

# Steenrod operations in motivic cohomology

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## Abstract

These notes outline the construction of the Steenrod reduced powers for mod- $l$  motivic cohomology defined over a field of characteristic zero, following the original construction proposed by Voevodsky.

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# 1 Introduction

The purpose of these notes is to give a construction of the Steenrod reduced power operations for mod- $l$  motivic cohomology over a field of characteristic zero, following the method outlined by Voevodsky [V2]; these are natural transformations on motivic cohomology, which are analogous to the classical Steenrod operations on singular cohomology theory in algebraic topology. A related approach to the construction of Steenrod operations has been applied in Chow theory [Br].

The motivic cohomology theory used here is the Zariski hypercohomology of certain complexes of sheaves, the motivic complexes of Suslin and Voevodsky, which are defined in terms of sheaves with transfers. The motivic cohomology of a smooth scheme  $X \in \mathcal{S}m/k$  over a field  $k$  can also be defined via Voevodsky's triangulated category  $DM_-^{\text{eff}}(k)$  of mixed motives. This category is a localization of the derived category of bounded below homological complexes of Nisnevich sheaves with transfers. The construction of Steenrod operations for motivic cohomology requires a *non-additive* homotopy category; this is supplied by the Morel-Voevodsky homotopy category  $\mathcal{H}(k)$ , which is a localization of the homotopy category of simplicial Nisnevich sheaves of sets on the category  $\mathcal{S}m/k$  of smooth  $k$ -schemes.

The initial step is to show that motivic cohomology is representable on the Morel-Voevodsky homotopy category. This depends on the following result of Voevodsky's:

**Theorem 1** *Suppose that  $k$  is a perfect field. There is an adjunction*

$$M[-] : \mathcal{H}(k) \rightarrow DM_-^{\text{eff}}(k) : K.$$

The proof of this result is sketched in the text as Theorem 4.2.3 - the details are available in [P].

The next step is to show that, when the field  $k$  has characteristic zero<sup>1</sup>, motivic cohomology is a *bigraded cohomology theory* (this is an appropriate generalization of what is usually meant by a cohomology theory in algebraic topology). The proof depends on the cancellation theorem of Friedlander, Lawson and Voevodsky [FV]; this can either be applied directly or be deduced using the above theorem from the corresponding result in  $DM_-^{\text{eff}}(k)$ , established in [V4].

When  $k$  is a sub-field of the complex numbers, there is a realization functor  $\mathcal{H}(k) \rightarrow \mathcal{H}$ , where  $\mathcal{H}$  denotes the standard homotopy category of topological spaces. The relation between motivic cohomology and singular cohomology in algebraic topology is established using the Suslin-Voevodsky version of the Dold-Thom theorem (see Theorem 6.1.21). In particular, this introduces symmetric products of schemes into the consideration of motivic cohomology; this is of importance in considering the classification of all possible cohomology operations

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<sup>1</sup>This restriction can be removed using Voevodsky's identification of motivic cohomology with Bloch's higher Chow groups

which requires the calculation of the motivic cohomology of the representing objects.

## 1.1 Steenrod operations in algebraic topology

Consider cohomology operations in the setting of algebraic topology. Let  $\tilde{E}^*$ ,  $\tilde{F}^*$  be reduced cohomology theories. (For the purposes of this introduction, a reduced cohomology theory can be understood to be a contravariant functor from the homotopy category of topological spaces to graded abelian groups which is equipped with a suspension isomorphism with respect to the suspension functor  $\Sigma$  on the homotopy category). A cohomology operation of type  $m, n$  is a natural transformation of sets  $\phi : \tilde{E}^m \rightarrow \tilde{F}^n$ . The set of isomorphism classes of natural transformations of this form has an abelian group structure and is denoted by  $(E, m, F, n)$ . A stable cohomology operation of degree  $d$  is a sequence of natural transformations  $\{\phi_n \in (E, n, F, n + d)\}$  which commute with the suspension isomorphisms:  $\sigma_E : \tilde{E}^* \rightarrow \tilde{E}^{*+1} \circ \Sigma$ ,  $\sigma_F : \tilde{F}^* \rightarrow \tilde{F}^{*+1} \circ \Sigma$ , where  $\Sigma$  is the reduced suspension functor. The stable cohomology operations are natural transformations of abelian groups, since the reduced suspension of a space is a co-group object in the homotopy category.

Taking  $E, F$  to be  $\tilde{H}^*(-; \mathbb{Z}/2)$ , reduced singular cohomology with  $\mathbb{Z}/2$  coefficients, one obtains the graded abelian group  $\mathcal{A}$  of stable cohomology operations, which has the structure of a non-commutative graded Hopf algebra, with product given by the composition of natural transformations. The algebra  $\mathcal{A}$  is the (mod-2) Steenrod algebra and the homogeneous component of degree  $i$ , consisting of operations of degree  $i$ , is written  $\mathcal{A}^i$ .

The Steenrod squares are cohomology operations  $\text{Sq}^i \in \mathcal{A}^i$ , for  $i \geq 0$ ; these can be given an explicit construction, using the fact that singular cohomology is represented by the Eilenberg-MacLane spaces. The operations  $\text{Sq}^i$  generate the algebra  $\mathcal{A}$ ; the proof of this fact requires the calculation of  $\mathcal{A}$ . The classic reference for the Steenrod operations is [SE]; for more recent algebraic considerations concerning the Steenrod algebra and applications, see [Sc].

**Theorem 1.1.1** *There are unique stable cohomology operations  $\text{Sq}^i \in \mathcal{A}^i$ ,  $i \geq 0$ , which satisfy the following axioms:*

1.  $\text{Sq}^0 = 1$ .
2. **Cartan formula:**  $\text{Sq}^k(xy) = \sum_{i=0}^k \text{Sq}^i(x)\text{Sq}^{k-i}(y)$ .
3. **Instability:** *Let  $x \in \tilde{H}^n(X, \mathbb{Z}/2)$  be a cohomology class, then:*

$$\text{Sq}^i x = \begin{cases} x^2 & i=n \\ 0 & i>n. \end{cases}$$

The Cartan formula corresponds to the cocommutative coproduct on the Hopf algebra  $\mathcal{A}$ :  $\Delta \text{Sq}^k = \sum_{i=0}^k \text{Sq}^i \otimes \text{Sq}^{k-i}$ . The dual of the Steenrod algebra  $\mathcal{A}_* := \text{Hom}_{\mathbb{Z}/2}(\mathcal{A}, \mathbb{Z}/2)$  has the structure of a commutative Hopf algebra. Milnor

established that the algebra  $\mathcal{A}_*$  is a polynomial algebra on classes  $\xi_i$  of degrees  $2^i - 1$ .

A consequence of the uniqueness and the explicit construction of the Steenrod operations is:

**Proposition 1.1.2**

1.  $\text{Sq}^1 = \beta$ , the mod-2 Bockstein operator.
2. The Steenrod squares satisfy the **Adem relations**. If  $0 < a < 2b$ , then

$$\text{Sq}^a \text{Sq}^b = \sum_{j=0}^{\lfloor a/2 \rfloor} \binom{b-1-j}{a-2j} \text{Sq}^{a+b-j} \text{Sq}^j.$$

The Adem relations allow the calculation of a basis of the Steenrod algebra in terms of monomials of iterated Steenrod operations, using the following terminology. A sequence of non-negative integers  $I = (i_1, \dots, i_k)$  is said to be *admissible* if either  $I = \emptyset$  or  $k \geq 1$ ,  $i_k \geq 1$  and  $i_{s-1} \geq 2i_s$ , for  $k \geq s \geq 2$ . If  $I$  is a sequence of non-negative integers, then  $\text{Sq}^I$  is written for  $\text{Sq}^{i_1} \dots \text{Sq}^{i_k}$ , with the convention that  $\text{Sq}^\emptyset = 1$ .

**Theorem 1.1.3** *The admissible monomials form a vector space basis for  $\mathcal{A}$ .*

## 1.2 The motivic Steenrod algebra

The motivic Steenrod algebra  $\mathcal{A}^{*,*}(k, \mathbb{Z}/l)$  is defined as the bigraded algebra of bistable cohomology operations for mod- $l$  motivic cohomology; here bistability means that the operations should commute with the two suspension isomorphisms for motivic cohomology which correspond to two different models for a circle in the Morel-Voevodsky homotopy category  $\mathcal{H}(k)$ . For example, the usual Bockstein operator,  $\beta$ , for motivic cohomology is a bistable cohomology operation which defines an element in  $\mathcal{A}^{1,0}(k, \mathbb{Z}/l)$ .

The principal theorem [V2] of Voevodsky's (stated for the case  $l = 2$ ) is the following:

**Theorem 2** *Let  $k$  be a field of characteristic zero. There exists a unique sequence of bistable cohomology operations  $\text{Sq}^{2^i} \in \mathcal{A}^{2^i, i}(k, \mathbb{Z}/2)$ ,  $i \geq 0$ , such that*

1.  $\text{Sq}^0 = \text{Id}$ .
2. **Cartan formula:** *Let  $X, Y$  be simplicial smooth schemes and let  $u \in H^{*,*}(X, \mathbb{Z}/2), v \in H^{*,*}(Y, \mathbb{Z}/2)$  be motivic cohomology classes. For all  $i \geq 0$ ,*

$$\text{Sq}^{2^i}(u \times v) = \sum_{a+b=i} \text{Sq}^{2^a}(u) \times \text{Sq}^{2^b}(v) + \tau \left( \sum_{a+b=i-2} \beta \text{Sq}^{2^a}(u) \times \beta \text{Sq}^{2^b}(v) \right)$$

*in  $H^{*,*}(X \times Y, \mathbb{Z}/2)$ .*

3. **Instability:** Let  $X$  be a simplicial smooth scheme and  $u \in H^{n,i}(X, \mathbb{Z}/2)$ ,
- $$\mathbb{S}q^{2i}(u) = \begin{cases} 0 & n < 2i \\ u^2 & n = 2i. \end{cases}$$

**Remark 1.2.1**

1. The construction produces Steenrod squaring operations  $\mathbb{S}q^j$  for all non-negative integers; there are relations  $\mathbb{S}q^{2i+1} = \beta\mathbb{S}q^{2i}$ , so that it is sufficient to establish uniqueness for the even operations  $\mathbb{S}q^{2j}$  as above. Theorem 2 is thus the direct analogue of Theorem 1.1.1.
2. The ‘coefficient ring’  $H^{*,*}(\text{Spec}(k); \mathbb{Z}/2)$  for motivic cohomology is not in general central in the motivic Steenrod algebra, since the Bockstein operator can act non-trivially; this contrasts with the situation in algebraic topology and implies that the motivic Steenrod algebra has a Hopf algebra structure rather than a Hopf algebra structure.

The second important result [V2] of Voevodsky’s concerns the calculation of the mod-2 motivic Steenrod algebra, and is the analogue of Theorem 1.1.3.

**Theorem 3** *The motivic Steenrod algebra  $\mathcal{A}^{*,*}(k, \mathbb{Z}/2)$  is a free left  $H^{*,*}(\text{Spec}(k), \mathbb{Z}/2)$  module on the admissible monomials  $\mathbb{S}q^f$ .*

**Remark 1.2.2** The proof of this theorem is based upon a calculation of the motivic cohomology of the spaces which represent motivic cohomology. This requires the consideration of the motivic cohomology of non-smooth schemes, namely the symmetric products of certain sub-schemes of projective spaces; for this, it is necessary to introduce the *cdh* topology.

An algebraic argument provides the determination of the ‘dual’ of the motivic Steenrod algebra (Theorem 9.5.10 in the text) which corresponds to Milnor’s calculation of the dual of the topological Steenrod algebra.

### 1.3 Organization of the paper

These notes are divided into three Parts, with the intention of making the material more accessible to the reader. The first part is devoted to the technical framework which is necessary to give the definition of motivic cohomology. The second part develops the basic theory of motivic cohomology which is required and states the main results. The final part concerns the explicit construction of the Steenrod operations.

**Remark 1.3.1** *These notes were mostly written in 2000, based on Voevodsky’s preliminary notes [V2] on the construction of the motivic Steenrod operations. The notes have not been revised to take into account the modified construction [V7] which has since appeared. In particular, the correspondence with Bloch’s higher Chow groups by Voevodsky allows the removal of certain of the hypotheses that the base field should have characteristic zero.*

*The author's original intention was to include the calculation of the motivic Steenrod algebra in these notes (indeed, this is the principle reason for the delay in their appearance). However, since this introduces issues involving motivic cohomology of non-smooth schemes and also the relation with  $qfh$  sheaves, which lie somewhat outside the theme of these notes, the author has preferred to write this up separately.*

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## Part I

# The homotopy theory of schemes and motivic cohomology

## 2 The homotopy category

The purpose of this section is to review the construction of a homotopy theory for the category of smooth schemes over a field  $k$ , in which the affine line  $\mathbb{A}^1$  becomes ‘contractible’ and hence plays the rôle of the interval  $[0, 1]$  in classical algebraic topology. Homotopy theories correspond to inverting a class of ‘weak equivalences’ in a suitable category  $\mathcal{C}$ ; the formal framework which establishes the existence of the associated homotopy category is provided by Quillen’s theory of model categories [Ho]. This necessitates working with a category which is both complete and cocomplete and hence requires the passage to presheaves. The approach followed here is that of Morel and Voevodsky [MV], who base their construction on the Joyal and Jardine model structure on the category of simplicial sheaves. This involves the choice of a Grothendieck topology on the category  $\mathcal{S}m/k$  of smooth schemes. Morel and Voevodsky show that taking the Nisnevich topology yields a homotopy category which has good properties and which is suitable for the study of the motivic cohomology of smooth schemes.

The reader should, however, be aware that there are variants of these constructions which give rise to an equivalent homotopy category (see [V3, W, M2], for example). One variant is to use simplicial presheaves, which have certain technical advantages when considering the change of Grothendieck topology.

It should be stressed that the main object of interest is the homotopy category which is obtained; the model structure which is used to obtain it plays a secondary rôle.

### 2.1 Simplicial sheaves and the Nisnevich topology

The following notation will be used throughout this paper:

**Notation 2.1.1** Let  $k$  be a field.

1. Let  $\mathcal{S}ch/k$  denote the category of separated schemes of finite type over the field  $k$ .
2. Let  $\mathcal{S}m/k$  denote the full subcategory of  $\mathcal{S}ch/k$  of separated smooth schemes of finite type over the field  $k$ .

Recall the definition of the Nisnevich topology on the category of smooth schemes; this lies strictly between the Zariski topology and the étale topology.

**Definition 2.1.2** [MV, 3.1.1] The Nisnevich topology on  $\mathcal{S}m/k$  is the Grothendieck topology which is generated by coverings of the form  $\{U_i \rightarrow X\}_i$ , a finite set

of étale morphisms such that, for any point  $x \in X$ , there exists a point  $u \in U_i$  over  $x$ , for some  $i$ , such that the induced morphism on residue fields is an isomorphism.

The Nisnevich topology is an example of a completely decomposed topology [V5]; in particular it is generated by coverings of the form  $\{U \xrightarrow{i} X, Y \xrightarrow{p} X\}$ , where  $i$  is an open immersion and  $p$  is an étale morphism which fits into an ‘elementary distinguished square’

$$\begin{array}{ccc} p^{-1}(U) & \longrightarrow & Y \\ \downarrow & & \downarrow p \\ U & \xrightarrow{i} & X \end{array}$$

such that  $p$  induces an isomorphism  $(Y - p^{-1}U)_{\text{red}} \rightarrow (X - U)_{\text{red}}$  (see [MV]).

**Notation 2.1.3**

1. Let  $\text{Shv}_{\text{Nis}}(\mathcal{S}m/k)$  denote the category of sheaves of sets for the Nisnevich topology on  $\mathcal{S}m/k$ .
2. Let  $\Delta^{\text{op}}\text{Shv}_{\text{Nis}}(\mathcal{S}m/k)$  denote the category of simplicial sheaves for the Nisnevich topology upon  $\mathcal{S}m/k$ , namely the category of simplicial objects in  $\text{Shv}_{\text{Nis}}(\mathcal{S}m/k)$ .

Suppose that  $x \in X$  is a point (not necessarily closed) of the smooth scheme  $X$  and let  $\mathcal{O}_{X,x}^h$  denote the Henselization of the local ring  $\mathcal{O}_{X,x}$  of  $X$  at  $x$  [Mi, I§4]; if  $F$  is a Nisnevich sheaf on  $\mathcal{S}m/k$  then let  $F(\text{Spec}(\mathcal{O}_{X,x}^h))$  denote the filtered colimit of the sets  $F(U)$  over the category of Nisnevich neighbourhoods of  $x$  (referred to as étale neighbourhoods in [Mi, I§4]).

**Lemma 2.1.4** *The functors  $F \mapsto F(\text{Spec}(\mathcal{O}_{X,x}^h))$  (for  $X$  belonging to a small skeleton of  $\mathcal{S}m/k$ ) form a conservative family of points for the category of sheaves with respect to the Nisnevich topology on  $\mathcal{S}m/k$ .*

The category of simplicial sheaves  $\Delta^{\text{op}}\text{Shv}_{\text{Nis}}(\mathcal{S}m/k)$  is the underlying category of the Morel-Voevodsky  $\mathbb{A}^1$ -local homotopy theory; this can be regarded as the analogue of the category of simplicial topological spaces in algebraic topology.

The following lemma recalls the rôle of the constant simplicial object functor:

**Lemma 2.1.5** *Let  $\Delta^{\text{op}}\mathcal{C}$  be the category of simplicial objects in the category  $\mathcal{C}$ . The constant simplicial structure functor  $\mathcal{C} \rightarrow \Delta^{\text{op}}\mathcal{C}$ , induced by the functor  $\Delta \rightarrow *$  to the discrete category with one object, is left adjoint to the functor  $\Delta^{\text{op}}\mathcal{C} \rightarrow \mathcal{C}$ ,  $X_* \mapsto X_0$ .*

The Nisnevich topology is sub-canonical, which means that the presheaf  $U \mapsto \text{Hom}_{\mathcal{S}m/k}(Y, X)$  represented by a smooth scheme  $X$  is a sheaf. Thus, the Yoneda functor defines an embedding  $\mathcal{S}m/k \hookrightarrow \text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$ , which extends to an embedding  $\mathcal{S}m/k \hookrightarrow \Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$  via the constant simplicial structure functor.

Smooth schemes are identified with their image in the category of simplicial sheaves. This gives the following standard objects in  $\Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$ :

1.  $\text{Spec}(k)$ , which plays the rôle of the basepoint in the theory; the associated sheaf is the constant sheaf  $U \mapsto *$ .
2. Affine space  $\mathbb{A}^n$ ,  $n \geq 0$ , obtained from the affine scheme  $\text{Spec}(k[x_1, \dots, x_n])$ .
3. Projective space  $\mathbb{P}^n$ ,  $n \geq 0$ , obtained from  $\text{Proj}(k[x_0, x_1, \dots, x_n])$ .
4. The multiplicative group  $\mathbb{G}_m \cong \mathbb{A}^1 - 0$ , obtained from the affine group scheme  $\text{Spec}(k[x, x^{-1}])$ .

Simplicial sets also yield objects of the category of simplicial sheaves, by the following result:

**Lemma 2.1.6** *The category  $\Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$  has the structure of a simplicial category.*

In particular, the constant sheaf functor  $\text{Set} \rightarrow \text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$  which sends a set  $K$  to the sheaf associated to the presheaf  $U \mapsto K$ , induces a functor

$$\Delta^{\text{op}}\text{Set} \rightarrow \Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k),$$

where  $\Delta^{\text{op}}\text{Set}$  is the category of simplicial sets.

## 2.2 The $\mathbb{A}^1$ -model structure

Model categories were developed by Quillen to provide a formal framework which guarantees the existence of the localization of a category by a class of weak equivalences. A model category is a category provided with a model structure of three classes of morphisms: cofibrations, weak equivalences and fibrations. These should satisfy axioms which allow a formal theory of ‘homotopy’ to be developed [Ho]; the class of fibrations is determined by the other two classes. The category of simplicial sets has the structure of a model category and the associated homotopy category is the usual homotopy category of algebraic topology. The category of simplicial sets plays a distinguished rôle in the theory of model categories and there is an enriched notion of a simplicial model category, for which the underlying category is a simplicial category and the simplicial structure and model structure satisfy compatibility conditions.

The category  $\Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$  has a simplicial model category structure, a case of a general result for simplicial sheaves over an essentially small site due to Joyal and Jardine. The weak equivalences are defined with respect to the points of the Grothendieck topology.

**Definition 2.2.1** A morphism  $f : \mathcal{X} \rightarrow \mathcal{Y}$  is a weak equivalence for the simplicial model category structure if, for any smooth scheme  $U \in \mathcal{S}m/k$  and any point  $u \in U$ , the corresponding morphism of simplicial sets:

$$\mathcal{X}(\mathrm{Spec}(\mathcal{O}_{U,u}^h)) \rightarrow \mathcal{Y}(\mathrm{Spec}(\mathcal{O}_{U,u}^h))$$

is a weak equivalence, where  $(-)^h$  denotes Henselization, and the values of the simplicial sheaves on  $\mathrm{Spec}(\mathcal{O}_{U,u}^h)$  are interpreted as the colimit over the category of Nisnevich neighbourhoods of  $u$ .

**Remark 2.2.2** The cofibrations in the Joyal and Jardine model structure are taken to be the monomorphisms in the category of simplicial sheaves. This implies that all objects in  $\Delta^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$  are cofibrant with respect to this structure.

**Notation 2.2.3** The homotopy category corresponding to the simplicial model category structure of Joyal and Jardine is written as  $\mathcal{H}_s(k)$ , where the  $s$  stands for ‘simplicial’.

This simplicial homotopy category  $\mathcal{H}_s(k)$  can be localized via the technique of Bousfield localization for model categories, so as to make the map  $\mathbb{A}^1 \rightarrow \mathrm{Spec}(k)$  a weak equivalence.

**Definition 2.2.4**

1. A simplicial sheaf  $\mathcal{X} \in \Delta^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$  is  $\mathbb{A}^1$ -local if the projection  $\mathcal{Y} \times \mathbb{A}^1 \rightarrow \mathcal{Y}$  induces a bijection  $\mathrm{Hom}_{\mathcal{H}_s(k)}(\mathcal{Y}, \mathcal{X}) \rightarrow \mathrm{Hom}_{\mathcal{H}_s(k)}(\mathcal{Y} \times \mathbb{A}^1, \mathcal{X})$ , for any  $\mathcal{Y} \in \Delta^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$ .
2. A morphism  $\mathcal{Y} \rightarrow \mathcal{Z}$  in  $\Delta^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$  is an  $\mathbb{A}^1$ -weak equivalence if, for any  $\mathbb{A}^1$ -local object  $\mathcal{X} \in \Delta^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$ , the induced map  $\mathrm{Hom}_{\mathcal{H}_s(k)}(\mathcal{Z}, \mathcal{X}) \rightarrow \mathrm{Hom}_{\mathcal{H}_s(k)}(\mathcal{Y}, \mathcal{X})$  is a bijection.

**Theorem 2.2.5** [MV, Theorem 2.2.5, Theorem 2.2.7] *The class of monomorphisms and  $\mathbb{A}^1$ -weak equivalences defines a proper model category structure on  $\Delta^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$ . The associated homotopy category is denoted  $\mathcal{H}(k)$ .*

These constructions have a pointed variant:

**Definition 2.2.6** A pointed simplicial sheaf is a morphism  $\mathrm{Spec}(k) \rightarrow F$  in  $\Delta^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$ , where  $\mathrm{Spec}(k)$  is the representable sheaf with constant simplicial structure.

**Notation 2.2.7**

1. Let  $\mathrm{Set}_\bullet$  denote the category of pointed sets and let  $\Delta^{\mathrm{op}}\mathrm{Set}_\bullet$  denote the category of pointed simplicial sets.
2. Let  $\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)_\bullet$  denote the category of pointed sheaves and let  $\Delta^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)_\bullet$  denote the category of pointed simplicial sheaves.

**Remark 2.2.8** The above notation is ambiguous but should not cause confusion: a pointed simplicial object can be regarded either as a simplicial object in the category of pointed objects or as a simplicial object which is pointed by the point taken with constant simplicial structure.

**Lemma 2.2.9** *The forgetful functor*

$$\Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)_{\bullet} \rightarrow \Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$$

has left adjoint  $(\cdot)_+$ , which is given by the canonical inclusion  $\mathcal{X}_+ := (\text{Spec}(k) \hookrightarrow \mathcal{X} \amalg \text{Spec}(k))$ .

The category  $\Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)_{\bullet}$  has a simplicial model category structure induced by the  $\mathbb{A}^1$ -model category structure on  $\Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$ , so that the cofibrations are the monomorphisms and the weak equivalences are  $\mathbb{A}^1$ -weak equivalences; the associated pointed homotopy category is denoted by  $\mathcal{H}_{\bullet}(k)$ . The functor  $(\cdot)_+$  and its right adjoint (the forgetful functor) induce an adjunction:

$$\mathcal{H}(k) \rightleftarrows \mathcal{H}_{\bullet}(k).$$

### 2.3 Monoidal structures

The category of pointed simplicial sheaves  $\Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)_{\bullet}$  has a symmetric monoidal structure with respect to the smash product, which induces a symmetric monoidal structure on the pointed  $\mathbb{A}^1$ -local homotopy category  $\mathcal{H}_{\bullet}(k)$ . Recall the definition of the smash product (see [MV, Section 2.2.5]):

**Definition 2.3.1** Let  $X, Y \in \Delta^{\text{op}}\text{Set}_{\bullet}$  be pointed simplicial sets and let  $\mathcal{X}, \mathcal{Y} \in \Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)_{\bullet}$  be pointed simplicial sheaves.

1. The wedge  $X \vee Y$  in  $\Delta^{\text{op}}\text{Set}_{\bullet}$  is the coequalizer of the diagram

$$* \rightrightarrows X \amalg Y$$

in the category of simplicial sets, pointed in the canonical way.

2. The smash product  $X \wedge Y$  in  $\Delta^{\text{op}}\text{Set}_{\bullet}$  is the quotient

$$(X \times Y)/(X \times * \vee * \times Y)$$

which is pointed by the image of  $(X \times * \vee * \times Y)$ . The unit for the smash product is the object  $(*)_+$ , where a disjoint basepoint is adjoined.

3. The smash product  $\mathcal{X} \wedge \mathcal{Y}$  in  $\Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)_{\bullet}$  is the sheaf associated to the presheaf:

$$U \mapsto \mathcal{X}(U) \wedge \mathcal{Y}(U).$$

The unit for the smash product is the object  $(\text{Spec}(k))_+$ .

In particular, the category  $\Delta^{\text{op}}\text{Set}_\bullet$  of pointed simplicial sets is a symmetric monoidal category with respect to the smash product.

**Notation 2.3.2** Let  $\mathcal{H}_\bullet$  denote the homotopy category derived from the category  $\Delta^{\text{op}}\text{Set}_\bullet$  of pointed simplicial sets, equipped with the usual notion of weak equivalences (see, for example, [Ho, Chapter 5]). The category  $\mathcal{H}_\bullet$  is equivalent to the classical homotopy category of pointed topological spaces.

The monoidal structures provided by the smash product carry over to the associated homotopy categories; the following result is a consequence of [Ho, Theorems 4.3.2, 4.3.4].

**Proposition 2.3.3** *Let  $k$  be a field.*

1. *The smash product on  $\Delta^{\text{op}}\text{Set}_\bullet$  induces a closed symmetric monoidal structure on  $\mathcal{H}_\bullet$ .*
2. *The smash product on  $\Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(Sm/k)_\bullet$  induces the structure of a closed  $\mathcal{H}_\bullet$ -algebra on  $\mathcal{H}_\bullet(k)$ .*

**Remark 2.3.4** The above result is a case of a general result for a pointed homotopy category associated to a pointed model category  $\mathcal{C}_\bullet$ . The homotopy category  $\text{Ho}(\mathcal{C}_\bullet)$  has the structure of a  $\mathcal{H}_\bullet$ -module which becomes an  $\mathcal{H}_\bullet$ -algebra structure if  $\mathcal{C}_\bullet$  has a suitable monoidal structure (see [Ho, Chapter 5]). The above situation is more straightforward, since the category  $\Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(Sm/k)_\bullet$  has a simplicial model category structure. These results show that the homotopy category  $\mathcal{H}_\bullet$  plays a distinguished rôle amongst the homotopy categories of pointed model categories.

## 2.4 Topological realization

Let  $k$  be a field, then  $\mathcal{H}_\bullet(k)$  has the structure of a  $\mathcal{H}_\bullet$ -algebra. In particular, there is a unit morphism:

$$\mathcal{H}_\bullet \rightarrow \mathcal{H}_\bullet(k).$$

When  $k$  is a sub-field of  $\mathbb{C}$ , then the general considerations of [MV] on derived functors between homotopy categories corresponding to continuous maps between sites shows that taking  $\mathbb{C}$ -points  $X \mapsto X(\mathbb{C})$  induces a retract to this functor.

**Proposition 2.4.1** *[MV] Let  $k$  be a field and let  $x : k \hookrightarrow \mathbb{C}$  be a  $\mathbb{C}$ -point, then there is a complex realization functor*

$$t_x^{\mathbb{C}} : \mathcal{H}_\bullet(k) \rightarrow \mathcal{H}_\bullet$$

*which is a morphism of  $\mathcal{H}_\bullet$ -algebras.*

**Remark 2.4.2** More general realization functors, which do not require a complex embedding, are provided by the étale homotopy type [F]. This theory is in the process of being developed from the point of view of  $\mathbb{A}^1$ -homotopy theory.

## 2.5 Spheres and suspension functors

There are standard objects in the category  $\Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)_{\bullet}$  which play the rôle of spheres in  $\mathbb{A}^1$ -homotopy theory; the justification for the terminology is that they yield spheres in the usual homotopy category  $\mathcal{H}_{\bullet}$  under topological realization and are independent of the field of definition.

### Definition 2.5.1

1. Let  $S_s^1 \in \Delta^{\text{op}}\text{Set}_{\bullet}$  denote the simplicial model  $(\Delta^1/\delta\Delta^1, \delta\Delta^1)$  for the pointed circle; the constant sheaf functor  $\Delta^{\text{op}}\text{Set}_{\bullet} \rightarrow \Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)_{\bullet}$  allows  $S_s^1$  to be regarded as an object of  $\Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)_{\bullet}$ .
2. Let  $S_t^1$  denote the pointed sheaf  $((\mathbb{A}^1 - \{0\}), 1)$ , the sheaf induced by the scheme  $\mathbb{G}_m$ , pointed by the rational point which corresponds to the identity. This is regarded as an object of  $\Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)_{\bullet}$  via the constant simplicial structure functor.
3. Let  $T$  denote the simplicial sheaf  $\mathbb{A}^1/(\mathbb{A}^1 - \{0\})$ , pointed by the image of  $(\mathbb{A}^1 - \{0\})$ ; again this will be regarded as an object of  $\Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)_{\bullet}$  via the constant simplicial structure functor.

The following statement concerning topological realization is clear:

**Proposition 2.5.2** *Let  $k$  be a sub-field of  $\mathbb{C}$  equipped with a complex embedding  $x : k \hookrightarrow \mathbb{C}$ . The complex realization functor  $t_x^{\mathbb{C}} : \mathcal{H}_{\bullet}(k) \rightarrow \mathcal{H}_{\bullet}$  induces isomorphisms in  $\mathcal{H}_{\bullet}$ :  $t_x^{\mathbb{C}}(S_s^1) \cong S^1 \cong t_x^{\mathbb{C}}(S_t^1)$  and  $t_x^{\mathbb{C}}(T) \cong S^2$ .*

The following result provides relations between these objects in  $\mathcal{H}_{\bullet}(k)$ .

**Lemma 2.5.3** *Let  $k$  be a field, then there are isomorphisms in  $\mathcal{H}_{\bullet}(k)$ :*

1.  $T \cong (\mathbb{P}^1, \infty)$ , where  $(\mathbb{P}^1, \infty)$  is the sheaf represented by the projective line, pointed by the rational point at infinity, equipped with constant simplicial structure.
2.  $T \cong S_s^1 \wedge S_t^1$ .

**Proof:** (1) There is a cocartesian square:

$$\begin{array}{ccc} (\mathbb{A}^1 - \{0\}) & \xrightarrow{0} & \mathbb{A}^1 \\ \infty \downarrow & & \downarrow \\ \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1 \end{array}$$

corresponding to the standard Zariski open cover of  $\mathbb{P}^1$ . This induces an  $\mathbb{A}^1$ -weak equivalence  $T \cong (\mathbb{P}^1, \infty)$ , by comparing cofibres and using the fact that  $\mathbb{P}^1/0 \rightarrow \mathbb{P}^1/\mathbb{A}^1$  is an  $\mathbb{A}^1$ -weak equivalence.

(2) There is an  $\mathbb{A}^1$ -weak equivalence  $T \cong S_s^1 \wedge \mathbb{G}_m$  [MV, 3.2.15], induced by the cofibration sequence which defines  $T$ . ■

There are suspension functors which correspond to taking the smash product with the objects defined in Definition 2.5.1:

**Notation 2.5.4** Let the functors  $\Sigma_s, \Sigma_t, \Sigma_T : \mathcal{H}_\bullet(k) \rightarrow \mathcal{H}_\bullet(k)$  respectively be induced by the smash product with the objects  $S_s^1, S_t^1, T$  respectively.

The simplicial suspension functor  $\Sigma_s$  is the canonical suspension functor which is associated to the  $\mathcal{H}_\bullet$  action on  $\mathcal{H}_\bullet(k)$ . The suspension functors are related by the following:

**Lemma 2.5.5**

1. There is a natural equivalence of functors  $\Sigma_T \cong \Sigma_s \circ \Sigma_t$ .
2. The transposition  $S_s^1 \wedge S_t^1 \rightarrow S_t^1 \wedge S_s^1$  induces a natural equivalence of functors  $\text{tr} : \Sigma_s \circ \Sigma_t \xrightarrow{\cong} \Sigma_t \circ \Sigma_s$ .

