MOTIVIC INVARIANTS OF p-ADIC FIELDS

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ABSTRACT. We provide a complete analysis of the motivic Adams spectral sequences converging to the bigraded coefficients of the 2-complete algebraic Johnson-Wilson spectra $\mathsf{BPGL}\langle n\rangle$ over p-adic fields. These spectra interpolate between integral motivic cohomology ($n = 0$), a connective version of algebraic K-theory ($n = 1$), and the algebraic Brown-Peterson spectrum ($n = \infty$). We deduce that, over p-adic fields, the 2-complete $BPGL\langle n\rangle$ splits over 2-complete $\mathsf{BPGL}\langle 0\rangle$, implying that the slice spectral sequence for $BPGL$ collapses.

This is the first in a series of two papers investigating motivic invariants of p-adic fields, and it lays the groundwork for an understanding of the motivic Adams-Novikov spectral sequence over such base fields.

CONTENTS

1. INTRODUCTION

This paper initiates a project to determine algebro-geometric invariants of p-adic fields via the methods of stable homotopy theory. The technology for such an endeavor resides in the Morel-Voevodsky motivic homotopy theory [MV99], and in the stabilizations thereof [Voe98, Hu03, Jar00, DRØ03]. The techniques here are natural generalizations of those used over an algebraically closed field in [HKO10], but the phenomena observed are more nuanced because of the arithmetically richer input.

Presently, we will concern ourselves with the bigraded coefficients of $\mathsf{BPGL}\langle n\rangle\widehat{_2}$ (the 2-complete n-th algebraic Johnson-Wilson spectrum at the prime 2, cf. Definition 3.1) over a p-adic field, $p > 2$. A sequel to this

Key words and phrases. motivic Adams spectral sequence, algebraic K-theory, algebraic cobordism. MSC2010: 14F42, 19D50.

This research was partially supported by NSF grant DMS-0602191.

work will use these results to provide information about a motivic Adams-Novikov spectral sequence converging to stable motivic homotopy groups of the 2-complete sphere spectrum over a p-adic field [Orm, Orm10]. Our main computational tool in all cases is the motivic Adams spectral sequence.

Our grading conventions will follow those in [HKO10], where the $(m+n\alpha)$-sphere $S^{m+n\alpha}$ is the smash product $(S^1)^{\wedge m} \wedge (\mathbb{A}^1 \smallsetminus 0)^{\wedge n}$. The wildcard $\star$ will refer to bigradings of the form $m+n\alpha$, $m, n \in \mathbb{Z}$, and if E is a motivic spectrum then its (bigraded) coefficients are $E_\star = \pi_\star E$.

In §3, we define and establish basic properties of $\mathsf{BPGL}\langle n\rangle$ and identify its mod 2 motivic homology as a comodule over the dual motivic Steenrod algebra $\mathcal{A}_\star$. In §4, we run appropriate filtration spectral sequences that determine the E_2-terms of the motivic Adams spectral sequences converging to 2-complete coefficients. We then analyze these spectral sequences in §5 in order to fully determine the bigraded coefficient rings $\pi_\star \mathsf{BPGL}\langle n\rangle\widehat{_2}$. The computation of $\pi_\star BPGL\widehat{_2}$, combined with motivic Landweber exactness, permits a description of the $\mathsf{BPGL}\widehat{_2}$ Hopf algebroid producing a computation of the E_2-term of the motivic Adams-Novikov spectral sequence over a p-adic field; see Theorem 5.12.

In order to follow this program, we use the rest of this introduction to review fundamental input from motivic cohomology. In §2, we describe our conventions for p-adic fields and review arithmetic input making explicit computations possible.

Motivic homology and the dual motivic Steenrod algebra over a field. Let k_*^M denote the mod 2 reduction of Milnor K-theory K_*^M; let H denote the mod 2 motivic cohomology spectrum. The main result of [Voe03a] determines the motivic cohomology of $\mathrm{Spec}(\mathsf{k})$, while [Voe03b] and [Voe] determine stable cohomology operations on mod 2 motivic cohomology.

Theorem 1.1 ([Voe03a])**.** *Mod 2 motivic cohomology of* $\mathrm{Spec}(\mathsf{k})$ *takes the form*

$$H^\star(\mathrm{Spec}(\mathsf{k}); \mathbb{Z}/2) = k_*^M(\mathsf{k})[\tau]$$

where $|k_1^M(\mathsf{k})| = \alpha$ *and* $|\tau| = -1+\alpha$. □

Theorem 1.2 ([Voe03b, Voe])**.** *The* motivic Steenrod algebra *is the algebra of stable operations on* H,

$$\mathcal{A}^\star := H^\star H.$$

The motivic Steenrod algebra is generated by β *and* P^i, $i \geq 0$. □

$\mathcal{A}^\star$ has the structure of a Hopf algebroid over $H^\star$. (See [Rav86, Appendix A1] for the theory of Hopf algebroids.) In this paper, we will be more concerned with the dual to the motivic Steenrod algebra, $\mathcal{A}_\star = H_\star H$ which is a Hopf algebroid over $H_\star = H^{-\star}$.

Theorem 1.3 ([Voe03b, Voe])**.** *The dual motivic Steenrod algebra is isomorphic to*

$$H_\star[\tau_0, \tau_1, \ldots, \xi_1, \xi_2, \ldots]/(\tau_i^2 - \tau\xi_{i+1} - \rho(\tau_{i+1} + \tau_0\xi_{i+1})).$$

Here τ is the generator of $H_{1-\alpha} = H^0(\mathrm{Spec}(\mathsf{k}); \mathbb{Z}/2(1))$, ρ is the class of -1 in $H_{-\alpha} = k_1^M(\mathsf{k}) = \mathsf{k}^\times/(\mathsf{k}^\times)^2$, $|\tau_i| = (2^i-1)(1+\alpha)+1$, and $|\xi_i| = (2^i-1)(1+\alpha)$.

The Hopf algebroid structure on $\mathcal{A}_\star$ is specified by the following: elements of $H_{0+*\alpha} = k_*^M(\mathsf{k})$ are primitive, and

$$\begin{aligned}
\eta_L\tau &= \tau \\
\eta_R\tau &= \tau + \rho\tau_0 \\
\Delta\xi_k &= \sum_{i=0}^{k} \xi_{k-i}^{2^i} \otimes \xi_i \qquad (1)\\
\Delta\tau_k &= \tau_k \otimes 1 + \sum_{i=0}^{k} \xi_{k-i}^{2^i} \otimes \tau_i. \quad \square
\end{aligned}$$

Remark 1.4. It follows that $\mathcal{A}^\star$ has a Milnor basis of elements of the form $Q_I(r_1, \ldots, r_n)$ as in topology with the degree shift $|Q_n| = 2^n(1+\alpha) - \alpha$; see [Bor03].

Certain quotient Hopf algebroids of $\mathcal{A}_\star$ will be useful in our analysis (see §3). The following definition is due to Mike Hill.

Definition 1.5 **([Hill]).** Let $\mathcal{E}(n)$, $0 \leq n < \infty$, denote the quotient Hopf algebroid

$$\begin{aligned}
\mathcal{E}(n) &:= \mathcal{A}_\star/(\xi_1, \xi_2, \ldots, \tau_{n+1}, \tau_{n+2}, \ldots) \\
&= H_\star[\tau_0, \ldots, \tau_n]/(\tau_i^2 - \rho\tau_{i+1} \mid 0 \leq i < n) + (\tau_n^2).
\end{aligned}$$

If $n = \infty$, let

$$\begin{aligned}
\mathcal{E}(\infty) &:= \mathcal{A}_\star/(\xi_1, \xi_2, \ldots) \\
&= H_\star[\tau_0, \tau_1, \ldots]/(\tau_i^2 - \rho\tau_{i+1} \mid 0 \leq i).
\end{aligned}$$

The Hopf algebroid $\mathcal{E}(n)$ is dual to the sub-Hopf algebroid of $\mathcal{A}^\star$ generated by the Milnor elements Q_i, $i \leq n$.

