A LEISURELY INTRODUCTION TO SIMPLICIAL SETS

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Abstract. Simplicial sets are introduced in a way that should be pleasing to the formally-inclined. Care is taken to provide both the geometric intuition and the categorical details, laying a solid foundation for the reader to move on to more advanced topological or higher categorical applications. The highlight, and the only possibly non-standard content, is a unified presentation of every adjunction whose left adjoint has the category of simplicial sets as its domain: geometric realization and the total singular complex functor, the ordinary nerve and its left adjoint, the homotopy coherent nerve and its left adjoint, and subdivision and extension are some of the many examples that fit within this framework.

1. Introduction

Simplicial sets, an extension of the notion of simplicial complexes, have applications to algebraic topology, where they provide a combinatorial model for the homotopy theory of topological spaces. There are functors between the categories of topological spaces and simplicial sets called the total singular complex functor and geometric realization, which form an adjoint pair and give a Quillen equivalence between the usual model structures on these categories.

More recently, simplicial sets have found applications in higher category theory because simplicial sets which satisfy a certain horn lifting criterion (see Definition 5.7) provide a model for $(\infty, 1)$-*categories*, in which every cell of dimension greater than one is invertible. These categories again provide a natural setting for homotopy theory; consequently, many constructions in this setting are motivated by algebraic topology as well.

These important applications will not be discussed here[1]. Instead, we aim to provide an elementary introduction to simplicial sets, accessible to anyone with a very basic familiarity with category theory. In §2, we give two equivalent definitions of simplicial sets, and in §3, we describe several examples. §4, which discusses several important adjunctions that involve the category of simplicial sets, is the heart of this paper. Rather than present the various adjunctions independently, as is commonly done, we use a categorical construction to show how all of these examples fit in the same general framework, greatly expediting the proof of the adjoint relationship. Finally, in §5, we define simplicial spheres and horns, which are the jumping off point for the applications to homotopy theory and higher category theory mentioned above.

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[1]Nor do we discuss the various model structures on the category of simplicial sets. See [DS95], [Hov99], or the original [Qui67] for an introduction to model categories and a description of the model structure relevant to topological spaces, and [Lur09] or [Rie08] for a description of the model structure relevant to $(\infty, 1)$-categories.

Because particular simplical sets called the standard simplices are represented functors, the Yoneda lemma plays an important role. We review several versions of this result in §3. The left adjoints in 4 are defined by means of a rather intricate colimit construction, though we give references for the more exotic colimits employed. Otherwise, comfort with the definitions of a category, functor, and natural transformation should suffice as prerequisites.

The standard references for the theory of simplicial sets and some of their many uses in algebraic topology are [GJ99] and [May67], though both sources quickly move on to more advanced topics. A wealth of excellent material can also be found in the classic [GZ67]. Less well known is a brief introduction given in [ML97, §7], which may serve better for the reader interested in only the most rudimentary definitions.

2. Definitions

Let $\mathbf{\Delta}$ be the category whose objects are finite, non-empty, totally ordered sets

$$[n] = \{0, 1, \dots, n\}$$

and morphisms are order preserving functions. Equivalently, $\mathbf{\Delta}$ is the full subcategory of **Cat** whose objects are the posets defining finite, non-empty ordinals, regarded as categories in the usual manner; here, we denote the ordinal $n+1$ (above) by $[n]$ to emphasize connections with topology, which will be expounded upon below.

Definition 2.1. A *simplicial set* is a contravariant functor from $\mathbf{\Delta}$ to **Set**. More generally, for any category $\mathcal{C}$, a *simplicial object* in $\mathcal{C}$ is a functor $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$.

Let $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$ be a simplicial set. It is standard to write X_n for the set $X[n]$ and call its elements *n-simplices*. We visualize an n-simplex as an n-dimensional tetrahedron with all of its lower-dimensional faces labeled by simplices of the appropriate dimension and whose $n+1$ vertices are ordered $0, \dots, n$. Unlike an abstract simplicial complex [Hat??], the vertices (0-dimensional simplices) need not be distinct nor necessarily determine the simplex spanning them.

We write **sSet** for the category of simplicial sets, which is simply the functor category $\mathbf{Set}^{\mathbf{\Delta}^{\mathrm{op}}}$. In particular, morphisms $f\colon X \to Y$ between simplicial sets are natural transformations: a map of simplicial sets consists of maps $X_n \to Y_n$ of n-simplices that commute with the actions by morphisms of $\mathbf{\Delta}$. In fact, it suffices to ask that these maps commute with the *face* and *degeneracy* maps defined in (2.3) below.

The combinatorial data of a simplicial set has a simpler presentation that exploits the fact that the category $\mathbf{\Delta}$ has a natural generating set of morphisms. For each $n \geq 0$ there are $n+1$ injections $d^i\colon [n-1] \to [n]$ called the *coface* maps and $n+1$ surjections $s^i\colon [n+1] \to [n]$ called the *codegeneracy* maps, defined as follows:

$$d^i(k) = \begin{cases} k, & k < i \\ k+1, & k \geq i \end{cases} \quad \text{and} \quad s^i(k) = \begin{cases} k, & k \leq i \\ k-1, & k > i \end{cases} \quad \text{for} \quad 0 \leq i \leq n.^2$$

The i-th coface map d^i misses the element i in the image, while the i-th codegeneracy map s^i sends two elements to i. These morphisms satisfy several obvious

[2] As is usual, the dependence on n is suppressed in this notation, as the codomain should be clear from the context.

relations:

$$
\begin{aligned}
d^j d^i &= d^i d^{j-1}, \quad i < j \\
s^j s^i &= s^i s^{j+1}, \quad i \leq j \\
s^j d^i &= \begin{cases} 1, & i = j, j+1 \\ d^i s^{j-1}, & i < j \\ d^{i-1} s^j, & i > j+1. \end{cases}
\end{aligned} \tag{2.2}
$$

It is not difficult to verify that every morphism of $\mathbf{\Delta}$ can be expressed as a composite of the coface and codegeneracy maps. If we impose further artificial requirements specifying the order in which the generating morphisms should occur, then each morphism of $\mathbf{\Delta}$ can be expressed uniquely as a composite of this particular form. Details are left to the reader (or see [GZ67] or [ML97, §7.5]).