**Remark 2.5.6** The spheres  $S_s^1, S_t^1$  are given bidegrees  $(1, 0), (1, 1)$  respectively, so as to be compatible with standard notions of degree and weight. The bidegree is extended to smash products by additivity; for example the sphere  $T$  has bidegree  $(2, 1)$ .

## 2.6 Cofibration sequences

The notion of a cofibre sequence arises naturally in the study of pointed model categories (see [Ho, Chapter 6]). Moreover, cofibrations sequences give rise to exact sequences when considering representable cohomology theories.

**Definition 2.6.1** Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a morphism in  $\Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)_\bullet$ .

1. The (simplicial) cone of  $f$ , written  $\text{Cone}(f)$ , is defined to be the pushout of the diagram:

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{i_0} & \mathcal{X} \wedge \Delta^1 \\ f \downarrow & & \downarrow \\ \mathcal{Y} & & \end{array}$$

where  $i_\epsilon : * \rightarrow \Delta^1$ , for  $\epsilon \in \{0, 1\}$  denote the two vertex morphisms and  $\Delta^1$  is pointed by  $i_1$ .

2. The cofibre sequence associated to  $f$  is the sequence which has the form:

$$\mathcal{X} \rightarrow \mathcal{Y} \rightarrow \text{Cone}(f) \rightarrow \Sigma_s \mathcal{X},$$

where  $\text{Cone}(f) \rightarrow \Sigma_s \mathcal{X}$  is the evident collapse map sending  $\mathcal{Y}$  to a point. (See [Ho, Chapter 6] for details).



**Remark 2.6.2** The simplicial cone construction can be regarded as the natural way of constructing cofibration sequences in simplicial model categories, in that it is ‘independent’ of the model structure used. With respect to the Morel-Voevodsky model structure, weaker notions can be used, since all monomorphisms are cofibrations.

## 2.7 Thom spaces

The Thom space of a vector bundle over a smooth scheme  $X$  is defined in the category  $\Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)_{\bullet}$ . The Thom space should be considered as a generalized iterated  $T$ -suspension of the base space, by considering the case of a trivial bundle (see Example 2.7.2).

**Definition 2.7.1** Let  $\xi$  be a vector bundle over a smooth scheme  $X \in \mathcal{S}m/k$ . The Thom space of  $\xi$  is the pointed sheaf:

$$\text{Th}(\xi) := E(\xi)/E(\xi)^{\times},$$

where  $E(\xi)^{\times}$  is the complement of the zero section of the total space  $E(\xi)$ .

### Example 2.7.2

1. The pointed sheaf  $T$  identifies with the Thom space of the trivial bundle  $\mathbb{A}^1$  over  $\text{Spec}(k)$ .
2. Let  $\xi, \eta$  be vector bundles over  $X, Y \in \mathcal{S}m/k$  respectively, then  $\text{Th}(\xi \boxplus \eta) \simeq \text{Th}(\xi) \wedge \text{Th}(\eta)$  [MV, 3.2.17 (i)].
3. Let  $\xi$  be a vector bundle over  $X \in \mathcal{S}m/k$  and write  $\theta$  for the trivial bundle of rank one on  $X$ ; there is a map

$$\mathbb{P}(\xi \oplus \theta)/\mathbb{P}(\xi) \rightarrow \text{Th}(\xi)$$

which is an  $\mathbb{A}^1$ -homotopy equivalence [MV, 3.2.17 (iii)].

## 2.8 Split Simplicial Sheaves

The class of split simplicial sheaves in  $\Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$  is useful in certain situations; in particular, any object in the Morel-Voevodsky homotopy category is weakly equivalent to a split simplicial sheaf. The basic facts concerning split simplicial sheaves are recalled here for the benefit of the reader:

### Definition 2.8.1

1. A simplicial sheaf  $\mathcal{X} \in \Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$  is split if, for all  $n \geq 0$ , there exists a set  $\{U_i | i \in \mathcal{I}_n\}$  of sheaves represented by smooth schemes  $U_i$  such that:

$$\mathcal{X}_n \cong (\mathcal{X}_n)^{\text{deg}} \amalg (\amalg_{i \in \mathcal{I}_n} U_i),$$

where  $(\mathcal{X}_n)^{\text{deg}}$  denotes the degenerate part of the simplicial structure.

2. A split simplicial sheaf is finite if the set  $\bigcup_n \mathcal{I}_n$  is finite.

**Remark 2.8.2** Suppose that  $\mathcal{X}$  is a split simplicial sheaf, with  $U_i \in \mathcal{S}m/k$  as in the definition, then, for each  $n$ , there is a cocartesian diagram in  $\Delta^{\text{op}}\text{Shv}_{\text{Nis}}(\mathcal{S}m/k)$ :

$$\begin{array}{ccc} \prod_{i \in \mathcal{I}_n} U_i \times \partial \Delta^n & \longrightarrow & \text{sk}_{n-1}(\mathcal{X}) \\ \downarrow & & \downarrow \\ \prod_{i \in \mathcal{I}_n} U_i \times \Delta^n & \longrightarrow & \text{sk}_n(\mathcal{X}), \end{array}$$

where  $\text{sk}_j(\mathcal{X})$  denotes the  $j$ -skeleton of the simplicial sheaf  $\mathcal{X}$  (see [GJ, Section VII.1] for basic notions concerning the skeleton and coskeleton of a simplicial object). This presentation makes it intuitively clear how split simplicial sheaves are constructed by attaching ‘cells’ which are labelled by smooth schemes. Moreover the ‘attaching’ morphisms  $U_i \times \partial \Delta^n \rightarrow \text{sk}_{n-1} \mathcal{X}$  are adjoint to morphisms  $U_i \rightarrow (\text{cosk}_{n-1}(\mathcal{X}))_n$ ; this establishes the connection with alternative treatments of split simplicial objects.

The following result is clear:

**Lemma 2.8.3** *Let  $\mathcal{X} \in \Delta^{\text{op}}\text{Shv}_{\text{Nis}}(\mathcal{S}m/k)$  be a split simplicial sheaf, then  $\mathcal{X}$  is the filtered colimit of the diagram of finite split sub-sheaves of  $\mathcal{X}$ .*

Resolutions can be formed on the left by split simplicial sheaves, by Proposition 2.8.5, which is a variant of [MV, Lemma 1.16]. The statement of the result involves the notion of local fibration in  $\Delta^{\text{op}}\text{Shv}_{\text{Nis}}(\mathcal{S}m/k)$ .

**Definition 2.8.4** A morphism  $\mathcal{X} \rightarrow \mathcal{Y}$  in  $\Delta^{\text{op}}\text{Shv}_{\text{Nis}}(\mathcal{S}m/k)$  is said to be a local fibration if  $x^* \mathcal{X} \rightarrow x^* \mathcal{Y}$  is a Kan fibration for any point  $x^*$  of the Nisnevich topology.

**Proposition 2.8.5** *There exists a functor  $\Phi : \Delta^{\text{op}}\text{Shv}_{\text{Nis}}(\mathcal{S}m/k) \rightarrow \Delta^{\text{op}}\text{Shv}_{\text{Nis}}(\mathcal{S}m/k)$  and a natural transformation  $\Phi \rightarrow 1$  such that, for all  $\mathcal{X} \in \Delta^{\text{op}}\text{Shv}_{\text{Nis}}(\mathcal{S}m/k)$ ,*

1.  $\Phi \mathcal{X}$  is a split simplicial sheaf.
2. The morphism  $\Phi \mathcal{X} \rightarrow \mathcal{X}$  is a simplicial weak equivalence and a local fibration.

## 2.9 Compact Objects

It is necessary to have an understanding of the compact (or *small*) objects in the  $\mathbb{A}^1$ -local homotopy category  $\mathcal{H}(k)$ , for technical reasons. Recall the following definition of a compact object (which can be generalized to consider higher ordinals):

**Definition 2.9.1** Let  $\mathcal{C}$  be a category which contains all small colimits. An object  $A$  in  $\mathcal{C}$  is compact if, for any small filtered diagram  $\mathcal{I} \rightarrow \mathcal{C}$ , the natural morphism:

$$\lim_{i \in \mathcal{I}} \mathrm{Hom}_{\mathcal{C}}(A, X_i) \rightarrow \mathrm{Hom}_{\mathcal{C}}(A, \lim_{i \in \mathcal{I}} X_i)$$

is a bijection.

The following lemma is straightforward:

**Lemma 2.9.2**

1. *The set of representable sheaves corresponding to a small skeleton of  $\mathcal{S}m/k$  forms a set of generators of the category  $\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$ .*
2. *The sheaf represented by  $X \in \mathcal{S}m/k$  is a compact object in the category  $\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$ .*

In fact, Kan observed that a stronger result holds: any sheaf  $F$  is canonically the colimit of a diagram of representable sheaves.

**Proposition 2.9.3** *Let  $\mathcal{X}$  be a finite split simplicial sheaf in  $\Delta^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$ . Then:*

1.  *$\mathcal{X}$  is a compact object in  $\Delta^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$ .*
2. *The image of  $\mathcal{X}$  in  $\mathcal{H}(k)$  is a compact object.*

*Moreover, an object in  $\mathcal{H}(k)$  is compact if and only if it is the retract in  $\mathcal{H}(k)$  of an object represented by a finite split simplicial sheaf.*

**Proof:** (1) This is straightforward.

(2) This is proved by using homotopy colimits, which are the correct homotopy theoretic form of colimits, and by applying [MV, Corollary 2.1.21].

Finally, suppose that  $C \in \Delta^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$  represents a compact object in  $\mathcal{H}(k)$ , then there is a resolution  $\mathcal{X} \xrightarrow{\cong} C$  of  $C$ , with  $\mathcal{X}$  a split simplicial sheaf, by Proposition 2.8.5. This morphism admits an inverse in the homotopy category  $C \cdots \cdots \rightarrow \mathcal{X}$ . Lemma 2.8.3 implies that  $\mathcal{X}$  is a filtered colimit in  $\Delta^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$  of objects in  $\mathcal{H}(k)$  represented by finite split simplicial sheaves, hence this morphism factors through an object represented by  $\mathcal{X}'$ , a finite split simplicial sheaf, by the compactness of  $C$ . In particular,  $C$  is a retract of  $\mathcal{X}'$  in  $\mathcal{H}(k)$ . The converse statement is clear. ■

**Remark 2.9.4** A pointed simplicial sheaf is said to be split if the underlying simplicial sheaf (forgetting the basepoint) is split. The above results adapt easily to the pointed situation.

### 3 Cohomology theories and cohomology operations

The purpose of this section is to give a systematic introduction to the notions of cohomology theory which are encountered in the study of the  $\mathbb{A}^1$ -local homotopy theory. This involves greater generality than is strictly necessary, so that this material could be omitted on first reading.

**Notation 3.0.5** Throughout this section,  $\mathcal{C}_\bullet$  will be used to denote a pointed model category, as defined in [Ho, Chapter 6]. The reader unfamiliar with this notion can consider the example  $\Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)_\bullet$  equipped with either the simplicial model structure or the  $\mathbb{A}^1$ -local model structure.

#### 3.1 Additive presheaves on homotopy categories

This section introduces the basic properties of additivity and homotopy invariance for presheaves  $\mathcal{H}_{s_\bullet}(k)^{\text{op}} \rightarrow \text{Set}$ .

**Definition 3.1.1** Let  $\mathcal{C}_\bullet$  be a pointed model category. An additive presheaf on the homotopy category  $\text{Ho}(\mathcal{C}_\bullet)$  is a presheaf

$$h : \text{Ho}(\mathcal{C}_\bullet)^{\text{op}} \rightarrow \text{Set},$$

such that, for any finite coproduct  $\bigvee X_i$  in  $\mathcal{C}_\bullet$ , the natural morphism:

$$h(\bigvee X_i) \rightarrow \prod h(X_i)$$

is a bijection.

An additive presheaf  $h$  is said to be non-trivial if there exists an object  $X \in \mathcal{C}_\bullet$  such that  $h(X) \neq \emptyset$ .

**Remark 3.1.2** The canonical functor  $\mathcal{C}_\bullet \rightarrow \text{Ho}(\mathcal{C}_\bullet)$  allows an additive presheaf to be considered as a presheaf on the category  $\mathcal{C}_\bullet$ . The universal property of the homotopy category states that a functor  $F : \mathcal{C}_\bullet \rightarrow \text{Set}$  factors across  $\mathcal{C}_\bullet \rightarrow \text{Ho}(\mathcal{C}_\bullet)$  if and only if  $F$  sends weak equivalences to bijections. This property is frequently termed the **homotopy axiom** and is subsumed in the above definition.

The following lemma is straightforward:

**Lemma 3.1.3** *Let  $\mathcal{C}_\bullet$  be a pointed model category and let  $h : \text{Ho}(\mathcal{C}_\bullet) \rightarrow \text{Set}$  be a non-trivial additive presheaf. Then  $h$  factors across the forgetful functor  $\text{Set}_\bullet \rightarrow \text{Set}$  and  $h(*)$  is the singleton set, where  $*$  denotes the point of  $\mathcal{C}_\bullet$ .*

**Definition 3.1.4** Let  $\mathcal{C}_\bullet$  be a pointed model category. A presheaf  $h$  on the homotopy category  $\text{Ho}(\mathcal{C}_\bullet)$  is representable if there exists an object  $Z \in \text{Ho}(\mathcal{C}_\bullet)$  together with a natural isomorphism:

$$\text{Hom}_{\text{Ho}(\mathcal{C}_\bullet)}(X, Z) \xrightarrow{\cong} h(X).$$

**Remark 3.1.5** A representable presheaf on the homotopy category  $\mathrm{Ho}(\mathcal{C}_\bullet)$  is necessarily an additive presheaf which is non-trivial.

Suppose that  $F : \mathcal{C}_\bullet \rightleftarrows \mathcal{D}_\bullet : G$  is a Quillen adjunction between pointed model categories (see [Ho] or consider simply the adjunction at the level of homotopy categories), so that there is a derived adjunction:

$$\mathbb{L}F : \mathrm{Ho}(\mathcal{C}_\bullet) \rightleftarrows \mathrm{Ho}(\mathcal{D}_\bullet) : \mathbb{R}G$$

at the level of the homotopy categories.

**Lemma 3.1.6** *Let  $F, G$  be as above and let  $h : \mathrm{Ho}(\mathcal{D}_\bullet)^{\mathrm{op}} \rightarrow \mathrm{Set}$  be an additive presheaf. Then*

1. *the composite functor  $h \circ \mathbb{L}F : \mathrm{Ho}(\mathcal{C}_\bullet)^{\mathrm{op}} \rightarrow \mathrm{Set}$  is an additive presheaf on  $\mathrm{Ho}(\mathcal{C}_\bullet)$ .*
2. *Moreover, if the additive presheaf  $h$  is represented by  $W \in \mathrm{Ho}(\mathcal{D}_\bullet)$ , then the additive presheaf  $h \circ \mathbb{L}F$  is represented by  $\mathbb{R}G(W)$ .*

In the special case that the adjunction corresponds to a Bousfield localization of model categories, there is an analogous result in the opposite direction. Recall the following definitions:

**Definition 3.1.7** Let  $F : \mathcal{C}_\bullet \rightleftarrows \mathcal{D}_\bullet : G$  be a Quillen adjunction between pointed model categories. The adjunction is said to be a Bousfield localization if the underlying categories of  $\mathcal{C}_\bullet$  and  $\mathcal{D}_\bullet$  are the same, the functors  $F, G$  are the identity, the class of cofibrations in  $\mathcal{C}_\bullet$  and  $\mathcal{D}_\bullet$  are the same whereas the class of weak equivalences in  $\mathcal{C}_\bullet$  is contained in the class of weak equivalences in  $\mathcal{D}_\bullet$ .

This is of special interest when the localization is obtained with respect to a set of morphisms  $f \in \mathcal{S}$  in the homotopy category  $\mathrm{Ho}(\mathcal{C}_\bullet)$ . Recall the relevant definitions:

**Definition 3.1.8** Let  $\mathcal{C}_\bullet$  be a pointed model category and let  $\mathcal{S}$  be a set of morphisms in  $\mathrm{Ho}(\mathcal{C}_\bullet)$ .

1. An object  $Z \in \mathrm{Ho}(\mathcal{C}_\bullet)$  is  $\mathcal{S}$ -local if, for any morphism  $f : A \rightarrow B$  in  $\mathcal{S}$ , the induced morphism  $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C}_\bullet)}(B, Z) \rightarrow \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C}_\bullet)}(A, Z)$  is a bijection.
2. A morphism  $g : X \rightarrow Y$  is an  $\mathcal{S}$ -weak equivalence if the induced morphism  $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C}_\bullet)}(Y, Z) \rightarrow \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C}_\bullet)}(X, Z)$  is a bijection, for any  $\mathcal{S}$ -local object  $Z$  in  $\mathrm{Ho}(\mathcal{C}_\bullet)$ .
3. The  $\mathcal{S}$ -localization of  $\mathcal{C}_\bullet$ , if it exists, is the model category with the same cofibrations as  $\mathcal{C}_\bullet$  and with weak equivalences taken to be the  $\mathcal{S}$ -weak equivalences.

The following is clear:

**Lemma 3.1.9** *Let  $\mathcal{C}_\bullet$  be a pointed model category and let  $\mathcal{S}$  be a set of morphisms in  $\mathrm{Ho}(\mathcal{C}_\bullet)$  for which the  $\mathcal{S}$ -localization  $\mathcal{C}_{\bullet, \mathcal{S}}$  exists. A representable additive presheaf  $h$  on  $\mathrm{Ho}(\mathcal{C}_\bullet)$  factors across an additive presheaf on  $\mathrm{Ho}(\mathcal{C}_{\bullet, \mathcal{S}})$  if and only if  $h$  sends the morphisms of  $\mathcal{S}$  to bijections. Moreover, in this case, the induced additive presheaf on  $\mathrm{Ho}(\mathcal{C}_{\bullet, \mathcal{S}})$  is representable.*

These general considerations apply to the  $\mathbb{A}^1$ -localization:

$$\mathcal{H}_{s_\bullet}(k) \rightleftarrows \mathcal{H}_\bullet(k)$$

**Definition 3.1.10** An additive presheaf  $\mathcal{H}_{s_\bullet}(k)^{\mathrm{op}} \rightarrow \mathrm{Set}$  is homotopy invariant if, for every  $\mathcal{X} \in \Delta^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)_\bullet$ , the projection morphism  $\mathcal{X} \wedge (\mathbb{A}^1)_+ \rightarrow \mathcal{X}$  induced by the projection  $\mathbb{A}^1 \rightarrow \mathrm{Spec}(k)$  induces a bijection:

$$h(\mathcal{X}) \rightarrow h(\mathcal{X} \wedge (\mathbb{A}^1)_+).$$

**Corollary 3.1.11** *A representable additive presheaf  $h : \mathcal{H}_{s_\bullet}(k)^{\mathrm{op}} \rightarrow \mathrm{Set}$  factors canonically across the  $\mathbb{A}^1$ -localization  $\mathcal{H}_{s_\bullet}(k) \rightarrow \mathcal{H}_\bullet(k)$  if and only if  $h$  is homotopy invariant.*

## 3.2 Brown representability

In practice, it is difficult to construct additive presheaves on a homotopy category which are *not* representable. In particular, there are results inspired by the Brown representability theorem of algebraic topology, which give sufficient conditions for a presheaf to be representable. In the current context, the point of view taken is that the existence of an underlying model category structure is accepted; the reader should however be aware for example that in the context of cohomological functors on triangulated categories, this is not always necessary.

**Remark 3.2.1** For technical reasons, Brown representability theorems are best formulated for compactly-generated homotopy categories and the additive presheaf should be restricted to the full subcategory of compact objects in the homotopy category.

**Notation 3.2.2** Let  $\mathrm{Ho}(\mathcal{C}_\bullet)^{\mathrm{comp}}$  denote the full sub-category of compact objects in  $\mathrm{Ho}(\mathcal{C}_\bullet)$ .

**Proposition 3.2.3** *Let  $\mathcal{C}_\bullet$  be a pointed model category and let  $h : \mathrm{Ho}(\mathcal{C}_\bullet)^{\mathrm{op}} \rightarrow \mathrm{Set}$  be an additive presheaf. Suppose that there exists a set  $\mathcal{I}$  of cofibrations in  $\mathcal{C}$  with compact domain and range such that the following conditions are satisfied:*

1. *For any compact object  $C \in \mathrm{Ho}(\mathcal{C}_\bullet)^{\mathrm{comp}}$ , there exists a morphism  $* \rightarrow X$  in  $\mathcal{I}$  such that  $C$  is a retract of  $X$  in  $\mathrm{Ho}(\mathcal{C}_\bullet)$ .*
2. **Mayer Vietoris axiom:** *For any pushout square in  $\mathcal{C}_\bullet$*

$$\begin{array}{ccc} A & \longrightarrow & E \\ \downarrow & & \downarrow \\ B & \longrightarrow & B \vee_A E, \end{array}$$

where  $i \in \mathcal{I}$ , and  $E$  is obtained by a finite sequence of pushouts of elements  $\mathcal{I}$ , the sequence of pointed sets

$$h(B \vee_A E) \longrightarrow h(B) \amalg h(E) \rightrightarrows h(A)$$

is exact in the middle.

Then the restriction of  $h$  to  $(\mathrm{Ho}(\mathcal{C}_\bullet)^{\mathrm{comp}})^{\mathrm{op}}$  is represented by an object of  $\mathrm{Ho}(\mathcal{C}_\bullet)$ .

**Example 3.2.4** The example which is of interest here is the case  $\mathrm{Ho}(\mathcal{C}_\bullet) = \mathcal{H}_{s_\bullet}(k)$ ; here  $\mathcal{I}$  should be chosen to be the set of inclusions  $A \hookrightarrow B$  of a sub split simplicial sheaf  $A$  in a finite split simplicial sheaf  $B$ .

**Remark 3.2.5** A more interesting problem is to consider when a functor defined on the underlying category  $\mathcal{C}_\bullet$  preserves weak equivalences. An interesting case is given by the Brown-Gersten descent condition.

### 3.3 Additive presheaves of abelian groups

The terminology introduced here is non-standard; the justification is that usage of ‘cohomology’ can be ambiguous. In particular, the following definition does not require any exactness property or suspension isomorphism; however, all cohomology theories here will be taken to be additive.

**Definition 3.3.1** Let  $\mathcal{C}_\bullet$  be a pointed model category. A reduced cohomology theory on the homotopy category  $\mathrm{Ho}(\mathcal{C}_\bullet)$  is an additive presheaf  $h : \mathrm{Ho}(\mathcal{C}_\bullet)^{\mathrm{op}} \rightarrow \mathit{Ab}$  which takes values in the category of abelian groups.

The usual way of obtaining a reduced cohomology theory is indicated by the general result:

**Lemma 3.3.2** Let  $\mathcal{C}_\bullet$  be a pointed model category equipped with a monoidal structure  $\wedge$  which is compatible with the model structure. Let  $h : \mathrm{Ho}(\mathcal{C}_\bullet)^{\mathrm{op}} \rightarrow \mathit{Set}$  be an additive presheaf and suppose that  $A$  is an abelian cogroup object in the category  $\mathrm{Ho}(\mathcal{C}_\bullet)$ . The presheaf:

$$X \mapsto h(A \wedge X)$$

defines a reduced cohomology theory.

In particular, this gives:

**Example 3.3.3** Let  $\mathcal{C}_\bullet$  be a pointed simplicial model category and let  $h : \mathrm{Ho}(\mathcal{C}_\bullet)^{\mathrm{op}} \rightarrow \mathit{Set}$  be an additive presheaf. The presheaf  $X \mapsto h(S^2 \wedge X)$  is a reduced cohomology theory, where  $S^2 \in \mathrm{Ho}(\mathcal{C}_\bullet)$  represents the pointed simplicial two-sphere.

### 3.4 Bigraded reduced cohomology theories

**Definition 3.4.1** A bigraded reduced cohomology theory on  $\mathcal{H}_\bullet(k)$  is a set of reduced cohomology theories:

$$h^{*,*} : \mathcal{H}_\bullet(k)^{\text{op}} \rightarrow \mathcal{A}b$$

indexed over  $\mathbb{Z} \times \mathbb{Z}$ , such that there exist natural suspension isomorphisms:

$$\begin{aligned} \sigma_s &: h^{*,*} \rightarrow h^{*+1,*} \circ \Sigma_s \\ \sigma_t &: h^{*,*} \rightarrow h^{*+1,*+1} \circ \Sigma_t \end{aligned}$$

which satisfy the **suspension coherence condition**:

For each  $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ , there exists a commutative diagram of natural transformations:

$$\begin{array}{ccc} h^{a,b} & \xrightarrow{\sigma_s} & h^{a+1,b} \circ \Sigma_s \\ \sigma_t \downarrow & & \downarrow \text{tr} \circ \sigma_t \\ h^{a+1,b+1} \circ \Sigma_t & \xrightarrow{\sigma_s} & h^{a+2,b+1} \circ \Sigma_s \circ \Sigma_t \end{array}$$

**Definition 3.4.2** A bigraded reduced cohomology theory on  $\mathcal{H}_\bullet(k)$  is **exact** if, for any cofibration sequence  $\mathcal{X} \rightarrow \mathcal{Y} \rightarrow \text{Cone}(f)$  the induced sequence of bigraded abelian groups:

$$h^{*,*}(\text{Cone}(f)) \rightarrow h^{*,*}(\mathcal{Y}) \rightarrow h^{*,*}(\mathcal{X})$$

is an exact sequence.

#### Remark 3.4.3

1. let  $h^{*,*}$  be a bigraded reduced cohomology theory which is exact, then the exact sequence of Definition 3.4.2 extends into a long exact sequence in cohomology, using the simplicial suspension isomorphism  $\sigma_s$ .
2. Let  $h^{*,*}$  be a bigraded reduced cohomology theory, which is representable in each bidegree, then the exactness condition is satisfied automatically.
3. The bidegrees of the suspension isomorphisms are compatible with the bidegrees of the spheres  $S_s^1$  and  $S_t^1$  given in Remark 2.5.6.

### 3.5 Unreduced cohomology theories

From the point of view of homotopy theory, reduced cohomology theories are the most natural objects for consideration. It is sometimes convenient to work with *unreduced* cohomology theories, especially when the objects being considered are not naturally pointed.



**Definition 3.5.1** An unreduced cohomology theory is an abelian presheaf

$$H : \mathcal{H}(k)^{\text{op}} \rightarrow \mathcal{A}b$$

which is additive: for each finite coproduct  $\coprod X_i$ , the natural morphism:

$$H(\coprod X_i) \rightarrow \prod H(X_i)$$

is an isomorphism in  $\mathcal{A}b$ .

The notions of reduced and unreduced cohomology theory are related via the adjunction:

$$(\cdot)_+ : \mathcal{H}(k) \rightleftarrows \mathcal{H}_\bullet(k) : \text{forget.}$$

The following lemma can be used to establish an equivalence between the classes of unreduced and reduced cohomology theories.

**Lemma 3.5.2**

1. Let  $h : \mathcal{H}_\bullet(k)^{\text{op}} \rightarrow \mathcal{A}b$  be a reduced cohomology theory, then the composite

$$\mathcal{H}(k)^{\text{op}} \xrightarrow{(\cdot)_+} \mathcal{H}_\bullet(k)^{\text{op}} \rightarrow \mathcal{A}b$$

is an unreduced cohomology theory.

2. Let  $H : \mathcal{H}(k)^{\text{op}} \rightarrow \mathcal{A}b$  be an unreduced cohomology theory, then the functor  $h : \mathcal{H}_\bullet(k)^{\text{op}} \rightarrow \mathcal{A}b$  defined by

$$h(\text{Spec}(k) \rightarrow \mathcal{X}) := \ker\{H(\mathcal{X}) \rightarrow H(\text{Spec}(k))\}$$

is a reduced cohomology theory.

**Notation 3.5.3** To avoid confusion between the reduced and unreduced version of a cohomology theory, a reduced cohomology theory will usually be decorated by a tilde, as in  $\tilde{H}$ .

### 3.6 Determining bigraded theories by $T$ -suspension theories

A bigraded reduced cohomology theory is determined by the reduced cohomology theories in bidegrees of the form  $(2*, *)$  together with a  $T$ -suspension isomorphism. This allows the theory of  $T$ -spectra, which gives a model for a ‘stable homotopy theory’, to be used to represent bigraded reduced cohomology theories [V1, M]; the theory of  $T$ -spectra is not necessary for the present purposes.

**Definition 3.6.1** Let  $k$  be a field; a  $T$ -suspension reduced cohomology theory is a sequence of reduced cohomology theories  $h^{[b]} : \mathcal{H}_\bullet(k)^{\text{op}} \rightarrow \mathcal{A}b$ , for  $b \in \mathbb{Z}$ , together with  $T$ -suspension natural isomorphisms:

$$\sigma_T : h^{[b]} \rightarrow h^{[b+1]} \circ \Sigma_s \circ \Sigma_t.$$

The category of  $T$ -suspension reduced cohomology theories is the category with morphisms given by natural transformations between  $T$ -suspension reduced cohomology theories which commute with the suspension isomorphism.

The following lemma is clear:

**Lemma 3.6.2** Let  $h^{*,*} : \mathcal{H}_\bullet(k)^{\text{op}} \rightarrow \mathcal{A}b$  be a bigraded reduced cohomology theory; let  $h^{[b]} := h^{2b,b}$ , for  $b \in \mathbb{Z}$  and define  $\sigma_T$  to be the composite:

$$h^{[b]} \cong h^{2b,b} \xrightarrow{\sigma_t} h^{2b+1,b} \circ \Sigma_t \xrightarrow{\sigma_s} h^{2(b+1),b+1} \circ \Sigma_s \circ \Sigma_t \cong h^{[b+1]} \circ \Sigma_s \circ \Sigma_t.$$

Then  $h^{[*]}$  forms a  $T$ -suspension reduced cohomology theory.

The converse is also true:

**Lemma 3.6.3** Let  $h^{[*]}$  be a  $T$ -suspension reduced cohomology theory, equipped with suspension isomorphism:

$$\sigma_T : h^{[b]} \rightarrow h^{[b+1]} \circ \Sigma_s \circ \Sigma_t.$$

Define reduced cohomology theories  $h^{m,n}$ , for  $(m,n) \in \mathbb{Z} \times \mathbb{Z}$  by:

$$h^{m,n}(\mathcal{X}) := \begin{cases} h^{[n]}(\Sigma_s^{2n-m} \mathcal{X}) & m \leq 2n \\ h^{[m-n]}(\Sigma_t^{m-2n} \mathcal{X}) & m \geq 2n. \end{cases}$$

Then there exist suspension isomorphisms  $\sigma_s, \sigma_t$  such that  $h^{*,*}$  forms a bigraded reduced cohomology theory.

**Proof:** (Indications) The technical point of the proof is that the suspension isomorphisms have to be defined so that the coherence condition is satisfied; this reduces to an exercise in permuting smash products of  $S_s^1$  and  $S_t^1$  in the category  $\mathcal{H}_\bullet(k)$ . ■

The bigraded reduced cohomology theories form a category, where the morphisms are the natural transformations which commute with the suspension isomorphisms. Lemmas 3.6.3 and 3.6.2 establish the following:

**Proposition 3.6.4** There is an equivalence between the category of bigraded reduced cohomology theories on  $\mathcal{H}_\bullet(k)$  and the category of  $T$ -suspension reduced cohomology theories on  $\mathcal{H}_\bullet(k)$ . In particular, a bigraded reduced cohomology theory  $h^{*,*}$  on  $\mathcal{H}_\bullet(k)$  is determined by the associated  $T$ -suspension reduced cohomology theory  $h^{[*]}$ , up to natural isomorphism.

### 3.7 Bistable cohomology operations

Throughout this paper, we are only concerned with *bistable* cohomology operations between bigraded reduced cohomology theories. The reader interested in general notions concerning generalized cohomology operations in algebraic topology can consult [Boa].

**Definition 3.7.1** Let  $h, g$  be bigraded reduced cohomology theories on  $\mathcal{H}_\bullet(k)$ . A bistable cohomology operation  $\theta$  of bidegree  $(p, q)$  from  $h$  to  $g$  is a collection of natural transformations (not *a priori* homomorphisms), for  $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ :

$$\theta^{a,b} : h^{a,b} \rightarrow g^{a+p,b+q}$$

which commute with the suspension isomorphisms  $\sigma_s, \sigma_t$ .

The simplicial circle  $S_s^1$  is a cogroup object in  $\mathcal{H}_\bullet(k)$ ; this implies (as in the ‘classical’ situation of algebraic topology):

**Lemma 3.7.2** Let  $h, g$  be bigraded reduced cohomology theories on  $\mathcal{H}_\bullet(k)$ . Suppose that  $\theta$  is a bistable cohomology operation between  $h$  and  $g$ , then the natural morphisms  $\theta^{a,b}(\mathcal{X})$  are morphisms of abelian groups, for all  $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ .

**Definition 3.7.3** Let  $h$  be a bigraded reduced cohomology theory on  $\mathcal{H}_\bullet(k)$  such that the isomorphism classes of cohomology operations of each bidegree  $(p, q)$  forms a set.

The bigraded algebra of bigraded cohomology operations is the set of isomorphism classes of bigraded cohomology operations, bigraded by the bidegree, provided with the structure of abelian group via addition in  $\mathcal{A}b$  and with ring structure (non-commutative) given by the composition of natural transformations.

