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A Cellular Construction of BP and Other Irreducible Spectra

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In this note we construct and derive the basic properties of the Brown-Peterson spectrum BP by attaching cells to the sphere spectrum S^0 (localized at a prime p) so as to kill the odd-dimensional homotopy groups. This procedure is, of course, entirely analogous to the construction of the Eilenberg-MacLane spectrum $K(\mathbb{Z}_{(p)})$ by killing the positive dimensional homotopy groups of S^0. One can thus view BP as lying "half-way" between S^0 and $K(\mathbb{Z}_{(p)})$. The advantage of our approach is that it avoids computations with Steenrod operations inherent in the original Postnikov tower construction [2]. Moreover, we obtain the homotopy and cohomology groups of BP immediately from the construction by a simple application of obstruction theory and the Adams spectral sequence.

Pedagogically, we have found this approach useful in introducing BP to students who have mastered homotopy theory including the Adams spectral sequence, as for example from the texts of Moser-Tangora [5] or Switzer [9].

This paper consists of four sections the first of which gives the construction of our candidate X for the BP spectrum after making precise the notion of attaching cells "non-trivially". $X \simeq \mathrm{BP}$ follows easily from the fact that a self map of a complex with cells attached non-trivially is an equivalence iff it is an equivalence on the bottom cell. In §2 we derive the main properties of X directly from the construction without assuming the existence of BP. The idea is to prove $H^*(X;\mathbb{Z}/p)$ is free over $A/(\beta)$, then use the Adams spectral sequence to simultaneously compute $\pi_* X$ and show $H^*(X;\mathbb{Z}/p)$ is monogenic. In §3 we analyze the individual spaces of our spectrum. They are found to be equivalent to certain spaces in Wilson's spectrum $\mathrm{BP}\langle n\rangle$. Finally in §4 we show that by attaching cells to kill the $(4k-1)$ dimensional homotopy groups of S^0 we obtain an interesting spectrum which is equivalent to MSp through the 30-skeleton.

Recollections. For convenience we recall the following properties of BP from [2]:

i) $\pi_* \mathrm{BP} = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$, $|v_i| = 2(p^i - 1)$

ii) $H^*(\mathrm{BP};\mathbb{Z}/p) = A/(\beta)$, where A is the mod-p Steenrod algebra and (β) is the two-sided ideal generated by the Bockstein.

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Conventions. Throughout this paper, we work in the category of p-local CW-complexes (spectra) with basepoint. All modules are finitely generated over $\mathbb{Z}_{(p)}$, the integers localized at the prime p.

§ 1. Construction

Definition. Given spaces $A \subset Z$, we say that Z *is obtained from* A *by non-trivially attaching* $(n+1)$-cells if $Z = A \cup \bigcup_{f_\alpha} e_\alpha^{n+1}$ for some set of attaching maps $\{f_\alpha \colon S^n \to A\}$ such that

$$\ker(\bigoplus_\alpha f_{\alpha *}) \subset p \cdot (\bigoplus_\alpha \pi_n S^n)$$

where

$$\bigoplus_\alpha f_{\alpha *} \colon \bigoplus_\alpha \pi_n S^n \to \pi_n A$$

is given by summation. Equivalently, if $\dim A \leqq n$, one can require that every element of $\pi_{n+1} Z$ have trivial Hurewicz image mod p (this point is elucidated in the proof of the theorem in § 2).

Similarly one can attach stable cells non-trivially in the stable category of p-local CW-spectra.

Construction. We can now define our candidate X for the BP spectrum. For $n \geqq 0$, let X_n be obtained from S^n by non-trivially attaching cells in dimensions $n+2, n+4, \ldots, n+2k, \ldots$ to kill the homotopy groups in dimensions $n+1$, $n+3, \ldots, n+2k-1, \ldots$. That is, if $(X_n)^k$ denotes the k-skeleton then $(X_n)^n = (X_n)^{n+1} = S^n$ and assuming $(X_n)^{n+2k-1}$ to be defined we attach $(n+2k)$-cells non-trivially to obtain $(X_n)^{n+2k}$ so that $\pi_{n+2k-1}((X_n)^{n+2k}) = 0$.

