

Algebraic K -theory of strict ring spectra

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Abstract. We view strict ring spectra as generalized rings. The study of their algebraic K -theory is motivated by its applications to the automorphism groups of compact manifolds. Partial calculations of algebraic K -theory for the sphere spectrum are available at regular primes, but we seek more conceptual answers in terms of localization and descent properties. Calculations for ring spectra related to topological K -theory suggest the existence of a motivic cohomology theory for strictly commutative ring spectra, and we present evidence for arithmetic duality in this theory. To tie motivic cohomology to Galois cohomology we wish to spectrally realize ramified extensions, which is only possible after mild forms of localization. One such mild localization is provided by the theory of logarithmic ring spectra, and we outline recent developments in this area.

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1. Strict ring spectra

First, let R be an abelian group. Ordinary singular cohomology with coefficients in R is a contravariant homotopy functor that associates to each based space X a graded cohomology group $\tilde{H}^*(X; R)$. It is stable, in the sense that there is a natural isomorphism $\tilde{H}^*(X; R) \cong \tilde{H}^{*+1}(\Sigma X; R)$, and this implies that it extends from the category of based spaces to the category of spectra. The latter is a category of space-like objects, where the suspension is invertible up to homotopy equivalence, and which has all colimits and limits. The extended cohomology functor becomes representable, meaning that there is a spectrum HR , called the Eilenberg–MacLane spectrum of R , and a natural isomorphism $\tilde{H}^*(X; R) \cong [X, HR]_{-*}$, where $[X, HR]_{-n}$ is the group of homotopy classes of morphisms $X \rightarrow \Sigma^n HR$.

Next, let R be a ring. Then the cohomology theory is multiplicative, meaning that there is a bilinear cup product $\tilde{H}^*(X; R) \times \tilde{H}^*(X; R) \rightarrow \tilde{H}^*(X; R)$. This is also representable in the category of spectra, by a morphism $\mu: HR \wedge HR \rightarrow HR$, where \wedge denotes the smash product of spectra. With the modern models for the category of spectra [26], [40], [48] we may arrange that μ is strictly unital and

associative, so that HR is a *strict ring spectrum*. Equivalent terms are A_∞ ring spectrum, S -algebra, symmetric ring spectrum and orthogonal ring spectrum.

If R is commutative, then the cup product is graded commutative, which at the representing level means that $\mu\tau \simeq \mu$, where $\tau: HR \wedge HR \rightarrow HR \wedge HR$ denotes the twist isomorphism. In fact, we may arrange that μ is strictly commutative, in the sense that $\mu\tau = \mu$ as morphisms of spectra, so that HR is a *strictly commutative ring spectrum*. Equivalent phrases are E_∞ ring spectrum, commutative S -algebra, commutative symmetric ring spectrum and commutative orthogonal ring spectrum. This leads to a compatible sequence of Σ_k -equivariant morphisms $E\Sigma_{k+} \wedge HR^{\wedge k} \rightarrow HR$ for $k \geq 0$. At the represented level these morphisms give rise to power operations in cohomology, including Steenrod's operations Sq^i for $R = \mathbb{F}_2$ and $\beta^\epsilon P^i$ for $R = \mathbb{F}_p$.

One now realizes that the Eilenberg–Mac Lane ring spectra HR exist as special cases within a much wider class of ring spectra. Each spectrum B represents a generalized cohomology theory $X \mapsto \tilde{B}^*(X) = [X, B]_{-*}$ and a generalized homology theory $X \mapsto \tilde{B}_*(X) = \pi_*(B \wedge X)$. Examples of early interest include the spectrum KU that represents complex topological K -theory, $KU^*(X) = [X_+, KU]_{-*}$, and the spectrum MU that represents complex bordism, $MU_*(X) = \pi_*(MU \wedge X_+)$. A fundamental example is given by the sphere spectrum S , which is the image of the based space S^0 under the stabilization functor from spaces to spectra. It represents stable cohomotopy $\pi_S^*(X) = \tilde{S}^*(X)$ and stable homotopy $\pi_*^S(X) = \tilde{S}_*(X)$. Each of these three examples, KU , MU and S , is naturally a strictly commutative ring spectrum, representing a multiplicative cohomology theory with power operations, etc. Furthermore, there are interesting multiplicative morphisms connecting these ring spectra to the Eilenberg–Mac Lane ring spectra previously considered, as in the diagram

$$\begin{array}{ccccccc}
 & & & & KU & \longrightarrow & H\mathbb{Q} \\
 & & & & \uparrow & & \uparrow \\
 S & \longrightarrow & MU & \longrightarrow & ku & \longrightarrow & H\mathbb{Z},
 \end{array}$$

where $ku = KU[0, \infty)$ denotes the connective cover of KU .

By placing the class of traditional rings inside the wider realm of all strict ring spectra, a new world of possibilities opens up. Following Waldhausen [47, p. xiii] we may refer to strict ring spectra as “brave new rings”. If we think in algebro-geometric terms, where commutative rings appear as the rings of functions on pieces of geometric objects, then strictly commutative ring spectra are the functions on affine pieces of brave new geometries, more general than those realized by ordinary schemes.

How vast is this generalization? In the case of connective ring spectra B , i.e., those with $\pi_i(B) = 0$ for i negative, there is a natural ring spectrum morphism $B \rightarrow H\pi_0(B)$ that induces an isomorphism on π_0 . This behaves for many purposes like a topologically nilpotent extension, and in geometric terms, B can be viewed as the ring spectrum of functions on an infinitesimal thickening of $\text{Spec } \pi_0(B)$.

This infinitesimal thickening can be quite effectively controlled in terms of diagrams of Eilenberg–Mac Lane spectra associated with simplicial rings. The

Hurewicz map $B \cong S \wedge B \rightarrow H\mathbb{Z} \wedge B$ is 1-connected, and there is an equivalence $H\mathbb{Z} \wedge B \simeq HR_\bullet$ for some simplicial ring R_\bullet . The square

$$\begin{array}{ccc} S \wedge S \wedge B & \longrightarrow & H\mathbb{Z} \wedge S \wedge B \\ \downarrow & & \downarrow \\ S \wedge H\mathbb{Z} \wedge B & \longrightarrow & H\mathbb{Z} \wedge H\mathbb{Z} \wedge B \end{array}$$

induces a 2-connected map from $B \cong S \wedge S \wedge B$ to the homotopy pullback. More generally, for each $n \geq 1$ there is an n -dimensional cubical diagram that induces an n -connected map from the initial vertex B to the homotopy limit of the remainder of the cube, and the terms in that remainder have the form HR_\bullet for varying simplicial rings R_\bullet . Dundas [24] used a clever strengthening of this statement to prove that relative algebraic K -theory is p -adically equivalent to relative topological cyclic homology for morphisms $A \rightarrow B$ of connective strict ring spectra, under the assumption that $\pi_0(A) \rightarrow \pi_0(B)$ is a surjection with nilpotent kernel. He achieved this by reducing to the analogous statement for homomorphisms $R_\bullet \rightarrow T_\bullet$ of simplicial rings, which had been established earlier by McCarthy [49]. This confirmed a conjecture of Goodwillie [33], motivated by a similar result for rational K -theory and (negative) cyclic homology [32].

How about the case of non-connective ring spectra? Those that arise as homotopy fixed points $B^{hG} = F(EG_+, B)^G$ for a group action may be viewed as the functions on an orbit stack for the induced G -action on the geometry associated to B . Those that arise as smashing Bousfield localizations $L_E B$, with respect to a homology theory E_* , may be viewed as open subspaces in a finer topology than the one derived from the Zariski topology on $\text{Spec } \pi_0(B)$ [63, §9.3]. In the general case the connection to classical geometry is less clear.

2. Automorphisms of manifolds

Why should we be interested in brave new rings and their ring-theoretic invariants, like algebraic K -theory [55], [26, Ch. VI], other than for the sake of generalization? One good justification comes from the tight connection between the geometric topology of high-dimensional manifolds and the algebraic K -theory of strict ring spectra. This connection is given by the higher simple-homotopy theory initiated by Hatcher [36], which was fully developed by Waldhausen in the context of his algebraic K -theory of spaces [82] and the stable parametrized h -cobordism theorem [83]. On the geometric side, this theory concerns the fundamental problem of finding a parametrized classification of high-dimensional compact manifolds, up to homeomorphism, piecewise-linear homeomorphism or diffeomorphism, as appropriate for the respective geometric category. The set of path components of the resulting moduli space corresponds to the set of isomorphism classes of such manifolds, and each individual path component is a classifying space for the automorphism group $\text{Aut}(X)$ of a manifold X in the respective isomorphism class.

This parametrized classification is finer than the one provided by the Browder–Novikov–Sullivan–Wall surgery theory [21], [84], which classifies manifolds up to h -cobordism (or s -cobordism), and whose associated moduli space has path components that classify the block automorphism groups of manifolds, rather than their actual automorphism groups. The difference between these two classifications is controlled by the space $H(X)$ of h -cobordisms $(W; X, Y)$ with a given manifold X at one end. Here W is a compact manifold with $\partial W = X \cup Y$, and the inclusions $X \rightarrow W$ and $Y \rightarrow W$ are homotopy equivalences. More precisely, there is one h -cobordism space $H^{\text{Cat}}(X)$ for each of the three flavors of manifolds mentioned, namely $\text{Cat} = \text{Top}$, PL or Diff .

