

# Homotopy theory of symmetric powers

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**ABSTRACT.** We introduce the symmetricity notions of symmetric h-monoidality, symmetroidality, and symmetric flatness. As shown in our paper arXiv:1410.5675, these properties lie at the heart of the homotopy theory of colored symmetric operads and their algebras. In particular, they allow one to equip categories of algebras over operads with model structures and to show that weak equivalences of operads induce Quillen equivalences of categories of algebras. We discuss these properties for elementary model categories such as simplicial sets, simplicial presheaves, and chain complexes. Moreover, we provide powerful tools to promote these properties from such basic model categories to more involved ones, such as the stable model structure on symmetric spectra.

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## 1. INTRODUCTION

Model categories provide an important framework for homotopy-theoretic computations. Algebraic structures such as monoids, their modules, and more generally operads and their algebras provide means to concisely encode multiplication maps and their properties such as unitality, associativity, and commutativity. Homotopy coherent versions of such algebraic structures form the foundation of a variety of mathematical areas, such as algebraic topology, homological algebra, derived algebraic geometry, higher category theory, and derived differential geometry. This motivates the following question: what conditions on a monoidal model category  $(\mathcal{C}, \otimes)$  are needed for a meaningful homotopy theory of monoids, modules, etc.? The first answer to this type of question was given by Schwede and Shipley's *monoid axiom*, which guarantees that for a monoid  $R$  in  $\mathcal{C}$ , the category  $\mathbf{Mod}_R(\mathcal{C})$  of  $R$ -modules carries a model structure transferred from  $\mathcal{C}$ , see [SS00]. The monoid axiom asks that transfinite compositions of pushouts of maps of the form

$$Y \otimes s,$$

where  $s$  is an acyclic cofibration and  $Y$  is any object are again weak equivalences. Moreover, given two weakly equivalent monoids  $R \xrightarrow{\sim} S$ , the categories  $\mathbf{Mod}_R$  and  $\mathbf{Mod}_S$  are Quillen equivalent if

$$Y \otimes X \rightarrow Y' \otimes X$$

is a weak equivalence for any weak equivalence  $Y \rightarrow Y'$  and any cofibrant object  $X$ .

This paper is devoted to a thorough study of the homotopy-theoretic behavior of more general algebraic expressions in a model category, such as

$$(1.0.1) \quad X_{\Sigma_n}^{\otimes n}, \quad Y \otimes_{\Sigma_n} X^{\otimes n}, \quad Z \otimes_{\Sigma_{n_1} \times \dots \times \Sigma_{n_e}} (X_1^{\otimes n_1} \otimes \dots \otimes X_e^{\otimes n_e}),$$

where  $X, Y, Z \in \mathcal{C}$ ,  $Y$  has an action of  $\Sigma_n$ ,  $Z$  has an action of  $\prod \Sigma_{n_i}$ , and the subscripts denote coinvariants by the corresponding group actions. More specifically, we introduce *symmetricity properties* for a symmetric monoidal model category  $\mathcal{C}$ : *symmetric h-monoidality*, *symmetroidality*, and *symmetric flatness*.

Symmetric h-monoidality requires, in particular, that for any object  $Y$  as above and any acyclic cofibration  $s$  in  $\mathcal{C}$ , the map

$$(1.0.2) \quad Y \otimes_{\Sigma_n} s^{\square n}$$

is a couniversal weak equivalence, i.e., a map whose cobase changes are weak equivalences. Here  $s^{\square n}$  is the  $n$ -fold pushout product of  $s$ , which is a monoidal product on morphisms. Symmetric h-monoidality is a natural enhancement of h-monoidality introduced by Batanin and Berger in [BB13].

Symmetric flatness requires that for any  $\Sigma_n$ -equivariant map  $y$  whose underlying map in  $\mathcal{C}$  is a weak equivalence and any cofibration  $s \in \mathcal{C}$ , the map

$$(1.0.3) \quad y \square_{\Sigma_n} s^{\square n}$$

is a weak equivalence. This implies that  $y \otimes_{\Sigma_n} X^{\otimes n}$  is a weak equivalence for any cofibrant object  $X$ . Among other things this means that the  $\Sigma_n$ -quotients in (1.0.1) are also homotopy quotients. See 4.2.7, 4.2.2 for the precise definitions.

Expressions as in (1.0.1) are of paramount importance for handling monoids and, more generally, algebras over colored symmetric operads. Indeed, a free commutative monoid, more generally, a free algebra over a (colored) symmetric operad, involves such terms. In [PS14a], we show that symmetric h-monoidality ensures the existence of a transferred model structure on algebras over *any* symmetric colored operad, while symmetric flatness yields a Quillen equivalence of algebras over weakly equivalent operads. We also introduce *symmetroidality* in this paper, which can be used to govern the behavior of cofibrant algebras over operads.

Up to transfinite compositions present in the monoid axiom, which we treat separately, symmetric h-monoidality and symmetric flatness can be regarded as natural enhancements of the above conditions of Schwede and Shipley. However, it turns out to be hard to establish the symmetric h-monoidality, symmetroidality, and symmetric flatness for a given model category  $\mathcal{C}$  directly. Therefore, in this paper, we also provide a powerful and convenient set of tools that enable us to quickly promote these properties through various constructions on model categories.

**Theorem 1.0.4.** (See Theorem 4.3.9 for the precise statement.) *To check that  $\mathcal{C}$  is symmetric h-monoidal or symmetric flat it is enough to consider (1.0.2) and (1.0.3) for generating cofibrations  $s$ .*

**Theorem 1.0.5.** (See Theorem 5.2.6 for the precise statement.) *Given an adjunction of symmetric monoidal model categories,*

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G,$$

*which is sufficiently compatible with the monoidal products, such as  $\mathcal{D} = \mathbf{Mod}_R(\mathcal{C})$ , where  $R$  is a commutative monoid in  $\mathcal{C}$ , the symmetric h-monoidality and symmetric flatness of  $\mathcal{C}$  imply the one of  $\mathcal{D}$ .*

**Theorem 1.0.6.** (See Theorem 6.2.2 for the precise statement.) *Given a monoidal left Bousfield localization*

$$\mathcal{C} \rightleftarrows \mathcal{D} = \mathbf{L}_S^\otimes(\mathcal{C}),$$

*the symmetric h-monoidality and symmetric flatness of  $\mathcal{C}$  imply the one of  $\mathcal{D}$ .*

As an illustration of these principles, consider the problem of establishing the symmetric h-monoidality, symmetroidality, and symmetric flatness for the monoidal model category of simplicial symmetric spectra. This allows one to establish the homotopy theory of operads and their algebras in spectra, such as commutative ring spectra or  $E_\infty$ -ring spectra. First, by direct inspection (Subsection 7.1) one establishes these properties for the generating (acyclic) cofibrations of simplicial sets, i.e.,  $\partial\Delta^n \rightarrow \Delta^n$  and  $\Lambda_k^n \rightarrow \Delta^n$ . By Theorem 4.3.9, this shows that  $\mathbf{sSet}$  is symmetric h-monoidal, symmetroidal, and flat. Next, again by direct inspection, one can show that positive cofibrations of symmetric sequences (i.e., cofibrations that are isomorphisms in degree 0) form a symmetric h-monoidal, symmetric flat class. Via Theorem 5.2.6 these properties can be transferred to modules over a (fixed) commutative monoid in symmetric sequences (specifically, the sphere spectrum), equipped with the positive unstable (i.e., transferred) model structure. Finally, by applying Theorem 6.2.2, one establishes them for the left Bousfield localization of the positive unstable model structure with respect to the stabilizing maps, which gives the positive stable model structure on simplicial symmetric spectra. These steps are carried out in detail for spectra in an abstract model category in [PS14b].

After recalling some basic notions pertaining to model categories in Section 2, we embark on a systematic study of the arrow category  $\mathbf{Ar}(\mathcal{C})$  of a monoidal model category  $\mathcal{C}$ . Equipped with the pushout product of morphisms, we show that  $\mathbf{Ar}(\mathcal{C})$  is again a monoidal model category (Subsection 3.1). We then recall the notion of h-monoidality due to Batanin and Berger [BB13], and the concept of flatness, which is well-known and has been independently studied by Hovey, for example, see [Hov14]. In Section 4, we define the above-mentioned symmetricity concepts. This extends the work of Lurie [Lur] and Gorchinskiy and Guletskiĭ [GG09]. An important technical key is Theorem 4.3.9, which shows the stability of these properties under weak saturation. This extends a similar statement of Gorchinskiy and Guletskiĭ [GG09, Theorem 5] about stability under weak saturation of a special case of symmetroidality (which we also prove in 4.3.9). Simplified expository accounts of this result were later given by White [Whi14a, Appendix A] and Pereira [Per14, §4.2]. Our proof uses similar ideas, but is shorter. The stability of the symmetricity and various other model-theoretic properties under transfers and left Bousfield localizations is shown in §5 and §6. Given that these two methods are the most commonly used tools to construct model structures, the main results of these sections (5.2.1, 5.2.6, 6.2.1, 6.2.2) should be useful to establish the symmetricity for many other model categories not considered in this paper. For example, the combination of h-monoidality and flatness allows to carry through the monoid axiom to a left Bousfield localization. This is illustrated in Section 7, where we discuss the symmetricity properties of model categories such as simplicial sets, simplicial modules, and simplicial (pre)sheaves, as well as topological spaces and chain complexes.

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## 2. MODEL CATEGORIES

In this section we recall parts of the language of model categories [Hov99], [Hir03], [MP12, Part 4] that is used throughout this paper. A *model category* is a complete and cocomplete category  $\mathcal{C}$  equipped with a *model*

*structure*: a class  $W$  of morphisms (called weak equivalences) satisfying the 2-out-of-3 property together with a pair of weak factorization systems  $(C, AF)$  (cofibrations and acyclic fibrations) and  $(AC, F)$  (acyclic cofibrations and fibrations) such that  $AC = C \cap W$  and  $AF = F \cap W$ .

An object  $X$  in a model category  $\mathcal{C}$  is *cofibrant* if the canonical map  $\emptyset \rightarrow X$  from the initial object to  $X$  is a cofibration. The class of cofibrant objects is denoted  $CO$ . Likewise, an object  $Y$  is *fibrant* if the canonical map  $Y \rightarrow 1$  to the terminal object is a fibration. A model category is *pointed* if the unique map  $\emptyset \rightarrow 1$  is an isomorphism.

Different model structures on the same category are distinguished using superscripts. The weakly saturated class generated by some class  $M$  of morphisms is denoted  $\text{cof}(M)$ . The class of maps having the right lifting property with respect to all maps in  $M$  is denoted  $\text{inj}(M)$ .

**Definition 2.0.1.** A model category is *cofibrantly generated* [Hir03, Definition 11.1.2] if its cofibrations and acyclic cofibrations are generated by sets (as opposed to proper classes) that permit the small object argument, *quasi-tractable* if its (acyclic) cofibrations are contained in the weak saturation of (acyclic) cofibrations with cofibrant source (and target), *combinatorial* [Lur09, Definition A.2.6.1] if it is locally presentable and cofibrantly generated, *tractable* [Bar10, Definition 1.21] if it is combinatorial and quasi-tractable.

Combinatoriality or alternatively *cellularity* [Hir03, Definition 12.1.1] is the key assumption used to guarantee the existence of Bousfield localizations.

**Definition 2.0.2.** A model category  $\mathcal{C}$  is *pretty small* if there is a cofibrantly generated model category structure  $\mathcal{C}'$  on the same category as  $\mathcal{C}$  such that  $W_{\mathcal{C}} = W_{\mathcal{C}'}$ ,  $C_{\mathcal{C}} \supset C_{\mathcal{C}'}$  and the domains and codomains  $X$  of some set of generating cofibrations of  $\mathcal{C}'$  are compact, i.e.,  $\text{Mor}(X, -)$  preserves filtered colimits.

Pretty smallness is stable under transfer and localization (Propositions 5.1.2(v) and 6.1.3). Lemma 2.0.3 implies that weak equivalences are stable under colimits of chains in a pretty small model category. Pretty smallness is a fairly mild condition: it is satisfied for all basic model categories in Section 7 except for topological spaces, which can be treated by a more narrowly tailored compactness condition.

**Lemma 2.0.3.** *Let  $\lambda$  be an ordinal and  $f: \lambda \rightarrow \mathbf{Ar}(\mathcal{C})$  a cocontinuous chain of morphisms in a model category, i.e., a sequence of commutative squares*

$$\begin{array}{ccc} X_i & \xrightarrow{x_i} & X_{i+1} \\ \downarrow f_i & & \downarrow f_{i+1} \\ Y_i & \xrightarrow{y_i} & Y_{i+1} \end{array}$$

*indexed by  $i \in \lambda$  such that  $f_i = \text{colim}_{j < i} f_j$  for all limit ordinals  $i \in \lambda$ . Set  $f_\infty = \text{colim } f_i$ .*

- (i) [CS02, Proposition I.2.6.3] *If every  $f_i$  (equivalently, only  $f_0$ ) and every map  $X_{i+1} \sqcup_{X_i} Y_i \rightarrow Y_{i+1}$  is an (acyclic) cofibration, then so is  $f_\infty$ .*
- (ii) *If cofibrations in  $\mathcal{C}$  are generated by cofibrations with compact domain and codomain and every  $f_i$  is an acyclic fibration, then so is  $f_\infty$ .*
- (iii) *If  $\mathcal{C}$  is pretty small and every  $f_i$  is a weak equivalence, then so is  $f_\infty$ . In particular, colimits of chains are homotopy colimits. The same is true for arbitrary filtered colimits.*
- (iv) *If  $\mathcal{C}$  is pretty small then weak equivalences are stable under transfinite compositions, i.e., for any cocontinuous chain  $X: \lambda \rightarrow \mathcal{C}$  of weak equivalences the map  $X_0 \rightarrow \text{colim } X$  is also a weak equivalence.*

*Proof.* (ii): Following the proof of [Hov99, Corollary 7.4.2], consider the lifting diagram

$$\begin{array}{ccc} A & \rightarrow & X_s \\ \downarrow & & \downarrow \\ B & \rightarrow & Y_s, \end{array}$$

where  $A \rightarrow B$  is a generating cofibration and  $s = \infty$ . The horizontal maps factor through some stage  $X_\alpha$ , and  $Y_\beta$ . We can take  $\alpha = \beta$ , increasing them if necessary. By further increasing  $\alpha$  we can make the above diagram commutative for  $s = \alpha$ . Since  $X_\alpha \rightarrow Y_\alpha$  is an acyclic fibration, we have a lifting  $B \rightarrow X_\alpha$ , which gives a lifting of the original diagram after postcomposing with  $X_\alpha \rightarrow X_\infty$ .

(iii): We may assume that  $\mathcal{C}$  is such that its generating cofibrations have compact (co)domains. Suppose  $Qf \rightarrow f$  is a cofibrant replacement of  $f$  in the projective structure. Part (i) shows that the transfinite composition of  $Qf$  is a weak equivalence, whereas part (ii) shows that the filtered colimit of the maps  $QX_i \rightarrow X_i$  and  $QY_i \rightarrow Y_i$  is a weak equivalence. (iv) is a particular case of (iii).  $\square$

The notion of h-cofibrations due to Batanin and Berger recalled below is the basis of (symmetric) h-monoidality (Definitions 3.2.1, 4.2.7), which a key condition in the admissibility results of a subsequent paper [PS14a, Theorem 5.10]. There is a similar concept of *i-cofibrations*. By definition, an *i-cofibration* is a map along which pushouts are homotopy pushouts. In a left proper model category, this is the same as being an h-cofibration. In a

non-left proper model category i-cofibrations behave better than h-cofibrations. For example the left properness assumptions in Theorem 6.2.2(ii) and Lemma 3.2.7 is unnecessary if one uses i-cofibrations instead. Moreover, acyclic i-cofibrations, i.e., maps that are i-cofibrations and weak equivalences, coincide with couniversal weak equivalences in any (not necessarily left proper) model category, as can be shown. However, our main supply of h-cofibrations (or i-cofibrations) comes from h-monoidal (or i-monoidal) categories, which are automatically left proper (Lemma 3.2.2), so the two concepts agree in this case. In particular, there is no difference between h-monoidality and i-monoidality (or their symmetric versions). Hence we do not pursue a separate study of i-cofibrations in this paper.

**Definition 2.0.4.** [BB13, Definition 1.1] A map  $f: X \rightarrow X'$  in a model category  $\mathcal{C}$  is an *h-cofibration* if for any pushout diagram

$$\begin{array}{ccccc} X & \longrightarrow & A & \xrightarrow{g} & B \\ f \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & A' & \xrightarrow{g'} & B' \end{array}$$

with a weak equivalence  $g, g'$  is also a weak equivalence. An *acyclic h-cofibration* is a map that is both an h-cofibration and a weak equivalence.

**Example 2.0.5.** In the category  $\mathbf{sSet}$ , equipped with its standard model structure, a map is an (acyclic) cofibration if and only if it is an (acyclic) h-cofibration. By 2.0.6(v), we only need to prove the if-part. Suppose a noninjective map  $f: A \rightarrow B$  is an h-cofibration. Then  $A$  has two nondegenerate simplices  $a, a' \in A_n$  with  $f(a) = f(a')$ . Since any cofibration is an h-cofibration and h-cofibrations are stable under composition by 2.0.6(ii), we may first replace  $A$  by the union of all faces of  $a$  and  $a'$  and then by  $S^n \vee S^n$ , using the pushout along the map  $A \rightarrow S^n \vee S^n$  collapsing all proper faces of  $a$  and  $a'$  to the base point. The pushout of  $B \sqcup_{S^n \vee S^n} S^n$  (using the obvious collapsing map) is isomorphic to  $B$ . If  $B$  was also the homotopy pushout, there was a homotopy fiber square of derived mapping spaces

$$\begin{array}{ccc} \underline{\mathbf{R}}\mathrm{Map}(S^n \vee S^n, K(\mathbf{Z}, n)) & \longleftarrow & \underline{\mathbf{R}}\mathrm{Map}(S^n, K(\mathbf{Z}, n)) \\ f^* \uparrow & & \uparrow \\ \underline{\mathbf{R}}\mathrm{Map}(B, K(\mathbf{Z}, n)) & \longleftarrow_{\mathrm{id}} & \underline{\mathbf{R}}\mathrm{Map}(B, K(\mathbf{Z}, n)), \end{array}$$

contradicting the fact that the path components of these spaces are  $\mathbf{Z} \oplus \mathbf{Z}, \mathbf{Z}$ , and  $H^n(B, \mathbf{Z})$ , respectively.

Usually, h-cofibrations form a strictly larger class than cofibrations, though. We don't know an effective criterion characterizing h-cofibrations.

**Lemma 2.0.6.** *Suppose  $\mathcal{C}$  is a model category.*

- (i) *If  $\mathcal{C}$  is left proper, a map is an h-cofibration if and only if pushouts along it are homotopy pushouts.*
- (ii) *(Acyclic) h-cofibrations in  $\mathcal{C}$  are stable under composition, pushouts and retracts.*
- (iii) *If weak equivalences are stable under colimits of chains (e.g., if  $\mathcal{C}$  is pretty small, see Lemma 2.0.3(iii)), then so are (acyclic) h-cofibrations. In particular, they are closed under transfinite composition, so they form a weakly saturated class.*
- (iv) *Couniversal weak equivalences are acyclic h-cofibrations. The converse is true if  $\mathcal{C}$  is left proper.*
- (v) *Any acyclic cofibration is an acyclic h-cofibration. If  $\mathcal{C}$  is left proper, any cofibration is an h-cofibration.*

*Proof.* Parts (i), (ii), (iv) are due to Batanin and Berger [BB13, Proposition 1.5, Lemmas 1.3, 1.6].