Structure maps $\varepsilon_n \colon X_n \to \Omega X_{n+1}$ are easy to define. By construction $\pi_{n+2k-1} \Omega X_{n+1} = \pi_{n+2k} X_{n+1} = 0$, hence $H^{n+2k}(X_n; \pi_{n+2k-1} \Omega X_{n+1}) = 0$ and so by obstruction theory there exists a map $\varepsilon_n \colon X_n \to \Omega X_{n+1}$ extending the identity map on S^n. Thus $X = \{X_n, \varepsilon_n\}$ is a spectrum.

Presently in § 2, we shall show X has the desired properties without assuming BP exists. For now, we show we have made the proper construction.

Theorem. *X is homotopy equivalent to* BP.

Definition. A space (spectrum) Z is called *irreducible* if it is obtained from a sphere (sphere spectrum) S^N by successively attaching cells (stable cells) non-trivially.

Proposition. *Suppose Z is irreducible (and simply connected if Z is a space). Then $g \colon Z \to Z$ is a homotopy equivalence if $g|_{S^N}$ is a homotopy equivalence.*

Proof. We inductively assume $g \colon Z^n \to Z^n$ is an equivalence. Consider the commutative diagram derived from the cofibration $\bigvee_\alpha S^n \to Z^n \to Z^{n+1}$ ($H_* = H_*(\cdot\,; \mathbb{Z}_{(p)})$)

$$\begin{array}{ccccccccccc}
0=H_{n+1}Z^n & \longrightarrow & H_{n+1}Z^{n+1} & \longrightarrow & H_n(\bigvee_\alpha S^n) & \longrightarrow & H_n(Z^n) & \longrightarrow & H_n(Z^{n+1}) & \longrightarrow & 0\\
\uparrow g_1 & & \uparrow g_2 & & \uparrow g_3 & & \uparrow \approx g_4 & & \uparrow g_5 & & \\
0=H_{n+1}Z^n & \longrightarrow & H_{n+1}Z^{n+1} & \longrightarrow & H_n(\bigvee_\alpha S^n) & \longrightarrow & H_n(Z^n) & \longrightarrow & H_n(Z^{n+1}) & \longrightarrow & 0
\end{array}$$

where g_i is induced by g. Since Z is simply connected, it suffices to show g_2 is an automorphism. By assumption g_4 and hence g_5 is an automorphism (for g_5 we use the well known fact that for a finitely generated module any epic endomorphism is bijective). Thus g_2 is bijective iff g_3 is. Using the Hurewicz homomorphism we are reduced to showing g_3 is an automorphism in homotopy. Consider the commutative diagram where g_4 is bijective by assumption.

$$\begin{array}{ccc}
\pi_n(\bigvee_\alpha S^n) & \xrightarrow{\oplus f_{\alpha *}} & \pi_n(Z^n)\\
\uparrow g_3 & & \uparrow \approx g_4\\
\pi_n(\bigvee_\alpha S^n) & \xrightarrow{\oplus f_{\alpha *}} & \pi_n(Z^n)
\end{array}$$

Thus we have

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker(\bigoplus_\alpha f_{\alpha *}) & \xrightarrow{i} & \bigoplus_\alpha \pi_n S^n & \longrightarrow & \operatorname{Im}(\bigoplus_\alpha f_{\alpha *}) & \longrightarrow & 0\\
 & & \uparrow & & \uparrow g_3 & & \uparrow \approx g_4 & & \\
0 & \longrightarrow & \ker(\bigoplus_\alpha f_{\alpha *}) & \xrightarrow{i} & \bigoplus_\alpha \pi_n S^n & \longrightarrow & \operatorname{Im}(\bigoplus_\alpha f_{\alpha *}) & \longrightarrow & 0
\end{array}$$

Tensoring with $\mathbb{Z}/p$ and using the definition of irreducible we have $i\otimes\mathbb{Z}/p=0$. Thus $g_3\otimes\mathbb{Z}/p$ is bijective. By Nakayama's lemma g_3 is bijective, which completes the proof.

