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MULTIPLICATIONS ON THE MOORE SPECTRUM

Dedicated to Professor Minoru Nakaoka on his sixtieth birthday

By

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Statements of Results

The Moore spectrum studied here is the suspension spectrum $\boldsymbol{M}_q=\{M_q^n\}$ which consists of, as the n-th space for $\boldsymbol{M}_q$, the Moore spaces $M_q^n=S^n \cup e^{n+1}$ with attaching map which is of degree q. It has non-trivial homology group $\mathbf{Z}/q$ only in dimension 0. For a given homology theory, the theory with coefficient group $\mathbf{Z}/q$ is given by forming smash product with $\boldsymbol{M}_q$, cf. [1], [7].

It is a classical result of M. Barratt [3] that the order of $1_{\boldsymbol{M}}$, the stable class of the identity map of $\boldsymbol{M}_q$, is q when $q\not\equiv 2$ mod 4, and $2q$ otherwise. Also, $q1_{\boldsymbol{M}}=0$ is equivalent to the existence of a multiplication μ_q: $\boldsymbol{M}_q\wedge\boldsymbol{M}_q\to\boldsymbol{M}_q$ with unit i_q: $\boldsymbol{S}\to\boldsymbol{M}_q$, that is,

$$\mu_q(i_q\wedge 1_{\boldsymbol{M}})=1_{\boldsymbol{M}},\quad \mu_q(1_{\boldsymbol{M}}\wedge i_q)=1_{\boldsymbol{M}},$$

where $\boldsymbol{S}=\{S^n\}$ is the sphere spectrum and i_q is the inclusion map $S^n\to M_q^n$ at n-th level. The multiplicative structure on $\boldsymbol{M}_q$ is usuful in studying the structure of the graded ring $\boldsymbol{M}_q{}^*(\boldsymbol{M}_q)$ which consists of stable self-maps of $\boldsymbol{M}_q$, cf. [6], [4], [5], [8]. It is easy to see that μ_q is commutative and associative if q is relatively prime to 6, [6]. B. Gray proved in [4] that μ_q is not commutative when $q=4$ and H. Toda proved in [11] that μ_q, q odd, is not associative if and only if $q\equiv 0$ mod 3 and $\not\equiv 0$ mod 9. B. Gray also stated in [4, Th. 13], [5, Prop. 3.3] without proof that $\boldsymbol{M}_q$ has an associatvie multiplication if and only if $q\not\equiv 3$ mod 9*) (at least when q is odd).

The purpose of this note is to give a complete proof for the question wheather or not $\boldsymbol{M}_q$ admits a commutative/asscociative multiplication. Since we are working in the stable homotopy category, a map is identified with its stable

*) Appearently the statement is not true when $q\equiv 6$ mod 9, for example, $\boldsymbol{M}_{15}=\boldsymbol{M}_3\vee\boldsymbol{M}_5$ cannot be associative because of the non-associativity of $\boldsymbol{M}_3$.

homotopy class; "commutative" and "associative" mean homotopy commutative and homotopy associative.

Let δ_q: $\boldsymbol{M}_q \to \Sigma \boldsymbol{M}_q$ be the coboundary map, where Σ denotes the suspension functor.

DEFINITION 1. *A multiplication μ_q on $\boldsymbol{M}_q$ is said to be regular if*

$$\mu_q(\delta \wedge 1_{\boldsymbol{M}}) + \mu_q(1_{\boldsymbol{M}} \wedge \delta_q) = \delta_q \mu_q \quad in \quad [\boldsymbol{M}_q \wedge \boldsymbol{M}_q, \Sigma \boldsymbol{M}_q],$$

where the square brackets denote the additive group consisting of stable maps.

If we regard δ_q as a cohomology operation on a multiplicative cohomology theory with coefficient $\boldsymbol{Z}/q$ [1], the regularity means that the cohomology operation δ_q behaves as a derivation. Our main result is then stated as follows:

THEOREM 2. (a) *$\boldsymbol{M}_q$ has a multiplication if and only if $q \not\equiv 2 \bmod 4$, whence the number of (homotopy classes of) multiplications is 4 when $q \equiv 0 \bmod 4$, and 1 when q is odd.*

(b) *Suppose $q \not\equiv 2 \bmod 4$. Then $\boldsymbol{M}_q$ has always a regular multiplication. When $q \equiv 0 \bmod 4$, the number of regular multiplications on $\boldsymbol{M}_q$ is 2.*

(c) *$\boldsymbol{M}_q$ has a commutative multiplication if and only if $q \equiv 0 \bmod 8$ or q is odd, whence all the multiplications (including non-regular ones) are commutative.*

(c′) *Suppose $q \equiv 0 \bmod 4$. Let T: $\boldsymbol{M}_q \wedge \boldsymbol{M}_q \to \boldsymbol{M}_q \wedge \boldsymbol{M}_q$ be the map switch ing factors, j_q: $\boldsymbol{M}_q \to \Sigma \boldsymbol{S}$ the map represented by the projection $M_q^n \to S^{n+1}$, and η^2: $\Sigma^2 \boldsymbol{S} \to \boldsymbol{S}$ the generator of $\pi_2^S = [\Sigma^2 \boldsymbol{S}, \boldsymbol{S}] = \boldsymbol{Z}/2$. Then, for any μ_q,*

$$\mu_q T = \mu_q + (q/4) i_q \eta^2 (j_q \wedge j_q)$$

and $i_q \eta^2 (j_q \wedge j_q)$ is of order 2.

(d) *$\boldsymbol{M}_q$ has an associative multiplication if and only if $q \not\equiv 2 \bmod 4$ and $q \not\equiv \pm 3 \bmod 9$, whence all the multiplications are associative.*

(d′) *Suppose $q \not\equiv 2 \bmod 4$ and $q \equiv 0 \bmod 3$. Let α be a generator of the 3-primary part of $\pi_3^S = [\Sigma^3 \boldsymbol{S}, \boldsymbol{S}] = \boldsymbol{Z}/24$. Then, for any μ_q,*

$$\mu_q(\mu_q \wedge 1_{\boldsymbol{M}}) = \mu_q(1_{\boldsymbol{M}} \wedge \mu_q) + (q/3) i_q \alpha (j_q \wedge j_q \wedge j_q)$$

and the element $i_q \alpha (j_q \wedge j_q \wedge j_q)$ is of order 3.

Constructing multilication starts with pre-multiplication, which is a left $\boldsymbol{M}_q$-module multiplication on $\boldsymbol{M}_q$ (not necessarily associative), that is, which has i_q as left unit.

DEFINITION 3. *A map* μ_q: $M_q \wedge M_q \to M_q$ *is said to be a pre-multiplication if* $\mu_q(i_q \wedge 1_M) = 1_M$ *in* $[M_q, M_q]$.

Appearently, a pre-multiplication μ_q is a multiplication if and only if $\mu_q(1_M \wedge i_q) = 1_M$. The regularity and the associativity of a pre-multiplication are defined in the same way as the case of multiplication. If q is odd, the premultiplication is still unique, and if $q \equiv 2 \bmod 4$, the non-existence still holds. If $q \equiv 0 \bmod 4$, there are just eight pre-multiplications. However, no more associativity can arise.