(iii): We use the notation of Lemma 2.0.3. For an object  $S$  under  $X_\infty$ , there is a functorial isomorphism  $S \sqcup_{X_\infty} Y_\infty = \mathrm{colim} S \sqcup_{X_i} Y_i$ . Therefore, the pushout of a weak equivalence  $s: S \rightarrow S'$  under  $X_\infty$  along  $f_\infty$  is the filtered colimit of the pushouts of  $s \sqcup_{X_i} Y_i$ . Each of those is a weak equivalence since  $f_i$  is an h-cofibration. By assumption, their colimit is also a weak equivalence, so  $f_\infty$  is an h-cofibration. For acyclic h-cofibrations, use Lemma 2.0.3(iii) one more time.

(v): The acyclic part is immediate from (iv). The nonacyclic part is [BB13, Lemma 1.2].  $\square$

**Lemma 2.0.7.** *If  $G: \mathcal{D} \rightarrow \mathcal{C}$  is a functor between model categories that creates weak equivalences (for example, if the model structure on  $\mathcal{D}$  is transferred from  $\mathcal{C}$ ) and preserves pushouts along a map  $d \in \mathrm{Mor}(\mathcal{D})$  and  $G(d)$  is an (acyclic) h-cofibration then  $d$  is an (acyclic) h-cofibration.*

*Proof.* Given a pushout  $f'$  in  $\mathcal{D}$  of a weak equivalence  $f$  under  $\mathrm{dom}(d)$ , we apply  $G$  and get a pushout in  $\mathcal{C}$ . As  $G(d)$  is an h-cofibration,  $G(f')$  is a weak equivalence, hence  $f'$  is a weak equivalence and therefore  $d$  is an h-cofibration. The acyclic part is similar, using that  $G$  detects weak equivalences.  $\square$

## 3. MONOIDAL MODEL CATEGORIES

In this section, we study certain properties of monoidal model categories. We first review the standard definitions of a monoidal model category and, more generally, a model category with a (left module) action of a monoidal category. In Subsection 3.2, we recall the concepts of h-monoidality (due to Batanin and Berger) and flatness (due to Hovey). In the case of a symmetric monoidal model category, these notions will be refined in Section 4.

**Definition 3.0.1.** [Hov99, Definitions 4.1.6, 4.2.6] A (*symmetric*) *monoidal category*  $(\mathcal{C}, \otimes, 1)$  is a (commutative) 2-monoid in the (large) bicategory of categories, functors, and natural transformations. For a monoidal category  $\mathcal{C}$ , a *left  $\mathcal{C}$ -module*  $\mathcal{C}'$  over  $\mathcal{C}$  is a left module over  $\mathcal{C}$  regarded as a 2-monoid. The functor

$$\otimes: \mathcal{C} \times \mathcal{C}' \rightarrow \mathcal{C}'$$

will be referred to as the *scalar product*. To simplify the notation, Mac Lane's coherence theorem for monoidal categories will implicitly be used.

A (*symmetric*) *monoidal model category* is a closed (symmetric) monoidal category  $\mathcal{C}$  such that

$$\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$$

is a left Quillen bifunctor, i.e.,

$$c \square d: C_1 \otimes D_2 \sqcup_{C_1 \otimes D_1} C_2 \otimes D_1 \rightarrow C_2 \otimes D_2$$

is a cofibration in  $\mathcal{C}$  for any two cofibrations  $c: C_1 \rightarrow C_2$  and  $d: D_1 \rightarrow D_2$  in  $\mathcal{C}$ , which is moreover acyclic if  $c$  or  $d$  is acyclic. This is also referred to as the *pushout product axiom*.

If a left  $\mathcal{C}$ -module  $\mathcal{C}'$  (but not necessarily  $\mathcal{C}$  itself) carries a model structure, we call  $\mathcal{C}'$  it a *left  $\mathcal{C}$ -module with a model structure*.

A left  $\mathcal{C}$ -module  $\mathcal{C}'$  with a model structure satisfies the *monoid axiom* if the class  $\text{cof}(\mathcal{C} \otimes \text{AC}_{\mathcal{C}'})$  consists of weak equivalences in  $\mathcal{C}'$  [SS00, Definition 3.3].

In the definition of a monoidal model category, we do not require the unit axiom (which asks that  $(Q(1) \rightarrow 1) \otimes X$  is a weak equivalence, where  $X$  is any cofibrant object and the map is the cofibrant replacement of 1). It is a special case of flatness (Definition 3.2.3).

Suppose  $\mathcal{V}$  is a symmetric monoidal model category. A  $\mathcal{V}$ -*enriched model category* [Bar10, Definition 1.27.4.1] is a  $\mathcal{V}$ -enriched category  $\mathcal{C}$  that is tensored and cotensored over  $\mathcal{V}$  and such that the tensor functor  $\mathcal{V} \times \mathcal{C} \rightarrow \mathcal{C}$  is a left Quillen bifunctor. We also assume the unit axiom for the  $\mathcal{V}$ -module  $\mathcal{C}$ , i.e., that for some (equivalently, any) cofibrant replacement  $Q(1_{\mathcal{V}}) \rightarrow 1_{\mathcal{V}}$  of the monoidal unit,  $Q(1_{\mathcal{V}}) \otimes X \rightarrow X$  is a weak equivalence for all cofibrant objects  $X$ . (This requirement is used in Proposition 4.3.5.) Two important examples of enriching categories for us are the categories of simplicial sets  $\mathbf{sSet}$ , which gives us simplicial model categories, and connective chain complexes of abelian groups  $\mathbf{Ch}_+$ , which gives us differential graded model categories. Chain complexes of various kinds are not enriched over simplicial sets, which necessitates considering different enriching categories. In both cases, 1 is cofibrant, so the unit axiom is trivial.

To ensure that  $\mathcal{V}$ -enriched left Bousfield localizations exist, we require the enriching model category  $\mathcal{V}$  to be tractable or at least quasi-tractable (see Proposition 6.1.3). Both of the above examples are tractable.

**3.1. The pushout product.** In this section, we define an endofunctor  $\mathbf{Ar}$  on the bicategory of cocomplete monoidal categories, cocontinuous strong monoidal functors, and monoidal natural transformations. Roughly speaking,  $\mathbf{Ar}$  sends a category  $\mathcal{C}$  to its category of morphisms equipped with a new monoidal structure, the *pushout product*. The underlying category of  $\mathbf{Ar}(\mathcal{C})$  is the category of functors  $\mathbf{Fun}(2, \mathcal{C})$ , where  $2 := \{0 \rightarrow 1\}$  is the walking arrow category. Its objects are morphisms in  $\mathcal{C}$  and its morphisms are commutative squares in  $\mathcal{C}$ . If  $\mathcal{C}$  is (co)complete, then  $\mathbf{Ar}(\mathcal{C})$  is also (co)complete, because (co)limits in categories of functors are computed componentwise. In this section we study the monoidal structure of  $\mathbf{Ar}(\mathcal{C})$  given by the pushout product and the projective model structure on  $\mathbf{Ar}(\mathcal{C})$ .

**Definition 3.1.1.** Given a cocomplete monoidal category  $\mathcal{C}$ , its (cocomplete) category  $\mathbf{Ar}(\mathcal{C})$  of morphisms can be endowed with a monoidal structure (the *pushout product*) as follows. Interpret an object in  $\mathbf{Ar}(\mathcal{C})$  as a functor  $2 \rightarrow \mathcal{C}$ . A finite family  $f: I \rightarrow \mathbf{Ar}(\mathcal{C})$  of objects in  $\mathbf{Ar}(\mathcal{C})$  (i.e., morphisms  $f_i: X_i \rightarrow Y_i$  in  $\mathcal{C}$ ) gives a functor  $2^I \rightarrow \mathcal{C}^I \rightarrow \mathcal{C}$ , where  $\mathcal{C}^I \rightarrow \mathcal{C}$  is the monoidal product on  $\mathcal{C}$ . We interpret this functor as a cocone on the category  $2^I \setminus \{1^I\}$  (observe that  $1^I$  is the terminal object of the category  $2^I$ ) and the monoidal product of  $f$  is defined to be the universal map  $\square f_i: \square f_i \rightarrow \otimes_i Y_i$  associated to this cocone, interpreted as an object in  $\mathbf{Ar}(\mathcal{C})$ . This defines a monoidal structure on  $\mathbf{Ar}(\mathcal{C})$ .

For example, the pushout product of two morphisms  $f_1$  and  $f_2$  is

$$f_1 \square f_2: f_1 \square f_2 = X_1 \otimes Y_2 \sqcup_{X_1 \otimes X_2} Y_1 \otimes X_2 \rightarrow Y_1 \otimes Y_2.$$

We obtain a bifunctor

$$(3.1.2) \quad \square: \mathbf{Ar}(\mathcal{C}) \times \mathbf{Ar}(\mathcal{C}) \rightarrow \mathbf{Ar}(\mathcal{C}).$$

*Remark 3.1.3.* If  $(\mathcal{C}, \otimes)$  is braided or symmetric, then so is  $(\mathbf{Ar}(\mathcal{C}), \square)$ . Moreover, if  $\otimes$  preserves colimits of a certain type (e.g., sifted colimits) in one or both variables, then so does  $\square$ . For example, if  $\mathcal{C}$  is a closed monoidal category, then so is  $\mathbf{Ar}(\mathcal{C})$ , with the internal hom  $\mathrm{Hom}(f_1, f_2)$  (which one can call the *pullback hom* from  $f_1$  to  $f_2$ ) being the morphism  $\mathrm{Hom}(Y_1, X_2) \rightarrow \mathrm{Hom}(Y_1, Y_2) \times_{\mathrm{Hom}(X_1, Y_2)} \mathrm{Hom}(X_1, X_2)$ . For brevity of the exposition, we only spell out the nonsymmetric, nonclosed case in the sequel.

**Proposition 3.1.4.** *A cocontinuous strong monoidal functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  between cocomplete monoidal categories induces a cocontinuous strong monoidal functor  $\mathbf{Ar}(F): \mathbf{Ar}(\mathcal{C}) \rightarrow \mathbf{Ar}(\mathcal{D})$ .*

*Proof.* The functor  $\mathbf{Ar}(F)$  is cocontinuous because colimits of diagrams are computed componentwise. To prove strong monoidality, suppose  $f: I \rightarrow \mathbf{Ar}(\mathcal{C})$  is a finite family of objects in  $\mathbf{Ar}(\mathcal{C})$ . The diagram

$$\begin{array}{ccccc} 2^I & \xrightarrow{f} & \mathcal{C}^I & \xrightarrow{\otimes} & \mathcal{C} \\ \downarrow \mathrm{id} & & \downarrow F^I & & \downarrow F \\ 2^I & \xrightarrow{F(f)} & \mathcal{D}^I & \xrightarrow{\otimes} & \mathcal{D}. \end{array}$$

is commutative, meaning the left square is strictly commutative and the right square is commutative up to the canonical natural isomorphism coming from the monoidal structure on the functor  $F$ . The pushout product  $\square f$  is the universal map associated to the cocone  $2^I \xrightarrow{f} \mathcal{C}^I \xrightarrow{\otimes} \mathcal{C}$  with the apex  $1^I \in 2^I$ , and similarly for  $\square \mathbf{Ar}(F)(f)$ . Since  $F$  is cocontinuous, it preserves universal maps associated to cocones. Thus the image of the universal morphism associated to the cocone  $2^I \rightarrow \mathcal{C}^I \rightarrow \mathcal{C}$  is also the universal morphism associated to the cocone  $2^I \rightarrow \mathcal{C}^I \rightarrow \mathcal{C} \rightarrow \mathcal{D}$ . The latter cocone is canonically isomorphic to the cocone  $2^I \rightarrow \mathcal{D}^I \rightarrow \mathcal{D}$ , which is the cocone defining  $\square \mathbf{Ar}(F)(f)$ .  $\square$

**Definition 3.1.5.** A morphism in the category  $\mathbf{Ar}(\mathcal{C})$  for some monoidal category  $\mathcal{C}$  is a *pushout morphism* if the corresponding commutative square in  $\mathcal{C}$  is cocartesian.

**Proposition 3.1.6.** *For any cocomplete closed monoidal category  $\mathcal{C}$  pushout morphisms in  $\mathbf{Ar}(\mathcal{C})$  are closed under the pushout product.*

*Proof.* A pushout morphism can be presented as a functor  $2 \times 2 \rightarrow \mathcal{C}$ , where the first 2 is responsible for the morphism direction in  $\mathbf{Ar}(\mathcal{C})$  and the second 2 is responsible for the morphism direction in  $\mathcal{C}$ . Schematically, we denote this by the commutative diagram

$$\begin{array}{ccc} 00 & \rightarrow & 10 \\ \downarrow & & \downarrow \\ 01 & \rightarrow & 11. \end{array}$$

A finite family of pushout morphisms  $f: I \rightarrow \mathrm{Mor}(\mathbf{Ar}(\mathcal{C}))$  gives a functor  $(2 \times 2)^I \rightarrow \mathcal{C}^I$ , which we compose with the monoidal product  $\mathcal{C}^I \rightarrow \mathcal{C}$  to obtain a functor  $F: (2 \times 2)^I \rightarrow \mathcal{C}$ . Consider now the category DC of all full subcategories  $A$  of  $(2 \times 2)^I$  that are *downward closed*: if  $Y \in A$  and  $X \rightarrow Y$  is a morphism in  $(2 \times 2)^I$ , then also  $X \in A$ . Morphisms in DC are inclusions of subcategories. Taking the colimit of the functor  $F$  restricted to the given full subcategory  $A$  yields a cocontinuous functor  $Q: \mathrm{DC} \rightarrow \mathcal{C}$ . In particular, the set of all inclusions  $A \rightarrow B$  in DC that are mapped to isomorphisms by  $Q$  forms a subcategory of DC closed under cobase changes of the underlying sets.

Suppose that  $B \in \mathrm{DC}$  is obtained from  $A \in \mathrm{DC}$  by adding an element  $W \times 11$  and taking the downward closure, where  $W \in (2 \times 2)^{I \setminus i}$  for some  $i \in I$  is such that  $W \times \{00, 01, 10\} \subset A$ . The resulting inclusion  $A \rightarrow B$  gives an isomorphism after we apply  $Q$  because the commutative square  $2 \times 2 \xrightarrow{\times W} (2 \times 2)^I \xrightarrow{F} \mathcal{C}$  is a cocartesian square because each  $f_i$  is a cocartesian square and the monoidal product with a fixed object preserves cocartesian squares. This uses the closedness of the monoidal product.

Consider the following commutative square in DC, whose right entries are obtained by taking the left entries, replacing 0 in the first components by 1, and downward closing:

$$\begin{array}{ccc} \{00, 01\}^I \setminus \{01\}^I & \rightarrow & \{00, 01, 10, 11\}^I \setminus \{01, 11\}^I \\ \downarrow & & \downarrow \\ \{00, 01\}^I & \rightarrow & \{00, 01, 10, 11\}^I. \end{array}$$

The pushout product  $\square f_i$  is obtained by applying  $Q$  to the following map:

$$\{00, 01, 10, 11\}^I \setminus \{01, 11\}^I \sqcup_{\{00, 01\}^I \setminus \{01\}^I} \{00, 01\}^I \rightarrow \{00, 01, 10, 11\}^I.$$

We present this morphism in DC as a composition of pushouts of generating maps explained in the previous paragraph, which implies that the map itself is sent to an isomorphism by  $Q$ . Such a presentation can be obtained by using the rule explained above to add all elements of  $\{01, 11\}^I \setminus \{01\}^I$  to the source by induction on the number of 11's. If there are no 11's, the element  $\{01\}^I$  belongs to the bottom left corner, proving our claim.

By induction, assuming that all tuples with less than  $k$  elements equal to 11 have already been added, take any tuple with exactly  $k$  components equal to 11 and observe that by replacing this component with 00, 01, or 10 we obtain a tuple already present in our set. Thus we can also add the tuple under consideration to our set.  $\square$

The elementary proof of the following lemma is left to the reader. Together with Proposition 3.1.6, it can be rephrased by saying that  $x \square -$  preserves finite cellular maps.

**Lemma 3.1.7.** *Given two composable maps  $y$  and  $z$ , and another map  $x$ ,  $x \square (y \circ z)$  is the composition of the pushout of  $x \square z$  along  $x \square z \rightarrow x \square (y \circ z)$ , followed by  $x \square y$ .*

We now extend the formation of arrow categories to monoidal model categories. A *strong monoidal left Quillen functor* between monoidal model categories is a left Quillen functor  $F$  that is also equipped with the structure of a strong monoidal functor, i.e., functorial isomorphisms  $F(X \otimes Y) \cong F(X) \otimes F(Y)$  compatible with the unit and associativity of  $\otimes$ . Monoidal model categories, strong monoidal left Quillen functors, and monoidal natural transformations form a bicategory. (As in Remark 3.1.3, there are obvious variants for (symmetric) monoidal model categories, which we will not spell out explicitly.)

The following proposition was shown independently by Hovey under the additional assumption that  $\mathcal{C}$  is cofibrantly generated [Hov14, Proposition 3.1].

**Proposition 3.1.8.** *The functor  $\mathbf{Ar}$  described in Definition 3.1.1 and Proposition 3.1.4 descends to the bicategory of closed monoidal model categories, as described in the proof below.*

*Proof.* Given a closed monoidal model category  $\mathcal{C}$ , the monoidal category  $\mathbf{Ar}(\mathcal{C})$  is complete and cocomplete. We equip  $\mathbf{Ar}(\mathcal{C})$  with the projective model structure, which coincides with the Reedy model structure, where the nonidentity arrow  $0 \rightarrow 1$  in  $\mathbf{2}$  is declared to be positive. In particular, the projective model structure on  $\mathbf{Ar}(\mathcal{C})$  exists. Fibrations and weak equivalences are defined componentwise. (Acyclic) cofibrations  $f: g \rightarrow h$  are commutative squares

$$\begin{array}{ccc} W & \xrightarrow{p} & Y \\ \downarrow g & & \downarrow h \\ X & \xrightarrow{q} & Z \end{array}$$

such that  $p$  and the universal map  $Y \sqcup_W X \rightarrow Z$  are both (acyclic) cofibrations, hence  $q$  is also an (acyclic) cofibration. In particular, cofibrant objects in  $\mathbf{Ar}(\mathcal{C})$  are morphisms  $g: W \rightarrow X$  such that  $W$  is cofibrant and  $g$  is a cofibration in  $\mathcal{C}$ .

We now prove the pushout product axiom for  $\mathbf{Ar}(\mathcal{C})$  from the one of  $\mathcal{C}$  (Definition 3.0.1). Actually, we show that the pushout product of a finite nonempty family  $f: I \rightarrow \text{Mor}(\mathbf{Ar}(\mathcal{C}))$  of cofibrations in  $\mathbf{Ar}(\mathcal{C})$  is a cofibration, and if one of the cofibrations is acyclic, then the resulting cofibration is also acyclic. The infrastructure of the following proof is the same as in the proof of Proposition 3.1.6. Just like there we get a functor  $F: (2 \times 2)^I \rightarrow \mathcal{C}$  and a cocontinuous functor  $Q: \text{DC} \rightarrow \mathcal{C}$ . Let

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \downarrow a & & \downarrow a' \\ B & \longrightarrow & B' \end{array}$$

be a cocartesian square in DC, i.e.,  $B' = A' \cup_A B$ . If  $Q(a)$  is a cofibration, then so is  $Q(a')$ . Suppose that for every  $i \in I$  we select one of the morphisms  $\{00\} \rightarrow \{00, 10\}$  or  $\{00, 01, 10\} \rightarrow \{00, 01, 10, 11\}$  in  $\text{DC}(2 \times 2)$ . Then the pushout product of these morphisms belongs to the above subcategory because of the pushout product axiom for  $\mathcal{C}$ . The first morphism above expresses the fact that the top arrow of a cofibration in  $\mathbf{Ar}(\mathcal{C})$  is itself a cofibration and the second morphism corresponds to the canonical map from the pushout to the bottom right corner, which is also a cofibration. The pushout product mentioned above always has the form  $A \setminus \{x\} \rightarrow A$ , where the individual components of  $x$  are 10 respectively 11, according to the choice made above.