If X is a simplicial set, we write

$$
\begin{aligned}
d_i &= X d^i \colon\ X_n \to X_{n-1} \qquad 0 \leq i \leq n \\
s_i &= X s^i \colon\ X_n \to X_{n+1} \qquad 0 \leq i \leq n
\end{aligned} \tag{2.3}
$$

and call these the *face* and *degeneracy* maps respectively. The d_i and s_i will then satisfy relations dual to (2.2). The face maps assign, to each $x \in X_n$, $n+1$ $(n-1)$-simplices $d_0(x), \ldots, d_n(x) \in X_{n-1}$. By convention, the face $d_i(x)$ is the one not containing the i-th vertex of x. The relation $d_i d_j = d_{j-1} d_i$ for all $i < j$ expresses the fact that, as faces of a common n-simplex, the i-th and $(j-1)$-th faces of $d_j(x)$ and $d_i(x)$ agree.

To each $x \in X_n$, the degeneracy maps associate $n{+}1$ $(n{+}1)$-simplices $s_0(x), \ldots, s_n(x) \in X_{n+1}$. The $(n+1)$-simplex $s_i(x)$ has x as its i-th and $(i+1)$-th faces; the intuition is that the projection that collapses the edge from the i-th to the $(i+1)$-th vertex to a point returns the n-simplex x. We will be more precise about these projections in the next section. We say a simplex $x \in X_n$ is *degenerate* if it is the image of some degeneracy map s_i and *non-degenerate* otherwise. By the Eilenberg-Zilber lemma [GZ67, pp 26-27], each degenerate $x \in X_n$ is uniquely expressible as $(X\epsilon)y$ for some non-degenerate $y \in X_m$ and epimorphism $\epsilon \colon [n] \twoheadrightarrow [m]$ in $\mathbf{\Delta}$.

In fact, the data of a simplicial set is completely specified by the sets X_n and the maps d_i, s_i in the sense of the following alternative definition:

Definition 2.4. A *simplicial set* X is a collection of sets X_n for each integer $n \geq 0$ together with functions $d_i \colon X_n \to X_{n-1}$ and $s_i \colon X_n \to X_{n+1}$ for all $0 \leq i \leq n$ and for each n satisfying the following relations:

$$
\begin{aligned}
d_i d_j &= d_{j-1} d_i, \quad i < j \\
s_i s_j &= s_{j+1} s_i, \quad i \leq j \\
d_i s_j &= \begin{cases} 1, & i = j, j+1 \\ s_{j-1} d_i, & i < j \\ s_j d_{i-1}, & i > j+1. \end{cases}
\end{aligned} \tag{2.5}
$$

This is the first definition given in [May67]. In practice, one usually specifies a simplicial set X by describing the sets X_n of n-simplices and then defining the required face and degeneracy maps. Mercifully, the required relations are often obvious, and even if they are not, it is still advisable to assert that they are, after privately verifying that they do in fact hold.

3. Yoneda and Examples

The simplest examples of simplicial sets are the *standard simplices* Δ^n for each $n \geq 0$, which are the represented functors for each object $[n] \in \mathbf{\Delta}$. Their universal properties will follow immediately from the Yoneda lemma, which we briefly review.

For any small category $\mathcal{C}$ and any object $c \in \mathcal{C}$, there is a *represented functor*

$$\mathcal{C}(-,c)\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$$

that takes an object $b \in \mathcal{C}$ to the set of arrows $\mathcal{C}(b,c)$. A morphism $g\colon a \to b$ in $\mathcal{C}$ induces a function $g^*\colon \mathcal{C}(b,c) \to \mathcal{C}(a,c)$ defined by pre-composition. This defines the functor $\mathcal{C}(-,c)$.

Furthermore, a morphism $f\colon c \to d$ in $\mathcal{C}$ defines a natural transformation

$$f_*\colon \mathcal{C}(-,c) \to \mathcal{C}(-,d)$$

defined pointwise by post-composition with f. This defines a functor

$$\mathcal{C} \hookrightarrow \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$$

called the *Yoneda embedding*. One version of the Yoneda lemma says that this functor is full and faithful, which means that natural transformations between represented functors uniquely correspond to morphisms between the representing objects.

Returning to simplicial sets, for each $[n] \in \mathbf{\Delta}$, we denote its image under the Yoneda embedding

$$y\colon \mathbf{\Delta} \hookrightarrow \mathbf{Set}^{\mathbf{\Delta}^{\mathrm{op}}} = \mathbf{sSet} \quad \text{by} \quad \Delta^n := y[n] = \mathbf{\Delta}(-,[n]).$$

This is the simplicial set representing the *standard n-simplex*. From the definition, $\Delta^n_k = \mathbf{\Delta}([k],[n])$, i.e., k-simplices in Δ^n are maps $[k] \to [n]$ in $\mathbf{\Delta}$. The face and degeneracy maps d_i and s_i are given by pre-composition in $\mathbf{\Delta}$ by d^i and s^i, respectively. Explicitly:

$$d_i\colon \Delta^n_k \to \Delta^n_{k-1} \text{ is the function } [k] \xrightarrow{f} [n] \mapsto [k-1] \xrightarrow{d^i} [k] \xrightarrow{f} [n] \text{ and}$$

$$s_i\colon \Delta^n_k \to \Delta^n_{k+1} \text{ is the function } [k] \xrightarrow{f} [n] \mapsto [n+1] \xrightarrow{s^i} [k] \xrightarrow{f} [n].$$

The simplicial set Δ^n has a unique non-degenerate n-simplex, which corresponds to the identity map at $[n]$. More generally, the non-degenerate k-simplices of Δ^n are precisely the injective maps $[k] \to [n]$ in $\mathbf{\Delta}$.