The original h -cobordism theorem enumerates the isomorphism classes of h -cobordisms with X at one end, i.e., the set $\pi_0 H(X)$ of path components of the h -cobordism space of X , in terms of an algebraic K -group of the integral group ring $\mathbb{Z}[\pi]$, where $\pi = \pi_1(X)$ is the fundamental group of X . One defines the Whitehead group as the quotient $\text{Wh}_1(\pi) = K_1(\mathbb{Z}[\pi]) / (\pm\pi)$, and associates a Whitehead torsion class $\tau(W; X, Y) \in \text{Wh}_1(\pi)$ to each h -cobordism on X .

Theorem 2.1 (Smale [73], Barden, Mazur, Stallings). *Let X be a compact, connected n -manifold with $n \geq 5$ and $\pi = \pi_1(X)$. The Whitehead torsion defines a bijection*

$$\pi_0 H(X) \cong \text{Wh}_1(\pi).$$

These constructions involve using Morse functions or triangulations to choose a relative CW complex structure on the pair (W, X) , or equivalently, a π -equivariant relative CW complex structure on the pair of universal covers (\tilde{W}, \tilde{X}) , and to study the associated cellular chain complex $C_*(\tilde{W}, \tilde{X})$ of free $\mathbb{Z}[\pi]$ -modules. This works fine as long as one is only concerned with a classification of h -cobordisms up to isomorphism, but for the parametrized problem, i.e., the study of the full homotopy type of $H(X)$, the passage from a CW complex structure to the associated cellular chain complex loses too much information. One should remember the actual attaching maps from the boundaries of cells to the preceding skeleta, not just their degrees. In a stable range this amounts to working with maps from spheres to spheres and coefficients in the sphere spectrum S , rather than with degrees and coefficients in the integers \mathbb{Z} . Likewise, the passage to the π -equivariant universal cover $\tilde{X} \rightarrow X$ should be replaced to a passage to a G -equivariant principal fibration $P \rightarrow X$, where P is contractible and $G \simeq \Omega X$ is a topological group that is homotopy equivalent to the loop space of $X \simeq BG$. To sum up, the parametrized analog of the Whitehead torsion must take values in a Whitehead space that is built from the algebraic K -theory of the spherical group ring $S[G] = S \wedge G_+$, a strict ring spectrum, rather than that of its discrete reduction, the integral group ring $\pi_0(S[G]) \cong \mathbb{Z}[\pi]$.

Waldhausen’s algebraic K -theory of spaces, traditionally denoted $A(X)$, was first introduced without reference to strict ring spectra [80], but can be rewritten as the algebraic K -theory $A(X) = K(S[G])$ of the strict ring spectrum $S[G]$, cf. [81] and [26, Ch. VI]. This point of view is convenient for the comparison of algebraic K -theory with other ring-theoretic invariants.

In the case of differentiable manifolds, the Whitehead space $\mathrm{Wh}^{\mathrm{Diff}}(X)$ is defined to sit in a split homotopy fiber sequence of infinite loop spaces

$$\Omega^\infty(S \wedge X_+) \xrightarrow{\iota} K(S[G]) \longrightarrow \mathrm{Wh}^{\mathrm{Diff}}(X).$$

In the topological case there is a homotopy fiber sequence of infinite loop spaces

$$\Omega^\infty(K(S) \wedge X_+) \xrightarrow{\alpha} K(S[G]) \longrightarrow \mathrm{Wh}^{\mathrm{Top}}(X),$$

where α is known as the assembly map. The piecewise-linear Whitehead space is the same as the topological one. The stable parametrized h -cobordism theorem reads as follows.

Theorem 2.2 (Waldhausen–Jahren–Rognes [83, Thm. 0.1]). *Let X be a compact Cat manifold, for $\mathrm{Cat} = \mathrm{Top}, \mathrm{PL}$ or Diff . There is a natural homotopy equivalence*

$$\mathcal{H}^{\mathrm{Cat}}(X) \simeq \Omega \mathrm{Wh}^{\mathrm{Cat}}(X),$$

where $\mathcal{H}^{\mathrm{Cat}}(X) = \mathrm{colim}_k H^{\mathrm{Cat}}(X \times I^k)$ is the stable Cat h -cobordism space of X .

When combined with connectivity results about the dimensional stabilization map $H(X) = H^{\mathrm{Cat}}(X) \rightarrow \mathcal{H}^{\mathrm{Cat}}(X)$, and here the main result is Igusa’s stability theorem for smooth pseudoisotopies [42], knowledge of $K(S)$ and $K(S[G])$ gives good general results on the h -cobordism space $H(X)$ and the automorphism group $\mathrm{Aut}(X)$ of a high-dimensional manifold X .

Example 2.3. When G is trivial, so that $S[G] = S$ and X is contractible, the π_0 -isomorphism and rational equivalence $S \rightarrow H\mathbb{Z}$ induces a rational equivalence $K(S) \rightarrow K(\mathbb{Z})$. Here $\pi_* K(\mathbb{Z}) \otimes \mathbb{Q}$ was computed by Borel [20], so

$$\pi_i \mathrm{Wh}^{\mathrm{Diff}}(*) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & \text{for } i = 4k + 1 \neq 1, \\ 0 & \text{otherwise.} \end{cases}$$

For $X = D^n$, Farrell–Hsiang [27] used this to show that

$$\pi_i \mathrm{Diff}(D^n) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & \text{for } i = 4k - 1, n \text{ odd,} \\ 0 & \text{otherwise,} \end{cases}$$

for i up to approximately $n/3$, where $\mathrm{Diff}(D^n)$ denotes the group of self-diffeomorphisms of D^n that fix the boundary. For instance, $\pi_3 \mathrm{Diff}(D^{13})$ is rationally nontrivial. By contrast, the group $\mathrm{Top}(D^n)$ of self-homeomorphisms of D^n that fix the boundary is contractible. Similar results follow for n -manifolds that are roughly $n/3$ -connected.

The case of spherical space forms, when G is finite with periodic cohomology, has been studied by Hsiang–Jahren [41]. For closed, non-positively curved manifolds X , Farrell–Jones [28] showed that $\mathrm{Wh}^{\mathrm{Diff}}(X)$ can be assembled from copies of $\mathrm{Wh}^{\mathrm{Diff}}(*)$ and $\mathrm{Wh}^{\mathrm{Diff}}(S^1)$, indexed by the points and the closed geodesics in X , respectively. These correspond to the special cases G trivial and G infinite

cyclic, respectively, so $K(S)$ and $K(S[\mathbb{Z}])$ are of fundamental importance for the parametrized classification of this large class of Riemannian manifolds. In this paper we shall focus on the case of $K(S)$, but see Hesselholt's paper [37] for the case of $K(S[\mathbb{Z}])$, and see Weiss–Williams [86] for a detailed survey about automorphisms of manifolds and algebraic K -theory.

Remark 2.4. More recent papers of Madsen–Weiss [46], Berglund–Madsen [12] and Galatius–Randal-Williams [29] give precise results about automorphism groups of manifolds of a fixed even dimension $n = 2d \neq 4$, at the expense of first forming a connected sum with many copies of $S^d \times S^d$. The latter results are apparently not closely related to the algebraic K -theory of strict ring spectra.

3. Algebraic K -theory of the sphere spectrum

We can strengthen the rational results about $A(X) = K(S[G])$, $\mathrm{Wh}^{\mathrm{Diff}}(X)$ and $\mathrm{Diff}(X)$ to integral results, or more precisely, to p -adic integral results for each prime p . From here on it will be convenient to think of algebraic K -theory as a spectrum-valued functor, and likewise for the Whitehead theories, so that there are homotopy cofiber sequences of spectra

$$\begin{array}{ccc} S \wedge X_+ & \xrightarrow{\iota} & K(S[G]) \longrightarrow \mathrm{Wh}^{\mathrm{Diff}}(X) \\ K(S) \wedge X_+ & \xrightarrow{\alpha} & K(S[G]) \longrightarrow \mathrm{Wh}^{\mathrm{Top}}(X), \end{array}$$

and the first one is naturally split.

A key tool for this study is the cyclotomic trace map $\mathrm{trc}: K(B) \rightarrow TC(B; p)$ from algebraic K -theory to the topological cyclic homology of Bökstedt–Hsiang–Madsen [16]. The latter invariant of the strict ring spectrum B can sometimes be calculated by analyzing the S^1 -equivariant homotopy type of the topological Hochschild homology spectrum $THH(B)$. Its power is illustrated by the following previously mentioned theorem.

Theorem 3.1 (Dundas [24]). *Let B be a connective strict ring spectrum. The square*

$$\begin{array}{ccc} K(B) & \longrightarrow & K(\pi_0(B)) \\ \mathrm{trc} \downarrow & & \downarrow \mathrm{trc} \\ TC(B; p) & \longrightarrow & TC(\pi_0(B); p) \end{array}$$

becomes homotopy Cartesian upon p -completion.

In the basic case $B = S$, when $K(S) \simeq S \vee \mathrm{Wh}^{\mathrm{Diff}}(*)$ determines $\mathrm{Diff}(D^n)$ for large n , this square takes the form below. Three of the four corners are quite well

understood, but for widely different reasons.

$$\begin{array}{ccc} K(S) & \longrightarrow & K(\mathbb{Z}) \\ \text{trc} \downarrow & & \downarrow \text{trc} \\ TC(S; p) & \longrightarrow & TC(\mathbb{Z}; p). \end{array}$$

These reasons were tied together by the author for $p = 2$ in [61], and for p an odd regular prime in [62], to compute the mod p cohomology

$$H^*(K(S); \mathbb{F}_p) \cong \mathbb{F}_p \oplus H^*(\text{Wh}^{\text{Diff}}(*); \mathbb{F}_p)$$

as a module over the Steenrod algebra \mathcal{A} of stable mod p cohomology operations. This sufficed to determine the E_2 -term of the Adams spectral sequence

$$E_2^{s,t} = \text{Ext}_{\mathcal{A}}^{s,t}(H^*(K(S); \mathbb{F}_p), \mathbb{F}_p) \implies \pi_{t-s}K(S)_p$$

in a large range of degrees, and to determine the homotopy groups $\pi_i K(S)_p$ and $\pi_i \text{Wh}^{\text{Diff}}(*)_p$ in a smaller range of degrees.