The pushout product of  $f$  is the functor  $Q$  applied to the commutative square

$$\begin{array}{ccc} \{00, 01, 10, 11\}^I \setminus \{10, 11\}^I \setminus \{01, 11\}^I & \longrightarrow & \{00, 01, 10, 11\}^I \setminus \{01, 11\}^I \\ \downarrow & & \downarrow \\ \{00, 01, 10, 11\}^I \setminus \{10, 11\}^I & \longrightarrow & \{00, 01, 10, 11\}^I \end{array}$$

It remains to prove that  $Q$  applied to the top map and the map from the pushout of the left and top arrows (i.e., the union of all corners except for the bottom right corner) to the bottom right corner is a cofibration. We present the morphism in DC under consideration as a composition of pushouts of generating maps explained in the previous paragraph. This implies that the map itself is sent by  $Q$  to a cofibration.

For the top map, such a presentation can be obtained by using the rule explained above to add all elements of  $\{10, 11\}^I \setminus \{11\}^I$  to the source by induction on the number of 11's. Assume that all tuples with less than  $k$  11's have already been added and take any tuple with exactly  $k$  11's. By applying the rule explained in the

previous paragraph to the family of maps that are either  $\{00\} \rightarrow \{00, 10\}$  if the corresponding component is 10 or  $\{00, 01, 10\} \rightarrow \{00, 01, 10, 11\}$  if the corresponding component is 11 we can conclude that the tuple under consideration can be added to our set.

For the map from the pushout of the top and left arrows to the bottom right corner observe that we only need to add the element  $\{11\}^I$ , which is possible because the conditions for the corresponding rule are satisfied.

For acyclic cofibrations observe that the rule in the previous paragraph now guarantees that the resulting map is an acyclic cofibration after we apply  $Q$ , precisely because the pushout product in  $\mathcal{C}$  of a family of cofibrations, at least one of which is acyclic, is again an acyclic cofibration. The rest of the proof is exactly the same, because the category of acyclic cofibrations is also closed under pushouts.

Finally,  $\mathbf{Ar}$  descends to strong monoidal left Quillen functors: if  $F: \mathcal{C} \rightarrow \mathcal{D}$  is such a functor, then the induced functor  $\mathbf{Ar}(F): \mathbf{Ar}(\mathcal{C}) \rightarrow \mathbf{Ar}(\mathcal{D})$  is cocontinuous and strong monoidal (Proposition 3.1.4). It is a left Quillen functor because  $F$  preserves (acyclic) cofibrations and pushouts.  $\square$

**3.2. H-monoidality and flatness.** In this section, we discuss the notion of h-monoidality and flatness of a left module  $\mathcal{C}'$  with a model structure over a monoidal category  $\mathcal{C}$ .

H-monoidality was introduced by Batanin and Berger [BB13, Definition 1.7]. Essentially, h-monoidality ensures that category of modules over some monoid  $R \in \mathcal{C}$  carries a model structure. This statement is referred to as the *admissibility* of the monoid  $R$ . The admissibility of monoids is also guaranteed by the monoid axiom [SS00, Theorem 4.1], which is a combination of two weak saturation properties, namely weak saturation by transfinite compositions and by pushouts. In this paper, we focus on admissibility conditions using pretty smallness and h-monoidality, which individually govern the homotopical behavior of transfinite compositions and of (certain) pushouts, respectively. Basic model categories are usually h-monoidal by Lemmas 3.2.4 and 3.2.5. On the other hand, h-monoidality is very robust since is stable under transfer and localization (5.2.5(i), 6.2.1(iii)). We don't know a similar statement for the monoid axiom (without the detour via pretty smallness and h-monoidality).

**Definition 3.2.1.** A class  $S$  of (acyclic) cofibrations in a left  $\mathcal{C}$ -module with a model structure (over a monoidal category  $\mathcal{C}$ ) is *(acyclic) h-monoidal* if for any any object  $C \in \mathcal{C}$  and any  $s: S_1 \rightarrow S_2$  in  $S$ , the map

$$C \otimes s: C \otimes S_1 \rightarrow C \otimes S_2$$

is an (acyclic) h-cofibration (Definition 2.0.4). The category  $\mathcal{C}'$  is *h-monoidal* if the classes of (acyclic) cofibrations are (acyclic) h-monoidal.

**Lemma 3.2.2.** [BB13, Lemma 1.8] *Any h-monoidal model category is left proper.*

We now define flatness, which is the main condition in rectification of modules over monoids. Its symmetric strengthening, symmetric flatness, plays the corresponding role for algebras over symmetric operads [PS14a, Theorem 7.5].

**Definition 3.2.3.** A class  $S$  of cofibrations in a left module  $\mathcal{C}'$  over a model category  $\mathcal{C}$  is *flat* if for all weak equivalences  $y: Y_1 \rightarrow Y_2$  in  $\mathcal{C}$  and all  $s: S_1 \rightarrow S_2$  in  $S$ , the following map is a weak equivalence:

$$y \square s: Y_2 \otimes S_1 \sqcup_{Y_1 \otimes S_1} Y_1 \otimes S_2 \rightarrow Y_2 \otimes S_2$$

The category  $\mathcal{C}'$  is *flat* if the class of all cofibrations is flat.

For example, if  $\mathcal{C}'$  is flat then for any cofibrant object  $X \in \mathcal{C}'$  and any weak equivalence  $y \in \mathcal{C}$ , the map  $y \otimes X$  is a weak equivalence. In this slightly weaker form, flatness is independently due to Hovey [Hov14, Definition 2.4]. Actually, the notion appears already in [SS00, Theorem 4.3]. We use the above slightly stronger definition since it is stable under weak saturation of  $S$  (Theorem 3.2.8(ii)). This is useful to show the stability of flatness under transfer (Proposition 5.2.1(ii)) and localization (Proposition 6.2.1(i)).

In general, we avoid cofibrancy hypotheses where possible, in particular, we do not in general assume that the monoidal unit 1 is cofibrant. The combination of the following two lemmas is useful to establish h-monoidality and flatness in practice, though.

**Lemma 3.2.4.** *Let  $\mathcal{C}$  be a model category in which all objects are cofibrant. Then  $\mathcal{C}$  is left proper and quasi-tractable. Moreover, tractability follows from combinatoriality, while h-monoidality and flatness follow from monoidality.*

*Proof.* See [Hir03, Corollary 13.1.3] for left properness, [SS00, Remark 3.4] for flatness and [BB13, Lemma 1.8] for h-monoidality.  $\square$

**Lemma 3.2.5.** *Assume that there are two model structures  $\mathcal{C}$  and  $\mathcal{C}_1$  on the same underlying category such that  $W_{\mathcal{C}} = W_{\mathcal{C}_1}$  and  $C_{\mathcal{C}} \subset C_{\mathcal{C}_1}$ . Then the left properness of  $\mathcal{C}_1$  implies the one of  $\mathcal{C}$ . If  $\mathcal{C}$  is equipped with a monoidal structure, the same is true for monoidality, h-monoidality, and flatness.*

*Proof.* This follows from the definitions. For the h-monoidality, note that (acyclic) h-cofibrations only depend on weak equivalences.  $\square$

**Lemma 3.2.6.** (cf. [BB13, Proposition 2.5]) *If  $\mathcal{C}'$  is an h-monoidal left  $\mathcal{C}$ -module with a model structure (over a monoidal category  $\mathcal{C}$ ) and its acyclic h-cofibrations are stable under transfinite compositions (for example,  $\mathcal{C}'$  is pretty small, see Lemma 2.0.6(iii)), then  $\mathcal{C}'$  satisfies the monoid axiom.*

*Proof.* The monoid axiom says  $\text{cof}(\mathcal{C} \otimes \text{AC}_{\mathcal{C}'}) \subset \text{W}_{\mathcal{C}'}$ , i.e., the weak saturation of the monoidal product of any object of  $\mathcal{C}$  with acyclic cofibrations, consists of weak equivalences. For  $C \in \mathcal{C}$  and  $f \in \text{AC}_{\mathcal{C}'}$ , the morphism  $C \otimes f$  is an acyclic h-cofibration in  $\mathcal{C}'$  by assumption, hence so are its cobase changes. Acyclic h-cofibrations are stable under transfinite compositions, again by assumption. Finally, acyclic h-cofibrations are always stable under retracts.  $\square$

We finish this section with two weak saturation properties. A slightly weaker statement than Theorem 3.2.8(ii) is independently due to Hovey [Hov14, Theorem A.2]. The following lemma is the basis of the interaction of h-monoidality and flatness, see for example the proof of 3.2.8(ii).

**Lemma 3.2.7.** *Let  $\mathcal{C}'$  be a left proper model category that is a left module over a monoidal category  $\mathcal{C}$ . Let*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow a & & \downarrow b \\ A' & \longrightarrow & B' \end{array}$$

*be a cocartesian square in  $\mathcal{C}'$ . Let  $y: Y \rightarrow Y' \in \mathcal{C}$  be any morphism such that  $y \square a$  is a weak equivalence in  $\mathcal{C}'$ , and both  $Y \otimes a$  and  $Y' \otimes a$  are h-cofibrations (Definition 2.0.4). Then  $y \square b$  is a weak equivalence.*

*Proof.* Consider the commutative diagram

$$\begin{array}{ccccc} Y \otimes A & \xrightarrow{y \otimes a} & Y' \otimes A & & \\ Y \otimes a \downarrow & & \downarrow \alpha & \searrow^{Y' \otimes a} & \\ Y \otimes A' & \longrightarrow & y \square a & \xrightarrow{y \square a} & Y' \otimes A'. \end{array}$$

As usual,  $\square$  denotes the domain of the pushout product  $\square$ . By assumption,  $Y \otimes a$  is an h-cofibration, hence so is  $\alpha$  by Lemma 2.0.6. Likewise,  $Y' \otimes a$  is an h-cofibration. Hence the top square and the outer rectangle in the diagram below are homotopy pushouts (Lemma 2.0.6(i)). Hence so is the bottom square. The map  $y \square b$  is therefore also a weak equivalence:

$$\begin{array}{ccc} Y' \otimes A & \longrightarrow & Y' \otimes B \\ \downarrow \alpha \text{ h-cofib.} & & \downarrow \\ \text{h-cofib.} \left( \begin{array}{ccc} y \square a & \longrightarrow & y \square b \\ \downarrow \sim & & \downarrow y \square b \\ Y' \otimes A' & \longrightarrow & Y' \otimes B' \end{array} \right. \end{array}$$

$\square$

**Theorem 3.2.8.** *Let  $\mathcal{C}$  be a monoidal model category and let  $\mathcal{C}'$  be a pretty small left  $\mathcal{C}$ -module with a model structure. We say some property of a class  $S$  of morphisms in  $\mathcal{C}'$  is stable under saturation if it also holds for the weak saturation  $\text{cof}(S)$ .*

- (i) *If the scalar product  $\otimes: \mathcal{C} \times \mathcal{C}' \rightarrow \mathcal{C}'$  preserves all colimits in  $\mathcal{C}'$ , then the property of  $S$  of being (acyclic) h-monoidal is stable under saturation.*
- (ii) *Suppose the scalar product  $\otimes$  preserves filtered colimits in  $\mathcal{C}'$ . If  $S$  is h-monoidal then flatness of  $S$  is stable under saturation. In particular, if some class of generating cofibrations in  $\mathcal{C}$  is flat and h-monoidal, then  $\mathcal{C}$  is flat.*

*Proof.* (i): The stability of (acyclic) h-monoidality of  $S$  under weak saturation follows from Lemma 2.0.6(iii) and the preservation by  $\mathcal{C}'$  of colimits in  $\mathcal{C}'$ .

(ii): For a weak equivalence  $y: Y \rightarrow Y'$  in  $\mathcal{C}$  and any  $s \in S$ ,  $y \square s$  is a weak equivalence by assumption. By h-monoidality of  $S$ ,  $Y \otimes s$  and  $Y' \otimes s$  are h-cofibrations. Thus for any pushout  $s'$  of  $s$ ,  $y \square s'$  is a weak equivalence by Lemma 3.2.7. For a transfinite composition  $s_\infty$  of maps  $s_i$ ,  $y \square s_\infty$  is the transfinite composition of  $y \square s_i$  by preservation of filtered colimits in the second variable. Therefore it is again a weak equivalence using pretty smallness (Lemma 2.0.3). As usual, retracts are clear.  $\square$

## 4. SYMMETRICITY PROPERTIES

In this section we study three properties of a symmetric monoidal model category  $\mathcal{C}$ : symmetric h-monoidality, symmetroidality and symmetric flatness. As the name indicates, these involve the formation of pushout powers, i.e., expressions of the form

$$\square_n f := f^{\square n} := \underbrace{f \square \cdots \square f}_{n \text{ times}}.$$

After settling preliminaries on objects with a finite group action, these properties are defined in Subsection 4.2. The main result of Subsection 4.3 is Theorem 4.3.9 which shows the stability of these notions under weak saturation. This is a key step in showing that the properties also interact well with transfer and localization of model structures. Examples of model categories satisfying these properties are given in Section 7.

**4.1. Objects with a finite group action.** We first examine model-theoretic properties of objects with an action of a finite group, for example the permutation action of  $\Sigma_n$  on  $f^{\square n}$ . Given a finite group  $G$ , considered as a category with one object, and any category  $\mathcal{C}$ , define

$$(4.1.1) \quad G\mathcal{C} := \mathbf{Fun}(G, \mathcal{C}).$$

This is the category of objects in  $\mathcal{C}$  with a left  $G$ -action. It is symmetric monoidal if  $\mathcal{C}$  is, by letting  $G$  act diagonally on the monoidal product. Given some  $X \in G\mathcal{C}$  and any subgroup  $H \subset G$ , we write  $X_H = \operatorname{colim}_H X$  for the coinvariants.

For any  $X \in \mathcal{C}$  we define  $G/H \cdot X := \coprod_{G/H} X \in G\mathcal{C}$  on which  $G$ -acts by the left  $G$ -action on  $G/H$ . More generally, given any  $X \in H\mathcal{C}$  and any morphism of groups  $H \rightarrow G$ , we define

$$G \cdot_H X := (G \cdot X)_H,$$

where  $H$  acts on the right on  $G$  and on the left on  $X$ .

**Lemma 4.1.2.** *Suppose  $\mathcal{C}$  is a cocomplete category and  $H$  is a subgroup of a finite group  $G$ . Any choice of a partition  $G = \coprod_i H \cdot g_i$  of  $G$  into  $H$ -cosets induces a natural isomorphism*

$$\varphi(G \cdot_H -) \rightarrow (G/H) \cdot \varphi(-)$$

of functors  $H\mathcal{C} \rightarrow \mathcal{C}$ , where  $\varphi$  denotes the forgetful functor to  $\mathcal{C}$ .

*Proof.* The canonical projection  $G \cdot \varphi X \rightarrow G/H \cdot \varphi X$  factors over  $\varphi(G \cdot_H X)$ . Conversely, given  $g \in G$ , the partition gives a unique  $h \in H$  and  $i$  such that  $g = hg_i$ . Define  $G/H \cdot \varphi X \rightarrow G \cdot_H \varphi X$  by  $x_{gH} \mapsto (h^{-1}x)_{g_i}$ .  $\square$

**Proposition 4.1.3.** *Suppose  $\mathcal{C}$  is a cofibrantly generated model category. The category  $G\mathcal{C}$  carries the projective model structure, denoted  $G^{\operatorname{pro}}\mathcal{C}$ , whose weak equivalences and fibrations are precisely those maps in  $G\mathcal{C}$  that are mapped to weak equivalences respectively fibrations in  $\mathcal{C}$  by the forgetful functor  $G\mathcal{C} \rightarrow \mathcal{C}$ . The cofibrations of  $G^{\operatorname{pro}}\mathcal{C}$  are generated by the maps of the form  $G \cdot f$ , where  $f$  runs over generating cofibrations of  $\mathcal{C}$ .*

*Given a morphism of groups  $H \rightarrow G$ , there is a Quillen adjunction*

$$(4.1.4) \quad G \cdot_H - : H^{\operatorname{pro}}\mathcal{C} \rightleftarrows G^{\operatorname{pro}}\mathcal{C} : R,$$

where the right adjoint functor is the restriction.

*Finally, suppose  $\mathcal{C}$  is a symmetric monoidal model category. Given two groups  $G$  and  $H$ , the monoidal product on  $\mathcal{C}$  induces a left Quillen bifunctor*

$$(4.1.5) \quad G^{\operatorname{pro}}\mathcal{C} \times H^{\operatorname{pro}}\mathcal{C} \rightarrow (G \times H)^{\operatorname{pro}}\mathcal{C}.$$

*Proof.* The existence of this model structure is standard, see, for example, Hirschhorn [Hir03, Theorem 11.6.1]. The adjunction (4.1.4) is seen to be a Quillen adjunction by looking at the right adjoint. The functor (4.1.5) is a left Quillen bifunctor because  $(G \cdot I_{\mathcal{C}}) \square (H \cdot I_{\mathcal{C}}) = (G \times H) \cdot (I_{\mathcal{C}} \square I_{\mathcal{C}}) \subset (G \times H) \cdot C_{\mathcal{C}}$ , using the cocontinuity and monoidality of the functor  $G \cdot -$  and the pushout product axiom for  $\mathcal{C}$ .  $\square$

**Proposition 4.1.6.** *The functor  $-\square^n : \mathbf{Ar}\mathcal{C} \rightarrow \Sigma_n \mathbf{Ar}\mathcal{C}$  preserves filtered colimits.*

*Proof.* The functor  $-\square^n$  is the composition  $\mathbf{Ar}(\mathcal{C}) \xrightarrow{\Delta} \Sigma_n \mathbf{Ar}(\mathcal{C})^n \xrightarrow{\square} \Sigma_n \mathbf{Ar}(\mathcal{C})$ . The monoidal product  $\mathbf{Ar}(\mathcal{C})^n \rightarrow \mathbf{Ar}(\mathcal{C})$  is separately cocontinuous because the monoidal structure is closed, so  $-\square^n$  evaluated on  $\operatorname{colim} D$  for some filtered diagram  $D : I \rightarrow \mathbf{Ar}(\mathcal{C})$  can be computed as  $\operatorname{colim} D^n$ , where  $D^n : I^n \rightarrow \mathbf{Ar}(\mathcal{C})$  is obtained by composing the  $n$ th cartesian power  $I^n \rightarrow \mathbf{Ar}(\mathcal{C})^n$  of  $D$  with the monoidal product  $\mathbf{Ar}(\mathcal{C})^n \rightarrow \mathbf{Ar}(\mathcal{C})$ . For a filtered category  $I$  the diagonal  $I \rightarrow I^n$  is a cofinal functor, thus the last colimit can be computed as  $\operatorname{colim}_I \square^n D$ .  $\square$

**Proposition 4.1.7.** [Har09, Proposition 6.13] *Suppose  $h : f \rightarrow g$  is a pushout morphism in  $\mathbf{Ar}(\mathcal{C})$ . Then  $h^{\square n} : f^{\square n} \rightarrow g^{\square n}$  is also a pushout morphism.*

*Proof.* This follows immediately from Proposition 3.1.6.  $\square$

**4.2. Definitions.** We now define three properties of (morphisms in) a symmetric monoidal model category  $\mathcal{C}$ : symmetric flatness, symmetric h-monoidality and symmetroidality. They are appropriate strengthenings of flatness (Definition 3.2.3), h-monoidality (Definition 3.2.1) and the pushout product axiom. Symmetric flatness is the key condition required to obtain a rectification result for operadic algebras [PS14a, Theorem 7.1]. Approximately, it says that for any cofibrant object  $X \in \mathcal{C}$ , the map

$$y \otimes_{\Sigma_n} X^{\otimes n} : Y \otimes_{\Sigma_n} X^{\otimes n} \rightarrow Y' \otimes_{\Sigma_n} X^{\otimes n}$$

is a weak equivalence for any weak equivalence  $y : Y \rightarrow Y'$  in  $\Sigma_n \mathcal{C}$ . Slightly more accurately, the definition is phrased in terms of more general cofibrations  $s$  using instead

$$y \square_{\Sigma_n} s^{\square n}.$$

For  $s : \emptyset \rightarrow X$  this gives back the previous expressions. In order to ensure that the three symmetricity properties are stable under weak saturation (Theorem 4.3.9), we actually define them for a class of morphisms instead of a single morphism. In such cases, we use the following notational conventions.