Proposition 3.6.4 implies that a bigraded reduced cohomology theory  $h^{*,*}$  is determined by its associated  $T$ -suspension reduced cohomology theory. In this context, bistable cohomology operations are interpreted as  $T$ -stable cohomology operations, as follows:

**Definition 3.7.4** Let  $h^{[*]}, g^{[*]}$  be  $T$ -suspension reduced cohomology theories on  $\mathcal{H}_\bullet(k)$ . A  $T$ -stable cohomology operation of bidegree  $(p, q)$  from  $h$  to  $g$  is a sequence of natural transformations, for  $b \in \mathbb{Z}$ :

$$\begin{aligned} h^{[b]} &\rightarrow g^{[b+q]} \circ \Sigma_s^{2q-p} & 2q \geq p \\ h^{[b]} &\rightarrow g^{[b+p-q]} \circ \Sigma_t^{p-2q} & p \geq 2q \end{aligned}$$

which commute with the  $T$ -suspension isomorphisms  $\sigma_T$ .

## 4 Sheaves with transfers and the adjunction between $\mathcal{H}(k)$ and $DM_-^{\text{eff}}(k)$

The point of view which is adopted in this paper is that motivic cohomology is a bigraded representable functor from the opposite of the homotopy category  $\mathcal{H}(k)$  of Morel and Voevodsky, defined with respect to the Nisnevich topology on the category  $\mathcal{S}m/k$  of smooth schemes, to the category of abelian groups. In the case that the field  $k$  admits resolution of singularities<sup>2</sup>, this defines a cohomology theory in the sense of Section 3.4. This viewpoint is in accordance with the philosophy that motivic cohomology should be defined as a sheaf hypercohomology theory.

The proof that motivic cohomology is a bigraded cohomology theory depends on two ingredients:

- The duality theorem of Friedlander, Lawson and Voevodsky [FV, Theorems 7.1, 7.4].
- The usage of Nisnevich sheaves with transfers,  $\mathcal{N}_k^{\text{tr}}$ , which are used to establish the existence of localization sequences.

Heuristically, one can regard the transfers as performing a further localization of the homotopy category  $\mathcal{H}(k)$ , by applying [V4, Theorem 4.1.2].

Motivic cohomology factors across Voevodsky's triangulated category of motives,  $DM_-^{\text{eff}}(k)$ ; when the ground field  $k$  is perfect, the category  $DM_-^{\text{eff}}(k)$  is equivalent to the  $\mathbb{A}^1$ -localization of the derived category  $\mathcal{D}_- \mathcal{N}_k^{\text{tr}}$  of Nisnevich sheaves with transfers. The factorization is provided by the derived functor  $\mathcal{H}(k) \rightarrow DM_-^{\text{eff}}(k)$  of the free presheaf with transfers functor,  $\mathbb{Z}_{\text{tr}}[-]$ , which forms part of an adjunction between  $\mathcal{H}(k)$  and  $DM_-^{\text{eff}}(k)$ , when the field  $k$  is perfect.

**Remark 4.0.1** The proof that motivic cohomology defines a bigraded cohomology theory does not require passage through the category  $DM_-^{\text{eff}}(k)$ . Alternatively, this fact can be regarded as being a consequence of the cancellation theory in  $DM_-^{\text{eff}}(k)$ , [V4, Theorem 4.3.1], once the adjunction between  $\mathcal{H}(k)$  and  $DM_-^{\text{eff}}(k)$  has been established. The ideas which go into the proof of the cancellation theorem are the same as indicated above.

The purpose of this section is to recall the definition of the category of Nisnevich sheaves with transfers and then to indicate the construction of the adjunction between  $\mathcal{H}(k)$  and  $DM_-^{\text{eff}}(k)$ .

### 4.1 Nisnevich sheaves with transfers

The definition of motivic cohomology uses Nisnevich sheaves with transfers, which are defined in terms of the category of smooth correspondences.

<sup>2</sup>Voevodsky has established that this restriction is unnecessary.

**Definition 4.1.1** [SV2, §1] Let  $X, Y \in \mathcal{S}m/k$  be smooth schemes over the field  $k$ . Let  $C(X, Y)$  denote the set of closed, integral subschemes  $Z \hookrightarrow X \times Y$  which are finite and surjective over a component of  $X$  and let  $c(X, Y)$  the free abelian group  $\mathbb{Z}[C(X, Y)]$  generated by  $C(X, Y)$ .

**Definition 4.1.2** [SV2, §1] The category  $\mathcal{S}mCor/k$  of smooth correspondences over the field  $k$  is the category with objects  $[X]$ , where  $X \in \mathcal{S}m/k$ , and morphisms  $\text{Hom}_{\mathcal{S}mCor/k}([X], [Y])$  given by the abelian group  $c(X, Y)$ . The composition of morphisms is given by intersection of cycles as follows. Suppose that  $S \in c(X, Y)$ ,  $T \in c(Y, Z)$ , then the cycles  $S \times Z$  and  $X \times T$  intersect properly on  $X \times Y \times Z$  and each component of the intersection cycle  $(S \times Z) \bullet (X \times T)$  is finite and surjective on a component of  $X$ . The composite  $T \circ S \in c(X, Z)$  is defined as the push-forward  $(p_{XZ})_*((S \times Z) \bullet (X \times T))$ , where  $p_{XZ}$  denotes the projection  $X \times Y \times Z \rightarrow X \times Z$ .

The category  $\mathcal{S}mCor/k$  is an additive category, with the direct sum of the objects  $[X]$  and  $[Y]$  given by  $[X \amalg Y]$ . There is a functor

$$\mathcal{S}m/k \rightarrow \mathcal{S}mCor/k$$

which is the association  $X \mapsto [X]$  on objects and which sends a morphism  $f : X \rightarrow Y$  to the cycle  $\Gamma_f \in c(X, Y)$  given by the graph of  $f$ .

**Notation 4.1.3**

1. Let  $\mathcal{N}_k$  denote the category of abelian sheaves for the Nisnevich topology on  $\mathcal{S}m/k$ .
2. Let  $\mathcal{P}_k^{\text{tr}}$  be the category of abelian presheaves with transfers, namely additive functors  $\mathcal{S}mCor/k^{\text{op}} \rightarrow \text{Ab}$ . (A functor is additive if, for any  $X, Y \in \mathcal{S}m/k$ ,  $F(X \amalg Y) \cong F(X) \oplus F(Y)$ ).
3. Let  $\mathcal{N}_k^{\text{tr}}$  denote the full subcategory of  $\mathcal{P}_k^{\text{tr}}$  of sheaves for the Nisnevich topology. The category  $\mathcal{N}_k^{\text{tr}}$  is abelian and the inclusion  $\mathcal{N}_k^{\text{tr}} \hookrightarrow \mathcal{P}_k^{\text{tr}}$  has an exact left adjoint functor, which is given on the underlying presheaves by  $F \mapsto (F)_{\text{Nis}}$ , the sheafification functor [V4, 3.1.4].

**Example 4.1.4** The sheaf  $\mathbb{G}_m$  is an abelian sheaf for the étale topology, hence belongs to  $\mathcal{N}_k$ . It is canonically equipped with the structure of a presheaf with transfers, hence  $\mathbb{G}_m \in \mathcal{N}_k^{\text{tr}}$ . (This result generalizes to any sheaf represented by an affine abelian variety).

**Definition 4.1.5** Let  $Y$  be a smooth scheme in  $\mathcal{S}m/k$ . The free presheaf with transfers  $\mathbb{Z}_{\text{tr}}[Y]$  generated by  $Y$  is the abelian presheaf on  $\mathcal{S}m/k$  given by  $X \mapsto c(X, Y)$ .

The main properties of the functor  $\mathbb{Z}_{\text{tr}}[-]$  are summarized in the following result:

**Proposition 4.1.6** [SV2, §1] *Let  $Y \in \mathcal{S}m/k$  be a smooth  $k$ -scheme.*

1.  $\mathbb{Z}_{\text{tr}}[Y]$  is a sheaf for the étale topology and, in particular, is a Nisnevich sheaf.
2.  $\mathbb{Z}_{\text{tr}}[Y]$  is a presheaf with transfers and  $\mathbb{Z}_{\text{tr}}[-]$  defines a functor  $\mathcal{S}m/k \rightarrow \mathcal{N}_k^{\text{tr}}$ .
3. The functor  $\mathbb{Z}_{\text{tr}}[-]$  extends by left Kan extension to a functor  $\mathbb{Z}_{\text{tr}}[-] : \text{Shv}_{\mathcal{N}_{is}}(\mathcal{S}m/k) \rightarrow \mathcal{N}_k^{\text{tr}}$ , which is left adjoint to the forgetful functor  $\mathcal{N}_k^{\text{tr}} \rightarrow \text{Shv}_{\mathcal{N}_{is}}(\mathcal{S}m/k)$ .

The following basic structure is also required:

**Lemma 4.1.7** *Let  $X, Y \in \mathcal{S}m/k$  be smooth schemes.*

1. There is a natural morphism  $\mathbb{Z}_{\text{tr}}[X] \times \mathbb{Z}_{\text{tr}}[Y] \rightarrow \mathbb{Z}_{\text{tr}}[X \times Y]$  in the category  $\mathcal{N}_k^{\text{tr}}$ , induced by the exterior product of cycles.
2. The graph morphism induces a morphism of sheaves of sets  $X \rightarrow \mathbb{Z}_{\text{tr}}[X]$ .

To be explicit concerning the product structure, let  $U, X, Y \in \mathcal{S}m/k$  be smooth schemes; the exterior product of cycles defines a map:

$$C(U, X) \times C(U, Y) \rightarrow \mathbb{Z}[C(U \times U, X \times Y)]$$

which extends by linearity to  $\mathbb{Z}_{\text{tr}}[X](U) \times \mathbb{Z}_{\text{tr}}[Y](U) \rightarrow \mathbb{Z}_{\text{tr}}[X \times Y](U \times U)$ . The interior product is obtained by composing with the morphism  $\mathbb{Z}_{\text{tr}}[X \times Y](U \times U) \rightarrow \mathbb{Z}_{\text{tr}}[X \times Y](U)$  which is induced by the diagonal  $U \hookrightarrow U \times U$ . This corresponds to taking the intersection product with the diagonal.

The abelian sheaf  $\mathbb{Z}_{\text{tr}}[X]$  can be regarded as a pointed sheaf of sets pointed by zero, by forgetting the abelian structure. The above morphisms induce respectively morphisms of pointed sheaves of sets:

$$\begin{aligned} \mathbb{Z}_{\text{tr}}[X] \wedge \mathbb{Z}_{\text{tr}}[Y] &\rightarrow \mathbb{Z}_{\text{tr}}[X \times Y] \\ X_+ &\rightarrow \mathbb{Z}_{\text{tr}}[X]. \end{aligned}$$

**Notation 4.1.8** Suppose that  $F$  is a pointed sheaf, with structure map  $\text{Spec}(k) \rightarrow F$ , then write:

$$\mathbb{Z}_{\text{tr}}(F) := \mathbb{Z}_{\text{tr}}[F]/\mathbb{Z}_{\text{tr}}[*],$$

where the cokernel is taken in the category of abelian sheaves. If  $\mathcal{X}$  is an unpointed sheaf, then  $\mathcal{X}_+$  is the sheaf  $\mathcal{X} \amalg \text{Spec}(k)$ , pointed by the canonical inclusion; additivity of  $\mathbb{Z}_{\text{tr}}$  implies that  $\mathbb{Z}_{\text{tr}}(\mathcal{X}_+) \cong \mathbb{Z}_{\text{tr}}[\mathcal{X}]$ .

## 4.2 The adjunction between $\mathcal{H}(k)$ and $DM_-^{\text{eff}}(k)$

**Notation 4.2.1** Suppose that  $\mathcal{A}$  is an abelian category, then let  $\mathcal{C}_-\mathcal{A}$  denote the category of complexes (with differential of degree  $-1$ ) which are bounded below and let  $\mathcal{C}_{\geq 0}\mathcal{A}$  denote the subcategory of complexes which are concentrated in non-negative degree.

The derived category of  $\mathcal{C}_-\mathcal{A}$  will be denoted by  $\mathcal{D}_-\mathcal{A}$ .

Recall the following definitions:

**Definition 4.2.2** [V4, 3.1.10]

1. A presheaf with transfers,  $F \in \mathcal{P}_k^{\text{tr}}$ , is homotopy invariant if, for all  $X \in \mathcal{S}m/k$ , the morphism  $F(X) \rightarrow F(X \times \mathbb{A}^1)$  induced by the projection morphism  $X \times \mathbb{A}^1 \rightarrow X$  is an isomorphism.
2. A Nisnevich sheaf with transfers is homotopy invariant if the underlying presheaf with transfers is homotopy invariant.
3. A complex  $C_* \in \mathcal{C}_-\mathcal{N}_k^{\text{tr}}$  is motivic if its homology sheaves are homotopy invariant.
4. The triangulated category  $DM_-^{\text{eff}}(k)$  of motivic complexes is the full subcategory of  $\mathcal{D}_-\mathcal{N}_k^{\text{tr}}$  generated by objects represented by motivic complexes.

The Dold-Kan equivalence [W2, §8.4] provides an adjunction and equivalence of categories:

$$C_*^N : \Delta^{\text{op}}\mathcal{A}b \rightleftarrows \mathcal{C}_{\geq 0}\mathcal{A}b : K,$$

where  $C_*^N$  denotes the normalized chain complex. Under this correspondence, the homotopy of a simplicial abelian group is the homology of the associated chain complex in the category  $\mathcal{C}_{\geq 0}\mathcal{A}b$ .

The Dold-Kan equivalence is used to establish the adjunction between the homotopy category  $\mathcal{H}(k)$  and the category  $DM_-^{\text{eff}}(k)$ . This is analogous to the adjunction between the usual topological homotopy category  $\mathcal{H}$  and the derived category  $\mathcal{D}_-\mathcal{A}b$  of bounded below chain complexes of abelian groups.

**Theorem 4.2.3** *Suppose that  $k$  is a perfect field. There is an adjunction:*

$$M[-] : \mathcal{H}(k) \rightleftarrows DM_-^{\text{eff}}(k) : K.$$

This is proved in two steps; the first is Proposition 4.2.4, the abelian sheaf with transfers version of the Dold-Kan correspondence. The second step is to show that this induces an adjunction on passage to  $\mathbb{A}^1$ -localization.

**Proposition 4.2.4** *There is an adjunction*

$$\mathbb{L}\mathbb{Z}_{\text{tr}}[-] : \mathcal{H}_s(k) \rightleftarrows \mathcal{D}_-\mathcal{N}_k^{\text{tr}} : K.$$

The left adjoint  $\mathbb{Z}_{\text{tr}}[-] : \text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k) \rightarrow \mathcal{N}_k^{\text{tr}}$  to the forgetful functor  $\mathcal{N}_k^{\text{tr}} \rightarrow \text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$  gives rise to a diagram of adjunctions:

$$\Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k) \rightleftarrows \Delta^{\text{op}}\mathcal{N}_k^{\text{tr}} \rightleftarrows \mathcal{C}_{\geq 0}\mathcal{N}_k^{\text{tr}} \rightleftarrows \mathcal{C}_-\mathcal{N}_k^{\text{tr}}$$

where the adjunction  $C_*^N : \Delta^{\text{op}}\mathcal{N}_k^{\text{tr}} \rightleftarrows \mathcal{C}_{\geq 0}\mathcal{N}_k^{\text{tr}} : K$  is the Dold-Kan equivalence and the adjunction  $\mathcal{C}_{\geq 0}\mathcal{N}_k^{\text{tr}} \rightleftarrows \mathcal{C}_-\mathcal{N}_k^{\text{tr}}$  corresponds to the truncation of chain complexes. The composite adjunction is written:

$$\mathbb{Z}_{\text{tr}}[-] : \Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k) \rightleftarrows \mathcal{C}_-\mathcal{N}_k^{\text{tr}} : K.$$

By construction, the functor  $K : \mathcal{C}_-\mathcal{N}_k^{\text{tr}} \rightarrow \Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$  sends quasi-isomorphisms to simplicial weak equivalences, hence induces a derived functor:  $K : \mathcal{D}_-\mathcal{N}_k^{\text{tr}} \rightarrow \mathcal{H}_s(k)$ . However, the functor  $C_*^N\mathbb{Z}_{\text{tr}}[-]$  is not known to send simplicial weak equivalences to quasi-isomorphisms; it is therefore necessary to use a suitable resolution functor to define the derived functor.

Recall the definition of a split simplicial sheaf from Definition 2.8.1. Split simplicial sheaves permit inductive arguments over the skeletal filtration; at each stage the simplices which are attached are disjoint unions of sheaves represented by smooth schemes. In particular, the functor  $\mathbb{Z}_{\text{tr}}[-]$  is calculated on a split simplicial sheaf by applying the functor  $\mathbb{Z}_{\text{tr}}[-]$  termwise to (coproducts of) representable sheaves.

**Lemma 4.2.5**

1. *The functor  $C_*^N\mathbb{Z}_{\text{tr}}[-]$  sends simplicial weak equivalences between split simplicial sheaves to quasi-isomorphisms.*
2. *If  $\mathcal{X}$  is split and  $C_* \in \mathcal{C}_-\mathcal{N}_k^{\text{tr}}$ , then there is an isomorphism:*

$$[C_*^N\mathbb{Z}_{\text{tr}}[\mathcal{X}], C_*]_{\mathcal{D}_-\mathcal{N}_k^{\text{tr}}} \cong [\mathcal{X}, KC_*]_{\mathcal{H}_s(k)}.$$

**Proof:** (Indications)

(1) A morphism  $\mathcal{X} \rightarrow \mathcal{Y}$  between split simplicial sheaves induces a morphism  $C_*^N\mathbb{Z}_{\text{tr}}[\mathcal{X}] \rightarrow C_*^N\mathbb{Z}_{\text{tr}}[\mathcal{Y}]$  in  $\mathcal{C}_-\mathcal{N}_k^{\text{tr}}$ ; it is sufficient, by the Dold-Kan theorem, to show that this is an isomorphism in  $\mathcal{D}_-\mathcal{N}_k^{\text{tr}}$ . One reduces to showing that the morphism  $C_*^N\mathbb{Z}[\mathcal{X}] \rightarrow C_*^N\mathbb{Z}[\mathcal{Y}]$  is an isomorphism in  $\mathcal{D}_-\mathcal{N}_k$ , by using an induction upon the skeletal filtration of the split simplicial sheaves together with [V4, Proposition 3.1.9]. The morphism  $C_*^N\mathbb{Z}[\mathcal{X}] \rightarrow C_*^N\mathbb{Z}[\mathcal{Y}]$  is an isomorphism in  $\mathcal{D}_-\mathcal{N}_k$  since the functor  $C_*^N\mathbb{Z}[-]$  commutes with the passage to points and the functor  $C_*^N\mathbb{Z}[-]$  sends weak equivalences of simplicial sets to quasi-isomorphisms of chain complexes, by [GJ, III.2.16].

(2) This is essentially a formal consequence of (1), using the adjunction between the homotopy category of simplicial sheaves and the derived category  $\mathcal{D}_-\mathcal{N}_k$ , which is induced by  $(C_*^N\mathbb{Z}[-], K)$ . (Compare with [B] and [MV, 2.1.26]).

■



**Proof of Proposition 4.2.4:** Let  $\Phi$  denote the resolution functor by split simplicial sheaves constructed in Proposition 2.8.5, which is equipped with a natural weak equivalence  $\Phi\mathcal{X} \rightarrow \mathcal{X}$ . Lemma 4.2.5(1) implies that the functor  $C_*^N \mathbb{Z}_{\text{tr}} \Phi$  sends simplicial weak equivalences to quasi-isomorphisms. Moreover, Lemma 4.2.5(2) implies that there is a natural isomorphism, for  $\mathcal{X} \in \Delta^{\text{op}} \text{Shv}_{\mathcal{N}^{\text{tr}}}(\mathcal{S}m/k)$  and  $C_* \in \mathcal{C}_- \mathcal{N}_k^{\text{tr}}$ :

$$[\mathcal{X}, KC_*]_{\mathcal{H}_s(k)} \cong [C_*^N \mathbb{Z}_{\text{tr}}[\Phi\mathcal{X}], C_*]_{\mathcal{D}_- \mathcal{N}_k^{\text{tr}}}.$$

The result follows.  $\blacksquare$

Theorem 4.2.3 is proved by passage to  $\mathbb{A}^1$ -localization, using Proposition 4.2.9 together with the following result.

**Proposition 4.2.6** [MV, 2.2.5] *The localization functor  $\mathcal{H}_s(k) \rightarrow \mathcal{H}(k)$  is left adjoint to the functor  $\mathcal{H}(k) \rightarrow \mathcal{H}_s(k)$  induced by the inclusion of  $\mathbb{A}^1$ -local objects.*

The hypothesis that  $k$  be perfect is required since Theorem [V4, 3.1.12] is used in the proof of the following result which compares the concepts of motivic complex and  $\mathbb{A}^1$ -local simplicial sheaf via the Dold-Kan correspondence.

**Proposition 4.2.7** *Let  $k$  be a perfect field and let  $C_*$  be a complex in  $\mathcal{C}_- \mathcal{N}_k^{\text{tr}}$ . The following conditions are equivalent:*

1. *The complex  $C_*$  is motivic.*
2. *The simplicial sheaf  $K(C_*[p])$  is  $\mathbb{A}^1$ -local, for all  $p \in \mathbb{Z}$ .*
3. *For all  $U \in \mathcal{S}m/k$  the morphism  $U \times \mathbb{A}^1 \rightarrow U$  induces an isomorphism of Nisnevich hypercohomology groups:  $\mathbb{H}_{\text{Nis}}^i(U, C_*) \cong \mathbb{H}_{\text{Nis}}^i(U \times \mathbb{A}^1, C_*)$ , for all  $i$ .*

Recall the definition of the singular complex functor (which could be defined more generally on the category of abelian presheaves):

**Definition 4.2.8** [SV2, §1] Let  $\underline{C}_* : \mathcal{C}_- \mathcal{N}_k^{\text{tr}} \rightarrow \mathcal{C}_- \mathcal{N}_k^{\text{tr}}$  denote the singular complex functor  $D_* \in \mathcal{C}_- \mathcal{N}_k^{\text{tr}} \mapsto \text{Tot}\{D_*(- \times \Delta_{\text{alg}}^*)\}$ , where  $\Delta_{\text{alg}}^*$  denotes the standard cosimplicial object in  $\mathcal{S}m/k$ , where the object  $\Delta_{\text{alg}}^n$  is represented by affine space  $\mathbb{A}^n$ .

When  $k$  is a perfect field:

**Proposition 4.2.9** ([V4, Proposition 3.2.3] and [SV2, Lemma 1.4]). *Let  $k$  be a perfect field. The singular complex functor  $\underline{C}_*$  induces a functor  $\mathcal{D}_- \mathcal{N}_k^{\text{tr}} \rightarrow DM_-^{\text{eff}}(k)$  which is left adjoint to the inclusion  $DM_-^{\text{eff}}(k) \hookrightarrow \mathcal{D}_- \mathcal{N}_k^{\text{tr}}$ .*

**Proof of Theorem 4.2.3:** Suppose that the field  $k$  is perfect and write  $l : \mathcal{H}_s(k) \rightleftarrows \mathcal{H}(k) : r$  for the adjunction of Proposition 4.2.6. There is a diagram of adjunctions:

$$\begin{array}{ccccc}
 & & \xrightarrow{1_{\mathcal{H}(k)}} & & \\
 \mathcal{H}(k) & \xrightarrow{r} & \mathcal{H}_s(k) & \xrightarrow{l} & \mathcal{H}(k) \\
 & \searrow & \downarrow \text{LZ}_{\text{tr}}(-) & \uparrow K & \swarrow \\
 & & \mathcal{D}\text{-}\mathcal{N}_k^{\text{tr}} & & \\
 & \swarrow G & \downarrow & \uparrow & \searrow F \\
 & & \mathcal{D}M_-^{\text{eff}}(k) & & 
 \end{array}$$

where the functor  $G$  exists because  $KC_*$  is  $\mathbb{A}^1$ -local when  $C_*$  is a motivic complex, by Proposition 4.2.7. The factorization  $F$  exists by the universal property of  $\mathbb{A}^1$ -localization.

The functors  $F : \mathcal{H}(k) \rightleftarrows \mathcal{H}_s(k) : G$  define an adjunction, by a formal argument applied to the above diagram. ■

## Part II

# Motivic cohomology and Steenrod operations

## 5 Motivic cohomology

Throughout this section, let  $k$  be a perfect field. The following notation is used:

**Notation 5.0.10** Let  $M[-] : \mathcal{S}m/k \rightarrow DM_{-}^{\text{eff}}(k)$  denote the composite of the functor  $\mathcal{S}m/k \rightarrow \mathcal{H}(k)$  induced by the Yoneda embedding and the functor  $M[-] : \mathcal{H}(k) \rightarrow DM_{-}^{\text{eff}}(k)$  which is the  $\mathbb{A}^1$ -localization of the derived functor  $\mathbb{L}\mathbb{Z}_{\text{tr}}[-]$  of the functor  $\mathbb{Z}_{\text{tr}}[-]$ . This functor coincides with the functor defined by Voevodsky in [V4].

The (Beilinson) motivic cohomology of a smooth scheme  $X \in \mathcal{S}m/k$  with coefficients in an arbitrary motivic complex  $C_{\bullet}$  is defined to be:

$$H_{\mathcal{M}}^i(X, C_{\bullet}) := [M[X], C_{\bullet}[i]]_{DM_{-}^{\text{eff}}(k)}.$$

The adjunction between the Morel-Voevodsky homotopy category  $\mathcal{H}(k)$  and  $DM_{-}^{\text{eff}}(k)$  implies that there is an isomorphism:  $H_{\mathcal{M}}^i(X, C_{\bullet}) \cong [X, K(C_{\bullet}[i])]_{\mathcal{H}(k)}$ . Moreover, [SV2, Theorem 1.5] implies that there is an isomorphism with the Nisnevich hypercohomology group  $\mathbb{H}_{\text{Nis}}^i(X, C^{-\bullet})$ ; this corresponds to the passage between the simplicial homotopy category  $\mathcal{H}_s(k)$  and the  $\mathbb{A}^1$ -local homotopy category  $\mathcal{H}(k)$ , using the fact that the object  $K(C_{\bullet}[i])$  is  $\mathbb{A}^1$ -local.

In addition, Voevodsky's results on the cohomology of presheaves with transfers [V] allow these groups to be related to Zariski hypercohomology, as suggested by Beilinson's approach to defining motivic cohomology. (See for example [SV2, §3]).

From this point of view, it is necessary to choose suitable *motivic chains*  $\mathbb{Z}(n) \in \mathcal{C}_{-}\mathcal{N}_k^{\text{tr}}$ , for  $n \geq 0$ , which give rise to a bigraded cohomology theory.

### 5.1 Sheaves of relative cycles

The most direct way of defining integral motivic cohomology from the homotopy theoretic point of view is to follow the approach of Friedlander and Voevodsky [FV] in terms of sheaves of relative cycles. See Section 5.3 for the definition via the motivic chains of Suslin-Voevodsky.

**Definition 5.1.1** [FV, §2] Let  $U \in \mathcal{S}m/k$  be a smooth  $k$ -scheme and let  $X \in \mathcal{S}ch/k$  be a  $k$ -scheme. For an integer  $r \geq 0$ , let  $z_{\text{equi}}(X, r)(U)$  denote the free abelian group generated by cycles  $Z \hookrightarrow X \times U$  for which the composite  $p$  in the

diagram

$$\begin{array}{ccc} Z & \hookrightarrow & X \times U \\ & \searrow p & \downarrow p_U \\ & & U \end{array}$$

is of finite type of relative dimension  $r$ , dominant over a connected component of  $U$ .

The above construction defines presheaves on the category of smooth schemes:

**Proposition 5.1.2** [FV, §2] *Let  $X \in \mathcal{S}ch/k$  be a  $k$ -scheme, then the cycle morphisms associated to pullback induce a presheaf structure:*

$$z_{\text{equi}}(X, r) : (\mathcal{S}m/k)^{\text{op}} \rightarrow \mathcal{A}b.$$

The following result summarizes some of the basic properties of the relative cycle construction:

**Proposition 5.1.3** *Let  $r$  be a non-negative integer.*

1. [V4, §4.2] *Let  $X \in \mathcal{S}ch/k$  be a scheme, then  $z_{\text{equi}}(X, r)$  is naturally a Nisnevich sheaf with transfers.*
2. *Let  $f : X \rightarrow Y$  be a proper morphism in  $\mathcal{S}ch/k$ , then proper push-forward of cycles induces a morphism of sheaves:*

$$f_* : z_{\text{equi}}(X, r) \rightarrow z_{\text{equi}}(Y, r).$$

3. *Let  $q : W \rightarrow X$  be a flat morphism in  $\mathcal{S}ch/k$  of relative dimension  $n$ , then flat pull-back induces morphisms of sheaves:*

$$q^* : z_{\text{equi}}(X, r) \rightarrow z_{\text{equi}}(W, n + r).$$

A key property is the localization sequence which is a consequence of the following result:

**Proposition 5.1.4** [FV, Theorem 5.11] *Let  $k$  be a field admitting resolution of singularities. Let  $X \in \mathcal{S}ch/k$  be a scheme with closed subscheme  $Z \hookrightarrow X$ . For any non-negative integer  $r$ , there is an exact sequence of Nisnevich sheaves with transfers:*

$$0 \rightarrow z_{\text{equi}}(Z, r) \rightarrow z_{\text{equi}}(X, r) \rightarrow z_{\text{equi}}(U, r) \rightarrow Q \rightarrow 0$$

*such that the sheaf with transfers satisfies  $\underline{C}_*Q \simeq 0$  in  $\mathcal{C}\text{-}\mathcal{N}_k^{\text{tr}}$ .*

Of particular importance is the following Corollary, which uses the fact that  $\mathbb{P}^1$  is proper:

**Corollary 5.1.5** *Let  $k$  be a field admitting resolution of singularities and let  $X \in \text{Sch}/k$  be a scheme, then there is a weak equivalence in  $\mathcal{C}\text{-}\mathcal{N}_k^{\text{tr}}$ :*

$$\underline{\mathcal{C}}_* z_{\text{equi}}(X \times \mathbb{P}^1, r) \simeq \underline{\mathcal{C}}_* z_{\text{equi}}(X, r) \oplus \underline{\mathcal{C}}_* z_{\text{equi}}(X \times \mathbb{A}^1, r).$$

The sheaves of cycles which are of interest here are given by the case  $r = 0$ ; this gives rise to a version of motives ‘with compact supports’, a name which is justified by the following Lemma and the existence of a localization sequence.

**Lemma 5.1.6** *Let  $X \in \text{Sm}/k$  be a smooth, proper  $k$ -scheme, then there is a natural isomorphism  $z_{\text{equi}}(X, 0) \cong \mathbb{Z}_{\text{tr}}[X]$ .*

However, it should be noted that the proof that integral motivic cohomology defines a bigraded cohomology theory in the sense of Section 3.4 requires passage through  $z_{\text{equi}}(X \times -, 1)$ .

## 5.2 Integral motivic cohomology

The following is taken as the definition of integral motivic cohomology, restricted to the bidegrees of the form  $(2*, *)$ .

**Definition 5.2.1** Let  $X \in \text{Sm}/k$  be a smooth scheme and let  $n \geq 0$  be an integer. The integral motivic cohomology of bidegree  $(2n, n)$  is defined by:

$$H^{2n, n}(X, \mathbb{Z}) := [X, KC_* z_{\text{equi}}(\mathbb{A}^n, 0)]_{\mathcal{H}_s(k)}$$

where the morphisms are taken in the unpointed homotopy category and  $K$  denotes the Kan functor from chain complexes to simplicial objects.

The fact that this extends to a bigraded cohomology theory is equivalent to the following statement:

**Theorem 5.2.2** *Let  $k$  be a field admitting resolution of singularities and let  $X \in \text{Sm}/k$  be a smooth scheme. For any integer  $n \geq 0$ , there is a natural isomorphism:*

$$H^{2(n+1), (n+1)}(X \times \mathbb{P}^1, \mathbb{Z}) \cong H^{2n, n}(X, \mathbb{Z}) \oplus H^{2(n+1), (n+1)}(X, \mathbb{Z}).$$

**Proof:** (Indications) Throughout the proof, the Kan functor  $K$  is omitted from the notation. Consider the cohomology group

$$\mathcal{G} := [X \times \mathbb{P}^1, \underline{\mathcal{C}}_* z_{\text{equi}}(\mathbb{A}^{n+1}, 0)]_{\mathcal{H}_s(k)}.$$

The key step is to use a Brown-Gersten descent argument and the Friedlander-Lawson-Voevodsky duality theorem [FV, Theorem 7.4] to deduce (as in the proof of [FV, Theorem 8.2]) that  $\mathcal{G}$  is isomorphic to

$$[X, \underline{\mathcal{C}}_* z_{\text{equi}}(\mathbb{P}^1 \times \mathbb{A}^{n+1}, 1)]_{\mathcal{H}_s(k)}.$$

The localization property (Corollary 5.1.5) of the sheaves of relative cycles yields a weak equivalence in  $\mathcal{C}_-\mathcal{N}_k^{\text{tr}}$ :

$$\underline{\mathcal{C}}_*z_{\text{equi}}(\mathbb{P}^1 \times \mathbb{A}^{n+1}, 1) \simeq \underline{\mathcal{C}}_*z_{\text{equi}}(\mathbb{A}^{n+1}, 1) \oplus \underline{\mathcal{C}}_*z_{\text{equi}}(\mathbb{A}^1 \times \mathbb{A}^{n+1}, 1).$$

Applying the duality argument in reverse yields an isomorphism:

$$\mathcal{G} \cong [X \times \mathbb{A}^1, \underline{\mathcal{C}}_*z_{\text{equi}}(\mathbb{A}^n, 0) \oplus \underline{\mathcal{C}}_*z_{\text{equi}}(\mathbb{A}^{n+1}, 0)]_{\mathcal{H}_s(k)}.$$

The terms on the right hand side are  $\mathbb{A}^1$ -local, hence this implies the result.  $\blacksquare$

### 5.3 Motivic chains and Eilenberg MacLane spaces

**Definition 5.3.1** let  $n$  be a non-negative integer and let  $k$  be a field. The motivic integral Eilenberg MacLane space  $K_{\mathcal{M}}(\mathbb{Z}, n)$  is the isomorphism class of objects in the  $\mathbb{A}^1$ -local pointed homotopy category  $\mathcal{H}_{\bullet}(k)$  which represent reduced motivic cohomology  $\tilde{H}^{2n, n}(-, \mathbb{Z})$ .