THEOREM 2. (d″) *Suppose* $q \equiv 0 \bmod 4$. *A pre-multiplication which is not a multiplication is always non-associative.*

In [9], we have developped the theory of pre-multiplications on arbitrary (finite) suspension spectra, which will be usuful in our discussion, in particular, in discussing assiciativity. In case $q \equiv 2 \bmod 4$, $2q1_M = 0$ implies the existence of an M_{2q} pre-multiplication m_q: $M_{2q} \wedge M_q \to M_q$, i.e., a map m_q with $m_q(i_{2q} \wedge 1_{M_q}) = 1_{M_q}$. M_q is said to be an associative M_{2q}-module spectrum, if it admits an M_{2q}-pre-multiplication m_q which is associative with respect to some pre-multiplication μ_{2q} on M_{2q} in the usual sense: $m_q(1_{M_{2q}} \wedge m_q) = m_q(\mu_{2q} \wedge 1_{M_q})$.

THEOREM 4. *Suppose* $q \equiv 2 \bmod 4$. *If* $q \equiv \pm 3 \bmod 9$, *then* M_q *can not be an associative* M_{2q}*-module spectrum. If* $q \not\equiv \pm 3 \bmod 9$, *then there are exactly six pairs* (μ_{2q}, m_q) *of pre-multiplications which satisfy*

$$m_q(1_{M_{2q}} \wedge m_q) = m_q(\mu_{2q} \wedge 1_{M_q}) + a'_q i_q \nu j_q (j_{2q} \wedge j_{2q} \wedge 1_{M_q})$$

for some $a'_q \in \mathbf{Z}/2$ *not depending on the pair* (μ_{2q}, m_q), *where* ν *is a generator of the 2-primary part of* π_3^S. *The* μ_{2q}*'s of these pairs are associative, regular multiplications, and none of the other pairs of pre-multiplications satisfies the associativity.*

Let r be a positive divisor of q. The commutative square

$$\begin{array}{ccc} S & \xrightarrow{\;q\;} & S \\ \downarrow{\scriptstyle q/r} & & \| \\ S & \xrightarrow{\;r\;} & S, \end{array}$$

where n: $S \to S$ is the map of degree n, then determines a map

$$\rho_{q,r}: M_q \longrightarrow M_r,$$

which is of degree 1 on the bottom cells, i.e., $\rho_{q,r}i_q=i_r$, and of q/r on the top cells. Since the commutative square holds at the level of honest maps with no homotopy, there can be a canonical choice of $\rho_{q,r}$. Especially $\rho_{q,q}=1_{M_q}$, $\rho_{r,s}\rho_{q,r}=\rho_{q,s}$: $M_q \to M_r \to M_s$ for a divisor s of r. Let μ_q, μ_r be pre-multiplications on M_q, M_r. Then $\rho_{q,r}$ is said to be multiplicative with respect to μ_q, μ_r, if

$$\mu_r(\rho_{q,r}\wedge\rho_{q,r})=\rho_{q,r}\mu_q \quad \text{in} \quad [M_q\wedge M_q, M_r],$$

in other words, $\rho_{q,r}$: $(M_q, \mu_q)\to(M_r, \mu_r)$ is a map of ring spectra (when μ_q and μ_r are multiplications). The following lemma is a key result in proving the main theorem.

LEMMA 5. *Let r be a positive divisor of q and suppose $q, r\not\equiv 2 \bmod 4$. For a given regular multiplication μ_r on M_r, there exists a regular multiplication μ_q on M_q so that $\rho_{q,r}$ is multiplicative with respect to μ_q, μ_r.*

This paper is organized as follows: in §1 we prove the parts (a), (b) of Theorem 2. The key lemma, Lemma 5, is proved in §2. The obstruction to the commutativity is analized in §3 and the parts (c), (c') are proved. In §4, the obstruction to the associativity is analized and, in §§4–5, the parts (d), (d'), (d'') of Theorem 2 are proved. The M_{2q}-module structure on M_q with $q\equiv 2 \bmod 4$ is discussed in §6.

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§1. Enumerating multiplications

We consider the usual cofibration

$$S \xrightarrow{i_q} M_q \xrightarrow{j_q} \Sigma S. \tag{6}$$

Then the coboundary δ_q is the composite

$$\delta_q=i_q j_q\colon M_q \longrightarrow \Sigma S \longrightarrow \Sigma M_q.$$

Let r be a divisor of q. As mentioned before, the map $\rho_{q,r}$ satisfies

$$\rho_{q,r}i_q=i_r,\quad j_r\rho_{q,r}=(q/r)j_q,\quad \rho_{q,q}=1_M.$$

In the same manner as ρ, there is canonically defined map

$$\lambda_{r,q}\colon M_r \longrightarrow M_q$$

which satisfies

$$\lambda_{r,q} i_r = (q/r) i_q, \quad j_q \lambda_{r,q} = j_r, \quad \lambda_{q,q} = 1_{\boldsymbol{M}}.$$

Let $\eta\colon \Sigma \boldsymbol{S} \to \boldsymbol{S}$ be the stable class of the Hopf map $S^3 \to S^2$ which is the generator of $\pi_1^S = [\Sigma \boldsymbol{S}, \boldsymbol{S}] = \mathbf{Z}/2$. Since $2\eta = 0$, there exist elements

$$\bar{\eta}_2\colon \Sigma \boldsymbol{M}_2 \longrightarrow \boldsymbol{S}, \quad \tilde{\eta}_2\colon \Sigma^2 \boldsymbol{S} \longrightarrow \boldsymbol{M}_2$$

such that

$$\bar{\eta}_2 i_2 = \eta, \quad j_2 \tilde{\eta}_2 = \eta.$$

When q is even, we define the elements

$$\bar{\eta}_q\colon \Sigma \boldsymbol{M}_q \longrightarrow \boldsymbol{S}, \quad \tilde{\eta}_q\colon \Sigma^2 \boldsymbol{S} \longrightarrow \boldsymbol{M}_q$$

to be the composites

$$\bar{\eta}_q = \bar{\eta}_2 \rho_{q,2}, \quad \tilde{\eta}_q = \lambda_{2,q} \tilde{\eta}_2,$$

which satisfy the same relations

$$\bar{\eta}_q i_q = \eta, \quad j_q \tilde{\eta}_q = \eta.$$

The elements $\bar{\eta}_q$ and $\tilde{\eta}_q$ are of order 2 when $q \equiv 0 \mod 4$, and of order 4 when $q \equiv 2 \mod 4$ [3], [1]. We shall quote, from [1] (originally due to [3]), the following computational result, which is obtained by the standard method using cofibration (6) together with the following results on stable homotopy groups of spheres:

$$\pi_1^S = \mathbf{Z}/2\{\eta\}, \quad \pi_2^S = \mathbf{Z}/2\{\eta^2\},$$

where $\eta^2 = \eta\eta = \{2, \eta, 2\}$ (Toda bracket, cf. [10]) and we indicate generators inside $\{\ \}$.

LEMMA 7.

