# THE MODULI VARIETY FOR FORMAL GROUPS

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Fix an algebraically closed field $k$. A (one-parameter, commutative) formal group law over $k$ is an element $F \in k[[X, Y]]$ such that

1. $F(X, Y) = F(Y, X) = X + Y +$ higher order terms,
2. $F(F(X, Y), Z) = F(X, F(Y, Z))$.

Let $\Lambda$ denote the set of such formal group laws. By a theorem of Lazard [4] there exists a ring $L$ carrying a *universal* formal group law; consequently $\Lambda$ can be identified with the set of ring homomorphisms $L \to k$. This may be used to enrich the structure of the set $\Lambda$; it gains a Zariski topology, and a sheaf of rings making it into a ringed space; in fact with this data, $\Lambda$ would be an algebraic space [7] in the sense of Serre, except that it is not noetherian.

Now let $\Gamma$ denote the group of formal power series $f \in k[[T]]$ of the form $f(T) = uT +$ higher order terms, $u \neq 0$, the group operation being composition. This is a (nonnoetherian) algebraic group, which acts on $\Lambda$ by changing coordinates: if $f \in \Gamma$, $F \in \Lambda$, then we define $F^f(X, Y) = f^{-1}F(fX, fY)$. It is not hard to see that $(f, F) \mapsto F^f$ defines a morphism $\Gamma \times \Lambda \to \Lambda$ of (pro-)algebraic spaces of Serre. In this note we discuss this group action when the characteristic of $k$ is positive (the char 0 case being trivial). In a succeeding note we will apply these results to the study of complex cobordism, via Quillen's theorem, which identifies the ring $L$ with the complex cobordism ring of a point [6]. We remark here that $\Gamma$ is the proalgebraic group underlying the Landweber–Novikov algebra of operations for cobordism.

**Theorem 1.** *$\Lambda$ is stratified into orbits $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_\infty$, such that*

(a) $\overline{\Lambda}_n = \bigcup_{m \geq n} \Lambda_m$*; $\Lambda_1$ is open, and $\Lambda_\infty$ is closed.*
(b) $\overline{\Lambda}_n$ *is a complete intersection of hyperplanes.*
(c) $\Lambda_n$ *is homogeneous under $\Gamma$; for finite $n$, there exists a $p$-adic Lie group $G_n$ such that $\Gamma/G_n \xrightarrow{\sim} \Lambda_n$.*
(d) *The normal bundle of $\Lambda_n$ in $\Lambda$ is given by an $(n-1)$-dimensional representation of $G_n$ over $k$.*

*Remark.* The $\Lambda_n$ can be described explicitly in terms of Milnor's generators of the complex cobordism ring: $\Lambda_n$ is the locus where $p = p_1 = \cdots = p_{n-1} = 0$ and $p_n$ is a unit, where $p_k$ is a Milnor generator of dimension $2(p^k - 1)$. A theorem of Landweber [3] is a corollary: the ideals $(p, p_1, \ldots, p_n)$ in the complex cobordism ring are the only prime ideals invariant under the Landweber–Novikov algebra.

To complete the description of the orbit structure we must identify the groups $G_n$ and the representations in (d). For this we recall that central simple division algebras over the $p$-adic numbers $\mathbb{Q}_p$ are completely classified by the rank (as $\mathbb{Q}_p$-vector spaces) and Brauer invariant, which lies in $\mathbb{Q}/\mathbb{Z}$.

Let $D_n$ be such a division algebra of rank $n$ and Brauer invariant $1/n$. There is a natural valuation on $D_n$, $v \colon D_n^* \to \mathbb{Q}_p^* \to \mathbb{Z}$, the former arrow being the norm, and the latter being $p$-adic valuation. Let $E_n = \{x \in D_n \mid v(x) \geq 0\}$ denote the ring of integers of $D_n$.