The object  $\underline{\mathcal{C}}_*z_{\text{equi}}(\mathbb{A}^n, 0) \in \mathcal{C}_{\geq 0}\mathcal{N}_k^{\text{tr}}$  yields a pointed simplicial sheaf in  $\Delta^{\text{op}}\text{Shv}_{\mathcal{N}_{\text{is}}}(\mathcal{S}m/k)_{\bullet}$  by applying the Kan functor; this is one model for the motivic Eilenberg-MacLane space  $K_{\mathcal{M}}(\mathbb{Z}, n)$ . There exist other models, defined in the category  $DM_{-}^{\text{eff}}(k)$ , which are also useful. For the convenience of the reader, the definition of the Suslin-Voevodsky motivic chains is recalled:

**Definition 5.3.2** [SV2, §3]

1. For  $n \geq 0$ , define  $\mathbb{Z}_{\text{tr}}(\mathbb{G}_m^{\wedge n})$  to be the abelian sheaf quotient  $\mathbb{Z}_{\text{tr}}[\mathbb{G}_m^{\times n}]/\mathcal{D}_n$ , where  $\mathcal{D}_n$  is the sum of the images of the homomorphisms  $\mathbb{Z}_{\text{tr}}[\mathbb{G}_m^{\times n-1}] \rightarrow \mathbb{Z}_{\text{tr}}[\mathbb{G}_m^{\times n}]$ , induced by the embeddings  $\mathbb{G}_m^{\times n-1} \hookrightarrow \mathbb{G}_m^{\times n}$ , given by  $* \rightarrow \mathbb{G}_m, * \mapsto 1$  on the  $i^{\text{th}}$  factor. The sheaf  $\mathcal{D}_n$  is a direct summand of  $\mathbb{Z}_{\text{tr}}[\mathbb{G}_m^{\times n}]$ , so that  $\mathbb{Z}_{\text{tr}}(\mathbb{G}_m^{\wedge n})$  is a direct summand of  $\mathbb{Z}_{\text{tr}}[\mathbb{G}_m^{\times n}]$ .
2. The motivic complex  $\mathbb{Z}(n)$  of weight  $n$  is the complex of (étale) abelian sheaves:

$$\mathbb{Z}(n) := \{\underline{\mathcal{C}}_*\mathbb{Z}_{\text{tr}}(\mathbb{G}_m^{\wedge n})\}[-n]$$

By construction,  $\mathbb{Z}(n) \in \mathcal{C}_-\mathcal{N}_k^{\text{tr}}$  can be regarded as an object of  $DM_{-}^{\text{eff}}(k)$ .

**Proposition 5.3.3** *Let  $k$  be a field admitting resolution of singularities. The following Nisnevich sheaves with transfers give rise to isomorphic objects in  $DM_{-}^{\text{eff}}(k)$  under the composite functor*

$$\mathcal{N}_k^{\text{tr}} \hookrightarrow \mathcal{C}_-\mathcal{N}_k^{\text{tr}} \xrightarrow{\underline{\mathcal{C}}_*} DM_{-}^{\text{eff}}(k)$$

where the first functor is the inclusion as a complex concentrated in degree zero.

1.  $z_{\text{equi}}(\mathbb{A}^n, 0)$

2.  $\mathbb{Z}_{\text{tr}}[\mathbb{P}^n]/\mathbb{Z}_{\text{tr}}[\mathbb{P}^{n-1}]$
3.  $\mathbb{Z}_{\text{tr}}[\mathbb{A}^n]/\mathbb{Z}_{\text{tr}}[(\mathbb{A}^n - \{0\})]$ .

These objects are isomorphic to  $\mathbb{Z}(n)[2n] \cong \{\underline{C}_* \mathbb{Z}_{\text{tr}}(\mathbb{G}_m^{\wedge n})\}[n]$  in  $DM_-^{\text{eff}}(k)$ .

**Proof:** The equivalence of the objects  $\underline{C}_* z_{\text{equi}}(\mathbb{A}^n, 0)$  and  $\underline{C}_*(\mathbb{Z}_{\text{tr}}[\mathbb{P}^n]/\mathbb{Z}_{\text{tr}}[\mathbb{P}^{n-1}])$  follows from the localization sequence associated to the inclusion  $\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$  as follows. There is an exact triangle in  $DM_-^{\text{eff}}(k)$ :

$$\underline{C}_* z_{\text{equi}}(\mathbb{P}^{n-1}, 0) \rightarrow \underline{C}_* z_{\text{equi}}(\mathbb{P}^n, 0) \rightarrow \underline{C}_* z_{\text{equi}}(\mathbb{A}^n, 0) \rightarrow .$$

Moreover, since  $\mathbb{P}^{n-1}, \mathbb{P}^n$  are proper, the first morphism identifies with the morphism  $\underline{C}_* \mathbb{Z}_{\text{tr}}[\mathbb{P}^{n-1}] \rightarrow \underline{C}_* \mathbb{Z}_{\text{tr}}[\mathbb{P}^n]$ ; the cofibre of this morphism is isomorphic to  $\underline{C}_*(\mathbb{Z}_{\text{tr}}[\mathbb{P}^n]/\mathbb{Z}_{\text{tr}}[\mathbb{P}^{n-1}])$  in  $DM_-^{\text{eff}}(k)$ .

The equivalence of the objects  $\underline{C}_*(\mathbb{Z}_{\text{tr}}[\mathbb{P}^n]/\mathbb{Z}_{\text{tr}}[\mathbb{P}^{n-1}])$  and  $\underline{C}_*(\mathbb{Z}_{\text{tr}}[\mathbb{A}^n]/\mathbb{Z}_{\text{tr}}[(\mathbb{A}^n - \{0\})])$  follows from an excision argument. There is a cartesian square

$$\begin{array}{ccc} (\mathbb{A}^n - \{0\}) & \longrightarrow & \mathbb{A}^n \\ \downarrow & & \downarrow \\ (\mathbb{P}^n - \{0\}) & \longrightarrow & \mathbb{P}^n \end{array}$$

which is derived from the evident Zariski open covering of  $\mathbb{P}^n$ . This induces a cocartesian square in  $\mathcal{N}_k^{\text{tr}}$ :

$$\begin{array}{ccc} \mathbb{Z}_{\text{tr}}[(\mathbb{A}^n - \{0\})] & \longrightarrow & \mathbb{Z}_{\text{tr}}[\mathbb{A}^n] \\ \downarrow & & \downarrow \\ \mathbb{Z}_{\text{tr}}[(\mathbb{P}^n - \{0\})] & \longrightarrow & \mathbb{Z}_{\text{tr}}[\mathbb{P}^n] \end{array}$$

in particular the cofibres of the horizontal morphisms are isomorphic in  $\mathcal{N}_k^{\text{tr}}$ . Finally, the projection  $(\mathbb{P}^n - \{0\}) \rightarrow \mathbb{P}^{n-1}$  has the structure of a vector bundle for the Zariski topology, hence induces an isomorphism  $\underline{C}_* \mathbb{Z}_{\text{tr}}[(\mathbb{P}^n - \{0\})] \xrightarrow{\cong} \underline{C}_* \mathbb{Z}_{\text{tr}}[\mathbb{P}^{n-1}]$  in  $DM_-^{\text{eff}}(k)$ ; the result follows.

The equivalence with the object  $\mathbb{Z}(n)[n]$  in  $DM_-^{\text{eff}}(k)$  is proved by using the object  $\underline{C}_*(\mathbb{Z}_{\text{tr}}[\mathbb{P}^n]/\mathbb{Z}_{\text{tr}}[\mathbb{P}^{n-1}])$ . Consider the standard Zariski open cover  $\mathcal{U}$  of  $\mathbb{P}^n$  by  $U_i \cong \mathbb{A}^n$ , for  $0 \leq i \leq n+1$ , where in terms of co-ordinates for points over an algebraically closed field, the open set corresponds to  $\{[x_0, \dots, x_n] | x_i \neq 0\}$ . Consider  $U_0$  as the distinguished open, and write  $\mathcal{U}'$  for the restricted open covering of  $(\mathbb{P}^n - \{0\})$ , where 0 is the point corresponding to  $[1, 0, \dots, 0]$ .

Mayer-Vietoris (associated to the Zariski open cover  $\mathcal{U}$ ) for the sheaves  $\mathbb{Z}_{\text{tr}}[U]$  (regarded as functorial in  $U$ ) implies that there is a Čech complex  $\mathbb{Z}_{\text{tr}}[\check{\mathcal{U}}] \in \mathcal{C}_- \mathcal{N}_k^{\text{tr}}$  associated to the open cover, which becomes trivial in  $DM_-^{\text{eff}}(k)$ .

The cover  $\mathcal{U}'$  of  $(\mathbb{P}^n - \{0\})$  likewise gives rise to a complex  $\mathbb{Z}_{\text{tr}}[\check{\mathcal{U}}'] \in \mathcal{C}_- \mathcal{N}_k^{\text{tr}}$  which becomes trivial in  $DM_-^{\text{eff}}(k)$  and there is a morphism of complexes:

$$\mathbb{Z}_{\text{tr}}[\check{\mathcal{U}}'] \rightarrow \mathbb{Z}_{\text{tr}}[\check{\mathcal{U}}]$$

and the cokernel again induces a trivial object in  $DM_-^{\text{eff}}(k)$ .

The quotient complex can be identified explicitly as a complex:

$$\begin{aligned} \mathbb{Z}_{\text{tr}}[\mathbb{G}_m^{\times n}] &\rightarrow \bigoplus \mathbb{Z}_{\text{tr}}[\mathbb{A}^1 \times \mathbb{G}_m^{\times n-1}] \rightarrow \bigoplus \mathbb{Z}_{\text{tr}}[\mathbb{A}^2 \times \mathbb{G}_m^{\times n-2}] \rightarrow \dots \\ &\dots \rightarrow \mathbb{Z}[\mathbb{A}^n] \rightarrow \mathbb{Z}_{\text{tr}}[\mathbb{P}^n]/\mathbb{Z}_{\text{tr}}[\mathbb{P}^{n-1}]. \end{aligned}$$

After passage to  $DM_-^{\text{eff}}(k)$ , there is an equivalence  $\underline{C}_* \mathbb{Z}_{\text{tr}}[\mathbb{A}^t \times \mathbb{G}_m^{\times n-t}] \cong \underline{C}_* \mathbb{Z}_{\text{tr}}[\mathbb{G}_m^{\times n-t}]$ ; the result follows by a filtration argument by using the decomposition [SV2, Equation (3.0), §3] of the products  $\mathbb{Z}_{\text{tr}}[\mathbb{G}_m^{\times n-t}]$ .  $\blacksquare$

**Remark 5.3.4** Heuristically, this result follows directly from the weak equivalence

$$B\mathbb{G}_m \simeq_{\mathcal{H}_\bullet(k)} \mathbb{P}^\infty$$

in the  $\mathbb{A}^1$ -local homotopy category, which is given by [MV, Proposition 4.3.7]. (Here  $B$  denotes the simplicial classifying space construction applied to the sheaf  $\mathbb{G}_m$  considered as an abelian monoid).

The technical point is that the derived functor of  $\mathbb{Z}_{\text{tr}}$  is obtained with respect to the simplicial structure and not the  $\mathbb{A}^1$ -local structure, so this is not an immediate consequence of the material which has been presented here.

## 5.4 The product structure

The product structure in motivic cohomology is induced by the intersection product of cycles. In the case of the model  $\underline{C}_* z_{\text{equi}}(\mathbb{A}^n, 0)$  for the Eilenberg-MacLane space  $K_{\mathcal{M}}(\mathbb{Z}, n)$ , this product is derived from:

**Proposition 5.4.1** [FV, page 117] *Let  $k$  be a field, let  $X, X' \in \text{Sch}/k$  be schemes over  $k$  and let  $U \in \text{Sm}/k$  be a smooth scheme. There is a pairing of abelian presheaves:*

$$\times : z_{\text{equi}}(X, r) \otimes z_{\text{equi}}(X', r') \rightarrow z_{\text{equi}}(X \times X', r + r')$$

which sends a pair of cycles  $Z \hookrightarrow X \times U$ ,  $Z' \hookrightarrow X' \times U$  to the cycle associated to  $Z \times_U W \hookrightarrow X \times Y \times U$ .

In particular, this induces a morphism:

$$z_{\text{equi}}(\mathbb{A}^m, 0) \otimes z_{\text{equi}}(\mathbb{A}^n, 0) \rightarrow z_{\text{equi}}(\mathbb{A}^{m+n}, 0)$$

which is the basis for the product structure.

**Lemma 5.4.2** *Let  $k$  be a field. There are product morphisms:*

$$(K\underline{C}_* z_{\text{equi}}(\mathbb{A}^m, 0)) \wedge (K\underline{C}_* z_{\text{equi}}(\mathbb{A}^n, 0)) \rightarrow K\underline{C}_* z_{\text{equi}}(\mathbb{A}^{m+n}, 0)$$

in  $\Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\text{Sm}/k)_\bullet$  which induce morphisms:

$$K_{\mathcal{M}}(\mathbb{Z}, m) \wedge K_{\mathcal{M}}(\mathbb{Z}, n) \rightarrow K_{\mathcal{M}}(\mathbb{Z}, m + n)$$

in  $\mathcal{H}_\bullet(k)$  which are associative in the obvious sense.



These structure morphisms in the homotopy category yield the product structure for motivic cohomology as follows:

**Definition 5.4.3** The external product for integral motivic cohomology is induced by the product morphisms of Lemma 5.4.2. Let  $\mathcal{X}, \mathcal{Y}$  be pointed simplicial sheaves in  $\Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)_{\bullet}$ , then there is an external product:

$$\tilde{H}^{2m,m}(\mathcal{X}, \mathbb{Z}) \otimes \tilde{H}^{2n,n}(\mathcal{Y}, \mathbb{Z}) \rightarrow \tilde{H}^{2(m+n),(m+n)}(\mathcal{X} \wedge \mathcal{Y}, \mathbb{Z})$$

which is induced by sending a pair of morphisms  $\mathcal{X} \rightarrow K_{\mathcal{M}}(\mathbb{Z}, m)$  and  $\mathcal{Y} \rightarrow K_{\mathcal{M}}(\mathbb{Z}, n)$  in the pointed homotopy category  $\Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)_{\bullet}$  to the composite

$$\mathcal{X} \wedge \mathcal{Y} \rightarrow K_{\mathcal{M}}(\mathbb{Z}, m) \wedge K_{\mathcal{M}}(\mathbb{Z}, n) \rightarrow K_{\mathcal{M}}(\mathbb{Z}, m+n).$$

The product extends to other bidegrees by the usual suspension techniques.

The product can also be obtained via other models for the motivic Eilenberg-MacLane spaces. For technical reasons it is sometimes useful to work with the model provided by the sheaf  $\mathbb{Z}_{\text{tr}}[\mathbb{A}^n]/\mathbb{Z}_{\text{tr}}[(\mathbb{A}^n - \{0\})]$ . There is a morphism

$$\mathbb{Z}_{\text{tr}}[\mathbb{A}^m] \times \mathbb{Z}_{\text{tr}}[\mathbb{A}^n] \rightarrow \mathbb{Z}_{\text{tr}}[\mathbb{A}^{m+n}]$$

which is given by Lemma 4.1.7.

**Lemma 5.4.4** *The product of cycles induces a product morphism:*

$$\mathbb{Z}_{\text{tr}}[\mathbb{A}^m]/\mathbb{Z}_{\text{tr}}[(\mathbb{A}^m - \{0\})] \wedge \mathbb{Z}_{\text{tr}}[\mathbb{A}^n]/\mathbb{Z}_{\text{tr}}[(\mathbb{A}^n - \{0\})] \rightarrow \mathbb{Z}_{\text{tr}}[\mathbb{A}^{m+n}]/\mathbb{Z}_{\text{tr}}[(\mathbb{A}^{m+n} - \{0\})]$$

in the category  $\Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)_{\bullet}$ .

**Proof:** It suffices to check that the sub-sheaves  $\mathbb{Z}_{\text{tr}}[(\mathbb{A}^m - \{0\})] \times \mathbb{Z}_{\text{tr}}[\mathbb{A}^n]$  and  $\mathbb{Z}_{\text{tr}}[\mathbb{A}^m] \times \mathbb{Z}_{\text{tr}}[(\mathbb{A}^n - \{0\})]$  map to  $\mathbb{Z}_{\text{tr}}[(\mathbb{A}^{m+n} - \{0\})] \hookrightarrow \mathbb{Z}_{\text{tr}}[\mathbb{A}^{m+n}]$  under the product map; this is straightforward.  $\blacksquare$

**Remark 5.4.5** It is necessary to check that the product obtained is equivalent in the homotopy category  $\mathcal{H}_{\bullet}(k)$  to the product defined in Definition 5.4.3.

## 5.5 The tensor product in $DM_{-}^{\text{eff}}(k)$

The category  $DM_{-}^{\text{eff}}(k)$  is equipped with a tensor product, which is the localization of a derived tensor product defined on the category  $\mathcal{D}_{-}\mathcal{N}_k^{\text{tr}}$ .

**Lemma 5.5.1** [SV2, Lemma 2.6] *Let  $k$  be a field. There exists a symmetric monoidal structure  $\otimes_{\text{tr}}^L : \mathcal{D}_{-}\mathcal{N}_k^{\text{tr}} \times \mathcal{D}_{-}\mathcal{N}_k^{\text{tr}} \rightarrow \mathcal{D}_{-}\mathcal{N}_k^{\text{tr}}$  which is the derived functor of  $\otimes_{\text{tr}} : \mathcal{N}_k^{\text{tr}} \times \mathcal{N}_k^{\text{tr}} \rightarrow \mathcal{N}_k^{\text{tr}}$  which satisfies the identity  $\mathbb{Z}_{\text{tr}}[X] \otimes_{\text{tr}} \mathbb{Z}_{\text{tr}}[Y] \rightarrow \mathbb{Z}_{\text{tr}}[X \times Y]$ .*

Suslin and Voevodsky establish that this symmetric monoidal structure passes to the  $\mathbb{A}^1$ -localization; the field  $k$  is assumed to be perfect, since Proposition 4.2.7 is applied.

**Proposition 5.5.2** [SV2, Proposition 2.8] *Let  $k$  be a perfect field. There exists a symmetric monoidal structure  $(\otimes, \mathbb{Z})$  on  $DM_-^{\text{eff}}(k)$  which satisfies the following properties:*

1. *Let  $A \in DM_-^{\text{eff}}(k)$  be a motivic complex. The functors  $A \otimes -$  and  $- \otimes A$  take exact triangles in  $DM_-^{\text{eff}}(k)$  to exact triangles.*
2. *The functors  $A \otimes -$  and  $- \otimes A$  preserve the direct sums which exist in  $DM_-^{\text{eff}}(k)$ .*
3. *The localization functor defines a functor of symmetric monoidal categories:*

$$\underline{C}_* : (\mathcal{D}_- \mathcal{N}_k^{\text{tr}}, \otimes_{\text{tr}}^L) \rightarrow (DM_-^{\text{eff}}(k), \otimes)$$

*in the sense that, for  $A_\bullet, B_\bullet \in \mathcal{C}_- \mathcal{N}_k^{\text{tr}}$ , there is a natural isomorphism:*

$$\underline{C}_*(A_\bullet \otimes_{\text{tr}}^L B_\bullet) \xrightarrow{\cong} \underline{C}_*(A_\bullet) \otimes \underline{C}_*(B_\bullet).$$

**Corollary 5.5.3** *Let  $k$  be a perfect field. The derived functor  $M[-] : \mathcal{H}(k) \rightarrow DM_-^{\text{eff}}(k)$  is a functor of symmetric monoidal categories: for each  $\mathcal{X}, \mathcal{Y} \in \Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$ , there is a natural isomorphism in  $DM_-^{\text{eff}}(k)$ :*

$$M[\mathcal{X} \times \mathcal{Y}] \rightarrow M[\mathcal{X}] \otimes M[\mathcal{Y}].$$

**Proof:** This is a straightforward consequence of the explicit construction of the derived functor of  $\mathbb{Z}_{\text{tr}}[-]$  by using split simplicial sheaves.  $\blacksquare$

**Remark 5.5.4** It should be stressed that the morphism  $K : DM_-^{\text{eff}}(k) \rightarrow \mathcal{H}(k)$  will not respect the symmetric monoidal structures. (See the remark on page 210 of [V4]).

The tensor product in  $DM_-^{\text{eff}}(k)$  can also be used to construct the product in motivic cohomology, using the following result:

**Lemma 5.5.5** [SV2, Lemma 3.2] *Let  $m, n$  be non-negative integers, then there is a natural quasi-isomorphism in  $DM_-^{\text{eff}}(k)$ :*

$$\mathbb{Z}(m) \otimes \mathbb{Z}(n) \cong \mathbb{Z}(m+n).$$

From this point of view, the product of classes in motivic cohomology of simplicial sheaves  $\mathcal{X}, \mathcal{Y} \in \Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$  given by morphisms in  $DM_-^{\text{eff}}(k)$

$$\begin{aligned} \alpha_1 : M[\mathcal{X}_1] &\rightarrow \mathbb{Z}(a_1)[b_1] \\ \alpha_2 : M[\mathcal{X}_2] &\rightarrow \mathbb{Z}(a_2)[b_2] \end{aligned}$$

is given by the composite:

$$M[\mathcal{X}_1 \times \mathcal{X}_2] \xrightarrow{\cong} M[\mathcal{X}_1] \otimes M[\mathcal{X}_2] \xrightarrow{\alpha_1 \otimes \alpha_2} \mathbb{Z}(a_1)[b_1] \otimes \mathbb{Z}(a_2)[b_2] \cong \mathbb{Z}(a_1+a_2)[b_1+b_2].$$

**Remark 5.5.6** It is again necessary to verify that this product is compatible with that given in Definition 5.4.3.

## 6 The algebraic Dold-Thom theorem

The explicit connection between motivic cohomology and singular cohomology in algebraic topology is provided by the algebraic Dold-Thom theorem, which was established by Suslin and Voevodsky. This material is essential for the *calculation* of the motivic Steenrod algebra.

### 6.1 Symmetric products and group completion

Recall the following basic notions concerning the quotients of schemes by the action of a finite group:

**Definition 6.1.1** Let  $X \in \mathcal{S}ch/k$  be a scheme which admits an action by the finite group  $G$ ; the action is said to be admissible if every orbit of points of  $X$  is contained in some  $G$ -invariant open affine sub-scheme of  $X$ .

The following is standard:

**Lemma 6.1.2** *Suppose that the finite group  $G$  acts admissibly upon the scheme  $X \in \mathcal{S}ch/k$  and that  $H \subset G$  is a sub-group, then:*

1. *the quotient schemes  $X/G$  and  $X/H$  exist*
2. *There is an induced morphism  $X/H \rightarrow X/G$  which is finite and surjective.*

**Example 6.1.3** Suppose that the symmetric group  $\mathfrak{S}_d$  acts admissibly upon the product  $X^d$ , where  $X \in \mathcal{S}ch/k$  (for example this happens if  $X$  is a quasi-projective scheme). Then the quotient scheme  $X^d/\mathfrak{S}_d$  exists and is denoted  $S^d(X)$ ; this is the  $d^{\text{th}}$  symmetric product of  $X$ .

The following result is the algebraic version of the Dold-Thom theorem [DT]:

**Theorem 6.1.4** [SV, Theorem 6.8] *Let  $Z \in \mathcal{S}ch/k$  be a scheme such that any finite subset of  $Z$  is contained in an open affine set. Let  $S$  be a normal, connected scheme, then there is an isomorphism:*

$$\mathbb{Z}_{\text{tr}}[Z](S)\left[\frac{1}{p}\right] \cong \text{hom}(S, \Pi_{d \geq 0} S^d(Z))^+ \left[\frac{1}{p}\right],$$

where the  $(\cdot)^+$  denotes the group completion of the monoid structure induced by the monoid structure of  $\Pi_{d \geq 0} S^d(Z)$  and  $p$  is the exponential characteristic of  $k$ .

In particular, this implies:

**Corollary 6.1.5** *Let  $Z \in \mathcal{S}ch/k$  be a scheme such that any finite subset of  $Z$  is contained in an open affine set and let  $k$  be a field of characteristic zero, then there is an isomorphism of Zariski sheaves on  $\mathcal{S}m/k$ :*

$$\mathbb{Z}_{\text{tr}}[Z] \xrightarrow{\cong} (\Pi_{d \geq 0} S^d(Z))^+$$

where the group completion is formed in the category of sheaves.

The group completion of a monoid in the category of simplicial sheaves is considered in [MV, §4.1]; the simplicial category structure on the category of simplicial sheaves is used to define the group completion, so that the argument reduces, with respect to the *simplicial* model structure, to the classical case of group completion of a simplicial monoid [Q] by passage to the points of the site. However, the connectivity hypothesis is too restrictive, and the subtle part of the argument is the passage from the simplicial case [MV, Proposition 4.1.9] to the  $\mathbb{A}^1$ -local result [MV, Theorem 4.1.10].

This requires the notion of  $\mathbb{A}^1$ -connectivity, which is the  $\mathbb{A}^1$ -local version of connectivity. In particular, it will be necessary to know that the sheaves represented by symmetric products of  $\mathbb{P}^n$  are  $\mathbb{A}^1$ -connected.

**Definition 6.1.6**

1. The functor  $\pi_0 : \Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k) \rightarrow \text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$  is the left adjoint to the constant simplicial structure functor  $\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k) \rightarrow \Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$ . Explicitly the sheaf  $\pi_0\mathcal{X}$  associated to  $\mathcal{X} \in \Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$  is the sheaf colimit of the diagram  $\mathcal{X}_1 \rightrightarrows \mathcal{X}_0$ ; equivalently,  $\pi_0\mathcal{X}$  is the sheaf associated to the presheaf  $U \mapsto \pi_0(\mathcal{X}(U))$ , where  $\pi_0$  here refers to the usual functor on the category of simplicial sets.
2. A simplicial sheaf  $\mathcal{X}$  is said to be connected if  $\pi_0\mathcal{X}$  is the trivial sheaf (namely the constant sheaf which is a singleton set on each scheme).
3. A scheme  $X \in \mathcal{S}m/k$  is said to be connected if the simplicial sheaf which it represents is connected.

**Remark 6.1.7**

1. Suppose that  $\mathcal{X}$  is a simplicial sheaf with constant simplicial structure, then  $\pi_0\mathcal{X} \cong \mathcal{X}_0$ ; in particular,  $\mathcal{X}$  is connected if and only if  $\mathcal{X}$  is the trivial sheaf.
2. There is a natural surjection of sheaves  $\mathcal{X}_0 \rightarrow \pi_0\mathcal{X}$ .

**Notation 6.1.8** Let  $\mathcal{X} \rightarrow L_{\mathbb{A}^1}\mathcal{X}$  denote a functorial  $\mathbb{A}^1$ -local resolution in the category  $\Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$ , so that  $L_{\mathbb{A}^1}\mathcal{X}$  is  $\mathbb{A}^1$ -local and the morphism is an  $\mathbb{A}^1$ -weak equivalence.

**Definition 6.1.9** [MV, §3.2.1] A simplicial sheaf  $\mathcal{X} \in \Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$  is  $\mathbb{A}^1$ -connected if the  $\mathbb{A}^1$ -localization  $L_{\mathbb{A}^1}\mathcal{X}$  is connected. If  $X$  is a smooth scheme then  $X$  is said to be  $\mathbb{A}^1$ -connected if the sheaf represented by  $X$  with constant simplicial structure is  $\mathbb{A}^1$ -connected.

It is straightforward to check that the definition is independent of the choice of  $\mathbb{A}^1$ -local resolution. In particular, an  $\mathbb{A}^1$ -local sheaf is connected if and only if it is connected.

**Example 6.1.10** The sheaf  $\mathbb{G}_m$  is  $\mathbb{A}^1$ -local, since  $\mathcal{O}^*$  is homotopy invariant for regular schemes. It follows that  $\mathbb{G}_m$  is **not**  $\mathbb{A}^1$ -connected.

An understanding of an explicit construction of an  $\mathbb{A}^1$ -localization functor is necessary to obtain examples of  $\mathbb{A}^1$ -connected sheaves. Recall the following definition, which is the analogue of Definition 4.2.8:

**Definition 6.1.11** [MV, §2.3.2] Let  $\text{Sing}_*^{\mathbb{A}^1} : \Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k) \rightarrow \Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$  denote the singular functor defined by

$$\mathcal{X} \mapsto \text{diag}\{\text{Hom}(\Delta_{\text{alg}}^*, \mathcal{X})\}$$

where  $\Delta_{\text{alg}}^*$  denotes the standard algebraic cosimplicial scheme with  $\Delta_{\text{alg}}^n$  represented by affine space  $\mathbb{A}^n$  and  $\text{Hom}$  is the internal hom in the category of sheaves applied degreewise.

There is a canonical monomorphism  $\mathcal{X} \rightarrow \text{Sing}_*^{\mathbb{A}^1}(\mathcal{X})$ , which is an  $\mathbb{A}^1$ -weak equivalence [MV, Corollary 2.3.8]. This is a first approximation to an  $\mathbb{A}^1$ -local resolution; however, even if  $\mathcal{X}$  is fibrant,  $\text{Sing}_*^{\mathbb{A}^1}(\mathcal{X})$  need not be  $\mathbb{A}^1$ -local [MV, Example 3.2.7]. The following result is a consequence of the explicit construction of an  $\mathbb{A}^1$ -fibrant resolution functor.

**Proposition 6.1.12** [MV, Corollary 2.3.22] Suppose that  $\mathcal{X} \in \Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$  is a simplicial sheaf and that  $\mathcal{X} \rightarrow \mathcal{X}'$  is an  $\mathbb{A}^1$ -weak equivalence, with  $\mathcal{X}'$   $\mathbb{A}^1$ -local, then  $\pi_0\mathcal{X} \rightarrow \pi_0\mathcal{X}'$  is an epimorphism of sheaves.

**Corollary 6.1.13** Let  $\mathcal{X} \in \Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$  be a simplicial sheaf. Suppose that  $\text{Sing}_*^{\mathbb{A}^1}(\mathcal{X})$  is connected, then  $\mathcal{X}$  is  $\mathbb{A}^1$ -connected.

The sheaf  $\pi_0\text{Sing}_*^{\mathbb{A}^1}(\mathcal{X})$  is the sheaf associated to the presheaf:

$$U \mapsto \text{colim}\{\mathcal{X}_1(U \times \mathbb{A}^1) \rightrightarrows \mathcal{X}_0(U)\},$$

where the morphisms are the diagonal morphisms induced by the simplicial structure of  $\mathcal{X}$  and the morphisms induced by the diagram  $U \rightrightarrows U \times \mathbb{A}^1$ , corresponding to the rational points 0, 1 of  $\mathbb{A}^1$ . In particular, if  $\mathcal{X}$  is represented by a scheme  $X$ , with constant simplicial structure, then  $\pi_0\text{Sing}_*^{\mathbb{A}^1}(\mathcal{X})$  is the sheaf colimit of the diagram:

$$\text{Hom}(\mathbb{A}^1, X) \rightrightarrows X$$

where the objects are interpreted as sheaves.

**Definition 6.1.14** A simplicial sheaf  $\mathcal{X}$  is  $\mathbb{A}^1$ -contractible if:

1.  $\mathcal{X}$  is pointed by  $\text{Spec}(k) \rightarrow \mathcal{X}$ .
2. There is an elementary  $\mathbb{A}^1$ -homotopy [MV, §2.3] between  $1_{\mathcal{X}}$  and the composite  $\mathcal{X} \rightarrow \text{Spec}(k) \rightarrow \mathcal{X}$ .

**Example 6.1.15** Let  $L/k$  be a finite separable extension of fields. Observe that the scheme  $\text{Spec}(L)$  is not  $\mathbb{A}^1$ -contractible, since it does not have a rational point, although  $\text{Spec}(L)$  is a one point scheme. This is reasonable, since base change to  $\text{Spec}(L)$  yields a non-connected scheme.

The following result is a consequence of [MV, 2.3.4, 2.3.8]:

**Lemma 6.1.16** *Let  $\mathcal{X}$  be an  $\mathbb{A}^1$ -contractible simplicial sheaf, then:*

1. *There is a simplicial weak equivalence  $\mathrm{Sing}_*^{\mathbb{A}^1}(\mathcal{X}) \simeq *$ .*
2.  *$\mathcal{X}$  is  $\mathbb{A}^1$ -connected.*

This yields the following criterion for  $\mathbb{A}^1$ -connectivity:

**Proposition 6.1.17** *Let  $k$  be a field which admits resolution of singularities. Let  $X \in \mathrm{Sch}/k$  be an irreducible scheme such that:*

1. *Rational points are dense in  $X$ .*
2. *Every point of  $X$  admits an  $\mathbb{A}^1$ -contractible open neighbourhood.*

*Then the simplicial sheaf represented by  $X$  is  $\mathbb{A}^1$ -connected.*

**Proof:** (Sketch) The result is required for non-smooth schemes and, in particular, requires Mayer-Vietoris squares which are homotopy cocartesian in the homotopy category; the  $\mathbb{A}^1$ -connectivity should be treated with respect to the *cdh*-topology. This is the origin of the hypothesis that  $k$  should admit resolution of singularities.

