

## MODULES OVER THE STEENROD ALGEBRA

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### §1. INTRODUCTION

IN [1], Adams and Margolis prove the following theorem about modules over the mod 2 Steenrod algebra: Let  $M$  be a module which is bounded below, then  $M$  is a free module if and only if  $H(M, a_i) = 0$ , where  $\{a_i\}$  is a collection of elements in the Steenrod algebra with  $(a_i)^2 = 0$ .

In this paper we generalize this theorem to the mod  $p$  Steenrod algebra. To do this there are several problems to be overcome. Firstly, a correct theorem must be formulated. This is done by finding elements  $a_i$  in the Steenrod algebra with  $(a_i)^p = 0$  and defining  $H(M, a_i) = \text{Ker } a_i / \text{Im}(a_i)^{p-1}$ . Secondly, the proof given in [1] does not seem to generalize to the mod  $p$  Steenrod algebra and hence we had to find a proof that did. Finally, the reduced power operations only generate a subalgebra of the mod  $p$  Steenrod algebra and we had to extend our results from this subalgebra to the full Steenrod algebra.

In Section 2 we state our results precisely and give the proofs in the following sections.

### §2. STATEMENTS OF RESULTS

Let  $\mathcal{A}$  denote the mod  $p$  Steenrod algebra (including  $p = 2$ ). Let  $'\mathcal{A} \subset \mathcal{A}$  be the subalgebra generated by the reduced power operations. Let  $P_i(r) = \mathcal{P}^{(0, \dots, r, 0, \dots)}$ , with the  $r$  in the  $i$ th place, using Milnor's notation [4]. Let  $P_i^s = P_i(p^s)$ . It is easy to check that  $(P_i^s)^p = 0$  if  $s < t$ . We can now state our first main theorem, which is the same as the theorem of Adams and Margolis when  $p = 2$ .

**THEOREM 2.1.** *Let  $M$  be a graded left  $'\mathcal{A}$ -module which is bounded below. Then  $M$  is a free  $'\mathcal{A}$ -module if and only if  $H(M, P_i^s) = 0$  for all  $s < t$ .*

Let  $p > 2$ . Let  $Q_i \in \mathcal{A}$  be defined by  $Q_0 = \beta$  and  $Q_{i+1} = [\mathcal{P}^{p^i}, Q_i]$ . Let  $E \subset \mathcal{A}$  be the exterior algebra on the  $Q_i$ 's.  $\mathcal{A}$  is generated by the subalgebras  $'\mathcal{A}$  and  $E$  (see [4]). Our second main theorem is the following one.

**THEOREM 2.2.** *Let  $M$  be a graded left  $\mathcal{A}$ -module which is bounded below. Then  $M$  is a free  $\mathcal{A}$ -module if and only if  $M$  is a free  $'\mathcal{A}$ -module and a free  $E$ -module if and only if  $H(M, P_i^s) = 0$  for all  $s < t$  and  $H(M, Q_i) = 0$ , all  $i \geq 0$ .*

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As an application of our results, we note that the analogue of the main theorem of [6] concerning coalgebras over the mod  $p$  Steenrod algebra is an immediate corollary of Theorem 2.1 or 2.2 and the proof given in [6].

Let  $\Lambda' = \Lambda_t'$  be the subalgebra of  $\mathcal{A}$  generated by  $P_t^0, P_t^1, \dots, P_t^{t-1}$ , and  $P_{2t}^0$ . Let  $\Lambda = \Lambda_t$  be the subalgebra of  $\mathcal{A}$  generated by  $\Lambda'$  and  $P_t^t$ . The structure of  $\Lambda'$  and  $\Lambda$  is given by the following proposition.

**PROPOSITION 2.3.**

- (a) If  $r_1 < p^t$ , then  $P_t(r_1)P_t(r_2) = [(r_1 + r_2)!/r_1!r_2!]P_t(r_1 + r_2)$ ,
- (b) if  $r_1$  and  $r_2 < p^t$ , then  $[P_t(r_1), P_t(r_2)] = 0$ ,
- (c) if  $r < p^t$ , then  $(P_t(r))^p = 0$ ,
- (d) if  $r \leq p^t$ , then  $[P_t(r), P_{2t}^0] = 0$ ,
- (e) if  $1 \leq r < p^t$ , then  $[P_t^t, P_t(r)] = P_t(r - 1)P_{2t}^0$ ,

and

- (f)  $(P_t^t)^p = P_t(p^t - (p - 1))(P_{2t}^0)^{p-1}$ .

Relation (f) for  $p$  odd will be proved in §6; the rest of Proposition 2.3 is an easy exercise in manipulation with Milnor matrices [4].

It follows immediately from Proposition 2.3 that  $\Lambda_t' \approx Z_p[P_t^0, \dots, P_t^{t-1}, P_{2t}^0]/(P_t^0)^p, \dots, (P_t^{t-1})^p, (P_{2t}^0)^p$ , and that the extension  $0 \rightarrow \Lambda' \rightarrow \Lambda_t$  is determined by relations (d), (e), and (f). Note that  $\Lambda_t/\Lambda_t' \approx Z_p[P_t^t]/(P_t^t)^p$ . The algebra  $\Lambda_1$  has been studied by Liulevicius [2]. The following theorem is the main step in the proof of Theorem 2.1.

