic closure \( \bar{k} \). Quillen proved [Qui72 §11] that \( H_i(BGL(k); \mathbb{F}_p) = 0 \) for \( i > 0 \), so that \( K(\bar{k})_p \simeq H\mathbb{Z}_p \). Furthermore, he deduced that \( \pi_*, K(\bar{k})_p \cong \pi_* K(\bar{k})^{hG_k} \) for \( * \geq 0 \), where the absolute Galois group \( G_k \) acts continuously on \( K(\bar{k}) \), so \( K(\bar{k})_p \simeq H\mathbb{Z}_p \). Multiplication by \( p \) acts injectively on \( \pi_*, K(\bar{k})_p \), hence also on \( \pi_* K(\bar{k})_p \). Think of \( p \) as a lift of \( p = v_0 \in \pi_* BP \), where \( BP \) is the Brown–Peterson ring spectrum with \( \pi_* BP = \mathbb{Z}[v_1 | v_1 \geq 1] \).

For a separably closed field \( \bar{F} \) of characteristic \( \neq p \) (including 0), Lichtenbaum conjectured that \( \pi_* K(\bar{F})_p \) is \( \mathbb{Z}_p \) for \( t \geq 0 \) even and 0 for \( t \) odd. This was proved by Suslin [Sus84 Cor. 3.13], and implies that \( K(\bar{F})_p \simeq ku_p \) and \( L_1 K(\bar{F}) \simeq KU_p \). Here \( ku \) is the connective cover of the complex topological \( K \)-theory ring spectrum \( KU \), and \( L_n = L_{K(n)} \) denotes Bousfield localization [Bou79] with respect to the Morava \( K \)-theory ring spectrum \( K(n) \). Multiplication by the Bott element \( u \in \pi_2 ku_p \) acts bijectively on \( \pi_* K(\bar{F})_p \), for \( * \geq 0 \).

Let \( F \) be a number field, with a ring of \( S \)-integers \( A \).

\[
\begin{array}{c}
A \longrightarrow F \\
\downarrow \quad \downarrow \\
\mathbb{Z} \longrightarrow \mathbb{Z}[1/p] \longrightarrow \mathbb{Q}
\end{array}
\]

Quillen conjectured [Qui75 §9] that there is a spectral sequence
\[ E^2_{s,t} = H^s_{et}(Spec A; \mathbb{Z}_p(t/2)) \implies \pi_{s+t} K(A)_p \]
converging in total degrees \( \geq 1 \). Here \( H^s_{et}(-) \) denotes étale cohomology, which is only well-behaved if \( 1/p \in A \), and \( \mathbb{Z}_p(t/2) \cong \pi_t K(\bar{F})_p \). For \( A = F \) this means that \( \pi_*, K(F)_p \cong \pi_* K(\bar{F})^{hG_F} \) for \( * \geq 1 \), where \( G_F \) is the absolute Galois group. The general case requires the more elaborate framework of étale homotopy types. Passing to mod \( p \) homotopy, a lift \( \beta \in \pi_{2p-2}(S/p) \) of \( u^{p-1} \in \pi_{2p-2}(ku; \mathbb{Z}/p) \) would act bijectively on \( \pi_* (K(A); \mathbb{Z}/p) \), for \( * \geq 1 \). Think of \( \beta = v_1 \) as a lift of \( v_1 \in \pi_*(BP; \mathbb{Z}/p) \).

Thomason [Tho85 Thm. 4.1] proved Quillen’s conjecture, up to the localization given by inverting \( \beta \). In particular, \( \pi_*(K(F); \mathbb{Z}/p)[1/\beta] \cong \pi_*(K(\bar{F})^{hG_F}; \mathbb{Z}/p) \) for \( * \geq 2 \). It remained to show that \( \pi_*(K(A); \mathbb{Z}/p) \to \pi_*(K(A); \mathbb{Z}/p)[1/\beta] \) is an isomorphism for \( * \geq 2 \). Waldhausen [Wald84 p. 193] noted that this amounts to asking that \( K(A) \to L_1 K(A) \) is a \( p \)-adic equivalence, in sufficiently high degrees. Here
Let \( L_n = L_{E(n)} \) denotes Bousfield localization with respect to the Johnson–Wilson ring spectrum \( E(n) \), or equivalently with respect to \( BP[1/v_n] \).

