MOONSHINE ELEMENTS IN ELLIPTIC COHOMOLOGY

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ABSTRACT. This is a historical talk about the recent confluence of two lines of research in equivariant elliptic cohomology, one concerned with **connected** Lie groups, the other with the **finite** case. These themes come together in (what seems to me remarkable) work of N. Ganter, relating replicability of McKay-Thompson series to the theory of exponential cohomology operations.

INTRODUCTION

Moonshine, the Monster, conformal field theory, and elliptic cohomology have all been linked since birth. This survey foregrounds the last of these topics; its theme is the emergence, under the influence of Hopkins and Miller and their coworkers, of a coordinate-free perspective on the moduli stacks of bundles over elliptic curves.

I could never have approached this subject but for the untiring interest of John McKay, who has often seemed to me an emissary from some advanced Galactic civilization, sent here to speed up our evolution. I also want to acknowledge many helpful conversations about this material with Matt Ando, Andy Baker, Jorge Devoto, and Nora Ganter.

1. 'CLASSICAL' ELLIPTIC COHOMOLOGY

1.1 The orbifold

$$\overline{\mathcal{M}}_{\rm ell} = (\mathfrak{h} \cup \mathbb{Q})/\mathrm{PSl}_2(\mathbb{Z})$$

($\mathfrak{h}$ is the complex upper half-plane) has one point with isotropy $\mathbb{Z}/2$, one with isotropy $\mathbb{Z}/3$, and a third (the cusp) with (infinite) isotropy $\mathbb{Z}$; it's a good model over $\mathbb{C}$ for the stack $\overline{\mathcal{M}}_{1,1}(\mathbb{Z})$ of stable genus zero curves with one marked point, which is what the real experts work with. These moduli objects are **not** affine; there is a canonical line bundle ω over $\overline{\mathcal{M}}_{\rm ell}$, defined by the cotangent line $T^*_{E,0}$ at the origin of the elliptic curve $E \ni 0$. Modular forms are sections of powers of this bundle; they provide a graded substitute for an affine coordinate ring.

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I want to discuss a cohomology theory [23, 30] taking values in the abelian category of quasicoherent sheaves over $\mathcal{M}_{\rm ell}$, such that for example

$$\mathbf{Ell}(S^{2n}) = \omega^{\otimes n} \;.$$

Regarding ω_E as dual to the Lie algebra of E leads to a refinement of this picture which incorporates odd-dimensional spheres naturally, but that's one of the many topics I'll leave aside in this talk.

There is an associated cohomology theory Ell^* of more classical type, taking values in modules over a certain graded ring. Away from small primes (two and three) its coefficient ring is a graded algebra

$$\mathrm{Ell}^*(\mathrm{pt}) \;\cong\; \oplus_{n \geq 0} \Gamma(\mathcal{M}_{\rm ell}, \omega^{\otimes n}) \quad (\otimes \mathbb{Z}[\tfrac{1}{6}])$$

of classical modular forms, but in general this isomorphism must be replaced by a spectral sequence

$$R^*\Gamma(\mathcal{M}_{\rm ell}, \omega^{*/2}) \Rightarrow \mathrm{tmf}^*(\mathrm{pt})$$

with the derived functor of global sections as its E_2 term. Away from six these derived functors vanish; but in general the spectral sequence is complicated, with **lots** of two- and three-torsion, which may look strange to arithmetic geometers but turns out to be quite familiar to algebraic topologists. At the prime two, for example, the Hurewicz homomorphism from the stable homotopy ring of spheres to the resulting cohomology theory is injective up to dimension fifty or so [25].

There are other differences, which leads homotopy theorists to distinguish the abutment of this spectral sequence, which they call the ring of **topological** modular forms, from its classical analog. The discriminant Δ, for example, is a modular form but not a topological one: it supports a nontrivial (torsion-valued) differential in the spectral sequence. However, both 24Δ and Δ^{24} survive, and **do** represent elements of $\mathrm{tmf}^*(\mathrm{pt})$. These facts seem to be related to some congruences of Borcherds which are beyond my depth [10, 23 Theorem 5.10]; I mention them only to indicate the arithmetic power of the theory of topological modular forms.