**Definition 4.2.1.** Let  $v := (v_1, \dots, v_e)$  be a finite family of morphisms. For any sequence of nonnegative integers  $n := (n_i)_{i \leq e}$ , we write  $\Sigma_n := \prod \Sigma_{n_i}$ ,  $v^{\square n} := v_1^{\square n_1} \square \dots \square v_e^{\square n_e}$ , and  $v^{\otimes n} := v_1^{\otimes n_1} \otimes \dots \otimes v_e^{\otimes n_e}$ . We write  $m \leq n$  if  $m_i \leq n_i$  for all  $i$  and  $m < n$  if  $m \leq n$  and  $m \neq n$ . Given a class  $S$  of morphisms, we write  $v \subset S$  if all  $v_i$  are in  $S$ . Given another sequence of integers  $(m_i)_{i=1}^e$ , we write  $mn := \sum m_i n_i$  and  $\Sigma_m^n := \prod \Sigma_{m_i}^{n_i}$  and  $\Sigma_n \times \Sigma_m^n := \prod \Sigma_{n_i} \times \Sigma_{m_i}^{n_i}$ .

**Definition 4.2.2.** A class  $S$  of cofibrations in  $\mathcal{C}$  is called *symmetric flat with respect to some class  $\mathcal{Y} = (\mathcal{Y}_n)$*  of morphisms  $\mathcal{Y}_n \subset \text{Mor } \Sigma_n \mathcal{C}$  if

$$y \square_{\Sigma_n} s^{\square n} := (y \square s^{\square n})_{\Sigma_n}$$

is a weak equivalence in  $\mathcal{C}$  for any  $y \in \mathcal{Y}_n$ , any finite multi-index  $n \geq 1$  and any  $s \in S$ . We say  $S$  is *symmetric flat* if it is symmetrically flat with respect to the classes  $\mathcal{Y}_n = (\text{W}_{\Sigma_n^{\text{pro}} \mathcal{C}})$  of projective weak equivalences (i.e., those maps in  $\Sigma_n \mathcal{C}$  which are weak equivalences after forgetting the  $\Sigma_n$ -action). We say  $\mathcal{C}$  is *symmetric flat* if the class of cofibrations is symmetric flat.

**Example 4.2.3.** A class  $S$  is symmetric flat (i.e., with respect to  $\text{W}_{\Sigma_n^{\text{pro}} \mathcal{C}}$ ) if and only if  $y \square_{\Sigma_n} s^{\square n}$  is a weak equivalence for a single map  $s \in S$ , i.e., no multi-indices are necessary in this case. The reader is encouraged to mainly think of this case.

The following definition is necessary to ensure that the small object argument can be applied to construct a model structure on operadic algebras [PS14a, Theorem 5.10]. Recall from [Hir03, Definition 10.4.1] or [Hov99, Definition 2.1.3] that an object  $A \in \mathcal{C}$  is *small relative to some subcategory  $\mathcal{D} \subset \mathcal{C}$*  if there is some cardinal  $\lambda$  such that for any  $\lambda$ -sequence  $X_0 \rightarrow X_1 \rightarrow \dots \rightarrow X_\beta \rightarrow \dots$  ( $\beta < \lambda$ ) in  $\mathcal{D}$ , the canonical map of Hom-sets

$$\text{colim}_{\beta < \lambda} \text{Hom}_{\mathcal{C}}(A, X_\beta) \rightarrow \text{Hom}_{\mathcal{C}}(A, \text{colim}_{\beta < \lambda} X_\beta)$$

is an isomorphism. We will often apply this to  $\mathcal{D} = \text{cell}(I)$ , the closure of a class  $I$  of maps under pushouts and transfinite composition. Also recall that, by definition, any object in a combinatorial model category is small with respect to all maps of  $\mathcal{C}$ , so is automatically admissibly generated in the sense below. Topological spaces are a non-combinatorial, but admissibly generated model category (Subsection 7.5).

**Definition 4.2.4.** A symmetric monoidal model category  $\mathcal{C}$  is *admissibly generated* relative to a class  $S$  of morphisms in  $\mathcal{C}$  if all cofibrant objects in  $\mathcal{C}$  are small with respect to the subcategory

$$(4.2.5) \quad \text{cell}(Y \otimes_{\Sigma_n} s^{\square n})$$

for any finite family  $s \subset S$ , any multi-index  $n > 0$ , and any object  $Y \in \Sigma_n \mathcal{C}$ . We call  $\mathcal{C}$  *admissibly generated* if it is cofibrantly generated and admissibly generated relative to the cofibrations  $\mathcal{C}_{\mathcal{C}}$ .

**Lemma 4.2.6.** [Hir03, Proposition 10.4.9] *For  $\mathcal{C}$  to be admissibly generated relative to  $S$  it is enough that the (co)domains of some set of generating cofibrations are small with respect to (4.2.5).*

The notions of symmetric h-monoidal maps (respectively, symmetroidal maps) presented next are designed to ultimately address the (strong) admissibility of operads ([PS14a, Theorem 5.10]).

**Definition 4.2.7.** A class  $S$  of morphisms in a symmetric monoidal category  $\mathcal{C}$  is called *(acyclic) symmetric h-monoidal* if for any finite family  $s \subset S$  and any multi-index  $n \neq 0$ , and any object  $Y \in \Sigma_n \mathcal{C}$  the morphism  $Y \otimes_{\Sigma_n} s^{\square n}$  is an (acyclic) h-cofibration. We say  $\mathcal{C}$  is *symmetric h-monoidal* if the class of (acyclic) cofibrations is (acyclic) symmetric h-monoidal.

The notion of power cofibrations presented next is due to Lurie [Lur, Definition 4.5.4.2] and Gorchinskiy and Guletskiĭ [GG09, Section 3], who also introduced symmetrizable maps.

**Definition 4.2.8.** Let  $\mathcal{Y} = (\mathcal{Y}_n)_{n>0}$  be a collection of classes  $\mathcal{Y}_n$  of morphisms in  $\Sigma_n\mathcal{C}$ , where  $n > 0$  is any finite multi-index. We suppose that for  $y \in \mathcal{Y}_n$ ,  $y \square -$  preserves *injective* (acyclic) cofibrations in  $\Sigma_n\mathcal{C}$ , i.e., those maps which are (acyclic) cofibrations in  $\mathcal{C}$ .

A class  $S$  of morphisms in a symmetric monoidal category  $\mathcal{C}$  is called *(acyclic)  $\mathcal{Y}$ -symmetroidal* if for all multi-indices  $n > 0$  and all maps  $y \in \mathcal{Y}_n$ , the morphism

$$(4.2.9) \quad y \square_{\Sigma_n} s^{\square n}$$

is an (acyclic) cofibration in  $\mathcal{C}$  for all  $s \in S$ . If  $\mathcal{Y}_n = \mathcal{C}_{\Sigma_n^{\text{in}}}\mathcal{C}$ , we say that  $S$  is *(acyclic) symmetroidal*. For  $\mathcal{Y}_n = \{\emptyset \rightarrow 1_{\mathcal{C}}\}$ , we say  $S$  is *(acyclic) symmetrizable*.

A map  $f \in \mathcal{C}$  is called an *(acyclic) power cofibration* if the morphism  $f^{\square n}$  is an (acyclic) cofibration in  $\Sigma_n^{\text{pro}}\mathcal{C}$  for all integers  $n > 0$  (i.e., a projective cofibration with respect to the  $\Sigma_n$ -action).

The category  $\mathcal{C}$  is called symmetric h-monoidal/ $\mathcal{Y}$ -symmetroidal/freely powered if the class of all (acyclic) cofibrations is (acyclic) symmetric h-cofibrant/(acyclic)  $\mathcal{Y}$ -symmetroidal/(acyclic) power cofibration.

*Remark 4.2.10.* In the definition of power cofibrations, no multi-indices are necessary: for power cofibrations  $s_i$  and any any multi-index  $n = (n_i)$ ,  $s^{\square n} := \square_i s^{\square n_i}$  is a  $\Sigma_n := \prod \Sigma_{n_i}$  projective cofibration by the pushout product axiom.

Unlike the definition of power cofibrations in [Lur], we exclude the case  $n = 0$ , for this would require  $1$  to be cofibrant, which is not always satisfied. In fact, it is *never* satisfied for the positive model structures on symmetric spectra which is a main motivating example for us [PS14b].

We have the following implications (where symmetroidality is with respect to the classes  $\mathcal{Y}_n$  of injective cofibrations in  $\Sigma_n\mathcal{C}$ ):

$$(4.2.11) \quad \begin{array}{ccccc} \text{power cofibration} & \Longrightarrow & \text{symmetroidal map} & \Longrightarrow & \text{cofibration} \\ & & \vdots & & \Downarrow \\ & & \text{symmetric h-cofibration} & \Longrightarrow & \text{h-cofibration.} \end{array}$$

The vertical implication holds if  $\mathcal{C}$  is left proper. The dotted arrow is not an implication in the strict sense unless all objects in  $\mathcal{C}$  are cofibrant. A symmetroidal map  $x$  is such that for all cofibrant objects  $Y \in \Sigma_n^{\text{in}}\mathcal{C}$ , the map  $Y \otimes_{\Sigma_n} x^{\square n}$  is a cofibration and therefore (again if  $\mathcal{C}$  is left proper) an h-cofibration. Being a symmetric h-cofibration demands the latter for any object  $Y \in \Sigma_n\mathcal{C}$ . Every power cofibration is a symmetrizable cofibration since the coinvariants  $\Sigma_n^{\text{pro}}\mathcal{C} \rightarrow \mathcal{C}$  are a left Quillen functor. The implications in (4.2.11) are in general strict: in a monoidal model category  $\mathcal{C}$  with cofibrant monoidal unit or, more generally, one satisfying the strong unit axiom, every object is h-cofibrant [BB13, Proposition 1.17], but of course not necessarily cofibrant. In the category  $\mathbf{sSet}$  of simplicial sets every cofibration is a symmetrizable cofibration, but not a power cofibration (see Subsection 7.1).

The homotopy orbit  $\text{hocolim}_{\Sigma_n} X^{\otimes n}$  can be computed by applying the derived functor of the either of the following two left Quillen bifunctors to  $(1_{\mathcal{V}}, X^{\otimes n})$  [Gam10, Theorem 3.2 and Theorem 3.3]:

$$(4.2.12) \quad \Sigma_n^{\text{op, in}}\mathcal{V} \times \Sigma_n^{\text{pro}}\mathcal{C} \xrightarrow{\otimes} \mathcal{C},$$

$$(4.2.13) \quad \Sigma_n^{\text{op, pro}}\mathcal{V} \times \Sigma_n^{\text{in}}\mathcal{C} \xrightarrow{\otimes} \mathcal{C}.$$

Here  $\mathcal{V}$  denotes the symmetric monoidal model category used for the enrichment and the monoidal unit  $1_{\mathcal{V}} \in \mathcal{V}$  is equipped with the trivial  $\Sigma_n$ -action. If  $\mathcal{C}$  is freely powered, then for any cofibrant object  $X \in \mathcal{C}$ ,  $X^{\otimes n}$  is projectively cofibrant, i.e., cofibrant in  $\Sigma_n^{\text{pro}}\mathcal{C}$ . Thus, the homotopy orbit is given by  $(X^{\otimes n})_{\Sigma_n}$ , provided that  $1_{\mathcal{V}}$  is cofibrant [Lur, Lemma 4.5.4.11]. However, most model categories appearing in practice are not freely powered, so that  $X^{\otimes n}$  needs to be *projectively* cofibrantly replaced to compute the homotopy colimit. This is usually a difficult task. On the other hand, when using (4.2.13), one needs to cofibrantly replace  $1$  in  $\Sigma_n^{\text{op, pro}}\mathcal{V}$ , but no cofibrant replacement has to be applied to  $X^{\otimes n}$ , provided that  $X$  is cofibrant in  $\mathcal{C}$ . This makes the second approach to computing homotopy colimits much more easily applicable. This observation is used in Lemma 4.3.4 below, which in its turn is the key technical step in establishing the compatibility of symmetric h-monoidality and Bousfield localizations (Theorem 6.2.2(ii)).

**4.3. Basic properties and weak saturation.** In this section, we provide a few elementary facts concerning the symmetricity notions defined in Subsection 4.2. After this, we show the main theorem of this section (4.3.9), which asserts that the symmetricity notions behave well with respect to weak saturation.

The following two results have a similar spirit: we show that symmetric flatness can be reduced to (projective) acyclic fibrations, and that the class  $\mathcal{Y}$  appearing in the definition of  $\mathcal{Y}$ -symmetroidality can be weakly saturated.

**Lemma 4.3.1.** *If  $S$  is symmetric flat with respect to  $\mathcal{Y}$ , it is also symmetric flat with respect to the class  $\mathcal{Z}$ , where  $\mathcal{Z}_n$  consists of compositions  $z = y \circ c$ , with  $y \in \mathcal{Y}_n$  and  $c \in \text{AC}_{\Sigma_n^{\text{pro}}}\mathcal{C}$ , i.e., an acyclic projective cofibration. In particular, any class of cofibrations is symmetric flat with respect to  $\text{AC}_{\Sigma_n^{\text{pro}}}\mathcal{C}$ . Moreover, being symmetric flat is equivalent to being symmetric flat with respect to the acyclic projective fibrations  $\text{AF}_{\Sigma_n^{\text{pro}}}\mathcal{C}$ .*

*Proof.* Any acyclic projective cofibration  $c$  lies in the weak saturation of  $\Sigma_n \cdot \text{AC}_{\mathcal{D}}$ . For  $h \in \text{AC}_{\mathcal{C}}$  and  $c = \Sigma_n \cdot h$ , we get an isomorphism in  $\mathcal{C}$ ,

$$(\Sigma_n \cdot h) \square_{\Sigma_n} s^{\square n} \cong h \square s^{\square n}$$

which is an acyclic cofibration in  $\mathcal{C}$  by the pushout product axiom, using that  $s \in S$  is a cofibration. The assignment  $c \mapsto c \square_{\Sigma_n} s^{\square n}$  is cocontinuous, so that the latter expression is an acyclic cofibration in  $\mathcal{C}$  for all  $c \in \text{AC}_{\Sigma_n^{\text{pro}} \mathcal{C}}$ . The pushout product  $z \square_{\Sigma_n} s^{\square n}$  is the composition of a pushout of  $c \square_{\Sigma_n} s^{\square n}$ , which is an acyclic cofibration, followed by  $y \square_{\Sigma_n} s^{\square n}$  which is a weak equivalence by assumption.  $\square$

**Lemma 4.3.2.** *Let  $S, \mathcal{Y}, \mathcal{C}$  be as in Definition 4.2.8. If  $S$  is  $\mathcal{Y}$ -symmetroidal, it is also  $\text{cof}(\mathcal{Y})$ -symmetroidal.*

*Proof.* For a fixed  $s \in S$ , the functor  $F_s: y \mapsto y \square_{\Sigma_n} s^{\square n}$  is cocontinuous. In particular  $F_s(\text{cof}(\mathcal{Y})) \subset \text{cof}(F_s(\mathcal{Y})) \subset \text{cof}(\mathcal{C})_{\mathcal{C}} = \mathcal{C}_{\mathcal{C}}$  and likewise for acyclic  $\mathcal{Y}$ -symmetroidal maps.  $\square$

**Definition 4.3.3.** The cofibrant replacement of 1 in  $\Sigma_n^{\text{op,pro}} \mathcal{V}$  is denoted by  $\text{E}\Sigma_n$ . (For  $\mathcal{V} = \mathbf{sSet}$ , this coincides with the usual definition of  $\text{E}\Sigma_n$  as a weakly contractible simplicial set with a free  $\Sigma_n$ -action.)

Proposition 4.3.5 is a key step in the proof of stability of symmetric h-monoidality and symmetroidality under left Bousfield localizations. It relies on the following technical lemma.

**Lemma 4.3.4.** *Suppose  $\mathcal{C}$  is a symmetric monoidal, h-monoidal, flat model category,  $y \in \Sigma_n \mathcal{C}$  is any map,  $s$  is a finite family of acyclic cofibrations with cofibrant domain that lies in some symmetric flat class  $S$ , and  $y \square s^{\square n}$  is a weak equivalence in  $\mathcal{C}$  for some multiindex  $n > 0$ . Then  $y \square_{\Sigma_n} s^{\square n}$  is also a weak equivalence.*

*Proof.* Let

$$\begin{array}{ccc} A' & \xrightarrow{\sim a} & A \\ y' \downarrow & & \downarrow y \\ B' & \xrightarrow{\sim b} & B \end{array}$$

be the functorial cofibrant replacement of  $y: A \rightarrow B \in \mathbf{Ar}(\mathcal{C})$  (in the projective model structure, so that  $y'$  is a cofibration with a cofibrant domain). Functoriality and the fact that  $y \in \mathbf{Ar}(\Sigma_n \mathcal{C})$  imply that  $y' \in \mathbf{Ar}(\Sigma_n \mathcal{C})$ . We claim that  $y' \square s^{\square n}$  is a cofibrant replacement of  $y \square s^{\square n}$  in  $\mathbf{Ar}(\mathcal{C})$ . Let  $t := s^{\square n}: T \rightarrow S$ . The map  $b \otimes S$  is a weak equivalence by the flatness assumption. To see that  $B' \otimes T \sqcup_{A' \otimes T} A' \otimes S \rightarrow B \otimes T \sqcup_{A \otimes T} A \otimes S$  is a weak equivalence we first note that these pushouts are homotopy pushouts by Lemma 2.0.6(i) since  $A \otimes t$  is an h-cofibration. Thus it suffices that the three individual terms in the pushouts are weakly equivalent, which again follows from flatness. The claim is shown.