**THEOREM 2.4.** *Let  $M$  be a left  $\Lambda_t$ -module which is bounded below. If  $M$  is free as a  $\Lambda_t'$ -module, then  $M$  is free as a  $\Lambda_t$ -module.*

D. W. Anderson (unpublished) has given a general method for proving theorems like 2.4. This can be used to prove Theorem 2.4 for  $p = 2$ , but as yet we have not been able to use this method to prove 2.4 for odd primes.

Let  $K$  be a finite dimensional Hopf algebra over a field. Let  $K'$  be a Hopf subalgebra. Assume  $K'$  is  $(n - 1)$ -connected and that  $K$  is generated by  $K'$  and  $z \in K$ . Assume  $z^q = 0$ , and  $z^{q-1} \neq 0$ . The following theorem is the main step in the proof of Theorem 2.2.

**THEOREM 2.5.** *Assume  $|z^{q-1}| < n$ . Let  $M$  be a  $K$ -module which is bounded below. If  $M$  is free as a  $K'$ -module and  $H(M, z) = 0$ , then  $M$  is free as a  $K$ -module.*

In §3 we give some elementary properties of differentials of height  $q$ , in §4 we prove Theorem 2.4, in §5 we prove Theorems 2.5, 2.1, and 2.2, and in §6 we prove 2.3(f).

**§3. HOMOLOGY**

Let  $k$  be a field, and let  $A = k[d]/(d^q)$ . If  $M$  is a left module over  $A$ , define, for

$$1 \leq i \leq q - 1,$$

$$H_n(M, d^i) = \text{Ker}(d^i: M_n \rightarrow M_{n+i|d|})/\text{Im}(d^{q-i}: M_{n-(q-i)|d|} \rightarrow M_n).$$

The following four propositions, which show that these homology groups are similar to the usual ones, are all elementary and their proofs are left to the reader.

PROPOSITION 3.1. *Let  $M$  be a left  $A$ -module which is bounded below. The following four conditions are equivalent:*

- (a)  $M$  is a free  $A$ -module,
- (b)  $H_n(M, d) = 0$  for all  $n$ ,
- (c)  $H_n(M, d^i) = 0$  for some  $i$  and all  $n$ ,
- (d)  $H_n(M, d^i) = 0$  for all  $i$  and all  $n$ .

PROPOSITION 3.2. *Let  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  be a short exact sequence of left  $A$ -modules. Then the following is a long exact sequence:*

$$\begin{aligned} \cdots \rightarrow H_n(M', d^i) \rightarrow H_n(M, d^i) \rightarrow H_n(M'', d^i) \\ \rightarrow H_{n+i|d|}(M', d^{i-1}) \rightarrow H_{n+i|d|}(M, d^{i-1}) \rightarrow \cdots \end{aligned}$$

PROPOSITION 3.3. *Assume  $\text{char } k|q$ . Let  $f: M \rightarrow N$  be a map of left  $A$ -modules. Then  $f_*: H_n(M, d) \rightarrow H_n(N, d)$  is an isomorphism for all  $n$  if and only if  $f_*: H_n(M, d^i) \rightarrow H_n(N, d^i)$  is an isomorphism for all  $n$  and all  $i, 1 \leq i \leq q - 1$ .*

§4. PROOF OF THEOREM 2.4

We first do the case  $p = 2$ . The proof for  $p$  odd is a generalization of this but is somewhat complicated due to certain technicalities, but the basic ideas already appear in the proof for  $p = 2$ .

We first prove a combinatorial lemma and a corollary.

LEMMA 4.1. *Let  $\{A_\alpha\}$  be a collection of non-empty subsets of  $\{0, 1, \dots, t - 1\}$ .  $A_\alpha = i_{\alpha_1} < \dots < i_{\alpha_s}$ . Then there is a non-empty subset  $I = i_1 < \dots < i_s$  of  $\{0, 1, \dots, t - 1\}$  such that  $I \cap A_\alpha \neq \emptyset$  all  $\alpha$  and there is a  $\beta$  such that  $I \cap A_\beta = i_{\beta_1} = i_1$ .*

PROPOSITION 3.4. *Let  ${}^iA = A/d^iA, 0 < i < q$ . Then*

$$\text{Tor}_{2s, t}^A({}^iA, M) = H_{t-ps|d|}(M, d^{p-i}) \quad s > 0$$

and

$$\text{Tor}_{2s+1, t}^A({}^iA, M) = H_{t-(ps+i)|d|}(M, d^i), \quad s \geq 0.$$

*Proof.* Put all  $A_\alpha$  with  $|A_\alpha| = 1$  into  $I_1$ . Assume we have constructed  $I_n$  such that  $I_n \cap A_\alpha \neq \emptyset$  for all  $\alpha$  with  $|A_\alpha| \leq n$  and that  $I_n \cap A_{\beta_n} = i_{1n} = i_{\beta_{n1}}$  for some  $\beta_n$  with  $|A_{\beta_n}| \leq n$ .  $I_1$  satisfies this for  $n = 1$ . To show that we can construct  $I_{n+1}$ . Consider all  $A_\alpha$  such that  $|A_\alpha| = n + 1$  and  $I_n \cap A_\alpha = \emptyset$ . If  $i_{\alpha_1} > i_{1n}$ , then find  $i_{\alpha'} \notin A_{\beta_n}$  and add to  $I_n$  to form  $I_n'$ . Do this for all such  $A_\alpha$ . Now consider all  $A_\alpha$  such that  $I_n' \cap A_\alpha = \emptyset$  and  $|A_\alpha| = n + 1$ . Let  $\bar{A}_\alpha = A_\alpha \cap \{0, 1, \dots, i_{n1} - 1\} \subset \{0, 1, \dots, i_{n1} - 1\}$ . By induction on  $t$ , we can find  $\bar{I}$  and  $\bar{A}_\beta$  satisfying the lemma. Let  $I_{n+1} = I_n' \cup \bar{I}$  and let  $\beta_{n+1} = \beta$ . This induction works if  $i_{n1} < t$ , i.e. if  $I_n \neq \emptyset$ . If  $I_n = \emptyset$  and there exist  $A_\alpha$  with  $|A_\alpha| = n + 1$ , let  $I_{n+1}$  be the smallest  $i_{\alpha_1}$  in these  $A_\alpha$  and some element from each of the other  $A_\alpha$  which is not in this  $A_\beta$ . This can be done because of the equal cardinality.