The argument is an induction based upon the following: suppose that  $X$  admits a Zariski covering by  $\mathbb{A}^1$ -connected schemes  $U, V$ , then there is a homotopy cartesian diagram

$$\begin{array}{ccc} U \cap V & \longrightarrow & U \\ \downarrow & & \downarrow \\ V & \longrightarrow & X. \end{array}$$

The  $\mathbb{A}^1$ -localization functor applied to this diagram should yield a homotopy cocartesian diagram:

$$\begin{array}{ccc} L_{\mathbb{A}^1}(U \cap V) & \longrightarrow & L_{\mathbb{A}^1}U \\ \downarrow & & \downarrow \\ L_{\mathbb{A}^1}V & \longrightarrow & L_{\mathbb{A}^1}X. \end{array}$$

where, by hypothesis  $\pi_0 L_{\mathbb{A}^1}U = * = \pi_0 L_{\mathbb{A}^1}V$  and  $\pi_0 L_{\mathbb{A}^1}(U \cap V)$  is non-trivial. It follows that  $\pi_0 L_{\mathbb{A}^1}X = *$ , as required. ■

**Corollary 6.1.18** *Let  $k$  be a field which admits resolution of singularities and let  $t \geq 1$  be an integer. The simplicial sheaves represented by the symmetric products  $S^t(\mathbb{P}^n)$ ,  $n \geq 0$  are  $\mathbb{A}^1$ -connected.*

**Example 6.1.19** The simplicial sheaf represented by  $\mathbb{P}^n$  is  $\mathbb{A}^1$ -connected, for  $n \geq 0$  (without any hypothesis on the base field).

With this understanding of  $\mathbb{A}^1$ -connectivity, we return to the refined version of group completion.

**Definition 6.1.20** (Compare with [MV, Definition 4.1.8]). Suppose that any finite subset of the scheme  $Z \in \mathcal{S}ch/k$  is contained in an open affine set and that  $Z$  is pointed by a choice of a rational point  $*$ . Multiplication by  $*$  induces a morphism  $S^d(Z) \rightarrow S^{d+1}(Z)$  and hence a morphism in the category of sheaves:

$$\coprod_{d \geq 0} S^d(Z) \rightarrow \coprod_{d \geq 0} S^d(Z).$$

Define  $S^\infty(Z)$  to be the colimit in the category of sheaves of the associated direct system.

The  $\mathbb{A}^1$ -local version of the group completion theorem [MV, Theorem 3.1.10] has the following consequence in this context:

**Theorem 6.1.21** *Let  $k$  be a field of characteristic zero and let  $Z$  be a quasi-projective scheme, such that the schemes  $S^t(Z)$  are  $\mathbb{A}^1$ -connected for each  $t \geq 0$ , then there are  $\mathbb{A}^1$ -weak equivalences:*

$$\begin{array}{ccc} & \mathbb{Z}_{\text{tr}}[Z] & \\ & \downarrow \simeq_{\mathbb{A}^1} & \\ \mathbb{Z} \times S^\infty(Z) & \xrightarrow{\simeq_{\mathbb{A}^1}} & R\Omega_s^1 B(\coprod S^d(Z)), \end{array}$$

where  $R\Omega_s^1$  indicates the derived functor of the simplicial loop functor  $\Omega_s^1$ , which is right adjoint to the simplicial suspension  $\Sigma_s$ .

In particular, Corollary 6.1.18 implies the following result, which is the version of the algebraic Dold-Thom theorem which is used.

**Corollary 6.1.22** *Let  $k$  be a field of characteristic zero and suppose that  $n \geq 0$ , then there is an equivalence in the  $\mathbb{A}^1$ -local homotopy category:*

$$\mathbb{Z}_{\text{tr}}[\mathbb{P}^n] \simeq_{\mathbb{A}^1} \mathbb{Z} \times S^\infty(\mathbb{P}^n).$$

## 6.2 Consequences of the algebraic Dold-Thom theorem

The algebraic Dold-Thom theorem, Theorem 6.1.4, is the basis of the comparison theorem of Suslin-Voevodsky between the algebraic singular cohomology of a separated variety (non necessarily smooth) over the complex numbers  $\mathbb{C}$  and the ordinary singular cohomology of its complex realization.

**Theorem 6.2.1** [SV, Theorem 8.3] *Let  $Z$  be a separated variety over  $\mathbb{C}$  and let  $n \geq 0$  be an integer, then there is a natural isomorphism:*

$$H^*(Z(\mathbb{C}), \mathbb{Z}/n) \rightarrow H_{\text{sing}}^*(Z, \mathbb{Z}/n).$$



In the homotopical approach to motivic cohomology, there is the following analogous result, using the complex realization functor of Section 2.4:

**Theorem 6.2.2** *Let  $x : k \hookrightarrow \mathbb{C}$  be a complex embedding of fields and let  $t_x^{\mathbb{C}} : \mathcal{H}_{\bullet}(k) \rightarrow \mathcal{H}_{\bullet}$  be the complex realization functor to the classical pointed homotopy category of topological spaces, then there is an isomorphism:*

$$t_x^{\mathbb{C}}(K_{\mathcal{M}}(\mathbb{Z}/n, m)) \cong K(\mathbb{Z}/n, 2m)$$

for any integers  $m, n \geq 0$ .

**Proof:** (Indications) This is a consequence of the classical Dold-Thom theorem in algebraic topology together with the algebraic version, Theorem 6.1.21. The proof requires working with non-smooth schemes and hence relies on the passage to the *cdh*-topology together with the comparison theorem of [V6]. ■

**Corollary 6.2.3** *Let  $\mathcal{X}$  be a simplicial sheaf and let  $x : k \hookrightarrow \mathbb{C}$  be a complex embedding of the field  $k$ . There is a morphism of  $\mathbb{Z}/l$ -algebras:*

$$(t_x^{\mathbb{C}})^* : H^{d,n}(\mathcal{X}, \mathbb{Z}/l) \rightarrow H^d(t_x^{\mathbb{C}}(\mathcal{X}), \mathbb{Z}/l),$$

where the right hand side denotes the singular cohomology of the realization  $t_x^{\mathbb{C}}(\mathcal{X})$  of  $\mathcal{X}$ .

The morphism  $(t_x^{\mathbb{C}})^*$  is a morphism of  $H^{*,*}(\text{Spec}(k), \mathbb{Z}/l)$ -algebras, via the induced morphism of algebras

$$(t_x^{\mathbb{C}})^* : H^{d,n}(\text{Spec}(k), \mathbb{Z}/l) \rightarrow H^d(*, \mathbb{Z}/l) \cong \mathbb{Z}/l,$$

where  $\mathbb{Z}/l$  is concentrated in bidegree  $(0, 0)$ .

**Proof:** (Indications) This follows directly from Theorem 6.2.2; the statement about the multiplicative structure follows since the realization functor  $t_x^{\mathbb{C}}$  respects the monoidal structure. ■

## 7 Basic properties of motivic cohomology

The purpose of this section is to collect some basic properties of motivic cohomology which are used in this paper.

### 7.1 Motivic cohomology of small weights and vanishing conditions

The following result is a straightforward consequence of the explicit identification of the complexes of sheaves  $\mathbb{Z}, \mathbb{Z}(1)$  in  $\mathcal{C}\text{-}\mathcal{N}_k^{\text{tr}}$ .

**Proposition 7.1.1** [SV2, Corollary 3.2.1] *Let  $k$  be a field and let  $X \in \text{Sm}/k$  be a smooth scheme over  $k$ . There are identifications of integral motivic cohomology in weights zero and one:*

$$H^{i,0}(X, \mathbb{Z}) = \begin{cases} H_{\text{Zar}}^0(X, \mathbb{Z}) & i = 0 \\ 0 & i \neq 0 \end{cases}$$

$$H^{i,1}(X, \mathbb{Z}) = \begin{cases} H_{\text{Zar}}^0(X, \mathcal{O}^*) & i = 1 \\ \text{Pic}(X) & i = 2 \\ 0 & i \notin \{1, 2\} \end{cases}.$$

The connection with Bloch's higher Chow groups is made explicit by the following result, which is a *cohomological* reformulation of [V4, Theorem 4.2.9], using the duality theorem [V4, Theorem 4.3.7]:

**Theorem 7.1.2** *Let  $k$  be a field admitting resolution of singularities and let  $X$  be a smooth quasi-projective scheme which is equidimensional of dimension  $n$ , then there is a canonical isomorphism:*

$$H^{2m-t,m}(X, \mathbb{Z}) \cong CH^m(X, t)$$

where the groups on the right hand side denote Bloch's higher Chow groups.

In particular, there is the following identification of the motivic cohomology of a point in bidegrees of the form  $(n, n)$ .

**Theorem 7.1.3** [SV2, Theorem 3.4] *Let  $k$  be a field and let  $n$  be a non-negative integer. The motivic cohomology groups  $H^{n,n}(\text{Spec } k, \mathbb{Z})$  are isomorphic to the Milnor  $K$ -groups  $K_n^M(k)$ .*

It is also useful to have vanishing results for motivic cohomology:

**Proposition 7.1.4** [V1, Corollary 2.2] *Let  $k$  be a field admitting resolution of singularities and let  $\mathcal{X}$  be a simplicial sheaf in  $\Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\text{Sm}/k)$ , then the motivic cohomology groups  $H^{p,q}(\mathcal{X}, \mathbb{Z})$  are trivial in the following cases:*

1.  $q < 0$
2.  $q = 0$  and  $p < 0$

3.  $q = 1$  and  $p \leq 0$ .

When  $\mathcal{X}$  has constant simplicial structure and is represented by a smooth scheme, then the following applies:

**Proposition 7.1.5** [V1, Corollary 2.3] *Let  $k$  be a field admitting resolution of singularities and let  $X$  be a smooth scheme over  $k$ , then the motivic cohomology group  $H^{2m-t,m}(X, \mathbb{Z})$  is zero for:*

1.  $t < 0$
2.  $m - t > \dim X$ .

## 7.2 Motivic cohomology with coefficients

The definition of motivic cohomology with coefficients in an abelian group is straightforward.

**Definition 7.2.1** Let  $A \in \mathcal{A}b$  be an abelian group; the motivic cohomology of a simplicial sheaf  $\mathcal{X} \in \Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$  with coefficients in  $A$  is defined in bidegrees  $(2n, n)$  as the group of homomorphisms:

$$[\mathcal{X}, \underline{C}_*(z_{\text{equi}}(\mathbb{A}^n, 0) \otimes A)]_{\mathcal{H}_s(k)}$$

where the tensor product can be interpreted either in  $DM_-^{\text{eff}}(k)$  or in  $\mathcal{D}_-\mathcal{N}_k$ .

In particular, this definition applies when  $l$  is a prime and  $A = \mathbb{Z}/l$ . The proof that integral motivic cohomology defines a bigraded cohomology theory extends to the case of motivic cohomology with coefficients to give:

**Theorem 7.2.2** *Let  $k$  be a field admitting resolution of singularities, then motivic cohomology with coefficients  $\mathbb{Z}/l$  is a bigraded cohomology theory on  $\mathcal{H}(k)$ .*

**Definition 7.2.3** Let  $l$  be a rational prime and let  $n$  be a non-negative integer. The mod- $l$  motivic Eilenberg-MacLane space  $K_{\mathcal{M}}(\mathbb{Z}/l, n)$  is the isomorphism class in  $\Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)_{\bullet}$  of objects which represent mod- $l$  motivic cohomology in bidegree  $(2n, n)$ .

For the purposes of constructing the Steenrod reduced power operations, the following notation is fixed for an explicit model for the mod- $l$  Eilenberg-MacLane space:

**Notation 7.2.4** Let  $n$  be a positive integer.

1. Let  $\mathbb{Z}_{\text{tr}}(\mathbb{A}^n / (\mathbb{A}^n - \{0\}))$  denote the abelian Nisnevich sheaf with transfers

$$\mathbb{Z}_{\text{tr}}[\mathbb{A}^n] / \mathbb{Z}_{\text{tr}}[(\mathbb{A}^n - \{0\})],$$

where the quotient is taken in the category of abelian Nisnevich sheaves with transfers.

2. Let  $\mathbb{Z}_{\text{tr}}(\mathbb{A}^n/(\mathbb{A}^n - \{0\}), \mathbb{Z}/l)$  denote the abelian Nisnevich sheaf with transfers  $\mathbb{Z}_{\text{tr}}(\mathbb{A}^n/(\mathbb{A}^n - \{0\})) \otimes \mathbb{Z}/l$ . Thus,  $\mathbb{Z}_{\text{tr}}(\mathbb{A}^n/(\mathbb{A}^n - \{0\}), \mathbb{Z}/l)$  gives a model for  $K_{\mathcal{M}}(\mathbb{Z}/l, n)$ .

The definition of the product structure on integral motivic cohomology carries over to the case with coefficients; in particular, for  $\mathcal{X}, \mathcal{Y} \in \Delta^{\text{op}}\text{Shv}_{\text{Nis}}(\mathcal{S}m/k)$ , there is a natural exterior product:

$$H^{*,*}(\mathcal{X}, \mathbb{Z}/l) \otimes H^{*,*}(\mathcal{Y}, \mathbb{Z}/l) \rightarrow H^{*,*}(\mathcal{X} \times \mathcal{Y}, \mathbb{Z}/l).$$

In particular, taking  $\mathcal{X}$  to be the sheaf represented by  $\text{Speck}$ , this induces:

$$H^{*,*}(\text{Speck}, \mathbb{Z}/l) \otimes H^{*,*}(\mathcal{Y}, \mathbb{Z}/l) \rightarrow H^{*,*}(\mathcal{Y}, \mathbb{Z}/l).$$

**Lemma 7.2.5** *Let  $k$  be a field, then the mod- $l$  motivic cohomology  $H^{*,*}(\text{Spec}(k), \mathbb{Z}/l)$  has the structure of a bigraded  $\mathbb{F}_l$ -algebra (non-commutative).*

**Definition 7.2.6** Let  $k$  be a field. The Bockstein operator  $\beta : H^{*,*}(-, \mathbb{Z}/l) \rightarrow H^{*+1,*}(-, \mathbb{Z}/l)$  is the natural transformation which is induced by the exact triangle in  $\mathcal{D}_-\mathcal{N}_k$ :

$$\mathbb{Z}/l(n) \rightarrow \mathbb{Z}/l^2(n) \rightarrow \mathbb{Z}/l(n) \rightarrow .$$

The following is clear:

**Lemma 7.2.7** *Let  $k$  be a field admitting resolution of singularities, then the Bockstein operator is a bistable cohomology operation for mod- $l$  motivic cohomology.*

The algebra  $H^{*,*}(\text{Speck}, \mathbb{Z}/l)$  is the *coefficient ring* of motivic cohomology with coefficients  $\mathbb{Z}/l$ ; the following results give information on its structure:

**Proposition 7.2.8** *Let  $k$  be a field which admits resolution of singularities, then:*

1.  $H^{p,q}(\text{Spec}(k), \mathbb{Z}/l) = 0$  if one of the following conditions holds:

$$\left\{ \begin{array}{l} p > q \\ q < 0 \\ q = 0, p \neq 0 \\ q = 1, p \neq 0, 1. \end{array} \right.$$

2.  $H^{p,p}(\text{Spec}(k), \mathbb{Z}/l) = K_p^M(k)/l$ , mod- $l$  Milnor  $K$ -theory.

3.  $H^{0,1}(\text{Spec}(k), \mathbb{Z}/l) = \mu_l(k)$ , where  $\mu_l$  is the sheaf of  $l^{\text{th}}$  roots of unity.

The following example defines elements which are of importance in the construction of the Steenrod reduced power operations.

**Example 7.2.9** Let  $k$  be a field and let  $l = 2$ ; there are canonical elements:

1.  $\tau \in H^{0,1}(\mathrm{Spec}(k), \mathbb{Z}/2)$ , corresponding to the element  $-1 \in \mu_2(k)$ .
2.  $\rho \in H^{1,1}(\mathrm{Spec}(k), \mathbb{Z}/2)$ , corresponding to the class of  $-1 \in k^*/k^{*2}$ .

**Lemma 7.2.10** *Let  $k$  be a field and let  $l = 2$  then the Bockstein yields a relation  $\beta\tau = \rho$ .*

In the case that  $k$  has characteristic zero, one has the following:

**Theorem 7.2.11 (Voevodsky)** *Let  $k$  be a field of characteristic zero and let  $l = 2$ , then there is an isomorphism:*

$$H^{*,*}(\mathrm{Spec}(k), \mathbb{Z}/2) \cong (K^M(k)/2)[\tau],$$

where the elements of  $K_p^M(k)/2$  are placed in bidegree  $(p, p)$ .

### 7.3 The projective bundle theorem and the Thom isomorphism theorem

Motivic cohomology satisfies a projective bundle theorem; this is the basis for the theory of Chern classes for motivic cohomology and the Thom isomorphism theorem. An analogue of these results holds for any multiplicative cohomology theory which is induced from a ring spectrum with a  $\mathbb{P}^\infty$ -orientation in the stable homotopy category of  $T$ -spectra [M]. (The notion of a spectrum with  $\mathbb{P}^\infty$ -orientation is a generalization of that of a complex-oriented ring spectrum in the usual stable homotopy category of algebraic topology).

Let  $\xi$  be a rank  $n$  vector bundle over a smooth scheme  $X \in \mathcal{S}m/k$ ; write  $\mathbb{P}\xi$  for the associated projective bundle over  $X$  and  $\lambda_\xi$  for the canonical line bundle over  $\mathbb{P}\xi$ . Motivic cohomology of weight one is known, by Proposition 7.1.1; there is an isomorphism  $H^{2,1}(X, \mathbb{Z}) \cong \mathrm{Pic}(X)$ , hence the line bundle  $\lambda_\xi$  is classified by a cohomology class  $c(\lambda_\xi) \in H^{2,1}(\mathbb{P}\xi, \mathbb{Z})$ . The cup product defines classes  $c(\lambda_\xi)^i \in H^{2i,i}(\mathbb{P}\xi, \mathbb{Z})$ .

**Theorem 7.3.1** [V4, 3.5.1] *Let  $k$  be a field which admits resolution of singularities and let  $\xi$  be a vector bundle of rank  $n$  over a smooth scheme  $X \in \mathcal{S}m/k$ . Suppose that  $A$  is a commutative ring, then  $H^{*,*}(\mathbb{P}\xi, A)$  is a free  $H^{*,*}(X, A)$ -module on the classes  $\{c(\lambda_\xi)^i | 0 \leq i \leq n - 1\}$ .*

#### Remark 7.3.2

1. The hypothesis that  $k$  admits resolution of singularities is necessary since the proof uses the fact that motivic cohomology has a suspension isomorphism with respect to suspension by  $\mathbb{P}^1$ .<sup>3</sup>
2. Coefficients are taken in a commutative ring so that a product structure is defined.

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<sup>3</sup>This requirement can now be removed, according to Voevodsky.

Theorem 7.3.1 implies formally that motivic cohomology has a theory of Chern classes: there are cohomology classes  $c_i(\xi) \in H^{*,*}(X)$ , defined by the equation:

$$c(\lambda_\xi)^n = \sum_{i=1}^n (-1)^{i-1} c_i(\xi) c(\lambda_\xi)^{n-i}.$$

$c_i(\xi)$  is the  $i^{\text{th}}$  Chern class of the vector bundle  $\xi$ . These classes satisfy standard properties which are deduced axiomatically by using the splitting principle (compare with [Gro]).

The projective bundle theorem also gives a proof of the Thom isomorphism theorem for motivic cohomology. Let  $\theta$  denote the trivial bundle of rank one on the scheme  $X \in \mathcal{S}m/k$ ; there is a natural morphism of pointed simplicial sheaves

$$\mathbb{P}(\xi \oplus \theta)/\mathbb{P}(\xi) \rightarrow \text{Th}(\xi),$$

which is an  $\mathbb{A}^1$ -homotopy equivalence (see Example 2.7.2). This allows the description of the Thom class  $u_\xi$  of a vector bundle  $\xi$ :

**Theorem 7.3.3** *Let  $k$  be a field which admits resolution of singularities and let  $\xi$  be a vector bundle of rank  $n$  over the scheme  $X \in \mathcal{S}m/k$ . There exists a unique cohomology class  $u_\xi \in H^{2n,n}(\text{Th}(\xi))$  such that, for any closed point  $\text{Spec}(k(x)) \rightarrow X$ , the restriction  $u_\xi|_{\text{Spec}(k(x))} = u^{\cup n}$ , where  $u \in H^{2,1}(\mathbb{P}^1)$  is the canonical class and  $u^{\cup n} \in H^{*,*}(\text{Th}(\xi|_{\text{Spec}(k(x))})) \cong H^{*,*}(\mathbb{P}^1 \wedge^n)$ .*

**Proof:** The existence of the Thom class follows formally from the properties of Chern classes, as follows: the sequence  $\mathbb{P}(\xi) \rightarrow \mathbb{P}(\xi \oplus \theta) \rightarrow \text{Th}(\xi)$  is equivalent to a cofibration sequence and induces a short exact sequence in cohomology

$$\tilde{H}^{*,*}(\text{Th}(\xi)) \rightarrow H^{*,*}(\mathbb{P}(\xi \oplus \theta)) \rightarrow H^{*,*}(\mathbb{P}(\xi)),$$

since  $H^{*,*}(\mathbb{P}(\xi \oplus \theta)) \rightarrow H^{*,*}(\mathbb{P}(\xi))$  is surjective, by the projective bundle theorem. Consider the class

$$\sum_{i=0}^n (-1)^{i-1} c_i(\xi \oplus \theta) c(\lambda_{\xi \oplus \theta})^{n-i} \in H^{*,*}(\mathbb{P}(\xi \oplus \theta)).$$

By construction, this class is sent to zero in  $H^{*,*}(\mathbb{P}(\xi))$  and hence arises from a class  $u_\xi$  in  $\tilde{H}^{*,*}(\text{Th}(\xi))$ .

The uniqueness of the Thom class follows from the construction above and the proof of the projective bundle theorem.  $\blacksquare$

The Thom class gives rise to the Thom isomorphism theorem for motivic cohomology:

**Theorem 7.3.4** [V1, Theorem 3.20] *Let  $k$  be a field which admits resolution of singularities and let  $\xi$  be a vector bundle of rank  $n$  over  $X \in \mathcal{S}m/k$ . The cup product with the Thom class  $u_\xi$  induces an isomorphism:  $H^{*,*}(X) \rightarrow \tilde{H}^{*+2n,*+n}(\text{Th}(\xi))$ .*

## 7.4 The Gysin sequence

There are Gysin sequences for motivic cohomology, which are deduced from the following homotopy purity theorem, together with the Thom isomorphism for motivic cohomology.

**Theorem 7.4.1** [MV, Theorem 3.2.23] *Let  $i : Z \hookrightarrow X$  be a closed embedding of smooth  $k$ -schemes and let  $\nu_{X,Z}$  denote the normal bundle to  $i$  over  $Z$ . There is a canonical isomorphism in  $\mathcal{H}_\bullet(k)$ :*

$$X/(X - iZ) \cong \mathrm{Th}(\nu_{X,Z}).$$

**Remark 7.4.2** This result depends on the fact that the the Grothendieck topology is at least as strong as the Nisnevich topology and the fact that the schemes are smooth.

The Thom isomorphism theorem implies the following result:

**Corollary 7.4.3** *Let  $i : Z \hookrightarrow X$  be a closed embedding of smooth  $k$ -schemes which is everywhere of codimension  $d$ , then the cofibration sequence*

$$(X - iZ) \rightarrow X \rightarrow \mathrm{Th}(\nu_{X,Z})$$

*induces a long exact sequence in motivic cohomology with coefficients in a commutative ring  $A$ :*

$$\dots \rightarrow H^{*-2d,*-d}(Z, A) \xrightarrow{\cup_{c_d}(\nu)} H^{*,*}(X, A) \rightarrow H^{*,*}(X - iZ, A) \rightarrow \dots$$

**Remark 7.4.4** Corollary 7.4.3 generalizes immediately to any multiplicative cohomology theory which satisfies a Thom isomorphism theorem.

In the case that the closed immersion is the zero section of a vector bundle of constant rank over a smooth scheme, one obtains the following:

**Proposition 7.4.5** [V4, 3.5.4] *Let  $k$  be a field which admits resolution of singularities and let  $\xi$  be a vector bundle of rank  $d$  on  $X \in \mathcal{S}m/k$ . Suppose that  $A$  is a commutative ring, then there is an exact sequence of  $H^{*,*}(X, A)$  modules:*

$$\dots \rightarrow H^{*-2d,*-d}(X, A) \xrightarrow{\cup_{c_d}(\xi)} H^{*,*}(X, A) \rightarrow H^{*,*}(E(\xi)^\times, A) \rightarrow \dots$$

## 8 Classifying spaces

In algebraic topology, the classifying space  $B\mathbb{Z}/2$  represents the singular cohomology group  $H^1(-, \mathbb{Z}/2)$ ; in particular  $B\mathbb{Z}/2$  has an ‘infinite loop space structure’ and the space  $B\mathbb{Z}/2$  plays a fundamental rôle in studying singular cohomology with  $\mathbb{Z}/2$ -coefficients. This is exemplified by the work of Lannes which yields his proof to the Sullivan conjecture [La]. Similar statements hold for motivic cohomology with  $\mathbb{Z}/l$ -coefficients, for  $l$  an odd prime.

The purpose of this section is to introduce the analogous notions of classifying space in  $\mathbb{A}^1$ -homotopy theory. In particular, the construction of the Steenrod squaring operations given here follows from the construction of the total Steenrod power map; this depends on the usage of the geometric classifying space of  $\mathbb{Z}/2$ .

### 8.1 Classifying spaces in $\mathbb{A}^1$ -homotopy theory

The geometric classifying space of a linear algebraic group is defined following a standard construction of the classifying space of a group in algebraic topology. The proof that this has a suitable homotopical meaning introduces the étale classifying space [MV, §4.2].

**Definition 8.1.1** [MV, §4.2] Let  $G$  be a linear algebraic group over  $k$ , namely a closed subgroup of  $GL_m(k)$ , for some  $m$ , with given closed embedding  $i : G \hookrightarrow GL_m(k)$ .

For  $t \geq 1$ , let  $U_t$  denote the open subscheme of  $\mathbb{A}^{mt}$  on which the diagonal action of  $G$  induced by the embedding  $i : G \hookrightarrow GL_m(k)$  is free. Let  $V_t := U_t/G$  denote the quotient scheme, which is a smooth scheme which identifies with the image of  $U_t$  in  $\mathbb{A}^{mt}/G$ .

The embeddings  $\mathbb{A}^{mt} \hookrightarrow \mathbb{A}^{m(t+1)}$  induced by  $\mathbb{A}^t \hookrightarrow \mathbb{A}^{t+1}$ ,  $(x_1, \dots, x_t) \mapsto (x_1, \dots, x_t, 0)$  induce closed embeddings  $U_t \hookrightarrow U_{t+1}$  and  $V_t \hookrightarrow V_{t+1}$ . Define

$$\begin{aligned} E_{\text{gm}}(G, i) &:= \text{colim}_{t \in \mathbb{N}} U_t \\ B_{\text{gm}}(G, i) &:= \text{colim}_{t \in \mathbb{N}} V_t \end{aligned}$$

where the colimit is taken in the category of sheaves. The sheaf  $B_{\text{gm}}(G, i)$  is the geometric classifying space of  $G$ .

The étale classifying space of a sheaf of groups is defined below and has a canonical description in the homotopy category  $\mathcal{H}(k)$ . Write  $(\mathcal{S}m/k)_T$  for the category of sheaves on  $\mathcal{S}m/k$  with respect to the topology  $T$ .

**Definition 8.1.2** Let  $G \in (\mathcal{S}m/k)_{\text{Nis}}$  be a sheaf of groups and let  $E(G) \rightarrow B(G)$  denote the universal  $G$ -torsor [MV, 4.1.1], so that  $B(G)$  denotes the associated ‘classifying’ simplicial sheaf. The morphism of sites  $\pi : (\mathcal{S}m/k)_{\text{et}} \rightarrow (\mathcal{S}m/k)_{\text{Nis}}$ , induces adjoint functors  $\pi^* \dashv \mathbb{R}\pi_*$  between the corresponding homotopy categories with respect to the *simplicial* model structure.

The étale classifying space of  $G$ ,  $B_{\text{et}}G$ , is defined as  $\mathbb{R}\pi_*\pi^*(B(G)) \in \mathcal{H}_s(k)$ .



If  $G$  is an étale sheaf of groups, then the étale classifying space  $B_{\text{ét}}G$  is represented by  $\mathcal{B}_{\text{ét}}G$ , where  $B(G) \rightarrow \mathcal{B}_{\text{ét}}G$  is a fibrant model with respect to the simplicial model structure for the étale topology on  $\Delta^{\text{op}}\text{Shv}_{\text{ét}}(\mathcal{S}m/k)$ .

The following result can be interpreted as showing that the class of the geometric classifying space  $B_{\text{gm}}(G, i)$  in the homotopy category  $\mathcal{H}(k)$  is independent of the embedding  $i$ .

**Proposition 8.1.3** [MV, Proposition 4.2.6] *Let  $k$  be an infinite field and let  $G$  be a linear algebraic group over  $k$ . The geometric classifying space  $B_{\text{gm}}(G, i)$  is isomorphic in  $\mathcal{H}(k)$  to the étale classifying space  $B_{\text{ét}}G$ .*

**Proof:** (Indications). It is sufficient to check that the hypotheses of [MV, Proposition 4.2.6] are satisfied for the group  $G$  and the construction in Definition 8.1.1. The sheaf of groups is representable, hence is an étale sheaf; moreover [MV, Example 4.2.2] shows that  $B_{\text{gm}}(G, i)$  is constructed from an admissible gadget (in the terminology of *loc. cit.*). It remains to check that the conditions (1),(2),(3) of [MV, Definition 4.2.4] are satisfied. The field  $k$  is assumed to be infinite hence condition (3) is satisfied, by the remark before [MV, Example 4.2.10]. Moreover, the condition (2) is satisfied, by construction; the proof is completed by verifying that condition (1) of [MV, Definition 4.2.4] is satisfied. ■

In particular, this Proposition shows that the following terminology makes sense in the homotopy category.

**Notation 8.1.4** Let  $G$  be a linear algebraic group over an infinite field  $k$ ; the isomorphism class in the homotopy category  $\mathcal{H}(k)$  of the geometric classifying space  $B_{\text{gm}}(G, i)$  associated to a closed immersion  $i : G \hookrightarrow GL_m(k)$  is denoted by  $B_{\text{gm}}G$ .

**Remark 8.1.5** The motivation for introducing the étale classifying space into the discussion of the geometric classifying space is to establish étale descent, allowing the functor  $H_{\text{ét}}^1(-, G)$  to be related to the functor represented by  $B_{\text{gm}}G$  on the homotopy category  $\mathcal{H}(k)$ .

One has the following:

**Proposition 8.1.6** *Let  $k$  be a field of characteristic zero, then the geometric classifying space  $B_{\text{gm}}\mathbb{Z}/2$  represents motivic cohomology with  $\mathbb{Z}/2$ -coefficients  $H^{1,1}(-, \mathbb{Z}/2)$ .*

**Proof:** (Indications) This follows from Hilbert's Theorem 90 (Lemma 8.3.1), Proposition 8.1.3 together with the results of Morel-Voevodsky upon étale classifying spaces [MV, Section 4.1.4] and [MV, Proposition 4.3.1]. ■

## 8.2 The lens space $B_{\text{gm}}\mu_l$

The group scheme  $\mu_l$  of  $l^{\text{th}}$  roots of unity is the affine group scheme  $\text{Spec}(k[T]/T^l - 1)$ , which is a linear algebraic group via the closed embedding  $\mu_l \hookrightarrow \mathbb{G}_m$  which is induced by the surjection of Hopf algebras  $k[T, T^{-1}] \rightarrow k[T]/T^l - 1$ . The standard action of  $\mathbb{G}_m$  upon  $\mathbb{A}^1$  restricts to an action of  $\mu_l$  upon  $\mathbb{A}^1$ . This induces a diagonal action of  $\mu_l$  upon  $\mathbb{A}^n$ , for  $n \geq 1$  an integer, which restricts to an action upon  $(\mathbb{A}^n - \{0\})$ .

**Lemma 8.2.1** *Let  $n \geq 1$  be an integer; the morphism  $(\mathbb{A}^n - \{0\}) \rightarrow \mathbb{P}^{n-1}$  has the structure of a (Zariski)  $\mathbb{G}_m$ -torsor and the restriction of the  $\mathbb{G}_m$ -action on  $(\mathbb{A}^n - \{0\})$  to  $\mu_l$  coincides with the above action.*

**Definition 8.2.2** Let  $n \geq 1$  be an integer.

1. Let  $W_n$  denote the quotient scheme  $(\mathbb{A}^n - \{0\})/\mu_l$ .
2. Let  $W_n \hookrightarrow W_{n+1}$  denote the closed embedding induced by the embedding  $\mathbb{A}^n \rightarrow \mathbb{A}^{n+1}$ ,  $\{x_1, \dots, x_n\} \mapsto \{x_1, \dots, x_n, 0\}$
3. Let  $B_{\text{gm}}\mu_l$  denote the colimit

$$B_{\text{et}}\mu_l := \varinjlim_{n \rightarrow \infty} W_n,$$

where the colimit is taken in the category of sheaves.