The motivic Adams spectral sequence. Our primary means of computation is the motivic Adams spectral sequence (mASS). This spectral sequence first appeared in Morel's work on connectivity and stable motivic π_0 [Mor05, Mor04] and has since been used over algebraically closed fields by Hu-Kriz-Ormsby [HKO10] (more accurately, their work focuses on the application of the motivic Adams-Novikov spectral sequence) and, independently, Dugger-Isaksen [DI10]. Hopkins-Morel (unpublished) knew that the motivic Adams spectral sequence (at the prime 2) converges to $(2,\eta)$-completions and Hu-Kriz-Ormsby [HKO] both prove this result and show that η-completion is unnecessary when $cd_2(\mathsf{k}[i]) < \infty$. This condition holds for $\mathsf{k} = F$ a p-adic field, so we have the following:

Theorem 1.6 **([HKO]).** *Fix a p-adic field F (see §2) and let X be a cell spectrum of finite type. Then the E_2-term of the mASS is*

$$E_2^{*,\star} = \mathrm{Ext}_{\mathcal{A}_\star}(H_\star, H_\star X)$$

and the mASS converges to $\pi_\star X\,\widehat{_2}$ *where permanent cycles in tri-degree* $(s, m + n\alpha)$ *represent elements of* $\pi_{m+n\alpha-s} X\,\widehat{_2}$. □

A word on the grading of the mASS will make computations easier to follow: The mASS is tri-graded. We denote the r-th page of the mASS by $E_r^{*,\star}$ where the first $*$ is an integer called the *homological degree*, and the second $\star$ is a bigrading of the form $m + n\alpha$ called the *motivic degree*. For a tri-grading $(s, m+n\alpha)$, we call the bigrading $m+n\alpha-s = (m-s)+n\alpha$ the *total motivic degree* or *Adams grading*; sometimes Adams grading will also refer to the tri-degree $(s, m + n\alpha - s)$. The differentials in the mASS take the form

$$d_r : E_r^{s,m+n\alpha} \to E_r^{s+r,m+n\alpha+r-1}.$$

In other words, the r-th differential increases homological degree by r and decreases total motivic degree by 1.

Remark 1.7. This paper concerns itself with mASS computations of the bigraded coefficients of the 2-complete $\mathsf{BPGL}\langle n\rangle$ over p-adic fields. Its sequel [Orm] analyzes the motivic ANSS over p-adic fields, in particular a motivic analogue of the alpha family in that setting.

Acknowledgments. This paper represents the first half of my thesis and it is a genuine pleasure to thank my advisor, Igor Kriz, for his input and help. I would also like to thank Mike Hill and Paul Arne Østvær for their encouragement and interest over the summer of 2009. Finally, I would like to thank the anonymous referee for numerous stylistic improvements, a correction to the proof of Theorem 3.8, a strengthening of Proposition 4.2, and a streamlined method of proof for Theorem 5.8 that avoided dependence on K-theory computations.

2. Arithmetic input from p-adic fields

A *p-adic field* is a complete discrete valuation field of characteristic 0 with finite characteristic p residue field. It is well-known that every p-adic field is a finite extension of the p-adic rationals $\mathbb{Q}_p$. A good reference for the basic theory is [Cas86].

Let $v : F \to \mathbb{Z}\cup\infty$ denote the valuation on F. F has a *ring of integers* $\mathcal{O} := \{x \in F \mid v(x) \geq 0\}$. $\mathcal{O}$ is a domain with $F = \operatorname{Frac}\mathcal{O}$, the field of fractions of $\mathcal{O}$. Moreover, $\mathcal{O}$ is a local ring with maximal ideal $\mathfrak{m} := \{x \in F \mid v(x) \geq 1\}$. A *uniformizer* of F is an element $\pi \in F$ such that $v(\pi) = 1$; note that for any choice of uniformizer π, $(\pi) = \mathfrak{m}$.

The *residue field* of F is

$$\mathbb{F} := \mathcal{O}/\mathfrak{m}.$$

Let $q = p^m = |\mathbb{F}|$ denote the *residue order* of F.

As a consequence of Hensel's lemma (see, e.g., [Cas86, Lemma 3.1]), the units of a p-adic field F are equipped with a *Teichmüller lift* $\mathbb{F}^\times \hookrightarrow F^\times$. Identifying $\mathbb{F}^\times$ with its image in $F^\times$, we have

$$F^\times = \pi^{\mathbb{Z}} \times \mathbb{F}^\times \times (1 + \mathfrak{m}).$$

Corollary 2.1. *Let F be a p-adic field, $p > 2$, with chosen uniformizer π and choose u to be a nonsquare in the Teichmüller lift $\mathbb{F}^\times$. Then*

$$F^\times/(F^\times)^2 = \pi^{\mathbb{Z}/2} \times u^{\mathbb{Z}/2}.$$

When $q = |\mathbb{F}^\times| \equiv 3 \ (4)$, we may choose u to be -1; when $q \equiv 1 \ (4)$, the image of -1 in $F^\times/(F^\times)^2$ is zero. (If $p = 2$, then $(1+\mathfrak{m})/(1+\mathfrak{m})^2 \neq 0$.) □

Remark 2.2. The structure of p-adic fields differs in the cases $p = 2$ and $p > 2$: for instance, $|\mathbb{Q}_2^\times/(\mathbb{Q}_2^\times)^2| = 8$ while p-adic fields have $|F^\times/(F^\times)^2| = 4$ for every $p > 2$. In order to avoid a great many minor modifications, we will only deal with p-adic fields for which $p > 2$ in this paper. Henceforth, the term p-adic field will only refer to *nondyadic* p-adic fields; moreover, the letter F will always refer to a p-adic field unless stated otherwise.

Every discretely valued field (E, v) with residue field $\mathbb{E}$ comes equipped with a *tame symbol*

$$\left(\frac{\cdot,\cdot}{E}\right) : E^\times \times E^\times \to \mathbb{E}^\times$$

defined by the formula

$$\left(\frac{x,y}{E}\right) = (-1)^{v(x)v(y)} x^{v(y)} y^{-v(x)} \mod \mathfrak{m}.$$

Lemma 2.3 ([Mil71, Lemma 11.5])**.** *The tame symbol is a Steinberg symbol and hence induces a homomorphism $K_2^M(E) \to \mathbb{E}^\times = K_1^M(\mathbb{E})$.* □

As a consequence of Lemma 2.3 and [Mil70, Example 1.7], we can determine the mod 2 Milnor K-theory of a p-adic field, a result presumably well-known to those who study such objects.

Proposition 2.4. *Fix a p-adic field F, a uniformizer π, and a nonsquare $u \in \mathbb{F}^\times$. As a $\mathbb{Z}$-graded $\mathbb{Z}/2$-algebra,*

$$k_*^M(F) = \begin{cases} \mathbb{Z}/2[\{u\}, \{\pi\}]/(\{u\}^2, \{\pi\}^2) & \text{if } q \equiv 1 \quad (4), \\ \mathbb{Z}/2[\{u\}, \{\pi\}]/(\{u\}^2, \{\pi\}(\{u\} - \{\pi\})) & \text{if } q \equiv 3 \quad (4) \end{cases} \tag{2}$$

where $|\{\pi\}| = |\{u\}| = 1$.

Proof. Abusing notation, we will write x for $\{x\} \in K_1^M(F)$ or $k_1^M(F)$ whenever the context does not admit confusion.

Since $k_1^M(F) = F^\times/(F^\times)^2$, Corollary 2.1 implies that

$$k_1^M(F) = \pi^{\mathbb{Z}/2} \times u^{\mathbb{Z}/2}.$$

Moreover, in [Mil70, Example 1.7(2)] Milnor shows that $k_2^M(F)$ has dimension 1 as a $\mathbb{Z}/2$-vector space. By the same reference, $k_n^M(F) = 0$ for all $n \geq 3$.