Proof of the Theorem. Since $\pi_{2k-1}X=\pi_{2k-1}\mathrm{BP}=0=H^{2k-1}\mathrm{BP}=H^{2k-1}X$, obstruction theory gives maps $g\colon X\to \mathrm{BP}$, $h\colon \mathrm{BP}\to X$ extending the identity on S^0. By the proposition, $hg\colon X\to X$ is therefore an equivalence. Since $H^*(\mathrm{BP};\mathbb{Z}/p)=A/(\beta)$ is monogenic, $gh\colon \mathrm{BP}\to\mathrm{BP}$ is also an equivalence.

§ 2. Properties

In this section we derive the main properties of the spectrum X directly from the construction of § 1.

Theorem. i) $\pi_* X=\mathbb{Z}_{(p)}[v_1,v_2,\ldots]$, $|v_i|=2(p^i-1)$,
ii) $H^*(X;\mathbb{Z}/p)=A/(\beta)$.

First we observe that X is a ring spectrum with unit since by obstruction theory there exists maps $X_m\wedge X_n\to X_{m+n}$ extending the usual homeomorphism $S^m\wedge S^n\to S^{m+n}$.

Lemma. *$H^*(X;\mathbb{Z}/p)$ is free over $A/(\beta)$.*

We give two proofs of this fact. The first uses the cohomology of the complex cobordism spectrum MU; the second is direct from the construction using the Nishida formula.

First Proof. Since the cohomology of MU is even dimensional, obstruction theory gives a map $f: MU \to X$ extending the identity on S^0. Let $\iota: X \to K(\mathbb{Z}_{(p)})$ classify $1 \in H^0 X$. Since $H^*(MU;\mathbb{Z}/p)$ is free over $A/(\beta)$ (see [1; II 8.4]) the composite $MU \xrightarrow{f} X \xrightarrow{\iota} K(\mathbb{Z}_{(p)})$ induces a monomorphism in mod-p cohomology. Since ι is a map of ring spectra the Milnor-Moore theorem shows $H^*(X;\mathbb{Z}/p)$ is free over $A/(\beta)$.

Second Proof. We shall show that evaluation on the unit $A/(\beta) \to H^*(X;\mathbb{Z}/p)$ is injective. Let $\alpha_1 \in \pi_{2p-3} S^0$ be a generator and consider $Y = S^0 \bigcup_{\alpha_1} e^{2p-2} \to X$ extending the unit map $S^0 \to X$ ($\alpha_1 = \eta$ if $p=2$). By obstruction theory there are maps $D_p(X_{2n}) = (E\Sigma_{p^+}) \bigwedge_{\Sigma_p} (X_{2n})^{(p)} \to X_{2pn}$ extending the ring structure maps $(X_{2n})^{(p)} \to X_{2pn}$ (the cohomology of $D_p(X_{2n})$ is even dimensional). Hence we may form the composite $D_p(\ldots(D_p Y)\ldots) \to D_p(\ldots(D_p X)\ldots) \to X$ and by naturality we are reduced to checking that Milnor's element P^{Δ_i} ($Sq^{2\Delta_i}$ if $p=2$) is non-zero on the bottom class of $D_p(\ldots(D_p Y)\ldots)$. But this verification follows using the Nishida formula as in Proposition 3.4 of [6].