The following result is standard:

**Lemma 8.2.3** *Let  $k$  be a field which contains a primitive  $l^{\text{th}}$  root of unity. The (étale) sheaf represented by  $\mu_l$  is (non-canonically) isomorphic to the sheaf defined by the discrete abelian group  $\mathbb{Z}/l$ .*

**Proposition 8.2.4** *Let  $k$  be a field which contains a primitive  $l^{\text{th}}$  root of unity, then there is a (non-canonical) isomorphism*

$$B_{\text{gm}}\mu_l \cong B_{\text{gm}}\mathbb{Z}/l$$

in  $\Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$ . In particular,  $B_{\text{gm}}\mu_l$  is a model for the étale classifying space of  $\mathbb{Z}/l$ .

The following result is the key to performing the calculation of the motivic cohomology of  $B_{\text{gm}}\mu_l$ .

**Proposition 8.2.5** *Let  $k$  be a field,  $n \geq 1$  be an integer and let  $\lambda$  denote the canonical line bundle over  $\mathbb{P}^{n-1}$ . The scheme  $W_n$  is a smooth  $k$ -scheme, which is isomorphic to  $E(\lambda^{\otimes l})^\times$ , the complement of the zero section of the line bundle  $\lambda^{\otimes l}$  over  $\mathbb{P}^{n-1}$ .*

**Proof:** (Indications). This is a case of a more general result. The total space  $E(\lambda^{\otimes l})$  identifies with  $(\mathbb{A}^1)^\sim \times_{\mathbb{G}_m} (\mathbb{A}^n - \{0\})$ , where  $\mathbb{G}_m$  acts on  $(\mathbb{A}^1)^\sim$  through the map  $\mathbb{G}_m \xrightarrow{(-)^l} \mathbb{G}_m$  and the standard  $\mathbb{G}_m$ -action. The complement of the zero section then identifies with

$$(\mathbb{G}_m)^\sim \times_{\mathbb{G}_m} (\mathbb{A}^n - \{0\})$$

and hence with  $(\mathbb{A}^n - \{0\})/\mu_l$ . ■

**Corollary 8.2.6** *Let  $X$  be a smooth scheme over the field  $k$ ; the scheme  $X \times W_n$  is smooth and identifies with  $E(X \times \lambda^{\otimes l})^\times$ , where  $X \times \lambda^{\otimes l}$  denotes the pullback of  $\lambda^{\otimes l}$  to  $X \times \mathbb{P}^{n-1}$ , via the projection  $X \times \mathbb{P}^{n-1} \rightarrow X$ .*

### 8.3 The cohomology of $\mathcal{X} \wedge B_{\text{gm}}\mu_{l+}$

There are canonical elements  $a \in H^{1,1}(W_n, \mathbb{Z}/l)$  and  $b \in H^{2,1}(W_n, \mathbb{Z}/l)$ , which permit the description of  $H^{*,*}(W_n, \mathbb{Z}/l)$ . Their description requires the following result:

**Lemma 8.3.1** *Let  $k$  be a field.*

1. (Hilbert 90) *There is an isomorphism  $H^{1,1}(X, \mathbb{Z}/l) \cong H_{\text{et}}^1(X, \mu_l)$ .*
2. *If  $X \in \text{Sm}/k$ , there is a commutative diagram:*

$$\begin{array}{ccc} H^{1,1}(X, \mathbb{Z}/l) & \xrightarrow{B} & H^{2,1}(X, \mathbb{Z}) \\ \downarrow & & \downarrow \\ H_{\text{et}}^1(X, \mu_l) & \rightarrow & H_{\text{et}}^1(X, \mathbb{G}_m) \end{array}$$

*in which  $B$  denotes the integral Bockstein operator and the vertical arrows are isomorphisms.*

#### Definition 8.3.2

1. Let  $b \in H^{2,1}(W_n, \mathbb{Z}/l)$  denote the image of  $c_1(\lambda) \in H^{2,1}(\mathbb{P}^{n-1}, \mathbb{Z}/l)$  under the map  $H^{2,1}(\mathbb{P}^{n-1}, \mathbb{Z}/l) \rightarrow H^{2,1}(W_n, \mathbb{Z}/l)$  induced by  $W_n \rightarrow \mathbb{P}^{n-1}$ .
2. Let  $a \in H^{1,1}(W_n, \mathbb{Z}/l)$  correspond to the class in  $H_{\text{et}}^1(X, \mu_l)$  classifying the  $\mu_l$ -torsor defining  $W_n$ , via Lemma 8.3.1(i).

Lemma 8.3.1(2) implies:

**Lemma 8.3.3** *The classes  $a, b$  are linked by the Bockstein operation:  $\beta(a) = b$ .*

The Gysin sequence of Proposition 7.4.5, applied using Corollary 8.2.6, implies the following result:

**Lemma 8.3.4** *Let  $X \in \mathcal{S}m/k$  be a smooth  $k$ -scheme and let  $n \geq 1$  be an integer. The Gysin sequence for  $E(X \times \lambda^{\otimes l})^\times$  induces a short exact sequence of  $H^{*,*}(X \times \mathbb{P}^{n-1})$ -modules:*

$$H^{*,*}(X \times \mathbb{P}^{n-1}, \mathbb{Z}/l) \rightarrow H^{*,*}(X \times W_n, \mathbb{Z}/l) \rightarrow H^{*-1, *-1}(X \times \mathbb{P}^{n-1}, \mathbb{Z}/l).$$

**Proof:** The first Chern class for motivic cohomology is additive with respect to tensor products of line bundles, hence the class  $c_1(\lambda^{\otimes l})$  identifies with  $lc_1(\lambda)$ . It follows that cup product with the class  $c_1(X \times \lambda^{\otimes l})$  is trivial in motivic cohomology with  $\mathbb{Z}/l$ -coefficients. The result therefore follows directly from Proposition 7.4.5.  $\blacksquare$

The calculation (Lemma 8.3.5) of the cohomology  $H^{*,*}(X \times \mathbb{P}^{n-1}, \mathbb{Z}/l)$  as an  $H^{*,*}(X, \mathbb{Z}/l)$ -module is a special case of the projective bundle theorem, Theorem 7.3.1.

**Lemma 8.3.5** *Let  $k$  be a field which admits resolution of singularities, let  $n \geq 1$  be an integer and let  $X \in \mathcal{S}m/k$  be a smooth scheme; then  $H^{*,*}(X \times \mathbb{P}^{n-1}, \mathbb{Z}/l)$  is a free  $H^{*,*}(X, \mathbb{Z}/l)$ -module on classes  $c, \dots, c^{n-1}$ , where  $c$  is the mod- $l$  reduction of the first Chern class of the pullback of  $\lambda$ .*

**Proposition 8.3.6** *Let  $k$  be a field of characteristic zero and let  $X \in \mathcal{S}m/k$  be a smooth scheme. There is an isomorphism of  $H^{*,*}(X, \mathbb{Z}/l)$ -modules:*

$$H^{*,*}(X \times B_{\text{et}}\mu_l, \mathbb{Z}/l) \cong H^{*,*}(X, \mathbb{Z}/l)[a, b] / \sim,$$

where  $\sim$  denotes the relation  $a^2 = 0$  if  $l \neq 2$  and  $a^2 = \tau b + \rho a$ , for  $l = 2$ .

The proof of this result requires the calculation of the relation involving the class  $a^2$ . This can be established using the complex realization argument of Corollary 6.2.3.

**Proof of Proposition 8.3.6:** The Gysin (short) exact sequence given by Lemma 8.3.4 together with Lemma 8.3.5 establish that  $H^{*,*}(X \times W_n, \mathbb{Z}/l)$  is a free  $H^{*,*}(X, \mathbb{Z}/l)$ -module on the classes  $ab^i, b^i, 0 \leq i \leq n-1$ . The cohomology of  $X \times B_{\text{et}}\mu_l$  is obtained as the inverse limit of the tower of surjections associated to  $W_n \hookrightarrow W_{n+1}$  (As usual in considering the cohomology of a colimit, care should be taken with the  $\lim_{\leftarrow}^1$  term).

Suppose that  $l = 2$ ; there is a relation involving  $a^2$ , which is easily seen to be of the form  $a^2 = \epsilon\tau b + \eta a$ , where  $\epsilon \in \{0, 1\}$  and  $\eta \in H^{0,1}(\text{Spec}(k), \mathbb{Z}/2)$ .

Suppose that  $k = \mathbb{C}$ , so that  $\eta_{\mathbb{C}} = 0$ ; thus the relation has the form  $a^2 = \epsilon_{\mathbb{C}}\tau b$ , and it suffices to show that  $\epsilon_{\mathbb{C}}$  is non-trivial. Consider the Complex realization functor  $t^{\mathbb{C}} : \mathcal{H}_{\bullet}(\mathbb{C}) \rightarrow \mathcal{H}_{\bullet}$ . The topological space  $t^{\mathbb{C}}(B_{\text{gm}}\mu_l)$  identifies with  $\mathbb{R}P^{\infty}$ , which has cohomology algebra  $\mathbb{Z}/2[u]$  on a class  $u$  of degree one which corresponds to  $a$ . In particular,  $u^2$  is non-trivial, hence  $a^2$  must be non-trivial, by Corollary 6.2.3. The case of a general field is treated by a functoriality argument.<sup>4</sup>

<sup>4</sup>Voevodsky has shown that usage of complex realization can be avoided.

Thus, the relation has the form  $a^2 = \tau b + \eta a$ . The Bockstein operator is a derivation and  $\beta\tau = \rho$ ,  $\beta a = b$ ; moreover, for reasons of bidegree,  $\beta\eta = 0$ . Therefore there is an equality  $\beta(a^2) = 0 = \rho b + \eta b$ , which yields  $\eta = \rho$ , as required.

The case  $l$  odd is treated similarly. ■

**Remark 8.3.7** The calculation of Proposition 8.3.6 using the Gysin sequence in motivic cohomology can be generalized, by passing to the category  $DM_-^{\text{eff}}(k)$  and using the Gysin sequence in this setting. Essentially, there is a Künneth-style formula for the cohomology of any object of the form  $\mathcal{X} \times B_{\text{gm}}\mu_l$ , since the reduced motive of  $B_{\text{gm}}\mu_l$  is a direct sum of motives of the form  $\mathbb{Z}/l(*_1)[*_2]$ . This implies the following result.

**Proposition 8.3.8** *Let  $k$  be a field of characteristic zero and let  $\mathcal{X} \in \Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$  be a simplicial sheaf. There is a natural isomorphism of  $H^{*,*}(\mathcal{X}, \mathbb{Z}/l)$ -modules:*

$$H^{*,*}(\mathcal{X} \times B_{\text{et}}\mu_l, \mathbb{Z}/l) \cong H^{*,*}(\mathcal{X}, \mathbb{Z}/l)[a, b] / \sim,$$

where  $\sim$  denotes the relation  $a^2 = 0$  if  $l \neq 2$  and  $a^2 = \tau b + \rho a$ , for  $l = 2$ .

## 9 Properties of the motivic Steenrod algebra

The purpose of this section is to discuss the basic properties of the motivic Steenrod algebra and of the Steenrod squaring operations.

### 9.1 The motivic Steenrod algebra

Throughout this section, let  $l$  be a fixed prime and let  $k$  be a field admitting resolution of singularities.

**Definition 9.1.1** The mod- $l$  motivic Steenrod algebra,  $\mathcal{A}^{*,*}(k, \mathbb{Z}/l)$ , is the algebra of bistable cohomology operations for motivic cohomology  $H^{*,*}(-, \mathbb{Z}/l)$ .

#### Example 9.1.2

1. The Bockstein operator  $\beta$  defines a bistable cohomology operation in  $\mathcal{A}^{1,0}(k, \mathbb{Z}/l)$ , by Lemma 7.2.7.
2. Multiplication by an element of  $H^{*,*}(\text{Spec}(k), \mathbb{Z}/l)$  defines a bistable cohomology operation, hence there is a bi-homogeneous inclusion

$$H^{*,*}(\text{Spec}(k), \mathbb{Z}/l) \hookrightarrow \mathcal{A}^{*,*}(k, \mathbb{Z}/l),$$

which is a homomorphism of algebras over  $\mathbb{Z}/l$ .

**Remark 9.1.3** The action of the Bockstein operation given in Lemma 7.2.10 shows that the image of  $H^{*,*}(\text{Spec}(k), \mathbb{Z}/l)$  in  $\mathcal{A}^{*,*}(k, \mathbb{Z}/l)$  is not central. This is contrary to the situation in algebraic topology.

Motivic cohomology with coefficients  $\mathbb{Z}/l$  of bidegree  $(2n, n)$  is represented by the motivic Eilenberg-MacLane space  $K_{\mathcal{M}}(\mathbb{Z}/l, n)$  and the suspension morphism is induced by the structure morphism:

$$\mathbb{P}^1 \wedge K_{\mathcal{M}}(\mathbb{Z}/l, n) \rightarrow K_{\mathcal{M}}(\mathbb{Z}/l, n+1).$$

These morphisms induce an inverse system of bigraded abelian groups:

$$\dots \rightarrow H^{*+2n, *+n}(K_{\mathcal{M}}(\mathbb{Z}/l, n), \mathbb{Z}/l) \rightarrow H^{*+2(n-1), *+n-1}(K_{\mathcal{M}}(\mathbb{Z}/l, n-1), \mathbb{Z}/l) \rightarrow \dots$$

and the motivic Steenrod algebra is related to the inverse limit of this system.

**Proposition 9.1.4** *Let  $l$  be a rational prime and let  $k$  be a field which admits resolution of singularities, then there is an isomorphism:*

$$\mathcal{A}^{*,*}(k, \mathbb{Z}/l) \cong \varprojlim H^{*+2n, *+n}(K_{\mathcal{M}}(\mathbb{Z}/l, n), \mathbb{Z}/l).$$

**Proof:** (Indications) The result follows by a standard Milnor exact sequence argument by showing that  $\lim_{\leftarrow}^1 H^{**+2n, **+n}(K_{\mathcal{M}}(\mathbb{Z}/l, n), \mathbb{Z}/l)$  vanishes; this requires an understanding of the motivic cohomology of the motivic Eilenberg-MacLane spaces. ■

When  $k$  is a sub-field of the complex numbers, there is a morphism to the topological Steenrod algebra, constructed by using topological realization. Recall that a complex embedding  $x : k \hookrightarrow \mathbb{C}$  induces an associated complex realization functor  $t_x^{\mathbb{C}} : \mathcal{H}(k) \rightarrow \mathcal{H}$  from the Morel-Voevodsky  $\mathbb{A}^1$ -local homotopy category to the homotopy category of topological spaces.

**Notation 9.1.5** Let  $\mathcal{A}^*(\mathbb{Z}/l)$  denote the mod- $l$  topological Steenrod algebra.

**Proposition 9.1.6** *Let  $x : k \hookrightarrow \mathbb{C}$  be an embedding of fields, then there is a morphism of  $H^{*,*}(\text{Spec}(k), \mathbb{Z}/l)$ -algebras:*

$$(t_x^{\mathbb{C}})^* : \mathcal{A}^{*,*}(k, \mathbb{Z}/l) \rightarrow \mathcal{A}^*(\mathbb{Z}/l).$$

**Proof:** The result follows from Proposition 9.1.4 and Corollary 6.2.3. ■

## 9.2 The motivic Steenrod squares

Henceforth, the prime  $l$  will be taken to be two; analogous results hold when the prime  $l$  is odd. The motivic Steenrod squaring operations are bistable cohomology operations which are characterized by the following result (Theorem 2 of the Introduction):

**Theorem 9.2.1** *Let  $k$  be a field of characteristic zero. There exists a unique sequence of bistable cohomology operations  $\mathbb{S}q^{2i} \in \mathcal{A}^{2i,i}(k, \mathbb{Z}/2)$ ,  $i \geq 0$ , such that*

1.  $\mathbb{S}q^0 = Id$ .

2. **Cartan formula:** *Let  $X, Y$  be simplicial smooth schemes and  $u \in H^{*,*}(X, \mathbb{Z}/2), v \in H^{*,*}(Y, \mathbb{Z}/2)$ . For all  $i \geq 0$ ,*

$$\mathbb{S}q^{2i}(u \times v) = \sum_{a+b=i} \mathbb{S}q^{2a}(u) \times \mathbb{S}q^{2b}(v) + \tau \left( \sum_{a+b=i-2} \beta \mathbb{S}q^{2a}(u) \times \beta \mathbb{S}q^{2b}(v) \right)$$

*in  $H^{*,*}(X \times Y, \mathbb{Z}/2)$ .*

3. **Instability:** *Let  $X$  be a simplicial smooth scheme and  $u \in H^{n,i}(X, \mathbb{Z}/2)$ ,*

$$\mathbb{S}q^{2i}(u) = \begin{cases} 0 & n < 2i \\ u^2 & n = 2i. \end{cases}$$

### Remark 9.2.2

1. For the uniqueness statement of Theorem 9.2.1, it is essential to consider the motivic cohomology of simplicial schemes and not just of representable sheaves with constant simplicial structure.

2. The Cartan formula is a consequence of the multiplicative structure of motivic cohomology. This is illustrated by the situation for the topological Steenrod algebra,  $\mathcal{A}(\mathbb{Z}/2)$ ; the multiplicative structure of singular cohomology induces a Hopf algebra structure over  $\mathbb{Z}/2$  on  $\mathcal{A}(\mathbb{Z}/2)$  and the diagonal acts on the topological Steenrod squares by  $\Delta \text{Sq}^n = \sum_{i=0}^n \text{Sq}^i \otimes \text{Sq}^{n-i}$ ; this expression corresponds to the Cartan formula for the topological Steenrod algebra. An analogous explanation holds in the motivic setting, which relies on establishing a flatness property of the motivic Steenrod algebra. This is used in Section 9.5, where the dual of the motivic Steenrod algebra is considered; the diagonal induces a product upon the dual.

The construction of the motivic Steenrod squaring operations is given in Sections 10 and 11; an indication of the proof that these satisfy the given conditions is also given. The uniqueness statement of the theorem is related to the calculation of the motivic Steenrod algebra and can be proved by showing that all bistable cohomology operations can be detected upon products of  $B_{\text{gm}}\mathbb{Z}/2$ 's.

**Notation 9.2.3** Write  $\text{Sq}^{2i+1}$  for the bistable cohomology operation  $\beta \text{Sq}^{2i} \in \mathcal{A}^{2i+1,i}(k, \mathbb{Z}/2)$  where  $i$  is a non-negative integer.

**Remark 9.2.4** The construction of the Steenrod squaring operations yields operations  $\text{Sq}^{2i+1}$ ; the identification implicit in Notation 9.2.3 is one of the motivic Adem relations.

The fact that the Bockstein operation acts non-trivially via  $\beta\tau = \rho$  greatly increases the algebraic complexity of the motivic Steenrod algebra. For instance, one has the Cartan formula for the odd squaring operations:

$$\text{Sq}^{2i+1}(u \times v) = \sum_{c+d=2i+1} \text{Sq}^c(u) \times \text{Sq}^d(v) + \rho \left( \sum_{a+b=i-2} \text{Sq}^{2a+1}(u) \times \text{Sq}^{2b+1}(v) \right).$$

The term with coefficient  $\rho$  is not present in the analogous topological Cartan formula.

The motivic Steenrod operations are related to the topological Steenrod operations under topological realization, via the characterization of the topological Steenrod operations:

**Corollary 9.2.5** *Let  $x : k \hookrightarrow \mathbb{C}$  be a complex embedding, then the induced morphism of  $H^{*,*}(\text{Spec}(k), \mathbb{Z}/2)$ -algebras,  $(t_x^{\mathbb{C}})^* : \mathcal{A}^{*,*2}(k, \mathbb{Z}/2) \rightarrow \mathcal{A}^{*1}(\mathbb{Z}/2)$  satisfies  $(t_x^{\mathbb{C}})^* \text{Sq}^i = \text{Sq}^i$ .*

### 9.3 Detection of bistable cohomology operations

The objects  $B_{\text{gm}}\mathbb{Z}/2 \in \mathcal{H}(k)$  play a distinguished role in the study of motivic cohomology with  $\mathbb{Z}/2$ -coefficients; in particular, the motivic cohomology of products of these spaces can be used to detect certain additive cohomology operations. This is the basis for recent work on the structure of the category of unstable modules over the topological Steenrod algebra (see [Sc], for example).



**Notation 9.3.1** Following the standard notation from algebraic topology, let  $H^{*,*}$  denote the coefficient ring  $H^{*,*}(\mathrm{Spec}(k), \mathbb{Z}/2)$ .

The motivic cohomology algebras of projective space  $\mathbb{P}^\infty$  and of the classifying space  $B_{\mathrm{gm}}\mathbb{Z}/2$  are known; this extends to a calculation of the cohomology algebras of products:

**Proposition 9.3.2** *Let  $k$  be a field admitting resolution of singularities, then there are isomorphisms of  $H^{*,*}$ -algebras:*

$$\begin{aligned} H^{*,*}((\mathbb{P}^\infty)^{\times n}, \mathbb{Z}/2) &\cong H^{*,*}[b_1, \dots, b_n] \\ H^{*,*}((B_{\mathrm{gm}}\mathbb{Z}/2)^{\times n}, \mathbb{Z}/2) &\cong H^{*,*}[a_1, \dots, a_n, b_1, \dots, b_n] / a_i^2 = \tau b_i + \rho a_i. \end{aligned}$$

Moreover, the canonical morphism  $B_{\mathrm{gm}}\mathbb{Z}/2 \rightarrow \mathbb{P}^\infty$  in cohomology induces the morphism of  $H^{*,*}$ -algebras given by  $b_i \mapsto b_i$  and  $a_i \mapsto 0$ .

These algebras afford useful representations of the motivic Steenrod algebra; the action of the motivic Steenrod squares upon  $H^{*,*}(B_{\mathrm{gm}}\mathbb{Z}/2, \mathbb{Z}/2)$  is given by the following result:

**Lemma 9.3.3** *The Steenrod squares  $\mathbb{S}q^i$  act on the classes  $ab^i, b^i$  of the cohomology  $H^{*,*}(B_{\mathrm{gm}}\mathbb{Z}/2, \mathbb{Z}/2)$  as follows:*

1.

$$\begin{aligned} \mathbb{S}q^{2i}(b^j) &= \binom{2j}{2i} b^{i+j} \\ \mathbb{S}q^{2i+1}(b^j) &= 0 \end{aligned}$$

2.

$$\begin{aligned} \mathbb{S}q^{2i}(ab^j) &= \binom{2j}{2i} ab^{i+j} \\ \mathbb{S}q^{2i+1}(ab^j) &= \binom{2j}{2i} b^{i+j+1} \end{aligned}$$

**Proof:** The Bockstein operation acts by  $\beta a = b$  and  $\beta b = 0$ ; the instability condition implies that  $\mathbb{S}q^i a = 0$  for  $i > 1$  and that  $\mathbb{S}q^2(b) = b^2$ , whereas  $\mathbb{S}q^i b = 0$  for  $i > 2$ . The calculation follows from the Cartan formula.  $\blacksquare$

The action of the Steenrod squares upon  $H^{*,*}(\mathbb{P}^\infty, \mathbb{Z}/2)$  follows immediately from this result; the calculation extends to the cohomology algebras  $H^{*,*}((\mathbb{P}^\infty)^{\times n}, \mathbb{Z}/2)$  and  $H^{*,*}((B_{\mathrm{gm}}\mathbb{Z}/2)^{\times n}, \mathbb{Z}/2)$  by applying the Cartan formula.

**Example 9.3.4** Lemma 9.3.3 suggests a formal similarity between the action of the motivic Steenrod operations upon products of  $B_{\mathrm{gm}}\mathbb{Z}/2$  and the action of the topological Steenrod operations upon the singular cohomology of products

of  $B\mathbb{Z}/2$ . However, the non-trivial action of the Bockstein upon the coefficients  $H^{*,*}$  introduces a certain complexity. For example, consider the class  $a_1a_2a_3$  in  $H^{3,3}((B_{\text{gm}}\mathbb{Z}/2)^{\times 3}, \mathbb{Z}/2)$ ; the Cartan formula yields the following:

$$\mathbb{S}q^3(a_1a_2a_3) = \tau(b_1b_2b_3) + \rho(b_1b_2a_3 + b_1a_2b_3 + a_1b_2b_3).$$

By contrast, the action of the motivic Steenrod operations upon the motivic cohomology of products of  $\mathbb{P}^\infty$  is formally similar to the corresponding action in topology, apart from the non-trivial action of the Bockstein upon the coefficient ring.

**Notation 9.3.5** Let  $F(n)$  denote the  $\mathcal{A}^{*,*}(k, \mathbb{Z}/l)$ -module which is generated as a sub-module of  $H^{*,*}((B_{\text{gm}}\mathbb{Z}/2)^{\times n}, \mathbb{Z}/2)$  by the class  $\prod_{i=1}^n a_i$  of bidegree  $(n, n)$ .

**Remark 9.3.6** This notation is consistent via topological realization with that used in the study of unstable modules over the topological Steenrod algebra [Sc]; in the topological case it is straightforward to see that the action of the Steenrod squares on the modules of the form  $H^*((B\mathbb{Z}/2)^{\times m}, \mathbb{Z}/2)$  is determined by the structure of the modules  $F(n)$ , as  $n$  varies. In particular, in the topological setting, all cohomology operations which can be detected upon products of  $B\mathbb{Z}/2$  can be detected using the modules  $F(n)$ .

**Example 9.3.7** The additional complexity in the motivic setting is exhibited already in the case  $n = 2$ ; one has:

$$\begin{aligned} \mathbb{S}q^2(a_1a_2) &= \tau b_1b_2 \\ \mathbb{S}q^3(a_1a_2) &= \rho b_1b_2 \end{aligned}$$

There is no bistable cohomology operation  $\theta$  such that  $\theta(a_1, a_2) = b_1b_2$ ; for suppose otherwise, then  $\theta$  would satisfy the relation  $\theta(a^2) = b^2$ , by naturality. The relation  $a^2 = \tau b + \rho a$  would imply that  $\theta(a^2)$  belongs to the submodule of  $H^{*,*}((B_{\text{gm}}\mathbb{Z}/2)^{\times 2}, \mathbb{Z}/2)$  given by the product with the ideal of elements of bidegree not equal to  $(0, 0)$  in  $H^{*,*}$  (this argument supposes that a form of diagonal morphism extending the Cartan formula is available). This is a contradiction.

It follows that  $F(2)$  is not a free  $H^{*,*}$ -module; however, the sub- $H^{*,*}$ -module generated by elements of bidegrees  $(*_1, *_2)$  with  $*_1 \leq 2$  is free.

The module  $F(n)$  is cyclic, hence there is a morphism of  $\mathcal{A}^{*,*}(k, \mathbb{Z}/2)$ -modules:

$$f(n) : \mathcal{A}^{*,*}(k, \mathbb{Z}/2) \rightarrow \Sigma^{-n, -n} F(n),$$

where  $\Sigma^{-n, -n}$  indicates the shift of degree of the underlying bigraded abelian group, which defines a functor on the category of  $\mathcal{A}^{*,*}(k, \mathbb{Z}/2)$ -modules in the obvious way (no sign considerations occur working over  $\mathbb{Z}/2$ ).

The morphism  $f(n)$  is used to consider the detection of bistable cohomology operations using the motivic cohomology of products of  $B_{\text{gm}}\mathbb{Z}/2$ . Two issues arise:

1. The identification of  $F(n)$ , using the fact that bistable cohomology operations are additive. The problem is to consider divisibility arising from powers of  $\tau$ .
2. To show that the morphism  $f(n)$  is bijective through a certain range of bidegrees.

These questions are intimately linked to the calculation of the motivic Steenrod algebra. A related algebraic result is given in Lemma 9.4.7

## 9.4 The Adem relations and their consequences

The motivic Steenrod algebra satisfies Adem relations which correspond under topological realization to those for the topological Steenrod algebra.

**Proposition 9.4.1** *The operations  $\mathbb{S}q^i$  satisfy relations for  $0 < a < 2b$ :*

$$\mathbb{S}q^a \mathbb{S}q^b = \begin{cases} \sum_{j=0}^{\lfloor a/2 \rfloor} \binom{b-1-j}{a-2j} \mathbb{S}q^{a+b-j} \mathbb{S}q^j & a \text{ odd} \\ \sum_{j=0}^{\lfloor a/2 \rfloor} \tau^{\epsilon_j} \binom{b-1-j}{a-2j} \mathbb{S}q^{a+b-j} \mathbb{S}q^j + \rho \sum_{j=1, j \equiv b(2)}^{\lfloor a/2 \rfloor} \binom{b-1-j}{a-2j} \mathbb{S}q^{a+b-j-1} \mathbb{S}q^j & a \text{ even,} \end{cases}$$

$$\text{where } \epsilon_j = \begin{cases} 1 & b \text{ even, } j \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

In particular, setting  $\tau = 1$  and  $\rho = 0$  and formally identifying  $\mathbb{S}q^i$  with the topological Steenrod square  $\text{Sq}^i$ , one obtains the Adem relations for the topological Steenrod algebra. In the case that the field  $k$  admits a complex embedding, this is clear by Corollary 9.2.5.

As a special case, the Adem relations give:

**Corollary 9.4.2** *There is a relation  $\beta \mathbb{S}q^{2i} = \mathbb{S}q^{2i+1}$ .*

The Adem relations permit the calculation of a basis for the sub-algebra of the motivic Steenrod algebra which is generated by the motivic Steenrod squaring operations; this is a generalization of the usual result for the topological Steenrod algebra [SE].

In order to be algebraically explicit, introduce the following algebra:

### Notation 9.4.3

1. Let  $\mathcal{H}$  denote the sub-algebra of  $H^{*,*}(\text{Spec}(k), \mathbb{Z}/2)$  generated by the elements  $\tau, \rho \in H^{*,*}(\text{Spec}(k), \mathbb{Z}/2)$ . Thus  $\mathcal{H} \cong \mathbb{Z}/2[\tau, \rho]$  is a commutative polynomial algebra.
2. Let  $\mathcal{B}$  denote the sub- $\mathcal{H}$ -algebra of  $\mathcal{A}^{*,*}(k, \mathbb{Z}/2)$  generated by the motivic squaring operations  $\mathbb{S}q^i$ , for  $i \geq 0$ .

The algebras considered in Section 9.3 have analogues in the study of  $\mathcal{B}$ :

**Lemma 9.4.4** *The  $\mathcal{H}$ -algebra  $\mathcal{H}[a_1, \dots, a_n, b_1, \dots, b_n]/a_i^2 = \tau b_i + \rho a_i$  has the structure of a  $\mathcal{B}$ -module.*

**Definition 9.4.5**

1. A sequence of non-negative integers  $I = (i_1, \dots, i_k)$  is said to be *admissible* of length  $k$  if either  $I = \emptyset$  or  $k \geq 1$ ,  $i_k \geq 1$  and  $i_{s-1} \geq 2i_s$ , for all  $k \geq s \geq 2$ .
2. Suppose that  $I$  is a sequence of non-negative integers, then let  $\mathbb{S}q^I$  denote the composite cohomology operation  $\mathbb{S}q^{i_1} \dots \mathbb{S}q^{i_k}$ , with the convention that  $\mathbb{S}q^\emptyset = 1$ .

**Proposition 9.4.6** *The  $\mathcal{H}$ -algebra  $\mathcal{B}$  is a free left  $\mathcal{H}$ -module on the basis*

$$\{\mathbb{S}q^I \mid I \text{ is an admissible sequence}\}.$$

**Proof:** (Indications) The Adem relations imply that the operations  $\mathbb{S}q^I$ , for  $I$  admissible, generate  $\mathcal{B}$  as an  $\mathcal{H}$ -module. The representations of  $\mathcal{B}$  on algebras of the form  $\mathcal{H}[a_1, \dots, a_n, b_1, \dots, b_n]/a_i^2 = \tau b_i + \rho a_i$  of Lemma 9.4.4 can be used to show that  $\mathcal{B}$  is free on the given basis. ■

The argument used in the proof of Proposition 9.4.6 can be made more precise as follows (compare with the discussion in Section 9.3):

**Lemma 9.4.7** *Let  $F_{\mathcal{B}}(n)$  denote the sub- $\mathcal{B}$ -module of  $\mathcal{H}[a_1, \dots, a_n, b_1, \dots, b_n]/a_i^2 = \tau b_i + \rho a_i$  generated by  $\prod_{i=1}^n a_i$ , then the induced morphism of  $\mathcal{B}$ -modules:*

$$\mathcal{B} \rightarrow \Sigma^{-n, -n} F_{\mathcal{B}}(n)$$

*is an isomorphism in bidegrees  $(*_1, *_2)$  with  $*_1 \leq n$ .*

The calculation of the stable motivic cohomology of the motivic Eilenberg MacLane spaces establishes that the operations  $\mathbb{S}q^j$  generate the algebra  $\mathcal{A}^{*,*}(k, \mathbb{Z}/2)$  over the coefficient ring:

**Theorem 9.4.8** *Let  $k$  be a field of characteristic zero. The motivic Steenrod algebra  $\mathcal{A}^{*,*}(k, \mathbb{Z}/2)$  is a free left  $H^{*,*}(\text{Spec}(k), \mathbb{Z}/2)$ -module on basis:*

$$\{\mathbb{S}q^I \mid I \text{ is an admissible sequence}\}.$$

## 9.5 Algebraic Duality

The structure of  $\mathcal{A}^{*,*}(k, \mathbb{Z}/2)$  can be studied algebraically by duality. The dual situation is related to the study of homology operations in the associated motivic homology theory; the general issues involved in duality theory lie outside the theme of this paper. The algebraic result, Theorem 9.5.10, is modelled on Milnor's calculation of the dual of the topological Steenrod algebra [Mil], paying attention to the fact that  $\mathcal{H}$  is not central in  $\mathcal{B}$ .