COROLLARY 4.2. *Let  $M'$  be a free  $\Lambda_t'$ -module with all elements of degree  $\geq 0$ . Assume  $M' \subset \overline{M}$ , a  $\Lambda_t$ -module. Let  $m_1'$  be a non-zero decomposable in  $M'$  such that  $|m_1'| < |P_{2t}^0|$ . Then there is a  $\lambda \in \Lambda_t'$  such that  $\lambda P^t m_1' \neq 0 \in M'$ .*

*Proof.*  $m_1' = \sum \lambda_x m_x$ , with  $|\lambda_x| > 0$ . Assume  $\lambda_x$  are nominals  $\lambda_x = P_t^{i_{x1}} \cdots P_t^{i_{xs}}$  and let  $A_x = \{i_{x1}, \dots, i_{xs}\}$ . Note  $P_{2t}^0$  does not appear as  $|m_1'| < |P_{2t}^0|$ . Here  $m_x$  are a  $\Lambda_t$ -base for  $M'$ . Apply the lemma and let  $\lambda = P_t^{i_1} \cdots P_t^{i_s}$ . Then  $\lambda P_t^t m_1' = P_t^{i_1} \cdots P_t^{i_s} P_t^t (\sum \lambda_x m_x) = P_t^{i_1} (2^{i_1} - 1) P_t^{i_2} \cdots P_t^{i_s} P_{2t}^0 \lambda_\beta m_\beta + P_t^{i_2} \cdots P_t^{i_s} P_t^t (P_t^{i_1})^2 \cdots m_\beta + \text{other terms} \neq 0$  in  $M'$  as all terms with  $P_t^t$  in them contain  $(P_t^t)^2$  for some  $j$ .

*Proof of 2.4 for  $p = 2$ .* Let  $M$  be a  $\Lambda_t$ -module which is free as a  $\Lambda_t'$ -module. We must show  $M$  is  $\Lambda_t$ -free. Embed  $M$  in its injective envelope (see [3]). That is,  $M \subset \overline{M}$  with  $\overline{M}$  a  $\Lambda_t$ -module which is injective and the extension is essential; i.e. if  $N \subset \overline{M}$  and  $N \neq 0$ , then  $N \cap M \neq 0$ . We wish to show that  $M = \overline{M}$  as injectives over  $\Lambda_t$  are free [5]. Since  $\overline{M}$  is free,  $M$  is bounded below, and  $\Lambda_t$  is finite dimensional, we see that  $\overline{M}$  is also bounded below. As  $\Lambda_t'$ -modules,  $M$  is free hence injective, hence  $\tilde{M} = M \oplus M'$  as  $\Lambda_t'$ -modules with all three modules free over  $\Lambda_t'$ . Let  $m' \neq 0$  be an element of minimal dimension in  $M'$ .  $P_t^t(m') = m_1 \oplus m_1'$ . Note  $m_1' \neq 0$  or we would have  $(P_t^t)^2 m' = P_t^t m_1 \in M$  and  $(P_t^t)^2 m' = P_t^t(2^t - 1)P_{2t}^0 m' \neq 0 \in M'$  as  $m'$  must generate a free  $\Lambda_t'$ -module in  $M'$ . Further,

$$P_t(2^t - 1)P_{2t}^0 P_t^t(m') = P_t(2^t - 1)P_{2t}^0 m_1 \oplus P_t(2^t - 1)P_{2t}^0 m_1'$$

(because  $P_t(2^t - 1)P_{2t}^0 \in \Lambda_t$ ). This must be in  $M$  as  $P_t(2^t - 1)P_{2t}^0 P_t^t$  is the top class of  $\Lambda_t$  and (some multiple of  $m') \in M$  by the essential extension property. Hence,  $m_1'$  is  $\Lambda_t'$ -decomposable in  $M'$ . By Corollary 4.2, there is an element  $\lambda \in \Lambda_t'$ ,  $|\lambda| > 0$ , such that  $\lambda P_t^t m_1' \in M$  and is non-zero. Then  $\lambda P_t^t P_t^t m' = \lambda P_t^t m_1 \oplus \lambda P_t^t m_1' = \lambda P_t(2^t - 1)P_{2t}^0 m' = 0$ , a contradiction. Thus  $m'$  did not exist and  $M = \overline{M}$ .

*Proof of 2.4 for  $p$  odd.* Let  $M \subset \overline{M}$  be the  $\Lambda_t$ -injective envelope of  $M$ . As  $\Lambda_t'$ -modules,  $\overline{M} = M \oplus M'$  and we can write  $(P_t^t)^{p-1} m' = m_1 \oplus m_1'$ , where  $m'$  is a lowest dimensional element of  $M'$ .  $m_1' \neq 0$  as before and

$$P_t(p^t - 1)(P_{2t}^0)^{p-1} (P_t^t)^{p-1} m' = P_t(p^t - 1)(P_{2t}^0)^{p-1} m_1 \oplus P_t(p^t - 1)(P_{2t}^0)^{p-1} (m_1') \in M$$

by the essential extension property. Hence  $m_1'$  is decomposable and non-zero. The following combinatorial lemma is now needed.