The structure of the algebraic K -theory of the integers, $K(\mathbb{Z})$, was predicted by the Lichtenbaum–Quillen conjectures [55], which were confirmed for $p = 2$ by Voevodsky [78] with contributions by Rognes–Weibel [67], and for p odd by Voevodsky [79] with contributions by Rost and Weibel. For $p = 2$ or p a regular odd prime, this led to a p -complete description of the spectrum $K(\mathbb{Z})$ in terms of topological K -theory spectra, which in turn led to an explicit description of the spectrum cohomology $H^*(K(\mathbb{Z}); \mathbb{F}_p)$ as an \mathcal{A} -module.

The topological cyclic homology of the integers, $TC(\mathbb{Z}; p)$, was computed for odd primes p by Bökstedt–Madsen [17], [18] and for $p = 2$ by the author [57], [58], [59], [60], in papers that start with knowledge of the mod p homotopy of the S^1 -spectrum $THH(\mathbb{Z})$ and inductively determine the mod p homotopy of the C_{p^n} -fixed points $THH(\mathbb{Z})^{C_{p^n}}$ for $n \geq 1$. It is then possible to recognize the p -completed spectrum level structure by comparisons with known models, using [56] for p odd, and to obtain the \mathcal{A} -module $H^*(TC(\mathbb{Z}; p); \mathbb{F}_p)$ from this.

The topological cyclic homology of the sphere spectrum, $TC(S; p)$, was determined in the original paper [16]. There is an equivalence of spectra $TC(S; p) \simeq S \vee \Sigma \mathbb{C}P_{-1}^{\infty}$ after p -completion, where $\Sigma \mathbb{C}P_{-1}^{\infty}$ is the homotopy fiber of the dimension-shifting S^1 -transfer map $t: \Sigma \mathbb{C}P_{+}^{\infty} \rightarrow S$. The mod p cohomology $H^*(\Sigma \mathbb{C}P_{-1}^{\infty}; \mathbb{F}_p)$ is well known as an \mathcal{A} -module. For $p = 2$ it is cyclic with

$$H^*(\Sigma \mathbb{C}P_{-1}^{\infty}; \mathbb{F}_2) \cong \Sigma^{-1} \mathcal{A}/C,$$

where the ideal $C \subset \mathcal{A}$ is generated by the admissible Sq^I where $I = (i_1, \dots, i_n)$ with $n \geq 2$ or $I = (i)$ with i odd. The determination of the homotopy groups of $TC(S; p)$ is of comparable difficulty to the computation of the homotopy groups of S , due to our extensive knowledge about the attaching maps in the usual CW spectrum structure on $\Sigma \mathbb{C}P_{-1}^{\infty}$, cf. Mosher [52].

The linearization map $TC(S; p) \rightarrow TC(\mathbb{Z}; p)$ is only partially understood [45], but for p regular the cyclotomic trace map $K(\mathbb{Z}) \rightarrow TC(\mathbb{Z}; p)$ can be controlled by an appeal to global Tate–Poitou duality [76, Thm. 3.1], see [62, Prop. 3.1]. This leads to the following conclusion for $p = 2$. See [62, Thm. 5.4] for the result at odd regular primes.

Theorem 3.2 ([61, Thm. 4.5]). *The mod 2 cohomology of the spectrum $\mathrm{Wh}^{\mathrm{Diff}}(*)$ is given by the unique non-trivial extension of \mathcal{A} -modules*

$$0 \rightarrow \Sigma^{-2}C/\mathcal{A}(Sq^1, Sq^3) \longrightarrow H^*(\mathrm{Wh}^{\mathrm{Diff}}(*); \mathbb{F}_2) \longrightarrow \Sigma^3\mathcal{A}/\mathcal{A}(Sq^1, Sq^2) \rightarrow 0.$$

Using the Adams spectral sequence and related methods, the author obtained the following explicit calculations. Less complete information, in a larger range of degrees, is provided in the cited references. Previously, Bökstedt–Waldhausen [19, Thm. 1.3] had computed $\pi_i \mathrm{Wh}^{\mathrm{Diff}}(*)$ for $i \leq 3$.

Theorem 3.3 ([61, Thm. 5.8], [62, Thm. 4.7]). *The homotopy groups of $\mathrm{Wh}^{\mathrm{Diff}}(*)$ in degrees $i \leq 18$ are as follows, modulo p -power torsion for irregular primes p .*

i	0	1	2	3	4	5	6	7	8	9
$\pi_i \mathrm{Wh}^{\mathrm{Diff}}(*)$	0	0	0	$\mathbb{Z}/2$	0	\mathbb{Z}	0	$\mathbb{Z}/2$	0	$\mathbb{Z} \oplus \mathbb{Z}/2$
i	10			11	12	13	14			
$\pi_i \mathrm{Wh}^{\mathrm{Diff}}(*)$	$\mathbb{Z}/8 \oplus (\mathbb{Z}/2)^2$			$\mathbb{Z}/6$	$\mathbb{Z}/4$	\mathbb{Z}	$\mathbb{Z}/36 \oplus \mathbb{Z}/3$			
i	15		16		17		18			
$\pi_i \mathrm{Wh}^{\mathrm{Diff}}(*)$	$(\mathbb{Z}/2)^2$		$\mathbb{Z}/24 \oplus \mathbb{Z}/2$		$\mathbb{Z} \oplus (\mathbb{Z}/2)^2$		$\mathbb{Z}/480 \oplus (\mathbb{Z}/2)^3$			

Example 3.4. For $X = D^n$ with n sufficiently large, it follows that $\pi_{4p-4} \mathrm{Diff}(D^n)$ or $\pi_{4p-4} \mathrm{Diff}(D^{n+1})$ contains an element of order p , for each regular $p \geq 5$, and that $\pi_9 \mathrm{Diff}(D^n)$ or $\pi_9 \mathrm{Diff}(D^{n+1})$ contains an element of order 3, see [62, Thm. 6.4]. To get more precise results one needs to investigate the canonical involution on $\mathrm{Wh}^{\mathrm{Diff}}(*)$ and apply Weiss–Williams [85, Thm. A].

Remark 3.5. It would be interesting to extend these results to irregular primes. Dwyer–Mitchell [25] described the spectrum $K(\mathbb{Z})_p$ in terms of the p -primary Iwasawa module of the rationals. It should be possible to turn this into a description of the \mathcal{A} -module $H^*(K(\mathbb{Z}); \mathbb{F}_p)$. Next one must control the cyclotomic trace map $K(\mathbb{Z}) \rightarrow TC(\mathbb{Z}; p)$, or the closely related completion map $K(\mathbb{Z}) \rightarrow K(\mathbb{Z}_p)$, whose behavior is governed by special values of p -adic L -functions, cf. Soulé [74, Thm. 3].

4. Algebraic K -theory of topological K -theory

The calculations reviewed in the previous section extracted detailed information about $\pi_* K(S) \cong \pi_*(S) \oplus \pi_* \mathrm{Wh}^{\mathrm{Diff}}(*)$ from our knowledge of $\pi_*(\mathbb{C}P_1^\infty)$. However, this understanding was not presented to us in as conceptual a way as the

understanding we have of $K(\mathbb{Z})$, say in terms of Quillen's localization sequence

$$K(\mathbb{F}_p) \longrightarrow K(\mathbb{Z}) \longrightarrow K(\mathbb{Z}[1/p])$$

and the étale descent property

$$\pi_i K(\mathbb{Z}[1/p])_p \xrightarrow{\cong} K_i^{\text{ét}}(\mathbb{Z}[1/p]; \mathbb{Z}_p)$$

for $i > 0$, cf. [54, §5] and [55, §9]. It would be desirable to have a similarly conceptual understanding of $K(S)_p$ in terms of a comparison with $K(B)_p$ for suitably local strict ring spectra B , a descent property describing $K(B)_p$ as a homotopy limit of $K(C)_p$ for appropriate extensions $B \rightarrow C$, and a simple description of $K(\Omega)_p$ for a sufficiently large such extension $B \rightarrow \Omega$.

To explore this problem, we first simplify the number theory involved by working with the p -adic integers \mathbb{Z}_p in place of the rational integers \mathbb{Z} , and then seek a conceptual understanding of $K(B)_p$ for some of the strictly commutative ring spectra B that are closest to $H\mathbb{Z}_p$, namely the p -complete connective complex K -theory spectrum ku_p and its Adams summand ℓ_p . Here $\pi_*(ku_p) = \mathbb{Z}_p[u]$ and $\pi_*(\ell_p) = \mathbb{Z}_p[v_1]$, with $|u| = 2$ and $|v_1| = 2p - 2$. Let KU_p and L_p denote the associated periodic spectra, with $\pi_*(KU_p) = \mathbb{Z}_p[u^{\pm 1}]$ and $\pi_*(L_p) = \mathbb{Z}_p[v_1^{\pm 1}]$. There are multiplicative morphisms

$$\begin{array}{ccccc} & & L_p & \xrightarrow{\phi} & KU_p & & (1) \\ & & \uparrow & & \uparrow & & \\ S_p & \longrightarrow & \ell_p & \xrightarrow{\phi} & ku_p & \longrightarrow & H\mathbb{Z}_p \end{array}$$

of strictly commutative ring spectra, where $\phi_*(v_1) = u^{p-1}$. The group $\Delta \cong \mathbb{F}_p^\times$ of p -adic roots of unity acts by Adams operations on KU_p , and $\phi: L_p \rightarrow KU_p$ is a Δ -Galois extension in the sense of [63, p. 3].