Thus we have

$$\text{hocolim}_{\Sigma_n} (y \square s^{\square n}) = (\text{E}\Sigma_n \otimes y' \square s^{\square n})_{\Sigma_n} \sim y \square_{\Sigma_n} s^{\square n}.$$

The last weak equivalence holds by symmetric flatness of  $S$  since  $\text{E}\Sigma_n \otimes y' \rightarrow y' \rightarrow y$  is a weak equivalence by the unit axiom for the  $\mathcal{V}$ -enrichment (note that the cofibrant replacement  $\text{E}\Sigma_n \rightarrow 1$  in  $\Sigma_n^{\text{pro}} \mathcal{V}$  is in particular a cofibrant replacement in  $\mathcal{V}$ ). Finally,  $y \square s^{\square n}$  is a weak equivalence in  $\mathcal{C}$  by assumption. Therefore, the above homotopy colimit is a weak equivalence in  $\mathcal{C}$ .  $\square$

**Proposition 4.3.5.** *The class of acyclic power cofibrations coincides with the intersection of  $\mathbf{W}$  with the class of power cofibrations.*

*A  $\mathcal{Y}$ -symmetroidal class  $S$  which consists of acyclic cofibrations with cofibrant source is acyclic  $\mathcal{Y}$ -symmetroidal, provided that  $\mathcal{C}$  is h-monoidal and flat and  $S$  is symmetric flat in  $\mathcal{C}$ .*

*Proof.* The first claim follows from the pushout product axiom.

For any  $s \in S$  and any map  $y \in \mathcal{Y}_n \subset \text{Mor}(\Sigma_n \mathcal{C})$ ,  $y \square s^{\square n}$  is a weak equivalence in  $\Sigma_n \mathcal{C}$  by assumption on the class  $\mathcal{Y}$  (see Definition 4.2.8). Now apply Lemma 4.3.4.  $\square$

We now establish the compatibility of the three symmetricity properties with weak saturation. Parts (iv) and (v) of Theorem 4.3.9 are due to Gorchinskiy and Guletskiĭ [GG09, Theorem 5]. Part (ii) extends arguments in [GG11, Theorem 9], which shows a weak saturation property for symmetrically cofibrant objects in a stable model category. Of course, it also extends the analogous statement for nonsymmetric flatness (Theorem 3.2.8(ii)). Likewise, (iii) extends the weak saturation property of h-cofibrations (see Lemma 2.0.6). The proof of the closure under transfinite compositions in (iv) is reminiscent of §4 of Gorchinskiy and Guletskiĭ [GG09]. See also the expository accounts by White [Whi14a, Appendix A] and Pereira [Per14, §4.2]. In the proof of the theorem, we will need a combinatorial lemma that we establish first. Recall the conventions for multiindices in Definition 4.2.1.

**Lemma 4.3.6.** *Let  $X_0^{(i)} \xrightarrow{v_0^{(i)}} X_1^{(i)} \xrightarrow{v_1^{(i)}} X_2^{(i)}$ ,  $1 \leq i \leq e$  be a finite family of composable maps in a symmetric monoidal category. For a pair of multiindices  $0 \leq k \leq n$  of length  $e$ , we set*

$$(4.3.7) \quad m_k := \Sigma_n \cdot_{\Sigma_{n-k} \times \Sigma_k} v_0^{\square n-k} \square v_1^{\square k}.$$

- (i) The map  $(v_1 v_0)^{\square n} : \square^n(v_1 v_0) \rightarrow X_2^{\otimes n}$  is the composition of pushouts (with the attaching maps constructed in the proof) of the maps  $m_k$  ( $0 \leq k < n$ ), and the map  $m_n = v_1^{\square n}$ .
- (ii) The map  $\kappa : \square^n(v_1 v_0) \sqcup_{\square^n v_0} X_1^{\otimes n} \rightarrow X_2^{\otimes n}$  is the composition of pushouts of  $m_k$  for  $1 \leq k < n$ , and the map  $m_n$ . (Here  $1$  denotes the multiindex whose components are all equal to 1.)

*Proof.* We interpret the composable pair  $(v_0, v_1)$  as a functor  $v : 3 = \{0 \rightarrow 1 \rightarrow 2\} \rightarrow \mathcal{C}^I$ , where  $I = \{1, \dots, e\}$ . Let  $\mathcal{E}$  be the category of posets  $C$  lying over  $3^n = \prod_i 3^{n_i}$  and let  $\Sigma_n \mathcal{E}$  be those posets with a  $\Sigma_n$ -action which is compatible with the  $\Sigma_n$ -action on  $3^n$ . For all posets considered below, the map to  $3^n$  will be obvious from the context. Consider the following functor:

$$Q : \Sigma_n \mathcal{E} \rightarrow \Sigma_n \mathcal{C}, \quad (C \rightarrow 3^n) \mapsto \operatorname{colim} \left( C \longrightarrow 3^n \xrightarrow{v^n} \mathcal{C}^n \xrightarrow{\otimes} \mathcal{C} \right).$$

Being the composition of the two cocontinuous functors

$$\operatorname{posets}/3^n \longrightarrow \operatorname{posets}/\mathcal{C} \xrightarrow{\operatorname{colim}} \mathcal{C},$$

$Q$  is also cocontinuous. The map  $(v_1 v_0)^{\square n}$  is obtained by applying  $Q$  to the map

$$\iota : \{0, 1, 2\}^n \setminus \{1, 2\}^n \rightarrow \{0, 1, 2\}^n$$

which adds all tuples containing only 1's and 2's. It is the composition of the maps

$$\iota_k : \{0, 1, 2\}^n \setminus \{1, 2\}^n \cup \{\Sigma_n 1^* 2^{<k}\} \rightarrow \{0, 1, 2\}^n \setminus \{1, 2\}^n \cup \{\Sigma_n 1^* 2^{\leq k}\},$$

for  $0 \leq k \leq n$ , with  $\prod_i (n_i + 1)$  maps in total. The superscript  $*$  means that one adds as many elements as needed to get an  $n$ -multituple. For multiindices the above statements should be interpreted separately for each component. The map  $\iota_k$  adds the  $\Sigma_n$ -orbit  $O$  consisting of tuples with  $k$  2's and  $n - k$  1's, i.e.,  $\Sigma_n 1^{n-k} 2^k$ . The cardinality of  $O$  is  $\binom{n}{k}$ . For  $o \in O$ , consider the downward closure  $D_o$  of  $o$  and  $C_o := D_o \setminus \{o\}$ .

There is a pushout diagram in  $\Sigma_n \mathcal{E}$

$$(4.3.8) \quad \begin{array}{ccc} A := \prod_{o \in O} C_o & \xrightarrow{\alpha_k} & \{0, 1, 2\}^n \setminus \{1, 2\}^n \cup \{\Sigma_n 1^* 2^{<k}\} \\ \downarrow \mu_k & & \downarrow \iota_k \\ B := \prod_{o \in O} D_o & \longrightarrow & \{0, 1, 2\}^n \setminus \{1, 2\}^n \cup \{\Sigma_n 1^* 2^{\leq k}\}. \end{array}$$

(For  $k = n$  the top horizontal row is an identity, so  $\iota_n = \mu_n$  in this case.) Any  $o \in O$  determines a partition of  $\prod_i n_i$  into  $\prod_i \{1 \leq j \leq n_i \mid o_{i,j} = 1\}$  and  $\prod_i \{1 \leq j \leq n_i \mid o_{i,j} = 2\}$ . Using this partition, we have  $D_o = \Sigma_{n-k} 0^* 1^* \times \Sigma_k 0^* 1^* 2^*$  and  $C_o = \Sigma_{n-k} 0^* 1^{<n-k} \times \Sigma_k 0^* 1^* 2^* \cup \Sigma_{n-k} 0^* 1^* \times \Sigma_k 0^* 1^* 2^{<k}$ . Thus the map  $Q(C_o \rightarrow D_o)$  is just  $v_0^{\square n-k} \square v_1^{\square k}$ . Using the cocontinuity of  $Q$ , this shows  $Q(\mu_k) = m_k$ .

The second part now follows immediately from the above once we observe that the codomain of  $\iota_0$  is precisely the domain of the map under consideration.  $\square$

**Theorem 4.3.9.** *Let  $S$  be a class of morphisms in a symmetric monoidal model category  $\mathcal{C}$ . We say some property of  $S$  is stable under saturation if it also holds for the weak saturation  $\operatorname{cof}(S)$ .*

- (i) *The property of being admissibly generated relative to  $S$  (Definition 4.2.4) is stable under saturation. Therefore, if  $\mathcal{C}$  is cofibrantly generated and admissibly generated relative to some set of generating cofibrations, it is admissibly generated.*
- (ii) *If  $\mathcal{C}$  is pretty small and  $S$  is symmetric h-monoidal, then symmetric flatness of  $S$  relative to a class  $\mathcal{Y} = (\mathcal{Y}_n)$  of weak equivalences in  $\Sigma_n \mathcal{C}$  is stable under saturation. In particular, if some class of generating cofibrations in  $\mathcal{C}$  is symmetric flat and symmetric h-monoidal, then  $\mathcal{C}$  is symmetric flat.*
- (iii) *If  $\mathcal{C}$  is pretty small, then the property of being (acyclic) symmetric h-monoidal is stable under saturation. In particular, if some class of generating (acyclic) cofibrations consists of (acyclic) symmetric h-cofibrations, then  $\mathcal{C}$  is symmetric h-monoidal.*
- (iv) *Being  $\mathcal{Y}$ -symmetroidal (Definition 4.2.8) is stable under saturation. In particular, if some class of generating (acyclic) cofibrations is (acyclic)  $\mathcal{Y}$ -symmetroidal, then  $\mathcal{C}$  is  $\mathcal{Y}$ -symmetroidal.*
- (v) *The same statement holds for power cofibrations.*

*Proof.* For a finite family of maps  $v = (v^{(1)}, \dots, v^{(e)})$  we use the multi-index notation of Definition 4.2.1. We prove the statements by cellular induction, indicating the necessary arguments for each statement individually in each step. The acyclic parts of (iii) and (iv) are the same as the nonacyclic parts, so they will be omitted. Fix an object  $Y \in \Sigma_n \mathcal{C}$ , respectively a map  $y \in \mathcal{Y}_n \subset \operatorname{Mor} \Sigma_n \mathcal{C}$ . For (ii) and (iv), respectively (i) and (iii), we write

$$g(v, n) := y \square_{\Sigma_n} v^{\square n}, \quad \text{respectively, } g(v, n) := Y \otimes_{\Sigma_n} v^{\square n}.$$

By Proposition 4.1.7,  $g(-, n)$  preserves pushout morphisms  $\varphi : v \rightarrow v'$  (in the sense that, say,  $\varphi^{(1)}$  is a pushout morphism and all other  $\varphi^{(j)}$ 's are identities) and retracts. Thus, if  $g(v, n)$  is an (acyclic) h-cofibration or (acyclic) cofibration, so is  $g(v', n)$ . This shows the stability of the properties of being symmetric h-monoidal and symmetroidal under cobase changes. For (ii), we additionally observe that  $Y \otimes_{\Sigma_n} v^{\square n}$  is an h-cofibration and

similarly with  $Y'$  since  $S$  is symmetric h-monoidal by assumption. By Lemma 3.2.7 (more precisely, replace  $\otimes$  there by  $\otimes_{\Sigma_n}$ ), applied to  $a = v^{\square n}$  and  $b = v'^{\square n}$ , we see that  $g(v', n)$  is a weak equivalence since  $g(v, n)$  is one. For (i), we also use here and below that an object  $X$  is small relative to some class  $\text{cell}(T)$  if and only if it is small relative to its weak saturation [Hir03, Proposition 10.5.13].

We now show the stability of the three symmetricity properties and being admissibly generated relative to a class under transfinite composition: suppose  $v^{(1)}$  is the transfinite composition

$$v^{(1)}: X_0^{(1)} \xrightarrow{v_0^{(1)}} \dots \rightarrow X_i^{(1)} \xrightarrow{v_i^{(1)}} X_{i+1}^{(1)} \rightarrow \dots \rightarrow X_\infty^{(1)} = \text{colim } X_i^{(1)},$$

whose maps are obtained as pushouts

$$(4.3.10) \quad \begin{array}{ccc} A & \xrightarrow{s \in S} & A' \\ \downarrow & (*) & \downarrow \\ X := X_i^{(1)} & \xrightarrow{x := v_i^{(1)}} & X' := X_{i+1}^{(1)}. \end{array}$$

For the statements (ii), (iii), respectively (iv) we need to show that  $g(v, n) = g((v^{(1)}, \dots, v^{(e)}, n))$  is a weak equivalence, h-cofibration, or cofibration, respectively, provided that

$$\{v_i^{(1)}, i \leq \infty, v^{(2)}, \dots, v^{(e)}\}$$

is a symmetric flat, symmetric h-monoidal, respectively symmetroidal class. Applying this argument  $e$  times gives the desired stability under transfinite compositions. We write  $r_i^{(1)}: X_0^{(1)} \rightarrow X_i^{(1)}$  for the (finite) compositions of the  $v_i^{(1)}$ . Consider

$$(4.3.11) \quad \text{id}_{(X_0^{(1)})^{\otimes n}} = (r_0^{(1)})^{\square n} \rightarrow (r_1^{(1)})^{\square n} \rightarrow \dots \rightarrow (v^{(1)})^{\square n}.$$

As an object of  $\Sigma_n \mathbf{Ar}(\mathcal{C})$ ,

$$(4.3.12) \quad g(v, n) = \text{colim}_i g(\underbrace{(r_i^{(1)}, v^{(2)}, \dots, v^{(e)})}_{=: v_i}, n) = \text{colim}_i g(v_i, n),$$

since  $-\square n$  preserves filtered colimits (Proposition 4.1.6). We now show that  $v_i$  is a symmetric flat (respectively symmetric h-monoidal or symmetroidal) family, so that  $g(v_i)$  is a weak equivalence (h-cofibration, cofibration, respectively). We consider the composition of two morphisms  $r_0^{(1)}$  and  $r_1^{(1)}$  only and leave the similar case of a finite composition of more than two maps to the reader. By Lemmas 3.1.7 and 4.3.6,  $v_1^{\square n}$  is the (finite) composition of pushouts of  $\Sigma_n \cdot_{\Sigma_m} w^{\square m}$ , where  $w = (r_0^{(1)}, r_1^{(1)}, v^{(2)}, \dots, v^{(e)})$ , and  $m$  runs through multi-indices of length  $e+1$  such that  $0 \leq m^{(1)} \leq n^{(1)}$ ,  $m^{(1)} + m^{(2)} = n^{(1)}$ , and  $m^{(k)} = n^{(k-1)}$  for  $2 \leq k \leq e+1$ .

For (iii), each  $g(w, m) = y \square_{\Sigma_m} w^{\square m}$  is an h-cofibration. Hence so is  $g(v_1, n)$  since h-cofibrations are stable under pushouts and (finite) compositions by Lemma 2.0.6. By Lemma 2.0.6(iii),  $g(v, n)$  is also an h-cofibration then.

Similarly, for (iv), each  $g(w, m)$  is a cofibration, so that  $g(v_1, n)$  is a cofibration. By Lemma 4.3.6,  $(v_1^{(1)} \circ v_0^{(1)})^{\square n}$  is the composition of a pushout of  $(v_0^{(1)})^{\square n}$  and the map

$$\begin{array}{c} \square \\ \bullet \end{array} \xrightarrow{n^{(1)}} (v_1^{(1)} \circ v_0^{(1)}) \sqcup_{\square_{n^{(1)}}(v_0^{(1)})} (X_1^{(1)})^{\otimes n} \rightarrow (X_2^{(1)})^{\otimes n}.$$

Here, as usual,  $\square^{n^{(1)}}$  denotes the domain of the  $-\square n^{(1)}$ . The latter map is the composition of pushouts of the maps  $g(w, m)$ , where  $w$  and  $m$  are as above, except that now  $0 \leq m^{(1)} < n^{(1)}$ . Again, these are cofibrations, so the above map is a cofibration. By Lemma 2.0.3(i),  $g(v, n)$  is therefore a cofibration.

For (ii), each  $g(w, m)$  is a weak equivalence. The map  $g(v_1, n)$  is the composition of pushouts of  $g(w, m)$  along  $Y \otimes_{\Sigma_n} \Sigma_n \cdot_{\Sigma_m} w^{\square m} = Y \otimes_{\Sigma_m} w^{\square m}$ . The latter map (and similarly for  $Y'$ ) instead of  $Y$  is an h-cofibration by the symmetric h-monoidality assumption. Thus the pushouts of  $g(w, m)$ , the compositions of which are  $g(v_1, n)$ , are weak equivalences by Lemma 3.2.7 (again, replace  $\otimes$  by  $\otimes_{\Sigma_n}$  there). We have shown that  $g(v_1, n)$  is a weak equivalence. By Lemma 2.0.3(iii),  $g(v, n)$  is then also weak equivalence.

For (i), we again use that  $g(v_1, n)$  is in the weak saturation of maps  $g(w, m)$  and the above-mentioned stability of smallness under weak saturation.

(v) can be shown using the same argument but considering  $g(v) := v^{\square n} \in \Sigma_n \mathcal{C}$  instead. By Remark 4.2.10 it is unnecessary to use multi-indices in this proof.  $\square$

## 5. TRANSFER OF MODEL STRUCTURES

In this section, we fix an adjunction

$$(5.0.1) \quad F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

such that  $\mathcal{C}$  is a model category and  $\mathcal{D}$  is complete and cocomplete. One can ask whether it is possible to construct a model structure on  $\mathcal{D}$  from this data. The following definitions turn out to be convenient in practice.

**Definition 5.0.2.** A model structure on  $\mathcal{D}$  is *transferred along  $G$*  if the weak equivalences and fibrations in  $\mathcal{D}$  are those morphisms which are mapped by  $G$  to weak equivalences and fibrations in  $\mathcal{C}$ , respectively.

If a transferred model structure on  $\mathcal{D}$  exists, it is unique, so we also speak of *the* transferred model structure.

**5.1. Existence and basic properties.** The existence of the transferred model structure is addressed by the following proposition. Note that the condition that  $G$  maps  $F(J)$ -cellular maps (i.e., transfinite compositions of pushouts of maps in  $F(J)$ ) to weak equivalences is necessary because  $F$  is a left Quillen functor, in particular it maps  $J$  to acyclic cofibrations in  $\mathcal{D}$ , which are closed under cobase changes and transfinite compositions.

**Proposition 5.1.1.** [Hir03, Theorem 11.3.2] *Suppose that  $\mathcal{C}$  is a cofibrantly generated model category and  $\mathcal{D}$  is a complete and cocomplete category. Fix some sets  $I$  and  $J$  of generating cofibrations and acyclic cofibrations in  $\mathcal{C}$ . Suppose that the functor  $G$  maps  $F(J)$ -cellular maps to weak equivalences in  $\mathcal{C}$ . The transferred model structure on  $\mathcal{D}$  exists if  $F(I)$  and  $F(J)$  permit the small object argument [Hir03, Definition 10.5.15]. For example, the latter condition is satisfied if  $\mathcal{D}$  is locally presentable, in which case  $\mathcal{D}$  is a combinatorial model category.*

The next proposition describes basic properties of transferred model structures. Part (vi) can be applied to adjunctions of the form  $\mathcal{C} \rightleftarrows \mathbf{Mod}_R$ , where  $R$  is a commutative monoid which is cofibrant as an object of the underlying symmetric monoidal model category  $\mathcal{C}$ . It is a special case of much more general left properness results by Batanin and Berger [BB13].