Using topological cyclic homology, Hesselholt–Madsen \([HM03\) Thm. A] confirmed Quillen’s conjecture for valuation rings in local number fields, after special cases were treated by Bökstedt–Madsen \([BM94\), [BM95\] and Rognes \([Rog99\), [Rog99b\].

Finally, Voevodsky’s proof \([Voe03\), [Voe11\] of the Milnor and Bloch–Kato conjectures confirmed Quillen’s conjecture for rings of integers in global number fields.

3. Redshift in the \( K \)-theory of ring spectra

We continue with examples of chromatic redshift in the context of algebraic \( K \)-theory of structured ring spectra.

Let \( L = E(1) \) be the Adams summand of \( KU(p) \), and \( \ell = BP(1) \) its connective cover. Using topological cyclic homology, Ausoni–Rognes \([AR02\) Thm. 0.4] computed \( V(1)_* K(\ell_p) \), and Ausoni \([Aus10\) Thm. 1.1] computed \( V(1)_* K(ku_p) \), where \( p \geq 5 \) and \( V(1) = S/(p, v_1) \) is the Smith-Toda spectrum of chromatic type 2. Using a localization sequence of Blumberg–Mandell \([BM08\) p. 157], this also calculates \( V(1)_* K(L_p) \) and \( V(1)_* K(KU_p) \). In each case, a lift \( v_2 \in \pi_{2p-2} V(1) \) of \( v_2 \in V(1)_* BP \) acts bijectively on the answer \( V(1)_* K(B) \), for \( * \geq 2p - 2 \).

The results are compatible with the existence of a spectral sequence

\[
E^2_{s,t} = H_{mot}^{-s}(\text{Spec } B; \mathbb{F}_p^2(t/2)) \Rightarrow V(1)_{s+t} K(B)
\]

for suitable “\( \ell \)-algebras of \( S \)-integers” \( B \), converging in sufficiently high total degrees. Here \( H_{mot}^{-s}(\_\) refers to a hypothetical form of motivic cohomology for commutative structured ring spectra, and \( \mathbb{F}_p^2(t/2) \cong V(1)_t E_2 \) where \( E_2 \) is the \( K(2) \)-local Lubin–Tate ring spectrum with \( \pi_* E_2 = WF_p^{[[u]]}[[u]] \).

The appearance of the field \( \mathbb{F}_p^2 \) is needed to account for the sign in Ausoni’s relation \( b^p - 1 = -v_2 \) in \( V(1)_* K(ku_p) \), since if \( b \) is represented by \( \alpha u^{p+1} \) and \( v_2 \) by \( u^{p-1} \) then \( \alpha^{p-1} = -1 \), which cannot be satisfied for \( \alpha \in \mathbb{F}_p \).

4. An analogue of the Lichtenbaum–Quillen conjectures

Consider a Galois extension \( L_p[1/p] \to M \), like in \([Rog08\) §4]. By an \( \ell \)-algebra of integers in \( M \) we mean a connected (only trivial idempotents) commutative \( \ell \)-algebra \( B \), with a structure map to \( M \), such that \( B \) is semi-finite (retract of a finite cell module), or perhaps dualizable, as an \( \ell \)-module:

\[
\begin{array}{ccc}
\Omega_1 & \rightarrow & M \\
\downarrow & & \downarrow G \\
B & \rightarrow & M \\
\downarrow & & \downarrow \ell_p \\
\ell_p & \rightarrow & L_p & \rightarrow & L_p[1/p] \\
\downarrow & & \downarrow \ell_p & & \downarrow \ell_p \\
& & J_p & &
\end{array}
\]
For $S$-integers we may allow localizations that invert $p$ or $v_1$. Let $\Omega_1$ be the $p$-completed homotopy colimit of all such $B$, i.e., the $\ell_p$-algebraic integers.