There are at present two approaches to constructing topological modular forms, both based a systematic rigidifications of Landweber's exact functor theorem. The original approach of Hopkins and Miller builds a homotopy limit of spectra whose connective cover is tmf, while Lurie's more recent approach constructs this object as the global sections of a sheaf of spectra over a certain kind of derived moduli stack. The subject is very much under construction (see §2.4 below), and both approaches require deep new ideas which change the way we think of both homotopy theory and algebraic geometry; but in this talk I will take these ideas to some extent for granted, concentrating instead on their applications to Moonshine and the Monster.

1.2 In practice these constructions have many variants, associated to related moduli problems: away from two, for example, curves with $\Gamma_0(2)$ - level structures correspond to Jacobi/Igusa quadrics

$$Y^2 = 1 - 2\delta X^2 + \epsilon X^4 \ ,$$

which was the example which really crystallized this whole subject [22, 29]. Away from two, the associated modular forms have Fourier coefficients in $\mathbb{Z}[\frac{1}{2}]$ and poles only at the cusp $\tau \to i\infty,\ q = e^{2\pi i \tau} \to 0$, where Tate's elliptic curve over $\mathbb{Z}[[q]]$ lives. Over the localization $\mathbb{Z}((q))$ its formal group law has multiplicative type, and restriction to a (formal, **affine**) neighborhood of $i\infty$ defines a natural transformation

$$\mathbf{Ell} \to K^{\mathrm{Tate}}$$

of cohomology theories, which on coefficients becomes the q-expansion map

$$\mathrm{Ell}^*(\mathrm{pt}) \sim \{\text{modular forms}\} \to \mathbb{Z}((q)) \ .$$

The target is a variant of classical K-theory, defined by 'extension of scalars' from $\mathbb{Z}$ to $\mathbb{Z}((q))$, suitably oriented: this means that the Chern class of a complex line bundle L in K^{Tate} is essentially the Weierstrass σ-function [4, 34].

It is possible to think of K-theory as the truncation of Ell defined by taking the leading term of a modular form [5, 26, 29]. Some of the more interesting calculations in K-theory involve Bernoulli numbers; in Ell their analogs involve the Eisenstein series having those numbers as constant term [7]. The natural operations in K-theory are the Adams operations; their analogs in Ell are Hecke operators [1,6], as we shall see.

1.3 Physicists (Witten in particular) became interested in these matters because Ell^* is the best approximation by a cohomology theory

$$\begin{array}{ccc} K^*_{\mathbb{T}}(LM) & \longrightarrow & \mathrm{Ell}^*(M) \\ & \searrow & \Big\downarrow {\scriptstyle q-\mathrm{exp'n}} \\ & & K^{\mathrm{Tate}}(M) \end{array}$$

to the $\mathbb{T}$-equivariant K-theory of a free loopspace: that construction does not preserve cofibrations, so the group on the left is very complicated. The completion

$$u \mapsto 1 - q : K_{\mathbb{T}}(\mathrm{pt}) = \mathbb{Z}[u, u^{-1}] \to \mathbb{Z}[[q]] \to K^{\mathrm{Tate}}(\mathrm{pt})$$

restricts to the neighborhood of the fixed point set and so simplifies things greatly [27]; on the other hand, the resulting theory has a coefficient ring much larger than Ell^*.

Witten studied the index of an analog of the Dirac operator on LM, and showed that when the loopspace possesses the proper analog of a spin structure, the resulting invariant is a modular form. This precipitated an enormous amount of research; in particular, such a spin structure is now understood as the natural form of orientation [4] for elliptic cohomology. Witten further conjectured the rigidity of his invariant, under actions on the manifold of a **connected** Lie group G. This is now understood in topological terms [3] as a property of the Thom isomorphism in a suitable form of equivariant elliptic cohomology.

2. The equivariant picture

2.1 Sheaf-valued cohomology theories defined over non-affine objects are unfamiliar in algebraic topology, but working with them is usually routine, perhaps involving higher derived functors of global sections but not too much more. Extending this framework to the equivariant context, however, seems to lead us into a genuinely new world: it poses deep **conceptual** questions. Ganter's recent progress on these questions is the real focus of this talk.