Proof of the Theorem. We shall prove i) and ii) simultaneously using the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(H^*(X;\mathbb{Z}/p), \mathbb{Z}/p) \Rightarrow \pi_{t-s} X.$$

By the lemma, $H^*(X;\mathbb{Z}/p)$ is free over $A/(\beta)$ on *even* dimensional generators say $\{g_\gamma\}$. Thus $H^*(X;\mathbb{Z}/p) = \bigoplus_\gamma [A/(\beta)]\, g_\gamma$. According to Milnor [4], $A/(\beta) = A \otimes_E \mathbb{Z}/p$ where $E = E[Q_0, Q_1, \ldots]$, $Q_0 = \beta$, $Q_{i+1} = [\mathscr{P}^{p^i}, Q_i]$, $|Q_i| = 2p^i - 1$. Thus by change of rings and the standard formula for the cohomology of an exterior algebra we have

$$\begin{aligned} E_2 &= \bigoplus_\gamma \mathrm{Ext}_A^{s,t}(A/(\beta), \mathbb{Z}/p) \cdot g_\gamma = \bigoplus_\gamma \mathrm{Ext}_E^{s,t}(\mathbb{Z}/p, \mathbb{Z}/p) \cdot g_\gamma \\ &= \bigoplus_\gamma \mathbb{Z}/p[v_0, v_1, v_2, \ldots] \cdot g_\gamma \quad \text{with } |v_i| = 2(p^i - 1). \end{aligned}$$

Also $E_2 = E_\infty$ by even dimensionality. Now we claim there is only one generator g_γ and it is in dimension zero. Suppose g_γ is a generator of minimal positive dimension $2n$. Then $g_\gamma \in \mathrm{Ext}^{0,2n}(H^*(X;\mathbb{Z}/p), \mathbb{Z}/p)$ represents a class in $\pi_{2n} X$ which has non-zero Hurewicz image mod p. Since $\pi_{2n} X \approx \pi_{2n} X^{2n}$ we have $hj(g_\gamma) \neq 0$ in

$$\begin{array}{ccccc} \pi_{2n} X^{2n} & \xrightarrow{j} & \pi_{2n}(X^{2n}, X^{2n-1}) & \xrightarrow{\partial} & \pi_{2n-1}(X^{2n-1}) \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} & & \\ H_{2n}(X^{2n};\mathbb{Z}/p) & \xrightarrow[\approx]{j} & H_{2n}(X^{2n}, X^{2n-1};\mathbb{Z}/p). & & \end{array}$$

However $\partial j(g_\gamma)=0$ by exactness so $\ker\partial \not\subset p\cdot\pi_{2n}(X^{2n},X^{2n-1})$ contradicting the irreducibility of X. Thus $H^*(X;\mathbb{Z}/p)=A/(\beta)$ and $\pi_* X=\mathbb{Z}_{(p)}[v_1,v_2,\dots]$ since v_0 corresponds to multiplication by p. Q.E.D.

§ 3. The Spaces X_n

In this section we identify the individual spaces X_n of the spectrum X of § 1.

Using the techniques of Baas-Sullivan, Wilson [10] has constructed Ω-spectra $\mathrm{BP}\langle n\rangle=\{\mathrm{BP}\langle n\rangle_k\}$ for $0\leqq n\leqq\infty$ such that $\mathrm{BP}\langle 0\rangle=K(\mathbb{Z}_{(p)})$, $\mathrm{BP}\langle\infty\rangle=\mathrm{BP}$ and $\pi_*\mathrm{BP}\langle n\rangle=\mathbb{Z}_{(p)}[v_1,v_2,\dots,v_n]$.

Lemma. (Wilson [10; 5.1, 6.9]). i) *for* $k\leqq(p^n+\dots+p+1)$, $H^*(\mathrm{BP}\langle n\rangle_{2k};Z_{(p)})$ *is a polynomial algebra on even dimensional classes.*

ii) *for* $k>(p^{n-1}+\dots+p+1)$, *any map* $\mathrm{BP}\langle n\rangle_k\to\mathrm{BP}\langle n\rangle_k$ *which induces an isomorphism on* π_k *is a homotopy equivalence.*

Proposition. *If* $2(p^{n-1}+\dots+p+1)<2k\leqq 2(p^n+\dots+p+1)$ *then* X_{2k} *is homotopy equivalent to* $\mathrm{BP}\langle n\rangle_{2k}$.

