HOMOTOPY THEORY FOR BEGINNERS

JESPER M. MØLLER

Abstract. This note contains comments to Chapter 0 in Allan Hatcher's book [5].

Contents

1. Notation and some standard spaces and constructions

In this section we fix some notation and recollect some standard facts from general topology.

1.1. **Standard topological spaces.** We will often refer to these standard spaces:

- $\mathbf{R}$ is the real line and $\mathbf{R}^n = \mathbf{R} \times \cdots \times \mathbf{R}$ is the n-dimensional real vector space
- $\mathbf{C}$ is the field of complex numbers and $\mathbf{C}^n = \mathbf{C} \times \cdots \times \mathbf{C}$ is the n-dimensional complex vector space
- $\mathbf{H}$ is the (skew-)field of quaternions and $\mathbf{H}^n = \mathbf{H} \times \cdots \times \mathbf{H}$ is the n-dimensional quaternion vector space
- $S^n = \{x \in \mathbf{R}^{n+1} \mid |x| = 1\}$ is the unit n-sphere in $\mathbf{R}^{n+1}$
- $D^n = \{x \in \mathbf{R}^n \mid |x| \leq 1\}$ is the unit n-disc in $\mathbf{R}^n$
- $I = [0,1] \subset \mathbf{R}$ is the unit interval
- $\mathbf{R}P^n$, $\mathbf{C}P^n$, $\mathbf{H}P^n$ is the topological space of 1-dimensional linear subspaces of $\mathbf{R}^{n+1}$, $\mathbf{C}^{n+1}$, $\mathbf{H}^{n+1}$.
- $M_g = (S^1 \times S^1)\# \cdots \#(S^1 \times S^1)$ is the orientable and $N_g = \mathbf{R}P^2 \# \cdots \# \mathbf{R}P^2$ the nonorientable compact surface of genus $g \geq 1$.

We shall meet these spaces many times later on. How do we define the topology on the projective spaces (and the surfaces)?

Date: September 9, 2015.

/home/moller/underv/algtop/notes/chp0/comments.tex.

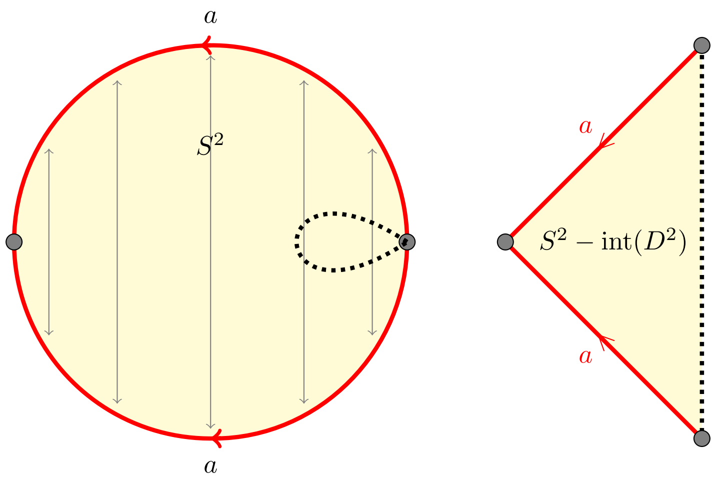


FIGURE 1. The 2-sphere $S^2 = M_0$ and the punctured S^2 (2-disc D^2)

1.2. **The quotient topology.** If X and Y are topological spaces a **quotient map** (General Topology, 2.76) is a surjective map $p\colon X \to Y$ such that

$$\forall V \subset Y\colon V \text{ is open in } Y \iff p^{-1}(V) \text{ is open in } X$$

The map $p\colon X \to Y$ is continuous and the topology on Y is the finest topology making p continuous. If $f\colon X \to Z$ is a continuous map from X into a topological space Z then

$$f \text{ is constant on the fibres of } p \iff f \text{ factors through } p$$

where we say that f factors through p if there exists a continuous map $\bar{f}\colon Y \to Z$ such that the diagram

$$X = \coprod_{y \in Y} p^{-1}y \qquad \begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Z \\ & Y \arrow[ur, dotted, "\bar{f}"'] & \end{tikzcd}$$

commutes. This means that a map defined on Y is the same thing as a map defined on X and constant on the fibres of p.

If X is a topological space, Y is a *set* (with no topology), and $p\colon X \to Y$ a surjective map, the **quotient topology** on Y is the topology $\{V \subset Y \mid p^{-1}V \text{ is open in } X\}$ that makes $p\colon X \to Y$ a quotient map (General Topology, 2.74).

Example 1.1. Here are three examples of quotient topologies and quotient maps:

- **$X/\sim$:** If $\sim$ is a relation on X, let $X/\sim$ be the set of equivalence classes for the smallest equivalence relation containing the relation $\sim$. We give $X/\sim$ the quotient topology for the surjective map $p\colon X \to X/\sim$ taking points in X to their equivalence classes. This means that a set of equivalence classes is open in $X/\sim$ if and only if their union is an open set in X. The universal property of quotient maps says that a continuous map $g\colon X/\sim \to Z$ is the same thing as a continuous map $f\colon X \to Z$ that respects the relation in the sense that $x_1 \sim x_2 \implies f(x_1) = f(x_2)$ for all $x_1, x_2 \in X$.
- **X/A:** If A is a closed subspace of X, the quotient space X/A is the *set* $(X - A) \cup \{A\}$ with the quotient topology for the map $p\colon X \to X/A$ taking points of $X - A$ to points of $X - A$ and points of A to $\{A\}$. A map $X/A \to Z$ is the same thing as a map $X \to Z$ constant on A.
- **X/G:** If G is a group that acts on a topological space X then X/G is the set of G-orbits. A map $X/G \to Z$ is the same thing as a map $X \to Z$ that is constant of all G-orbits.

Example 1.2. (a) The projective spaces have the quotient topology for the surjective maps from the unit spheres

$$p_n\colon S^n = S(\mathbf{R}^{n+1}) \to \mathbf{R}P^n, \qquad p_n\colon S^{2n+1} = S(\mathbf{C}^{n+1}) \to \mathbf{C}P^n, \qquad p_n\colon S^{4n+3} = S(\mathbf{H}^{n+1}) \to \mathbf{H}P^n \tag{1.3}$$