It seems very reasonable to expect that for a compact Lie group G,

- there is an equivariant version $X \mapsto \mathbf{Ell}_G(X)$ of elliptic cohomology taking values in quasicoherent sheaves over some moduli stack

$$\mathcal{M}_{\mathrm{ell}}(G) = \{G - \text{bundles over } \check{E} \mid E \in \mathcal{M}_{\mathrm{ell}}\} \to \mathcal{M}_{\mathrm{ell}}$$

of **principal** G**-bundles** over (the dual of) the 'universal' elliptic curve.

[The appearance here of the dual Abelian variety is technical, having to do with naturality. When $G = A$ is finite abelian, A-bundles over $\check{E}$ are closely related to extensions of $\check{E}$ by A, and thus (by some Cartier duality or Cartan exchange isomorphism) to homomorphisms from the Pontrjagin dual $\hat{A}$ to E; this is related to another approach [20] to equivariance.]

2.2 For example, when $G = \mathbb{T}$ is the circle group, bundles over $\check{E}$ are classified by elements of the Picard group

$$\mathrm{Pic}(\check{E}) \cong E\ .$$

A $\mathbb{T}$-space X thus defines a sheaf $\mathbf{Ell}_{\mathbb{T}}(X)$ over the universal curve $\mathbf{E} = \mathcal{M}_{\mathrm{ell}}(\mathbb{T})$.

For **connected** groups this program was pioneered by Grojnowski and Ginzburg-Kapranov-Vasserot, followed by Ando and others. For example, the two-sphere $\mathbb{C} \cup \infty$ with $z \in \mathbb{T}$ acting as multiplication by z^n has (reduced) equivariant elliptic cohomology defined by the sheaf

$$\widetilde{\mathbf{Ell}}_{\mathbb{T}}(S^2(n)) = \mathcal{O}_{\mathbf{E}}(-\sum E[n])$$

[41 §3.11]; where $\sum E[n]$ represents the divisor of n-torsion points on $\mathbf{E}$ (equal as such to $n^2 \cdot [0]$), and $\mathcal{O}_{\mathbf{E}}$ is the structure sheaf of the universal curve.

The grading of a classical G-equivariant cohomology theory E extends naturally from $\mathbb{Z}$ to a Grothendieck group of E-orientable linear representations of G, but sheaf-valued cohomology theories over non-affine objects are naturally graded by the Picard group of line bundles over the base object. When G is connected and (for example) simply-connected, Grojnowkski [21 §3.3] and Ando [2 §9.20, 10.11] use earlier work of Looijenga to construct a line bundle $\mathcal{L}$ over $\mathcal{M}_{\rm ell}(G)$ such that the $\mathbb{Z}((q))$-module of cusp expansions of sections of $\mathcal{L}^{\otimes k}$ can be identified with the corresponding module generated by graded characters of level k representations of the loop group LG.

2.3 In this talk, however, I want to focus on recent developments in the case of G **finite**. In spite of deep early work of Devoto, this seems to have received less attention than for G connected: perhaps because of Witten's rigidity conjecture, and perhaps because of its intrinsic subtlety.

We can think of G-bundles over $\check{E}$ as classified by elements of the (zero-dimensional!) orbifold

$$\mathrm{Pairs}_G = \mathrm{Hom}(\mathbb{Z}^2, G)/G^{\rm conj} \cong \mathrm{Hom}(\pi_1(\check{E}), G)/G^{\rm conj} \cong H^1(\check{E}, G)$$

of conjugacy classes of commuting pairs of elements in G [19]. There is an action of $\mathrm{Sl}_2(\mathbb{Z})$ on such pairs, by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} [h, g] = [h^a g^b, h^c g^d] \; ,$$

and we can take [see [16 §2.1] for a more precise statement]

$$\mathcal{M}_{\rm ell}(G) \sim \mathrm{Pairs}_G \times_{\mathrm{Sl}_2(\mathbb{Z})} \mathfrak{h} \to \mathrm{pt} \times_{\mathrm{PSl}_2(\mathbb{Z})} \mathfrak{h} \sim \mathcal{M}_{\rm ell}$$

as a model for the stack of G-bundles. Experienced Moonshiners will recognize the space on the left as a natural habitat for Norton's work.