Definition 4.1. Let $V(1) = S \cup_p e^1 \cup_{\alpha_1} e^{2p-1} \cup_p e^{2p}$ be the type 2 Smith–Toda complex, defined as the mapping cone of the Adams self-map $v_1: \Sigma^{2p-2}S/p \rightarrow S/p$ of the mod p Moore spectrum $S/p = S \cup_p e^1$. It is a ring spectrum up to homotopy for $p \geq 5$, which we now assume. We write $V(1)_*B = \pi_*(V(1) \wedge B)$ for the “mod p and v_1 homotopy” of any spectrum B . It is naturally a module over the polynomial ring $\mathbb{F}_p[v_2]$, where $v_2 \in \pi_{2p^2-2}V(1)$.

The mod p and v_1 homotopy of the topological cyclic homology of the connective Adams summand ℓ was computed by Ausoni and the author, by starting with knowledge of $V(1)_*THH(\ell)$ from [51] and inductively determining the mod p and v_1 homotopy of the fixed points $THH(\ell)^{C_p^n}$ for $n \geq 1$. The calculations were later extended to the full connective complex K -theory spectrum ku by Ausoni. To avoid introducing too much notation, we only describe the most striking features of the answers, referring to the original papers for more precise statements.

Theorem 4.2 (Ausoni–Rognes [5, Thm. 0.3, Thm. 0.4]). $V(1)_*TC(\ell; p)$ is a finitely generated free $\mathbb{F}_p[v_2]$ -module on $4(p+1)$ generators, which are located in degrees $-1 \leq * \leq 2p^2 + 2p - 2$. There is an exact sequence of $\mathbb{F}_p[v_2]$ -modules

$$0 \rightarrow \Sigma^{2p-3}\mathbb{F}_p \rightarrow V(1)_*K(\ell_p) \xrightarrow{\text{trc}} V(1)_*TC(\ell; p) \rightarrow \Sigma^{-1}\mathbb{F}_p \rightarrow 0.$$

Theorem 4.3 (Ausoni [4, Thm. 7.9, Thm. 8.1]). $V(1)_*TC(ku; p)$ is a finitely generated free $\mathbb{F}_p[v_2]$ -module on $4(p-1)(p+1)$ generators, which are located in degrees $-1 \leq * \leq 2p^2 + 2p - 2$. There is an exact sequence of $\mathbb{F}_p[v_2]$ -modules

$$0 \rightarrow \Sigma^{2p-3}\mathbb{F}_p \rightarrow V(1)_*K(ku_p) \xrightarrow{\text{trc}} V(1)_*TC(ku; p) \rightarrow \Sigma^{-1}\mathbb{F}_p \rightarrow 0,$$

and the natural map $K(\ell_p) \rightarrow K(ku_p)^{h\Delta}$ is a p -adic equivalence.

Blumberg–Mandell [14] constructed homotopy cofiber sequences

$$K(\mathbb{Z}_p) \rightarrow K(\ell_p) \rightarrow K(L_p) \tag{2}$$

and $K(\mathbb{Z}_p) \rightarrow K(ku_p) \rightarrow K(KU_p)$, which lead to calculations of $V(1)_*K(L_p)$ and $V(1)_*K(KU_p)$, cf. [4, Thm. 8.3]. The natural map $K(L_p) \rightarrow K(KU_p)^{h\Delta}$ is also a p -adic equivalence, which confirms the étale descent property for algebraic K -theory in this particular case.

Remark 4.4. The examples discussed above are the case $n = 1$ of a series of approximations to S associated with the Lubin–Tate spectra E_n , with coefficient rings $\pi_*E_n = \mathbb{W}\mathbb{F}_{p^n}[[u_1, \dots, u_{n-1}]][u^{\pm 1}]$, which are known to be strictly commutative ring spectra by the Goerss–Hopkins–Miller obstruction theory [31]. There are multiplicative morphisms

$$\begin{array}{ccccc} \hat{L}_n S & \longrightarrow & \hat{L}_n E(n) & \xrightarrow{\phi} & E_n \\ \uparrow & & \uparrow & & \uparrow \\ L_n S & \longrightarrow & BP\langle n \rangle & \longrightarrow & e_n \longrightarrow H\pi_0(e_n), \end{array}$$

where L_n and \hat{L}_n denote Bousfield localization with respect to the Johnson–Wilson spectrum $E(n)$ and the Morava K -theory spectrum $K(n)$, respectively, and $BP\langle n \rangle$ is the truncated Brown–Peterson spectrum. The n -th extended Morava stabilizer group \mathbb{G}_n acts on E_n , and $\hat{L}_n E(n) \rightarrow E_n$ is an H -Galois extension for $H \cong \mathbb{F}_{p^n}^\times \rtimes \mathbb{Z}/n$. We write e_n for the connective cover of E_n .

The algebraic K -theory computations above provide evidence for the chromatic redshift conjecture, see [5, p. 7] and [6], predicting that the algebraic K -theory $K(B)$ of a purely v_n -periodic strictly commutative ring spectrum B , such as E_n , is purely v_{n+1} -periodic in sufficiently high degrees.

5. Motivic truncation and arithmetic duality

The proven Lichtenbaum–Quillen conjectures subsume a spectral sequence

$$E_{s,t}^2 = H_{\text{ét}}^{-s}(R; \mathbb{Z}_p(t/2)) \implies \pi_{s+t}K(R)_p,$$

which converges for reasonable R and $s+t$ sufficiently large. Here $H_{\text{ét}}^*$ denotes étale cohomology, R is a commutative $\mathbb{Z}[1/p]$ -algebra, and $\mathbb{Z}_p(t/2) = \pi_t(KU_p)$ is $\mathbb{Z}_p(m)$ when $t = 2m$ is even, and 0 otherwise. For instance, we may take $R = \mathcal{O}_F[1/p]$ to be the ring of p -integers in a number field F , or R may be a p -adic field, i.e., a finite extension of \mathbb{Q}_p .

The proven Beilinson–Lichtenbaum conjectures, cf. [75] and [30], provide a more precise convergence statement. For each field F containing $1/p$ there is a spectral sequence

$$E_{s,t}^2 = H_{\text{mot}}^{-s}(F; \mathbb{Z}(t/2)) \implies \pi_{s+t}K(F),$$

converging in all degrees, and similarly with mod p coefficients. Here H_{mot}^* denotes motivic cohomology, which satisfies

$$H_{\text{mot}}^r(F; \mathbb{Z}/p(m)) \cong \begin{cases} H_{\text{ét}}^r(F; \mathbb{Z}/p(m)) & \text{for } 0 \leq r \leq m, \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

In terms of Bloch’s higher Chow groups [13], the vanishing of these groups for $r > m$ expresses the fact that there are no codimension r subvarieties of affine m -space over $\text{Spec } F$. Conversely,

$$H_{\text{ét}}^r(F; \mathbb{Z}/p(*)) \cong v_1^{-1}H_{\text{mot}}^r(F; \mathbb{Z}/p(*)) \quad (4)$$

with $v_1 \in H_{\text{mot}}^0(F; \mathbb{Z}/p(p-1))$. We refer to the aspects (3) and (4) of the Beilinson–Lichtenbaum conjectures as the *motivic truncation property* for the field F .

The following prediction expresses a similar conceptual description of $K(B)$ for some strictly commutative ring spectra, and should in particular apply for $B = \ell_p$, L_p , ku_p and KU_p .

Conjecture 5.1. *For purely v_1 -periodic strictly commutative ring spectra B there is a spectral sequence*

$$E_{s,t}^2 = H_{\text{mot}}^{-s}(B; \mathbb{F}_{p^2}(t/2)) \implies V(1)_{s+t}K(B),$$

converging for $s+t$ sufficiently large.

Here H_{mot}^* denotes a currently undefined form of motivic cohomology for strictly commutative ring spectra. The coefficient $\mathbb{F}_{p^2}(t/2)$ may be interpreted as $V(1)_t E_2$, where E_2 is the Lubin–Tate ring spectrum [31] with $\pi_* E_2 = \mathbb{W}\mathbb{F}_{p^2}[[u_1]][[u^{\pm 1}]]$.

More generally one might consider purely v_n -periodic ring spectra B , replace $V(1)$ by any type $n+1$ finite spectrum $F(n+1)$, see [39], and replace $V(1)_t E_2$ and $V(1)_{s+t}K(B)$ by $F(n+1)_t E_{n+1}$ and $F(n+1)_{s+t}K(B)$, respectively.

Example 5.2. Based on the detailed calculations behind Theorem 4.2, it is fairly evident that the E^2 -term for the spectral sequence conjectured to converge to $V(1)_*K(\ell_p)$ will be concentrated in the four columns $-3 \leq s \leq 0$, and that the free $\mathbb{F}_p[v_2]$ -module generators are located in the groups $H_{\text{mot}}^r(\ell_p; \mathbb{F}_{p^2}(m))$ where $0 \leq r \leq 3$ and $r \leq m < r + p^2 + p - 1$. This presumes that the spectral sequence collapses at the E^2 -term, for $p \geq 5$. In addition, there is a sporadic copy of \mathbb{F}_p in $V(1)_{2p-3}K(\ell_p)$.