By analogy with Quillen's conjecture/Voevodsky's theorem we predict that $v_2$ acts bijectively on $V(1)_*K(B)$, for $* \gg 0$. By analogy with Lichtenbaum's conjecture/Suslin's theorem, we predict that $V(1)_*K(\Omega_1) \cong V(1)_*E_2$, in all sufficiently high degrees, and that $\hat{L}_2K(\Omega_1) \cong E_2$.

In the case when $B \to \Omega_1$ is an unramified $G$-Galois extension, the hypothetical motivic cohomology would reduce to group cohomology, and $V(1)_*K(B) \cong V(1)_*K(\Omega_1)^{hG}$ for $* \gg 0$. The general case would require a more elaborate construction than the familiar homotopy fixed points. Even establishing the existence of a ring spectrum map $K(ku) \to E_2$ seems to be an open problem.

Similarly, for $n \geq 1$ let $E_n$ be the $K(n)$-local Lubin–Tate ring spectrum, and let $e_n$ be its connective cover, so that $E_n = e_n[1/u]$. Consider Galois extensions $E_n[1/p] \to M$ and connected commutative $e_n$-algebras $B$, with a structure map to $M$, such that $B$ is semi-finite as an $e_n$-module:

$$
\begin{array}{cccc}
\Omega_n & \longrightarrow & M \\
B & \longrightarrow & E_n & \longrightarrow & E_n[1/p] \\
\downarrow e_n & & \downarrow & & \\
\hat{L}_nS & & & &
\end{array}
$$

Let $\Omega_n$ be the $p$-completed homotopy colimit of all such $B$, i.e., the $e_n$-algebraic integers.

Let $F$ be a finite $p$-local spectrum admitting a $v_{n+1}$ self map $v : \Sigma^dF \to F$, cf. Hopkins–Smith [HS98, Def. 8]. The finite localization functor $L_{n+1}^f$, which annihilates all finite $E(n+1)$-acyclic spectra [Mil92, Thm. 4], is a smashing localization such that $F_*L_{n+1}^fX \cong F_*X[1/v]$ for all spectra $X$.

I stated something like the following at Schloß Ringberg in January 1999 and in Oberwolfach in September 2000:

**Conjecture 4.1.** Let $B \to \Omega_n$ and $(F,v)$ be as above.

(a) Multiplication by $v$ acts bijectively on $F_*K(B)$ for $* \gg 0$, and $K(B) \to L_{n+1}^fK(B)$ is a $p$-adic equivalence in sufficiently high degrees.

(b) There are isomorphisms $F_*K(\Omega_n) \cong F_*E_{n+1}$ for $* \gg 0$, and $\hat{L}_{n+1}K(\Omega_n) \cong E_{n+1}$.

The cases $n = -1$ and $n = 0$ correspond to Quillen’s results and the proven Lichtenbaum–Quillen conjectures, respectively.

5. The cyclotomic trace map

We can detect chromatic redshift in algebraic $K$-theory using the cyclotomic trace map to topological cyclic homology, or one of its variants.
The topological Hochschild homology THH(B) of a commutative S-algebra B is an $S^1$-equivariant spectrum whose underlying spectrum with $S^1$-action can be constructed as $B \otimes S^1$, where $\otimes$ refers to the tensored structure of commutative $S$-algebras over spaces. Let

$$\text{THH}(B)^hS^1 = F(ES^1_+, \text{THH}(B))^{S^1}$$

be the $S^1$-homotopy fixed points of THH(B), and let

$$\text{THH}(B)^tS^1 = [ES^1 \wedge F(ES^1_+, \text{THH}(B))]^{S^1}$$

be its $S^1$-Tate construction, also denoted $t_{S^1} \text{THH}(B)^{S^1}$ or $\hat{\text{THH}}(S^1, \text{THH}(B))$. Here $ES^1$ is a free contractible $S^1$-space, and $\tilde{ES}^1$ is the mapping cone of the collapse map $ES^1_+ \to S^0$. Homotopy fixed point spectra model group cohomology, and the Tate construction models Tate cohomology.

