THE MODULI VARIETY FOR FORMAL GROUPS

JACK MORAVA

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Fix an algebraically closed field $k$. A (one-parameter, commutative) formal group law over $k$ is an element $F \in k[[X,Y]]$ such that

1. $F(X,Y) = F(Y,X) = X + Y + \text{higher order terms}$,
2. $F(F(X,Y),Z) = F(X,F(Y,Z))$.

Let $\Lambda$ denote the set of such formal group laws. By a theorem of Lazard [4] there exists a ring $L$ carrying a universal formal group law; consequently $\Lambda$ can be identified with the set of ring homomorphisms $L \to k$.

This may be used to enrich the structure of the set $\Lambda$; it gains a Zariski topology, and a sheaf of rings making it into a ringed space; in fact with this data, $\Lambda$ would be an algebraic space [7] in the sense of Serre, except that it is not noetherian.

Now let $\Gamma$ denote the group of formal power series $f \in k[[T]]$ of the form $f(T) = uT + \text{higher order terms}$, $u \neq 0$, the group operation being composition. This is (a nonnoetherian) algebraic group, which acts on $\Lambda$ by changing coordinates: if $f \in \Gamma$, $F \in \Lambda$, then we define $F^f(X,Y) = f^{-1}F(fX,fY)$. It is not hard to see that $(f,F) \mapsto F^f$ defines a morphism $\Gamma \times \Lambda \to \Lambda$ of (pro-)algebraic spaces of Serre. In this note we discuss this group action when the characteristic of $k$ is positive (the char 0 case being trivial). In a succeeding note we will apply these results to the study of complex cobordism, via Quillen’s theorem, which identifies the ring $L$ with the complex cobordism ring of a point [6]. We remark here that $\Gamma$ is the proalgebraic group underlying the Landweber–Novikov algebra of operations for cobordism.

**Theorem 1.** $\Lambda$ is stratified into orbits $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_\infty$, such that

(a) $\Lambda_n = \bigcup_{m \geq n} \Lambda_m$; $\Lambda_1$ is open, and $\Lambda_\infty$ is closed.
(b) $\Lambda_n$ is a complete intersection of hyperplanes.
(c) $\Lambda_n$ is homogeneous under $\Gamma$; for finite $n$, there exists a $p$-adic Lie group $G_n$ such that $\Gamma/G_n \cong \Lambda_n$.
(d) The normal bundle of $\Lambda_n$ in $\Lambda$ is given by an $(n-1)$-dimensional representation of $G_n$ over $k$.

**Remark.** The $\Lambda_n$ can be described explicitly in terms of Milnor’s generators of the complex cobordism ring: $\Lambda_n$ is the locus where $p = p_1 = \cdots = p_{n-1} = 0$ and $p_n$ is a unit, where $p_k$ is a Milnor generator of dimension $2(p^k - 1)$. A theorem of Landweber [3] is a corollary: the ideals $(p,p_1,\ldots,p_n)$ in the complex cobordism ring are the only prime ideals invariant under the Landweber–Novikov algebra.

To complete the description of the orbit structure we must identify the groups $G_n$ and the representations in (d). For this we recall that central simple division algebras over the $p$-adic numbers $Q_p$ are completely classified by the rank (as $Q_p$-vector spaces) and Brauer invariant, which lies in $Q/\mathbb{Z}$.

Let $D_n$ be such a division algebra of rank $n$ and Brauer invariant $1/n$. There is a natural valuation on $D_n$, $v$: $D_n^* \to Q_p^* \to \mathbb{Z}$, the former arrow being the norm, and the latter being $p$-adic valuation. Let $E_n = \{x \in D_n \mid v(x) \geq 0\}$ denote the ring of integers of $D_n$.