We still must determine the multiplicative structure of $k_*^M(F)$, which amounts to determining the products $u^2, u\pi, \pi^2 \in k_2^M(F)$. First note that

$$\left(\frac{u,\pi}{F}\right) = (-1)^0 u^1 \pi^0 = u \in K_1^M(\mathbb{F}),$$

which reduces to the nontrivial generator u of $k_1^M(\mathbb{F})$. By Lemma 2.3, it follows that $u\pi \neq 0 \in k_2^M(F)$.

The argument above also proves that, after reduction mod 2, the tame symbol is an isomorphism $k_2^M(F) \to k_1^M(\mathbb{F})$. Hence to compute the products u^2 and π^2, it suffices to compute

$$\left(\frac{u,u}{F}\right) \quad \text{and} \quad \left(\frac{\pi,\pi}{F}\right).$$

These symbols are 1 and -1, respectively, so $u^2 = 0 \in k_1^M(F)$ while π^2 is nontrivial iff $q \equiv 3 \ (4)$. This determines the multiplicative structure given in (2). □

Theorem 2.5. *Over a p-adic field F, the coefficients of mod 2 motivic homology are*

$$H_\star = k_*^M(F)[\tau]$$

where $|\tau| = 1-\alpha$, $|k_n^M(F)| = -n\alpha$, and $k_*^M(F)$ has the form given in Proposition 2.4.

The dual motivic Steenrod algebra has the form

$$\mathcal{A}_\star = H_\star[\tau_0, \tau_1, \ldots, \xi_1, \xi_2, \ldots]/(\tau_i^2 - \tau\xi_{i+1} - \rho(\tau_{i+1} + \tau_0\xi_{i+1})).$$

The class ρ is trivial iff $q \equiv 1 \ (4)$. In this case,

$$\mathcal{A}_\star = H_\star[\tau_0, \tau_1, \ldots, \xi_1, \xi_2, \ldots]/(\tau_i^2 - \tau\xi_{i+1}) \cong \mathcal{A}_\star^{\mathbb{C}} \otimes_{H_\star^{\mathbb{C}}} k_*^M(F)$$

where $(H_\star^{\mathbb{C}}, \mathcal{A}_\star^{\mathbb{C}})$ is the dual motivic Steenrod algebra over $\mathbb{C}$, which has the structure

$$H_\star^{\mathbb{C}} = \mathbb{Z}/2[\tau],$$
$$\mathcal{A}_\star^{\mathbb{C}} = \mathbb{Z}/2[\tau, \tau_0, \tau_1, \ldots, \xi_1, \xi_2, \ldots]/(\tau_i^2 - \tau\xi_{i+1}).$$

Proof. Most of the theorem is a concatenation of results in Theorems 1.1 and 1.3 and Proposition 2.4. The form of $(H_*^{\mathbb{C}}, \mathcal{A}_*^{\mathbb{C}})$ is obvious after noting that $k_*^M(\mathbb{C})$ is trivial outside of degree 0. The class ρ is trivial iff -1 is a square in $\mathbb{F}^\times = \mathbb{F}_q^\times$; it is standard that this is the case iff $q \equiv 1 \ (4)$. □

The structure of $H_\star$ over F is depicted in Figure 1. Here the horizontal axis measures the $\mathbb{Z}$-component of the motivic bigrading, while the vertical axis measures the $\mathbb{Z}\alpha$-component. Each "diamond" shape is a copy of $k_*^M(F)$, and the diagonal arrows of slope -1 represent τ-multiplication.

3. Comodoules over the dual motivic Steenrod algebra

In this section, we work over a general characteristic 0 field k. Recall the algebraic Brown-Peterson spectrum BPGL constructed by Hu-Kriz and Vezzosi in [HK01, Vez01]. (We only consider BPGL at the prime 2 in this paper.) There are canonical elements $v_1, v_2, \ldots \in \mathsf{BPGL}_\star$ that appear in dimensions $|v_i| = (2^i - 1)(1+\alpha)$. Let $v_0 = 2 \in \mathsf{BPGL}_0$. These elements are the images of $v_i \in \mathsf{BP}_{2(2^i-1)}$ under the Lazard ring isomorphism $\mathsf{MU}_* \to \mathsf{MGL}_{*(1+\alpha)}$.

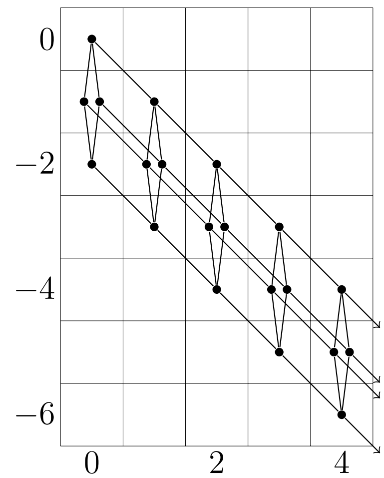

FIGURE 1. Mod 2 motivic homology over F

Definition 3.1. For $0 \leq n$, the *algebraic Johnson-Wilson spectra* are defined to be

$$\mathsf{BPGL}\langle n\rangle := \mathsf{BPGL}/(v_{n+1}, v_{n+2}, \ldots).$$

These quotients are well-defined since algebraic cobordism is E_∞. They fit into cofiber sequences

$$\Sigma^{|v_n|}\mathsf{BPGL}\langle n\rangle \xrightarrow{v_n} \mathsf{BPGL}\langle n\rangle \to \mathsf{BPGL}\langle n-1\rangle. \tag{3}$$

By convention, we write $\mathsf{BPGL}\langle\infty\rangle = \mathsf{BPGL}$.

The study of the algebraic Johnson-Wilson spectra should be motivated by the natural role they play when $n = 0, 1$, and ∞.

Theorem 3.2 (Hopkins-Morel)**.** *After 2-completion,* $\mathsf{BPGL}\langle 0\rangle$ *is the 2-complete integral motivic cohomology spectrum,*

$$\mathsf{BPGL}\langle 0\rangle\,\widehat{_2} = H\mathbb{Z}_2.$$

Hopkins defined $\mathsf{BPGL}\langle 1\rangle$ as a motivic analogue of connective K-theory, and we will sometimes write kgl instead of $\mathsf{BPGL}\langle 1\rangle$. Throughout the rest of this section, let KGL denote the 2-localization of the $(1+\alpha)$-periodic algebraic K-theory spectrum. The following theorem relates the coefficients of kgl with established objects of interest, the algebraic K-groups of the ground field.

Theorem 3.3. *Let* ${}_{v_1}\mathsf{kgl}_\star$ *denote the* v_1*-power torsion in the coefficients of* kgl*, i.e., the elements* $x \in \mathsf{kgl}_\star$ *such that there exists* $n \in \mathbb{N}$ *such that* $v_1^n x = 0 \in \mathsf{kgl}_\star$*. (We will refer to these elements simply as* v_1*-torsion.) Then there is an exact sequence*

$$0 \to {}_{v_1}\mathsf{kgl}_\star \to \mathsf{kgl}_\star \to \mathsf{KGL}_\star.$$

Moreover, if $\overline{\mathsf{KGL}_\star}$ *denotes the subalgebra of* $\mathsf{KGL}_\star$ *consisting of elements in degree* $m + n\alpha$, $m \geq 0$, *then there is a short exact sequence*

$$0 \to {}_{v_1}\mathsf{kgl}_\star \to \mathsf{kgl}_\star \to \overline{\mathsf{KGL}_\star} \to 0.$$

Proof. By the motivic Conner-Floyd theorem [SØ09], $\mathsf{KGL}_\star = v_1^{-1}\mathsf{kgl}_\star$. The first exact sequence is then a basic fact of localization.