Think of $B$ as a ring spectrum of functions on a brave new scheme $X$. Then $B \wedge \cdots \wedge B$ is the ring of functions on $X \times \cdots \times X$, so THH(B) plays the role of the ring of functions on the free loop space $\text{Map}(S^1, X) = \Lambda X$, and THH(B)$^{hS^1}$ is like the ring of functions on the Borel construction $ES^1_+ \wedge S^1 \Lambda X$. The Tate construction is a periodicized version of the Borel construction.

There is a natural trace map

$$K(B) \to \text{THH}(B)$$

that factors through the fixed point spectra $\text{THH}(B)^{C_r}$ for all finite subgroups $C_r \subset S^1$. In particular, there is a limiting map

$$K(B) \to TF(B; p) = \text{holim}_n \text{THH}(B)^{C_{p^n}}.$$ 

Continuing with the canonical map from fixed points to homotopy fixed points, the target of

$$\text{holim}_n \text{THH}(B)^{C_{p^n}} \to \text{holim}_n \text{THH}(B)^{hC_{p^n}}$$

is $p$-adically equivalent to THH(B)$^{hS^1}$. The cyclotomic structure of THH(B) gives a similar map

$$\text{holim}_n \text{THH}(B)^{C_{p^n}} \to \text{holim}_n \text{THH}(B)^{tC_{p^{n+1}}},$$

whose target is $p$-adically equivalent to THH(B)$^{tS^1}$.

The topological Hochschild construction itself does not introduce a redshift, since THH(B) is a commutative $B$-algebra. However, in all the computations made so far, any $v_{n+1}$-periodicity that is seen in the algebraic $K$-theory $K(B)$ has already been visible in the $S^1$-Tate construction THH(B)$^{tS^1}$.

Furthermore, it is possible to see in homological terms where the redshift arises, in terms of these $S^1$-equivariant constructions.

6. **Circle-equivariant redshift**

The algebra $H_*(e_n)$ appears to be unwieldy for $n \geq 2$, but there is a map $BP(n) \to e_n$ of (not necessarily commutative) $S$-algebras, covering the usual map $E(n) \to E_n$, and the augmentation $BP(n) \to H$ induces an identification

$$H_*(BP(n)) \cong P(\xi_k \mid k \geq 1) \otimes E(\bar{\tau}_k \mid k \geq n + 1)$$

of subalgebras of the dual Steenrod algebra

$$A_\ast = P(\xi_k \mid k \geq 1) \otimes E(\bar{\tau}_k \mid k \geq 0).$$
Forgetting some structure, we can therefore think of the homology $H_*(B)$ of a commutative $e_n$-algebra $B$ as a commutative $H_*(BP(n))$-algebra. This makes the Adams spectral sequence

$$E_2^{s,t}(B) = \text{Ext}_{S^1_*(B)}(\mathbb{F}_p, H_*(B)) \Longrightarrow \pi_{t-s}(B_p^\wedge)$$

an algebra over the Adams spectral sequence

$$E_2^{s,t} = \text{Ext}_{S_*(B)}(\mathbb{F}_p, H_*(BP(n))) \Longrightarrow \pi_{t-s}(BP(n)_p^\wedge)$$