[Note that this map is **not** the identity when $G = \{1\}$ is the trivial group: the object on the left is then an **un**reduced orbifold, with nontrivial isotropy coming from the central element -1 in $\mathrm{Sl}_2(\mathbb{Z})$ acting as the involution $[g, h] \mapsto [g^{-1}, h^{-1}]$ on the pair $[1, 1]$. This is part of an action of the multiplicative monoid $\mathbb{Z}^\times$ on pairs by 'Adams operations' $[g, h] \mapsto [g^n, h^n]$.]

Conjugacy classes in the centralizers $C_G(g) = \{h \in G \mid gh = hg\}$ form a kind of coordinate atlas on conjugacy classes of pairs, by the map

$$h \mapsto [h, g] : C_G(g)^{\rm conj} \to \mathrm{Pairs}_G$$

which extends to a parametrization

$$C_G(g) \times_{\mathbb{Z}} (\text{neighborhood of } i\infty) \to \mathrm{Pairs}_G \times_{\mathrm{Sl}_2(\mathbb{Z})} \mathfrak{h} = \mathcal{M}_{\rm ell}(G)$$

of the neighborhoods of the cusps in $\mathcal{M}_{\rm ell}(G)$; here $\mathbb{Z}$ acts as the subgroup

$$\left[\begin{array}{cc} 1 & * \\ 0 & 1 \end{array}\right]$$

of translations on pairs, sending $[h,g]$ to $[hg^n,g]$. Restriction to such neighborhoods of infinity in $\mathcal{M}_{\rm ell}(G)$ defines an equivariant lifting

$$\mathbf{Ell}_G({\rm pt}) \to K_G^{\rm Dev}({\rm pt})$$

of the q-expansion map, $K^{\rm Dev}$ being Ganter's reformulation [15 §3.1] of Devoto's equivariant version of $K^{\rm Tate}$, which accomodates some subtle Galois-theoretic properties of the coefficients of McKay-Thompson series, both classical and generalized [12]. I'll omit some details here, and describe that construction as the collection, indexed by conjugacy classes of elements $g \in G$, of functors of the form

$$X \mapsto K^*_{C_G(g)}(X^g)((q^{1/|g|}))|_0 \ ,$$

where X^g is the fixed point set of g, $|g|$ is the order of that element, and the symbol $|_0$ denotes the degree zero component of a certain auxiliary $|g|$-grading defined as follows:

There are orthogonal projections

$$P_k = |g|^{-1} \sum_{1 \leq n \leq |g|} e^{-2\pi i n k/|g|} g^n \ , \ |g| - 1 \geq k \geq 0 \ ,$$

in the complex group ring of the centralizer, which split any $C_G(g)$-equivariant bundle over X^g into a sum of eigenspaces in which g acts as multiplication by $\exp(2\pi i k/|g)$; this makes $K^*_{C_G(g)}(X^g)$ into a bigraded algebra (with one grading cyclic of order two, the other cyclic of order $|g|$). On the other hand $\mathbb{Z}((q^{1/|g|}))$ is also naturally $|g|$-graded, and the group above carries a tensor product grading. The boundary maps of a cofibration preserve this extra structure, so taking its degree zero component is again a cohomology theory.

At the 'classical' cusp (with $g = 1$, i.e. associated to pairs of the form $[h,1]$), this construction simplifies to $K_G(X)((q))$.