(a) $\pi_0(\boldsymbol{M}_q) = \boldsymbol{Z}/q\{i_q\}$.

$$\pi_1(\boldsymbol{M}_q) = \begin{cases} 0 & \text{for } q \text{ odd}, \\ \mathbf{Z}/2\{i_q\eta\} & \text{for } q \text{ even}, \end{cases} \quad \pi_2(\boldsymbol{M}_q) = \begin{cases} 0 & \text{for } q \text{ odd}, \\ \mathbf{Z}/4\{\tilde{\eta}_q\} & \text{for } q \equiv 2 \mod 4, \\ \mathbf{Z}/2\{\tilde{\eta}_q\} \oplus \mathbf{Z}/2\{i_q\eta^2\} & \\ \quad \text{for } q \equiv 0 \mod 4. \end{cases}$$

(b) *Let r be a divisor of q, and write* $\lambda=\lambda_{r,q}$, $\rho=\rho_{q,r}$.

$$[M_r, M_q]=\begin{cases} \mathbf{Z}/r\{\lambda\} & \text{for } r\equiv 1 \bmod 2,\\ \mathbf{Z}/2r\{\lambda\} & \text{for } r\equiv q\equiv 2 \bmod 4,\\ \mathbf{Z}/r\{\lambda\}\oplus\mathbf{Z}/2\{i_q\eta j_r\} & \text{for } r\equiv 0 \bmod 2,\ q\equiv 0 \bmod 4.\end{cases}$$

$$[M_q, M_r]=\begin{cases} \mathbf{Z}/r\{\rho\} & \text{for } r\equiv 1 \bmod 2,\\ \mathbf{Z}/2r\{\rho\} & \text{for } r\equiv q\equiv 2 \bmod 4,\\ \mathbf{Z}/r\{\rho\}\oplus\mathbf{Z}/2\{i_r\eta j_q\} & \text{for } r\equiv 0 \bmod 2,\ q\equiv 0 \bmod 4.\end{cases}$$

(c)

$$[\Sigma M_q, M_q]=\begin{cases} 0 & \text{for } q \text{ odd},\\ \mathbf{Z}/2\{i_q\bar{\eta}_q\}\oplus\mathbf{Z}/2\{\tilde{\eta}_q j_q\} & \text{for } q\equiv 2 \bmod 4,\\ \mathbf{Z}/2\{i_q\bar{\eta}_q\}\oplus\mathbf{Z}/2\{\tilde{\eta}_q j_q\}\oplus \mathbf{Z}/2\{i_q\eta^2 j_q\} & \text{for } q\equiv 0 \bmod 4.\end{cases}$$

Form the smash product of (6) with M_q to get the cofibration

$$M_q \xrightarrow{i_q\wedge 1_M} M_q\wedge M_q \xrightarrow{j_q\wedge 1_M} \Sigma M_q,$$

which is induced by $q1_M$.

Suppose $q\not\equiv 2 \bmod 4$. Then $q1_M=0$ by Lemma 7, (b), and the cofibration is trivial. The trivialization is given by maps

$$\mu_q\colon M_q\wedge M_q \longrightarrow M_q,\quad \hat{\mu}_q\colon \Sigma M_q \longrightarrow M_q\wedge M_q,$$

which satisfy the relations

(8–a) $\mu_q(i_q\wedge 1_M)=1_M$, i.e., μ_q *is a pre-multiplication.*

(8–b) $(j_q\wedge 1_M)\hat{\mu}_q=1_M$.

(8–c) $\mu_q\hat{\mu}_q=0$.

(8–d) $(i_q\wedge 1_M)\mu_q+\hat{\mu}_q(j_q\wedge 1_M)=1_{M\wedge M}$.

These relations determine uniquely $\hat{\mu}_q$ for a given pre-multiplication μ_q [9]. Henceforward, the index q in μ_q, $\hat{\mu}_q$ will be frequently dropped; the discussions with μ varied and q fixed are much more than those with q varied, which are Lemma 5, its analogue in case $q\equiv 2 \bmod 4$, and their consequences.

We have proved in [9, Th. 1.3] that, for given pre-multiplications μ, μ', there is a difference element

$$d(\mu', \mu) \in [\Sigma M_q, M_q]$$

uniquely associated to μ, μ' in such a way that

$$\mu' = \mu + d(\mu', \mu)(j_q \wedge 1_M), \quad \hat{\mu}' = \hat{\mu} - (i_q \wedge 1_M) d(\mu', \mu).$$

The correspondence $\mu' \mapsto d(\mu', \mu)$ with μ fixed is a bijection between the set of pre-multiplications and $[\Sigma M_q, M_q]$. Enumerating pre-multiplications as stated above Theorem 2, (d'') is then immediate from Lemma 7, (c).

Let μ be any pre-multiplication. We have $\mu(1_M \wedge i_q) i_q = i_q$, hence, by Lemma 7, (a), (b),

$$\mu(1_M \wedge i_q) = \begin{cases} 1_M & \text{for } q \text{ odd}, \\ 1_M + x i_q \eta j_q, \ x \in \mathbf{Z}/2, & \text{for } q \equiv 0 \bmod 4. \end{cases}$$

Therefore μ is a multiplication which is unique, when q is odd. When $q \equiv 0$ mod 4, the pre-multiplication μ' with $d(\mu', \mu) = x i_q \bar{\eta}_q$ becomes a multiplication, hence the multiplication always exists. The difference element

$$\bar{d}(\mu', \mu) \in \pi_2(M_q)$$

is then associated to given two multiplications μ, μ' in such a way that

$$d(\mu', \mu) = \bar{d}(\mu', \mu) j_q \in \mathbf{Z}/2\{\tilde{\eta}_q j_q\} \oplus \mathbf{Z}/2\{i_q \eta^2 j_q\},$$

which proves the part (a) *of Theorem* 2.

We shall give an equivalent condition for the regularity.

LEMMA 9. *A pre-multiplication μ is regular if and only if*

$$(1_M \wedge j_q)\hat{\mu} = \begin{cases} -1_M & \textit{when } \mu \textit{ is a multiplication}, \\ -1_M + i_q \eta j_q & \textit{otherwise}. \end{cases}$$

PROOF. We may put $\mu(1_M \wedge i_q) = 1_M + x i_q \eta j_q$, $x \in \mathbf{Z}/2$ ($x = 0$ if q is odd). Then we have

$$(*) \qquad \mu(\delta_q \wedge 1_M + 1_M \wedge \delta_q) = j_q \wedge 1_M + 1_M \wedge j_q + x i_q \eta (j_q \wedge j_q).$$

By (8–c), (8–d), the regularlity is equivalent to

$$\delta_q=\mu(\delta_q\wedge 1_M+1_M\wedge\delta_q)(i_q\wedge 1_M),$$

$$0=\mu(\delta_q\wedge 1_M+1_M\wedge\delta_q)\hat{\mu}.$$

The first condition is always valid. By (*) and (8–b), the second is equivalent to

$$0=1_M+(1_M\wedge j_q)\hat{\mu}+xi_q\eta j_q,$$

hence the lemma. Q. E. D.

Since the stable element j_q is of odd degree, the formula $(\alpha\wedge\beta)(\alpha'\wedge\beta')=(-1)^{\deg\beta\deg\alpha'}\alpha\alpha'\wedge\beta\beta'$ of smash product implies that

$$j_q(1_M\wedge j_q)=j_q\wedge j_q=-j_q(j\wedge 1_M). \tag{10}$$

By Lemma 7, (b) with $r=q$, and (10),

$$(1_M\wedge j_q)\mu=-1_M+yi_q\eta j_q,\quad y\in\mathbf{Z}/2,$$

where $y=0$, hence a regular multiplication exists and unique, for q odd. Suppose $q\equiv 0 \mod 4$. Then a pre-multiplication μ' with $d(\mu',\mu)=(x+y)\tilde{\eta}_q j_q$, where x is the same coefficient as above proof, is shown to be regular. Moreover there are exactly four regular pre-multiplications, which correspond with

$$\mathbf{Z}/2\{i_q\tilde{\eta}_q\}\oplus\mathbf{Z}/2\{i_q\eta^2 j_q\},$$

and there are exactly two regular multiplications, which correspond with

$$\mathbf{Z}/2\{i_q\eta^2 j_q\},$$

proving the part (b) *of Theorem* 2.