The Yoneda lemma tells us that y is full and faithful, which means that simplicial set maps $f\colon \Delta^n \to \Delta^m$ correspond bijectively to morphisms $f\colon [n] \to [m]$ in $\mathbf{\Delta}$; as before, the functions $f\colon \Delta^n_k \to \Delta^m_k$ are defined by post-composition by f. We drop the convention employed above and use the same notation for both maps. For instance, the coface maps $d^i\colon \Delta^{n-1} \to \Delta^n$ play an important role in defining spheres and horns in §5.

A more robust version of the Yoneda lemma is the following.

Lemma 3.1 (Yoneda lemma)**.** *Let $c \in \mathcal{C}$ and $X\colon \mathcal{C}^{op} \to \mathbf{Set}$. Natural transformations $\mathcal{C}(-,c) \to X$ correspond bijectively to elements of the set Xc and the correspondence is natural in both variables.*

In our case, the Yoneda lemma tells us that for any simplicial set X, there is a natural bijective correspondence between n-simplices of X and morphisms $\Delta^n \to X$ in $\mathbf{sSet}$. More explicitly, an n-simplex x can be regarded as a map $x\colon \Delta^n \to X$ that sends the unique non-degenerate n-simplex in Δ^n to x. Applying this notational convention, if x is an n-simplex, the $(n-1)$-simplex $d_i(x)$ equals xd^i:

$$x \in X_n \leftrightsquigarrow \Delta^n \xrightarrow{x} X \qquad\qquad d_i(x) \in X_{n-1} \leftrightsquigarrow \Delta^{n-1} \xrightarrow{d^i} \Delta^n \xrightarrow{x} X$$

This latter notation, which the author now prefers, can be a little tricky to become accustomed to but is then extremely convenient. Indeed, the author now prefers to think a simplicial set X as a graded set $\{X_n \mid n \in \mathbb{N}\}$ whose elements acted on the right by morphisms in $\mathbf{\Delta}$. This way, the natural notation for right action of maps in $\mathbf{\Delta}$ on simplices corresponds precisely to that of composition in the category **sSet** and one need only ever set notation for maps in $\mathbf{\Delta}$ and have the relations (2.2) in mind.

A simplicial set X has an associated *category of elements* $\mathbf{el}\, X$, in this case called the *category* of simplices, obtained by applying the Grothendieck construction to the functor $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$. Objects of $\mathbf{el}\, X$ are simplices $x \in X_n$ for some n. A morphism $x \in X_n \to y \in X_m$ is given by a map $f\colon [n] \to [m]$ in $\mathbf{\Delta}$ such that $yf = x$. The *density theorem*, dual in some sense to the Yoneda lemma, says that any simplicial set is a colimit of standard simplices indexed by its category of elements

$$\mathrm{colim}_{x \in X_n} \Delta^n \cong X.$$

The map $\Delta^n \to X$ from the object in the colimit diagram indexed by $x \in X_n$ is, not surprisingly, $x\colon \Delta^n \to X$.

For example, the map $s^i\colon \Delta^{n+1} \to \Delta^n$ induced by the order-preserving surjection $s^i\colon [n+1] \to [n]$ that sends to elements to $i \in [n]$ is a form of projection: the standard simplex is mapped to one of lower dimension in such a way that the ordering of the vertices is preserved. Hence, precomposing $x\colon \Delta^n \to X$ with this yields an $(n+1)$-simplex xs^i whose i-th and $(i+1)$-th faces xs^id^i and xs^id^{i+1} equal x by (2.2).

In the remainder of this section, we will give two further examples of simplicial sets, which will be reintroduced in §4.

Example 3.2 (nerve of a category). Let $\mathcal{C}$ be any small category. Define the *nerve* of $\mathcal{C}$ to be the simplicial set $N\mathcal{C}$ defined as follows:

$N\mathcal{C}_0 = \mathrm{ob}\, \mathcal{C}$
$N\mathcal{C}_1 = \mathrm{mor}\, \mathcal{C}$
$N\mathcal{C}_2 = \{\text{pairs of composable arrows } \to\to \text{ in } \mathcal{C}\}$
$\vdots$
$N\mathcal{C}_n = \{\text{strings of } n \text{ composable arrows } \to\to \cdots \to \text{ in } \mathcal{C}\}$.

The degeneracy map $s_i\colon N\mathcal{C}_n \to N\mathcal{C}_{n+1}$ takes a string of n composable arrows

$$c_0 \to c_1 \to \cdots \to c_i \to \cdots \to c_n$$

and obtains $n+1$ composable arrows by inserting the identity at c_i in the i-th spot. The face map $d_i\colon N\mathcal{C}_n \to N\mathcal{C}_{n-1}$ composes the i-th and $i+1$-th arrows if $0 < i < n$, and leaves out the first or last arrow for $i = 0$ or n respectively. One can verify directly that these maps satisfy the relations from §2. However, we decline to do so because this will become obvious once we redefine the nerve of a category in §4.