2.4 Relations of this form between Ell and K-theory are part of a more general picture: we can think of classical K_G as taking values in sheaves over the moduli space of principal G-bundles over the circle, which by reasoning like that sketched above is just the space

$$H^1(S^1,G) \cong {\rm Hom}(\pi_1(S^1),G) \cong G/G^{\rm conj} \cong {\rm Spec}\ (R(G) \otimes \mathbb{C})$$

of conjugacy classes in G. In fact Quillen showed, back in the 70's, that the classical cohomology of G can be naturally regarded as a sheaf over the category whose objects are the abelian subgroups of G, with morphisms $A \to A'$ being the homomorphisms induced by conjugation by elements of

G. There is a natural 'chromatic' filtration on the objects of this category (by their rank), with cyclic subgroups at the top. The elliptic picture sees the first **two** layers of this filtration; the general case is the concern of the generalized character theory of Hopkins, Kuhn, and Ravenel [24, 35]

The Hopkins-Miller theorem [19, 39] constructs topological modular forms in terms of a sheaf of E_∞-algebra spectra over the moduli stack of elliptic curves. Work of Lurie [30 §4] formulates this very naturally in terms of a general representability theorem in enriched (or derived) algebraic geometry; in particular, he constructs elliptic cohomology as a sheaf of spectra over a derived moduli stack of suitably oriented elliptic curves over E_∞-algebras. It seems likely that similar techniques [cf. also [12]] can be used to define an equivariant version of this theory as a sheaf of spectra over a derived moduli stack, now of G-torsors over Lurie's generalized oriented elliptic curves; but a precise definition of some such object has yet to appear.

3. Ganter's formulation of replicability

3.1 One concise (but historically misleading) way to tell the Moonshine story is to say that the graded character of the representation defined by the Frenkel-Lepowsky-Meurman vertex operator algebra defines a (McKay-Thompson) map J from conjugacy classes in the Monster to modular functions [9, 11]; on $1 \in \mathbb{M}$, for example, this construction yields the value $j(q)-744$. In the interpretation presented here, Norton's generalized Moonshine sees J as the restriction to the classical cusp of a section of ω^0 over $\mathcal{M}_{\rm ell}(\mathbb{M})$.

It is known, essentially by case-by-case verification, that the invariance group of a generalized Moonshine function is a genus zero subgroup of $\mathrm{Sl}_2(\mathbb{Z})$. This most mysterious property of McKay-Thompson series is believed (known, in the classical case) to be equivalent to a condition called **replicability** [11, 36]. The rest of this note is an introduction to recent work of Ganter, who shows that at the classical cusp, replicability can be expressed very naturally in terms of cohomology operations on a sheaf-valued theory.

3.2 This involves at least three separate issues, the first being the classification of line bundles over the G-bundle stack. Based on earlier work on Chern-Simons theory, Ganter [16 §2.3] defines a homomorphism

$$H^4(BG,\mathbb{Z}) \to \mathrm{Pic}(\mathcal{M}_{\rm ell}(G))$$

which associates to a degree four cohomology class in G, an element of the Picard group of line bundles over the moduli space of G-bundles. When G is connected and simple, this cohomology group is infinite cyclic, and its elements can be naturally identified with the **levels** which occur in the theory of positive-energy representations of loop groups [2, 13]. On the other hand this cohomology group is finite when G is; for example, there is reason to suspect [33 §11] that the fourth integral cohomology group of the Monster contains a nontrivial element of order 48.

She then constructs generalizations [16 §6.3]

$$T_k : \Gamma_{\mathcal{M}}(\mathcal{L}^\alpha) \to \Gamma_{\mathcal{M}}(\mathcal{L}^{k\alpha})$$

of Hecke operations on the line bundles $\mathcal{L}^\alpha$ corresponding to $\alpha \in H^4(BG, \mathbb{Z})$, which in the case $\alpha = 0$ restrict at the cusp to the classical form

$$T_k(f(\tau)) = \frac{1}{k} \sum_{ad=k, d>b\geq 0} \psi^a(f)(\frac{a\tau + b}{d}) \; ;$$

where $\psi^a(f)$ denotes the result of applying an Adams operation to the coefficients of the power series f.