Ending the section, we shall restate the enumeration we have made, as follows:

PROPOSITION 11. *Suppose* $q\equiv 0 \mod 4$. *Let* μ *be a multiplication. Then,* μ *is regular if and only if* $\mu'=\mu+i_q\eta^2(j_q\wedge j_q)$ *is regular. One of* μ *and* $\mu''=\mu+\tilde{\eta}_q(j_q\wedge j_q)$ *is regular and other is not. Write* $\mu'''=(\mu')''=(\mu'')'$, $\mu_{\text{pre}}=\mu+i_q\tilde{\eta}_q(j_q\wedge 1_M)$. *Then the set of pre-multiplications is*

$$\{\mu,\mu',\mu'',\mu''',\mu_{\text{pre}},\mu'_{\text{pre}},\mu''_{\text{pre}},\mu'''_{\text{pre}}\},$$

and the first four elements form the set of multiplications.

§2. The key lemma

Throughout this section, r will be a divisor of q and $p=q/r$. We consider the cofibration

$$(12) \qquad M_p \xrightarrow{\lambda} M_q \xrightarrow{\rho} M_r,\ \lambda=\lambda_{p,q},\ \rho=\rho_{q,p},$$

which is induced by

$$\delta'=i_p j_r\colon M_r \longrightarrow \Sigma M_p.$$

We put

$$\delta''=i_r j_q\colon M_q \longrightarrow \Sigma M_r$$

and $\delta_r=i_r j_r$ as before. Then we have

(13–a) $\qquad \rho_{r,p}\delta_r=\delta'$ *if* q *divides* r^2,

(13–b) $\qquad \delta_r\rho=p\delta''.$

PROOF OF LEMMA 5. Suppose $(p, r)=1$. Then $\delta'=0$ and hence M_q is homotopy equivalent to $M_p\vee M_r$. Let $\lambda'=\lambda_{r,q}\colon M_r\to M_q$, $\rho'=\rho_{q,p}\colon M_q\to M_p$. Then λ, ρ, λ' and ρ' give a splitting of M_q, because, by [2, Appendix, Th. 12.2],

$$\rho'\lambda=\begin{cases} r1_{M_q} & \text{for } p\not\equiv 2 \bmod 4,\\ 1_{M_q} & \text{for } p\equiv 2 \bmod 4,\end{cases}$$

$$\lambda'\rho=\begin{cases} p1_{M_r} & \text{for } r\not\equiv 2 \bmod 4,\\ 1_{M_r} & \text{for } r\equiv 2 \bmod 4.\end{cases}$$

For given multiplications μ_p on M_p and μ_r on M_r (since $(p, r)=1$, the assumption q, $r\not\equiv 2 \bmod 4$ implies $p\not\equiv 2 \bmod 4$, hence μ_p exists),

$$(M_p\vee M_r)\wedge(M_p\vee M_r)=(M_p\wedge M_p)\vee(M_p\wedge M_r)\vee(M_r\wedge M_p)\vee(M_r\wedge M_r)$$
$$\xrightarrow{\quad(\mu_p,0,0,\mu_r)\quad} M_p\vee M_r$$

defines a multiplication μ_q on M_q for which ρ is multiplicative. The element δ_q corresponds with $\delta_p\vee\delta_r$ via splitting, hence, if μ_p is also regular, so is μ_q. We may therefore reduce the lemma to the case when q is a prime power. Moreover, since $\rho_{b,c}\rho_{a,b}=\rho_{a,c}$, we may assume that p is a prime and r is a power of p. Then $q=pr$ divides r^2 and (13–a) is applicable.

By (13–a) and (13–b),

$$\delta'\mu_r(\rho\wedge\rho)=\rho_{r,p}\delta_r\mu_r(\rho\wedge\rho)=p\rho_{r,p}\mu_r(\delta''\wedge\rho)+p\rho_{r,p}\mu_r(\rho\wedge\delta'').$$

Since $p\rho_{r,p}=0$ (by Lemma 7, (b)), we have $\delta'\mu_r(\rho\wedge\rho)=0$. The exact sequence given by (12) then yields an element μ_q: $\boldsymbol{M}_q\wedge\boldsymbol{M}_q\to\boldsymbol{M}_q$ with the property

$$\mu_r(\rho\wedge\rho)=\rho\mu_q,$$

which implies the relation

$$\rho\mu_q(i_q\wedge 1_{\boldsymbol{M}})=\rho=\rho\mu_q(1_{\boldsymbol{M}}\wedge i_q).$$

An easy diagram chasing leads that there is a choice of μ_q which is a premultiplication. As was shown before, if p is odd, it is a regular multiplication.

Suppose $p=2$. Then $\mu_q(1_{\boldsymbol{M}}\wedge i_q)=1_{\boldsymbol{M}}+xi_q\eta j_q$, $x\in\mathbf{Z}/2$. The relation $\rho\mu_q(1_{\boldsymbol{M}}\wedge i_q)=\rho$ implies $x=0$. Thus, μ_q is a multiplication. The multiplication μ_q'' of Proposition 11 also makes ρ multiplicative because $\rho\tilde{\eta}_q=\rho\lambda\tilde{\eta}_2=0$. As mentioned in Proposition 11, either of μ_q, μ_q'' is regular. Q.E.D.

We mention that, if both of q, r are even (or odd), μ_q in Lemma 5 is unique.

§3. Commutative multiplications

Let T: $\boldsymbol{M}_q\wedge\boldsymbol{M}_q\to\boldsymbol{M}_q\wedge\boldsymbol{M}_q$ be the map switching the factors. For a multiplication μ, μT is also a multiplication; it is regular if μ is. Therefore if q is odd, $\mu T=\mu$, i.e., μ is commutative by the uniqueness. If $q\equiv 0 \bmod 4$,

$$\mu T=\mu+\bar{\gamma}(\mu)(j_q\wedge j_q)\quad\text{for some}\quad\bar{\gamma}(\mu)(=\bar{d}(\mu T,\mu))\in\mathbf{Z}/2\{i_q\eta^2\}.$$

Since $(j_q\wedge j_q)T=-(j_q\wedge j_q)$ and $2\bar{\gamma}(\mu)=0$, the element $\bar{\gamma}(\mu)$ is independent of the chice of multiplication μ. We call it $\bar{\gamma}_q$ and put $\gamma_q=\bar{\gamma}_q j_q$.

PROPOSITION 14. *If q is odd, the multiplication is commutative. If $q\equiv 0$* mod 4, *there is an element $\gamma_q\in\mathbf{Z}/2\{i_q\eta^2 j_q\}$ such that*

$$\mu T=\mu+\gamma_q(j_q\wedge 1_{\boldsymbol{M}})\quad\textit{for all multiplications.}$$

COROLLARY 15. *If a multiplication on $\boldsymbol{M}_q$ is commutative, all the multiplications on $\boldsymbol{M}_q$ are commutative.*

Suppose $q \equiv 0 \bmod 4$. We may put $\gamma_q = c_q i_q \eta^2 j_q$, $c_q \in \mathbf{Z}/2$. Let μ_4 and μ_q be regular multiplications on $\boldsymbol{M}_4$ and $\boldsymbol{M}_q$, respectively. Then

$$\mu_q T = \mu_q + c_q i_q \eta^2 (j_q \wedge j_q), \quad \mu_4 T = \mu_4 + c_4 i_4 \eta^2 (j_4 \wedge j_4).$$

Put $\rho = \rho_{q,4}$. By Lemma 5,

$$\begin{aligned} c_q i_4 \eta^2 (j_q \wedge j_q) &= \rho \mu_q T - \rho \mu_q \\ &= \mu_4 (\rho \wedge \rho) T - \mu_4 (\rho \wedge \rho) \\ &= (\mu_4 T - \mu_4)(\rho \wedge \rho) \\ &= (q/4)^2 c_4 i_4 \eta^2 (j_q \wedge j_q). \end{aligned}$$

A similar computation as in Lemma 7, (c) shows that $i_4 \eta^2 j_q \neq 0$. Hence,

$$c_q \equiv (q/4)^2 c_4 \equiv (q/4) c_4 \bmod 2.$$

B. Gray [4, Th. 10] proved that $c_4 \neq 0$. Therefore $c_q \equiv q/4 \bmod 2$, *proving the parts* (c) *and* (c′) *of Theorem* 2.