LEMMA 4.3. *Let  $\{A_\alpha\}$  be a collection of weighted nonempty, subsets of  $\{0, 1, \dots, t-1\}$ , with each  $i_\alpha$  having a weighting  $\leq p-1$ .  $A_\alpha = i_{\alpha_1}^{u_{\alpha_1}}, \dots, i_{\alpha_s}^{u_{\alpha_s}}$ . Then, there is a non-empty-weighted subset  $I = i_1^{u_1}, \dots, i_s^{u_s}$  of  $\{0, 1, \dots, t-1\}$  such that  $I \cup A_\alpha$  has an element of weight  $\geq p$  for all  $\alpha$  and there is a  $\beta$  such that  $I \cup A_\beta$  has only  $i_\beta = i_1$  of weight  $\geq p$  and that has weight  $p$ .*

*Proof.* The proof of Lemma 4.1 generalizes almost directly. After forming  $I_n'$ , consider those  $A_\alpha$  with  $|A_\alpha| = n+1$ ,  $I_n' \cup A_\alpha$  having weights  $\leq p-1$ , and  $i_{\alpha_1} = i_{1n}$ . If such exist, increase the weight of  $i_{1n}$  in  $I_n$  and change  $A_{\beta n}$ . Then continue the proof as before.

We now use the same argument as in the mod 2 case to see that there is a  $\lambda \in \Lambda'$  such that  $\lambda P_t^t m_1' \neq 0$  in  $M'$ . However, since  $P_t(p^t - (p-1)) = a P_t^0 (P_t^1 \cdots P_t^{t-1} P_{2t}^0)^{p-1}$  with

$a \neq 0$ , it may not be true that  $\lambda(P_t^i)^p m' = 0$ . Thus, we must discuss the case when  $\lambda = (P_t^0)^i, 1 \leq i \leq p-2$ . Recall that  $m_1' = \sum \lambda_x m_x$ . If  $\lambda_\beta$  has  $P_t^1, \dots, P_t^{t-1}$  with weight  $< p-1$  or  $P_{2t}^0$  with weight  $< p-2$ , then we can multiply  $\lambda$  by an appropriate  $P_t^i$  or  $P_{2t}^0$  so that  $P_t^i \lambda P_t^t m' = 0$  but  $P_t^i \lambda P_t^t m_1' \neq 0$ . Hence, this modification finishes the proof unless  $\lambda_\beta = (P_t^0)^{p-i} (P_t^1 \dots P_t^{t-1})^{p-1} (P_{2t}^0)^{p-2}$ . However, then  $|\lambda_\beta| > |(P_t^i)^{p-1}|$  unless  $i = p-2$ . Furthermore, if this  $\lambda_\beta$  appears in  $m_1'$ , we can use the argument on  $\beta'$  to reach our contradiction unless this  $\lambda_\beta$  is the only term in the sum.

Thus, we need only show a contradiction if

$$(P_t^i)^{p-1} m' = m_1 \oplus (P_t^0)^2 (P_t^1 \dots P_t^{t-1})^{p-1} (P_{2t}^0)^{p-2} m_\beta.$$

Apply  $(P_t^0)^{p-2} P_t^i$  to both sides of this equation to see that

$$(P_t^0)^{p-2} P_t^i (p' - (p-1)) (P_{2t}^0)^{p-1} m' = 2(P_t^0 P_t^1 \dots P_t^{t-1} P_{2t}^0)^{p-1} m_\beta.$$

Since  $P_t^i (p' - (p-1)) = a P_t^0 (P_t^1 \dots P_t^{t-1})^{p-1}$ , we see that  $am' = 2m_\beta$ ,  $a \neq 0$  as  $M'$  is a free  $\Lambda_t'$ -module and  $m'$  has minimal dimension.

Write  $P_t^i m' = m_2 \oplus m_2'$ . Then

$$\begin{aligned} P_t^i (p' - (p-1)) (P_{2t}^0)^{p-1} m' &= P_t^i (P_t^i)^{p-1} m' = \\ &P_t^i m_1 + \frac{a}{2} [(P_t^0)^2 (P_t^1 \dots P_t^{t-1})^{p-1} (P_{2t}^0)^{p-2} m_2 + \\ &(P_t^0)^2 (P_t^1 \dots P_t^{t-1})^{p-1} (P_{2t}^0)^{p-2} m_2' + 2P_t^0 (P_t^1 \dots P_t^{t-1} P_{2t}^0)^{p-1} m']. \end{aligned}$$

Looking at the term in the summand  $M'$ , we see that

$$(P_t^0)^2 (P_t^1 \dots P_t^{t-1})^{p-1} (P_{2t}^0)^{p-2} m_2' = 0.$$

Furthermore,  $m_2' \neq 0$  or otherwise we would have  $m_1' = 0$ . Since  $|P_{2t}^0| > |m_2'| - |m'|$ , we have  $(P_t^0)^2 (P_t^1 \dots P_t^{t-1})^{p-1} m_2' = 0$ . Under these conditions we prove below that there is a  $\lambda \in \Lambda'$  such that  $\lambda(P_t^i)^{p-2} m_2' \neq 0 \in M'$  and that

$$\lambda(P_t^0)^2 (P_t^1 \dots P_t^{t-1})^{p-1} (P_{2t}^0)^{p-2} = 0.$$

Applying  $\lambda(P_t^i)^{p-2}$  to the equation  $P_t^i m' = m_2 \oplus m_2'$ , and looking at the component in  $M'$ , we obtain  $\lambda(P_t^i)^{p-1} m' = 0 = \lambda(P_t^i)^{p-2} m_2' \neq 0$ ; a contradiction.