which collapses at the $E_2$-term

$$E_2^{r,s} = P(v_0, \ldots, v_n)$$

and converges to the homotopy

$$\pi_* BP(n)_p^\wedge \cong \mathbb{Z}_p[v_1, \ldots, v_n].$$

The Bökstedt spectral sequence

$$E_2^{s,t}(B) = HH_*(H_*(B))_t \Longrightarrow H_{s+t}(\text{THH}(B))$$

is then an algebra spectral sequence over

$$E_2^{r,s} = HH_*(H_*(BP(n))) \cong HH_*(BP(n)) \otimes E(\sigma \xi_k \mid k \geq 1) \otimes \Gamma(\sigma \check{\tau}_k \mid k \geq n + 1)$$

collapsing to $H_*(\text{THH}(BP(n)))$. Here $\sigma$ denotes the suspension operator, coming from the $S^1$-action on THH, and $\Gamma(x) = \mathbb{F}_p(\gamma_j x \mid j \geq 0)$ denotes the divided power algebra on $x$.

The Dyer–Lashof operations $Q^{p,k}(\check{\tau}_k) = \check{\tau}_{k+1}$ in $\mathcal{A}_*$ (coming from the commutative $S$-algebra structure on $H$), imply multiplicative extensions $(\sigma \check{\tau}_k)^p = \sigma \check{\tau}_{k+1}$, for $k \geq n + 1$, which in turn imply that the Bockstein images $\beta(\sigma \check{\tau}_{k+1}) = \sigma \xi_{k+1}$ vanish in the abutment. This argument, see Ausoni [Aus05, Lem. 5.3], implies differentials

$$d^{p-1}(\gamma_j \sigma \check{\tau}_k) \cong \sigma \check{\xi}_{k+1} \cdot \gamma_{j-p} \sigma \check{\tau}_k$$

for all $j \geq p$, which leave

$$E_2^{r,s} = E_\infty^{r,s} \cong H_*(BP(n)) \otimes E(\sigma \check{\xi}_1, \ldots, \sigma \check{\xi}_{n+1}) \otimes P(\sigma \check{\tau}_k \mid k \geq n + 1)$$

converging to

$$H_*(\text{THH}(BP(n))) \cong H_*(BP(n)) \otimes E(\sigma \check{\xi}_1, \ldots, \sigma \check{\xi}_{n+1}) \otimes P(\sigma \check{\tau}_{n+1}).$$

This will still have trivial $v_{n+1}$-periodic homotopy, but note how building in a circle action gives rise to the class $\sigma \check{\tau}_{n+1}$.

The homological Tate spectral sequence

$$E_2^{s,t}(B) = \check{H}^{-s}(S^1; H_*(\text{THH}(B))) \Longrightarrow H^s_{s+t}(\text{THH}(B)^{S^1})$$

converges to a limit that we call the continuous homology of $\text{THH}(B)^{S^1}$. It is an algebra spectral sequence over

$$E_2^{r,s} = \check{H}^{-s}(S^1; H_*(\text{THH}(BP(n)))) \cong P(t^{\pm 1}) \otimes H_*(\text{THH}(BP(n)))$$

converging to $H^c_*(\text{THH}(BP(n))^{S^1})$. Here

$$d^2(t^x) = t^{x+1} \cdot \sigma x$$

for all $x$, which leaves

$$E_3^{r,s} = P(t^{\pm 1}) \otimes P(\xi_1^{p-1}, \ldots, \xi_{n+1}^{p-1}, \xi_k \mid k \geq n + 2)$$

$$\otimes E(\gamma_k \mid k \geq n + 2) \otimes E(\xi_1^{p-1} \sigma \check{\xi}_1, \ldots, \xi_{n+1}^{p-1} \sigma \check{\xi}_{n+1})$$
where \( \tau'_k = \tilde{\tau}_k - \tilde{\tau}_{k-1}(\sigma \tilde{\tau}_{k-1})^p - 1 \) for each \( k \geq n + 2 \). Note that \( \tilde{\tau}_{n+1} \) supports a nontrivial \( d^2 \)-differential to \( t \cdot \sigma \tilde{\tau}_{n+1} \), and does not survive to the \( E^\infty \)-term, while the \( \tau'_k \) for \( k \geq n + 2 \) are \( d^2 \)-cycles, due to the known multiplicative extension.