Clearly, though, the map $\mathsf{kgl}_\star \to \mathsf{KGL}_\star$ is not surjective since $\mathsf{KGL}_\star$ is Bott $= v_1$-periodic. Note, though, that $\mathsf{KGL}_\star$ is generated by v_1 of dimension $1 + \alpha$ and elements of degree $m + n\alpha$, $m \geq 0$. (In fact, we could restrict the second collection of generators to degrees $0 + n\alpha$, $n \leq 0$.) Again by the motivic Conner-Floyd theorem, it is a straightforward combinatorial check that $\mathsf{kgl}_\star \to \mathsf{KGL}_\star$ is surjective in dimensions $m + n\alpha$, $m \geq 0$. □

Remark 3.4. The spectrum kgl is connective in the sense that $\mathsf{kgl}_{m+n\alpha} = 0$ for all $m < 0$. In general (though, we will see, not for p-adic fields) there is a rich class of v_1-torsion in $\mathsf{kgl}_\star$, so it is the case that $\mathsf{kgl}_\star$ is bigger than $\mathsf{KGL}_\star$ in its nonvanishing dimensional range. Still, producing computations of $\mathsf{kgl}_\star$ explicit enough to capture its v_1-torsion will determine $\mathsf{KGL}_\star$ in a meaningful dimensional range by the second exact sequence. In particular,

$$(\mathsf{kgl}_\star / {}_{v_1}\mathsf{kgl}_\star)_{m+0\alpha} = \mathsf{KGL}_{m+0\alpha}$$

for $m \geq 0$, and these groups match the 2-local Quillen K-groups of the base field.

We now turn to determining the $\mathcal{A}_\star$-comodule structure of $H_\star\mathsf{BPGL}\langle n\rangle$. To access these, we will determine the $\mathcal{A}^\star$-module structure of $H^\star\mathsf{BPGL}\langle n\rangle$.

Recall the Milnor elements $Q_i \in \mathcal{A}^\star$, $|Q_i| = 2^i(1+\alpha) - \alpha$ from §1. The following theorem of Borghesi should appear quite familiar to topologists.

Theorem 3.5 ([Bor03, Proposition 6])**.** *The mod 2 motivic cohomology of* MGL *takes the form*

$$H^\star\mathsf{MGL} = (\mathcal{A}^\star//E(Q_0, Q_1, \ldots))[m_i \mid i \neq 2^n - 1]$$

as an $\mathcal{A}^\star$*-module where* $|m_i| = i(1+\alpha)$. □

Corollary 3.6. *The mod 2 motivic cohomology of* BPGL *takes the form*

$$H^\star\mathsf{BPGL} = \mathcal{A}^\star//E(Q_0, Q_1, \ldots)$$

as an $\mathcal{A}^\star$*-module.* □

Recall Definition 1.5 which defines the $\mathcal{A}_\star$-algebras $\mathcal{E}(n)$, $0 \leq n \leq \infty$. In particular, we have

$$\begin{aligned}\mathcal{E}(\infty) &= \mathcal{A}_\star/(\xi_1, \xi_2, \ldots)\\ &= H_\star[\tau_0, \tau_1, \ldots]/(\tau_i^2 - \rho\tau_{i+1} \mid 0 \leq i),\\ \mathcal{E}(n) &= \mathcal{A}_\star/(\xi_1, \xi_2, \ldots, \tau_{n+1}, \tau_{n+2}, \ldots)\\ &= H_\star[\tau_0, \ldots, \tau_n]/(\tau_i^2 - \rho\tau_{i+1} \mid 0 \leq i \leq n) + (\tau_n^2).\end{aligned} \tag{4}$$

These algebras are dual to $E(Q_0, Q_1, \ldots)$ and $E(Q_0, \ldots, Q_n)$, respectively.

There is a general yoga of passing from $\mathcal{A}^\star$-module structure on cohomologies to $\mathcal{A}_\star$-comodule structure on homologies. Applied to the above situation, we get the following theorem describing the $\mathcal{A}_\star$-comodule structure on $H_\star\mathsf{BPGL}$.

Theorem 3.7. *As an $\mathcal{A}_\star$-comodule algebra,*

$$H_\star\mathsf{BPGL} = \mathcal{A}_\star \,\square_{\mathcal{E}(\infty)}\, H_\star.$$

□

To determine the $\mathcal{A}_\star$-comodule structure of $H_\star\mathsf{BPGL}\langle n\rangle$ we first determine $H^\star\mathsf{BPGL}\langle n\rangle$ as an $\mathcal{A}^\star$-module and then apply the same yoga. Our determination of $H^\star\mathsf{BPGL}\langle n\rangle$ is modeled on the topological calculation [Wil75]. (Since the first draft of this paper was written, a similar argument for the cohomology of kgl has appeared in Isaksen-Shkembi [IS, §5].)

Theorem 3.8. *As an $\mathcal{A}^\star$-module algebra,*

$$H^\star\mathsf{BPGL}\langle n\rangle = \mathcal{A}^\star//E(Q_0,\ldots,Q_n).$$

Proof. We use the cofiber sequence (3) and induction on n. By Theorem 3.2, we know the Theorem holds for $\mathsf{BPGL}\langle 0\rangle$. Assume it holds for some $n-1\geq 0$ and consider the long exact sequence in cohomology induced by (3). Following the exact argument of [Wil75, Proposition 1.7], it suffices to show that $Q_n(1)=0\in H^{2^n(1+\alpha)-\alpha}\mathsf{BPGL}\langle n\rangle$. To this end, note that $\mathsf{BPGL}\langle n\rangle$ is constructed from BPGL by killing off spheres of the form $S^{k+\ell\alpha}$ where $k,\ell\geq 2^{n+1}-1$. Invoking Morel's connectivity theorem and the long exact sequence in homotopy induced by $\mathsf{BPGL}\to\mathsf{BPGL}\langle n\rangle$, we see that this map induces an iso in degrees $m+*\alpha$, $m+1<2^{n+1}$. The same holds in cohomology, so Corollary 3.6 implies Q_n dies in $H^\star\mathsf{BPGL}\langle n\rangle$ since $2^n+1<2^{n+1}$. □

Since $\mathcal{E}(n)$ is dual to $E(Q_0,\ldots,Q_n)$, we have the following theorem.

Theorem 3.9. *As an $\mathcal{A}_\star$-comodule algebra,*

$$H_\star\mathsf{BPGL}\langle n\rangle = \mathcal{A}_\star \,\square_{\mathcal{E}(n)}\, H_\star.$$

□

4. Motivic Ext-algebras

Theorems 3.7 and 3.9 identify the homology of $\mathsf{BPGL}\langle n\rangle$, $0\leq n\leq\infty$, in the category of $\mathcal{A}_\star$-comodules. By Theorem 1.6, these data form the input to the E_2-term of the mASS for $\mathsf{BPGL}\langle n\rangle$. In fact, both E_2-terms take the form

$$\mathrm{Ext}_{\mathcal{A}_\star}(H_\star, \mathcal{A}_\star \,\square_{\mathcal{E}(n)}\, H_\star).$$

Theorem 4.1 ([Rav86, Theorem A1.3.12])**.** *For $0\leq n\leq\infty$, the map of Hopf algebroids $(H_\star,\mathcal{A}_\star)\to(H_\star,\mathcal{E}(n))$ induces an isomorphism*

$$\mathrm{Ext}_{\mathcal{A}_\star}(H_\star, \mathcal{A}_\star \,\square_{\mathcal{E}(n)}\, H_\star)\cong \mathrm{Ext}_{\mathcal{E}(n)}(H_\star,H_\star). \quad \square$$

Fix a p-adic field F (see Remark 2.2) with residue order q. In this section, we compute $\mathrm{Ext}_{\mathcal{E}(n)}(H_\star, H_\star)$ over F; this is the E_2-term for the mASS computing $\pi_\star \mathsf{BPGL}\langle n\rangle\,\widehat{2}$. This work was antecedent to Hill's paper [Hill] in which he performs similar computations over the field of real numbers $\mathbb{R}$.