**Definition 9.5.1** Let  $\Gamma$  denote the  $\mathcal{H}$ -module of morphisms of left  $\mathcal{H}$ -modules  $\text{Hom}_{\mathcal{H}}(\mathcal{B}, \mathcal{H})$ ; the  $\mathcal{H}$ -module structure is induced by the product on  $\mathcal{H}$ . The  $\mathcal{H}$ -algebra  $\Gamma$  is bigraded via  $\Gamma^{s,t} := \text{Hom}_{\mathcal{H}}^{-s,-t}(\mathcal{B}, \mathcal{H})$ , using the standard conventions for grading homomorphisms between graded modules.

The basis of admissible monomials  $\mathbb{S}q^I$  has dual basis  $\eta_I$  and  $\Gamma$  is a free  $\mathcal{H}$ -module on this basis.

**Notation 9.5.2** Write  $\mathcal{H}(B_{\text{gm}}\mathbb{Z}/2)$  for the commutative, bigraded, augmented  $\mathcal{H}$ -algebra  $\mathcal{H}[a, b]/_{a^2=\tau b+\rho a}$ , where  $|a| = (1, 1)$  and  $|b| = (2, 1)$ . The augmentation is induced by  $a, b \mapsto 0$ .

The algebra  $\mathcal{H}(B_{\text{gm}}\mathbb{Z}/2)$  is a free left  $\mathcal{H}$ -module on classes  $\{ab^i, b^i | i \geq 0\}$ ; moreover  $\mathcal{H}(B_{\text{gm}}\mathbb{Z}/2)$  is equipped with a Hopf algebra structure over  $\mathcal{H}$ , with diagonal  $\Delta$  given by

$$\begin{aligned} \Delta_{\mathcal{H}} : \mathcal{H}(B_{\text{gm}}\mathbb{Z}/2) &\rightarrow \mathcal{H}(B_{\text{gm}}\mathbb{Z}/2) \otimes_{\mathcal{H}} \mathcal{H}(B_{\text{gm}}\mathbb{Z}/2) \\ a &\mapsto a \otimes 1 + 1 \otimes a \\ b &\mapsto b \otimes 1 + 1 \otimes b \end{aligned}$$

The action of  $\mathcal{A}^{*,*}(k, \mathbb{Z}/2)$  on  $H^{*,*}(B_{\text{gm}}\mathbb{Z}/2, \mathbb{Z}/2)$  induces an action:

$$\mathcal{B} \otimes_{\mathcal{H}} \mathcal{H}(B_{\text{gm}}\mathbb{Z}/2) \rightarrow \mathcal{H}(B_{\text{gm}}\mathbb{Z}/2),$$

where  $\mathcal{B}$  denotes the sub-algebra of  $\mathcal{A}$  defined in Corollary 9.4.6.

**Definition 9.5.3** The coaction map

$$\lambda : \mathcal{H}(B_{\text{gm}}\mathbb{Z}/2) \rightarrow \mathcal{H}(B_{\text{gm}}\mathbb{Z}/2) \hat{\otimes}_{\mathcal{H}} \Gamma$$

is the bigraded map:  $x \mapsto \sum_I \mathbb{S}q^I(x) \otimes \eta_I$ , where the tensor product is completed with respect to the filtration of  $\mathcal{H}(B_{\text{gm}}\mathbb{Z}/2)$  by the powers of the augmentation ideal. (This is the analogue of the Milnor coaction for a module over the topological Steenrod algebra).

The Cartan formula defines a map  $\Delta : \mathcal{B} \rightarrow \mathcal{B} \otimes_{\mathcal{H}_L} \mathcal{B}$ , where the tensor product  $\otimes_{\mathcal{H}_L}$  is defined with respect to the left  $\mathcal{H}$ -module structures on both factors. This induces a dual product map:

$$\mu : \Gamma \otimes_{\mathcal{H}} \Gamma \rightarrow \Gamma.$$

In terms of the explicit basis  $\eta_I$  of  $\Gamma$ :

$$\eta_I \eta_J = \sum c_{IJ}^K \eta_K,$$

where  $c_{IJ}^K$  is the coefficient of  $\mathbb{S}q^I \mathbb{S}q^J$  in  $\Delta \mathbb{S}q^K$ . The following is clear:

**Lemma 9.5.4** *The product  $\mu$  provides  $\Gamma$  with the structure of a commutative, bigraded  $\mathcal{H}$ -algebra.*

The product map  $\mathcal{H}(B_{\text{gm}}\mathbb{Z}/2) \otimes \mathcal{H}(B_{\text{gm}}\mathbb{Z}/2) \rightarrow \mathcal{H}(B_{\text{gm}}\mathbb{Z}/2)$  is a map of  $\mathcal{B}$ -modules, where the domain is given the structure of a  $\mathcal{B}$ -module, via  $\Delta$ .

**Lemma 9.5.5** *The coaction  $\lambda$  is a multiplicative map.*

**Proof:** This follows directly from the Cartan formula and the definition of the product in  $\Gamma$ .  $\blacksquare$

The coaction  $\lambda$  is determined by  $\lambda(a), \lambda(b)$ , by the multiplicativity given by Lemma 9.5.5. These elements can be written in terms of the standard  $\mathcal{H}$ -basis for  $\mathcal{H}(B_{\text{gm}}\mathbb{Z}/2)$  as:

$$\begin{aligned}\lambda(a) &= \sum_{i \geq 0} (ab^i \otimes x_i + b^i \otimes y_i) \\ \lambda(b) &= \sum_{i \geq 0} (ab^i \otimes x'_i + b^i \otimes y'_i),\end{aligned}$$

for suitable homogeneous elements  $x_i, x'_i, y_i, y'_i \in \mathcal{B}^\vee$ .

**Definition 9.5.6** Define the morphism

$$\lambda_n : \mathcal{H}(B_{\text{gm}}\mathbb{Z}/2)^{\otimes n} \rightarrow \mathcal{H}(B_{\text{gm}}\mathbb{Z}/2)^{\otimes n} \hat{\otimes}_{\mathcal{H}} \Gamma$$

to be the composite of  $\lambda^{\otimes n}$  with the morphism induced by the product  $\Gamma^{\otimes n} \rightarrow \Gamma$ .

**Lemma 9.5.7** *The coaction  $\lambda$  is additive: there is a commutative diagram*

$$\begin{array}{ccc} \mathcal{H}(B_{\text{gm}}\mathbb{Z}/2) & \xrightarrow{\lambda} & \mathcal{H}(B_{\text{gm}}\mathbb{Z}/2) \hat{\otimes}_{\mathcal{H}} \Gamma \\ \Delta_{\mathcal{H}} \downarrow & & \downarrow \Delta_{\mathcal{H}} \otimes 1 \\ \mathcal{H}(B_{\text{gm}}\mathbb{Z}/2) \otimes_{\mathcal{H}} \mathcal{H}(B_{\text{gm}}\mathbb{Z}/2) & \xrightarrow[\lambda_2]{} & (\mathcal{H}(B_{\text{gm}}\mathbb{Z}/2) \otimes_{\mathcal{H}} \mathcal{H}(B_{\text{gm}}\mathbb{Z}/2)) \hat{\otimes}_{\mathcal{H}} \Gamma. \end{array}$$

**Proof:** The morphism  $\lambda_2$  is the adjoint to the action of  $\mathcal{B}$  on the tensor product  $\mathcal{H}(B_{\text{gm}}\mathbb{Z}/2) \otimes_{\mathcal{H}} \mathcal{H}(B_{\text{gm}}\mathbb{Z}/2)$  via the diagonal  $\Delta$  of  $\mathcal{B}$ . The lemma is thus a restatement of the fact that the product  $\mathcal{H}(B_{\text{gm}}\mathbb{Z}/2) \otimes_{\mathcal{H}} \mathcal{H}(B_{\text{gm}}\mathbb{Z}/2) \rightarrow \mathcal{H}(B_{\text{gm}}\mathbb{Z}/2)$  is a morphism of  $\mathcal{B}$ -modules.  $\blacksquare$

The following lemma defines the ‘Milnor elements’ in  $\Gamma$ :

**Lemma 9.5.8** *There exist unique elements  $\tau_j, \xi_j \in \Gamma$ , for  $j \geq 0$  such that:*

1. (a)  $\xi_0 = 1$   
(b)  $|\tau_j| = (1 - 2^{j+1}, 1 - 2^j)$   
(c)  $|\xi_j| = (2(1 - 2^j), 1 - 2^j)$ .
2. (a)  $\lambda(a) = a \otimes 1 + \sum_{j \geq 0} b^{2^j} \otimes \tau_j$

$$(b) \lambda(b) = \sum_{j \geq 0} b^{2^j} \otimes \xi_j.$$

3. The element  $\tau_0 \in \Gamma^{-1,0}$  is the dual of the Bockstein.

**Proof:** This follows directly from the additivity of  $\lambda$ ; the coefficients of the terms in  $a, b$  are determined by the fact that the  $\mathcal{B}$  action on  $\mathcal{H}(B_{\text{gm}}\mathbb{Z}/2)$  is unital together with the identifications  $\Gamma^{0,0} \cong \mathbb{F}_2, \Gamma^{1,0} = 0$ .

The identification of  $\tau_0$  follows for reasons of bidegree. ■

**Lemma 9.5.9**

1. The morphism  $\lambda$  is determined on  $\mathcal{H}$  by:

$$(a) \lambda(\rho) = \rho \otimes 1$$

$$(b) \lambda(\tau) = \tau \otimes 1 + \rho \otimes \tau_0.$$

2. For  $j \geq 0$  there is a relation in  $\Gamma$ :

$$\tau_j^2 = \tau \xi_{j+1} + \rho \tau_{j+1} + \rho \tau_0 \xi_j.$$

**Proof:** The image of  $\rho$  under  $\lambda$  is determined by the grading and unit condition.

The remaining statements are consequences of the relation  $a^2 = \tau b + \rho a$  in  $\mathcal{H}(B_{\text{gm}}\mathbb{Z}/2)$ . The unit condition implies that

$$\lambda(\tau) = \tau \otimes 1 + \rho \otimes \kappa,$$

for a suitable  $\kappa$  of bidegree  $(-1,0)$ . A straightforward calculation using the multiplicativity of  $\lambda$  gives:

$$\begin{aligned} \lambda(a^2) &= \lambda b \otimes 1 + \rho a \otimes 1 + \sum b^{2^{j+1}} \otimes \tau_j^2 \\ \lambda(\tau b) &= \sum \tau b^{2^j} \otimes \xi_j + \sum \rho b^{2^j} \otimes \kappa \xi_j \\ \lambda(\rho a) &= \rho a \otimes 1 + \sum \rho b^{2^j} \otimes \tau_j \end{aligned}$$

Consider the relation  $\lambda(a^2) = \lambda(\tau b) + \lambda(\rho a)$ . Equating the coefficients of  $b$  yields the relation  $\rho(\kappa + \tau_0) = 0$ ;  $\Gamma$  is free as a left  $\mathcal{H}$ -module, so this implies that  $\kappa = \tau_0$ , giving the stated value of  $\lambda(\tau)$ . The coefficient of  $a$  yields no new information.

It remains to consider the coefficient of  $b^{2^j}$ , for  $j \geq 0$ ; this yields the stated relation for  $\tau_j^2$ . ■

The following theorem is the motivic version of Milnor's calculation of the dual of the Steenrod algebra.

**Theorem 9.5.10** *The commutative  $\mathcal{H}$ -algebra  $\Gamma$  is isomorphic to*

$$\mathcal{H}[\tau_0, \tau_1, \dots, \xi_1, \xi_2, \dots] / \tau_j^2 = \tau \xi_{j+1} + \rho \tau_{j+1} + \rho \tau_0 \xi_j.$$

**Proof:** (Indications) The proof is similar to the calculation of Milnor [Mil] for the odd prime case of the dual of the topological Steenrod algebra. The relation corresponds in the motivic case to the relation given by Lemma 9.5.9. It remains to show that there are no further relations; this is a consequence of the fact that tensor products of the algebras  $\mathcal{H}(B_{\text{gm}}\mathbb{Z}/2)$  provide faithful representations of  $\mathcal{B}$  through certain ranges of bidegrees (compare with Lemma 9.4.7). ■

**Corollary 9.5.11** *The free  $\mathcal{H}$ -module  $\Gamma$  has a basis consisting of monomials of the form*

$$\prod_{i \geq 0} \tau_i^{\varepsilon_i} \prod_{j \geq 1} \xi_i^{\tau_j}.$$

This basis defines a dual basis on  $\mathcal{B}$  and extends to a  $H^{*,*}(k, \mathbb{Z}/2)$ -basis of the motivic Steenrod algebra  $\mathcal{A}^{*,*}(k, \mathbb{Z}/2)$ . In particular, this gives the definition of the operations used in the proof of the Milnor Conjecture [V1].

The algebra  $\Gamma$  admits further structure:

**Theorem 9.5.12** *The pair  $(\mathcal{H}, \Gamma)$  has the structure of a Hopf algebroid and the morphism  $\lambda$  gives  $\mathcal{H}(B_{\text{gm}}\mathbb{Z}/2)$  the structure of a comodule over the Hopf algebroid  $(\mathcal{H}, \Gamma)$ . In particular, the right unit is given by the restriction of the coaction map to  $\mathcal{H}$  and the diagonal  $\Delta_\Gamma : \Gamma \rightarrow \Gamma \otimes_{\mathcal{H}} \Gamma$  is determined by:*

1.  $\Delta_\Gamma \xi_k = \sum_{i+j=k} \xi_j^{2^i} \otimes \xi_i.$
2.  $\Delta_\Gamma \tau_k = \tau_k \otimes 1 + \sum_{i+j=k} \xi_j^{2^i} \otimes \tau_i.$
3.  $\Delta_\Gamma \rho = 1 \otimes \rho \cong \rho \otimes 1.$
4.  $\Delta_\Gamma \tau = \tau \otimes 1 \cong 1 \otimes \tau + \tau_0 \otimes \rho.$

**Proof:** The Hopf algebroid structure corresponds to a dual structure on  $(\mathcal{H}, \mathcal{B})$ ; the diagonal morphism is calculated by studying the coaction map  $\lambda$ , using the fact that  $\Gamma$  is free as an  $\mathcal{H}$ -module together with the coassociativity of the coaction. ■



## Part III

# Construction of the Steenrod squares

## 10 The total Steenrod power

The Steenrod reduced powers for mod- $l$  motivic cohomology are defined by constructing the *total Steenrod power* map in  $\mathcal{H}_\bullet(k)$ :

$$\mathbb{Z}_{\text{tr}}(\mathbb{A}^n/(\mathbb{A}^n - \{0\}), \mathbb{Z}/l) \wedge B_{\text{gm}}\mathbb{Z}/l_+ \rightarrow \mathbb{Z}_{\text{tr}}(\mathbb{A}^{ln}/(\mathbb{A}^{ln} - \{0\}), \mathbb{Z}/l),$$

using the notation of 7.2.4, where  $B_{\text{gm}}\mathbb{Z}/l$  is a geometric model for the étale classifying space  $B_{\text{et}}\mathbb{Z}/l$  for the finite group  $\mathbb{Z}/l$ . (See Section 8.1 for a discussion of classifying spaces).

The construction works more generally; for a finite sub-group  $G \subset \Sigma_m$ , there is a map in the homotopy category  $\mathcal{H}_\bullet(k)$ :

$$\mathbb{Z}_{\text{tr}}(\mathbb{A}^n/(\mathbb{A}^n - \{0\}), \mathbb{Z}/l) \wedge B_{\text{gm}}G_+ \rightarrow \mathbb{Z}_{\text{tr}}(\mathbb{A}^{mn}/(\mathbb{A}^{mn} - \{0\}), \mathbb{Z}/l).$$

**Remark 10.0.13** The construction of the total Steenrod power is a variant, using the theory of Thom classes, of the use of the quadratic construction (respectively the cyclic construction, for  $l \neq 2$ ) in algebraic topology [G, §27]. This should be compared with the construction of the operations for complex cobordism which Rudyak [Ru, §VII.7] calls the Steenrod-tom Dieck operations. In particular, the theory of Thom classes permits the construction to be carried out integrally<sup>5</sup>.

### 10.1 A review of Galois coverings

The passage from Galois coverings to vector bundles is used essentially in this construction of the Steenrod total power map; for the convenience of the reader, this is reviewed in this section.

**Definition 10.1.1** [Mi] A morphism of schemes  $Y \xrightarrow{\phi} X$  is a Galois covering with group  $G$  if:

1. The finite group  $G$  acts on the right upon the scheme  $Y$  and  $\phi$  is  $G$ -invariant.
2. The morphism  $\phi$  is locally of finite type and faithfully flat.
3. The canonical morphism  $Y \times G \rightarrow Y \times_X Y$  is an isomorphism of right  $G$ -schemes.

---

<sup>5</sup>Voevodsky's modified construction of the Steenrod squares no longer uses the Thom class.

**Proposition 10.1.2** [SGA1, Exposé V, Proposition 2.6] *Let  $Y \rightarrow X$  be a morphism of schemes and let the finite group  $G$  act on the right on  $Y$ . Suppose that the scheme  $X$  is locally Noetherian, then the following conditions are equivalent:*

1. *The scheme  $Y$  is finite over  $X$ ,  $X = Y/G$  and the inertia groups of  $G$  acting upon  $Y$  are trivial.*
2. *There exists a quasi-compact, faithfully flat base change  $X_1 \rightarrow X$  such that the scheme  $Y \times_X X_1$  is isomorphic to  $X_1 \times G$  as a right  $G$ -scheme.*
3. *There exists a finite, étale, surjective base change  $X_1 \rightarrow X$  such that the scheme  $Y \times_X X_1$  is isomorphic to  $X_1 \times G$  as a right  $G$ -scheme.*
4. *The morphism  $\phi$  is faithfully flat and quasi-compact and the canonical morphism  $Y \times G \rightarrow Y \times_X Y$  is an isomorphism.*

A morphism satisfying the above conditions is termed a principal covering with Galois group  $G$  in [SGA1].

**Definition 10.1.3** Let  $G$  be a finite group which acts on the right upon the locally Noetherian scheme  $Y \in \text{Sch}/k$ ; say that  $G$  acts admissibly upon  $Y$  if every orbit is contained in an open affine sub-scheme which is stable under the  $G$ -action.

**Corollary 10.1.4** *Let  $G$  be a finite group which acts admissibly upon the scheme  $Y \in \text{Sch}/k$ , so that the quotient  $Y \rightarrow Y/G =: X$  exists and is a finite morphism. Suppose that  $G$  acts freely upon  $Y$ , then the morphism  $\phi : Y \rightarrow X$  is a Galois covering.*

Galois coverings (and, more generally, principal homogeneous spaces - see [Mi, Section III.4]) are classified by the first Čech cohomology group of the base space with coefficients in the group  $G$ , defined with respect to the flat topology. When the group is commutative, the following lemma applies:

**Lemma 10.1.5** *Let  $G$  be a commutative finite group. There is a one-one correspondence between Galois coverings of the form  $Y \rightarrow X$  with group  $G$  and elements of the étale cohomology group  $H^1(X_{\text{ét}}, G)$ .*

**Proposition 10.1.6** *Let  $G$  be a finite group and let  $G \hookrightarrow GL_n(k)$  be a closed immersion. Suppose that  $Y \rightarrow X$  is a Galois covering with group  $G$ . Let  $G$  act upon  $\mathbb{A}^n$  via the morphism  $G \hookrightarrow GL_n(k)$  and act diagonally upon  $Y \times \mathbb{A}^n$ , then the scheme  $(Y \times \mathbb{A}^n)/G$  is canonically a vector bundle over the scheme  $X := Y/G$ .*

**Proof:** (Indications) This is a consequence of the standard technique of faithfully-flat descent, using Hilbert's theorem 90. ■

The following basic argument allows morphisms to be factored through the quotient by a group action.

**Lemma 10.1.7** [Mi, II.1.4] *Let  $Y \rightarrow X$  be a Galois covering with group  $G$  and let  $F$  be an étale sheaf, then the morphism  $F(X) \rightarrow F(Y)$  induces an isomorphism  $F(Y) \rightarrow F(X)^G$ .*

In particular, under the hypotheses of the lemma, if  $Y \rightarrow F$  is a morphism of sheaves which is  $G$ -equivariant when  $F$  is given the trivial  $G$ -action, then the morphism factors canonically through the morphism of representable sheaves  $Y \rightarrow X$ .

## 10.2 The geometric construction

Let  $X \in Sm/k$  be a smooth scheme and let  $G \subset \Sigma_m$  be a finite discrete group; fix a geometric model  $EG/G$  for the geometric classifying space  $B_{\text{gm}}G$  (as in 8.1.4).

The exterior product of cycles and the diagonal  $\mathbb{Z}_{\text{tr}}(X) \rightarrow \mathbb{Z}_{\text{tr}}(X)^{\wedge m}$  defines a composite:

$$\phi : \mathbb{Z}_{\text{tr}}[X] \rightarrow \mathbb{Z}_{\text{tr}}[X]^{\wedge m} \rightarrow \mathbb{Z}_{\text{tr}}[X^m].$$

**Remark 10.2.1** This morphism induces the  $m^{\text{th}}$  cup product in motivic cohomology, when  $X = \mathbb{A}^n$ , after passage to a suitable quotient. This can be compared with the situation in algebraic topology, as follows. Let  $K(\mathbb{Z}, s)$  denote the  $s^{\text{th}}$  integral Eilenberg MacLane space; the cup product of singular cohomology is induced by structure maps  $K(\mathbb{Z}, s) \wedge K(\mathbb{Z}, t) \rightarrow K(\mathbb{Z}, s+t)$  in the homotopy category. The  $m^{\text{th}}$  cup product is induced by the composite  $K(\mathbb{Z}, s) \xrightarrow{\text{diag}} K(\mathbb{Z}, s)^{\wedge m} \rightarrow K(\mathbb{Z}, ms)$ , where the second morphism is the iteration of the structure map.

The morphism  $\phi$  is *not* a morphism of abelian sheaves, just as the morphism  $K(\mathbb{Z}, s) \rightarrow K(\mathbb{Z}, ms)$  in algebraic topology is not an abelian group object morphism.

**Notation 10.2.2** Let  $X \in Sm/k$  be a smooth scheme, and let the group  $G \subset \Sigma_m$  act diagonally on  $X^m \times EG$ . For notational clarity, the quotient  $(X^m \times EG)/G$  is written (abusively) as  $X^m \times_G EG$ .

Consider the composite:

$$\mathbb{Z}_{\text{tr}}[X] \wedge EG_+ \xrightarrow{\phi \wedge \text{id}} \mathbb{Z}_{\text{tr}}[X^m] \wedge EG_+ \rightarrow \mathbb{Z}_{\text{tr}}[X^m \times EG] \rightarrow \mathbb{Z}_{\text{tr}}[X^m \times_G EG].$$

This map is  $G$ -equivariant and the right hand side is an étale sheaf, by Proposition 4.1.6, hence there is a factorization through the quotient morphism  $\mathbb{Z}_{\text{tr}}[X] \wedge EG_+ \rightarrow \mathbb{Z}_{\text{tr}}[X] \wedge B_{\text{gm}}G_+$ , via passage to the colimit using Lemma 10.1.7. Thus there is an induced morphism:

$$\mathbb{Z}_{\text{tr}}[X] \wedge B_{\text{gm}}G_+ \rightarrow \mathbb{Z}_{\text{tr}}[X^m \times_G EG].$$

Suppose that  $Z \hookrightarrow X$  is a closed immersion, then the above construction passes to the quotient:

$$\mathbb{Z}_{\text{tr}}[X]/\mathbb{Z}_{\text{tr}}[X-Z] \wedge B_{\text{gm}}G_+ \rightarrow \mathbb{Z}_{\text{tr}}[X^m \times_G EG]/\mathbb{Z}_{\text{tr}}[(X^m - Z^m) \times_G EG].$$

In particular, the closed immersion  $0 \hookrightarrow \mathbb{A}^n$  induces the morphism:

$$\alpha_{m,n,G} : \mathbb{Z}_{\text{tr}}(\mathbb{A}^n/(\mathbb{A}^n - \{0\})) \wedge B_{\text{gm}}G_+ \rightarrow \mathbb{Z}_{\text{tr}}[\mathbb{A}^{mn} \times_G EG]/\mathbb{Z}_{\text{tr}}[(\mathbb{A}^{mn} - 0) \times_G EG].$$

**Notation 10.2.3** Let  $\text{Th}(\xi)$  denote the Thom space  $E(\xi)/E(\xi)^\times$  of the vector bundle  $\xi$ . Write:

$$1. \mathbb{Z}_{\text{tr}}(\text{Th}(\xi)) := \text{Coker}\{\mathbb{Z}_{\text{tr}}[E(\xi)^\times] \rightarrow \mathbb{Z}_{\text{tr}}[E(\xi)]\},$$

$$2. \mathbb{Z}_{\text{tr}}(\text{Th}(\xi), \mathbb{Z}/l) := \text{Coker}\{\mathbb{Z}_{\text{tr}}(\text{Th}(\xi)) \xrightarrow{\times l} \mathbb{Z}_{\text{tr}}(\text{Th}(\xi))\},$$

where, in both cases, the cokernel is taken in the category of abelian sheaves.

The sheaf  $\mathbb{Z}_{\text{tr}}(\mathbb{A}^n/(\mathbb{A}^n - \{0\}))$  identifies with  $\mathbb{Z}_{\text{tr}}(\text{Th}(\mathbb{A}^n))$ , by regarding  $\mathbb{A}^n$  as the trivial vector bundle of rank  $n$  over  $\text{Spec}(k)$ . Moreover, the scheme  $\xi := (\mathbb{A}^{mn} \times_G EG)$  is canonically a vector bundle of rank  $mn$  over  $B_{\text{gm}}G$ , by Proposition 10.1.6. Thus, the morphism  $\alpha_{m,n,G}$  can be written as:

$$\alpha_{m,n,G} : \mathbb{Z}_{\text{tr}}(\text{Th}(\mathbb{A}^n)) \wedge B_{\text{gm}}G_+ \rightarrow \mathbb{Z}_{\text{tr}}(\text{Th}(\mathbb{A}^{mn} \times_G EG)).$$

### 10.3 Homotopical constructions with Thom classes

The second part of the construction is homotopical, calling upon the theory of Thom classes for motivic cohomology. The factorization properties which are used concerning Thom classes are made explicit in this section.

The adjunction provided by Theorem 4.2.3 factors across the pointed homotopy category as:

$$M(-) : \mathcal{H}_\bullet(k) \rightleftarrows DM_-^{\text{eff}}(k) : K$$

where  $K$  denotes the derived functor of the Kan functor and  $M(-)$  is a suitable derived functor of  $\mathbb{Z}_{\text{tr}}(-)$ .

**Lemma 10.3.1** *Let  $k$  be a perfect field, let  $\mathcal{X} \in \Delta^{\text{op}}\text{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)_\bullet$  be a split simplicial sheaf and let  $A$  be an abelian group. A motivic cohomology class  $u \in \tilde{H}^{n,i}(\mathcal{X}, A)$  is represented by a morphism in  $\mathcal{H}_\bullet(k)$ ,  $\mathcal{X} \rightarrow K(A(i)[n])$  which factors canonically in the homotopy category as*

$$\mathcal{X} \rightarrow \mathbb{Z}_{\text{tr}}(\mathcal{X}) \rightarrow K(A(i)[n]).$$

**Proof:** (Indications) The adjunction theorem yields a canonical factorization in  $\mathcal{H}_\bullet(k)$ :

$$\mathcal{X} \rightarrow M(\mathcal{X}) \rightarrow K(A(i)[n])$$

of the morphism which represents the cohomology class  $u$ . The construction of the derived functor of  $\mathbb{Z}_{\text{tr}}(-)$  provides a natural morphism  $M(\mathcal{X}) \rightarrow \underline{C}_* \mathbb{Z}_{\text{tr}}(\mathcal{X})$ , which induces an isomorphism in  $\mathcal{H}_\bullet(k)$ , since the simplicial sheaf is split. ■

**Example 10.3.2** The Lemma applies to the simplicial cone of a morphism  $Y \rightarrow X$  between representable sheaves. In particular, this applies when  $i : Y \hookrightarrow X$  is an immersion; moreover, in this case, the Dold-Kan correspondence relates the objects  $\mathbb{Z}_{\text{tr}}[\text{Cone}(i)]$  and the quotient (of abelian sheaves)  $\mathbb{Z}_{\text{tr}}[X]/\mathbb{Z}_{\text{tr}}[Y]$ .

Throughout the rest of this section, let  $k$  denote a field which admits resolution of singularities, so that Thom classes are defined. The Thom class  $u_\xi$  of a vector bundle of rank  $n$  on  $X \in \mathcal{S}m/k$  is represented by a morphism in the homotopy category which factors naturally as

$$\text{Th}(\xi) \rightarrow \mathbb{Z}_{\text{tr}}(\text{Th}(\xi)) \rightarrow \mathbb{Z}_{\text{tr}}(\text{Th}(\mathbb{A}^n)),$$

since  $\mathbb{Z}_{\text{tr}}(\text{Th}(\mathbb{A}^n))$  is a model for the motivic Eilenberg-MacLane space which represents motivic cohomology of bidegree  $(2n, n)$ . Hence, the Thom class induces a *natural* morphism in  $\mathcal{H}_\bullet(k)$  which will be denoted (abusively) by

$$u_\xi : \mathbb{Z}_{\text{tr}}(\text{Th}(\xi)) \rightarrow \mathbb{Z}_{\text{tr}}(\text{Th}(\mathbb{A}^n))$$

and which shall be referred to as the extended Thom class of  $\xi$ .

**Example 10.3.3** Let  $\eta := X \times \mathbb{A}^n$  be the trivial bundle of rank  $n$  over the smooth scheme  $X \in \mathcal{S}m/k$ ; then there is an  $\mathbb{A}^1$ -weak equivalence  $\text{Th}(\eta) \simeq \text{Th}(\mathbb{A}^n) \wedge X_+$ . The uniqueness of the Thom class implies that the extension of the Thom class  $\mathbb{Z}_{\text{tr}}(\text{Th}(\eta)) \rightarrow \mathbb{Z}_{\text{tr}}(\text{Th}(\mathbb{A}^n))$  is induced by the projection  $\text{Th}(\eta) \rightarrow \text{Th}(\mathbb{A}^n)$  given by  $X \rightarrow \text{Spec}(k)$ .

Let  $G$  be a finite group; a vector bundle  $\xi$  of finite rank over  $X \in \mathcal{S}m/k$  is said to be a  $G$ -vector bundle if there are  $G$ -actions on the total space and the base space such that the structure morphism is  $G$ -equivariant and the action of  $G$  is linear on each fibre. The following result is the key to proving basic properties of the Steenrod total power morphism.

**Lemma 10.3.4** *Let  $k$  be a field which admits resolution of singularities and let  $\xi$  be a  $G$ -vector bundle over  $X \in \mathcal{S}m/k$  of rank  $n$  such that the action of  $G$  on  $X$  is free and admissible and such that  $\xi/G$  has the structure of a vector bundle of rank  $n$  over  $X/G$ .*

*Suppose further that  $Y \in \mathcal{S}m/k$  admits a free admissible  $G$ -action and that there is a  $G$ -equivariant morphism of sheaves  $f : Y \rightarrow \mathbb{Z}_{\text{tr}}(\text{Th}(\xi))$ . Then the composite  $u_\xi \circ f$  factors naturally in the homotopy category  $\mathcal{H}_\bullet(k)$  as*

$$Y \rightarrow Y/G \rightarrow \mathbb{Z}_{\text{tr}}(\text{Th}(\mathbb{A}^n)).$$

**Proof:** The canonical factorization is induced by the extended Thom class of  $\xi/G$ , via the commutative diagram:

$$\begin{array}{ccccc} Y & \longrightarrow & \mathbb{Z}_{\text{tr}}(\text{Th}(\xi)) & \xrightarrow{u_\xi} & \mathbb{Z}_{\text{tr}}(\text{Th}(\mathbb{A}^n)) \\ \downarrow & & \downarrow & \nearrow & \\ Y/G & \longrightarrow & \mathbb{Z}_{\text{tr}}(\text{Th}(\xi/G)) & & \end{array}$$

$u_{\xi/G}$

The triangle on the right commutes in the homotopy category by uniqueness of the extended Thom class, whereas the square on the left is obtained by passage to the quotient, using the fact that  $\mathbb{Z}_{\text{tr}}(\text{Th}(\xi/G))$  is an étale sheaf, by Lemma 10.1.7.  $\blacksquare$

## 10.4 The total Steenrod power and its properties

Throughout this section, let  $l$  be a rational prime and let  $k$  be a field which admits resolution of singularities.

**Definition 10.4.1** Let  $G$  be a subgroup of the symmetric group  $\Sigma_m$ ; the general total Steenrod power map

$$\mathbb{Z}_{\text{tr}}(\mathbb{A}^n/(\mathbb{A}^n - \{0\}), \mathbb{Z}/l) \wedge B_{\text{gm}}G_+ \rightarrow \mathbb{Z}_{\text{tr}}(\mathbb{A}^{mn}/(\mathbb{A}^{mn} - \{0\}), \mathbb{Z}/l)$$

is the composite of the map

$$\alpha_{m,n,G} : \mathbb{Z}_{\text{tr}}(\text{Th}(\mathbb{A}^n)) \wedge B_{\text{gm}}G_+ \rightarrow \mathbb{Z}_{\text{tr}}(\text{Th}(\xi/G)),$$

where  $\xi$  denotes the vector bundle  $\xi := \mathbb{A}^{mn} \times EG$ , with the extended Thom class  $u_{\xi/G}$ , by passing to coefficients in  $\mathbb{Z}/l$ .