**Proposition 5.1.2.** *The following properties hold for a transferred model structure on  $\mathcal{D}$ . We write  $I$  (respectively  $J$ ) for a class of generating (acyclic) cofibrations of  $\mathcal{C}$ .*

- (i) *Suppose that  $\mathcal{V}$  is a symmetric monoidal model category and  $(F, G)$  is a  $\mathcal{V}$ -enriched adjunction of  $\mathcal{V}$ -enriched categories that are tensored and powered over  $\mathcal{V}$ . If  $\mathcal{C}$  is a  $\mathcal{V}$ -enriched model category, then so is  $\mathcal{D}$ .*
- (ii) *The class  $F(I)$  (respectively,  $F(J)$ ) generates (acyclic) cofibrations of  $\mathcal{D}$ .*
- (iii) *If  $\mathcal{C}$  is quasi-tractable, then so is  $\mathcal{D}$ .*
- (iv) *If  $\mathcal{C}$  is combinatorial or tractable, then so is  $\mathcal{D}$ , provided that  $\mathcal{D}$  is locally presentable.*
- (v) *Suppose that  $G$  preserves filtered colimits. If  $\mathcal{C}$  is pretty small, then so is  $\mathcal{D}$ , provided that  $\mathcal{D}$  is locally presentable, or, more generally,  $F(I')$  and  $F(J')$  permit the small object argument, where  $I'$  and  $J'$  come from pretty smallness.*
- (vi) *Suppose that  $G$  preserves pushouts along maps in  $F(I)$ . Also suppose that  $G$  preserves filtered colimits. Finally suppose that (a)  $G(F(I))$  consists of cofibrations or (b)  $\mathcal{C}$  is pretty small and  $G(F(I))$  consists of  $h$ -cofibrations. Then, if  $\mathcal{C}$  is left proper, so is  $\mathcal{D}$ .*
- (vii) *If  $G$  preserves filtered colimits and sends cobase changes of  $F(I)$  (respectively cobase changes of  $F(I)$  along maps with cofibrant targets) to cofibrations, then  $G$  preserves cofibrations (respectively, cofibrations with cofibrant source).*

*Proof.* (i): By [Hov99, Lemma 4.2.2] it suffices to check that for any cofibration  $j: K \rightarrow L$  in  $\mathcal{V}$  and any fibration  $\pi: E \rightarrow B$  in  $\mathcal{D}$  the natural map

$$\zeta: E^K \rightarrow E^L \times_{B^L} B^K$$

is a fibration in  $\mathcal{D}$  that is acyclic if either  $j$  or  $\pi$  is. The map  $G(\zeta)$  is an (acyclic) fibration because  $G$  preserves fiber products and  $\mathcal{V}$ -powers being a  $\mathcal{V}$ -enriched right adjoint.

(ii): By adjunction, a morphism  $f$  in  $\mathcal{D}$  has a right lifting property with respect to  $F(I)$  if and only if  $G(f)$  has a right lifting property with respect to  $I$ , which is true if and only if  $G(f)$  is an acyclic fibration in  $\mathcal{C}$ , equivalently  $f$  is an acyclic fibration in  $\mathcal{D}$ . Likewise for acyclic cofibrations.

(iii) The domains of  $F(I)$  are cofibrant because  $F$  is a left Quillen functor and the domains of  $I$  are cofibrant.

(iv): The combinatoriality of  $\mathcal{D}$  is immediate from (ii).

(v): By Definition 2.0.2, there is another model structure  $\mathcal{C}'$  on the underlying category of  $\mathcal{C}$  with the same weak equivalences and a smaller class of cofibrations that is generated by a set of morphisms with compact domains and codomains. By assumption  $F(\mathcal{C}_{\mathcal{C}'})$  permits the small object argument and similarly for acyclic cofibrations. This verifies the condition for the existence of the transfer of the model structure  $\mathcal{C}'$ . Thus the model structure  $\mathcal{C}'$  transfers to a model structure  $\mathcal{D}'$  on the category underlying  $\mathcal{D}$  and its cofibrations are a subset of cofibrations of  $\mathcal{D}$ . The (co)domains of the generating set of cofibrations  $F(I')$  are compact because  $G$  preserves filtered colimits and therefore  $F$  preserves compact objects.

(vi): We have to show that the pushout of any weak equivalence  $f_0: D_0 \rightarrow E_0$  along a  $\mathcal{D}$ -cofibration  $D_0 \rightarrow D$  is a weak equivalence. Every cofibration  $D_0 \rightarrow D$  is obtained as a retract of a transfinite composition  $d: D_0 \rightarrow D_1 \rightarrow \dots \rightarrow D_\infty = D$ , where every map  $d_i: D_i \rightarrow D_{i+1}$  is a cobase change of a map  $F(c_i)$  for some generating

cofibration  $c_i \in I_{\mathcal{C}}$ . Thus for each  $i$  we have the following diagram of cocartesian squares, where the objects  $E_i$  and the morphisms  $E_i \rightarrow E_{i+1}$  and  $D_i \rightarrow E_i$  are constructed inductively using pushouts and colimits:

$$\begin{array}{ccccc} C_i & \longrightarrow & D_i & \longrightarrow & E_i \\ \downarrow F(c_i) & & \downarrow d_i & & \downarrow \\ C_{i+1} & \longrightarrow & D_{i+1} & \longrightarrow & E_{i+1} \end{array}$$

All vertical maps are cofibrations in  $\mathcal{D}$ . Apply  $G$  to this diagram. The left square and the big rectangle in the resulting diagram are again cocartesian by assumption, hence the right square is also cocartesian.

If the morphism  $G(F(c_i))$  is an (h-)cofibration in  $\mathcal{C}$ , then so is its cobase change  $G(d_i)$  and therefore so is their transfinite composition  $G(D_0) \rightarrow G(D_\infty)$ : for h-cofibrations this is Lemma 2.0.6, using the assumption that  $\mathcal{C}$  is pretty small. For cofibrations this is true because cofibrations in any model category are weakly saturated. Cofibrations in a left proper model category are h-cofibrations. Thus in both cases under consideration the morphism  $G(D_0) \rightarrow G(D_\infty)$  is an h-cofibration. The latter morphism is isomorphic to  $G(d)$ , because  $G$  preserves filtered colimits. Pushouts along h-cofibrations are homotopy pushouts and therefore preserve weak equivalences. Thus  $D_\infty \rightarrow E_\infty$  is a weak equivalence, being the cobase change of the weak equivalence  $D_0 \rightarrow E_0$  along the h-cofibration  $D_0 \rightarrow D_\infty$ .

(vii): Cofibrations in  $\mathcal{D}$  are retracts of transfinite compositions of cobase changes of elements in  $F(I)$ . All three operations are preserved by the functor  $G$  by assumption. Thus it is sufficient to observe that  $G(F(I))$  consists of cofibrations in  $\mathcal{C}$ , which are weakly saturated, hence  $G$  preserves cofibrations. The preservation of cofibrations with cofibrant source is shown the same way.  $\square$

**5.2. Transfer of monoidal and symmetricity properties.** We now transfer monoidal properties along an adjunction of monoidal categories. We restrict to monoidal categories, as opposed to left modules, merely for notational convenience.

**Proposition 5.2.1.** *Let*

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

*be an adjunction between (symmetric) monoidal model categories. Suppose that the model structure on  $\mathcal{D}$  is transferred from  $\mathcal{C}$ , respectively, and that the left adjoint  $F$  is a strong (symmetric) monoidal functor between (symmetric) monoidal categories. If  $\mathcal{C}$  is a (symmetric) monoidal model category, then so is  $\mathcal{D}$ .*

*Proof.* By Proposition 5.1.2(ii), to prove the pushout product axiom it is enough to verify that  $F(C_{\mathcal{C}}) \square F(C_{\mathcal{C}}) \subset C_{\mathcal{C}}$  and similarly with acyclic cofibrations. This uses the preservation by  $\otimes_{\mathcal{D}}$  of colimits in both variables. Since  $F$  is strong monoidal and cocontinuous, we have  $F(C_{\mathcal{C}}) \square F(C_{\mathcal{C}}) = F(C_{\mathcal{C}} \square C_{\mathcal{C}}) = F(C_{\mathcal{C}}) \subset C_{\mathcal{D}}$ . Likewise for acyclic cofibrations.  $\square$

**Definition 5.2.2.** A *Hopf adjunction* is an adjunction between monoidal categories such that there is a functorial isomorphism for  $C \in \mathcal{C}$ ,  $D \in \mathcal{D}$ ,

$$(5.2.3) \quad G(F(C) \otimes D) \cong C \otimes G(D).$$

*Remark 5.2.4.* If the monoidal products  $\otimes_{\mathcal{C}}$  and  $\otimes_{\mathcal{D}}$  are closed, this is equivalent to  $G$  being strong closed, i.e., internal homs are preserved up to a coherent isomorphism.

**Proposition 5.2.5.** *Suppose the model structure on monoidal model category  $\mathcal{D}$  is transferred along a Hopf adjunction between monoidal model categories. Also suppose that  $G$  preserves pushouts along maps of the form  $D \otimes F(s)$ , where  $D \in \mathcal{D}$  is any object and  $s$  is any morphism in  $S$ . Let  $S$  be a class of cofibrations in  $\mathcal{C}'$ . We say that a property of the class  $S$  transfers, if the same property holds for  $F(S)$ .*

- (i) *Suppose  $\mathcal{C}$  and  $\mathcal{D}$  are left proper. Then the (acyclic) h-monoidality of  $S$  transfers. The h-monoidality of  $\mathcal{C}$  transfers to  $\mathcal{D}$  if  $\mathcal{D}$  is pretty small.*
- (ii) *The flatness of  $S$  transfers. The flatness of  $\mathcal{C}$  transfers to  $\mathcal{D}$  if  $\mathcal{D}$  is pretty small and h-monoidal.*
- (iii) *If  $G$  also preserves filtered colimits then the monoid axiom transfers from  $\mathcal{C}$  to  $\mathcal{D}$ .*

*Proof.* (i) and (ii) are shown exactly the same way as their symmetric counterparts, see Parts (ii) and (i) of Theorem 5.2.6, using Theorem 3.2.8 instead.

(iii): The preservation of colimits under  $\otimes_{\mathcal{D}}$  and Proposition 5.1.2(ii), the assumption that  $G$  preserves the weak saturation, the Hopf adjunction property, and the monoid axiom for  $\mathcal{C}$  give inclusions

$$\begin{aligned} G(\text{cof}(\mathcal{D} \otimes \text{AC}_{\mathcal{D}})) &\subset G(\text{cof}(\mathcal{D} \otimes F(\text{AC}_{\mathcal{C}}))) \subset \text{cof}(G(\mathcal{D} \otimes F(\text{AC}_{\mathcal{C}}))) \\ &= \text{cof}(G(\mathcal{D}) \otimes \text{AC}_{\mathcal{C}}) \subset \text{cof}(\mathcal{C} \otimes \text{AC}_{\mathcal{C}}) \subset W_{\mathcal{C}}. \end{aligned}$$

$\square$

The following theorem shows that the three symmetricity properties interact well with transfers. It is the symmetric counterpart of Proposition 5.2.5.

**Theorem 5.2.6.** *Let*

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

be a Quillen adjunction of symmetric monoidal model categories such that the model structure on  $\mathcal{D}$  is transferred from  $\mathcal{C}$ . We assume  $F$  is strong monoidal and, for parts (i), (ii), and (v) we also assume that (a) the adjunction is a Hopf adjunction; (b)  $G$  preserves pushouts along maps of the form  $D \otimes F(c)$ , where  $D \in \mathcal{D}$  is any object and  $c$  is any morphism in  $\mathcal{C}$ ; and (c) that  $G$  commutes with the coinvariants functor  $(-)\_{\Sigma_n}$  for all  $n$ .

Let  $S$  be a class of cofibrations in  $\mathcal{C}$ . We say that a property of the class  $S$  transfers, if the same property holds for  $F(S)$ .

- (i) *Symmetric flatness of  $S$  transfers. Moreover, the symmetric flatness of  $\mathcal{C}$  transfers to  $\mathcal{D}$  if, in addition,  $\mathcal{D}$  is pretty small and symmetric h-monoidal.*
- (ii) *Suppose  $\mathcal{C}$  and  $\mathcal{D}$  are left proper. Then the (acyclic) symmetric h-monoidality of  $S$  transfers. The symmetric h-monoidality of  $\mathcal{C}$  transfers if, in addition,  $\mathcal{D}$  is pretty small.*
- (iii) *For some class  $\mathcal{Y}$  of morphisms as in Definition 4.2.8, the  $\mathcal{Y}$ -symmetroidality of  $S$  transfers in the sense that  $\text{cof}(F(S))$  is  $F(\mathcal{Y})$ -symmetroidal. In particular, if  $\mathcal{C}$  is  $\mathcal{Y}$ -symmetroidal, then  $\mathcal{D}$  is  $\text{cof}(F(\mathcal{Y}))$ -symmetroidal.*
- (iv) *Then the property of being freely powered transfers. In particular, if  $\mathcal{C}$  is freely powered, then so is  $\mathcal{D}$ .*
- (v) *Suppose  $G$  preserves filtered colimits. If  $\mathcal{C}$  is admissibly generated, then so is  $\mathcal{D}$ .*

*Proof.* For all properties, the transfer for the given class  $S$  is proven using a specific argument. The transfer of the property from  $\mathcal{C}$  to  $\mathcal{D}$  follows from the fact that  $F(\mathcal{C}_C)$  generates the cofibrations of  $\mathcal{D}$  (Proposition 5.1.2(ii)), and likewise for acyclic cofibrations. Then, a weak saturation property (indicated below) is used. Let  $s \in S$  be any map.

(i): For any weak equivalence  $y$  in  $\Sigma_n \mathcal{D}$  we have to show that  $y \square_{\Sigma_n} F(s)^{\square n}$  is a weak equivalence. Indeed,  $G(y \square_{\Sigma_n} F(s)^{\square n})$  is isomorphic to  $G(y) \square_{\Sigma_n} s^{\square n}$  by the Hopf adjunction property, preservation of  $\Sigma_n$ -coinvariants by  $G$ , and the strong monoidality of  $F$  which ensures that  $F$  commutes with pushout products (Proposition 3.1.4). This is a weak equivalence since  $\mathcal{C}$  is symmetric flat. The symmetric flatness of  $\mathcal{C}$  transfers by Theorem 4.3.9(ii), using  $S = I_{\mathcal{C}}$ .

(ii): We need to show that  $Y \otimes_{\Sigma_n} F(s)^{\square n} = Y \otimes_{\Sigma_n} F(s^{\square n})$  is an h-cofibration for all  $Y \in \Sigma_n \mathcal{D}$ . By Lemma 2.0.7, this is true since  $G(Y \otimes_{\Sigma_n} F(s^{\square n})) = G(Y) \otimes_{\Sigma_n} s^{\square n}$  is an (acyclic) h-cofibration by the (acyclic) symmetric h-monoidality of  $S$ . The symmetric h-monoidality of  $\mathcal{C}$  transfers to  $\mathcal{D}$  by Theorem 4.3.9(iii).

(iii): As  $F$  is strongly monoidal and cocontinuous,  $F(y) \square_{\Sigma_n} F(s^{\square n}) = F(y \square_{\Sigma_n} s^{\square n})$ . This shows the  $F(\mathcal{Y})$ -symmetroidality since  $F$  preserves cofibrations and acyclic cofibrations. Then apply Lemma 4.3.2. The claim about the symmetroidality of  $\mathcal{D}$  follows from Theorem 4.3.9(iv).

(iv): Replace  $y \square_{\Sigma_n} s^{\square n}$  by  $s^{\square n}$  in (iii) and use Theorem 4.3.9(v).

(v): The cofibrant generation transfers to  $\mathcal{D}$  by Proposition 5.1.2(ii). By Lemma 4.2.6 and Theorem 4.3.9(i), we only have to show that  $(\text{co})\text{dom}(F(I))$  are small with respect to  $\text{cell}(Y \otimes_{\Sigma_n} s^{\square n})$ , where  $s = F(t)$  are finite families of *generating* cofibrations, i.e.,  $t$  are cofibrations in  $\mathcal{C}$ . By adjunction, this is equivalent to  $(\text{co})\text{dom}(I)$  being small with respect to

$$G(\text{cell}(Y \otimes_{\Sigma_n} F(t)^{\square n})) \subset \text{cell}(G(Y \otimes_{\Sigma_n} F(t)^{\square n})) = \text{cell}(G(Y) \otimes_{\Sigma_n} t^{\square n})$$

which holds by assumption.  $\square$

*Remark 5.2.7.* If  $\mathcal{C}$  is symmetroidal (i.e., symmetroidal with respect to the injective cofibrations in  $\Sigma_n \mathcal{C}$ ),  $\mathcal{D}$  need not be symmetroidal: for example, for  $\mathcal{C} = \mathbf{sSet}$  and  $\mathcal{D} = \mathbf{Mod}_R(\mathbf{sSet})$  with  $R = \mathbf{Z}/4$ , i.e., simplicial sets with an action of  $\mathbf{Z}/4$ . In this case,  $R$  has a  $\mathbf{Z}/2$ -action, so  $R$  is injectively cofibrant in  $\Sigma_2 \mathbf{Mod}_R$ , but  $R \otimes_{R, \Sigma_2} R^{\otimes_{R^2}} = R/2$  is not cofibrant as an  $R$ -module.

**5.3. Modules over a commutative monoid.** In this section we apply the criteria developed above to the case of the category of modules over a commutative monoid  $R$  in a symmetric monoidal model category  $\mathcal{C}$ . An example of this situation occurs in the construction of unstable model structures on symmetric spectra, which are by definition modules over a commutative monoid in symmetric sequences [HSS00, Theorem 5.1.2].

As  $R$  is commutative, the category  $\mathbf{Mod}_R$  of  $R$ -modules has a symmetric monoidal structure:

$$X \otimes_R Y := \text{coeq}(X \otimes R \otimes Y \rightrightarrows X \otimes Y).$$

The free-forgetful adjunction

$$F = R \otimes - : \mathcal{C} \rightleftarrows \mathbf{Mod}_R : U$$

has the following properties:  $R \otimes -$  is strong monoidal since  $(R \otimes X) \otimes_R (R \otimes Y) \cong R \otimes (X \otimes Y)$ . Moreover, it is a Hopf adjunction:  $(R \otimes C) \otimes_R D \cong C \otimes D$ . Finally,  $U$  also has a right adjoint, the internal hom functor  $\text{Hom}(R, -)$  (also known as the cofree  $R$ -module functor). In particular,  $U$  is cocontinuous.

The following theorem summarizes the properties of the transferred model structure on  $\mathbf{Mod}_R$ . The existence of the model structure is due to Schwede and Shipley [SS00, Theorem 4.1(2)]. As in Theorem 5.2.6, we say that some model-theoretic property *transfers* if it holds for  $\mathbf{Mod}_R$ , provided that it does for  $\mathcal{C}$ . The transfer of left properness to  $\mathbf{Mod}_R$  (and much more general algebraic structures) was established by Batanin and Berger under

the assumption that  $\mathcal{C}$  is strongly h-monoidal [BB13, Theorems 2.11, 3.1b]. The transfer of symmetric flatness, symmetric h-monoidality and symmetroidality is new.