To construct such a  $\lambda$ , let  $m_2' = \sum \lambda_x m_x$ , where  $\lambda_x$  are monomials in  $\Lambda'$  and  $m_x$  are a  $\Lambda'$ -basis of  $M'$ . We need the following formulae which are easily proved. Let

$$\lambda_x = (P_t^{i_1})^{u_1} \dots (P_t^{i_s})^{u_s}, i_1 < \dots < i_s.$$

Case 1. If  $i_1 > 0$ , then

$$(P_t^i)^{p-2} \lambda_x \equiv b(P_t^0)^2 (P_t^1 \dots P_t^{i_1-1})^{p-1} (P_t^{i_1})^{u_1-1} (P_t^{i_2})^{u_2} \dots (P_t^{i_s})^{u_s} (P_{2t}^0)^{p-2} \pmod{(P_t^0)^3},$$

where  $b \not\equiv 0(p)$ .

Case 2. If  $i_1 = 0, s > 1$ , and  $u_1 = p-1$ , then

$$(P_t^i)^{p-2} \lambda_x \equiv b P_t^0 (P_t^{i_2})^{u_2} \dots (P_t^{i_s})^{u_s} (P_{2t}^0)^{p-2} \pmod{(P_t^0)^2}.$$

Case 3. If  $i_1 = 0, s > 1$ , and  $0 < u_1 < p-1$ , then

$$\begin{aligned} (P_t^i)^{p-2} \lambda_x &\equiv b(P_t^0)^{p-2-u_1} (P_t^1 \dots P_t^{i_2-1})^{p-1} (P_t^{i_2})^{u_2-1} (P_t^{i_3})^{u_3} \dots \\ &(P_t^{i_s})^{u_s} (P_{2t}^0)^{p-2} \pmod{(P_t^0)^{p-1-u_1}}. \end{aligned}$$

Case 4. If  $i_1 = 0, s = 0$ , and  $u_1 = p - 1$ , then

$$(P_t^t)^{p-2} \lambda_x \equiv b P_t^0 (P_{2t}^0)^{p-2} \pmod{(P_t^0)^2}.$$

Case 5. If  $i_1 = 0, s = 0$ , and  $u_1 = p - 2$ , then  $(P_t^t)^{p-2} \lambda_x \equiv b (P_{2t}^0)^{p-2} \pmod{P_t^0}$ . If case 5 occurs in the sum  $m_2' = \sum \lambda_x m_x$ , choose  $\lambda = (P_t^0)^{p-1}$ . If case 5 does not occur, and case 3 with  $u_1 = p - 2$  does, choose  $\lambda = (P_t^0)^{p-1} P_t^{i_2}$ . If case 5 and case 3 with  $u_1 = p - 2$  do not occur, and case 2 or 4 do, choose  $\lambda = (P_t^0)^{p-2}$ . If cases 2, 4, 5, and case 3 with  $u_1 = p - 2$  do not occur, and case 3 with  $u_1 = p - 3$  does, choose  $\lambda = (P_t^0)^{p-2} P_t^{i_2}$ . If cases 2, 4, 5, and case 3 with  $u_1 \geq p - 3$  do not occur, and case 1 does, choose  $\lambda = (P_t^0)^{p-3} P_t^{i_1}$ . Finally, if only case 3 with  $u_1 < p - 3$  occurs, choose  $\lambda = (P_t^0)^{u_1+1} P_t^{i_2}$ , with the largest  $u_1$  for the different case 3's. This  $\lambda$  has the appropriate property and proves the theorem.

### §5. PROOFS OF THE MAIN THEOREMS

Before proving 2.1 and 2.2, we first prove Theorem 2.5 which is the main step in deducing 2.2 from 2.1.

*Proof of Theorem 2.5.* Recall we have  $K' \subset K$ , with  $K'(n - 1)$ -connected and  $z \in K$  with  $z^q = 0$  and  $|z^{q-1}| < n$ . Let  $M$  be a  $K$ -module. Following the style of the proof of Theorem 2.4, let  $M \subset \bar{M}$ , the  $K$ -injective envelope. As  $K$  modules,  $\bar{M} = M \oplus M'$  with all three modules free over  $K'$  and bounded below. Let  $m' \in M'$  be an element of minimal dimension.  $z^{q-1}(m') = m_1 \oplus m_1'$ . Let  $\kappa' \in K'$  be the top dimensional element. Then  $\kappa' z^{q-1}(m') \in M$  and  $\kappa'(m_1') = 0$ . Since  $M'$  is free over  $K'$ ,  $m_1'$  is  $K'$ -decomposable in  $M'$ . Since  $|z^{q-1}| < n$ ,  $m_1' = 0$ . Hence  $z^{q-1}(m') = m_1$  and  $z(m_1) = 0$ . Since  $H(M, z) = 0, m_1 = z^{q-1}(\tilde{m}), \tilde{m} \in M$ . Thus  $z^{q-1}(m' - \tilde{m}) = 0$  is  $\bar{M}$  free over  $K$ , hence  $H(\bar{M}, z) = 0$  and  $m' - \tilde{m} = z(\tilde{m}_1), \tilde{m}_1 \in M$  by dimensional reasons. Thus  $m' \in M$ , a contradiction.