Example 3.3 (total singular complex of a space). To begin, we define a *covariant* functor $\Delta\colon \mathbf{\Delta} \to \mathbf{Top}$ that sends $[n]$ to the standard topological n-simplex

$$\Delta_n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_0 + \cdots + x_n = 1, x_i \geq 0\}.$$

The morphisms $d^i\colon \Delta_{n-1} \to \Delta_n$ insert a zero in the i-th coordinate, while the morphisms $s^i\colon \Delta_{n+1} \to \Delta_n$ add the x_i and x_{i+1} coordinates. Geometrically, d^i inserts Δ_{n-1} as the i-th face of Δ_n and s^i projects the $n+1$ simplex Δ_{n+1} onto the n-simplex that is orthogonal to its i-th face.

Let Y be any topological space. We define a simplicial set SY by defining $SY_n = \mathbf{Top}(\Delta_n, Y)$ to be the set of continuous maps from the standard topological n-simplex to Y. Elements of this set are called n-simplices of Y in algebraic topology, which coincides with our terminology. Pre-composition by d^i induces a map of sets

$$d_i\colon \mathbf{Top}(\Delta_{n+1}, Y) \to \mathbf{Top}(\Delta_n, Y),$$

which obtains a singular n-simplex as the i-th face of a singular $n+1$-simplex. Similarly, pre-composition by s^i induces a map of sets

$$s_i\colon \mathbf{Top}(\Delta_{n-1}, Y) \to \mathbf{Top}(\Delta_n, Y),$$

whose image consists of the degenerate singular n-simplices, which are continuous functions $\Delta_n \to Y$ that factor through Δ_{n-1} along s^i. The morphisms d_i and s_i will satisfy (2.5) because the d^i and s^i satisfy the dual relations (2.2). This makes SY a simplicial set. It is called the *total singular complex* of a topological space.

In fact, both N and S are functors with a wide variety of applications. The functor S is implicit in the definition of singular homology, the n-th homology functor factoring as

$$H_n(-, \mathbb{Z})\colon \mathbf{Top} \xrightarrow{S} \mathbf{sSet} \xrightarrow{F_*} \mathbf{sAb} \xrightarrow{\sum_i (-1)^i d_i} \mathbf{Ch}_{\mathbb{Z}} \xrightarrow{H_n} \mathbf{Ab}.$$

The first functor associates the total singular complex to each space. The second maps this to the simplicial abelian group, with FSY_k the free abelian group on the k-simplices of SY. The third functor converts this into a chain complex with differentials defined to be the alternating sum of the face maps of the simplicial abelian group. The final functor takes homology of this chain complex, yielding the homology of the space Y.

The functor N can be used to model the classifying space of a group G. First, regard the group as a one object category (with the group elements as its morphisms), then take its nerve, and then its geometric realization, defined in §4. This construction is obviously functorial and also preserves finite products, making it a useful model in many contexts.

4. A unified presentation of all[3] adjunctions

In this section we will introduce several important pairs of adjoint functors with **sSet** the domain of the left adjoint. For all these examples, the same category-theoretic construction can be employed to define both left and right adjoints and to establish the adjunction. This construction works equally well with any small category in place of $\mathbf{\Delta}$, though we restrict ourselves to the level of generality needed here. After presenting the general proof, we will describe the examples of interest. First, recall

[3]whose left adjoint has domain **sSet**

Definition 4.1. An *adjunction* $L \dashv R$ consists of a pair of functors $L\colon \mathbf{sSet} \to \mathcal{E}$ and $R\colon \mathcal{E} \to \mathbf{sSet}$ together with hom-set bijections

$$\mathcal{E}(LX, e) \cong \mathbf{sSet}(X, Re)$$

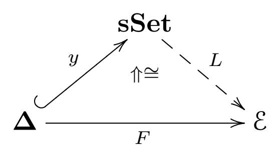
for all $x \in \mathbf{sSet}$ and $e \in \mathcal{E}$, natural in both X and e.

Let $\mathcal{E}$ be any cocomplete, locally small category and $F\colon \mathbf{\Delta} \to \mathcal{E}$ any covariant functor. Using F, we may define a functor $R\colon \mathcal{E} \to \mathbf{sSet}$ by setting Re, for each $e \in \mathcal{E}$, to be the simplicial set with n-simplices

$$Re_n = \mathcal{E}(F[n], e),$$

the set of morphisms in $\mathcal{E}$ from $F[n]$ to e. As in Example 3.3, $d_i\colon Re_n \to Re_{n-1}$ is given by pre-composition by Fd^i and $s_i\colon Re_n \to Re_{n+1}$ is given by pre-composition by Fs^i. By functoriality of F, the Fd^i and Fs^i satisfy (2.2), so by contravariant functoriality of $\mathcal{E}(-, e)$, the d_i and s_i will satisfy (2.5). Thus, Re is a simplicial set. Levelwise post-composition defines a map of simplicial sets for each $e \to e' \in \mathcal{E}$ and makes R a functor.

The functor R will be the right adjoint of a functor $L\colon \mathbf{sSet} \to \mathcal{E}$, which is most concisely described as the left Kan extension of $F\colon \mathbf{\Delta} \to \mathcal{E}$ along the Yoneda embedding.



Explicitly, the value of L at a simplicial set X is a particular type of colimit called a coend. As is the case for any colimit, coends can also be described as a coequalizer between a pair of maps between certain coproducts, but this author believes the coend description makes it easier to understand the conditions imposed by the colimiting diagram.