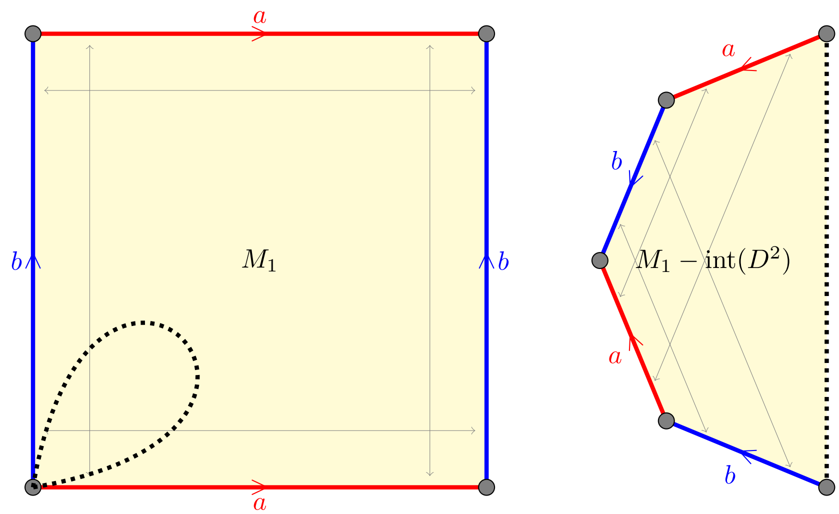


FIGURE 2. The torus $M_1 = S^1 \times S^1$ and the punctured torus

given by $p_n(x) = Fx \subset F^{n+1}$, $x \in S(F^{n+1})$, $F = \mathbf{R}, \mathbf{C}, \mathbf{H}$. We may also define the projective spaces

$$\mathbf{R}P^n = S(\mathbf{R}^{n+1})/S(\mathbf{R}), \qquad \mathbf{C}P^n = S(\mathbf{C}^{n+1})/S(\mathbf{C}), \qquad \mathbf{H}P^n = S(\mathbf{H}^{n+1})/S(\mathbf{H}),$$

as spaces of orbits for the actions of the unit spheres in $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$ on the unit spheres in $\mathbf{R}^{n+1}$, $\mathbf{C}^{n+1}$, $\mathbf{H}^{n+1}$. A map $\overline{f}\colon \mathbf{R}P^n \to Y$ is the same thing as a map $f\colon S^n \to Y$ so that $f(-x) = f(x)$ for all $x \in S^n$.

(b) $S^n = D^n/S^{n-1}$ for all $n \geq 1$. There is a bijective correspondence between maps $S^n \to Y$ and maps $D^n \to Y$ that take S^{n-1} to a point in Y.

(c) The 2-sphere S^2 is the quotient space of the 2-disc D^2 indicated in Figure 1. The punctured 2-sphere is a 2-disc. Thus $S^2 = (D^2 \amalg D^2)/S^1$ is the union of two 2-discs identified along their boundaries.

(d) The real projectivive plane $\mathbf{R}P^2$ is the quotient space of the 2-disc D^2 indicated in Figure 3. The puntured $\mathbf{R}P^2$ is a Möbius band. (Cut up $N_1 - \mathrm{int}(D^2)$ along a horizontal line and reassamble.) Thus $\mathbf{R}P^2 = (D^2 \amalg \mathrm{MB})/S^1$ is the union of a 2-disc and a Möbius band identified along their boundaries.

(e) The circle $S^1 = [-1,+1]/-1 \sim +1$ is a quotient of the interval $[-1,+1]$ and the torus $M_1 = S^1 \times S^1 = ([-1,+1]/-1 \sim +1) \times ([-1,+1]/-1 \sim +1)$ is a quotient of the square $[-1,1] \times [-1,1]$ as shown in Figure 2. (Note: The observant reader may wonder if the product space and the quotient space construction commute.) This description of $M_1 = S^1 \times S^1$ as a quotient of a 4-gon can be used to describe $M_g = D^2/a_1b_1a_1^{-1}b_1^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1}$ as a quotient space of a $4g$-gon (Example 4.2).

(f) Let D^2/a^2 denote the quotient space of Figure 3. Let $D^2 \to S^2$ be the inclusion of the disc D^2 as the digon of the upper hemisphere in S^2. Then $S^2 \to D^2 \to \mathbf{R}P^2$ respects the equivalence relation a^2. The universal property of quotient space shows that there is a commutative diagram

$$\begin{tikzcd} D^2 \arrow[r] \arrow[d] & S^2 \arrow[d, "x \sim -x"] \\ D^2/a^2 \arrow[r, "\simeq"] & \mathbf{R}P^2 \end{tikzcd}$$

where the bottom horizontal map is a homeomorphism. This description of $N_1 = \mathbf{R}P^2$ as a quotient space of a 2-gon can be used to describe the genus g nonorientable surface $N_g = D^2/a_1a_1\cdots a_ga_g$ as a quotient space of a $2g$-gon (Example 4.2).

1.3. **The category of topological spaces and continuous maps.** The **topological category**, **Top**, is the category where the objects are topological spaces and the morphisms are continuous maps between topological spaces. Two spaces are isomorphic in the topological category if they are homeomorphic. *Topology* is the study of the category **Top**.

In the following, **space** will mean topological space and **map** will mean continuous map.

2. Homotopy

Let X and Y be two (topological) spaces and $f_0, f_1 \colon X \to Y$ two (continuous) maps of X into Y.

Definition 2.1. *The maps f_0 and f_1 are* **homotopic**, $f_0 \simeq f_1$, *if there exists a map, a* **homotopy**, $F \colon X \times I \to Y$ *such that $f_0(x) = F(x,0)$ and $f_1(x) = F(x,1)$ for all $x \in X$.*

Homotopy is an equivalence relation on the set of maps $X \to Y$. We write $[X, Y]$ for the set of homotopy classes of maps $X \to Y$. Since homotopy respects composition of maps, in the sense that

$$f_0 \simeq f_1 \colon X \to Y \text{ and } g_0 \simeq g_1 \colon Y \to Z \Longrightarrow g_0 \circ f_0 \simeq g_1 \circ f_1 \colon X \to Z,$$

composition of maps induces composition $[X,Y] \times [Y,Z] \xrightarrow{\circ} [X,Z]$ of homotopy classes of maps.

$$\begin{tikzcd} X \arrow[r, "f_0 \simeq f_1"] \arrow[dr, dotted, "g_0 f_0 \simeq g_1 f_1"'] & Y \arrow[d, "g_0 \simeq g_1"] \\ & Z \end{tikzcd}$$

Example 2.2. The identity map $S^1 \to S^1$ and the map $S^1 \to S^1$ that takes $z \in S^1 \subset \mathbf{C}$ to z^2 are not homotopic (as we shall see). Indeed, none of the maps $z \to z^n$, $n \in \mathbf{Z}$, are homotopic to each other, so that the set $[S^1, S^1]$ is infinite.

Here is the homotopy class of the most simple map.

Definition 2.3. *A map is* **nullhomotopic** *if it is homotopic to a constant map.*

More explicitly: A map $f \colon X \to Y$ is nullhomotopic if there exist a point $y_0 \in Y$ and a homotopy $F \colon X \times I \to Y$ with $F(x,0) = f(x)$ and $F(x,1) = y_0$ for all $x \in X$.