**Theorem 5.3.1.** *Suppose  $\mathcal{C}$  is a cofibrantly generated symmetric monoidal model category that satisfies the monoid axiom and  $R$  is a commutative monoid in  $\mathcal{C}$ . The transferred model structure on  $\mathbf{Mod}_R$  exists and is a cofibrantly generated symmetric monoidal model category.*

*Combinatoriality, (quasi)tractability, admissible generation, pretty smallness,  $\mathcal{V}$ -enrichedness, and the property of being freely powered transfer from  $\mathcal{C}$  to  $\mathbf{Mod}_R$ . Moreover, if  $\mathcal{C}$  is symmetroidal with respect to some class  $\mathcal{Y}$  (Definition 4.2.8), then  $\mathbf{Mod}_R$  is symmetroidal with respect to  $\text{cof}(R \otimes \mathcal{Y})$ , the weak saturation of maps of free  $R$ -module maps generated by all  $y \in \mathcal{Y}$ .*

*If either  $R$  is a cofibrant object in  $\mathcal{C}$  or if  $\mathcal{C}$  is pretty small and h-monoidal, then left properness transfers.*

*If  $\mathcal{C}$  is pretty small and h-monoidal, then flatness, symmetric flatness, h-monoidality, symmetric h-monoidality, and the monoid axiom transfer from  $\mathcal{C}$  to  $\mathbf{Mod}_R$ .*

*Proof.* The existence of the transferred model structure follows from Proposition 5.1.1 after we observe that  $F(J) = R \otimes J$  and the class of  $F(J)$ -cellular maps consists of weak equivalences by the monoid axiom. It is symmetric monoidal by Proposition 5.2.1. The transfer of combinatoriality, (quasi)tractability, pretty smallness, enrichedness, and left properness were established in Proposition 5.1.2. The transfer of flatness, h-monoidality, and the monoid axiom is shown in Proposition 5.2.5, while their symmetric counterparts are treated in Theorem 5.2.6.  $\square$

## 6. LEFT BOUSFIELD LOCALIZATION

Left Bousfield localizations of various types (e.g., ordinary, enriched, monoidal) of model categories present reflective localizations of the corresponding locally presentable  $\infty$ -categories, i.e., they invert the reflective saturation of a given class of maps in a (homotopy) universal fashion. If the Bousfield localization of a given model category exists, it can be constructed as a model structure on the same underlying category, with a larger class of weak equivalences and the same class of cofibrations. Examples for left Bousfield localizations abound, e.g., local model structures on simplicial presheaves (see Section 7) and the stable model structure on symmetric spectra are left Bousfield localizations. (Right Bousfield localizations, which preserve fibrations and present coreflective localizations, are somewhat more rare.)

**6.1. Existence and basic properties.** Consider the following bicategories (specified by their objects, 1-morphisms, and 2-morphisms):

- model categories, left Quillen functors, and natural transformations;
- $\mathcal{V}$ -enriched model categories,  $\mathcal{V}$ -enriched left Quillen functors, and  $\mathcal{V}$ -enriched natural transformations ( $\mathcal{V}$  is a symmetric monoidal model category);
- (symmetric) monoidal model categories, strong (symmetric) monoidal left Quillen functors, and (symmetric) monoidal natural transformations;
- same as above, but  $\mathcal{V}$ -enriched.

There are obvious forgetful functors that discard enrichments or monoidal structures.

**Definition 6.1.1.** Fix one of the bicategories  $W$  defined above. Suppose  $\mathcal{C} \in W$  and  $S$  is a class of morphisms in  $\mathcal{C}$ . A *left Bousfield localization of  $\mathcal{C}$  with respect to  $S$*  is a 1-morphism  $j: \mathcal{C} \rightarrow L_S \mathcal{C}$  such that precomposition with  $j$  induces an equality between the category of morphisms  $L_S \mathcal{C} \rightarrow \mathcal{E}$  (note these are in particular left Quillen functors) and the category of morphisms  $\mathcal{C} \rightarrow \mathcal{E}$  whose left derived functors send elements of  $S$  to weak equivalences in  $\mathcal{E}$ .

In the case when objects of  $W$  are monoidal, we use the notation  $L^\otimes$  instead of  $L$  to remind the reader of this fact. The above definition can be located in the ordinary case in [Bar10, Definition 4.2] or [Hir03, Theorem 3.3.19], in the enriched case in [Bar10, Definition 4.42] (which also implicitly contains the unenriched monoidal case because any symmetric monoidal model category is enriched over itself), and in the enriched monoidal case implicitly in [Bar10, Proposition 4.47]. Gorchinskiy and Guletskiĭ [GG09, Lemma 26] give an explicit formula for the underlying model category of a monoidal Bousfield localization. The term “monoidal Bousfield localization” is due to White [Whi14b], who also gives an exposition of the existence of monoidal Bousfield localizations.

*Remark 6.1.2.* The above definition talks about equality of categories to ensure that the underlying category of a left Bousfield localization does not change. One can replace equality with isomorphism or equivalence, which would yield an isomorphic or equivalent underlying category.

**Proposition 6.1.3.** *Fix one of the bicategories  $W$  defined above. Suppose  $\mathcal{C} \in W$  and  $S$  is a set (as opposed to a proper class) of morphisms in  $\mathcal{C}$ . Suppose furthermore that  $\mathcal{C}$  is left proper and combinatorial (or cellular). If objects of  $W$  are  $\mathcal{V}$ -enriched or monoidal, assume that  $\mathcal{V}$  and  $\mathcal{C}$  are quasi-tractable. Then the left Bousfield localization  $L_S \mathcal{C}$  exists and is left proper and combinatorial (or cellular).*

- (i) If  $\mathcal{C}$  is tractable or pretty small, then so is  $L_S\mathcal{C}$ .
- (ii) If  $U: W \rightarrow W'$  is the forgetful functor that discards  $\mathcal{V}$ -enrichments, then  $U(L_S\mathcal{C}) = L_{S_{\mathcal{V}}}U(\mathcal{C})$ , where  $S_{\mathcal{V}}$  is the  $\mathcal{V}$ -enriched saturation of  $S$ , which consists of the derived tensor products of the elements of  $S$  and the objects of  $\mathcal{V}$  (or some class of homotopy generators of  $\mathcal{V}$ , e.g., the set of domains and codomains of some set of generating cofibrations of  $\mathcal{V}$ ).
- (iii) If  $U: W \rightarrow W'$  is the forgetful functor that discards monoidal structures, then  $U(L_S\mathcal{C}) = L_{S^{\otimes}}U(\mathcal{C})$ , where  $S^{\otimes}$  is the monoidal saturation of  $S$ , which consists of the derived monoidal products of the elements of  $S$  and the objects of  $\mathcal{C}$  (or some class of homotopy generators of  $\mathcal{C}$ , e.g., the set of domains and codomains of some set of generating cofibrations of  $\mathcal{C}$ ).

*Proof.* The ordinary localization exists by [Bar10, Theorem 4.7] (combinatorial case) and [Hir03, Theorem 4.1.1] (cellular case). The original proof is due to Smith and tractability is due to Hovey [Hov04, Proposition 4.3]. In the enriched case, existence and the statement about the underlying model category is proved in [Bar10, Theorem 4.46]. This also covers the unenriched monoidal case, because every symmetric monoidal model category is enriched over itself. For the enriched monoidal case, see [Bar10, Proposition 4.47]. Barwick's proofs also work for the cellular case, under the assumption of quasi-tractability.

By the formulas for enriched and monoidal localizations, it is enough to show the pretty smallness statement for the ordinary localization  $\mathcal{D} = L_S\mathcal{C}$ . Consider the localization  $\mathcal{D}' := L_S\mathcal{C}'$ , where  $\mathcal{C}'$  is the second model structure on  $\mathcal{C}$  (Definition 2.0.2). We have  $W_{\mathcal{D}'} = W_{\mathcal{D}}$  because both  $S$ -local objects and  $S$ -local weak equivalences only depend on  $S$  and weak equivalences. Thus  $\mathcal{D}$  is pretty small.  $\square$

*Remark 6.1.4.* Any left Bousfield localization of an  $\mathbf{sSet}$ -enriched model category is automatically  $\mathbf{sSet}$ -enriched [Hir03, Theorem 4.1.1(4)].

*Remark 6.1.5.* If  $\mathcal{C}$  is  $\mathcal{V}$ -enriched and monoidal and both  $\mathcal{C}$  and  $\mathcal{V}$  are quasi-tractable, then monoidal localizations and  $\mathcal{V}$ -enriched monoidal localizations agree: to show this we may replace the maps in  $S$  by weakly equivalent maps that are cofibrations with cofibrant source. Then the maps in  $S^{\otimes} = S \otimes (\text{co})\text{dom}(I_{\mathcal{C}})$  are weakly equivalent to  $S \otimes (\text{co})\text{dom}(I_{\mathcal{C}}) \otimes Q(1_{\mathcal{V}})$  by the unit axiom of the  $\mathcal{V}$ -enrichment. The latter class is contained in  $S_{\mathcal{V}}^{\otimes}$ . Vice versa,  $S_{\mathcal{V}}^{\otimes} = S \otimes (\text{co})\text{dom}(I_{\mathcal{V}}) \otimes (\text{co})\text{dom}(I_{\mathcal{C}})$  is contained in  $S \otimes (\text{co})\text{dom}(I_{\mathcal{C}})$  since  $\otimes: \mathcal{V} \times \mathcal{C} \rightarrow \mathcal{V}$  is a left Quillen bifunctor.

The standard description of fibrant objects and adjunctions of Bousfield localizations admit the following variants for monoidal localizations.

**Lemma 6.1.6.** *If  $\mathcal{D}$  is the monoidal left Bousfield localization  $L_{S^{\otimes}}\mathcal{C}$  of a monoidal model category  $\mathcal{C}$ , then fibrant objects in  $\mathcal{D}$  are those fibrant objects  $W$  in  $\mathcal{C}$  such that the derived internal Hom,*

$$\underline{\mathbf{R}}\text{Hom}_{\mathcal{C}}(\xi, W)$$

*is a weak equivalence in  $\mathcal{C}$  for any  $\xi \in S$ .*

*Proof.* By [Hir03, Proposition 3.4.1], fibrant objects in  $\mathcal{D}$  are those fibrant objects of  $\mathcal{C}$  such that the derived mapping space  $\underline{\mathbf{R}}\text{Map}_{\mathcal{C}}(\text{CO}_{\mathcal{C}} \otimes^{\mathbf{L}} \xi, W)$  or, equivalently,  $\underline{\mathbf{R}}\text{Map}_{\mathcal{C}}(\text{CO}_{\mathcal{C}}, \underline{\mathbf{R}}\text{Hom}(\xi, W))$  is a weak equivalence for any  $\xi \in S$ . The objects  $\text{CO}_{\mathcal{C}}$  are homotopy generators of  $\mathcal{C}$ , so this is equivalent to  $\underline{\mathbf{R}}\text{Hom}(\xi, W)$  being a weak equivalence [Hov01, Proposition 3.2].  $\square$

**Lemma 6.1.7.** *If  $F: \mathcal{C} \rightleftarrows \mathcal{C}': G$  is a Quillen adjunction of monoidal model categories such that  $F$  is strong monoidal, then there is a Quillen adjunction*

$$F: \mathcal{D} := L_{S^{\otimes}}\mathcal{C} \rightleftarrows \mathcal{D}' := L_{\underline{\mathbf{L}}F(S)^{\otimes}}\mathcal{C}': G,$$

*(assuming the left Bousfield localizations exist), which is a Quillen equivalence if  $\mathcal{C} \rightleftarrows \mathcal{C}'$  is one.*

*Proof.* The class  $F(\text{CO}_{\mathcal{C}})$  is a class of homotopy generators of  $\mathcal{C}'$ . Hence  $\mathcal{D}'$  can be computed as the (nonmonoidal) localization with respect to the class  $F(\text{CO}_{\mathcal{C}}) \otimes^{\mathbf{L}} \underline{\mathbf{L}}F(S) = F(\text{CO}_{\mathcal{C}} \otimes^{\mathbf{L}} S)$ . Thus, by [Hir03, Proposition 3.3.18, Theorem 3.3.20], the left Quillen functor  $\mathcal{C} \rightarrow \mathcal{C}' \rightarrow \mathcal{D}'$  factors over a left Quillen functor  $\mathcal{D} \rightarrow \mathcal{D}'$  since  $\underline{\mathbf{L}}F(\text{CO}_{\mathcal{C}} \otimes^{\mathbf{L}} S)$  consists of weak equivalences in  $\mathcal{D}'$ . Moreover,  $\mathcal{D} \rightleftarrows \mathcal{D}'$  is a Quillen equivalence if  $\mathcal{C} \rightleftarrows \mathcal{C}'$  is one.  $\square$

**6.2. Localization of monoidal and symmetricity properties.** Here is a tool to transport h-monoidality and flatness along a Bousfield localization. An example application in the context of symmetric spectra is given in [PS14b, Subsection 3.3]. The idea of combining h-monoidality and flatness was independently used by White [Whi14b].

**Proposition 6.2.1.** *Suppose  $\mathcal{V}$  is a symmetric monoidal model category,  $\mathcal{C}$  is a  $\mathcal{V}$ -enriched monoidal model category such that the monoidal left Bousfield localization  $\mathcal{D} := L_T^{\otimes}\mathcal{C}$  with respect to some class  $T$  exists. We say that a property of a class  $S$  of cofibrations in  $\mathcal{C}$  localizes if it holds for  $S$  regarded as a class of cofibrations in  $\mathcal{D}$ . Likewise, we say that some property of  $\mathcal{C}$  localizes, if it also holds for  $\mathcal{D}$ .*

- (i) *Flatness of  $S$  localizes. In particular, the flatness of  $\mathcal{C}$  localizes.*

- (ii) If  $\mathcal{C}$  and  $\mathcal{D}$  are left proper, any (acyclic) h-cofibration  $f$  in  $\mathcal{C}$  is also an (acyclic) h-cofibration in  $\mathcal{D}$ .
- (iii) If  $\mathcal{C}$  is left proper and  $\mathcal{D}$  is left proper, quasi-tractable, pretty small, and flat, then the h-monoidality of  $S$  or of  $\mathcal{C}$  localizes.
- (iv) If  $\mathcal{D}$  is pretty small and h-monoidal (which holds, for example, if  $\mathcal{C}$  is left proper, pretty small, h-monoidal, and flat), then  $\mathcal{D}$  also satisfies the monoid axiom.

*Proof.* (i): We have to show that  $y \square s$  is a weak equivalence in  $\mathcal{D}$  for all weak equivalences  $y$  in  $\mathcal{D}$  and  $s \in S$ . By the pushout product axiom (of  $\mathcal{D}$ ), we may assume  $y$  is a trivial fibration in  $\mathcal{D}$  or, equivalently, one in  $\mathcal{C}$ . Now invoke the flatness of  $S$  in  $\mathcal{C}$  and use  $W_{\mathcal{C}} \subset W_{\mathcal{D}}$ .

(ii): The acyclic part follows from the nonacyclic one and the inclusion  $W_{\mathcal{C}} \subset W_{\mathcal{D}}$ . Given a diagram  $A \leftarrow B \xrightarrow{f} C$ , where  $f$  is an h-cofibration in  $\mathcal{C}$ , we have to show by Lemma 2.0.6(i) that  $C \sqcup_B A$  is a homotopy pushout in  $\mathcal{D}$ . The identity functor  $\mathbf{Fun}(\bullet \leftarrow \bullet \rightarrow \bullet, \mathcal{C}) \rightarrow \mathbf{Fun}(\bullet \leftarrow \bullet \rightarrow \bullet, \mathcal{D})$  is a left Quillen functor if we equip both functor categories with the projective model structure. Since it also preserves all weak equivalences, it preserves homotopy colimits, i.e., sends the homotopy pushout  $C \sqcup_B A \sim C \sqcup_B^{h, \mathcal{C}} A$  to a homotopy pushout in  $\mathcal{D}$ .

(iii): As the cofibrations in  $\mathcal{C}$  and  $\mathcal{D}$  are the same, the nonacyclic part of the h-monoidality of  $\mathcal{D}$  follows from (ii). Acyclic h-cofibrations are weakly saturated by Lemma 2.0.6(iii). Therefore, it is enough to show  $f \otimes X \in W_{\mathcal{D}}$  for any  $f: Y \rightarrow Z \in J_{\mathcal{D}}$  and any object  $X$ . The quasi-tractability of  $\mathcal{D}$  (Proposition 6.1.3) allows us to assume that  $Y$  (hence  $Z$ ) is cofibrant. Writing  $Q(-)$  for the cofibrant replacement (equivalently in  $\mathcal{C}$  or  $\mathcal{D}$ ) we see that  $X \otimes f$  is a weak equivalence since  $Q(X) \otimes f$  is one (by the pushout product axiom for  $\mathcal{D}$ ) and  $q \otimes Y$  and  $q \otimes Z$  are weak equivalences in  $\mathcal{D}$  (by flatness).

(iv): Apply Lemma 3.2.6 to  $\mathcal{D}$ . □

The following proposition provides a method to transport the symmetricity notions to a Bousfield localization. It is the symmetric counterpart of Proposition 6.1.3.

**Theorem 6.2.2.** *Suppose  $\mathcal{V}$  is a symmetric monoidal model category,  $\mathcal{C}$  is a  $\mathcal{V}$ -enriched symmetric monoidal model category such that the  $\mathcal{V}$ -enriched symmetric monoidal left Bousfield localization  $\mathcal{D} := L_T^{\otimes} \mathcal{C}$  with respect to some class  $T$  of morphisms exists.*

*We say that a property of a class  $S$  of cofibrations in  $\mathcal{C}$  localizes if it holds for  $S$  regarded as a class of cofibrations in  $\mathcal{D}$ . Likewise, we say that some property of  $\mathcal{C}$  localizes, if it also holds for  $\mathcal{D}$ .*

- (i) Let  $\mathcal{Y} = (\mathcal{Y}_n)$  be some classes of morphisms in  $\Sigma_n \mathcal{C}$ . The property of  $S$  of being symmetric flat with respect to  $\mathcal{Y}$  localizes. In particular, the symmetric flatness of  $S$  and of  $\mathcal{C}$  localizes.
- (ii) If  $\mathcal{C}$  is left proper and  $\mathcal{D}$  is left proper, quasi-tractable, pretty small and symmetric flat, then the symmetric h-monoidality of  $S$  or of  $\mathcal{C}$  localizes.
- (iii) The property of  $S$  of being (acyclic)  $\mathcal{Y}$ -symmetroidal localizes provided that  $\mathcal{D}$  is flat and h-monoidal and provided that  $S$  consists of cofibrations with cofibrant source and is symmetric flat in  $\mathcal{D}$ . In particular if  $\mathcal{D}$  is h-monoidal and symmetric flat and  $\mathcal{C}$  is  $\mathcal{Y}$ -symmetroidal then  $\mathcal{D}$  is also  $\mathcal{Y}$ -symmetroidal.
- (iv) The property of being freely powered localizes.
- (v) Suppose  $\mathcal{D}$  is quasi-tractable. Then the property of being admissibly generated localizes.

*Proof.* (i): The  $\mathcal{Y}$ -symmetric flatness of  $S$  states that  $y \square_{\Sigma_n} s^{\square n}$  is a weak equivalence in  $\mathcal{C}$  for all  $y \in \mathcal{Y}_n$  and  $s \in S$ . Since weak equivalences of  $\mathcal{C}$  are contained in the ones of  $\mathcal{D}$  this property obviously localizes. The additional claims concern the symmetric flatness of  $S$  (or the class of all cofibrations on  $\mathcal{C}$ ) with respect to  $W_{\Sigma_n \text{pro} \mathcal{D}}$ . By Lemma 4.3.1, this is equivalent to symmetric flatness with respect to  $\text{AF}_{\Sigma_n \text{pro} \mathcal{D}} = \text{AF}_{\Sigma_n \text{pro} \mathcal{C}}$  which holds since  $S$  is symmetric flat with respect to  $W_{\Sigma_n \text{pro} \mathcal{C}}$  by assumption.

(ii): As (acyclic) h-cofibrations of  $\mathcal{C}$  are contained in the ones of  $\mathcal{D}$  (Proposition 6.2.1(ii)), a class  $S$  which is (acyclic) symmetric h-monoidal in  $\mathcal{C}$  is also (acyclic) symmetric h-monoidal in  $\mathcal{D}$ .