This spectral sequence often collapses at this stage \cite{BR05}, Prop. 6.1, and there can be \( \mathcal{A}_* \)-comodule extensions that combine \( p^{n+1} \) shifted copies of

\[
P(\tilde{\xi}_k, \ldots, \tilde{\xi}_{n+1}, \xi_k \mid k \geq n + 2) \otimes E(\tau'_k \mid k \geq n + 2)
\]
to a copy of \( P(\xi_k \mid k \geq 1) \otimes E(\tau'_k \mid k \geq n + 2) \cong H_*(BP(n+1)) \). The PhD theses of Sverre Lunøe-Nielsen \cite{LNR12, LNR11} and Knut Berg (to appear) address these questions. Note the transition from \( H_*(BP\langle n \rangle) \) to \( H_*(BP\langle n+1 \rangle) \), with non-trivial \( v_{n+1} \)-periodic homotopy groups. The typical result is that \( H_*^c(THH(B)t^{S^1}) \) is an algebra over \( H_*^c(THH(B)p^{S^1}) \), which has an associated graded of the form

\[
P(t^{\pm p^{n+1}}) \otimes H_*(BP(n+1)) \otimes E(\nu_1, \ldots, \nu_{n+1})
\]
where \( \nu_k \) is a \( t \)-power multiple of \( \tilde{\xi}_k^{p-1} \sigma \tilde{\xi}_k \), but that there is room for further \( \mathcal{A}_* \)-comodule extensions.

This implies that the inverse limit Adams spectral sequence

\[
E_2^{s,t}(B) = \text{Ext}_{\mathcal{A}_*}^s(F_p, H_*^c(THH(B)t^{S^1})) \Rightarrow \pi_{t-s} \text{THH}(B)p^{S^1}_t
\]
is an algebra over the Adams spectral sequence

\[
E_2^{s,t} = \text{Ext}_{\mathcal{A}_*}^{s,t}(F_p, H_*^c(THH(B)p^{S^1})) \Rightarrow \pi_{t-s} \text{THH}(B)p^{S^1}_t
\]
which contains factors like

\[
\text{Ext}_{\mathcal{A}_*}^{s,t}(F_p, H_*(BP\langle n+1 \rangle)) \cong P(v_0, \ldots, v_n, v_{n+1}).
\]
Due to the exterior factors \( E(\nu_1, \ldots, \nu_{n+1}) \) there is room for differentials that might truncate the periodic \( v_{n+1} \)-action visible above, but empirically this does not happen. A theory that explains the general picture is, however, currently lacking.

7. Beyond elliptic cohomology

Do \( K(tm\phi) \) and \( \text{THH}(tm\phi)t^{S^1} \) detect \( v_3 \)-periodic families? Work in progress for \( p = 2 \) with Bruner (2008).

8. Waldhausen’s localization tower

The chromatic localization functors \( (L_n \) and \( \hat{L}_n \) and the finite localizations functors \( L^f_n \) fit in a diagram of commutative structured ring spectra

\[
\begin{array}{ccccccc}
E_n & & & & KU_p \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\hat{L}_n S & \rightarrow & L^f_n S & \rightarrow & L^f_{n-1} S & \rightarrow & \cdots \rightarrow L^f_1 S & \rightarrow & L^f_0 S \\
S_{(p)} & \rightarrow & \cdots & \rightarrow & \hat{L}_n S & \rightarrow & L^f_n S & \rightarrow & \cdots \rightarrow L^f_1 S & \rightarrow & L^f_0 S
\end{array}
\]
where \( L^f_n S \to L_n S \) is an equivalence for \( n \leq 1 \), but probably not for \( n \geq 2 \), according to the wisdom concerning Ravenel’s telescope conjecture \[MRS01\]. Applying algebraic \( K \)-theory to the lower row one gets a telescopic localization tower