Example 2.4. The identity map of the circle S^1 is not nullhomotopic (it is **essential**). The equatorial inclusion $S^1 \hookrightarrow S^2$ is nullhomotopic. Indeed, any map of $S^1 \to S^2$ is nullhomotopic.

Definition 2.5. *The spaces X and Y are* **homotopy equivalent**, $X \simeq Y$, *if there are maps,* **homotopy equivalences**, $X \to Y$ *and* $Y \to X$, *such that the two compositions,* $X \to Y \to X$ *and* $Y \to X \to Y$, *are homotopic to the respective identity maps.*

More explicitly: X and Y are homotopy equivalent if there exist maps $f \colon X \to Y$, $g \colon Y \to X$, and homotopies $H \colon X \times I \to X$, $G \colon Y \times I \to Y$, such that $H(x,0) = x$, $H(x,1) = gf(x)$ for all $x \in X$ and $G(y,0) = y$, $G(y,1) = fg(y)$.

If two spaces are homeomorphic, then they are also homotopy equivalent because then there exist maps $f \colon X \to Y$, $g \colon Y \to X$ such that the two compositions are *equal* to the respective identity maps.

If $X \rightleftarrows Y$ are homotopy equivalences, then the induced maps $[T,X] \rightleftarrows [T,Y]$ and $[X,T] \rightleftarrows [Y,T]$ are bijections for any space T.

Homotopy equivalence is an equivalence relation on spaces. A **homotopy type** is an equivalence class of homotopy equivalent spaces.

Here is the homotopy type of the most simple space.

Definition 2.6. *A space is* **contractible** *if it is homotopy equivalent to a one-point space.*

Proposition 2.7. *The space X is contractible if and only if one of the following equivalent conditions holds:*

1. *The identity map 1_X of X is nullhomotopic*
2. *There is a point $x_0 \in X$ and a homotopy $C \colon X \times I \to X$ such that $C(x,0) = x$ and $C(x,1) = x_0$ for all $x \in X$.*

If X is contractible and T is any space, then $[T,X] = [T,*] = *$ and $[X,T] = [*,T]$ is the set of path-components of T.

Example 2.8. $\mathbf{R}^n$ is contractible, S^n is not contractible. The House with Two Rooms [5, p 4] and the House with One Room are contractible.

Any map $X \to \mathbf{R}^n$ is nullhomotopic. The standard inclusion $S^n \to S^{n+1}$ is nullhomotopic since it factors through the contractible space $S^{n+1} - \{*\} = \mathbf{R}^{n+1}$. The infinite dimensional sphere S^∞ is contractible (Example 5.10).

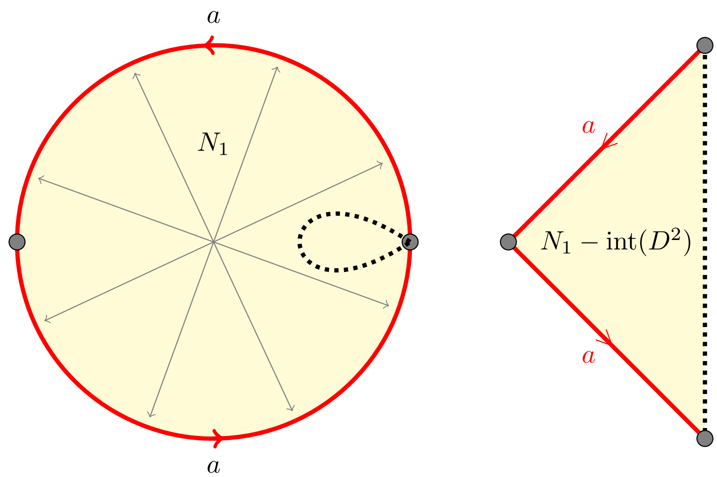


FIGURE 3. The real projective plane $\mathbf{R}P^2 = N_1$ and the punctured $\mathbf{R}P^2$ (Möbius band [5, Exercise 2.1.1])

The **homotopy category of spaces**, **hoTop**, is the category where the objects are topological spaces. The morphisms between two spaces X and Y is the set, $[X, Y]$, of homotopy classes of maps of X into Y. Composition in this category is composition of homotopy classes of maps. Two spaces are isomorphic in the homotopy category if they are homotopy equivalent. Algebraic topology is the study of the homotopy category of spaces, **hoTop**.

2.1. **Relative homotopy.** Let X be a space and $A \subset X$ a subspace. Suppose that $f_0, f_1 \colon X \to Y$ are maps that agree on A, ie $f_0(a) = f_1(a)$ for all $a \in A$.

Definition 2.9. *The maps f_0 and f_1 are homotopic relative to A, $f_0 \simeq f_1 \text{ rel } A$, if there exists a homotopy $F \colon X \times I \to Y$ from f_0 to f_1 such that $f_0(a) = F(a,t) = f_1(a)$ for all $a \in A$ and all $t \in I$.*

If two maps are homotopic rel A, then they are homotopic.

A **pointed space** is a pair (X, x_0) consisting of space X and one of its points $x_0 \in X$. The pointed topological category, $\mathbf{Top}_*$, is the category where the objects are pointed topological spaces and the morphisms are base-point preserving maps, based maps. The pointed homotopy category, $\mathbf{hoTop}_*$, is the category where the objects are pointed topological spaces and the morphisms are based homotopy classes of based maps.

2.2. **Retracts and deformation retracts.** Let X be a space, $A \subset X$ a subspace, and $i \colon A \to X$ the inclusion map. A deformation retraction of X onto A deforms X into A while keeping A fixed.

Definition 2.10. *A* **deformation retraction** *of X onto A is a homotopy $R \colon X \times I \to X$ such that $R(x, 0) = x$ and $R(x, 1) \in A$ for all $x \in X$ and $R(a, t) = a$ for all $a \in A$, $t \in A$. We say that A is a* **deformation retract** *of X if there exists a deformation retraction of X onto A.*

A deformation retraction is a homotopy relative to A between R_0, the identity X, and a map $r = R_1 \colon X \to A$, a retraction, that maps X into A and does not move the points of A. Since $ri = 1_A$ and $ir \simeq 1_A \text{ rel } A$, we see in particular that A and X are homotopy equivalent spaces.

Definition 2.11. *A* **retraction** *of X onto A is a map of X onto A such that $ri = 1_A$. We say that A is a* **retract** *of X if there exist a retraction of X onto A.*

A retraction of X onto A is a left inverse to the inclusion of A into X. A retraction can also be defined as an idempotent map, a map such that $r^j = r$ for all $j \geq 1$.