3.3 The third topic concerns power operations in generalized cohomology theories. In K-theory the formal sum

$$V \mapsto \Lambda_t(V) = \sum_{k\geq 0} \Lambda^k(V) t^k : K(X) \to (1 + tK(X)[[t]])^\times$$

of exterior powers of vector bundles defines a homomorphism, because the total exterior power satisfies the identity

$$\Lambda_t(V \oplus W) = \Lambda_t(V) \otimes \Lambda_t(W) \; .$$

The total symmetric power $S_t(V) = \sum_{k\geq 0} S^k(V) t^k$ is similarly exponential, in particular because these operations are related by a formal identity

$$\Lambda_{-t}(V) = S_t(V)^{-1}$$

(for example, in some ring of symmetric functions [31]). In that context, Newton's relations lead to Adams' identity

$$S_t(V) = \exp(\sum_{k\geq 1} \psi^k(V) \frac{t^k}{k}) \; .$$

Exponential operations are extremely important in algebraic topology, going back to work of Atiyah and Steenrod: in modern terms, an orbifold $[X/G]$ has an nth orbifold symmetric power $[X^n/(G \wr \Sigma_n)]$, and a good (multiplicative, equivariant) cohomology theory E^* will admit 'external' power operations

$$x \mapsto x^{[\otimes n]} : E^*_G(X) \to E^*_{G\wr\Sigma_n}(X^n)$$

which, restricted to the diagonal, yield 'internal' power operations: those of Steenrod and Adams, as well as Baker's Hecke operations and Ando's higher generalizations (involving sums over isogenies of formal groups).

Ganter defined exterior and symmetric power operations for Lubin-Tate theories in her thesis [14 §6.11, 7.15], and constructed Hecke operations satisfying the analog [§9.2] of the formula

$$\mathbf{S}_t(x) = \exp(\sum_{k\geq 1} T_k(x)\frac{t^k}{k})$$

of Newton and Adams. [At about the same time C. Rezk [40 §1.12], working in a related but different context, constructed a kind of universal **logarithmic** operation behaving much like an inverse to these symmetric powers.] Using the generalized Hecke operations mentioned above, Ganter extended these constructions to elliptic cohomology and established the

3.3 Theorem[16 §6.4]: Replicability of the classical Moonshine function J is equivalent to the equation

$$t(J(t) - J(q)) = \mathbf{\Lambda}_{-t}(J(q)) \in K^{\mathrm{Dev}}_{\mathbb{M}}(\mathrm{pt})[[t]] \; .$$

Proof: The argument uses the theory of Faber polynomials, which associates to a function

$$f(q) = q^{-1} + \sum_{k\geq 1} a_k q^k \in \mathbb{C}((q)) \; ,$$

the sequence of polynomials $P_{n,f}(X) = X^n - na_1X^{n-1} + \cdots + (-1)^n a_n$ characterized by the property

$$q^{-n} - P_{n,f}(f(q)) \in q\mathbb{C}[[q]] \; .$$

A generalization [32] of Newton's relations then imply the identity

$$\log \Big[q(f(q) - f(p))\Big] = \sum_{n\geq 1} P_{n,f}(f(p))\frac{q^n}{n} \; ;$$

the assertion of the theorem is thus equivalent to the original form

$$P_{k,J}(J(q)) = kT_k(J(\tau))$$

of the replicability condition. □

3.5 I will close with some remarks, perhaps too vague to be very useful:

The first concerns the beautifully simple consequence

$$q\mathbf{\Lambda}_{-t}(J(q)) = -t\mathbf{\Lambda}_{-q}(J(t))$$

of Ganter's formula, reminiscent in many ways of a product formula of Borcherds; which, however, at second sight becomes increasingly mysterious. It implies that the generalized exterior power of $J(q)$ is modular in the purely formal auxiliary variable t. The two sides of the equation are cusp expansions around very different points: the left-hand side of the equation lies in $K_{\mathbb{M}}(\mathrm{pt})((q))[[t]]$, while the right-hand side lies in $K_{\mathbb{M}}(\mathrm{pt})((t))[[q]]$, and their equality implies that both lie in $K_{\mathbb{M}}(\mathrm{pt})[[q,t]]$. A coordinate-free interpretation of this, or some similar relation [cf. [15], just after §5.15], might be very enlightening.

The second is that this replication formula, rewritten as

$$J(t) = J(q) + t^{-1}\mathbf{\Lambda}_{-t}(J(q)) \ ,$$

looks like some kind of evolution equation; but finding an infinitesimal version seems to be difficult, because of the presence of poles. Related integrable systems will be the subject of Devoto's talk.