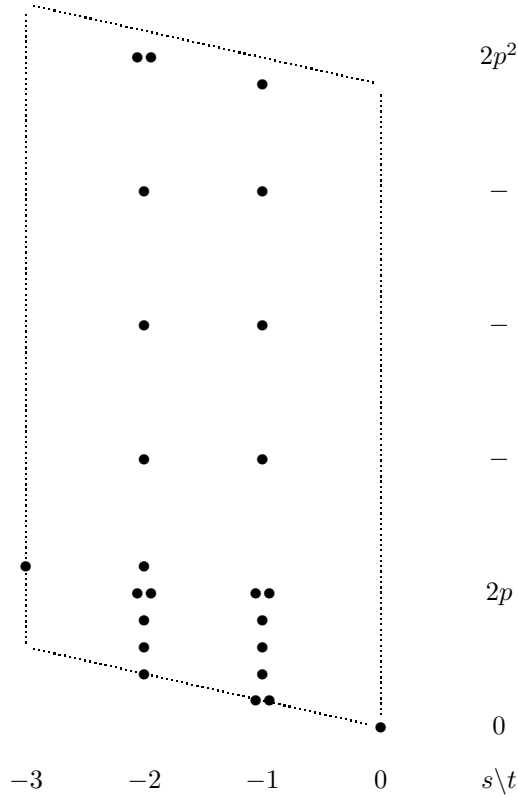


Figure 1. $\mathbb{F}_p[v_2]$ -generators of $E_{s,t}^2 = H_{\text{mot}}^{-s}(L_p; \mathbb{F}_{p^2}(t/2)) \implies V(1)_{s+t}K(L_p)$

The class $v_2 \in V(1)_{2p^2-2}K(\ell_p)$ is represented in bidegree $(s, t) = (0, 2p^2 - 2)$, corresponding to $(r, m) = (0, p^2 - 1)$. The presence of $\mathbb{F}_p[v_2]$ -module generators in the range $r + p^2 - 1 \leq m$ shows that $H_{\text{mot}}^r(\ell_p; \mathbb{F}_{p^2}(*))$ is *not* isomorphic to $v_2^{-1}H_{\text{mot}}^r(\ell_p; \mathbb{F}_{p^2}(*))$ in several bigradings (r, m) with $r \leq m < r + p$. In other words, the motivic truncation property fails for ℓ_p . However, this is to be expected, since ℓ_p has the residue ring spectrum $H\mathbb{Z}_p$ and should not behave as a field.

Example 5.3. Turning instead to $V(1)_*K(L_p)$, as determined from $V(1)_*K(\mathbb{Z}_p)$ and $V(1)_*K(\ell_p)$ by the homotopy cofiber sequence (2), free $\mathbb{F}_p[v_2]$ -module generators for the E^2 -term in Conjecture 5.1 would be concentrated in the groups $H_{\text{mot}}^r(L_p; \mathbb{F}_{p^2}(m))$ with $0 \leq r \leq 3$ and $r \leq m < r + p^2 - 1$. The (s, t) -bidegrees of these $4p + 4$ generators is displayed for $p = 5$ in Figure 1, lying in a fundamental domain in the shape of a parallelogram, of width 3 and height $2p^2 - 2$. In addition, there are sporadic copies of \mathbb{F}_p in $V(1)_{2p-3}K(L_p)$ and $V(1)_{2p-2}K(L_p)$.

In this case, the motivic truncation property for L_p is perfectly satisfied, in the

sense that

$$H_{\text{mot}}^r(L_p; \mathbb{F}_{p^2}(m)) \cong \begin{cases} H_{\text{ét}}^r(L_p; \mathbb{F}_{p^2}(m)) & \text{for } 0 \leq r \leq m, \\ 0 & \text{otherwise,} \end{cases}$$

where, by definition,

$$H_{\text{ét}}^r(L_p; \mathbb{F}_{p^2}(*)) = v_2^{-1} H_{\text{mot}}^r(L_p; \mathbb{F}_{p^2}(*))$$

is free over the Laurent polynomial ring $\mathbb{F}_p[v_2^{\pm 1}]$ on the same generators as in Figure 1. The (additive) Euler characteristic

$$\chi(L_p; \mathbb{F}_{p^2}(m)) = \sum_{r=0}^3 (-1)^r \dim_{\mathbb{F}_p} H_{\text{mot}}^r(L_p; \mathbb{F}_{p^2}(m))$$

is zero for each m , cf. [76, Thm. 2.2]. To the eyes of algebraic K -theory and the hypothetical motivic cohomology, the strictly commutative ring spectrum L_p behaves much like a brave new field. We discuss the role of its (non-commutative) residue ring L/p in the next section.

The étale cohomology of a p -adic field F satisfies local Tate–Poitou duality [76, Thm. 2.1]. In the case of mod p coefficients, this is a perfect pairing

$$H_{\text{ét}}^r(F; \mathbb{Z}/p(m)) \otimes H_{\text{ét}}^{2-r}(F; \mathbb{Z}/p(1-m)) \xrightarrow{\cup} H_{\text{ét}}^2(F; \mathbb{Z}/p(1)) \cong \mathbb{Z}/p$$

for each r and m . For general p -power torsion coefficients there is a perfect pairing taking values in the larger group $H_{\text{ét}}^2(F; \mathbb{Z}/p^\infty(1)) \cong \mathbb{Z}/p^\infty$, cf. [72, p. 130]. The multiplicative structure on $V(1)_*K(L_p)$ is compatible with an algebra structure on $H_{\text{mot}}^*(L_p; \mathbb{F}_{p^2}(*))$ such that the resulting multiplicative structure on $H_{\text{ét}}^*(L_p; \mathbb{F}_{p^2}(*))$ also satisfies *arithmetic duality*. This can be seen as a rotational symmetry about $(s, t) = (-3/2, p+1)$ in the variant of Figure 1 where v_2 has been inverted.

Conjecture 5.4. *For finite extensions B of L_p there is a perfect pairing*

$$H_{\text{ét}}^r(B; \mathbb{F}_{p^2}(m)) \otimes H_{\text{ét}}^{3-r}(B; \mathbb{F}_{p^2}(p+1-m)) \xrightarrow{\cup} H_{\text{ét}}^3(B; \mathbb{F}_{p^2}(p+1)) \cong \mathbb{Z}/p$$

for each r and m .

Remark 5.5. The dependence of the twist in $\mathbb{F}_{p^2}(p+1)$ on the prime p may be an artifact of the passage to mod p and v_1 coefficients. Let $E_2/(p^\infty, u_1^\infty)$ be the E_2 -module spectrum defined by the homotopy cofiber sequences $E_2 \rightarrow p^{-1}E_2 \rightarrow E_2/p^\infty$ and $E_2/p^\infty \rightarrow u_1^{-1}E_2/p^\infty \rightarrow E_2/(p^\infty, u_1^\infty)$. Its homotopy groups $\pi_*E_2/(p^\infty, u_1^\infty) = \mathbb{W}\mathbb{F}_{p^2}[[u_1]]/(p^\infty, u_1^\infty)[u^{\pm 1}]$ are Pontryagin dual to those of E_2 . Then

$$\mathbb{F}_{p^2}(p+1) = V(1)_{2p+2}E_2 \cong V(1)_{2p+3}(E_2/p^\infty) \cong V(1)_{2p+4}E_2/(p^\infty, u_1^\infty)$$

and

$$\begin{aligned} V(1)_{2p+4}E_2/(p^\infty, u_1^\infty) &\subset (S/p)_5E_2/(p^\infty, u_1^\infty) \\ &\subset \pi_4E_2/(p^\infty, u_1^\infty) = \mathbb{W}\mathbb{F}_{p^2}[[u_1]]/(p^\infty, u_1^\infty)(2). \end{aligned}$$

The conjectured arithmetic duality for mod p and v_1 coefficients may be a special case of a duality for p - and u_1 -power torsion coefficients, taking values in

$$H_{\text{ét}}^3(B; \mathbb{W}\mathbb{F}_{p^2}[[u_1]]/(p^\infty, u_1^\infty)(2)).$$

It would be desirable to find a canonical identification of this group, like the Hasse invariant in the classical case of p -adic fields and Kato's work [43] on the Galois cohomology of higher-dimensional local fields.

6. Fraction fields and ramified extensions

The étale cohomology of a field is, by construction, the same as its Galois cohomology, i.e., the continuous group cohomology of its absolute Galois group. There is no such direct description of $H_{\text{ét}}^r(L_p; \mathbb{F}_{p^2}(m))$, since according to Baker–Richter [8] the maximal connected pro-Galois extension of L_p is the composite

$$L_p \xrightarrow{\phi} KU_p \longrightarrow KU_p^{\text{nr}},$$

where $\pi_*(KU_p^{\text{nr}}) = \mathbb{W}\bar{\mathbb{F}}_p[[u^{\pm 1}]]$. The unramified extensions of $\pi_0(KU_p) = \mathbb{Z}_p$ are spectrally realized, using the methods of Schwänzl–Vogt–Waldhausen [71, Thm. 3] or Goerss–Hopkins–Miller [31], but the associated Galois group only has p -cohomological dimension 1, whereas L_p would have p -cohomological dimension 3. Likewise, the maximal connected pro-Galois extension of E_n is E_n^{nr} , with $\pi_*(E_n^{\text{nr}}) = \mathbb{W}\bar{\mathbb{F}}[[u_1, \dots, u_{n-1}]][[u^{\pm 1}]]$, of p -cohomological dimension 1 over E_n and $\hat{L}_n E(n)$.