Proposition 2.12. *Let X be a space and $A \subset X$ a subspace.*

$$A \text{ is a retract of } X \iff \text{Any map on } A \text{ extends to } X$$

$$A \text{ is a deformation retract of } X \iff \text{Any map on } A \text{ extends uniquely up to homotopy relative to } A \text{ to } X$$

Proof. First assertion:
$\Longrightarrow$: Let $r\colon X \to A$ be a retraction of X onto A. If $f\colon A \to Y$ is a map defined on A then $fr\colon X \to Y$ is an extension of f to X.
$\Longleftarrow$: The identity map of A extends to a map $r\colon X \to A$ defined on X.
Second assertion:
$\Longrightarrow$: Let $r\colon X \to A$ be a retraction of X onto A such that $ri = 1_A$ and $ir \simeq 1_X$ rel A. Let $f\colon A \to Y$ be a map defined on A. Since A is a retract of X, f extends to X. Suppose that $f_0, f_1\colon X \to Y$ are two extensions of f. Then $f_0 = f_0 \circ 1_X \simeq f_0 ir$ rel A and $f_1 = f_1 \circ 1_X \simeq f_1 ir$ rel A. As $f_0 i = f_1 i$, this says that $f_0 \simeq f_1$ rel A.
$\Longleftarrow$: The identity map of A extends to a map $r\colon X \to A$ defined on X and $ir \simeq 1_X$ rel A as both ir and 1_X are extensions of the inclusion of A into X. $\square$

$\boxed{\text{Retract}}$

$$\begin{tikzcd} A \arrow[r, equal, "ri = 1_A"] \arrow[d, hook, "i"'] & A \\ X \arrow[ur, "r"'] & \end{tikzcd}$$

$\boxed{\text{Deformation retract}}$

$$\begin{tikzcd} A \arrow[r, equal, "ri = 1_A"] \arrow[d, hook, "i"'] & A \arrow[d, hook, "i"] \\ X \arrow[ur, "r"'] \arrow[r, equal, "ir \simeq 1_X \text{ rel } A"'] & X \end{tikzcd}$$

We already noted that if A is a deformation retract of X, then the inclusion of A into X is a homotopy equivalence. The converse does not hold in general. If the inclusion map is a homotopy equivalence, there exists a map $r\colon X \to A$ such that $ri \simeq 1_A$ and $ir \simeq 1_X$ but r may not fix the points of A and, even if it does, the points in A may not be fixed under the homotopy from ri to the identity of A. Surprisingly enough, however, the converse does hold if the pair (X, A) is sufficiently nice (5.3.(2)).

Example 2.13. We'll later prove that S^1 is *not* a retract of D^2. S^1 is a retract, but not a deformation retract, of the torus $S^1 \times S^1$. $S^1 \vee S^1$, is a deformation retract of a punctured torus (see Figure 2). S^1 is a deformation retract of a punctured $\mathbf{R}P^2$, , a Möbius band (see Figure 3). If X deformation retracts onto one of its points then X is contractible, but there are contractible spaces that do not deformation retract to any of its points.

There are several other good examples of (deformation) retracts in [5].

Proposition 2.14. *Any retract A of a Hausdorff space X is closed.*

Proof. $A = \{x \in X \mid r(x) = x\}$ is the equalizer of two continuous maps [7, Ex 31.5]. $\square$

3. Constructions on topological spaces

We mention some standard constructions on topological spaces and maps.

Example 3.1 (Mapping cylinders, mapping cones, and suspensions)**.** Let X be a space. The **cylinder** on X is the product

$$X \times I$$

of X and the unit interval. The cylinder on X contains $X = X \times \{1\}$ as a deformation retract. What is the cylinder on the n-sphere S^n?

The **cone** on X

$$CX = \frac{X \times I}{X \times 1}$$

is obtained by collapsing one end of the cylinder on X. The cone is always contractible. What is the cone on S^n? A map $f\colon X \to Y$ is nullhomotopic if and only if it extends to the cone on X.

The (unreduced) **suspension** of X

$$SX = \frac{X \times I}{(X \times 0, X \times 1)}$$

is obtained by collapsing both ends of the cylinder on X. The suspension is a union of two cones. What is the suspension of the n-sphere S^n? (General Topology, 2.147) The cylinder, cone, and suspension are endofunctors of the topological category.

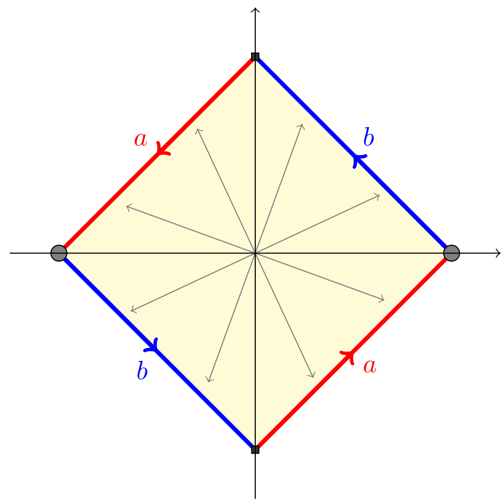


FIGURE 4. An alternative presentation of the projective plane $\mathbf{R}P^2$ (another interpretation of Figure 3)

Let $f\colon X \to Y$ be a map. The **cylinder** on f or **mapping cylinder** of f

$$M_f = \frac{(X \times I) \amalg Y}{(x,0) \sim f(x)}$$

is obtained by glueing one end of the cylinder on X onto Y by means of the map f. The mapping cylinder deformation retracts onto its subspace Y. (Set $r(x,t) = f(x)$ for $x \in X$, $t \in I$, and $r(y) = y$ for $y \in Y$.) What is the mapping cylinder of $z \to z^2\colon S^1 \to S^1$?

The **mapping cone** on f

$$C_f = \frac{CX \amalg Y}{(x,0) \sim f(x)}$$

is obtained by gluing the cone on X onto Y by means of the map f. What is the mapping cone of $aba^{-1}b^{-1}\colon S^1 \to S^1_a \vee S^1_b$ and $a^2\colon S^1 \to S^1_a$?

There is a sequence of maps

$$X \xrightarrow{f} Y \longrightarrow C_f \longrightarrow SX \xrightarrow{Sf} SY \longrightarrow C_{Sf} \longrightarrow SSX \longrightarrow \cdots$$

where the map $C_f \to SX$ is collapse of $Y \subset C_f$.

Proposition 3.2. *Any map factors as an inclusion map followed by a homotopy equivalence.*

Proof. For any map $f\colon X \to Y$ there is a commutative diagram using the mapping cylinder

$$\begin{tikzcd} & M_f \arrow[d, "\simeq"', "(x,t)\to f(x)"] \\ X \arrow[ur, hook, "x\to(x,1)"] \arrow[r, "f"] & Y \end{tikzcd}$$

where the slanted map is an inclusion map and the vertical map is a homotopy equivalence (the target is deformation retract of the mapping cylinder). □

Example 3.3 (Wedge sum and smash product of pointed spaces)**.** Let (X,x_0) and (Y,y_0) be pointed spaces. The **wedge sum** and the **smash product** of X and Y are

$$X \vee Y = X \times \{y_0\} \cup \{x_0\} \times Y \subset X \times Y, \qquad X \wedge Y = \frac{X \times Y}{X \vee Y}$$

The (reduced) **suspension** of the pointed space (X,x_0) is the smash product

$$\Sigma X = X \wedge S^1 = X \wedge I/\partial I = \frac{X \times I}{X \times \partial I \cup \{x_0\} \times I}$$

of X and a pointed circle $(S^1, 1) = (I/\partial I, \partial I/\partial I)$. (The last equality holds when X is a locally compact Hausdorff space (General Topology, 2.171).) Suspension is an endo-functor of $\mathbf{Top}_*$. How does the reduced suspension differ from the unreduced suspension?