Finally we mention that

(16) *If* $q \equiv 0 \bmod 4$, $T\hat{\mu} = -\hat{\mu} + (q/4)(i_q \wedge i_q)\eta^2 j_q$ *for any multiplication* μ, because $(\mu T)^{\wedge} = -T\hat{\mu}$.

§4. Obstructions to the assiciativity

LEMMA 17. *Let* μ *be a pre-multiplication on* $\boldsymbol{M}_q$.

(a)
$$\delta_q \mu = \begin{cases} j_q \wedge 1_M + 1_M \wedge j_q & \text{if } \mu \text{ is regular,} \\ j_q \wedge 1_M + 1_M \wedge j_q + i_q \eta (j_q \wedge j_q) & \text{otherwise.} \end{cases}$$

(b)
$$\hat{\mu} \delta_q = \begin{cases} 1_M \wedge i_q - i_q \wedge 1_M & \text{if } \mu \text{ is a multiplication,} \\ 1_M \wedge i_q - i_q \wedge 1_M + (i_q \wedge i_q)\eta j_q & \text{otherwise.} \end{cases}$$

(c)
$$(j_q \wedge 1_M)(\mu \wedge 1_M)(1 \wedge \hat{\mu})\hat{\mu} = \begin{cases} 0 & \text{if } \mu \text{ is regular,} \\ \eta \wedge 1_M & \text{otherwise.} \end{cases}$$

(d)
$$\mu(1_M \wedge \hat{\mu})(\hat{\mu} \wedge 1_M)(i_q \wedge 1_M) = \begin{cases} 0 & \text{if } \mu \text{ is a multiplication,} \\ \eta \wedge 1_M & \text{otherwise.} \end{cases}$$

(e) $$\mu(\mu \wedge 1_M)(1_M \wedge \hat{\mu})\hat{\mu} = \mu(1_M \wedge \mu)(\hat{\mu} \wedge 1_M)\hat{\mu}.$$

PROOF. We may put

$$\mu(1_M \wedge i_q) = 1_M + x i_q \eta j_q, \quad (1_M \wedge j_q)\mu = -1_M + y i_q \eta j_q,$$

where $x, y \in \mathbf{Z}/2$ ($x = y = 0$ if q is odd), $x = 0$ iff μ is a multiplication, and $y = 0$ iff μ is regular. Then,

$$\mu(1_M \wedge i_q - i_q \wedge 1_M + x(i_q \wedge i_q)\eta j_q) = 0,$$
$$(1_M \wedge j_q + j_q \wedge 1_M + y i_q \eta (j_q \wedge j_q))\hat{\mu} = 0.$$

By (8–c), (8–d),

$$\begin{aligned} 1_M \wedge i_q - i_q \wedge 1_M + x(i_q \wedge i_q)\eta j_q &= \hat{\mu}(j_q \wedge 1_M)(1_M \wedge i_q - i_q \wedge 1_M + x(i_q \wedge i_q)\eta j_q) \\ &= \hat{\mu}\delta_q, \end{aligned}$$
$$\begin{aligned} 1_M \wedge j_q + j_q \wedge 1_M + y i_q \eta (j_q \wedge j_q) &= (1_M \wedge j_q + j_q \wedge 1_M + y i_q \eta (j_q \wedge j_q))(i_q \wedge 1_M)\mu \\ &= \delta_q \mu, \end{aligned}$$

proving (a) and (b). To get (c),

$$\begin{aligned} (j_q \wedge 1_M)(\mu \wedge 1_M)(1_M \wedge \hat{\mu})\hat{\mu} &= \mu(i_q \wedge 1_M)(j_q \wedge 1_M)(\mu \wedge 1_M)(1_M \wedge \hat{\mu})\hat{\mu} \\ &= \mu(\delta_q \mu \wedge 1_M)(1_M \wedge \hat{\mu})\hat{\mu} \end{aligned}$$

and use (a). Similarly (b) implies (d). By (8–d),

$$\begin{aligned} \mu(1_M \wedge \mu)(1_M \wedge \hat{\mu})\hat{\mu} = \mu(1_M \wedge \mu)&(i_q \wedge 1_M \wedge 1_M)(\mu \wedge 1_M) \\ &+ (\hat{\mu} \wedge 1_M)(j_q \wedge 1_M \wedge 1_M))(1_M \wedge \hat{\mu})\hat{\mu} \end{aligned}$$

The left hand side is 0 (by (8–c)) and the right is the difference

$$\mu(\mu \wedge 1_M)(1_M \wedge \hat{\mu})\hat{\mu} - \mu(1_M \wedge \mu)(\hat{\mu} \wedge 1_M)\hat{\mu},$$

hence (e) is obtained. Q.E.D.

For given pre-multiplication μ, we define an element $\alpha(\mu) \in [\Sigma^2 M_q, M_q]$ to be either side of above (e):

(18) $$\alpha(\mu) = \mu(\mu \wedge 1_M)(1_M \wedge \hat{\mu})\hat{\mu} = \mu(1_M \wedge \mu)(\hat{\mu} \wedge 1_M)\hat{\mu},$$

which will be shown to be the obstruction to the associativity of μ.

LEMMA 19. *If μ is a multiplication, the element $\alpha(\mu)$ is independent of the choice of μ.*

PROOF. We shall show that $\alpha(\mu)=\alpha(\mu^0)$ for multiplications μ and μ^0. We may assume μ is regular. Put $\mu^0=\mu+d(j_q\wedge 1_M)$, $d=\bar{d}j_q$ as in §1. Then

$$di_q=0,\quad -d=d,\qquad j_qd=\begin{cases}0 & \text{if } \mu^0 \text{ is regular,}\\ \eta j_q & \text{otherwise,}\end{cases}\qquad d^2=\begin{cases}0 & \text{if } \mu^0 \text{ is regular,}\\ \bar{d}\eta j_q & \text{otherwise,}\end{cases}$$

from which we have, by computation, (we drop the indices: $1=1_M$, $i=i_q$, $j=j_q$)

$$\mu^0(1\wedge\mu^0)=\mu(1\wedge\mu)+\mu(1\wedge d)(1\wedge j\wedge 1)+d\mu(j\wedge 1\wedge 1)+d^2(j\wedge 1)(1\wedge j\wedge 1),$$

$$(\hat{\mu}^0\wedge 1)\hat{\mu}^0=(\hat{\mu}\wedge 1)\hat{\mu}+(i\wedge 1\wedge 1)(d\wedge 1)\hat{\mu}+(\hat{\mu}\wedge 1)(i\wedge 1)d,$$

and

$$\begin{aligned}\alpha(\mu^0)-\alpha(\mu)&=\mu(d\wedge 1)\hat{\mu}+\mu(1\wedge d)\hat{\mu}+d(j\wedge 1)(d\wedge 1)\hat{\mu}+3d^2\\ &=\mu(d\wedge 1\hat{\mu}+\mu(1\wedge d)\hat{\mu}.\end{aligned}$$

By Proposition 14 and (16),

$$\mu(d\wedge 1)\hat{\mu}=\mu T(1\wedge d)T\hat{\mu}=-\mu(1\wedge d)\hat{\mu}+\gamma_q d+d\gamma_q,$$

where $\gamma_q=(q/4)i\eta^2 j$ for $q\equiv 0 \bmod 4$, $\gamma_q=0$ for q odd. Then, it is enough to show $\gamma_q d+d\gamma_q=0$ for $q\equiv 4 \bmod 8$. The element $\gamma_q d+d\gamma_q$ is a multiple of $i\eta^3 j$, where $\eta^3=\eta\eta\eta$. There is a well known relation $\eta^3=4\nu$, cf. [10], hence η^3 is divisible by q and $i\eta^3 j=0$. Q.E.D.