Now consider the twisted polynomial algebra $k\langle\mathfrak{F}\rangle$ defined by the relation $\lambda^p \mathfrak{F} = \mathfrak{F}\lambda$, $\lambda \in k$. We abbreviate by $M_n$ the quotient ring modulo the (two-sided) ideal generated by $\mathfrak{F}^n$. It can be shown [1, p. 80] that $E_n/pE_n$ is isomorphic to $\mathbb{F}_q\langle\mathfrak{F}\rangle/(\mathfrak{F}^n)$, where $q = p^n$ and $\mathbb{F}_q\langle\mathfrak{F}\rangle$ is defined like $k\langle\mathfrak{F}\rangle$. Consequently $M_m$ is a *right* $E_n$-module, whenever $n \geq m$.

**Theorem 2.** *The stabilizer $G_n$ is isomorphic to the group of units of $E_n$. The representation of* (1d) *has $M_{n-1}$ as underlying $k$-vector space, with $G_n$-action given by*

$$g(v) = vg^{-1}, \qquad v \in M_{n-1}, g \in G_n \subset E_n .$$

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*Remarks.* $G_n$ is a profinite group over $\mathbb{F}_p$, but it can be given a $p$-adic analytic structure. As such it is a form, in the sense of Galois cohomology, of $\mathrm{GL}(n, \mathbb{Z}_p)$.

From the description of the normal bundle of $\Lambda_n$ in $\Lambda$ just given, it is possible to read off the normal representation of $\Lambda_n$ in $\Lambda_m$, $n \geq m$. It is also possible to identify the stabilizer of the infinite stratum: $\Lambda_\infty = \Gamma/G_\infty$, where $G_\infty$ is the group of units of $k\langle\langle\mathfrak{F}\rangle\rangle$, considered as a proalgebraic group over $\mathbb{F}_p$. This is just the group underlying the reduced Steenrod algebra (at the prime $p$).

*Notes on the proofs.* Theorem 1(a) is just a restatement of a theorem of Lazard [4]: two formal group laws over an algebraically closed field are isomorphic iff they are of the same height. Thus $\Lambda_n$ is the moduli variety of formal groups of height $n$, and part (b) is proved by applying Lazard's techniques to the moduli functor of formal groups of height $n$. Part (d) is trivial.

To prove (c), we use a theorem of Grothendieck [2, III, §3]. Let $G$, $X$ be respectively a groupscheme and a scheme upon which $G$ acts, both noetherian over $k$, with $X$ smooth. Then $X$ is a homogeneous space of $G$ iff $G(k)$ acts transitively on $X(k)$. (Here $G(k)$ and $X(k)$ are the $k$-valued points of $G$, $X$.)

Now the $\Lambda_n$ are represented by localizations of polynomial rings and are smooth, but neither they nor $\Gamma$ are noetherian. However the above result can be extended to *pro*algebraic group actions of a certain kind. Thus we let $\Lambda_n(\deg r)(A)$ denote the set of $r$-buds of a formal group over $A$, of height $n$ ($r \geq p^n$). Then $\Lambda_n = \operatorname{proj\,lim}_r \Lambda_n(\deg r)$, the maps being surjections for any $A$; it is not hard to see that $\Lambda_n(\deg r)$ is a smooth, noetherian scheme. Similarly, $\Gamma_r(A)$ is the set of invertible series in $A[T]/(T^{r+1})$, and acts on $\Lambda_n(\deg r)$, compatibly with truncations. Using these approximations systematically, we prove (c).

The identification of $G_n$ is due to Dieudonné and Lubin; see also [1, Theorem 3, p. 72]. To identify the normal representation, we show by direct computation, following [5], that the tangent space to $\Lambda$ at $F$ is the group $Z_s^2(F; k)$ of 2-cocycles of $F$, while the tangent space to $\Lambda_n$ at $F$, $n$ being the height of $F$, is the group $B_s^2(F; k)$ of 2-coboundaries. Thus the normal bundle is given by the 2-cohomology representation, $H_s^2(F; k)$. A basis for this group is approximately known, and one checks directly that the representation is as indicated.

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