To allow for ramification at p , one might simply invert that prime. However, the resulting strictly commutative ring spectrum $p^{-1}L_p$, with $\pi_*(p^{-1}L_p) = \mathbb{Q}_p[v_1^{\pm 1}]$, is an algebra over $p^{-1}S_p = H\mathbb{Q}_p$, so $V(1)_*K(p^{-1}L_p)$ is an algebra over $V(1)_*K(\mathbb{Q}_p)$, where v_2 acts trivially. Hence $H_{\text{ét}}^r(p^{-1}L_p; \mathbb{F}_{p^2}(*))$ would be zero.

A milder form of localization may be appropriate. By Waldhausen's localization theorem [82], the homotopy fiber of $K(L_p) \rightarrow K(p^{-1}L_p)$ is given by the algebraic K -theory of the category with cofibrations of finite cell L_p -modules with p -power torsion homotopy, equipped with the usual weak equivalences. We might instead step back to a category with cofibrations of coherent L/p^ν -modules (i.e., having degreewise finite homotopy groups, see Barwick–Lawson [9]), for some natural number ν , and suppose that these have the same algebraic K -theory as the category with cofibrations of finite cell L/p -modules. Here $L/p = K(1)$ is the first Morava K -theory, which by Angeltveit [2] is a strict ring spectrum, but not strictly commutative. By Davis–Lawson [22, Cor. 6.4] the tower $\{L/p^\nu\}_\nu$ is as commutative as possible in the category of pro-spectra:

$$L/p \leftarrow \{L/p^\nu\}_\nu \leftarrow L_p \longrightarrow p^{-1}L_p.$$

Definition 6.1. Let $K(\text{ff}L_p)$ be defined by the homotopy cofiber sequence

$$K(L/p) \xrightarrow{i_*} K(L_p) \longrightarrow K(\text{ff}L_p),$$

where i_* is the transfer map associated to $i: L_p \rightarrow L/p$.

We think of $K(\mathit{ff}L_p)$ as the algebraic K -theory of a hypothetical *fraction field* of L_p , intermediate between L_p and $p^{-1}L_p$, and similar to the 2-dimensional local field $\mathbb{Q}_p((u))$. Its mod p and v_1 homotopy groups can be calculated using the following result, in combination with the homotopy cofiber sequence $K(\mathbb{F}_p) \rightarrow K(\ell/p) \rightarrow K(L/p)$.

Theorem 6.2 (Ausoni–Rognes [7, Thm. 7.6, Thm. 7.7]). *$V(1)_*TC(\ell/p;p)$ is a finitely generated free $\mathbb{F}_p[v_2]$ -module on $2p^2 - 2p + 8$ generators, which are located in degrees $-1 \leq * \leq 2p^2 + 2p - 2$. There is an exact sequence of $\mathbb{F}_p[v_2]$ -modules*

$$0 \rightarrow V(1)_*K(\ell/p) \xrightarrow{\text{trc}} V(1)_*TC(\ell/p;p) \rightarrow \Sigma^{-1}\mathbb{F}_p \oplus \Sigma^{2p-2}\mathbb{F}_p \rightarrow 0.$$

Example 6.3. The expected E^2 -term for a spectral sequence

$$E_{s,t}^2 = H_{\text{mot}}^{-s}(\mathit{ff}L_p; \mathbb{F}_{p^2}(t/2)) \implies V(1)_{s+t}K(\mathit{ff}L_p)$$

is displayed for $p = 5$ in Figure 2. In addition there are four sporadic copies of \mathbb{F}_p , in degrees $2p - 3$, $2p - 2$, $2p - 2$ and $2p - 1$. The motivic truncation properties for $\mathit{ff}L_p$, analogous to (3) and (4), are clearly visible, and conjecturally there is now a perfect arithmetic duality pairing

$$H_{\text{ét}}^r(\mathit{ff}L_p; \mathbb{F}_{p^2}(m)) \otimes H_{\text{ét}}^{3-r}(\mathit{ff}L_p; \mathbb{F}_{p^2}(2-m)) \xrightarrow{\cup} H_{\text{ét}}^3(\mathit{ff}L_p; \mathbb{F}_{p^2}(2)) \cong \mathbb{Z}/p.$$

After such localization of L_p away from L/p , it may be possible to construct enough Galois extensions of $\mathit{ff}L_p$ to realize its v_2 -localized motivic cohomology as continuous group cohomology

$$H_{\text{ét}}^r(\mathit{ff}L_p; \mathbb{F}_{p^2}(m)) \cong H_{\text{gp}}^r(G_{\mathit{ff}L_p}; \mathbb{F}_{p^2}(m)),$$

for an absolute Galois group $G_{\mathit{ff}L_p}$ of p -cohomological dimension 3, corresponding to some maximal extension $\mathit{ff}L_p \rightarrow \Omega_1$. If each Galois extension of \mathbb{Q}_p can be lifted to an extension of $\mathit{ff}L_p$, we get a short exact sequence

$$1 \rightarrow I_{v_1} \rightarrow G_{\mathit{ff}L_p} \rightarrow G_{\mathbb{Q}_p} \rightarrow 1,$$

with I_{v_1} the inertia group over (v_1) . Here $G_{\mathbb{Q}_p}$ has p -cohomological dimension 2, and I_{v_1} will have p -cohomological dimension 1.

In the less structured setting of ring spectra up to homotopy it is possible to construct totally ramified extensions of KU_p , complementary to the unramified extension KU_p^{nr} , without inverting p . Torii [77, Thm. 2.5] shows that for each $r \geq 1$ the homotopy cofiber

$$F(B\mathbb{Z}/p_+^{r-1}, KU_p) \xrightarrow{\tau_r^\#} F(B\mathbb{Z}/p_+^r, KU_p) \rightarrow KU_p[\zeta_{p^r}],$$

of the map of function spectra induced by the stable transfer map $\tau_r: B\mathbb{Z}/p_+^r \rightarrow B\mathbb{Z}/p_+^{r-1}$, is a ring spectrum up to homotopy with $\pi_*KU_p[\zeta_{p^r}] \cong \mathbb{Z}_p[\zeta_{p^r}][u^{\pm 1}]$, where ζ_{p^r} denotes a primitive p^r -th root of unity. (He has similar realization

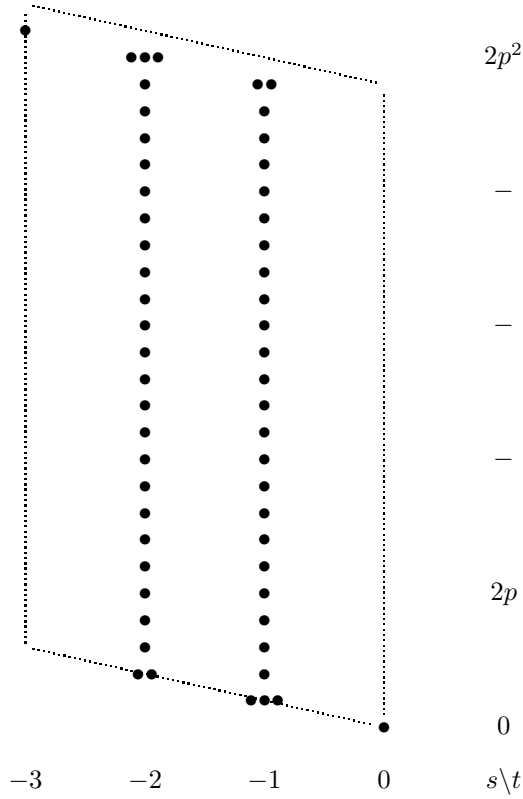


Figure 2. $\mathbb{F}_p[v_2]$ -generators of $E_{s,t}^2 = H_{\text{mot}}^{-s}(\text{ff}L_p; \mathbb{F}_{p^2}(t/2)) \implies V(1)_{s+t}K(\text{ff}L_p)$

results in the $K(n)$ -local category.) However, it does not make sense to talk about the algebraic K -theory of a ring spectrum up to homotopy, so these constructions are only helpful if they can be made strict.

It is not possible to realize $KU_p[\zeta_{p^r}]$ as a strictly commutative ring spectrum. Angeltveit [1, Rem. 5.18] uses the identity $\psi^p(x) = x^p + p\theta(x)$ among power operations in π_0 of a $K(1)$ -local strictly commutative ring spectrum to show that if $-p$ admits a k -th root in such a ring, with $k \geq 2$, then p is invertible in that ring. For $r = 1$ we have $\mathbb{Z}_p[\zeta_p] = \mathbb{Z}_p[\xi]$ where $\xi^{p-1} = -p$, so for p odd this proves that adjoining ζ_p to KU_p in a strictly commutative context will also invert p .

It is, however, possible to adjoin ζ_p to π_0 of the connective cover ku_p , in the category of strictly commutative ring spectra, without fully inverting p . Instead, one must make some positive power of the Bott element $u \in \pi_2(ku_p)$ singly divisible by p . If one thereafter inverts u , it follows that p has also become invertible. To achieve this, we modify Torii's construction for $r = 1$ by replacing the transfer

map with a norm map. This leads to the G -Tate construction

$$B^{tG} = t_G(B)^G = [\widetilde{EG} \wedge F(EG_+, i_* B)]^G$$

for a spectrum B with G -action, cf. Greenlees–May [35, p. 3]. This construction preserves strictly commutative ring structures, see McClure [50, Thm. 1].