As was seen in [9, §5] the element $\alpha(\mu)$ is the obstrauction for the multiplication μ to be associative. The following proposition is a version of [9, Th. 5.2].

PROPOSITION 20. *Let μ be a pre-multiplication on M_q.*

$$\mu(1_M\wedge\mu)=\mu(\mu\wedge 1_M)-\alpha(\mu)(j_q\wedge j_q\wedge 1_M)+A(\mu)\mu(j_q\wedge 1_M\wedge 1_M),$$

$$(1_M\wedge\hat{\mu})\hat{\mu}=-(\hat{\mu}\wedge 1_M)\hat{\mu}+(i_q\wedge i_q\wedge 1_M)\alpha(\mu)+(i_q\wedge 1_M\wedge 1_M)\hat{\mu}A'(\mu),$$

where

$$A(\mu)=\begin{cases}0 & \textit{if}\ \ \mu \textit{ is a multiplication,}\\ i_q\tilde{\eta}_q+\tilde{\eta}_q j_q & \textit{otherwise,}\end{cases}$$

$$A'(\mu)=\begin{cases} 0 & \text{if } \mu \text{ is regular,} \\ i_q\bar{\eta}_q+\bar{\eta}_q i_q & \text{otherwise.} \end{cases}$$

PROOF*). We prove the first equality. The second is similar and we omit the proof. We have

$$\begin{aligned}\alpha(\mu)(j\wedge j\wedge 1)&=-\mu(1\wedge\mu)(\hat{\mu}\wedge 1)\hat{\mu}(j\wedge 1)(j\wedge 1\wedge 1) \qquad \text{(by (18), (10))}\\ &=-\mu(1\wedge\mu)(\hat{\mu}\wedge 1)(j\wedge 1\wedge 1)+\mu(1\wedge\mu)(\hat{\mu}\wedge 1)(i\wedge 1)\mu(j\wedge 1\wedge 1)\\ &\qquad \text{(by (8–d))}\\ &=-\mu(1\wedge\mu)(\hat{\mu}\wedge 1)(j\wedge 1\wedge 1)+A(\mu)\mu(j\wedge 1\wedge 1)\\ &\qquad \text{(by Lemma 17, (d))},\end{aligned}$$

where we have used the relation $i_q\bar{\eta}_q+\bar{\eta}_q j_q=\eta\wedge 1_M$ in $[\Sigma M_q, M_q]$, q even [1], and

$$\begin{aligned}\mu(1\wedge\mu)(\hat{\mu}\wedge 1)(j\wedge 1\wedge 1)&=\mu(1\wedge\mu)-\mu(1\wedge\mu)(i\wedge 1\wedge 1)(\mu\wedge 1)\\ &\qquad \text{(by (8–d))}\\ &=\mu(1\wedge\mu)-\mu(\mu\wedge 1). \qquad \text{Q.E.D.}\end{aligned}$$

COROLLARY 21. *A pre-multiplication μ is associative if and only if $\alpha(\mu)=0$ and $A(\mu)=0$; in particular, a pre-multiplication which is not a multiplication is not associative. If a multiplication on M_q is associative, all the multiplications are associative.*

PROOF. The elements $j_q\wedge j_q\wedge 1_M$ and $\mu(j_q\wedge 1_M\wedge 1_M)$ are, together with two other elements $(j_q\wedge 1_M)(\mu\wedge 1_M)$, $\mu(\mu\wedge 1_M)$, the projections of splitting of $M_q\wedge M_q\wedge M_q$ into four copies of M_q. Therefore $-\alpha(\mu)(j_q\wedge j_q\wedge 1_M)+A(\mu)\mu(j_q\wedge 1_M\wedge 1_M)=0$ is equivalent to $\alpha(\mu)=0$ and $A(\mu)=0$. The last statement is immediate from Lemma 19. Q.E.D.

§5. Associative multiplications

According to Corollary 21, we can restrict ourselves the discussion of as-

*) In the proof given in [9, Th. 5.2], there is a mistake in the sign: the equality $a(m_X)(\pi\wedge\pi\wedge 1_X)=\theta(m_X)(\pi\wedge 1_M\wedge 1_X)$ below (5.3) in [9] should be $a(m_X)(\pi\wedge\pi\wedge 1_X)=-\theta(m_X)(\pi\wedge 1_M\wedge 1_X)$, because of the sign in $\pi\wedge\pi\wedge 1_X=-(\pi\wedge 1_X)(\pi\wedge 1_M\wedge 1_X)$ (see (10)). Therefore the first equality in (5.1) should be

$$m_X(1_M\wedge m_X)=m_X(m_M\wedge 1_X)-a(m_X)(\pi\wedge\pi\wedge 1_X).$$

The change of the sign does not affect the discussions in [9].

sociativity to the case of multiplications. We shall first show that the associator $\alpha(\mu)$ is a 3-primary element, that is,

PROPOSITION 22. *If μ is a multiplication*, $3\alpha(\mu)=0$.

We need some computations to get the proposition. We omit the index q in μ_q, $\hat{\mu}_q$, i_q, j_q, etc, and write $1=1_M$. By Lemma 19, we may assume that μ is regular. As before, let T be the twisting map of $M_q\wedge M_q$ and γ be the commutator: $\gamma=i\eta^2 j$ for $q\equiv 4 \bmod 8$, and $\gamma=0$ otherwise.

LEMMA 23. (a) $(i\wedge 1\wedge 1)(1\wedge\mu)(T\wedge 1)+(T\wedge 1)(1\wedge\hat{\mu})(j\wedge 1\wedge 1)=1\wedge 1\wedge 1$.
(b) $(1\wedge\mu)(T\wedge 1)(1\wedge T)=T(\mu\wedge 1)$.
(c) $(\mu\wedge 1)(1\wedge\hat{\mu})\hat{\mu}=(i\wedge 1)\alpha(\mu)$, $\mu(\mu\wedge 1)(1\wedge\hat{\mu})=\alpha(\mu)(j\wedge 1)$.
(d) $\mu(1\wedge\mu)(T\wedge 1)(1\wedge\hat{\mu})=2\alpha(\mu)(j\wedge 1)+\mu(\gamma\wedge 1)\hat{\mu}(j\wedge 1)$.