What is the smash product $X \wedge I$? What is the smash product $S^m \wedge S^n$? (General Topology, 2.171)

Example 3.4 (The mapping cylinder for the degree n map on the circle)**.** Let $n > 0$ and let $f \colon S^1 \to S^1$ be the map $f(z) = z^n$ where we think of the circle as the complex numbers of modulus 1. Let $C_n = \langle t \mid t^n \rangle$ be the cyclic group of order n. The mapping cylinder M_f of f is quotient space of $\left(\bigvee_{C_n} I\right) \times I$ by the equivalence relation $\sim$ that identifies $(s, x, 0) \sim (st, x, 1)$ for all $s \in C_n$, $x \in I$.

Example 3.5 (Adjunction spaces)**.** See (General Topology, 2.85). From the input $X \supset A \xrightarrow{\varphi} Y$ consisting of a map φ defined on a closed subspace A of X, we define the **adjunction space** $Y \cup_\varphi X$ as $(Y \amalg X)/\sim$ where $A \ni a \sim f(a) \in Y$ for all $a \in A$. The adjunction space sits in a commutative push-out diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, hook] & Y \arrow[d, hook] \\ X \arrow[r, "\Phi"] & Y \cup_\varphi X \end{tikzcd}$$

and it contains Y as a closed subspace with complement $X - A$. A map $Y \cup_\varphi X \to Z$ consists a map $h \colon Y \to Z$ and an extension $g \colon X \to Z$ to X of $h\varphi \colon A \to Z$.

Example 3.6 (The n-cellular extension of a space)**.** Let X be a space and $\phi \colon \coprod S^{n-1}_\alpha \to X$ a map from a disjoint union of spheres into X. The adjunction space $X \cup_\varphi \coprod D^n$ is the push-out of the diagram

$$\begin{tikzcd} \coprod S^{n-1}_\alpha \arrow[r, "\varphi"] \arrow[d, hook] & X \arrow[d, hook] \\ \coprod D^n_\alpha \arrow[r, "\Phi"] & X \cup_\varphi \coprod D^n_\alpha \end{tikzcd}$$

and it is called the **n-cellular extension** of X with **attaching map** φ and **characteristic map** Φ. (Alternatively, $X \cup_\phi \coprod D^n_\alpha$ is the mapping cone on φ.) Thi n-cellular extension of X consists of X as a closed subspace and the **open n-cells** $e^n_\alpha = \Phi(D^n_\alpha - S^{n-1}_\alpha)$. In fact, the complement to X is the disjoint union

$$\left(X \cup_\phi \coprod D^n_\alpha\right) - X = \coprod e^n_\alpha$$

so that the open n-cells are the path-connected components of the complement to X.

The compact surfaces of genus $g \geq 1$ are 2-cellular extensions

$$M_g = \bigvee_{i=1}^{g} S^1_{a_i} \vee S^1_{b_i} \cup_{\prod_{i=1}^g [a_i, b_i]} D^2, \qquad N_g = \bigvee_{i=1}^{g} S^1_{a_i} \cup_{\prod_{i=1}^g a_i^2} D^2$$

of wedges of circles and

$$\mathbf{R}P^n = \mathbf{R}P^{n-1} \cup_{p_{n-1}} D^n$$

is the n-cellular extension of $\mathbf{R}P^{n-1}$ along the quotient map $p_{n-1} \colon S^{n-1} \to \mathbf{R}P^{n-1}$ for $n \geq 1$.

Example 3.7. The **join** of two spaces X and Y is the homotopy push-out

$$\begin{tikzcd} X \times Y \arrow[r, "\mathrm{pr}_Y"] \arrow[d, "\mathrm{pr}_X"'] & Y \arrow[d] \\ X \arrow[r] & X * Y \end{tikzcd}$$

4. CW-complexes

J.H.C. Whitehead [8] decided in 1949 to 'abandon simplicial complexes in favor of cell complexes'. He introduced 'closure finite complexes with weak topology', abbreivated to 'CW-complexes'. They are a class of spaces particularly well suited for the methods of algebraic topology. CW-complexes are built inductively out of cells.

Definition 4.1. *A CW-complex is a space X with an ascending filtration of subspaces (called* **skeleta***)*

$$\emptyset = X^{-1} \subset X^0 \subset X^1 \subset \cdots \subset X^{n-1} \subset X^n \subset \cdots \subset X = \bigcup X^n$$

such that

- *X^0 is a discrete topological space*
- *X^n is (homeomorphic to) an n-cellular extension of X^{n-1} for $n \geq 1$*
- *The topology on X is coherent with the filtration in the sense that*

$$A \text{ is closed (open) in } X \iff A \cap X^n \text{ is closed (open) in } X^n \text{ for all } n$$

  for any subset A of X.

The second item of the definition means that for every $n \geq 0$ there are *attaching maps* $\varphi_\alpha \colon S^{n-1} \to X^{n-1}$ and *characteristic maps* $\Phi_\alpha \colon D^n \to X^n$ such that

- the n-skeleton

$$X^n = X^{n-1} \cup_{\coprod_\alpha \varphi_\alpha} \coprod_\alpha D^n_\alpha$$

  is the n-cellular extension of the $(n-1)$-skeleton by the attaching maps.
- The complement in the n-skeleton of the $(n-1)$-skeleton,

$$X^n - X^{n-1} = \coprod_\alpha e^n_\alpha \xleftarrow{\coprod \Phi_\alpha} \coprod_\alpha \operatorname{int} D^n$$

  is the disjoint union of its connected components $e^n_\alpha = \Phi_\alpha(\operatorname{int} D^n)$, the *open n-cells* of X. An open n-cell is open in X^n but maybe not in X.
- A CW-complex is the disjoint union

$$X = \bigcup_{n=-1}^{\infty} X^n = \bigcup_{n=0}^{\infty} (X^n - X^{n-1}) = \coprod_{n=0}^{\infty} \coprod_\alpha e^n_\alpha$$

  of its open cells. This is a disjoint union of sets (but usually not of topological spaces).
- The quotient of the n-skeleton by the $(n-1)$-skeleton,