*Proof of Theorem 2.1.* This theorem was proved for  $p = 2$  by Adams and Margolis [1]. We follow their proof with the following modifications for  $p$  odd. The results of their §2 carry over as is; in particular a  $\Lambda_t'$ -module  $M$  is free if and only if  $H(M, P_t^s) = 0$  for all  $s$  and for  $P_{2t}^0$  (this follows by induction from our Theorem 2.5). Most of their §3 goes over without change except for their Lemma 3.6 and its use in the proof of their Lemma 3.5. Let  $B$  be a Hopf subalgebra of  $\mathcal{A}$  such that  $P_t^t \in B$ . Then  $\Lambda_t \subset B$ . Assume  $M$  is a  $B$ -module such that  $M$  is a free  $\Lambda_t'$ -module. Then we must show that  $H(M/P_{2t}^0 M, P_t^t) = 0$  in order to complete the proof of their Lemma 3.5. By Theorem 2.4,  $M$  is a free  $\Lambda_t$ -module. Let  $E = Z_p[P_{2t}^0]/(P_{2t}^0)^p$ . Then  $\Lambda_t$  is a free  $E$ -module and hence so is  $M$ . Hence  $M/P_{2t}^0 M$  is a free module over  $\Lambda_t/\Lambda_t \bar{E} = Z_p[P_t^0, \dots, P_t^{t-1}, \bar{P}_t^t]/\{(P_t^0)^p, \dots, (P_t^{t-1})^p, (\bar{P}_t^t)^p\}$ , where  $\bar{P}_t^t$  is the image of  $P_t^t$ . Thus  $H(M/P_{2t}^0 M, P_t^t) = 0$ .

Let  ${}_1\Lambda_t$  be the subalgebra of  $\mathcal{A}$  generated by  $P_t^0, \dots, P_t^{t-1}$ , and  $E = E(Q_0, Q_1, \dots)$ . The structure of  ${}_1\Lambda_t$  is determined by Proposition 2.3 and the following relations (see [4]):  $[P_t^j, Q_i] = 0$  if  $i > j$  and  $[P_t^j, Q_i] = Q_{i+1} P_t^i (p^j - p^i)$  if  $i \leq j$ . Note that

$$|Q_i| = (2p - 2)(p^{i-1} + p^{i-2} + \dots + p + 1) + 1,$$

and  $|P_t^j| = p^j(2p - 2)(p^{t-1} + p^{t-2} + \dots + p + 1)$ .

In order to prove Theorem 2.2, we note that  $\mathcal{A} // E = ' \mathcal{A}$ . Let  $M$  be an  $\mathcal{A}$ -module which is free over  $E$  and free over  $' \mathcal{A}$ . By Proposition 2.2 of [1], it is enough to show that  $Z_p \otimes_E M$  is free over  $' \mathcal{A}$ . By Theorem 2.1, it is enough to show that  $Z_p \otimes_E M$  is free over  ${}_1\Lambda_t // E = Z_p[P_t^0, \dots, P_t^{t-1}] / ((P_t^0)^p, \dots, (P_t^{t-1})^p)$ . This follows immediately from the following proposition.

**PROPOSITION 5.1.** *Let  $M$  be a  ${}_1\Lambda_t$ -module which is bounded below. If  $M$  is free over  $E$  and  $H(M, P_t^j) = 0, j < t$ , then  $M$  is free over  ${}_1\Lambda_t$ .*

Before proving 5.1, we prove two lemmas, the first of which is analogous to Lemma 2.3 of [1].

**LEMMA 5.2.** *Let  $M$  be a  $Z_p[P] / P^p \otimes E(Q)$ -module which is bounded below where  $p | P | \neq 2 | Q |$ . If  $H(M, P) = 0$  and  $H(M, Q) = 0$ , then  $H(M/QM, P) = 0$ .*

*Proof.* Since  $H(M, Q) = 0, Q: M/QM \rightarrow QM$  is an isomorphism of degree  $|Q|$ . Thus,  $H_n(M/QM, P^i) \approx H_{n+|Q|}(QM, P^i)$ . Also,  $0 \rightarrow QM \rightarrow M \rightarrow M/QM \rightarrow 0$  is exact. By Proposition 3.2 we get a long exact sequence  $\dots \rightarrow H_n(QM, P^i) \rightarrow H_n(M, P^i) \rightarrow H_n(M/QM, P^i) \rightarrow H_{n+|P|}(QM, P^{p-i}) \rightarrow H_{n+|P|}(M, P^{p-i}) \rightarrow \dots$ . Since  $H_*(M, P) = 0$ , we obtain

$$H_n(M/QM, P^i) \approx H_{n+|P|}(QM, P^{p-i}).$$

Combining isomorphisms, we get

$$\begin{aligned} H_n(M/QM, P) &\approx H_{n+|P|}(QM, P^{p-1}) \approx H_{n+|P|-|Q|}(M/QM, P^{p-1}) \\ &\approx H_{n+|P|-|Q|+(p-1)|P|}(QM, P) \approx H_{n+|P|-2|Q|}(M/QM, P). \end{aligned}$$

Since  $p | P | - 2 | Q | \neq 0$  and  $M/QM$  is bounded below,  $H_*(M/QM, P) = 0$ .