PROOF. (a) Since $T^2=1$, it is enough to show

$$(T\wedge 1)(i\wedge 1\wedge 1)(1\wedge\mu)+(1\wedge\hat{\mu})(j\wedge 1\wedge 1)(T\wedge 1)=1\wedge 1\wedge 1.$$

The left hand side is equal to

$$(1\wedge i\wedge 1)(1\wedge\mu)+(1\wedge\hat{\mu})(1\wedge j\wedge 1)=1\wedge((i\wedge 1)\mu+\hat{\mu}(j\wedge 1)),$$

(b) Easy.
(c) By Lemma 17, (c), (18), (8–c) and (8–d).
(d) By Proposition 20, (10), Proposition 14 and Lemma 23, (c). Q. E. D.

PROOF OF PROPOSITION 22. By Proposition 14 and (8–c), $\gamma=\mu T\hat{\mu}$. We compute the element $D=\mu(1\wedge\gamma)\hat{\mu}=\mu(1\wedge\mu)(1\wedge T)(1\wedge\hat{\mu})\hat{\mu}$:

$$D=\mu(1\wedge\mu)(i\wedge 1\wedge 1)(1\wedge\mu)(T\wedge 1)(1\wedge T\hat{\mu})\hat{\mu}$$
$$+\mu(1\wedge\mu)(T\wedge 1)(1\wedge\hat{\mu})(j\wedge 1\wedge 1)(1\wedge T\hat{\mu})\hat{\mu} \qquad \text{(by Lemma 23, (a))}$$
$$=\mu(1\wedge\mu)(T\wedge 1)(1\wedge T)(1\wedge\hat{\mu})\hat{\mu}-\mu(1\wedge\mu)(T\wedge 1)(1\wedge\hat{\mu})T\hat{\mu} \quad \text{(by (8–a), (8–b))}$$
$$=\mu T(\mu\wedge 1)(1\wedge\hat{\mu})\hat{\mu}-2\alpha(\mu)(j\wedge 1)T\hat{\mu}-\mu(\gamma\wedge 1)\hat{\mu}(j\wedge 1)T\hat{\mu}$$
$$\text{(by Lemma 23, (b), (d))}$$
$$=\mu T(i\wedge 1)\alpha(\mu)+2\alpha(\mu)+\mu(\gamma\wedge 1)\hat{\mu} \qquad \text{(by Lemma 23, (c) and the regularity)}$$
$$=3\alpha(\mu)+\mu(\gamma\wedge 1)\hat{\mu}.$$

Since $\gamma^2=0$ and $2\gamma=0$, a similar computation as in the proof of Lemma 19 leads to $\mu(\gamma\wedge 1)\mu=\mu(1\wedge\gamma)\mu=D$. Hence $3\alpha(\mu)=0$. Q. E. D.

PROOF OF THE PARTS (d), (d′), (d″) OF THEOREM 2 (CONTINUED AFTER 20, 21).

A similar computation as in Lemma 7 shows that

$$\text{3-primary component of } [\Sigma^2 M_q, M_q]=\begin{cases} Z/3\{i_q\alpha j_q\} & \text{for } q\equiv 0 \bmod 3,\\ 0 & \text{for } q\not\equiv 0 \bmod 3,\end{cases}$$

where α is a generator of the 3-primary component $Z/3$ of $\pi_3^S=[\Sigma^3 S, S]$. Therefore $\alpha(\mu)=0$ for any multiplication μ_q on M_q when $q\not\equiv 0 \bmod 3$, by Proposition 22. Also we may put $\alpha(\mu_q)=a_q i_q\alpha j_q$, $a_q\in Z/3$, when $q\equiv 0 \bmod 3$. The coefficient a_q is independent of the choice of multiplication. Then

$$[A_q] \qquad \mu_q(1\wedge\mu_q)=\mu_q(\mu_q\wedge 1)-a_q i_q\alpha(j_q\wedge j_q\wedge j_q) \quad \textit{for} \quad q\equiv 0 \bmod 3.$$

Let $\rho=\rho_{q,3}$ and suppose that μ_q is regular. By Lemma 5, composing $[A_3]$ with $\rho\wedge\rho\wedge\rho$ from the right and $[A_q]$ with ρ from the left then implies that

$$a_q i_3\alpha j_q=(q/3)^3 a_3 i_3\alpha j_q=(q/3)a_3 i_3\alpha j_q\,.$$

Therefore $a_q\equiv(q/3)a_3 \bmod 3$. Toda proved in [11] that M_3 is not associative, that is, $a_3=1$ by replacing α by $-\alpha$ if necessary, completing the proof. Q. E. D.

§6. Remark on the case $q\equiv 2 \bmod 4$

Throughout the section, we assume $q\equiv 2 \bmod 4$, and denote simply 1_{M_q}, $1_{M_{2q}}$, i_q, j_q, i_{2q}, j_{2q} by 1, 1′, i, j, i', j', respectively. We shall try to make analoguous discussion for M_{2q}-module multiplications on M_q.

Since $2q1=0$ (by Lemma 7, (b) with $r=q$), the cofibration

$$M_q \xrightarrow{i'\wedge 1} M_{2q}\wedge M_q \xrightarrow{j'\wedge 1} \Sigma M_q$$

is trivial, hence there is an element, which we call *an M_{2q}-pre-multiplication on M_q*,

$$m_q\colon M_{2q}\wedge M_q \longrightarrow M_q$$

together with the associating element

$$\hat{m}_q\colon M_q \longrightarrow M_{2q}\wedge M_q$$

satisfying the same properties as (8–a)–(8–d) with μ_q, $\hat{\mu}_q$, i_q, j_q, $1_{M\wedge M}$ replaced by m_q, $\hat{m}_q$, i', j', $1'\wedge 1$. The difference element for two choices m_q, m_q^0 of M_{2q}-pre-multiplications

$$d(m_q^0,\ m_q)\in[\Sigma M_q,\ M_q]=\mathbf{Z}/2\{i\tilde{\eta}_q\}\oplus\mathbf{Z}/2\{\tilde{\eta}_q j\}$$

is similarly defined (notice that the third factor $\mathbf{Z}/2\{i\eta^2 j\}$ vanishes), hence there are exactly four M_{2q}-pre-multiplications on M_q. By a similar discussion in §1 of constructing a multiplication from pre-multiplication, there is an m_q which satisfies

(24) $$m_q(1'\wedge i)=\rho, \quad where \quad \rho=\rho_{2q,q}.$$

We call such an m_q *an M_{2q}-multiplication on M_q*. Enumerating M_{2q}-multiplications is also similar and one has that there are exactly two M_{2q}-multiplications on M_q and that, if m_q is one of them, the other is given by $m_q''=m_q+\tilde{\eta}_q j(j'\wedge 1)$.

The regularity of multiplications given in Definition 1 then corresponds with the following property:

(25) $$m_q(1'\wedge\delta_q)=\delta_q m_q,$$

that is, δ_q is an M_{2q}-module map, because, if a multiplication μ_q on M_q were constructed, m_q would be a composite $\mu_q(\rho\wedge 1)$. For an M_{2q}-multiplication m_q the property (25) is equivalent to

(25)′ $$(1'\wedge j)m_q=-\lambda, \quad where \quad \lambda=\lambda_{q,2q},$$

which is an analogue of Lemma 9. Then there exists uniquely an M_{2q}-multiplication on M_q with property (25) or (25)′, which we call *a regular M_{2q}-multiplication on M_q*. We shall stress that the lack of the factor $\mathbf{Z}/2\{i\eta^2 j\}$ in $[\Sigma M_q,\ M_q]$ implies the uniqueness.