Recall that when $q \equiv 1 \ (4)$, Theorem 2.5 implies that $\mathrm{Ext}_{\mathcal{E}(n)}(H_\star, H_\star)$ is easily computable in terms of its complex counterpart $\mathrm{Ext}_{\mathcal{E}(n)^{\mathbb{C}}}(H_\star^{\mathbb{C}}, H_\star^{\mathbb{C}})$. In fact, $(H_\star^F, \mathcal{E}(n)^F) = (H_\star^{\mathbb{C}}, \mathcal{E}(n)^{\mathbb{C}}) \otimes_{H_\star^{\mathbb{C}}} H_\star^F$ as Hopf algebroids, so, by change of base,

$$\mathrm{Ext}_{\mathcal{E}(n)^F}(H_\star^F, H_\star^F) = \mathrm{Ext}_{\mathcal{E}(n)^{\mathbb{C}}}(H_\star^{\mathbb{C}}, H_\star^{\mathbb{C}}) \otimes_{H_\star^{\mathbb{C}}} H_\star^F$$

when $q \equiv 1 \ (4)$. Moreover, since $\rho = 0$ over $\mathbb{C}$, $\mathcal{E}(n)^{\mathbb{C}} = \mathcal{E}(n)^{\mathrm{top}} \otimes_{\mathbb{Z}/2} H_\star^{\mathbb{C}}$. Here $\mathcal{E}(n)^{\mathrm{top}}$ is the analogous quotient of the topological dual Steenrod algebra, but degree-shifted so that elements usually in degree $2m$ appear in dimension $m(1+\alpha)$. Hence, again by change of base, we can compute the E_2-term of the mASS for $\mathsf{BPGL}\langle n\rangle^{\mathbb{C}}$. To be precise,

$$\begin{aligned}\mathrm{Ext}_{\mathcal{E}(n)^{\mathbb{C}}}(H_\star^{\mathbb{C}}, H_\star^{\mathbb{C}}) &= \mathrm{Ext}_{\mathcal{E}(n)^{\mathrm{top}}}(H_\star^{\mathrm{top}}, H_\star^{\mathrm{top}}) \otimes_{\mathbb{Z}/2} H_\star^{\mathbb{C}} \\ &= \mathbb{Z}/2[v_0, \ldots, v_n] \otimes_{\mathbb{Z}/2} H_\star^{\mathbb{C}}.\end{aligned}$$

(See [Rav86, Corollary 3.1.10] for the computation in topology.)

This yields, for $q \equiv 1 \ (4)$, the computation

$$\mathrm{Ext}_{\mathcal{E}(n)^F}(H_\star^F, H_\star^F) = H_\star^F[v_0, \ldots, v_n]$$

where $|v_i| = (1, (2^i - 1)(1+\alpha) + 1)$.

When $q \equiv 3 \ (4)$, $\mathcal{E}(n)$ does not split over $\mathcal{E}(n)^{\mathbb{C}}$. In order to deal with the extra complexity introduced by the relation $\tau_i^2 = \rho\tau_{i+1}$, we filter by powers of ρ and consider the associated filtration spectral sequence [Rav86, Theorem A1.3.9]. (In [Hill], Hill refers to this spectral sequence (over $\mathbb{R}$) as the "ρ-Bockstein spectral sequence.") Since $\mathcal{E}(n)_\star^F/(\rho) = \mathcal{E}(n)_\star^{\mathbb{C}} \oplus \pi\mathcal{E}(n)_\star^{\mathbb{C}}$, this spectral (in fact, long exact) sequence takes the form

$$E_1 = \begin{pmatrix} \mathrm{Ext}_{\mathcal{E}(n)^{\mathbb{C}}}(H_\star^{\mathbb{C}}, H_\star^{\mathbb{C}}) \\ \oplus \\ \pi\,\mathrm{Ext}_{\mathcal{E}(n)^{\mathbb{C}}}(H_\star^{\mathbb{C}}, H_\star^{\mathbb{C}}) \end{pmatrix} [\rho]/(\rho^2) \implies \mathrm{Ext}_{\mathcal{E}(n)^F}(H_\star^F, H_\star^F).$$

Since $\eta_R(\tau) - \eta_L(\tau) = \rho\tau_0$ in $\mathcal{E}(n)^F$, τ supports the d_1-differential

$$d_1\tau = \rho v_0.$$

The elements π and ρ are in the Hurewicz image of the sphere, and hence are permanent cycles; the v_i are represented in the cobar complex by the primitives $[\tau_i]$ and hence are permanent cycles. Thus we have determined the E_2 page of the filtration spectral sequence:

$$E_2 = \begin{matrix} k_*^M(F)[\tau^2, v_0, \ldots, v_n]/(\rho v_0) \\ \oplus \\ \rho\tau k_*^M(F)[\tau^2, v_0, \ldots, v_n] \end{matrix}.$$

Since $\rho^2 = 0$, the spectral sequence collapses here and $E_2 = E_\infty$.

In order to fully determine $\mathrm{Ext}_{\mathcal{E}(n)}(H_\star, H_\star)$ when $q \equiv 3 \ (4)$, we must address hidden extensions in $E_2 = E_\infty$.

Proposition 4.2. *There are no hidden extensions in the ρ-power filtration spectral sequence for $\mathrm{Ext}_{\mathcal{E}(n)}$.*

Proof. We only need to concern ourselves with the $q \equiv 3 \ (4)$ case. There is only one extension to consider since $\rho^2 = 0$. Elements in the ρ-divisible summand appear in lowest possible filtration and hence have their expected multiplicative structure. Thus it suffices to show that the v_is and τ^2 are free; this accomplished by considering the change-of-base map

$$\mathrm{Ext}_{\mathcal{E}(n)^F} \to \mathrm{Ext}_{\mathcal{E}(n)^{\mathbb{C}}}. \qquad \square$$

This, combined with the filtration spectral sequence computation, proves the following theorem.

Theorem 4.3. *Over a p-adic field F,*

$$\mathrm{Ext}_{\mathcal{A}_\star}(H_\star, H_\star \mathsf{BPGL}\langle n\rangle) = \begin{cases} k_*^M(F)[\tau, v_0, \ldots, v_n] & \text{if } q \equiv 1 \quad (4), \\ \begin{array}{c} k_*^M(F)[\tau^2, v_0, \ldots, v_n]/(\rho v_0) \\ \oplus \\ \rho\tau k_*^M(F)[\tau^2, v_0, \ldots, v_n] \end{array} & \text{if } q \equiv 3 \quad (4). \end{cases}$$

$\square$

Remark 4.4. Note that by Proposition 2.4, the algebra $k_*^M(F)$ takes the form

$$k_*^M(F) = \begin{cases} \mathbb{Z}/2[\pi, u]/(u^2, \pi^2) & \text{if } q \equiv 1 \quad (4), \\ \mathbb{Z}/2[\pi, \rho]/(\rho^2, \pi(\rho - \pi)) & \text{if } q \equiv 3 \quad (4) \end{cases}$$

so the above computation is completely explicit.

5. $\pi_\star \mathsf{BPGL}\langle n\rangle^\wedge_2$ via the motivic Adams spectral sequence

Theorem 3.7 determines the E_2-term of the mASS for $\mathsf{BPGL}\langle n\rangle$. We now determine the mASS for $\mathsf{BPGL}\langle 0\rangle$ and use this and the maps $\mathsf{BPGL}\langle n\rangle \to \mathsf{BPGL}\langle 0\rangle$ to compute the mASS for $\mathsf{BPGL}\langle n\rangle$, $0 < n \leq \infty$.

Let q be the residue order of our p-adic field F, let $a = \nu_2(q-1)$, and let $\lambda = \nu_2(q^2-1)$ where ν_2 is the 2-adic valuation of integers. Define the numbers w_i following [RW00]. Then

$$w_i = \begin{cases} 2^{a+\nu_2(i)} & \text{if } q \equiv 1 \quad (4), \\ 2^{\lambda-1+\nu_2(i)} & \text{if } q \equiv 3 \quad (4), i \text{ even}, \\ 2 & \text{if } q \equiv 3 \quad (4), i \text{ odd}. \end{cases}$$

(Note that $a = \lambda - 1$ when $q \equiv 1 \ (4)$.) By Theorem 3.2, the following lemma is a well-known computation in étale cohomology (see, for instance, [RW00, Corollary 2.10], apply the universal coefficient theorem, and then recall the relationship between étale and motivic cohomology of fields).