**LEMMA 5.3.** *Let  $M$  be a  ${}_1\Lambda_t // E \otimes E(Q_t, Q_{t+1}, \dots, Q_{t+r})$ -module which is bounded below. If  $H(M, P_t^j) = 0$  for  $j < t$  and  $H(M, Q_i) = 0$  for  $t \leq i \leq t + r$ , then  $M$  is a free module.*

*Proof.* Note that  $p | P_t^j | \neq 2 | Q_i |$ , if  $j < t$  and  $i \geq t$ . This lemma follows from Lemma 5.2, induction, Theorem 2.1 of [1] and Proposition 2.2 of [1] (see the proof of Theorem 2.1 of [1]).

*Proof of 5.1.* Let  ${}_1\Lambda_t^r \subset {}_1\Lambda_t$  be the subalgebra generated by  $P_t^0, \dots, P_t^{t-1}$ , and  $E^r = E(Q_0, Q_1, \dots, Q_{t+r})$  with  $r \geq t$ . In order to prove 5.1, it is enough to prove Proposition 5.1' which is the same as 5.1 with  $E$  replaced by  $E^r$  and  ${}_1\Lambda_t$  replaced by  ${}_1\Lambda_t^r$  (for example, see the "Proof of Theorem 3.1 from Theorem 3.2" in [1]). Proposition 5.1' now follows easily from Lemma 5.3 and theorem 2.5 by downward induction, because  $|Q_i| < |P_t^0| = 1 +$  the connectivity of  ${}_1\Lambda_t^r // E^r$ , if  $i < t$ .

§6. PROOF OF 2.3(f)

The algebra  $\Lambda_1$  was studied by Liulevicius [2] and 2.3(f) for  $t = 1$  was proved by him. Instead of calculating with Milnor matrices as it is easy to do to prove the other parts of Proposition 2.3, we work directly with  $\mathcal{A}^*$ , the dual of the Steenrod algebra.

Recall that in the dual basis to the monomial basis for  $\mathcal{A}^*, \mathcal{P}^{(r_1, r_2, \dots)}$  is dual to  $\xi_1^{r_1} \xi_2^{r_2} \dots$ . Hence  $P_t^t$  is dual to  $\xi_t^{p^t}, P_t(p^t - (p - 1))$  is dual to  $\xi_t^{p^t - (p - 1)}$ , and  $P_{2t}^0$  is dual to

$\xi_{2t}$ . By 2.3(a),  $(P_{2t}^0)^{p-1} = (p-1)!P_{2t}(p-1) = -P_{2t}(p-1)$  and so  $(P_{2t}^0)^{p-1}$  is dual to  $-\xi_{2t}^{p-1}$ .

$$P_t(p^t - (p-1))(P_{2t}^0)^{p-1} = -\mathcal{P}^{(0, \dots, p^t - (p-1), 0, \dots, p-1, 0, \dots)}$$

so  $P_t(p^t - (p-1))(P_{2t}^0)^{p-1}$  is dual to  $-\xi_t^{p^t - (p-1)}\xi_{2t}^{p-1}$ . Recall also that

$$\psi(\xi_t) = \sum_{i=0}^t \xi_{t-i}^{p^i} \otimes \xi_i.$$

Let  $\Psi = (1 \otimes \dots \otimes 1 \otimes \psi)(1 \otimes \dots \otimes \psi) \dots (1 \otimes \psi)\psi$  where there are  $(p-1)$  morphisms in the composition.  $\Psi: \alpha^* \rightarrow (\alpha^*)^p$ . To prove 2.3(f) we must prove the following proposition.

PROPOSITION 6.1.

- (a)  $\Psi(-\xi_t^{p^t - (p-1)}\xi_{2t}^{p-1}) = \xi_t^{p^t} \otimes \xi_{2t}^{p^t} \otimes \dots \otimes \xi_t^{p^t} + \text{other terms}$ .
- (b)  $\Psi(\text{other monomials}) = \text{other terms}$ . The following lemma is straightforward.

LEMMA 6.2

$$\Psi(\xi_t) = \sum_{i_1 + \dots + i_p = t} \xi_{i_1}^{p^{j_1}} \otimes \xi_{i_2}^{p^{j_2}} \otimes \dots \otimes \xi_{i_p}^{p^{j_p}}$$

where

$$j_1 = t - i_1 = i_2 + \dots + i_p, j_2 = t - i_1 - i_2 = i_3 + \dots + i_p, \dots, j_{p-1} = i_p, j_p = 0.$$

COROLLARY 6.3.

- (a)  $\Psi(\xi_t) = \Sigma 1 \otimes \dots \otimes \xi_t \otimes 1 \otimes \dots \otimes 1 + \text{terms involving } \xi_i, i \neq t$ .
- (b)  $\Psi(\xi_{2t}) = \Sigma 1 \otimes \dots \otimes \xi_t^{p^t} \otimes 1 \otimes \dots \otimes \xi_t \otimes \dots \otimes 1 + \text{terms involving } \xi_i, i \neq t$ .
- (c) if  $s \neq t$  or  $2t$ ,  $\Psi(\xi_s) = \text{terms involving } \xi_i, i \neq t \text{ or } \xi_t^{p^q} \text{ with } q > t$ .

Hence, to prove Proposition 6.1, we need only consider monomials in  $\xi_t$  and  $\xi_{2t}$  of the correct dimension, that is,  $\xi_t^{(s+1)p^t - p^{s+1}}\xi_{2t}^{p-1-s}$  with  $0 \leq s \leq p-1$ .