For any set S and object $e \in \mathcal{E}$, the *copower* or *tensor* of e by S, denoted $S \cdot e$ is simply the coproduct $\coprod_S e$ of copies of e indexed by S. In particular, if X is a simplicial set, we may form copowers

$$X_m \cdot F[n]$$

for any $n, m \in \mathbb{N}$. A morphism $f\colon [n] \to [m]$ of $\mathbf{\Delta}$ induces a map

$$f_*\colon X_m \cdot F[n] \to X_m \cdot F[m],$$

which applies Ff to the copy of $F[n]$ in the component corresponding to $x \in X_m$ and includes it in the component corresponding to x in $X_m \cdot F[m]$, and also a map

$$f^*\colon X_m \cdot F[n] \to X_n \cdot F[n],$$

which maps the component corresponding to $x \in X_m$ to the component corresponding to $xf \in X_n$.

Definition 4.2. Consider the diagram whose objects are copowers $X_m \cdot F[n]$ for $m, n \in \mathbb{N}$ and whose arrows consist of morphisms f_* and f^* for each $f \in \operatorname{mor} \mathbf{\Delta}$. A *wedge* under this diagram is an object e of $\mathcal{E}$ together with maps $\gamma_n\colon X_n \cdot F[n] \to e$

such that the squares

$$\begin{array}{ccc} X_m \cdot F[n] & \xrightarrow{f_*} & X_m \cdot F[m] \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle \gamma_m} \\ X_n \cdot F[n] & \xrightarrow[\gamma_n]{} & e \end{array} \tag{4.3}$$

commute for each f. The *coend*, written $\int^n X_n \cdot F[n]$, is defined to be a universal wedge. Equivalently, $\int^n X_n \cdot F[n]$ is a coequalizer of the diagram

$$\coprod_{f\colon [n]\to[m]} X_m \cdot F[n] \underset{f^*}{\overset{f_*}{\rightrightarrows}} \coprod_{[n]} X_n \cdot F[n] \dashrightarrow \int^n X_n \cdot F[n]$$

On objects, the functor $L\colon \mathbf{sSet} \to \mathcal{E}$ is given by this coend:

$$LX\colon = \int^n X_n \cdot F[n].$$

If $\alpha\colon X \to Y$ is a map of simplicial sets, then α induces a wedge from the diagram (4.3) for X to the object LY; hence, the universal property of the coend defines a map

$$L\alpha\colon LX \to LY.$$

This defines the functor L on morphisms. Uniqueness of the universal property will imply that L is functorial, as is always the case when one uses a colimit construction to define a functor.

It follows from the Yoneda lemma (or really, just from the fact that y is full and faithful) that $L\Delta^n \cong F[n]$. The map $\Delta^n_m \cdot F[m] \to F[n]$ is defined to be Ff at the component corresponding to $f\colon [m] \to [n]$. It follows from functoriality of F that these define a wedge to $F[n]$. Given another wedge $\gamma_m\colon \Delta^n_m \cdot F[m] \to e$ the component of γ_n and the identity $1\colon [n] \to [n]$ defines a map $\gamma\colon F[n] \to e$.

At the component corresponding to f, $\Delta^n_m \cdot F[m] \to F[n] \to e$ is $\gamma \cdot Ff$, which equals γ_m at f because the γ_m form a wedge. This is obviously the only thing that works at the identity component.

It remains to show that L is left adjoint to R. Immediately from the definitions, we have hom-set isomorphisms

$$\mathbf{sSet}(\Delta^n, Re) \cong Re_n = \mathcal{E}(F[n], e) \cong \mathcal{E}(L\Delta^n, e), \tag{4.4}$$

the first by an application of the Yoneda lemma. A companion result to the Yoneda lemma, the density theorem mention in §3, says that every simplicial set is canonically a colimit of the standard simplices. Since L is defined by a colimit formula, it commutes with colimits. Hence the functor L is completely determined by its value on the Δ^n, and the full adjoint correspondence follows from (4.4).

We can also describe the adjoint correspondence more constructively. A morphism $\gamma\colon X \to Re$ of simplicial sets consists of maps $\gamma_n\colon X_n \to \mathcal{E}(F[n], e)$ for each n. We can use this to define morphisms $\gamma_n\colon X_n \cdot F[n] \to e$ in $\mathcal{E}$ by applying $\gamma_n(x)$ to the component of the copower corresponding to the element $x \in X_n$. We claim that the γ_n form a wedge under the diagram for X to the element e; we must show that (4.3) commutes for each $f \in \operatorname{mor} \mathbf{\Delta}$. Unravelling the definitions, this is equivalent to commutativity of

$$\begin{array}{ccc} X_m & \xrightarrow{\gamma_m} & \mathcal{E}(F[m],e) \\ {\scriptstyle Xf}\downarrow & & \downarrow{\scriptstyle (Ff)^*} \\ X_n & \xrightarrow[\gamma_n]{} & \mathcal{E}(F[n],e) \end{array}$$

which is true by naturality of γ. The universal property of the coend, then gives us a map $\gamma^\flat\colon LX \to e$, which we use to define a map

$$\phi\colon \gamma \mapsto \gamma^\flat\colon \mathbf{sSet}(X, Re) \longrightarrow \mathcal{E}(LX, e).$$

Conversely, given an arrow $h\colon LX \to e$ in $\mathcal{E}$, we have maps

$$X_n \cdot F[n] \xrightarrow{\omega_n} LX \xrightarrow{h} e$$

for each n, where the ω_n are the maps of the coend wedge. We can use these to define functions

$$h_n^\sharp\colon X_n \to \mathcal{E}(F[n], e)$$

which take $x \in X_n$ to the map induced by $h\omega_n$ on the corresponding component of the coproduct. We claim that $h^\sharp\colon X \to Re$ is a map of simplicial sets. As noted above, the left hand square below commutes if and only if the right hand one does:

$$\begin{array}{ccc} X_m & \xrightarrow{h_m^\sharp} & \mathcal{E}(F[m],e) \\ {\scriptstyle Xf}\downarrow & & \downarrow{\scriptstyle (Ff)^*} \\ X_n & \xrightarrow[h_n^\sharp]{} & \mathcal{E}(F[n],e) \end{array} \qquad\qquad \begin{array}{ccc} X_m \cdot F[n] & \xrightarrow{f_*} & X_m \cdot F[m] \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle h\omega_m} \\ X_n \cdot F[n] & \xrightarrow[h\omega_n]{} & e \end{array}$$