Lemma 5.1. *The coefficients of* $\mathsf{BPGL}\langle 0\rangle\widehat{_2}$ *are*

$$\begin{aligned}\pi_{m+n\alpha}&\mathsf{BPGL}\langle 0\rangle\widehat{_2}\\ &= H_{\mathrm{mot}}^{-(m+n)}(F;\mathbb{Z}_2(-n))\\ &= \begin{cases}\mathbb{Z}_2 & \text{if } m=n=0,\\ \mathbb{Z}_2\oplus\mathbb{Z}/w_1 & \text{if } m=0, n=-1,\\ \mathbb{Z}/w_1 & \text{if } m=0, n=-2,\\ \mathbb{Z}/w_i & \text{if } m+n\alpha=(i-1)(1-\alpha)-\epsilon\alpha\\ & \text{for } i\ge 1, \epsilon=1 \text{ or } 2,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

□

Theorem 5.2. *The mASS for* $\mathsf{BPGL}\langle 0\rangle$ *is determined by the following differentials: If* $q\equiv 1\ (4)$*, then*

$$d_{a+s}\tau^{2^s} = u\tau^{2^s-1}v_0^{a+s}.$$

If $q\equiv 3\ (4)$*, then*

$$d_1\tau = \rho v_0,$$
$$d_{\lambda-1+s}\tau^{2^s} = \rho\tau^{2^s-1}v_0^{\lambda-1+s}.$$

Proof. Given Lemma 5.1 this is straightforward: differentials on τ-powers determine the spectral sequence since v_0 and elements of $k_*^M(F)$ are obviously permanent cycles. Now v_0 represents $1-\epsilon$ in general, but the 0-th coefficient group of $\mathsf{BPGL}\langle 0\rangle\widehat{_2}$ is $\mathbb{Z}_2$ so it represents 2 in this case. Hence the above differentials are necessary in order to produce the appropriate 2-torsion in $\pi_\star\mathsf{BPGL}\langle 0\rangle\widehat{_2}$. □

Remark 5.3. Upon noticing that the mASS for $\mathsf{BPGL}\langle 0\rangle$ is the same thing as the 2-Bockstein spectral sequence, we can also see the differentials of Theorem 5.2 by May's higher Leibniz rule [May70, Proposition 6.8]. This states that

$$d_{r+1}x^2 = xd_r(x)v_0$$

in Bockstein spectral sequences up to a correction term expressed by an (algebraic) power operation on $d_1(x)$ in the $r=1$ case. The first nontrivial differential is determined by the fact $\mathsf{BPGL}\langle 0\rangle_{-\alpha}=F^\times$ [Mor04] and the structure of $F^\times$. When $q\equiv 1\ (4)$, we avoid the correction term and this is enough to determine the spectral sequence. When $q\equiv 3\ (4)$ the correction term is quite important and implies that $d_2(\tau^2)=0$. One then must determine $d_?(\tau^2)$ by knowing the torsion in $\pi_{1-2\alpha}\mathsf{BPGL}\langle 0\rangle\widehat{_2}=H^1(F;\mathbb{Z}_2(2))$, at which point one may continue with the higher Leibniz argument.

Remark 5.4. The behavior of this spectral sequence is depicted in Figure 2. Here elements in degree $(s, m+n\alpha)$ are depicted in total motivic degree $m+n\alpha-s$ with the homological degree suppressed. (The horizontal axis measures $\mathbb{Z}$ while the vertical axis measures $\mathbb{Z}\alpha$.) It is convenient to imagine the supressed homological degree as coming out of the page, in which case there are v_0-towers sitting over every mark and differentials truncate these towers. In the dimensional range shown, $E_{\lambda+2} = E_\infty$. Note that the v_0-towers in all pictures actually come out of the page, as do the v_i-multiplication arrows.

We now use this information about the mASS for $\mathsf{BPGL}\langle 0\rangle$ to compute $\pi_\star \mathsf{BPGL}\langle n\rangle\,\widehat{}_2$. The surprising fact is that, for $i > 0$, the differentials have no v_i-components so that $E_r(\mathsf{BPGL}\langle n\rangle) = E_r(\mathsf{BPGL}\langle 0\rangle)[v_1, \ldots, v_n]$.

Theorem 5.5. *The differentials in the mASS for* $\mathsf{BPGL}\langle n\rangle$ *are identical to those for* $\mathsf{BPGL}\langle 0\rangle$ *in Theorem 5.2.*

Proof. Note that all the v_i and elements in $k_*^M(F)$ are permanent cycles. (This is obvious since their target ranges are trivial; see Figure 3.) Inductively, then, we can show that

$$E_r(\mathsf{BPGL}\langle n\rangle) = E_r(\mathsf{BPGL}\langle 0\rangle)[v_1, \ldots, v_n].$$

The result is true on E_2 pages; assume it is true on E_r and consider the map f of spectral sequences induced by $\mathsf{BPGL}\langle n\rangle \to \mathsf{BPGL}\langle 0\rangle$. By dimensional accounting, it is clear that the E_{r+1}-page is determined by the differential on the smallest surviving τ-power. The map f is an isomorphism on the target of this τ-power by dimensional accounting. Indeed, since $cd_2(F) = 2$ (i.e. $k_*^M(F)$ is only nonzero for $0 \leq * \leq 2$) no $v_{>0}$-terms can appear in the target. This proves that the τ-power supports the same differential as in the mASS for $\mathsf{BPGL}\langle 0\rangle$, from which we get the result on E_{r+1}. □

Remark 5.6. The mASS for $\mathsf{BPGL}\langle n\rangle$ is depicted in Figure 3. The grading convention described in Remark 5.4 is followed again. Note that the arrows with slope 1 represent multiplication by v_i's, $i > 0$, and they also come "out of the page" since v_i has homological degree 1. Note the obvious vanishing region with $k_*^M(F)$ and the v_i monomials on the boundary.

We now have the following unified description of the E_∞ page of the mASS for $\mathsf{BPGL}\langle n\rangle$, $0 \leq n \leq \infty$.

Theorem 5.7. *Let* $a = \nu_2(q-1)$ *and let* $\lambda = \nu_2(q^2-1)$. *The* E_∞*-term of the mASS for* $\mathsf{BPGL}\langle n\rangle$ *over a* p*-adic field* F *is*