Proof. By part i) of the lemma and obstruction theory there are maps $f\colon \mathrm{BP}\langle n\rangle_{2k}\to X_{2k}$ and $g\colon X_{2k}\to\mathrm{BP}\langle n\rangle_{2k}$ extending the identity on S^{2k}. Since X_{2k} is irreducible fg is an equivalence. Similarly gf is an equivalence by part ii) of the lemma.

Example. For $p=2$, we have $X_2\simeq\mathrm{BP}\langle 0\rangle_2\simeq\mathbb{C}P^\infty$, $X_4\simeq\mathrm{BP}\langle 1\rangle_4\simeq BSU$, $X_6\simeq\mathrm{BP}\langle 1\rangle_6\simeq BSU[6,\dots]$. The element v_n first appears on the 2^{n+1} sphere, i.e. $X_{2^{n+1}}\simeq\mathrm{BP}\langle n\rangle_{2^{n+1}}$ while $X_{2^{n+1}-2}\simeq\mathrm{BP}\langle n-1\rangle_{2^{n+1}-2}$.

§ 4. Other Irreducible Ring Spectra

In this section we consider the spectrum Y obtained from the sphere spectrum S^0 by killing π_{4k-1}, $k\geqq 1$. This spectrum is related to the symplectic cobordism spectrum MSp; however, as in the case of MSp, our information about Y is incomplete. Of course, X and Y are part of a family of irreducible ring spectra (one for each $N\geqq 1$) obtained from S^0 by killing π_{2^Nk-1}, $k\geqq 1$. These spectra approximate S^0 as $N\to\infty$ and so one expects their properties to become progressively more intractable.

Construction. The spaces Y_n of the spectrum Y are obtained from S^n by successively attaching cells non-trivially in dimensions $n+4$, $n+8,\dots,n+4k,\dots$ so as to kill the homotopy in dimensions $n+3$, $n+7,\dots,n+4k-1,\dots$. As in the case of X, obstruction theory allows us to define maps $Y_n\to\Omega Y_{n+1}$, $Y_m\wedge Y_n\to Y_{m+n}$ and so Y is a ring spectrum.

Remark. Since $\pi_7 S^4=\mathbb{Z}\oplus Z/12$, one can show $Y_4\simeq BSp$ for $p=2,3$.

Since $H^*(MSp;\mathbb{Z}/p)$ is zero in dimensions not divisible by 4, there is a map $f\colon MSp\to Y$ extending the identity on S^0.

Lemma. *$H^*(Y;\mathbb{Z}/p)$ is free over $A/(\beta)$ (over $A' = \mathbb{Z}/2[\xi_1^4, \xi_2^4, \ldots]^*$ if $p=2$).*

Proof. Let ι: $Y \to K(\mathbb{Z}_{(p)})$ classify $1 \in H^0 Y$. Then arguing as in the first proof of the lemma of § 2 and using the fact that $H^*(MSp; \mathbb{Z}/p)$ is free over $A/(\beta)$ (over A' if $p=2$) we have the desired result.

Corollary. *Y is homotopy equivalent to* BP *for $p > 2$.*

Henceforth we assume $p=2$. Now using results of Kochman [3], Ray [7], and Segal [8] we can compare Y and MSp more closely.

Proposition. *f: $MSp \to Y$ is a homotopy equivalence through dimension* 30.

Proof. MSp is irreducible through the 28 skeleton and $\pi_{4k-1} MSp = 0$ for $k < 8$ [8, 3]. Hence there is a map g: $Y^{30} \to MSp^{30}$ such that gf and fg are equivalences on the 30-skeleton. Q.E.D.

It is interesting to note the related facts: $\pi_{31} MSp \neq 0$ [7, 3] and MSp^{32} is not irreducible [3]. If $x_2 \in H^8(MSp; \mathbb{Z}/2)$ denotes an indecomposable element over A' then x_2^4 represents a non-zero element of $\pi_{32} MSp$ with non-zero Hurewicz image mod 2. In fact this is the first positive dimensional element with non-zero image in $\pi_* MSp \to \pi_* MO$; its image is $(\mathbb{R}P^2)^{16}$ [8].

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