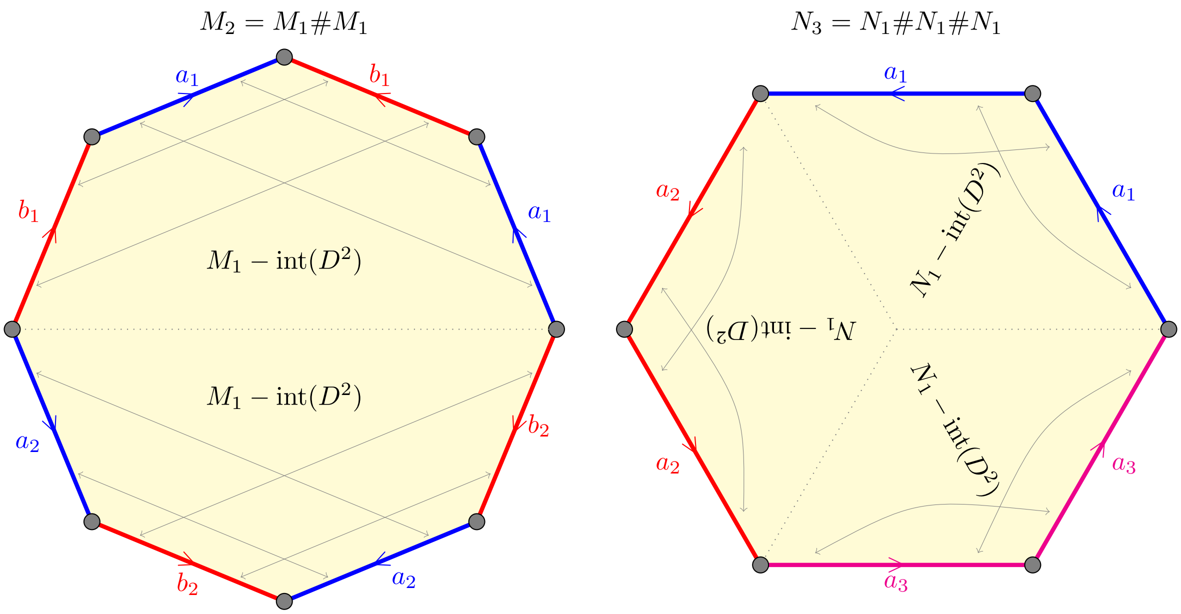
$$X^n/X^{n-1} = \bigvee_\alpha (D^n/S^{n-1}) = \bigvee_\alpha S^n$$

  is a wedge sum of n-spheres.

X^1 is a topological space since it is a 1-cellular extension of the topological space X^0. In fact, all the skeleta X^n are topological spaces and X^i is a closed subspace of X^j for $i \leq j$. The purpose of the third item of the definition is to equip the union of all the skeleta with the largest topology making all the inclusions continuous.

A CW-complex X is finite-dimensional if $X = X^n$ for some n. CW-decompositions are not unique; there are generally many CW-decompositions of a given space X – consider for instance $X = S^2$.

Example 4.2 (Compact surfaces as CW-complexes)**.** We consider the orientable and nonorientable compact surfaces.



The closed orientable surface $M_g = (S^1 \times S^1)\#\cdots\#(S^1 \times S^1)$ of genus $g \geq 1$ is a CW-complex

$$M_g = \bigvee_{1\leq i\leq g} S^1_{a_i} \vee S^1_{b_i} \cup_{\prod [a_i,b_i]} D^2$$

with 1 0-cell, $2g$ 1-cells, and 1 2-cell. (Picture of M_2.)

The closed nonorientable surface $N_h = \mathbf{R}P^2\#\cdots\#\mathbf{R}P^2$ of genus $h \geq 1$ is a CW-complex

$$N_h = \bigvee_{1\leq i\leq h} S^1_{a_i} \cup_{\prod a_i^2} D^2$$

with 1 0-cell, h 1-cells, and 1 2-cell.

Example 4.3 (Spheres as CW-complexes)**.** Points on the n-sphere $S^n \subset \mathbf{R}^{n+1} = \mathbf{R}^n \times \mathbf{R}$ have coordinates of the form (x,u). Let $D^n_\pm$ be the images of the embeddings $D^n \to S^n\colon x \to (x, \pm\sqrt{1-|x|^2})$. Then

$$S^n = S^{n-1} \cup D^n_+ \cup D^n_- = S^{n-1} \cup_{\mathrm{id}\amalg\mathrm{id}} (D^n \amalg D^n)$$

is obtained from S^{n-1} by attaching two n-cells. The infinite sphere S^∞ is an infinite dimensional CW-complex

$$S^0 \subset S^1 \subset \cdots \subset S^{n-1} \subset S^n \subset \cdots \subset S^\infty = \bigcup_{n=0}^{\infty} S^n$$

with two cells in each dimension. A subspace A of S^∞ is closed iff $A \cap S^n$ is closed in S^n for all n.

Example 4.4 (Projective spaces as CW-complexes)**.** The projective spaces are $\mathbf{R}P^n = S(\mathbf{R}^{n+1})/S(\mathbf{R})$, $\mathbf{C}P^n = S(\mathbf{C}^{n+1})/S(\mathbf{C})$, and $\mathbf{H}P^n = S(\mathbf{H}^{n+1})/S(\mathbf{H})$. In each case there are maps

$$D^n = D(\mathbf{R}^n) \xhookrightarrow{x \to (x,\sqrt{1-|x|^2})} S(\mathbf{R}^{n+1}) = S^n \xrightarrow{\quad p_n \quad} \mathbf{R}P^n$$

$$D^{2n} = D(\mathbf{C}^n) \xhookrightarrow{x \to (x,\sqrt{1-|x|^2})} S(\mathbf{C}^{n+1}) = S^{2n+1} \xrightarrow{\quad p_n \quad} \mathbf{C}P^n$$

$$D^{4n} = D(\mathbf{H}^n) \xhookrightarrow{x \to (x,\sqrt{1-|x|^2})} S(\mathbf{H}^{n+1}) = S^{4n+3} \xrightarrow{\quad p_n \quad} \mathbf{H}P^n$$

We note that each of the maps $D(\mathbf{F}^n) \hookrightarrow S(\mathbf{F}^{n+1}) \twoheadrightarrow \mathbf{F}P^n$ is

- surjective
- restricts to the projection $p_{n-1}\colon S(\mathbf{F}^n) \to \mathbf{F}P^{n-1}$ on the boundary $S(\mathbf{F}^n)$ of the disc $D(\mathbf{F}^n)$
- is injective on the interior $D(\mathbf{F}^n) - S(\mathbf{F}^n)$ of the disc