\[
K(S_{(p)}) \to \ldots \to K(L^f_1 S) \to K(L^f_{n-1} S) \to \ldots \to K(L_1 S) \to K(\mathbb{Q})
\]

similar to that of \[Wal84\, p. 174\], interpolating between the geometrically significant algebraic \( K \)-theory of spaces on the left hand side, and the arithmetically significant algebraic \( K \)-theory of number fields on the right hand side. Waldhausen worked with \( L_n \), and explicitly assumed that it is a finite localization functor, but we can work with \( L^f_n \) instead. This ensures that each finite cell \( L^f_n S \)-module is \( L^f_n \)-equivalent to a finite cell \( S \)-module, as can be proved by induction on the number of \( L^f_n S \)-cells.

Let \( \mathcal{C}_0 \) be the category of finite \( p \)-local spectra, and let \( w_n \mathcal{C}_0 \) be the subcategory of \( E(n)_\ast \)-equivalences, or equivalently of \( L^f_n \)-equivalences, for \( n \geq 0 \). Let \( \mathcal{C}_n = \mathcal{C}_0^{w_n^{-1}} \) denote the full subcategory of \( E(n-1)_\ast \)-acyclic spectra, i.e., the finite spectra of type \( \geq n \), for \( n \geq 1 \). Then \( K(\mathcal{C}_0, w_n) \simeq K(L^f_n S) \), and Waldhausen’s localization theorem \[Wal84\, \S3\] recognizes the homotopy fiber of \( K(L^f_n S) \to K(L^f_{n-1} S) \) as \( K(\mathcal{C}_n, w_n) \), i.e., the algebraic \( K \)-theory of finite spectra of type \( \geq n \), with respect to the \( E(n)_\ast \)-equivalences. We get a homotopy fiber sequence

\[
K(\mathcal{C}_n, w_n) \to K(L^f_n S) \to K(L^f_{n-1} S).\]

Let \( \mathcal{X}^{sm}_n \) be the category of small \( K(n) \)-local spectra, and let \( \mathcal{X}^r_n \) be the full subcategory of \( K(n) \)-localizations of finite spectra of type \( \geq n \). Hovey–Strickland \[HS99\, Thm. 8.5\] show that the inclusion \( \mathcal{X}^r_n \subset \mathcal{X}^{sm}_n \) is an idempotent completion, so the induced map \( K(\mathcal{X}^r_n) \to K(\mathcal{X}^{sm}_n) \) induces an isomorphism on \( \pi_i \) for each \( i \geq 1 \). The localization functors \( L_n \) and \( \hat{L}_n \) agree on \( \mathcal{C}_n \), hence induce an equivalence \( K(\mathcal{C}_n, w_n) \simeq K(\mathcal{X}^r_n) \). Thus we have a map

\[
K(\mathcal{C}_n, w_n) \to K(\mathcal{X}^{sm}_n),
\]

which induces a \( \pi_i \)-isomorphism for each \( i \geq 1 \). We view \( \mathcal{X}^{sm}_n \) as a category of suitably small \( \hat{L}_n S \)-modules.

Let \( \mathcal{E}^{df}_n \) be the category of \( E_n \)-module spectra that have degreewise finite homotopy groups. Base change along the \( K(n) \)-local pro-\( \mathbb{G}_n \)-Galois extension \( \hat{L}_n S \to E_n \) takes \( \mathcal{X}^{sm}_n \) to \( \mathcal{E}^{df}_n \), and conversely \[HS99\, Cor. 12.16\], so it is plausible that a Galois descent comparison map

\[
K(\mathcal{X}^{sm}_n) \to K(\mathcal{E}^{df}_n)^{h\mathbb{G}_n}
\]

is close to an equivalence. Finally, \( K(\mathcal{E}^{df}_n) \) is related to the algebraic \( K \)-theory of \( E_n \) and its various localizations. For \( n = 1 \) we have \( E_1 = KU_p \), and \( K(\mathcal{E}^{df}_1) \) is the algebraic \( K \)-theory of \( p \)-nilpotent finite cell \( KU_p \)-modules, which sits \[Bar13\, Prop. 11.15\] in a homotopy fiber sequence

\[
K(\mathcal{E}^{df}_1) \to K(KU_p) \to K(KU_{p}[1/p]).
\]