**Notation 10.4.2**

1. Write  $K_{\mathcal{M}}(\mathbb{Z}/l, n)$  for the motivic Eilenberg MacLane space, which shall be represented by the explicit model  $\mathbb{Z}_{\text{tr}}(\mathbb{A}^n/(\mathbb{A}^n - \{0\}), \mathbb{Z}/l)$ .
2. For  $s, t \geq 0$  let  $\mu$  denote the product morphism for the motivic Eilenberg-MacLane spaces:  $\mu : K_{\mathcal{M}}(\mathbb{Z}/l, s) \wedge K_{\mathcal{M}}(\mathbb{Z}/l, t) \rightarrow K_{\mathcal{M}}(\mathbb{Z}/l, s+t)$  which is induced by the product of cycles.

**Definition 10.4.3** The total Steenrod power in the homotopy category  $\mathcal{H}_{\bullet}(k)$ :

$$\mathcal{P}_n : K_{\mathcal{M}}(\mathbb{Z}/l, n) \wedge B_{\text{gm}}\mathbb{Z}/l_+ \rightarrow K_{\mathcal{M}}(\mathbb{Z}/l, ln).$$

is the morphism obtained by setting  $G = \mathbb{Z}/l$  and  $m = l$  in Definition 10.4.1.

The construction and Proposition 10.3.4 show that the Steenrod total power map satisfies the following basic properties:

**Proposition 10.4.4**

1. *There is a commutative diagram in  $\mathcal{H}_{\bullet}(k)$ :*

$$\begin{array}{ccc} K_{\mathcal{M}}(\mathbb{Z}/l, s) \wedge K_{\mathcal{M}}(\mathbb{Z}/l, t) \wedge B_{\text{gm}}\mathbb{Z}/l_+ & \longrightarrow & K_{\mathcal{M}}(\mathbb{Z}/l, s) \wedge B_{\text{gm}}\mathbb{Z}/l_+ \wedge K_{\mathcal{M}}(\mathbb{Z}/l, t) \wedge B_{\text{gm}}\mathbb{Z}/l_+ \\ \mu \wedge 1 \downarrow & & \downarrow \mathcal{P}_s \wedge \mathcal{P}_t \\ K_{\mathcal{M}}(\mathbb{Z}/l, s+t) \wedge B_{\text{gm}}\mathbb{Z}/l_+ & & K_{\mathcal{M}}(\mathbb{Z}/l, ls) \wedge K_{\mathcal{M}}(\mathbb{Z}/l, lt) \\ \mathcal{P}_{s+t} \downarrow & & \downarrow \mu \\ K_{\mathcal{M}}(\mathbb{Z}/l, l(s+t)) & \xlongequal{\quad\quad\quad} & K_{\mathcal{M}}(\mathbb{Z}/l, l(s+t)), \end{array}$$

in which the top horizontal arrow is induced by the diagonal on  $B_{\text{gm}}\mathbb{Z}/l$  and a permutation of the factors.

2. If  $\text{Spec}(k) \rightarrow B_{\text{gm}}\mathbb{Z}/l$  is a distinguished base point, then the composite:

$$K_{\mathcal{M}}(\mathbb{Z}/l, n) \rightarrow K_{\mathcal{M}}(\mathbb{Z}/l, n) \wedge B_{\text{gm}}\mathbb{Z}/l_+ \rightarrow K_{\mathcal{M}}(\mathbb{Z}/l, ln)$$

is the  $l^{\text{th}}$  power map.

**Proof:** It suffices to establish the proposition working integrally, so that the model for  $K_{\mathcal{M}}(\mathbb{Z}/l, n)$  is replaced by  $\mathbb{Z}_{\text{tr}}(\text{Th}(\mathbb{A}^n))$ .

1. There is a commutative diagram:

$$\begin{array}{ccc} \mathbb{Z}_{\text{tr}}[\mathbb{A}^s] \wedge \mathbb{Z}_{\text{tr}}[\mathbb{A}^t] \wedge EZ/l_+ & \rightarrow & \mathbb{Z}_{\text{tr}}[\mathbb{A}^s] \wedge \mathbb{Z}_{\text{tr}}[\mathbb{A}^t] \wedge (EZ/l \times EZ/l)_+ \\ \downarrow & & \downarrow \\ \mathbb{Z}_{\text{tr}}[\mathbb{A}^s \times \mathbb{A}^t \times EZ/l] & \rightarrow & \mathbb{Z}_{\text{tr}}[\mathbb{A}^s \times \mathbb{A}^t \times EZ/l \times EZ/l] \end{array}$$

where the horizontal arrows are induced by the diagonal  $EZ/l \rightarrow EZ/l \times EZ/l$ . This can be considered as a diagram of  $\mathbb{Z}/l$ -equivariant morphisms, where the group  $\mathbb{Z}/l$  acts diagonally via  $\mathbb{Z}/l \rightarrow \mathbb{Z}/l \times \mathbb{Z}/l$  on the right hand side of the diagram. The passage to the quotient  $EZ/l \times EZ/l \rightarrow \text{Spec}(k)$  will induce the Thom class for the respective trivial bundles (see Example 10.3.3), and there is a commutative diagram:

$$\begin{array}{ccc} \mathbb{Z}_{\text{tr}}[\mathbb{A}^s] \wedge \mathbb{Z}_{\text{tr}}[\mathbb{A}^t] \wedge EZ/l_+ & \rightarrow & \mathbb{Z}_{\text{tr}}[\mathbb{A}^s] \wedge \mathbb{Z}_{\text{tr}}[\mathbb{A}^t] \\ \downarrow & & \downarrow \\ \mathbb{Z}_{\text{tr}}[\mathbb{A}^s \times \mathbb{A}^t \times EZ/l] & \rightarrow & \mathbb{Z}_{\text{tr}}[\mathbb{A}^s \times \mathbb{A}^t]. \end{array}$$

The commutative diagram of the Proposition is obtained by passage to Thom spaces and taking the quotient by the group action, using the argument of Proposition 10.3.4.

2. Take a base point for  $B_{\text{gm}}\mathbb{Z}/l$  which is induced by a point  $\text{Spec}(k) \rightarrow EZ/l$ . The composite  $K_n \rightarrow K_n \wedge B_{\text{gm}}\mathbb{Z}/l_+ \rightarrow K_{ln}$  is induced by passage to quotient from the composite

$$\mathbb{Z}_{\text{tr}}[\mathbb{A}^n] \xrightarrow{\phi} \mathbb{Z}_{\text{tr}}[\mathbb{A}^{nl} \times \text{Spec}(k)] \rightarrow \mathbb{Z}_{\text{tr}}[\mathbb{A}^{nl} \times EZ/l] \rightarrow \mathbb{Z}_{\text{tr}}[\mathbb{A}^{nl} \times_{\mathbb{Z}/l} EZ/l]$$

together with the extended Thom class for  $\mathbb{A}^{nl} \times_{\mathbb{Z}/l} EZ/l$ . By Proposition 10.3.4, it suffices to consider the composite  $\mathbb{Z}_{\text{tr}}[\mathbb{A}^n] \xrightarrow{\phi} \mathbb{Z}_{\text{tr}}[\mathbb{A}^{nl} \times \text{Spec}(k)] \rightarrow \mathbb{Z}_{\text{tr}}[\mathbb{A}^{nl} \times EZ/l]$  together with the Thom class of  $\mathbb{A}^{nl} \times EZ/l$ . The Thom class of this trivial bundle is induced by the projection  $\mathbb{A}^{nl} \times EZ/l \rightarrow \mathbb{A}^{nl}$ , from which the result follows, by the definition of the product map in motivic cohomology. ■

The derivation of the Adem relations for motivic cohomology in Proposition 9.4.1 depends on the following result.

**Proposition 10.4.5** *The composite morphism:*

$$\mathcal{P}_{ln} \circ (\mathcal{P}_n \wedge 1) : K_{\mathcal{M}}(\mathbb{Z}/l, n) \wedge B_{\text{gm}}\mathbb{Z}/l_+ \wedge B_{\text{gm}}\mathbb{Z}/l_+ \rightarrow K_{\mathcal{M}}(\mathbb{Z}/l, l^2n)$$

is  $\mathbb{Z}/2$ -equivariant, where  $\mathbb{Z}/2$  transposes the factors  $B_{\text{gm}}\mathbb{Z}/l_+$  and acts trivially on the right hand side.

**Proof:** (Indications). For clarity, write  $G, H$  for the groups  $\mathbb{Z}/l$  in the proof. There is an evident commutative diagram in the category of sheaves of sets:

$$\begin{array}{ccccc}
\mathbb{Z}_{\text{tr}}[\mathbb{A}^n] \wedge EG_+ \wedge EH_+ & \longrightarrow & \mathbb{Z}_{\text{tr}}[\mathbb{A}^{nl} \times EG] \wedge EH_+ & \longrightarrow & \mathbb{Z}_{\text{tr}}[\mathbb{A}^{nl}] \wedge EH_+ \\
\downarrow & & & & \downarrow \\
& & & & \mathbb{Z}_{\text{tr}}[\mathbb{A}^{nl^2} \times EH] \\
& & & & \downarrow \\
\mathbb{Z}_{\text{tr}}[\mathbb{A}^{nl^2} \times EG \times EH] & \longrightarrow & & \longrightarrow & \mathbb{Z}_{\text{tr}}[\mathbb{A}^{nl^2}].
\end{array}$$

Passing to the quotient by the respective complements of zero sections yields a diagram which corresponds to the construction of the Steenrod total power *without* taking the quotient by the group actions.

Lemma 10.3.4 shows that first taking the quotient by the  $G$ -action and subsequently taking the quotient by the  $H$ -action, the composite around the top of this diagram induces the composite of two total Steenrod power maps.

Hence, it suffices to consider the passage to the quotient induced by the composite around the bottom of the diagram; using Lemma 10.3.4, this factors across the Thom class for the vector bundle  $\mathbb{A}^{nl^2} \times_{G \times H} EG \times EH$ . Hence, the induced map:

$$\mathbb{Z}_{\text{tr}}(\text{Th}(\mathbb{A}^n)) \wedge EG_+ \wedge EH_+ \rightarrow \mathbb{Z}_{\text{tr}}(\text{Th}(\mathbb{A}^{nl^2}))$$

is equivariant with respect to the  $\mathbb{Z}/2$ -action, by uniqueness of the Thom class of the bundle  $\mathbb{A}^{nl^2} \times_{G \times H} EG \times EH$ . ■



## 11 Definition of the Steenrod squares

Throughout this section, let  $l$  denote a rational prime. The total Steenrod power  $\mathcal{P}_n : K_{\mathcal{M}}(\mathbb{Z}/l, n) \wedge B_{\text{gm}}\mathbb{Z}/l_+ \rightarrow K_{\mathcal{M}}(\mathbb{Z}/l, ln)$  induces a natural transformation, which will be denoted (abusively) by  $\mathcal{P}_n^*$ :

$$\mathcal{P}_n^* : \tilde{H}^{2n,n}(\mathcal{X}, \mathbb{Z}/l) \rightarrow \tilde{H}^{2ln,ln}(\mathcal{X} \wedge B_{\text{gm}}\mathbb{Z}/l_+, \mathbb{Z}/l)$$

sending a cohomology class represented by  $\mathcal{X} \rightarrow K_{\mathcal{M}}(\mathbb{Z}/l, n)$  to the composite:

$$\mathcal{X} \wedge B_{\text{gm}}\mathbb{Z}/l_+ \rightarrow K_{\mathcal{M}}(\mathbb{Z}/l, n) \wedge B_{\text{gm}}\mathbb{Z}/l_+ \rightarrow K_{\mathcal{M}}(\mathbb{Z}/l, ln),$$

which represents a cohomology class in the given group.

The Steenrod reduced powers are defined by describing the map  $\mathcal{P}_n^*$  using the calculation of  $\tilde{H}^{2ln,ln}(\mathcal{X} \wedge B_{\text{gm}}\mathbb{Z}/l_+, \mathbb{Z}/l)$  given in Section 8.3 (when the field  $k$  contains a primitive  $l^{\text{th}}$  root of unity).

### 11.1 Definition of the Steenrod squares

Throughout this section, let  $k$  be a field which admits resolution of singularities and which contains a primitive  $l^{\text{th}}$  root of unity. (The latter hypothesis is vacuous if  $l = 2$  and can be removed by using transfer arguments). In this case  $B_{\text{gm}}\mathbb{Z}/l$  is equivalent to  $B_{\text{et}}\mu_l$ , hence the calculations of Section 8.3 are available. The modifications necessary in the general case are indicated in Section 11.6.

The definition will be given in terms of unreduced motivic cohomology; hence, if  $\mathcal{X}$  is a pointed simplicial sheaf, the reduced cohomology of  $\mathcal{X}_+$  is considered. The morphism  $\mathcal{P}_n^*$  identifies as:

$$H^{2n,n}(\mathcal{X}, \mathbb{Z}/l) \rightarrow H^{2ln,ln}(\mathcal{X} \times B_{\text{et}}\mu_l, \mathbb{Z}/l).$$

**Notation 11.1.1** Define classes  $u_k$ , for  $k \geq 0$ , by  $u_{2i} := b^i$  and  $u_{2i+1} := ab^i$ .

**Definition 11.1.2** Let  $x \in H^{2n,n}(X, \mathbb{Z}/l)$  be a class in motivic cohomology; for  $j$  an integer, let  $D_j(x) \in H^{*,*}(X, \mathbb{Z}/l)$  denote the unique classes such that:

$$\mathcal{P}_n^*(x) = \sum D_j(x)u_j. \tag{1}$$

(Proposition 7.1.4 implies that the sum above contains only finitely many non-zero terms).

**Lemma 11.1.3** Let  $k$  be a field of characteristic zero. The association  $x \mapsto D_j(x)$  define natural transformations of sets:

$$\begin{aligned} D_{2i} : H^{2n,n}(\mathcal{X}, \mathbb{Z}/l) &\rightarrow H^{2(ln-i), (ln-i)}(\mathcal{X}, \mathbb{Z}/l) \\ D_{2i+1} : H^{2n,n}(\mathcal{X}, \mathbb{Z}/l) &\rightarrow H^{2(ln-i)-1, (ln-i)-1}(\mathcal{X}, \mathbb{Z}/l) \end{aligned}$$

**Proof:** The naturality follows directly from the naturality of the isomorphism given in Proposition 8.3.6 and of its extension to the consideration of the cohomology of simplicial sheaves.  $\blacksquare$

The non-trivial  $D_j$  define the Steenrod reduced powers, up to relabelling and renormalization.

**Definition 11.1.4** Let  $k$  be a field of characteristic zero. The motivic Steenrod reduced square operations  $\mathbb{S}q^j$ , for  $j \in \mathbb{Z}$ , are defined on  $x \in H^{2n,n}(\mathcal{X}, \mathbb{Z}/2)$  by:

$$\begin{aligned}\mathbb{S}q^j(x) &:= 0 \text{ if } j > 2n \\ \mathbb{S}q^{2i}(x) &:= D_{2(n-i)}(x) \text{ if } i \leq n \\ \mathbb{S}q^{2i-1}(x) &:= D_{2(n-i)+1}(x) \text{ if } i \leq n\end{aligned}$$

**Remark 11.1.5**

1. The fact that the  $\mathbb{S}q^j$  define bistable cohomology operations is proven in Corollary 11.3.2.
2. It is not clear from the definition that the natural transformations  $\mathbb{S}q^j$  are trivial for  $j < 0$ .

## 11.2 The Weak Cartan formula

The Cartan formula calculates the Steenrod reduced power operations on products. A weak form of the Cartan formula can be proved immediately for cohomology classes in bidegrees of the form  $2*, *$ . The general Cartan formula will follow once it is known that the operations are bistable and that the operations  $\mathbb{S}q^j$  for negative  $j$  are trivial.

**Lemma 11.2.1** *Let  $k$  be a field of characteristic zero. Suppose that  $\mathcal{X}, \mathcal{Y}$  are simplicial sheaves and that  $u \in H^{2m,m}(\mathcal{X}, \mathbb{Z}/2)$ ,  $v \in H^{2n,n}(\mathcal{Y}, \mathbb{Z}/2)$  are motivic cohomology classes. For all  $i \geq 0$ , there are equalities:*

$$\begin{aligned}\mathbb{S}q^{2i}(u \times v) &= \sum_{a+b=i} \mathbb{S}q^{2a}(u) \times \mathbb{S}q^{2b}(v) + \tau \left\{ \sum_{a+b=i-2} \mathbb{S}q^{2a+1}(u) \times \mathbb{S}q^{2b+1}(v) \right\} \\ \mathbb{S}q^{2i+1}(u \times v) &= \sum_{j+k=2i+1} \mathbb{S}q^j(u) \times \mathbb{S}q^k(v) + \rho \left\{ \sum_{a+b=i-1} \mathbb{S}q^{2a+1}(u) \times \mathbb{S}q^{2b+1}(v) \right\}\end{aligned}$$

in  $H^{*,*}(\mathcal{X} \times \mathcal{Y}, \mathbb{Z}/2)$ .

**Proof:** The classes  $u, v$  are represented respectively by maps  $\mathcal{X} \rightarrow K_{\mathcal{M}}(\mathbb{Z}/l, m)$  and  $\mathcal{Y} \rightarrow K_{\mathcal{M}}(\mathbb{Z}/l, n)$  in the unpointed homotopy category. The product  $u \times v$  is represented by the composite:

$$\mathcal{X} \times \mathcal{Y} \rightarrow K_{\mathcal{M}}(\mathbb{Z}/l, m) \times K_{\mathcal{M}}(\mathbb{Z}/l, n) \rightarrow K_{\mathcal{M}}(\mathbb{Z}/l, m+n).$$

The class  $\mathcal{P}_{m+n}^*$  is represented by the map  $(\mathcal{X} \times \mathcal{Y}) \times B_{\text{gm}}\mathbb{Z}/l \rightarrow K_{\mathcal{M}}(\mathbb{Z}/l, l(m+n))$  which, by Proposition 10.4.4 (1), agrees with the composite

$$\begin{aligned} \mathcal{X} \times \mathcal{Y} \times B_{\text{gm}}\mathbb{Z}/l &\rightarrow \mathcal{X} \times \mathcal{Y} \times B_{\text{gm}}\mathbb{Z}/l \times B_{\text{gm}}\mathbb{Z}/l \rightarrow \\ (\mathcal{X} \times B_{\text{gm}}\mathbb{Z}/l) \times (\mathcal{Y} \times B_{\text{gm}}\mathbb{Z}/l) &\rightarrow K_{\mathcal{M}}(\mathbb{Z}/l, l(m+n)), \end{aligned}$$

where the map  $(\mathcal{X} \times B_{\text{gm}}\mathbb{Z}/l) \times (\mathcal{Y} \times B_{\text{gm}}\mathbb{Z}/l) \rightarrow K_{\mathcal{M}}(\mathbb{Z}/l, l(m+n))$  is the product of the classes  $\mathcal{P}_m^*(u)$  and  $\mathcal{P}_n^*(v)$ .

In unreduced cohomology, the map

$$\begin{array}{ccc} H^{*,*}(\mathcal{X} \times \mathcal{Y} \times B_{\text{gm}}\mathbb{Z}/l \times B_{\text{gm}}\mathbb{Z}/l, \mathbb{Z}/l) &\rightarrow & H^{*,*}(\mathcal{X} \times \mathcal{Y} \times B_{\text{gm}}\mathbb{Z}/l, \mathbb{Z}/l) \\ \parallel & & \parallel \\ H^{*,*}(\mathcal{X} \times \mathcal{Y}, \mathbb{Z}/l)[a, b, a', b'] / \sim &\rightarrow & H^{*,*}(\mathcal{X} \times \mathcal{Y}, \mathbb{Z}/l)[a, b] / \sim \end{array}$$

is induced by  $a, a' \mapsto a$  and  $b, b' \mapsto b$  and where the  $\sim$  indicates the equivalence relation derived from Proposition 8.3.6. The result is now straightforward by identification of cohomology classes.  $\blacksquare$

### 11.3 Bistability

The fact that the Steenrod reduced power operations extend to define bistable cohomology operations is a consequence of the projective bundle theorem together with the weak Cartan formula. Recall that the pointed simplicial sheaf  $T$  is equivalent to  $\mathbb{P}^1$  in the pointed homotopy category  $\mathcal{H}_\bullet(k)$ .

Let  $k$  be a field admitting resolution of singularities, and let  $\mathcal{X}$  be a simplicial sheaf, then the projective bundle theorem implies that there is an isomorphism:

$$H^{*,*}(\mathcal{X} \times T, \mathbb{Z}/2) \cong H^{*,*}(\mathcal{X}, \mathbb{Z}/2)[c]/c^2 = 0,$$

where  $c$  is the (mod-2 reduction of the) first Chern class of the canonical line bundle on  $\mathbb{P}^1$  in bidegree  $(2, 1)$ . The  $T$ -suspension  $\sigma_T$  is induced by cup product with the class  $c$ , hence the Cartan formula can be applied:

**Lemma 11.3.1** *The operation  $\mathbb{S}q^i$  commutes with  $\sigma_T$ .*

**Proof:** The motivic Steenrod squares act by  $\mathbb{S}q^i(c) = c$ , if  $i = 0$ , and 0 otherwise, for reasons of bidegree. The weak Cartan formula of Lemma 11.2.1 immediately implies the result.  $\blacksquare$

Hence:

**Corollary 11.3.2** *Let  $k$  be a field of characteristic zero. The operations  $\mathbb{S}q^i$  are bistable cohomology operations for  $\mathbb{Z}/2$ -motivic cohomology, with bidegrees:*

$$\begin{aligned} |\mathbb{S}q^{2i}| &= (2i, i) \\ |\mathbb{S}q^{2i+1}| &= (2i+1, i). \end{aligned}$$

## 11.4 The proof of Theorem 2

Let  $k$  be a field of characteristic zero and consider the prime  $l = 2$ . The cohomology operations  $\mathbb{S}q^j$  were constructed in Section 11.1, for integers  $j$ , and were shown to be bistable in Corollary 11.3.2. The  $\mathbb{S}q^j$  in degrees  $j \leq 1$  are identified by:

**Proposition 11.4.1** *Let  $k$  be a field of characteristic zero. The Steenrod squares  $\mathbb{S}q^j$  are trivial for  $j < 0$ . There are identifications:*

1.  $\mathbb{S}q^0 = \text{Id}$
2.  $\mathbb{S}q^1 = \beta$ .

**Proof:** (Indications) The proof is by the method of the universal example: namely the calculation of  $\tilde{H}^{*,*}(K_n, \mathbb{Z}/l)$  in degrees  $\leq 2(n+1)$ . (In fact, by Proposition 9.1.4, it is sufficient to consider the ‘stable cohomology’ of the motivic Eilenberg-MacLane spaces)<sup>6</sup>. ■

**Remark 11.4.2** In algebraic topology, there is a straightforward proof that the Steenrod squaring operation  $\mathbb{S}q^j$  is trivial for  $j < 0$ , based on the skeletal filtration of a simplicial set.

The Cartan formula is proved from the weak Cartan formula, Lemma 11.2.1, by using bistability.

**Remark 11.4.3** To state the Cartan formula in the form given in the Theorem requires the identification of the motivic Steenrod squares of odd degree:

$$\mathbb{S}q^{2i+1} = \beta \mathbb{S}q^{2i}.$$

This is one of the Adem relations; the proof of the Adem relations uses the Cartan formula but the argument is not circular, since the above identification is not used in the proof.

The instability condition is derived as follows: suppose that  $u \in H^{2i,i}(X, \mathbb{Z}/2)$  is a cohomology class represented by a morphism  $X \rightarrow K_{\mathcal{M}}(\mathbb{Z}/2, i)$ , then the class  $\mathbb{S}q^{2i}(u)$  is represented by the composite

$$X \rightarrow X \wedge B_{\text{gm}}\mathbb{Z}/2_+ \rightarrow K_{\mathcal{M}}(\mathbb{Z}/2, i) \wedge B_{\text{gm}}\mathbb{Z}/2_+ \rightarrow K_{\mathcal{M}}(\mathbb{Z}/2, 2i),$$

where the first map is induced by a point of  $B_{\text{gm}}\mathbb{Z}/2$ . Proposition 10.4.4(2) implies that this corresponds to the cup square of the class  $u$ . Suppose now that  $u \in H^{n,i}(X, \mathbb{Z}/2)$ , with  $n < 2i$ , then consider the simplicial suspension of  $u$ :

$$\sigma_s^{2i-n}(u) \in H^{2i,i}(\Sigma_s^{2i-n} X, \mathbb{Z}/2);$$

---

<sup>6</sup>This is non-trivial.

the operation  $\mathbb{S}q^{2i}$  is bistable and the cup product in the cohomology of a simplicial suspension is trivial, since  $\Sigma_s X$  has the structure of a cogroup object so that the diagonal  $\Sigma_s X \rightarrow \Sigma_s X \times \Sigma_s X$  factors up to homotopy across the inclusion of the wedge  $\Sigma_s X \vee \Sigma_s X \hookrightarrow \Sigma_s X \times \Sigma_s X$ . It follows that  $\mathbb{S}q^{2i}(u)$  is trivial in this case.

Uniqueness of the operations follows once that it is known that the  $\mathbb{S}q^j$  generate the motivic Steenrod algebra  $\mathcal{A}^{*,*}(k, \mathbb{Z}/2)$  over the coefficient ring  $H^{*,*}(\text{Spec}(k), \mathbb{Z}/2)$ . For example, the action of the Steenrod squares  $\mathbb{S}q^{2i}$  on the motivic cohomology of products of  $B_{\text{et}}\mathbb{Z}/2$  (respectively  $\mathbb{P}^\infty$ ) is determined by the properties given. Uniqueness is equivalent to the assertion that for any non-trivial bistable cohomology operation, there exists a cohomology class in  $H^{*,*}((B_{\text{gm}}\mathbb{Z}/2)^{\times m}, \mathbb{Z}/2)$ , for some  $m$ , on which the operation acts non-trivially. (See the discussion in Section 9.3).

## 11.5 A proof of the Adem Relations

The motivic Steenrod algebra satisfies Adem relations similar to those for the topological Steenrod algebra, which were stated in Proposition 9.4.1. The Adem relations can be derived from the fact that

$$\mathcal{P}_{ln} \circ (\mathcal{P}_n \wedge 1) : K_{\mathcal{M}}(\mathbb{Z}/l, n) \wedge B_{\text{et}}\mathbb{Z}/l_+ \wedge B_{\text{et}}\mathbb{Z}/l_+ \rightarrow K_{\mathcal{M}}(\mathbb{Z}/l, l^2 n)$$

is  $\mathbb{Z}/2$ -equivariant (see Proposition 10.4.5).

**Remark 11.5.1** It is also possible to derive the Adem relations using the representation of the motivic Steenrod algebra upon the motivic cohomology algebras  $H^{*,*}((B_{\text{gm}}\mathbb{Z}/2)^{\times n}, \mathbb{Z}/2)$ .

Recall that there is an isomorphism of  $H^{*,*}(X, \mathbb{Z}/2)$ -modules:

$$H^{*,*}(X \times_{B_{\text{et}}\mathbb{Z}/2} \times_{B_{\text{et}}\mathbb{Z}/2}, \mathbb{Z}/2) \cong H^{*,*}(X, \mathbb{Z}/2)[a, b, a', b'] / (a^2 = \tau b + \rho a, a'^2 = \tau b' + \rho a').$$

The Adem relations are given by considering the class  $\mathcal{P}^* \circ \mathcal{P}^* x$ , for a suitable universal choice of cohomology class, and equating the coefficients of  $a^{\epsilon_1} b^{i_1} a'^{\epsilon_2} b'^{i_2}$  and  $a^{\epsilon_2} b^{i_2} a'^{\epsilon_1} b'^{i_1}$ , for suitable choices of indices. To calculate the class  $\mathcal{P}^* \circ \mathcal{P}^* x$ , the Cartan formula is used together with the action of the Steenrod squares  $\mathbb{S}q^i$  on  $H^{*,*}(B_{\text{et}}\mathbb{Z}/2, \mathbb{Z}/2)$  (see Lemma 9.3.3).

Using the notation of Notation 11.1.1, the equations of Lemma 9.3.3 reduce to:

$$\mathbb{S}q^i u_k = \binom{k}{i} u_{k+i},$$

which do not involve  $\tau, \rho$ . The terms involving  $\rho, \tau$  arise from the Cartan formula and are therefore linear. The coefficient of  $\tau$  can be deduced by a weight argument from the topological Adem relations, hence it remains to calculate the term which is linear in  $\rho$ , which shall be called the *perturbation* from the topological Adem relations. In particular, setting  $\rho = 0$  and  $\tau = 1$ , one obtains the topological Adem relations (see Section 9.4). The calculation follows that

of Steenrod and Epstein [SE].

**Proof of Proposition 9.4.1:** It suffices to calculate the perturbation term in  $\rho$ , which arises from the Cartan formula for  $\mathbb{S}q^*$ , with  $*$  odd. Henceforth, it is sufficient to work only in terms of the degree of an element and not its weight.

Following [SE, p.119] (in particular using their notation  $k, c, s$ ), the Adem relation for  $\mathbb{S}q^k \mathbb{S}q^c$ , ( $0 < k < 2c$ ), is derived from  $\mathcal{P}^* \mathcal{P}^*(x)$ , for  $|x| = 2^s - 1 + c$ , where  $2^s > k$ . The relations are given by equating the coefficients of  $u_{2|x|-k} u'_{|x|-c}$  and  $u_{|x|-c} u'_{2|x|-k}$  in the expression

$$\mathcal{P}^* \mathcal{P}^*(x) = \sum_{j,l} \mathbb{S}q^l (\mathbb{S}q^j(x) u_{|x|-j}) u'_{2|x|-l}. \quad (2)$$

Consider the coefficient of  $u_{2|x|-k} u'_{|x|-c}$ ; these terms arise for  $2|x| - l = |x| - c$ , so that  $l = |x| + c = 2^s + 2c - 1$ , hence  $l$  is odd. For given  $j$ , the perturbation term from this expression is

$$\rho \sum_{m \text{ odd}} \mathbb{S}q^{l-m-1} \mathbb{S}q^j(x) \mathbb{S}q^m(u_{|x|-j}) u'_{|x|-c}.$$

and the terms in  $u_{2|x|-k} u'_{|x|-c}$ , are given by  $m = |x| - k + j$ . Hence, the perturbation term is:

$$\rho \sum_{\{j \mid |x|-k+j \text{ odd}\}} \mathbb{S}q^{l-m-1} \mathbb{S}q^j(x) \binom{|x|-j}{m} u_{2|x|-k} u'_{|x|-c}.$$

Similarly, consider the perturbation term in the coefficient of  $u_{|x|-c} u'_{2|x|-k}$  of equation (2); these terms arise for  $2|x| - k = 2|x| - l$ , so that  $l = k$ . Hence, if  $k$  is even, there is no perturbation term.

Thus, suppose that  $k$  is odd; proceeding as above, the perturbation term is:

$$\rho \sum_{n \text{ odd}} \mathbb{S}q^{l-n-1} \mathbb{S}q^j(x) \mathbb{S}q^n(u_{|x|-j}) u'_{2|x|-k},$$

where  $n + |x| - j = |x| - c$ , so that  $n = j - c$ . Now,  $\mathbb{S}q^n(u_{|x|-j}) = \binom{|x|-j}{n} u_{|x|-c}$  and  $|x| - j = |x| - (n + c) = 2^s - 1 - n$  is even, since  $n$  is odd. The binomial coefficient is thus trivial over  $\mathbb{Z}/2$ , so that the perturbation coefficient of  $u_{|x|-c} u'_{2|x|-k}$  is trivial.

Thus, the only additional term in the Adem relations is for  $k$  even

$$\rho \sum_{\{j \mid j \equiv c(2)\}} \binom{|x|-j}{|x|+j-k} \mathbb{S}q^{k+c-j-1} \mathbb{S}q^j(x).$$

Finally,  $|x| = 2^s - 1 + c$ , so that the binomial coefficient is equal to  $\binom{2^s-1+c-j}{k-2j} = \binom{c-j-1}{k-2j}$ , which completes the proof.  $\blacksquare$

## 11.6 Modifications for $l$ odd

When the prime  $l$  is odd, a similar method gives the construction of the motivic Steenrod reduced power operations. There are two additional issues to be addressed: the fact that the field  $k$  need not necessarily contain a primitive  $l^{\text{th}}$  root of unity and the fact that certain of the operations  $D_j$  are trivial. The first point is dealt with by using a transfer argument involving adjoining a root of unity to the base field  $k$ .

The second point is addressed as follows:

**Remark 11.6.1** A model for the classifying space of  $\mathbb{Z}/l$  in algebraic topology, is given by the topological realization of  $B_{\text{gm}}\mu_l$ . There is an analogous construction of the Steenrod reduced powers; in particular the classes  $D_j(x)$  are defined as in Section 11.1 (up to a sign). In algebraic topology, the classes  $D_{2i}(x)$  and  $D_{2i+1}(x)$  are shown to be trivial unless  $(l-1)|i$  by using coefficients  $\mathbb{Z}/l$  twisted by the action of  $\text{Aut}(\mathbb{Z}/l)$  [SE]. This corresponds to considering the non-trivial action of  $\text{Aut}(\mathbb{Z}/l)$  on  $B_{\text{gm}}\mathbb{Z}/l$ ; the argument in the motivic setting is completed by the following Proposition.

**Proposition 11.6.2** *The cohomology class in  $H^{2nl, nl}(K_{\mathcal{M}}(\mathbb{Z}/l, n) \times B_{\text{gm}}\mathbb{Z}/l, \mathbb{Z}/l)$  defined by the total Steenrod power  $\mathcal{P}_n$  is invariant under the action of  $\text{Aut}(\mathbb{Z}/l)$  on  $B_{\text{gm}}\mathbb{Z}/l$ .*

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