Henceforward, we reserve m_q for a regular M_{2q}-multiplication on M_q because of the uniqueness. As in Proposition 11, we name other M_{2q}-pre-multiplications:

$$m_q''=m_q+\tilde{\eta}_q j(j'\wedge 1),$$

$$m_{q,\mathrm{pre}}=m_q+i\tilde{\eta}_q(j'\wedge 1),$$

$$m_{q,\mathrm{pre}}''=m_q+(i\tilde{\eta}_q+\tilde{\eta}_q j)(j'\wedge 1).$$

An analogue of Lemma 5 is then stated as follows:

LEMMA 26. *There exists uniquely a regular multiplication μ_{2q} on M_{2q}*

such that $\rho\mu_{2q}=m_q(1'\wedge\rho)$, $\hat{\mu}_{2q}\lambda=(1'\wedge\lambda)\hat{m}_q$.

The proof is similar and we omit it. We mention that in Lemma 5 there hold more relations including $(\lambda\wedge\lambda)\hat{\mu}_r=\hat{\mu}_q\lambda$ [8, Lemma 2.1]. It is not hard to check the uniqueness of μ_{2q}, since $i\eta^2 j'\neq 0$. Summarizing the above discussion, we have

PROPOSITION 27. *There exist a regular* M_{2q}*-multiplication* m_q *on* M_q, (i.e., *a map with properties* $m_q(i'\wedge 1)=1$, (24) *and* (25)), *and a regular multiplication* μ_{2q} *on* M_{2q} *such that* $\rho\mu_{2q}=m_q(1'\wedge\rho)$, $\hat{\mu}_{2q}\lambda=(1'\wedge\lambda)m_q$. *All of these properties uniquely determine* m_q *and* μ_{2q}.

We also reserve the notation μ_{2q} for the unique regular multiplication in the proposition, and describe other pre-multiplications as in Proposition 11 with our μ_{2q}.

REMARK 28. *For each* q *with* $q\not\equiv 2 \bmod 4$, *we may uniquely specify a regular multiplication* μ_q *on* M_q *in the following way: for* q *odd,* μ_q *is always unique (Theorem* 2, (a)), *for* $q=2^n$, $n\geqq 2$, *Proposition* 27 *(in case* $n=2$) *and the remark at the end of* §2 *(in case* $n\geqq 3$) *give unique* μ_q, *and for arbitrary* $q=q_1q_2$ *with* $q_1=2^n$, q_2 *odd, the homotopy equivalence* $M_q=M_{q_1}\vee M_{q_2}$ *in* §2 *gives unique* μ_q.

In a similar manner as of (18), we define the associator of m_q as follows:

$$\alpha(m_q)=m_q(\mu_{2q}\wedge 1)(1'\wedge\hat{m}_q)\hat{m}_q=m_q(1'\wedge m_q)(\hat{\mu}_{2q}\wedge 1)\hat{m}_q,$$

which is an element of

(29) $[\Sigma^2 M_q, M_q]$

$$=\mathbf{Z}/2\{i\eta\bar{\eta}_q\}\oplus\mathbf{Z}/2\{\tilde{\eta}_q\eta j\}\oplus\mathbf{Z}/2\{i\nu j\}\oplus\begin{cases}\mathbf{Z}/3\{i\alpha j\} & \text{for } q\equiv 0 \bmod 3,\\ 0 & \text{for } q\not\equiv 0 \bmod 3,\end{cases}$$

where ν is a generator of the 2-primary part $\mathbf{Z}/8$ of π_3^S. The discussion in Proposition 20 with μ regular multiplication is applied to show

$$m_q(1'\wedge m_q)=m_q(\mu_{2q}\wedge 1)-\alpha(m_q)(j'\wedge j'\wedge 1), \tag{30}$$

$$(1'\wedge\hat{m}_q)\hat{m}_q=-(\mu_{2q}\wedge 1)\hat{m}_q+(i'\wedge i'\wedge 1)\alpha(m_q). \tag{30'}$$

Composing $1'\wedge 1'\wedge\rho$ to (30) from the right yields the relation

$$\alpha(m_q)\rho=\begin{cases}(2q/3)i\alpha j' & \text{for} \quad q\equiv 0 \mod 3,\\ 0 & \text{for} \quad q\not\equiv 0 \mod 3,\end{cases}$$

and similarly, from (30)$'$,

$$\lambda\alpha(m_q)=\begin{cases}(2q/3)i'\alpha j & \text{for} \quad q\equiv 0 \mod 3,\\ 0 & \text{for} \quad q\not\equiv 0 \mod 3.\end{cases}$$

From these relations, we get

(31) $$\alpha(m_q)=a_q i\alpha j+a'_q i\nu j,$$

where $a_q=0$ for $q\not\equiv 0 \mod 3$, $a_q=2q/3$ for $q\equiv 0 \mod 3$, and $a'_q\in \mathbf{Z}/2$.

Unfortunately an analogue of Proposition 22 can not be expected because of no commutator; we are not able to decide the coefficient a'_q. This is the reason why Theorem 4 has to be a weak form. To show Theorem 4, one needs computations similar to those in Lemma 19 (and Proposition 20). For the computations, one also needs the following relations in $[\Sigma^2 \boldsymbol{M}_q, \boldsymbol{M}_q]$:

$$m_q(1'\wedge\tilde{\eta}_q)\lambda=\rho(1'\wedge\bar{\eta}_q)\hat{m}_q=i\eta\bar{\eta}_q+\tilde{\eta}_q\eta j,$$

$$m_q(\tilde{\eta}_{2q}\wedge 1)=0,\ (\bar{\eta}_{2q}\wedge 1)\hat{m}_q=0,$$

which follow from the fact that

(32) *The homomorphisms*

$$i^*: [\Sigma^2\boldsymbol{M}_q, \boldsymbol{M}_q] \longrightarrow \pi_2(\boldsymbol{M}_q).$$
$$j_*: [\Sigma^2\boldsymbol{M}_q, \boldsymbol{M}_q] \longrightarrow [\Sigma\boldsymbol{M}_q, \boldsymbol{S}],$$
$$(\nu\wedge 1)_*: [\Sigma^2\boldsymbol{M}_q, \boldsymbol{M}_q] \longrightarrow [\Sigma^5\boldsymbol{M}_q, \boldsymbol{M}_q]$$

are monic, if they are restricted to the first, second, third factors $\mathbf{Z}/2$ *in* (29), *respectively. Moreover they are trivial on the complementary summands in* (29).

Now Theorem 4 follows at once from (30), (31). We shall here restate it more explicitly:

THEOREM 4$'$. *If* $q\equiv\pm 3 \mod 9$, $\boldsymbol{M}_q$ *can not be an associative* $\boldsymbol{M}_{2q}$*-module spectrum. If* $q\not\equiv\pm 3 \mod 9$, *the following pairs of pre-multiplication on* $\boldsymbol{M}_{2q}$ *and* $\boldsymbol{M}_{2q}$*-pre-multiplication on* $\boldsymbol{M}_q$ *are only possible pairs satisfying associativity*:

$$(\mu_{2q}, m_q), \quad (\mu''_{2q}, m''_{q,\mathrm{pre}}), \quad (\mu''_{2q}, m_q),$$

$$(\mu'_{2q}, m''_{q,\mathrm{pre}}), \quad (\mu'''_{2q}, m''_q), \quad (\mu'''_{2q}, m_{q,\mathrm{pre}}),$$

and either all of them or none of them satisfies the associativity. The pre-multiplications on M_{2q} appearing in these pairs are all associative, regular multiplications.

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