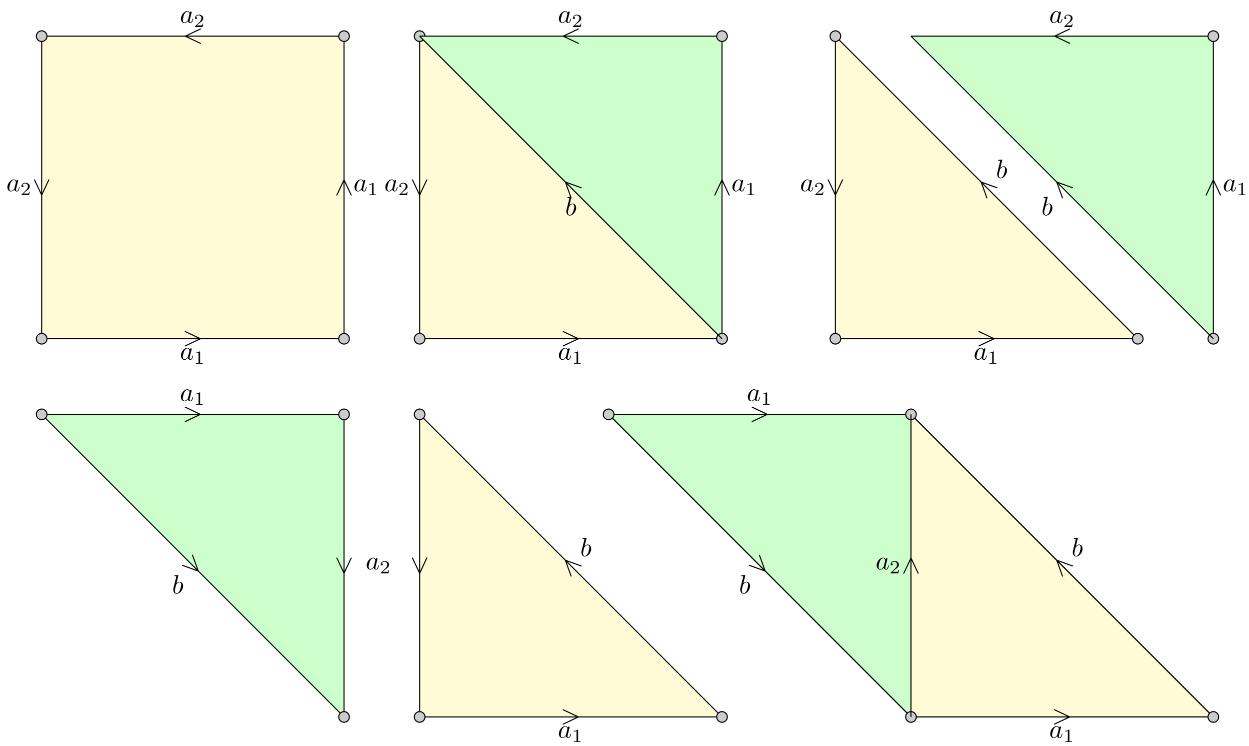


FIGURE 5. Two representations of the Klein bottle N_2

where $\mathbf{F} = \mathbf{R}, \mathbf{C}, \mathbf{H}$. To prove the first item observe that any point in projective space is represented by a point on the sphere with last coordinate ≥ 0. This means that $\mathbf{R}P^n$ consists of $\mathbf{R}P^{n-1}$ together with the n-disc $D(\mathbf{R}^n)$ with identifications on the boundary. In other words

$$\mathbf{R}P^n = \mathbf{R}P^{n-1} \cup_{p_{n-1}} D^n, \qquad \mathbf{C}P^n = \mathbf{C}P^{n-1} \cup_{p_{n-1}} D^{2n}, \qquad \mathbf{H}P^n = \mathbf{H}P^{n-1} \cup_{p_{n-1}} D^{4n},$$

Consequently, $\mathbf{R}P^n$ is a CW-complex with one cell in every dimension between 0 and n, $\mathbf{C}P^n$ is a CW-complex with one cell in every even dimension between 0 and $2n$, $\mathbf{H}P^n$ is a CW-complex with one cell in every dimension divisible by 4 between 0 and $4n$. In particular, $\mathbf{R}P^0 = *$, $\mathbf{C}P^0 = *$, $\mathbf{H}P^0 = *$, and $\mathbf{R}P^1 = S^1$, $\mathbf{C}P^1 = S^2$, $\mathbf{H}P^1 = S^4$. The *Hopf maps* are the projection maps

$$S^0 \to S^1 \xrightarrow{p_1} \mathbf{R}P^1 = S^1, \qquad S^1 \to S^3 \xrightarrow{p_1} \mathbf{C}P^1 = S^2, \qquad S^3 \to S^7 \xrightarrow{p_1} \mathbf{H}P^1 = S^4, \tag{4.5}$$

obtained when $n = 1$.

Definition 4.6. *Let A be any topological space. A relative CW-complex on A is a space X with an ascending filtration of subspaces (called* **skeleta***)*

$$A = X^{-1} \subset X^0 \subset X^1 \subset \cdots \subset X^{n-1} \subset X^n \subset \cdots \subset X = \bigcup X^n$$

such that

- *X^0 is the union of A and a discrete set of points*
- *X^n is (homeomorphic to) an n-cellular extension of X^{n-1} for $n \geq 1$*
- *The topology on X is coherent with the filtration in the sense that*

  $$B \text{ is closed (open) in } X \iff B \cap X^n \text{ is closed (open) in } X^n \text{ for all } n$$

  for any subset B of X.

4.1. **Topological properties of CW-complexes.** We shall see that CW-complexes have convenient topological properties.

Proposition 4.7. *Any CW-complex is a Hausdorff, even normal, topological space.*

Proof. See [7, Exercise 35.8] (solution) or [5, Proposition A.3]. □

Lemma 4.8. *The closure of the open n-cell $e^n_\alpha = \Phi_\alpha(\operatorname{int} D^n)$ is $\overline{e}^n_\alpha = \Phi_\alpha(D^n)$.*

Proof. The image $\Phi_\alpha(D^n)$ of the compact space D^n is compact and therefore closed in the Hausdorff space X. Thus $\overline{e}^n_\alpha \subset \Phi_\alpha(D^n)$. On the other hand, we have $\Phi_\alpha(D^n) = \Phi_\alpha(\overline{D^n - S^{n-1}}) \subset \overline{\Phi_\alpha(D^n - S^{n-1})} = \overline{e}^n_\alpha$ simply because Φ_α is continuous. □

Proposition 4.9. *Any compact subspace of a CW-complex X is contained in a skeleton.*

Proof. Let X be a CW-complex and C a compact subspace of X. Choose a point t_n in $C \cap (X^n - X^{n-1})$ for all n where this intersection is nonempty. Let $T = \{t_n\}$ be the subspace of these points. For all n, $T \cap X^n$ is finite and hence closed in X since points are closed in X (Proposition 4.7). Thus T is closed since X has the coherent topology. In fact, any subspace of T is closed by the same argument. In other words, T has the discrete topology. As a closed subspace of the compact space C, T is compact. Thus T is compact and discrete. Then T is finite. □

4.2. Subcomplexes.

We define what we mean by a subcomplex.