Example 6.4. Let

$$KU_p'[\xi] = (ku_p)^{t\mathbb{Z}/p}$$

denote the \mathbb{Z}/p -Tate construction on the spectrum ku_p with trivial \mathbb{Z}/p -action, and let $ku_p'[\xi] = KU_p'[\xi][0, \infty)$ be its connective cover. Additively, these are generalized Eilenberg–Mac Lane spectra, cf. Davis–Mahowald [23] and [35, Thm. 13.5]. Multiplicatively, $\pi_*(KU_p'[\xi]) \cong \mathbb{Z}_p[\xi][v^{\pm 1}]$ where $p + \xi^{p-1} = 0$ and $|v| = 2$. Furthermore,

$$\pi_*(ku_p'[\xi]) \cong \mathbb{Z}_p[\xi][v],$$

and a morphism $ku_p \rightarrow ku_p'[\xi]$ of strictly commutative ring spectra induces the ring homomorphism $\mathbb{Z}_p[u] \rightarrow \mathbb{Z}_p[\xi][v]$ that maps u to $\xi \cdot v$. There is no multiplicative morphism $KU_p \rightarrow KU_p'[\xi]$, but

$$KU_p \wedge_{ku_p} ku_p'[\xi] = u^{-1}ku_p'[\xi] \simeq KU\mathbb{Q}_p(\xi) = KU\mathbb{Q}_p(\zeta_p)$$

is a totally ramified extension of $KU\mathbb{Q}_p = p^{-1}KU_p$. We get a diagram of strictly commutative ring spectra

$$\begin{array}{ccccc} (H\mathbb{Z}_p)^{t\mathbb{Z}/p} & \longleftarrow & KU_p'[\xi] & \longrightarrow & KU\mathbb{Q}_p(\xi) \\ \uparrow & & \uparrow & & \uparrow \\ (H\mathbb{Z}_p)^{t\mathbb{Z}/p}[0, \infty) & \longleftarrow & ku_p'[\xi] & \longrightarrow & ku\mathbb{Q}_p(\xi) \\ \downarrow & & \downarrow & & \downarrow \\ H\mathbb{Z}/p & \longleftarrow & H\mathbb{Z}_p[\xi] & \longrightarrow & H\mathbb{Q}_p(\xi) \end{array}$$

with horizontal maps reducing modulo or inverting ξ , and vertical maps reducing modulo or inverting v . Here $\pi_*((H\mathbb{Z}_p)^{t\mathbb{Z}/p}) \cong \hat{H}^{-*}(\mathbb{Z}/p; \mathbb{Z}_p) = \mathbb{Z}/p[v^{\pm 1}]$. We view $ku_p'[\xi]$ as an integral model for a 2-dimensional local field close to $KU\mathbb{Q}_p(\xi)$, but note that $ku_p'[\xi]$ is not finite as a ku_p -module.

Example 6.5. Let $(\mathbb{Z}/p)^\times$ act on the group \mathbb{Z}/p by multiplication, hence also on the \mathbb{Z}/p -Tate construction $KU_p'[\xi] = (ku_p)^{t\mathbb{Z}/p}$. Let

$$KU_p' = (KU_p'[\xi])^{h(\mathbb{Z}/p)^\times}$$

be the homotopy fixed points, and let $ku_p' = KU_p'[0, \infty)$ be its connective cover. These are strictly commutative ring spectra, with $\pi_*(KU_p') \cong \mathbb{Z}_p[u, w^{\pm 1}]/(pw + u^{p-1})$ where $|w| = 2p - 2$, and $\pi_*(ku_p') \cong \mathbb{Z}_p[u, w]/(pw + u^{p-1})$. Multiplicative

morphisms $ku_p \rightarrow ku'_p \rightarrow ku'_p[\xi]$ induce the inclusions $\mathbb{Z}_p[u] \rightarrow \mathbb{Z}_p[u, w]/(pw + u^{p-1}) \rightarrow \mathbb{Z}_p[\xi][v]$ where w maps to v^{p-1} .

The morphism $KU'_p \rightarrow KU'_p[\xi]$ is a $(\mathbb{Z}/p)^\times$ -Galois extension in the sense of [63]. The morphism $ku'_p \rightarrow ku'_p[\xi]$ becomes $(\mathbb{Z}/p)^\times$ -Galois after inverting p or w . It remains to be determined whether $V(1)_*K(ku'_p)$ remains purely v_2 -periodic, i.e., whether the multiplicative approximation $ku_p \rightarrow ku'_p$ counters the additive splitting of ku'_p as a sum of suspended Eilenberg–Mac Lane spectra.

Example 6.6. More generally, for $r \geq 1$ let $G = \mathbb{Z}/p^r$, let \mathcal{P} be the family of proper subgroups of G , and let $KU_G[0, \infty)$ be the “brutal” truncation of G -equivariant periodic K -theory, cf. [34, p. 129]. Define

$$KU'_p[\zeta_{p^r}] = (KU_G[0, \infty))^{t\mathcal{P}} = [\widetilde{E\mathcal{P}} \wedge F(E\mathcal{P}_+, KU_G[0, \infty))]^G$$

to be the \mathcal{P} -Tate construction, as in Greenlees–May [35, §17], and let $ku'_p[\zeta_{p^r}]$ be its connective cover. Then $\pi_*(KU'_p[\zeta_{p^r}]) \cong \mathbb{Z}_p[\zeta_{p^r}][v^{\pm 1}]$ and

$$\pi_*(ku'_p[\zeta_{p^r}]) \cong \mathbb{Z}_p[\zeta_{p^r}][v].$$

The map $(KU_G[0, \infty))^{t\mathcal{P}} \rightarrow (KU_G)^{t\mathcal{P}}$ induces the inclusion $\mathbb{Z}_p[\zeta_{p^r}] \subset \mathbb{Q}_p[\zeta_{p^r}]$ in each even degree. To prove this, one can compute Amitsur–Dress homology for the family \mathcal{P} , use the generalized Tate spectral sequence from [35, §22], and then compare with the calculation in [35, §19] of the periodic case. The cyclotomic extension $\mathbb{Z}_p[\zeta_{p^r}]$ arises as $(R(G)/J'\mathcal{P})_{J'\mathcal{P}}^\wedge$, where $R(G)$ is the representation ring, $J'\mathcal{P}$ is the kernel of the restriction map $R(G) \rightarrow R(H)$, and $J'\mathcal{P}$ is the image of the induction map $R(H) \rightarrow R(G)$, where H is the index p subgroup in G .

7. Logarithmic ring spectra

The heuristics from the last two sections suggest that we should attempt to construct ramified finite extensions $B \rightarrow C$ of strictly commutative ring spectra B like ℓ_p , ku_p and e_n . The Goerss–Hopkins–Miller obstruction theory [31] for strictly commutative B -algebra structures on such spectra C has vanishing obstruction groups in the case of unramified extensions, but appears to be less useful in the case of ramification over (p) , due to the presence of nontrivial (topological) André–Quillen cohomology groups [11].

The same heuristics also suggest that we should approach the extension problem by passing to mildly local versions of B , intermediate between B and $p^{-1}B$. In arithmetic algebraic geometry, one such intermediary is provided by logarithmic geometry, cf. Kato [44]. An affine pre-log scheme $(\text{Spec } R, M)$ is a scheme $\text{Spec } R$, a commutative monoid M , and a homomorphism $\alpha: M \rightarrow (R, \cdot)$ to the underlying multiplicative monoid of R . More precisely, M and α live étale locally on $\text{Spec } R$. In this wider context, there is a factorization

$$\text{Spec } R[M^{-1}] \longrightarrow (\text{Spec } R, M) \longrightarrow \text{Spec } R$$

of the natural inclusion, and the right-hand map is often a well-behaved proper replacement for the composite open immersion. Logarithmic structures on valuation rings in p -adic fields were successfully used by Hesselholt–Madsen [38] to analyze the topological cyclic homology and algebraic K -theory of these classical rings.

A theory of logarithmic structures on strictly commutative ring spectra was started by the author in [64], and developed further in joint work with Sagave and Schlichtkrull. To present it, we take the category \mathcal{CSp}^Σ of commutative symmetric ring spectra [40], with the positive stable model structure [48, §14], as our model for strictly commutative ring spectra.

By the graded underlying space of a symmetric spectrum A we mean a diagram

$$\Omega^{\mathcal{J}}(A): (\mathbf{n}_1, \mathbf{n}_2) \mapsto \Omega^{n_2} A_{n_1}$$

of spaces, where $(\mathbf{n}_1, \mathbf{n}_2)$ ranges over the objects in a category \mathcal{J} . We call such a diagram a \mathcal{J} -space. Following Sagave, the natural category \mathcal{J} to consider turns out to be isomorphic to Quillen’s construction $\Sigma^{-1}\Sigma$, where Σ is the permutative groupoid of finite sets and bijections. Its nerve $B\mathcal{J} \cong B(\Sigma^{-1}\Sigma)$ is homotopy equivalent to $QS^0 = \Omega^\infty S$. For any \mathcal{J} -space X , the homotopy colimit $X_{h\mathcal{J}} = \text{hocolim}_{\mathcal{J}} X$ is augmented over $B\mathcal{J}$, so we say that X is QS^0 -graded. A map $X \rightarrow Y$ of \mathcal{J} -spaces is called a \mathcal{J} -equivalence if the induced map $X_{h\mathcal{J}} \rightarrow Y_{h\mathcal{J}}$ is a weak equivalence.