$$\Gamma'[v_1, \ldots, v_n]$$

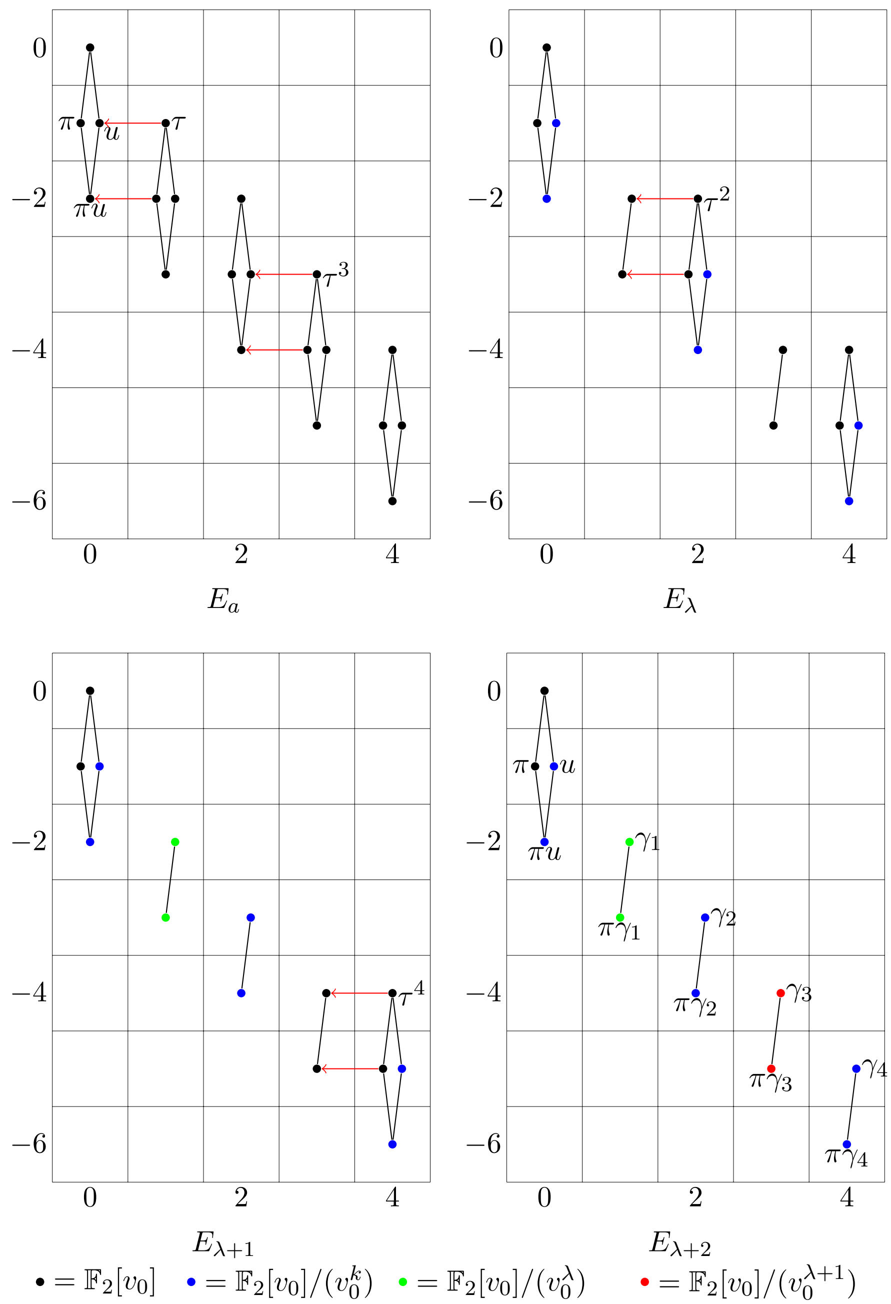

FIGURE 2. The mASS for BPGL$\langle 0\rangle$ over F

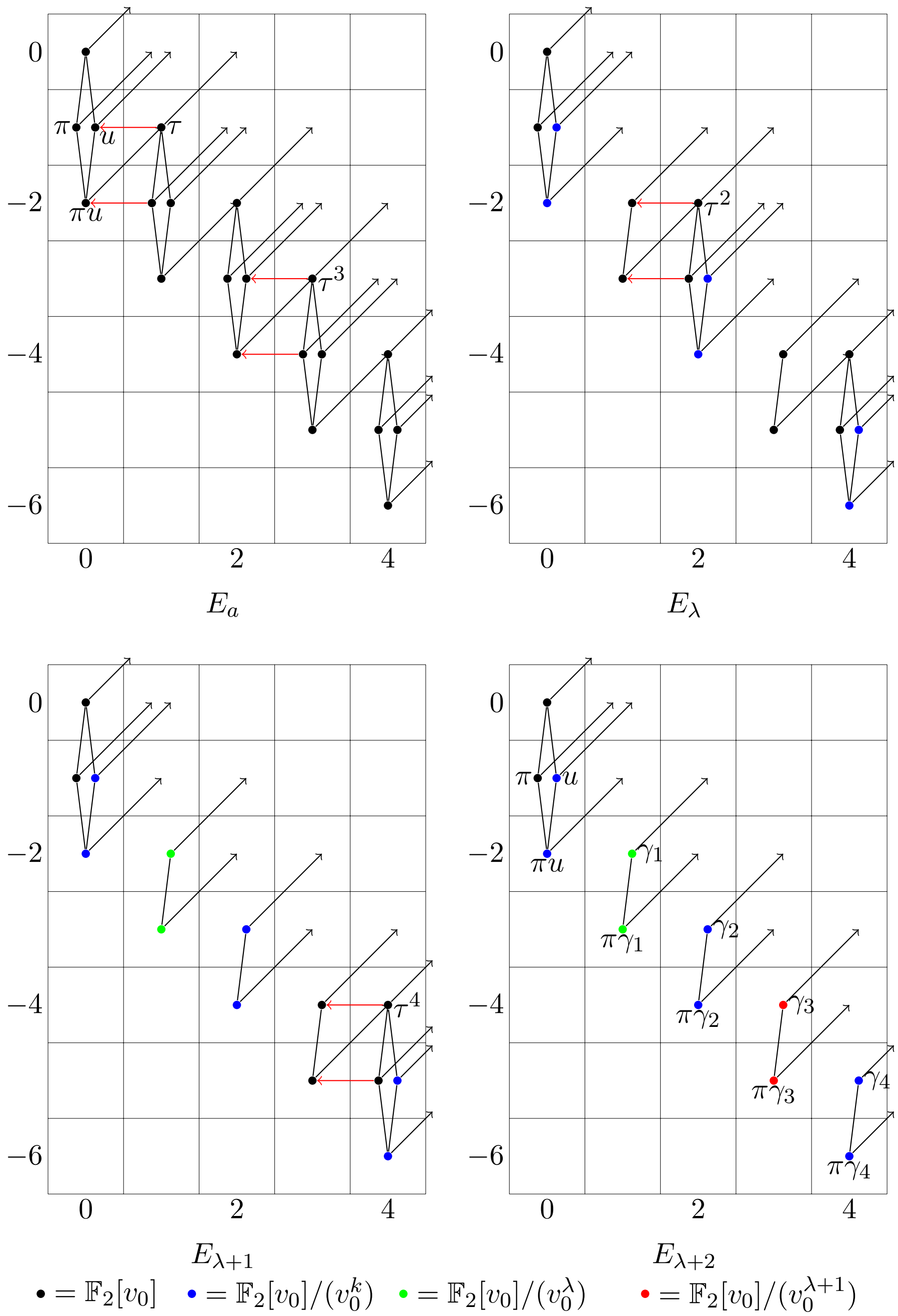

FIGURE 3. The mASS for BPGL$\langle n\rangle$ over F

where Γ' has additive structure

$$\Gamma' = \begin{cases} \mathbb{Z}/2[v_0] & \text{in dimension } 0, \\ \mathbb{Z}/2[v_0]\{\pi\} \oplus \mathbb{Z}/2[v_0]/v_0^a\{u\} & \text{in dimension } -\alpha, \\ \mathbb{Z}/2[v_0]/v_0^a\{\pi u\} & \text{in dimension } -2\alpha, \\ \mathbb{Z}/2[v_0]/v_0^{\lambda-1+\nu_2(i)}\{\gamma_i\} & \text{in dimension } i(1-\alpha)-\alpha \\ & \text{for } i \text{ odd}, \\ \mathbb{Z}/2[v_0]/v_0^{\lambda-1+\nu_2(i)}\{\pi\gamma_i\} & \text{in dimension } i(1-\alpha)-2\alpha \\ & \text{for } i \text{ odd}, \\ \mathbb{Z}/2[v_0]/v_0^a\{\gamma_i\} & \text{in dimension } i(1-\alpha)-\alpha \\ & \text{for } i \text{ even}, \\ \mathbb{Z}/2[v_0]/v_0^a\{\pi\gamma_i\} & \text{in dimension } i(1-\alpha)-2\alpha \\ & \text{for } i \text{ odd}, \\ 0 & \text{otherwise.} \end{cases}$$

The generators in each degree are indicated above. They satisfy the obvious multiplicative relations indicated by their notation while $u\gamma_i$ and $\gamma_i\gamma_j$ are 0. □

A quick inspection of tri-degrees reveals that there are no extensions except those created by v_0-multiplication. Indeed, the lines of slope 1 originating in the nontrivial dimensions of Γ' do not overlap, so we only need to worry about v_0-towers. Since v_0 represents 2 in $\pi_0\mathsf{BPGL}\langle n\rangle\,\widehat{_2}$, any copies of $\mathbb{Z}/2[v_0]$ produce copies of the 2-adic integers $\mathbb{Z}_2$, and any copies of $\mathbb{Z}/2[v_0]/v_0^b$ produce copies of $\mathbb{Z}/2^b$. This proves the following theorem.