So $h^\sharp$ is natural, since $h\omega$ is a wedge, and this defines a map

$$\psi\colon h \mapsto h^\sharp\colon \mathcal{E}(LX, e) \longrightarrow \mathbf{sSet}(X, Re).$$

It is easy to see that both ϕ and ψ are natural in X and e and are inverses. This shows that L is left adjoint to R, as desired.

Example 4.5. Post-composition turns the total singular complex defined in Example 3.3 into a functor $S\colon \mathbf{Top} \to \mathbf{sSet}$. Following the above prescription, its left adjoint $|-|\colon \mathbf{sSet} \to \mathbf{Top}$ is defined on objects by

$$|X| = \int^n X_n \times \Delta_n = \operatorname{colim}\left(\coprod_{f\colon [n]\to[m]} X_m \times \Delta_n \underset{f^*}{\overset{f_*}{\rightrightarrows}} \coprod_{[n]} X_n \times \Delta_n \right)$$

and is called the *geometric realization* of the simplicial set X. The copower $X_n \cdot \Delta_n$ of topological spaces is equivalent to the cartesian product $X_n \times \Delta_n$ where the set X_n is given the discrete topology. This coend is also knows as the *tensor product* of the functors $\mathbf{\Delta}^{\mathrm{op}} \overset{X}{\to} \mathbf{Set} \overset{D}{\to} \mathbf{Top}$ and $\mathbf{\Delta} \overset{\Delta}{\to} \mathbf{Top}$, which is commonly denoted by $X \otimes_{\mathbf{\Delta}} \Delta$, ignoring the discrete topological space functor D. Such a tensor product can be defined more generally for any pair of functors $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ and $G\colon \mathcal{C} \to \mathcal{D}$ where $\mathcal{D}$ has some monoidal product $\otimes$ to play the role of the cartesian product in $\mathbf{Top}$ and provided the desired copowers and coends exist in $\mathcal{D}$.

Example 4.6. The construction of the nerve of a category in Example 3.2 gives a functor **Cat** $\to$ **sSet**. Let $F\colon \mathbf{\Delta} \to \mathbf{Cat}$ be the functor that sends $[n]$ to the category

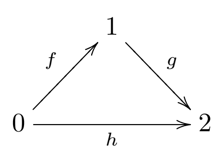
$$\underset{\sim}{n} = \cdot \to \cdot \to \cdots \to \cdot$$

with $n+1$ objects and n generating non-identity arrows, as well as their composites and the requisite identities. Each order-preserving map in $\mathbf{\Delta}$ uniquely determines a functor between categories of this type by prescribing the object function, and the object functions of all such functors give rise to order preserving maps. Thus F is actually an embedding $\mathbf{\Delta} \hookrightarrow \mathbf{Cat}$. In this example, the right adjoint is commonly called the *nerve* functor $N\colon \mathbf{Cat} \to \mathbf{sSet}$, with

$$N\mathcal{C}_n = \mathbf{Cat}(\underset{\sim}{n}, \mathcal{C}),$$

or, more colloquially, the set of strings of n composable arrows in $\mathcal{C}$. The map $s_i\colon N\mathcal{C}_n \to N\mathcal{C}_{n+1}$ inserts the appropriate identity arrow at the i-th place, and $d_i\colon N\mathcal{C}_n \to N\mathcal{C}_{n-1}$ leaves off an outside arrow if $i = 0$ or n and composes the i-th and $(i+1)$-th arrows otherwise.

The left adjoint is typically denoted $\tau_1\colon \mathbf{sSet} \to \mathbf{Cat}$ for first truncation. Surprisingly, the image of a simplicial set X under this functor is completely determined by its 0-, 1-, and 2-simplices and the maps between them. This becomes obvious when we give an alternate description of τ_1. Given a simplicial set X, define $\operatorname{ob}\tau_1 X$ to be X_0. Morphisms in $\tau_1 X$ are freely generated by the set X_1 subject to relations given by elements of X_2 in the sense described below. The degeneracy map $s_0\colon X_0 \to X_1$ picks out the identity morphism for each object. The face maps $d_1, d_0\colon X_1 \to X_0$ assign a domain and codomain, respectively, to each arrow. To obtain $\tau_1 X$, we begin by taking the free graph on X_0 generated by the arrows X_1 and then impose the relation $h = gf$ if there exists a 2-simplex $x \in X_2$ such that $xd^2 = f, xd^0 = g$, and $xd^1 = h$:



Composition is associative already in the free graph. The degenerate 2-simplices in the image of the maps s_0 and s_1 give witness to the fact that the identity arrows behave like identities with respect to composition. This makes $\tau_1 X$ a category.

It is not hard to verify explicitly that this definition of τ_1 gives a left adjoint to N; then the fact that left adjoints of a given functor must necessarily be naturally isomorphic gives an economical verification that this definition of τ_1 is compatible with the definition given above.

Example 4.7. This construction can also be used to show that **sSet** is *cartesian closed*, i.e., for every simplicial set Y, the functor $- \times Y\colon \mathbf{sSet} \to \mathbf{sSet}$ has a right adjoint. This is true for all categories of presheaves on a small category, and the construction of the right adjoint given here is the usual one.