The following lemma is well-known.

LEMMA 6.4. Let  $v_p(n) = \text{the highest power of } p \text{ which divides } n$ . Let

$$n = n_1 + \dots + n_k = \Sigma a_i p^i, n_j = \Sigma a_{ji} p^i.$$

Then  $v_p((n_1, \dots, n_k)) = (\Sigma a_{ji} - \Sigma a_i)/p - 1$ , where  $(n_1, \dots, n_k) = n!/n_1! \dots n_k!$ .

*Proof of 6.1(a).* By 6.3(b),  $\Psi(\xi_{2t}^{p-1}) = (1, \dots, 1)\xi_t^{p^t} \otimes \dots \otimes \xi_t^{p^t} \otimes \xi_t^{p-1} + \text{terms involving } \xi_i, i \neq t \text{ or } \xi_t^{p^q} \text{ with } q > t$ , where  $(1, \dots, 1) = (p-1)!/1! \dots 1! = (p-1)! - 1$ . By 6.3(a),  $\Psi(\xi_t^{p^t - (p-1)}) = 1 \otimes \dots \otimes 1 \otimes \xi_t^{p^t - (p-1)} + \text{other terms}$ . Hence

$$\Psi(\xi_t^{p^t - (p-1)}\xi_{2t}^{p-1}) = -\xi_t^{p^t} \otimes \dots \otimes \xi_t^{p^t} + \text{other terms}.$$

We now study  $\Psi(\xi_t^{(s+1)p^t - p^{s+1}}\xi_{2t}^{p-1-s})$  with  $1 \leq s \leq p-1$ .

LEMMA 6.5. If  $\Psi(\xi_{2t}^{p-1-s})$  has more than  $s$  ones in a term and only  $\xi_t^i$ 's in that term, then some exponent of  $\xi_t$  is  $> p^t$ .

*Proof.* This follows by dimensional reasons. If all exponents are  $\leq p^t$ , then the dimension of the term is

$$\leq (p-s-1)p^t(p^{t-1} + \dots + p + 1) < (p-1-s)(p^{2t-1} + \dots + p + 1) = |\xi_{2t}^{p-1-s}|.$$



LEMMA 6.6. Given an ordered  $(p - s)$ -tuple,  $(a_1, \dots, a_{p-s})$ , with all  $a_i \leq p^t$ , then the number of times terms of the form

$$1 \otimes \cdots \otimes \xi_t^{a_1} \otimes \cdots \otimes \xi_t^{a_2} \otimes \cdots \otimes \xi_t^{a_{p-s}} \otimes \cdots \otimes 1$$

appear in the expansion of  $\Psi(\xi_{2t})^{p-1-s}$  is  $\equiv 0(p)$  if  $1 \leq s \leq p - 1$ .

*Proof.* By 6.3(b), the only possible tuple that can appear is  $(p^t, p^t, \dots, p^t, p - s - 1)$  and this appears  $\binom{p}{s}$ -times, but  $\binom{p}{s} \equiv 0(p)$  if  $1 \leq s \leq p - 1$ .

LEMMA 6.7. Given an ordered  $p$ -tuple,  $(n_1, \dots, n_p)$ , with all  $n_j \leq p^t$  and fewer than  $s$   $p^t$ 's, then the number of times the term  $\xi_t^{n_1} \otimes \cdots \otimes \xi_t^{n_p}$  appears in the expansion of

$$\Psi(\xi_t^{(s+1)p^t - p + s + 1})$$

is  $\equiv 0(p)$  if  $1 \leq s \leq p - 1$ .

*Proof.* We must show  $v_p((n_1, \dots, n_p)) \geq 1$ , where

$$n_1 + \cdots + n_p = (s + 1)p^t - p + s + 1.$$

By Lemma 6.4,  $v_p((n_1, \dots, n_p)) \geq 1$ , unless  $\sum_j a_{ji} = a_i$  for all  $i$ . Assume  $\sum_j a_{ji} = a_i$ . We are given that  $a_{ji} = 0$  if  $i > t$ , hence  $a_i = 0$  if  $i > t$ , and that  $\sum_j a_{jt} = a_t < s$ . Then

$$n \leq (s - 1)p^t + (p^t - 1) = sp^t - 1 < (s + 1)p^t - p + s + 1 = n,$$

contradiction.

*Proof of 6.1(b).* Consider

$$\Psi(\xi_t^{(s+1)p^t - p + s + 1} \xi_{2t}^{p-1-s}) = \Psi(\xi_t^{(s+1)p^t - p + s + 1}) \Psi(\xi_{2t}^{p-1-s})$$

with  $1 \leq s \leq p - 1$ . In order to obtain a term  $\xi_t^{p^t} \otimes \cdots \otimes \xi_t^{p^t}$  in the expansion, the number of ones in expansion of  $\Psi(\xi_{2t}^{p-1-s})$  must be  $\leq s$  by Lemma 6.5. If the number of ones in this expansion is  $s$ , then the  $\xi_t^{p^t} \otimes \cdots \otimes \xi_t^{p^t}$  which appears would appear in a multiple of  $p$  times by Lemma 6.6 because  $\Psi(\xi_t^{(s+1)p^t - p + s + 1})$  is commutative. If the number of ones is less than  $s$ , then the corresponding term in  $\Psi(\xi_t^{(s+1)p^t - p + s + 1})$  is zero by Lemma 6.7. This proves 6.1 and hence 2.3(f).

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