The final remark concerns naturality. Once replicability is formulated in terms of cohomology operations, we can study its behavior under restriction; for example any cyclic subgroup of $\mathbb{M}$ pulls J back to a replicable element of $K^{\mathrm{Dev}}_{\mathbb{Z}/n\mathbb{Z}}(\mathrm{pt})$. [Of course the resulting element must be the restriction of a Moonshine class for the centralizer of $\mathbb{Z}/n\mathbb{Z}$. The $n = 3$ case, for example, is related to the Thompson group, and is studied in [22]. But even the pullback to a cyclic group is interesting.]

There is evidence [42] for the existence of analogs of J for other sporadic simple groups, and one might even speculate about symmetric groups. In that context the replicability equation calls to mind an equation

$$T(z) = z \cdot \exp(\sum_{k \geq 1} \frac{T(z^k)}{k})$$

satisfied by the generating function for unlabelled trees (originating with Cayley, but in this form due to Pólya). Current thinking [8] interprets equations of this sort as generalized **Dyson-Schwinger** equations, taking values in some combinatorially-defined Hopf algebra (eg, of trees). It is easier to draw a picture to explain where such equations come from, than to spell it out in words.

One might speculate that some such equation, in some ring of symmetric functions, might lie behind the integrable systems investigated in current work of Devoto and McKay.

Finally, though it seems to have no overt connection to Moonshine, I can't bring myself to end this talk without mentioning A. Ogg's observation [38] that the primes p dividing the order of $\mathbb{M}$ are precisely those for which every supersingular elliptic curve over a finite field of characteristic p has j-invariant in $\mathbb{F}_p$. This seems to be a property neither of $\mathbb{M}$ nor of $\mathcal{M}_{\mathrm{ell}}$ but rather of $\mathcal{M}_{\mathrm{ell}}(\mathbb{M})$; it deserves further attention.

References

1. M. Ando, Isogenies of formal group laws and power operations in the cohomology theories E_n, Duke Math. J. 79 (1995) 423–485.
2. ——, Power operations in elliptic cohomology and representations of loop groups, Trans. AMS 352 (2000) 5619–5666
3. ——, The sigma orientation for analytic circle-equivariant elliptic cohomology. Geom. Topol. 7 (2003) 91–153
4. ——, M. Hopkins, N. Strickland, Elliptic spectra, the Witten genus and the theorem of the cube, Invent. Math. 146 (2001) 595–687.
5. A. Baker, Operations and cooperations in elliptic cohomology, I: generalized modular forms and the cooperation algebra, NYJ Math 1 (1994) 39 74
6. ———, Hecke algebras acting on elliptic cohomology, in **Homotopy theory via algebraic geometry and group representations** 17–26, Contemp. Math AMS (1998)
7. ———, Hecke operations and the Adams E_2-term based on elliptic cohomology, Canad. Math. Bull. 42 (1999) 129–138
8. J.P. Bell, S.N. Burris, K.A. Yeats, Counting rooted trees: the universal law $t(n) \sim C$ $rho^{-n}n^{-3/2}$, Electronic J. Comb 13, Research paper R63 (2006); see `www.combinatorics.org`
9. R. Borcherds, Vertex algebras, Kac-Moody algebras, and the Monster, Proc. Natl. Acad. Sci. USA. 83 (1986) 3068–3071
10. ———, Monstrous moonshine and monstrous Lie superalgebras, Invent. Math. 109 (1992), 405–444.
11. C.J. Cummins, T. Gannon, Modular equations and the genus zero property of moonshine functions, Invent. Math. 129 (1997) 413–443.
12. J. Devoto, Equivariant elliptic homology and finite groups, Michigan Math. J. 43 (1996) 3–32.
13. D. Freed, M. Hopkins, C. Teleman, Twisted K-theory and loop group representations, available at `arXiv:math/0312155`
14. N. Ganter, Orbifold genera, product formulas and power operations, Adv. Math. 205 (2006) 84–133, available at `arXiv:math/0407021`
15. ———, Stringy power operations in Tate K-theory, available at `arXiv:math/0701565`
16. ———, Hecke operators in equivariant elliptic cohomology and generalized moonshine, available at `arXiv:0706.2898`
17. D. Gepner, Homotopy topoi and equivariant elliptic cohomology, UIUC dissertation (2005)
18. V. Ginzburg, M. Kapranov, E. Vasserot, Elliptic algebras and equivariant elliptic cohomology, available at `arXiv:q-alg/9505012`
19. P. Goerss, M. Hopkins, Moduli problems for structured ring spectra, book in progress
20. J. Greenlees, Equivariant formal group laws and complex oriented cohomology theories, in **Equivariant stable homotopy theory and related areas**, Homology Homotopy Appl. 3 (2001) 225–263
21. I. Grojnowski, Delocalized equivariant elliptic cohomology, in**Elliptic cohomology, geometry, and applications, and higher chromatic analogs**, LMS Lecture Notes 342, ed. H.R. Miller, D.C. Ravenel, CUP (2007)
22. F. Hirzebruch *et al*, **Manifolds and modular forms**, Aspects of Mathematics, Vieweg (1992)
23. M. Hopkins, Algebraic topology and modular forms, Proceedings of the 2002 ICM, Plenary Lectures and Ceremonies, 291-317 Beijing, Higher Education Press; available at`arXiv:math/0212397`
24. ——, N. Kuhn, D. Ravenel, Generalized group characters and complex oriented cohomology theories, Jour. AMS 13 (2000) 553–594