Fix a simplicial set Y and let $F\colon \mathbf{\Delta} \to \mathbf{sSet}$ be given on objects by

$$[n] \mapsto \Delta^n \times Y$$

and on morphisms by $f \mapsto f \times 1_Y$. Recall that we mentioned that the functor L is defined to be the left Kan extension of F along the Yoneda embedding. In this case, F is the composite of the Yoneda embedding with the functor $- \times Y$, so L is the functor $- \times Y$[4].

The right adjoint R is traditionally denoted $[Y, -]$ or $(-)^Y$ and sometimes referred to as an *internal hom*. For a simplicial set Z, the above construction gives

$$RZ_n = [Y, Z]_n = \mathbf{sSet}(\Delta^n \times Y, Z);$$

that is, n-simplices $[Y, Z]_n$ are natural transformations $\Delta^n \times Y \to Z$. $[Y, Z]$ is a simplicial set with face and degeneracy maps

$$d_i \colon [Y,Z]_n \to [Y,Z]_{n-1} \text{ given by pre-composition by } d^i \times 1 \colon \Delta^{n-1} \times Y \to \Delta^n \times Y$$
$$s_i \colon [Y,Z]_n \to [Y,Z]_{n+1} \text{ given by pre-composition by } s^i \times 1 \colon \Delta^{n+1} \times Y \to \Delta^n \times Y$$

These definitions give us the desired adjunction

$$\mathbf{sSet}(X \times Y, Z) \cong \mathbf{sSet}(X, [Y, Z]).$$

In fact, this bijection is natural in all three variables, which is easily verified.

There are several other examples of adjoint pairs of functors involving **sSet** that fit into the above template that we choose not to define here but list in case the reader may encounter these elsewhere. Descriptions of the first two can be found in [GJ99]; the third is defined in [Lur09].

Example 4.8. Groupoids can be regarded as categories where every morphism is an isomorphism; in fact the inclusion **Gpd** $\hookrightarrow$ **Cat** is reflective (i.e., has a left adjoint) and, more unusually, coreflective (has a right adjoint). Using inclusion and its left adjoint, the nerve functor for groupoids has the same form as above, yielding an adjunction

$$\Pi_1 \colon \mathbf{sSet} \underset{\perp}{\rightleftarrows} \mathbf{Gpd} \colon \mathbb{N}$$

The left adjoint Π_1 gives the fundamental groupoid of a simplicial set.

Example 4.9. Subdivision of a simplicial set gives a functor $\mathrm{sd} \colon \mathbf{sSet} \to \mathbf{sSet}$ such that $|\mathrm{sd}\, \Delta^n|$ is the barycentric subdivision of $|\Delta^n|$. This functor has a right adjoint

$$\mathrm{sd} \colon \mathbf{sSet} \underset{\perp}{\rightleftarrows} \mathbf{sSet} \colon \mathrm{Ex}$$

Example 4.10. Let **sCat** denote the category of small categories enriched in **sSet** and simplicially enriched functors. The simplicial nerve of a simplicially enriched category (which is different from the nerve functor of Example 4.6) has a left adjoint

$$\mathbb{C} \colon \mathbf{sSet} \underset{\perp}{\rightleftarrows} \mathbf{sCat} \colon \mathbb{N}$$

which, when restricted to the representable simplicial sets, can be regarded as a *simplicial thickening* or cofibrant replacement of the categories $\underset{\sim}{n}$. See [Rie11] for a considerably more detailed description of this adjunction.

[4]Alternatively, one can check directly that $X \times Y$ is a universal wedge under the diagram (4.3), with the limiting cone defined by the Yoneda lemma.

5. Spheres and Horns

We say a simplicial set Y is a subset of a simplicial set X if there is a monomorphism $Y \to X$, i.e., if $Y_n \subset X_n$ for all $[n] \in \mathbf{\Delta}$ and if

$$Xf\big|_{Y_n} = Yf$$

for all $f\colon [m] \to [n]$ in $\mathbf{\Delta}$. This second condition says that the subsets Y_n are closed under the right action by the face and degeneracy operations and furthermore that these operations agree with their definitions for X. In the presence of a simplicial set X, we often specify a simplicial subset by giving a set of *generators*, which will typically have the form of a subset $S \subset X_n$ for some n. The simplicial set *generated* by S is then the smallest simplicial subset of X that contains S. Its k-simplices will be those k-simplices of X that are in the image of S under the right action by some $f\colon [k] \to [n]$ in $\mathbf{\Delta}$.

There are a number of important simplicial subsets of the simplicial set Δ^n.

Definition 5.1. The i-th face $\partial_i \Delta^n$ of Δ^n is the simplicial subset generated by $d^i \in \Delta^n_{n-1}$. Equivalently,

$$\partial_i \Delta^n \cong \Delta^{n-1} \xrightarrow{d^i} \Delta^n.$$

Definition 5.2. The *simplicial n-sphere* $\partial\Delta^n$ is the simplicial subset of Δ^n given by the union of the faces $\partial_0\Delta^n, \ldots, \partial_n\Delta^n$. Equivalently, it is the simplicial subset generated by $\{d^0, \ldots, d^n\} \subset \Delta^n_{n-1}$. Alternatively, it is the colimit of the diagram consisting of the faces $\partial_i\Delta^n$ together with the morphisms that include each of the $(n-2)$-simplices that form *their* boundary into both of the faces in which each one is contained.