Now suppose that  $\mathcal{C}$  is symmetric h-monoidal. We want to show that (acyclic)  $\mathcal{D}$ -cofibrations form an (acyclic) symmetric h-monoidal class (in  $\mathcal{D}$ ). Again using the above fact, it is enough to show the acyclic part. Once again, we may restrict to generating acyclic cofibrations (4.3.9(iii)). Thus, let  $s$  be a finite family of generating acyclic cofibrations in  $\mathcal{D}$ . By quasi-tractability, we may assume they have cofibrant domains. Setting  $y: \emptyset \rightarrow Y$ , the pushout product  $y \square s^{\square n}$  is just  $Y \otimes s^{\square n}$ , which is a weak equivalence by the h-monoidality of  $\mathcal{D}$  ensured by Proposition 6.2.1(iii). Using the flatness and h-monoidality of  $\mathcal{D}$  (Proposition 6.2.1(i), (iii)), Lemma 4.3.4 applies to  $s$  and  $y$  and shows that  $Y \otimes_{\Sigma_n} s^{\square n}$  is a weak equivalence.

(iii): The stability of the nonacyclic part of  $\mathcal{Y}$ -symmetroidality is obvious. The acyclic part follows from Proposition 4.3.5, using the cofibrancy assumption and the symmetric flatness of  $S$  in  $\mathcal{D}$ . Similarly, by 4.3.9(iv), the symmetroidality of  $\mathcal{D}$  follows by using a set  $S$  of generating acyclic cofibrations (of  $\mathcal{D}$ ) with cofibrant domain, which is possible thanks to the tractability of  $\mathcal{D}$ .

(iv): This follows from Proposition 4.3.5.

(v): This is clear since  $C_{\mathcal{C}} = C_{\mathcal{D}}$ . □

## 7. EXAMPLES OF MODEL CATEGORIES

We discuss the model-theoretic properties of Section 2, Subsection 3.2, and Section 4 for simplicial sets, simplicial presheaves, simplicial modules, topological spaces, chain complexes, and symmetric spectra.

**7.1. Simplicial sets.** The most basic example of a monoidal model category is the category  $\mathbf{sSet}$  of simplicial sets equipped with the cartesian monoidal structure  $A \otimes B = A \times B$  and the Quillen model structure, see, e.g., [GJ99, Theorem I.11.3]. All objects are cofibrant, so  $\mathbf{sSet}$  is left proper, flat, and h-monoidal by Lemma 3.2.4.

Simplicial sets are symmetroidal: given any monomorphism  $y \in \Sigma_n \mathbf{sSet}$  and a finite family of monomorphisms  $v \in \mathbf{sSet}$ ,  $y \square_{\Sigma_n} v^{\square n}$  is a monomorphism. Indeed,  $y \square v^{\square n}$  is a  $\Sigma_n$ -equivariant monomorphism and passing to  $\Sigma_n$ -orbits preserves monomorphisms. By Theorem 4.3.9(iv), the acyclic part of symmetroidality follows if  $y \square_{\Sigma_n} v^{\square n}$  is a weak equivalence for any  $y$  as above and any finite family of horn inclusions  $v: \Lambda_k^m \rightarrow \Delta^m$  (where  $m$  and  $k$  are multiindices). To this end we first construct a homotopy  $h: \Lambda \times \Delta^m \rightarrow \Delta^m$  from the identity map  $\Delta^m \rightarrow \Delta^m$  to the composition  $\Delta^m \rightarrow \Delta^0 \xrightarrow{k} \Delta^m$  such that  $\Lambda_k^m \subset \Delta^m$  is preserved by the homotopy. Here  $\Lambda$  is the 2-horn, which can be depicted as  $0 \rightarrow 1 \leftarrow 2$ . We parametrize  $h$  by  $\Lambda$  and not by the usual  $\Delta^1$  since  $\Delta^m$  is not fibrant. The map  $h$  is uniquely specified by its value on vertices, i.e.,  $\{0, 1, 2\} \times \{0, \dots, m\} \rightarrow \{0, \dots, m\}$ . We have  $(0, i) \mapsto i$ ,  $(1, i) \mapsto \max(k, i)$ ,  $(2, i) \mapsto k$ . Thus we have constructed a simplicial deformation retraction  $\Lambda \times (\Lambda_k^m \rightarrow \Delta^m) \rightarrow (\Lambda_k^m \rightarrow \Delta^m)$  that contracts the inclusion  $\Lambda_k^m \rightarrow \Delta^m$  to the identity map  $\Delta^0 \rightarrow \Delta^0$ . (Morphisms of maps are commutative squares, as usual.) The map  $h$  gives rise to a simplicial deformation retraction

$$\Lambda \times (y \square_{\Sigma_n} v^{\square n}) \xrightarrow{\Delta} (\Lambda^{\times n} \times (y \square v^{\square n}))_{\Sigma_n} \cong y \square_{\Sigma_n} (\Lambda \times v)^{\square n} \xrightarrow{h} y \square_{\Sigma_n} v^{\square n}$$

using the fact that the diagonal  $\Delta: \Lambda \rightarrow \Lambda^{\times n}$  is  $\Sigma_n$ -equivariant. It contracts the map  $y \square_{\Sigma_n} v^{\square n}$  to the map  $y \square_{\Sigma_n} (\text{id}_{\Delta^0})^{\square n}$ . For  $n > 0$  the latter map is the identity map on the codomain of  $y$ , in particular, a weak equivalence, hence so is  $y \square_{\Sigma_n} v^{\square n}$ .

Symmetroidality and cofibrancy of all objects implies that  $\mathbf{sSet}$  is symmetric h-monoidal.

The category  $\mathbf{sSet}$  is far from freely powered: the map  $(\partial \Delta^1 \rightarrow \Delta^1)^{\square 2}$  is not a  $\Sigma_2$ -projective cofibration, since  $\Sigma_2$  does not act freely on the complement of the image.

Simplicial sets are not symmetric flat:  $E\Sigma_n \rightarrow *$  is  $\Sigma_n$ -equivariant and a weak equivalence of the underlying simplicial sets, but  $B\Sigma_n := (E\Sigma_n)_{\Sigma_n} \rightarrow *$  is not a weak equivalence: recall that  $B\Sigma_2$  is weakly equivalent to  $\mathbf{RP}^\infty$ , the infinite real projective space.

Similar statements hold for pointed simplicial sets equipped with the smash product.

The category  $\mathbf{sSet}$  also carries the Joyal model structure [Lur09, Theorem 2.2.5.1]. It is an interesting question whether it is symmetric h-monoidal.

**7.2. Simplicial presheaves.** A more general example than simplicial sets is the category

$$\mathbf{sPSh}(S) = \mathbf{Fun}(S^{\text{op}}, \mathbf{sSet})$$

of simplicial presheaves on some site  $S$ . The *projective model structure* on this category is transferred from the Quillen model structure on  $\mathbf{sSet}$  along

$$(7.2.1) \quad \prod_{X \in S} \mathbf{sSet} \rightleftarrows \mathbf{sPSh}(S).$$

It is pretty small by 5.1.2(v) and left proper by 5.1.2(vi). The monoid axiom, h-monoidality, flatness, and symmetric h-monoidality follow from the corresponding properties of the injective model structure by Lemma 3.2.5. Alternatively, even though (7.2.1) is not a Hopf adjunction, the arguments of Proposition 5.2.5 can be generalized to (7.2.1). The projective model structure is not in general symmetroidal (for  $X \in S$ ,  $(X^n)_{\Sigma_n}$  is in general not projectively cofibrant).

In the *injective model structure* on  $\mathbf{sPSh}(S)$ , weak equivalences and cofibrations are checked pointwise. It is combinatorial [Lur09, Proposition A.2.8.2] and therefore tractable. It is pretty small (as the second model structure in Definition 2.0.2, take the projective structure), left proper, h-monoidal and flat (Lemma 3.2.4). The symmetric monoidality, symmetric h-monoidality and symmetroidality (with respect to injective cofibrations  $\mathcal{Y}_n = \mathbf{C}_{\Sigma_n^{\text{in}}} \mathbf{sPSh}(S)$ ) follows from the one of  $\mathbf{sSet}$ .

There are various *intermediate model structures* on  $\mathbf{sPSh}(S)$ , such as Isaksen's flasque model structure [Isa05]. They also have pointwise weak equivalences but other choices of cofibrations which lie between projective and injective cofibrations. For such intermediate model structures, monoidality, h-monoidality, symmetric h-monoidality, symmetroidality, the monoid axiom, and flatness follow from the injective case and pretty smallness follows from the projective case.

The properties mentioned above are stable under *Bousfield localization*. For example, given some Grothendieck topology  $\tau$  on the site  $S$ , the  $\tau$ -local projective model structure is the left Bousfield localization of the projective model structure with respect to  $\tau$ -hypercovers [DHI04, Theorem 6.2]. Since hypercovers are stable under product with any  $X \in S$  by [DHI04, Proposition 3.1], this is a monoidal localization. It is also  $\mathbf{sSet}$ -enriched by Remark 6.1.4. By Proposition 6.2.1, the localized model structure is again left proper, tractable, monoidal and h-monoidal, pretty small, flat, and satisfies the monoid axiom. It is symmetric h-monoidal at least

if  $\tau$  has enough points, for in this case local weak equivalences are maps which are stalkwise weak equivalences [Jar87, page 39].

**7.3. Simplicial modules.** Let  $R$  be a commutative simplicial ring and consider the transferred model structure on simplicial  $R$ -modules via the free-forgetful adjunction

$$(7.3.1) \quad R[-] : \mathbf{sSet} \rightleftarrows \mathbf{sMod}_R : U.$$

The model category  $\mathbf{sMod}_R$  is pretty small by Proposition 5.1.2. As for chain complexes,  $\mathbf{sMod}_R$  is flat, but not symmetric flat (unless  $R$  is a rational algebra).

Simplicial  $R$ -modules are symmetric h-monoidal. The nonacyclic part follows from the fact that monomorphisms, i.e., *injective cofibrations*, of simplicial  $R$ -modules are h-cofibrations.

We reduce the acyclic part of symmetric h-monoidality of  $\mathbf{sMod}_R$  to the one of  $\mathbf{sSet}$  using the cocontinuous strong monoidal functor  $R[-] : (\mathbf{sSet}, \times) \rightarrow (\mathbf{sMod}_R, \otimes)$ , which preserves weak equivalences. Given any object  $Y \in \Sigma_n \mathbf{sMod}_R$  and any finite family  $w$  of generating cofibrations of  $\mathbf{sMod}_R$ , i.e.,  $w = R[v]$ , we have a deformation retraction

$$R[\Lambda] \otimes (Y \otimes_{\Sigma_n} R[v]^{\square n}) \xrightarrow{R[\Delta]} (R[\Lambda]^{\otimes n} \otimes Y \otimes_{\Sigma_n} R[v]^{\square n})_{\Sigma_n} \cong Y \otimes_{\Sigma_n} (R[\Lambda \times v])^{\square n} \xrightarrow{R[h]} Y \otimes_{\Sigma_n} R[v]^{\square n}$$

of  $Y \otimes_{\Sigma_n} w^{\square n}$  to a weak equivalence, which shows that the former is also a weak equivalence.

Simplicial  $R$ -modules are symmetroidal with respect to the class  $\mathcal{Y} = (\mathcal{Y}_n) = (R[C_{\Sigma_n^+} \mathbf{sSet}])$ , which follows immediately from the symmetroidality of simplicial sets and cocontinuity and strong monoidality of  $R[-]$ . Note that  $\mathbf{sMod}_R$  is not symmetroidal, as can be shown as in Remark 5.2.7.

**7.4. Chain complexes.** The category  $\mathbf{Ch}(\mathbf{Mod}_R)$  of unbounded chain complexes of  $R$ -modules, for some commutative ring  $R$ , carries the *projective model structure* whose weak equivalences are the quasiisomorphisms and fibrations are the degreewise epimorphisms. It is enriched over  $\mathbf{Ch}(\mathbf{Mod}_{\mathbf{Z}})$  (equipped with the projective model structure). The generating (acyclic) cofibrations are given by all shifts of the canonical inclusion  $[0 \rightarrow R] \rightarrow [R \xrightarrow{\text{id}} R]$  ( $[0 \rightarrow 0] \rightarrow [R \xrightarrow{\text{id}} R]$ , respectively) [Hov99, Definition 2.3.3, Theorem 2.3.11]. In particular, the model structure is tractable and pretty small. It is flat, as can be seen using Theorem 3.2.8(ii). The category is h-monoidal by [BB13, Corollary 1.14].

It is *not* symmetric flat, for the same reason as  $\mathbf{sSet}$  above. Moreover, it is neither symmetric h-monoidal nor symmetroidal: for the chain complex  $A = [\mathbf{Z} \xrightarrow{\text{id}} \mathbf{Z}]$  in degrees 1 and 0, we have

$$A^{\otimes 2} = [\mathbf{Z} \xrightarrow{(1,-1)} \mathbf{Z} \oplus \mathbf{Z} \xrightarrow{+} \mathbf{Z}],$$

where from left to right we have the sign representation, the regular and the trivial representation of  $\Sigma_2$ . However,  $(A^{\otimes 2})_{\Sigma_2} = [\mathbf{Z}/2 \rightarrow \mathbf{Z} \xrightarrow{\text{id}} \mathbf{Z}]$  is not exact nor cofibrant.

By the Dold-Kan correspondence  $N : (\mathbf{sMod}_R, \times) \rightleftarrows (\mathbf{Ch}_R^+, \otimes)$  between simplicial  $R$ -modules and connective chain complexes of  $R$ -modules, the projective model structures correspond to each other. However,  $N$  fails to be a strong symmetric monoidal functor. Instead,  $\times$  corresponds to the *shuffle* tensor product  $\tilde{\otimes}$  of chain complexes, which is much bigger than the usual tensor product. According to Subsection 7.3,  $(\mathbf{Ch}_R^+, \tilde{\otimes})$  is symmetric h-monoidal. The reason why a similar argument fails for  $\otimes$  is that the (smaller) ordinary tensor product fails to allow for a  $\Sigma_n$ -equivariant diagonal map for an interval object.

If  $R$  contains  $\mathbf{Q}$ , the picture changes drastically: every  $R$ -module  $M$  with a  $\Sigma_n$ -action is projective as an  $R$ -module if and only if it is projective as an  $R[\Sigma_n]$ -module (Maschke's theorem). Thus, the projective and injective model structure (with respect to the  $\Sigma_n$ -action) on  $\Sigma_n \mathbf{Ch}(\mathbf{Mod}_R)$  agree. Therefore,  $\mathbf{Ch}(\mathbf{Mod}_R)$  is symmetric flat and freely powered (and therefore symmetroidal and symmetric h-monoidal).

With appropriate additional assumptions, the statements above can be generalized to chain complexes in a Grothendieck abelian category  $\mathcal{A}$ . For example, flatness and h-monoidality of  $\mathbf{Ch}(\mathcal{A})$  require that projective objects  $P \in \mathcal{A}$  are flat, i.e.,  $P \otimes -$  is an exact functor.

**7.5. Topological spaces.** The category  $\mathbf{Top}$  of compactly generated weakly Hausdorff topological spaces carries the Quillen model structure which is transferred from  $\mathbf{sSet}$  via the singular simplicial set functor. This model category is left proper [Hir03, Theorem 13.1.10], monoidal [Hov99, Corollary 4.2.12], and h-monoidal [BB13, Example 1.15]. It is cellular [Hir03, Propositions 4.1.4], though not locally presentable and therefore not combinatorial. However, it is admissibly generated. Alternatively, one can use Smith's  $\Delta$ -generated topological spaces, which are combinatorial.

Topological spaces are *not* pretty small. However, since closed inclusions are stable under  $\Sigma_n$ -coinvariants, products with arbitrary spaces and pushout products, and compact spaces are compact relative to closed inclusions [Hov99, Lemma 2.4.1], they satisfy the following property:

**Definition 7.5.1.** A symmetric monoidal model category  $\mathcal{C}$  is *strongly admissibly generated* if (co)domains of its generating cofibrations are compact relative to the class (4.2.5).

- Proposition 7.5.2.** (i) *The weak saturation of (symmetric) h-monoidality and (symmetric) flatness (Theorem 3.2.8(ii), (i), Theorem 4.3.9(ii), (iii)) holds if  $\mathcal{C}$  is strongly admissibly generated instead of pretty small.*
- (ii) *(Symmetric) flatness and (symmetric) h-monoidality are stable under transfers and monoidal left Bousfield localizations as in Proposition 5.2.5, Theorem 5.2.6, Proposition 6.2.1, and Theorem 6.2.2, provided that one replaces “pretty small” in these statements by “strongly admissibly generated”.*
- (iii) *The stability of left properness under transfer as in Proposition 5.1.2(vi) is true if one replaces pretty smallness in loc. cit. by the condition that (co)domains of a set  $I$  of generating cofibrations of  $\mathcal{C}$  are compact relative to pushouts of maps  $F(I)$ .*

*Proof.* Analogously to Lemma 2.0.3(iii), a filtered colimit  $f_\infty$  of weak equivalences  $f_i$  is a weak equivalence, provided that (co)domains of the generating cofibrations of  $\mathcal{C}$  are compact relative to the class spanned by the acyclic cofibrations and the transition maps  $x_i, y_i$ . Similarly, if this size condition is satisfied,  $f_\infty$  is an h-cofibration provided that the  $f_i$  and the maps  $X_{i+1} \sqcup_{X_i} Y_i \rightarrow Y_{i+1}$  are h-cofibrations. This refines Lemma 2.0.6(iii).

(i): To show the weak saturation of symmetric h-monoidality as in 4.3.9(iii) using only that  $\mathcal{C}$  is strongly admissibly generated, we use (cf. the proof of 4.3.9(iv)) that the transition maps appearing in the proof of 4.3.9(iii) are precisely of the form as in (4.2.5).

As for the stability of symmetric flatness under weak saturation (4.3.9(ii)), it is enough to show that for a transfinite composition  $s$  of symmetric flat maps  $s_j$ , and a weak equivalence  $y$ , the filtered colimit  $y \square_{\Sigma_n} s^{\square n} = \text{colim}_i y \square_{\Sigma_n} t_i^{\square n}$  is also a filtered homotopy colimit, where  $t_i = s_i \circ \cdots \circ s_0$  are the (finite) compositions of  $s_j$ . By the above variant of Lemma 2.0.3(iii), this is true if the (co)domains of generating cofibrations are compact relative to the transition maps of this filtered colimit. By Lemma 4.3.6 and its proof, especially (4.3.8), these transition maps are given by  $y \square_{\Sigma_n} Q(\alpha_k)$ , so this is true again since  $\mathcal{C}$  is strongly admissibly generated.

(ii): The indicated statements use pretty smallness only to invoke Theorem 4.3.9.

(iii): This follows from the above variant of Lemma 2.0.6 and the proof of Proposition 5.1.2(vi).  $\square$

By Proposition 7.5.2, flatness and (symmetric) h-monoidality of **Top** only needs to be checked for generating cofibrations, which is easy. Hence **Top** is flat and symmetric h-monoidal (but not symmetric flat).

**7.6. Symmetric spectra.** The positive stable model structure on symmetric spectra with values in an abstract model category  $\mathcal{C}$  is both symmetric flat and symmetric h-monoidal. With a careful choice of the model structure on symmetric sequences, it is also symmetroidal. As a special case, this shows that any model category is Quillen equivalent to one which is symmetric flat and symmetroidal. For this, only mild conditions on  $\mathcal{C}$  are necessary (such as flatness and h-monoidality, but not their symmetric counterparts). See [PS14b, Theorem 3.3.4] for the precise statement.

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