If A is a commutative symmetric ring spectrum, then $\Omega^{\mathcal{J}}(A)$ is a commutative monoid with respect to a convolution product in the category of \mathcal{J} -spaces. The category $\mathcal{CS}^{\mathcal{J}}$ of commutative \mathcal{J} -space monoids has a positive projective model structure [70, §4], with the \mathcal{J} -equivalences as the weak equivalences, and is Quillen equivalent to a category of E_∞ spaces over $B\mathcal{J}$. The functor $\Omega^{\mathcal{J}}: \mathcal{CSp}^\Sigma \rightarrow \mathcal{CS}^{\mathcal{J}}$ admits a left adjoint $M \mapsto S^{\mathcal{J}}[M]$, and $(S^{\mathcal{J}}[-], \Omega^{\mathcal{J}})$ is a Quillen adjunction.

There is a commutative submonoid of graded homotopy units $\iota: GL_1^{\mathcal{J}}(A) \subset \Omega^{\mathcal{J}}(A)$. A *pre-log ring spectrum* (A, M, α) is a commutative symmetric ring spectrum A with a pre-log structure (M, α) , i.e., a commutative \mathcal{J} -space monoid M and a map $\alpha: M \rightarrow \Omega^{\mathcal{J}}(A)$ in $\mathcal{CS}^{\mathcal{J}}$. If the pullback $\alpha^{-1}(GL_1^{\mathcal{J}}(A)) \rightarrow GL_1^{\mathcal{J}}(A)$ is a \mathcal{J} -equivalence we call (M, α) a log structure and (A, M, α) a *log ring spectrum*. We often omit α from the notation.

In order to classify extensions $(A, M) \rightarrow (B, N)$ of pre-log ring spectra, one is led to study infinitesimal deformations and derivations in this category. Derivations are corepresented by a logarithmic version $TAQ(A, M)$ of topological André–Quillen homology, defined by a pushout

$$\begin{array}{ccc} A \wedge_{S^{\mathcal{J}}[M]} TAQ(S^{\mathcal{J}}[M]) & \xrightarrow{\psi} & A \wedge \gamma(M) \\ \alpha \downarrow & & \downarrow \\ TAQ(A) & \longrightarrow & TAQ(A, M) \end{array}$$

of A -module spectra, cf. [64, Def. 11.19] and [69, Def. 5.20]. Here $TAQ(A)$ is the ordinary topological André–Quillen homology, as defined by Basterra [10], and $\gamma(M)$ is the connective spectrum associated to the E_∞ space $M_{h\mathcal{J}}$. A morphism

$(A, M) \rightarrow (B, N)$ is formally log étale if $B \wedge_A \text{TAQ}(A, M) \rightarrow \text{TAQ}(B, N)$ is an equivalence.

Let $j: e \rightarrow E$ be a fibration of commutative symmetric ring spectra. The direct image of the trivial log structure on E is the log structure $(j_*GL_1^{\mathcal{J}}(E), \alpha)$ on e given by the pullback

$$\begin{array}{ccc} j_*GL_1^{\mathcal{J}}(E) & \longrightarrow & GL_1^{\mathcal{J}}(E) \\ \alpha \downarrow & & \downarrow \iota \\ \Omega^{\mathcal{J}}(e) & \xrightarrow{\Omega^{\mathcal{J}}(j)} & \Omega^{\mathcal{J}}(E) \end{array}$$

in $\mathcal{CS}^{\mathcal{J}}$. Applying this natural construction to the vertical maps in (1), we get the following example. Note that $\ell_p \rightarrow ku_p$ is not étale, while $L_p \rightarrow KU_p$ is Δ -Galois, hence étale.

Theorem 7.1 (Sagave [69, Thm. 6.1]). *The morphism*

$$\phi: (\ell_p, j_*GL_1^{\mathcal{J}}(L_p)) \longrightarrow (ku_p, j_*GL_1^{\mathcal{J}}(KU_p))$$

is log étale.

In order to approximate algebraic K -theory, one is likewise led to study logarithmic topological Hochschild homology and logarithmic topological cyclic homology. The former is defined by a pushout

$$\begin{array}{ccc} S^{\mathcal{J}}[B^{\text{cy}}(M)] & \xrightarrow{\rho} & S^{\mathcal{J}}[B^{\text{rep}}(M)] \\ \alpha \downarrow & & \downarrow \\ THH(A) & \longrightarrow & THH(A, M) \end{array}$$

in \mathcal{CSp}^{Σ} , cf. [65, §4]. Here $THH(A)$ is the ordinary topological Hochschild homology of A , given by the cyclic bar construction of A in \mathcal{CSp}^{Σ} . The cyclic bar construction $B^{\text{cy}}(M)$ is formed in $\mathcal{CS}^{\mathcal{J}}$, and is naturally augmented over M . The *replete bar construction* $B^{\text{rep}}(M)$ can be viewed as a fibrant replacement of $B^{\text{cy}}(M)$ over M in a group completion model structure on $\mathcal{CS}^{\mathcal{J}}$, cf. [68, Thm. 1.6], but also has a more direct description as the (homotopy) pullback in the right hand square below:

$$\begin{array}{ccccc} B^{\text{cy}}(M) & \xrightarrow{\rho} & B^{\text{rep}}(M) & \longrightarrow & B^{\text{cy}}(M^{\text{gp}}) \\ \epsilon \downarrow & & \downarrow & & \downarrow \epsilon \\ M & \xrightarrow{=} & M & \xrightarrow{\eta} & M^{\text{gp}}. \end{array}$$

Here $\eta: M \rightarrow M^{\text{gp}}$ is a group completion in $\mathcal{CS}^{\mathcal{J}}$, which means that $(M^{\text{gp}})_{h\mathcal{J}}$ is a group completion of the E_{∞} space $M_{h\mathcal{J}}$. The role of repletion in homotopy theory is similar to that of working within the subcategory of fine and saturated logarithmic structures in the discrete setting [44, §2].

A morphism $(A, M) \rightarrow (B, N)$ is formally log thh-étale if $B \wedge_A THH(A, M) \rightarrow THH(B, N)$ is an equivalence. The following theorem strengthens the previous result.

Theorem 7.2 (Rognes–Sagave–Schlichtkrull [66, Thm. 1.5]). *The morphism*

$$\phi: (\ell_p, j_* GL_1^{\mathcal{J}}(L_p)) \longrightarrow (ku_p, j_* GL_1^{\mathcal{J}}(KU_p))$$

is log thh-étale.

Remark 7.3. These results harmonize with the classical correspondence between tamely ramified extensions and log étale extensions. By Noether’s theorem [53], tame ramification corresponds locally to the existence of a normal basis. This conforms with the observation that ku_p is a retract of a finite cell $\ell_p[\Delta]$ -module, so that $\ell_p \rightarrow ku_p$ is tamely ramified. By contrast, ku_2 is not a retract of a finite cell $ko_2[C_2]$ -module, e.g. because $(ku_2)^{tC_2}$ is nontrivial, so $ko_2 \rightarrow ku_2$ is wildly ramified.

We say that a commutative symmetric ring spectrum E is d -periodic if d is the minimal positive integer such that $\pi_*(E)$ contains a unit in degree d .

Theorem 7.4 (Rognes–Sagave–Schlichtkrull [65, Thm. 1.5]). *Let E in \mathcal{CSp}^{Σ} be d -periodic, with connective cover $j: e \rightarrow E$. There is a natural homotopy cofiber sequence*

$$THH(e) \xrightarrow{\rho} THH(e, j_* GL_1^{\mathcal{J}}(E)) \xrightarrow{\partial} \Sigma THH(e[0, d]),$$

where $e[0, d]$ is the $(d - 1)$ -th Postnikov section of e .

These results allow us to realize the strategy outlined in [3, §10] to compute the $V(1)$ -homotopy of $THH(ku_p)$ by way of $THH(\ell_p)$, $THH(\ell_p, j_* GL_1^{\mathcal{J}}(L_p))$ and $THH(ku_p, j_* GL_1^{\mathcal{J}}(KU_p))$. The details are given in [66, §7, §8].

When $e[0, d] = H\pi_0(e)$ with $\pi_0(e)$ regular, Blumberg–Mandell [15, Thm. 4.2.1] have constructed a map of horizontal homotopy cofiber sequences

$$\begin{array}{ccccc} K(\pi_0(e)) & \xrightarrow{i_*} & K(e) & \xrightarrow{j^*} & K(E) \\ \downarrow & & \downarrow & & \downarrow \\ THH(\pi_0(e)) & \xrightarrow{i_*} & THH(e) & \xrightarrow{j^*} & WTHH^{\Gamma}(e|E) \end{array}$$

where the vertical arrows are trace maps.

Conjecture 7.5. *There is an equivalence of cyclotomic spectra*

$$THH(e, j_* GL_1^{\mathcal{J}}(E)) \simeq WTHH^{\Gamma}(e|E),$$

compatible with the maps from $THH(e)$ and to $\Sigma THH(\pi_0(e))$.

The author hopes that a logarithmic analog of the Goerss–Hopkins–Miller obstruction theory [31] can be developed to classify log extensions of log ring spectra,

and that in the case of log étale extensions the obstruction groups will vanish in such a way as to enable the construction of interesting examples. The underlying strictly commutative ring spectra should then provide novel examples of tamely ramified extensions, and realize a larger part of motivic cohomology as a case of Galois cohomology.

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