The sphere $\partial\Delta^n$ has the property that $(\partial\Delta^n)_k = \Delta^n_k$ for all $k < n$; all higher simplices of $\partial\Delta^n$ are degenerate. In otherwords, $\partial\Delta^n$ is the $(n-1)$-*skeleton* of Δ^n.

More generally, a *n-sphere* in X is a map $\partial\Delta^n \to X$ of simplical sets.

Definition 5.3. The *simplicial horn* Λ^n_k is the union of all of the faces of Δ^n except for the k-th face; equivalently, it is the simplicial subset of Δ^n generated by the set $\{d^0, \ldots, d^{k-1}, d^{k+1}, \ldots, d^n\}$. Alternatively, it can be described as a colimit in **sSet**, analogous to that for $\partial\Delta^n$ described above, except with the face $\partial_k\Delta^n$ left out of the colimiting diagram.

The horn Λ^n_k has the property that $(\Lambda^n_k)_j = \Delta^n_j$ for $j < n-1$ and $(\Lambda^n_k)_{n-1} = \Delta^n_{n-1} \backslash \{d^k\}$, with higher simplices again being degenerate.

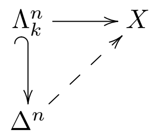
More generally, a horn in X is a map $\Lambda^n_k \to X$ of simplicial sets.

Remark 5.4. For each of these simplicial sets, their geometric realization is the topological object suggested by their name; $|\partial_i\Delta^n|$ is the i-th face of the standard topological n-simplex $\Delta_n = |\Delta^n|$, $|\partial\Delta^n|$ is its boundary, and $|\Lambda^n_k|$ is the union of all faces but the k-th.

There are certain special types of simplicial sets, which are defined by various "horn filling" conditions. We give only the basic definitions and list a few sources for the interested reader.

Definition 5.5. A *Kan complex* is a simplicial set X such that every horn has a filler (which is not assumed to be unique). This means that for each horn $\Lambda^n_k \to X$

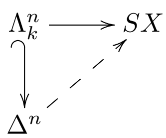
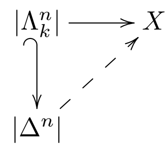
in X there exists an extension along the inclusion $\Lambda^n_k \hookrightarrow \Delta^n$ as shown



By the Yoneda lemma, the map $\Delta^n \to X$ identifies an n-simplex in X whose faces agree with those specified by the horn.

Lemma 5.6. *If X is a topological space, then SX is a Kan complex.*

Proof. By the adjunction (4.5), the diagram

 in **sSet** corresponds to  in **Top**.

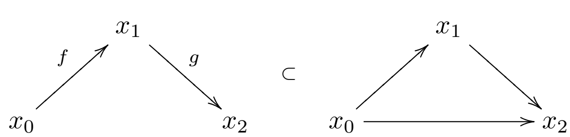
A topological (n, k)-horn is a deformation retract of the standard n-simplex $\Delta_n = |\Delta^n|$, so the lift on the right hand side exists. The adjunct to this map gives us the desired lift on the left. □

Kan complexes play an important role in studying the homotopy theory of **sSet**, which is closely linked to the homotopy theory of **Top** via the adjunction (4.5). For more details see [GJ99] or [May67].

Definition 5.7. A *quasi-category* or ∞-*category* is a simplicial set X such that every *inner* horn, i.e., horn Λ^n_k with $0 < k < n$, has a filler.

Example 5.8. For any category $\mathcal{C}$, its nerve $N\mathcal{C}$ is a quasi-category. In fact, it is a quasi-category with the special property that every inner horn has an *unique* filler. Conversely, any quasi-category such that every inner horn has a unique filler is isomorphic to the nerve of a category.

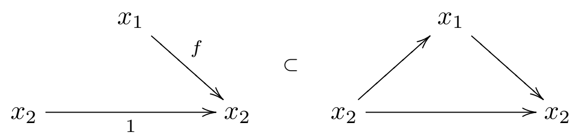
We won't give formal proofs of these facts here (instead see [Lur09]) but we will at least provide some intuition for why the nerve of a category has a unique filler for horns $\Lambda^2_1 \to N\mathcal{C}$. This horn is often represented by the following picture:



Here $f, g \in N\mathcal{C}_1$ are morphisms in $\mathcal{C}$ and $x_0, x_1, x_2 \in N\mathcal{C}_0$ are objects in $\mathcal{C}$. $fd_1 = x_0$ and $fd_0 = x_1$, colloquially, x_0 is the domain of f and x_1 is its codomain, and similarly for g. The essential point that this picture communicates is that if f and g are the generating 1-simplices of a horn $\Lambda^2_1 \to N\mathcal{C}$, then f and g are a composable pair of arrows in $\mathcal{C}$. The statement that this horn can be filled then simply expresses the fact that this pair necessarily has a composite gf. Composition is unique in a category, so this horn can be filled uniquely[5].

[5]However, in many higher categorical settings, composition is not required to be unique. This example gives a glimpse of why horn filling conditions and quasi-categories in particular are so useful for studying higher categories.

However, it will not necessarily be true that *outer* horns $\Lambda^2_0 \to N\mathcal{C}$ and $\Lambda^2_2 \to N\mathcal{C}$ will have fillers. Indeed, asking the $(2,2)$-horn depicted below has a filler is equivalent to asking that the arrow f have a right inverse, which is certainly not true in general.



It turns out that if $\mathcal{C}$ is a *groupoid*, then these outer horns will have fillers. In fact *all* outer horns will have fillers, which says that the nerve of a groupoid is a Kan complex.

Quasi-categories were first defined by Boardman and Vogt in [BV73] under the name *weak Kan complexes*. In recent years, their theory has been developed extensively by André Joyal (see [Joy02] and, if you can find it, [Joy08]) and Jacob Lurie [Lur09].

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