In general, this fiber sequence is replaced by an \( n \)-dimensional cubical diagram. Note that the transfer map \( K(KU/p) \to K(\mathcal{E}^{df}_1) \) associated to \( KU_p \to KU/p \) is far from an equivalence, by the calculations of \[ART12\, Cor. 1.3\], so there does not appear to be any easy way to describe the algebraic \( K \)-theory of degreewise finite \( E_n \)-modules in terms of dévissage, cf. \[Wal84\, p. 188\].
Conjecture [43] about the structure of the algebraic $K$-theory of $E_n$ (and various localizations) is therefore also a statement about $K(E_{nf})$, and conjecturally about $K(K_{smn})$, which rather precisely measures the difference between $K(L^n_S)$ and $K(L^n_{n-1}S)$.

9. THE SPHERICAL CASE

Calculations of $TC(S;p)$, $K(Z)$ and $TC(Z;p)$ were assembled to a calculation of $K(S)$ at $p = 2$ in [Rog02] and at odd regular primes in [Rog03]. These results describe the cohomology of $K(S)$ as an $A$-module in all degrees (up to an extension in the odd case), and lead to Adams spectral sequence calculations in a finite range of degrees.

The algebraic $K$-groups of $S$ are at least as complicated as those of its stable homotopy groups. The complex cobordism spectrum $MU$ has turned out to be a convenient halfway house

$$S \rightarrow MU \rightarrow H$$

between homology and homotopy. The Thom equivalence $MU \wedge MU \simeq MU \wedge BU_+$ makes $S \rightarrow MU$ a Hopf–Galois extension [Rog08 §12], and the cosimplicial Amitsur resolution

$$[q] \mapsto MU \wedge MU^{\wedge q}$$

of $S$ is equivalent to the cobar resolution $[q] \mapsto MU \wedge BU_+^{\wedge q}$ for the $S[BU] = \Sigma^{\infty}(BU_+)$-comodule algebra $MU$. Applying algebraic $K$-theory, an analogue of Quillen’s conjecture would predict that $K(S)$ is well approximated by the totalization of the cosimplicial spectrum

$$[q] \mapsto K(MU \wedge MU^{\wedge q})$$

rewriteable as $[q] \mapsto K(MU \wedge BU_+^{\wedge q})$. If, by analogy with the Galois case, there are compatible maps $K(MU \wedge BU_+^{\wedge q}) \rightarrow K(MU) \wedge BU_+^{\wedge q}$, then this might in turn be approximated by the totalization of the cobar resolution $[q] \mapsto K(MU) \wedge BU_+^{\wedge q}$ for an $S[BU]$-comodule algebra structure on $K(MU)$.

Conceivably, this leads to a more conceptual understanding of $\pi_*, K(S)$ in terms of $\pi_* K(MU)$ and Hopf–Galois descent, by analogy with the Adams–Novikov spectral sequence description of $\pi_* S$ in terms of $\pi_* MU$ and its $H_*(BU)$-coaction. This has been a motivating factor for the study of $K(MU)$, advertised in [BR05] and [Rog09], and pursued in [LNR11].
10. Higher redshift

For a Lie group $G$ of rank $k$, consider $(B \otimes G)^{hG}$ or something like $(B \otimes G)^{tG}$. If $B$ is $v_n$-periodic but not $v_{n+1}$-periodic, then apparently $(B \otimes G)^{tG}$ is $v_{n+k}$-periodic. Tested for $B = H$ and $G = T^k$ for small $k$, as well as for $G = SO(3)$ and $G = S^3$. Work in progress (Rognes, 2008–2011) and in Torleif Veen’s PhD thesis (2013).

References


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