Now consider the twisted polynomial algebra $k(\mathfrak{g})$ defined by the relation $\lambda^p \mathfrak{g} = \mathfrak{g} \lambda$, $\lambda \in k$. We abbreviate by $M_n$ the quotient ring modulo the (two-sided) ideal generated by $\mathfrak{g}^n$. It can be shown [1, p. 80] that $E_n/pE_n$ is isomorphic to $F_q(\mathfrak{g})/(\mathfrak{g}^n)$, where $q = p^n$ and $F_q(\mathfrak{g})$ is defined like $k(\mathfrak{g})$. Consequently $M_n$ is a right $E_n$-module, whenever $n \geq m$.

**Theorem 2.** The stabilizer $G_n$ is isomorphic to the group of units of $E_n$. The representation of (1d) has $M_n-1$ as underlying $k$-vector space, with $G_n$-action given by

$$g(v) = vg^{-1}, \quad v \in M_n-1, g \in G_n \subset E_n.$$
Remarks. \( G_n \) is a profinite group over \( \mathbb{F}_p \), but it can be given a \( p \)-adic analytic structure. As such it is a form, in the sense of Galois cohomology, of \( \text{GL}(n, \mathbb{Z}_p) \).

From the description of the normal bundle of \( \Lambda_n \) in \( \Lambda_m \), \( n \geq m \). It is also possible to identify the stabilizer of the infinite stratum: \( \Lambda_\infty = \Gamma/G_\infty \), where \( G_\infty \) is the group of units of \( k\langle\langle \mathbb{F}_p \rangle\rangle \), considered as a proalgebraic group over \( \mathbb{F}_p \). This is just the group underlying the reduced Steenrod algebra (at the prime \( p \)).

Notes on the proofs. Theorem 1(a) is just a restatement of a theorem of Lazard [4]: two formal group laws over an algebraically closed field are isomorphic iff they are of the same height. Thus \( \Lambda_n \) is the moduli variety of formal groups of height \( n \), and part (b) is proved by applying Lazard’s techniques to the moduli functor of formal groups of height \( n \). Part (d) is trivial.

To prove (c), we use a theorem of Grothendieck [2, III, §3]. Let \( G, X \) be respectively a groupscheme and a scheme upon which \( G \) acts, both noetherian over \( k \), with \( X \) smooth. Then \( X \) is a homogeneous space of \( G \) iff \( G(k) \) acts transitively on \( X(k) \). (Here \( G(k) \) and \( X(k) \) are the \( k \)-valued points of \( G, X \).)

Now the \( \Lambda_n \) are represented by localizations of polynomial rings and are smooth, but neither they nor \( \Gamma \) are noetherian. However the above result can be extended to proalgebraic group actions of a certain kind. Thus we let \( \Lambda_n(\text{deg } r)(A) \) denote the set of \( r \)-buds of a formal group over \( A \), of height \( n \) \( (r \geq p^n) \). Then \( \Lambda_n = \text{proj lim } \Lambda_n(\text{deg } r) \), the maps being surjections for any \( A \); it is not hard to see that \( \Lambda_n(\text{deg } r) \) is a smooth, noetherian scheme. Similarly, \( \Gamma_r(A) \) is the set of invertible series in \( A[T]/(T^{r+1}) \), and acts on \( \Lambda_n(\text{deg } r) \), compatibly with truncations. Using these approximations systematically, we prove (c).

The identification of \( G_n \) is due to Dieudonné and Lubin; see also [1, Theorem 3, p. 72]. To identify the normal representation, we show by direct computation, following [5], that the tangent space to \( \Lambda \) at \( F \) is the group \( \mathbb{Z}_2^2(F; k) \) of \( 2 \)-cocycles of \( F \), while the tangent space to \( \Lambda_n \) at \( F \), \( n \) being the height of \( F \), is the group \( B_2^2(F; k) \) of \( 2 \)-coboundaries. Thus the normal bundle is given by the \( 2 \)-cohomology representation, \( H_2^2(F; k) \). A basis for this group is approximately known, and one checks directly that the representation is as indicated.

References


School of Mathematics, Institute for Advanced Study, Princeton, New Jersey 08540