25. ——, M. Mahowald: From elliptic curves to homotopy theory, available at `http://hopf.math.purdue.edu/`
26. N.M. Katz, Higher congruences between modular forms, Ann. Math. 101 (1975) 332–367
27. N. Kitchloo, J. Morava, Thom prospectra for loopgroup representations, in **Elliptic cohomology, geometry, and applications, and higher chromatic analogs**, LMS Lecture Notes 342, ed. H.R. Miller, D.C. Ravenel, CUP (2007); available at`arXiv:math/040454`
28. P. Landweber, **Elliptic cohomology and modular forms**, Springer LNM 1326 (1988)
29. G. Laures, The topological q-expansion principle, Topology 38 (1999) 387– 425
30. J. Lurie, A survey of elliptic cohomology, available at `www-math.mit.edu/~lurie/papers/survey.pdf`
31. I. MacDonald, **Symmetric functions and Hall polynomials**, Oxford (1996)
32. J. McKay, The essentials of Monstrous Moonshine, in **Groups and combinatorics (in memory of Michio Suzuki)** 347–353, Adv. Stud. Pure Math.32 (2001)
33. G. Mason, Orbifold conformal field theory and cohomology of the Monster, available at `http://www.newton.cam.ac.uk/programmes/NST/Mason.pdf`
34. H. Miller, The elliptic character and the Witten genus, in **Algebraic topology** 281–289 Contemp. Math., 96 AMS (1989)
35. J. Morava, HKR characters and higher twisted sectors , in **Gromov-Witten theory of spin curves and orbifolds** 143–152, Contemp. Math., 403, Amer. Math. Soc., Providence, RI, 2006, available at `arXiv:math/0208235`
36. S. Norton, More on moonshine, in **Computational group theory (Durham, 1982)** 185–193 Academic Press (1984)
37. ———, Appendix to G. Mason, Finite groups and modular functions, in Proc. Sympos. Pure Math., 47, Part 1, **The Arcata Conference on Representations of Finite Groups** 181–210 AMS (1987)
38. A. Ogg, Modular functions, in **The Santa Cruz Conference on Finite Groups** 521–532, Proc. Sympos. Pure Math. 37, AMS (1980)
39. C. Rezk, Notes on the Hopkins-Miller theorem, in **Homotopy theory via algebraic geometry and group representations** 313–366, Contemp. Math. 220, AMS (1998)
40. ——, The units of a ring spectrum and a logarithmic cohomology operation, Jour. AMS 19 (2006) 969–1014, available at `arXiv:math/0407022`
41. I. Rosu, Equivariant elliptic cohomology and rigidity, available at `arXiv:math/9912089`
42. C. Thomas, **Elliptic cohomology**, Kluwer Academic/Plenum (1999)

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