Theorem 5.8. *Let $a = \nu_2(q-1)$, $\lambda = \nu_2(q^2-1)$ and set $w_i = \mathbb{Z}/2^a$ for i odd, $w_i = 2^{\lambda-1+\nu(i)}$ for i even. The coefficients of the 2-complete algebraic Johnson-Wilson spectra $\mathsf{BPGL}\langle n\rangle\,\widehat{_2}$ over a p-adic field F are*

$$\pi_\star\mathsf{BPGL}\langle n\rangle\,\widehat{_2} = (\pi_\star\mathsf{BPGL}\langle 0\rangle\,\widehat{_2})[v_1, v_2, \ldots]$$

where $|v_i| = (2^i-1)(1+\alpha)$ and, additively,

$$\pi_\star\mathsf{BPGL}\langle 0\rangle\,\widehat{_2} = \begin{cases} \mathbb{Z}_2 & \text{in dimension } 0, \\ \mathbb{Z}_2 \oplus \mathbb{Z}/w_1 & \text{in dimension } -\alpha, \\ \mathbb{Z}/w_1 & \text{in dimension } -2\alpha, \\ \mathbb{Z}/w_i & \text{in dimension } (i-1)(1-\alpha)-\epsilon\alpha \\ & \text{for } i \geq 1 \text{ and } \epsilon = 1 \text{ or } 2, \\ 0 & \text{otherwise.} \end{cases}$$

Multiplicative structure and generator names for $\pi_\star\mathsf{BPGL}\langle 0\rangle\,\widehat{_2}$ are the same as in Theorem 5.7. □

Corollary 5.9. *The coefficients of the 2-complete algebraic cobordism spectrum over a p-adic field F are*

$$\pi_\star\mathsf{MGL}\,\widehat{_2} = \Gamma[x_1, x_2, \ldots]$$

where $|x_i| = i(1+\alpha)$. □

Corollary 5.10. *Over a p-adic field, the slice spectral sequences for* $\mathsf{MGL}\widehat{_2}$ *and* $\mathsf{BPGL}\langle n\rangle\widehat{_2}$ *($0 \le n \le \infty$) collapse.*

Proof. Hopkins-Morel show that the slice associated gradeds for MGL and $\mathsf{BPGL}\langle n\rangle$ are $H\mathbb{Z}_\star[x_1, x_2, \ldots]$, $H\mathbb{Z}_\star[v_1, \ldots, v_n]$, respectively, and Theorem 5.8 implies that there are no differentials in the slice spectral sequences for their 2-completions. □

Corollary 5.11. *The coefficients of 2-complete* kgl *are* $(\pi_\star \mathsf{BPGL}\langle 0\rangle)[v_1]$. *In particular, there is no v_1-torsion and we recover the 2-complete algebraic K-theory of F in degrees $m + 0\alpha$.*

We conclude by discussing some easy corollaries of this work that highlight the importance of the algebraic Brown-Peterson spectra and have important applications to the motivic ANSS [Orm10, Orm]. See [HKO10, HKO, DI10] for discussions of the motivic Adams-Novikov spectral sequence.

For convenience, let $\Gamma := \pi_\star \mathsf{BPGL}\langle 0\rangle\widehat{_2} = \pi_\star H\mathbb{Z}_2$. For typographical simplicity, we drop the 2-completion $(\ )\widehat{_2}$ from our notation in the rest of this section.

Theorem 5.12. *Fix a p-adic field F and work in the 2-complete stable motivic homotopy category over F. Then the Hopf algebroid for* BPGL *splits as*

$$(\mathsf{BPGL}_\star, \mathsf{BPGL}_\star\mathsf{BPGL}) = (\mathsf{BP}_*, \mathsf{BP}_*\mathsf{BP}) \otimes_{\mathbb{Z}_2} \Gamma.$$

Moreover, the E_2-term of the motivic ANSS in homological degree s is

$$\mathrm{Ext}^s_{\mathsf{BPGL}_\star\mathsf{BPGL}}(\mathsf{BPGL}_\star, \mathsf{BPGL}_\star) = \begin{array}{c} {}^{\mathrm{top}}E_2^s \otimes_{\mathbb{Z}_2} \Gamma \\ \oplus \\ \mathrm{Tor}_1^{\mathbb{Z}_2}({}^{\mathrm{top}}E_2^{s+1}, \Gamma) \end{array}.$$

Here ${}^{\mathrm{top}}E_2 = \mathrm{Ext}_{\mathsf{BP}_*\mathsf{BP}}(\mathsf{BP}_*, \mathsf{BP}_*)$ *with degrees shifted so that elements appearing in degree $(s, 2m)$ in topology appear in degree $(s, m(1+\alpha))$ motivically.*

Proof. The second statement is an easy consequence of the first via the cobar resolution computing $\mathrm{Ext}_{\mathsf{BPGL}_\star\mathsf{BPGL}}(\mathsf{BPGL}_\star, \mathsf{BPGL}_\star)$ and the universal coefficient theorem.

As a consequence of motivic Landweber exactness, Naumann-Østvær-Spitzweck [NSØ09] deduce a splitting of the MGL Hopf algebroid as

$$(\mathsf{MGL}_\star, \mathsf{MGL}_\star\mathsf{MGL}) = (\mathsf{MU}_*, \mathsf{MU}_*\mathsf{MU}) \otimes_{\mathsf{MU}_*} \mathsf{MGL}_\star$$

where MU_* is the coefficients of (topological) complex cobordism, the Lazard ring. This splitting passes to BPGL, so

$$\begin{aligned}(\mathsf{BPGL}_*, \mathsf{BPGL}_*\mathsf{BPGL}) &= (\mathsf{BP}_*, \mathsf{BP}_*\mathsf{BP}) \otimes_{\mathsf{BP}_*} \mathsf{BPGL}_\star \\ &= (\mathsf{BP}_*, \mathsf{BP}_*\mathsf{BP}) \otimes_{\mathsf{BP}_*} (\mathsf{BP}_* \otimes_{\mathbb{Z}_2} \Gamma) \\ &= (\mathsf{BP}_*, \mathsf{BP}_*\mathsf{BP}) \otimes_{\mathbb{Z}_2} \Gamma.\end{aligned}$$

□

This description of the E_2-term of the motivic ANSS over F already pays dividends in the form of a graded algebra with infinitely many nonzero components previously undiscovered in the stable stems $\pi_\star \mathbb{1}\,\widehat{_2}$ of the 2-complete sphere spectrum.

Theorem 5.13. *The algebra Γ survives to E_∞ of the motivic Adams-Novikov spectral sequence and represents a copy of $(H\mathbb{Z}_2)_\star$ in $\pi_\star \mathbb{1}\,\widehat{_2}$.*

Proof. The elements of $\Gamma = E_2^{0,0} \otimes \Gamma$ are in filtration 0 and hence are not the targets of differentials. We must show that elements of Γ do not support differentials. For an element of degree $(s, m+n\alpha)$ in E_2, call $m+n-s$ the classical Adams degree. Since Γ is concentrated in classical Adams degrees $0, -1$, and -2, Theorem 5.12 gives a vanishing line $E_2^{s,m+n\alpha} = 0$ for $s > m+n-s+2$. Differentials in the motivic ANSS decrease classical Adams degree by 1 and increase homological degree by at least 2. Since Γ has $s = 0$ and classical Adams degree $0, -1$, and -2, we see that it does not support differentials. □

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