Definition 4.10. *A subcomplex of a CW-complex is a closed subspace that is a union of open cells.*

If A is subcomplex then the closure of any open cell in A is still in A since A is closed.
If A is a subcomplex of the CW-complex X then

- A is a CW-complex with n-skeleton $A^n = A \cap X^n$
- (X, A) is a relative CW-complex
- (X, A) has the homotopy extension property
- X/A is a CW-complex and the quotient map $X \to X/A$ is cellular

Example 4.11. The n-skeleton of X is always a subcomplex of X.
Consider $X = S^1 \vee S^2$ as a CW-complex with one 0-cell, one 1-cell, and one 2-cell attached at a point different from the 0-cell. Then closed subspace S^1 is subcomplex of X. The closed subspace S^2 is not a subcomplex since it is not the union of open cells.

4.3. Products of CW-complexes.

We shall now discuss the product of two CW-complexes. A slight complication will arise because product topologies and infinite union topologies do not in general commute. The product of two pairs of spaces, (X, A) and (Y, B), is defined as

$$(X, A) \times (Y, B) = (X \times Y, A \times Y \cup X \times B)$$

where $(X \times Y) - (A \times Y \cup X \times B) = (X - A) \times (Y - B)$. For example, if I^n is the unit cube in $\mathbf{R}^n$ then clearly

$$(I^n, \partial I^n) = (I^i, \partial I^i) \times (I^j, \partial I^j)$$

whenever $i, j \geq 0$ and $i + j = n$. Since (D^n, S^{n-1}) and $(I^n, \partial I^n)$ are homeomorphic pairs, we have just seen that

$$(D^n, S^{n-1}) = (D^i, S^{i-1}) \times (D^j, S^{j-1})$$

where the equality sign means that the two sides are homeomorphic. We make this observation because by convention we build CW-complexes from discs rather than cubes.

Let $X = A \cup_\varphi D^i$, $Y = B \cup_\psi D^j$, be an i-cellular and a j-cellular extension with characteristic maps $\Phi\colon (D^i, S^{i-1}) \to (X, A)$, $\Psi\colon (D^j, S^{j-1}) \to (Y, B)$ and open cells $e^i = X - A$ and $f^j = Y - B$. The product $X \times Y$ is an $(i + j)$-cellular extension (see below Definition 4.12)

$$X \times Y = (A \times Y \cup X \times B) \cup_\varphi (D^i \times D^j)$$

with one open cell $X \times Y - (A \times Y \cup X \times B) = (X - A) \times (Y - B) = e^i \times f^j$ that is the product of the open cells in X and Y. The characteristic map of $X \times Y$ is the product $\Phi \times \Psi\colon (D^i, S^{i-1}) \times (D^j, S^{j-1}) \to (X, A) \times (Y, B)$ of the characteristic maps and the attaching map $\varphi\colon D^i \times S^{j-1} \cup S^{i-1} \times D^j \to X \times B \cup A \times Y$ is the restriction of $\Phi \times \Psi$ to the sphere $S^{i+j-1} = D^i \times S^{j-1} \cup S^{i-1} \times D^j$.

Definition 4.12. *Let X and Y be CW-complexes with characteristc maps $\Phi_\alpha \colon (D^i, S^{i-1}) \to (X^i, X^{i-1})$ and $\Psi_\beta \colon (D^j, S^{j-1}) \to (Y^j, Y^{j-1})$. The product CW-complex has n-skeleton*

$$(X \times_{\mathrm{CW}} Y)^n = \bigcup_{i+j=n} X^i \times Y^j$$

The characteristic maps for the n-cells are products of characteristc maps

$$\Phi_\alpha \times \Psi_\beta \colon (D^i, S^{i-1}) \times (D^j, S^{j-1}) \to (X^i, X^{i-1}) \times (Y^j, Y^{j-1}) \subset ((X \times_{\mathrm{CW}} Y)^n, (X \times_{\mathrm{CW}} Y)^{n-1})$$

for all $i, j \geq 0$ and $i + j = n$. The attaching maps for the n-cells are the restrictions

$$D^i \times S^{j-1} \cup S^{i-1} \times D^j \to X^i \times Y^{j-1} \cup X^{i-1} \times Y^j \subset (X \times Y)^{n-1}$$

of the characteristic maps. $X \times_{\mathrm{CW}} Y)$ has the topology coherent with the skeleta.

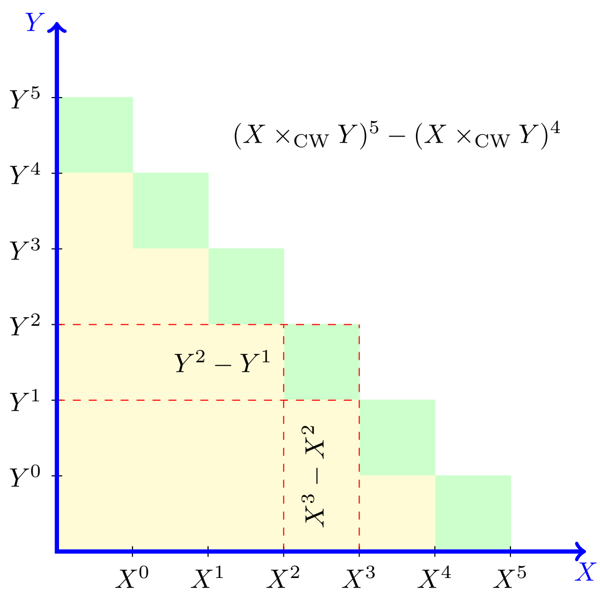
There is a commutative diagram

$$\begin{tikzcd}
(X \times_{\mathrm{CW}} Y)^{n-1} \amalg \coprod_{i+j=n} (D^i_\alpha \times D^j_\beta) \arrow[r, "\mathrm{incl} \amalg \coprod (\Phi^i_\alpha \times \Psi^j_\beta)"] \arrow[d] & (X \times_{\mathrm{CW}} Y)^n \\
(X \times_{\mathrm{CW}} Y)^{n-1} \cup_\rho \coprod_{i+j=n} (D^i_\alpha \times D^j_\beta) \arrow[ur, dotted] &
\end{tikzcd}$$

The horizontal map is closed and the slanted map, produced by the universal property, is a homeomorphism (because it is a closed continuous bijection). This shows that $(X \times_{\mathrm{CW}} Y)^n$ is an n-cellular extension of $(X \times_{\mathrm{CW}} Y)^{n-1}$. Thus $X \times_{\mathrm{CW}} Y$ is a CW-complex as in Definition 4.1. The open n-cells of the product CW-complex,

$$(X \times_{\mathrm{CW}} Y)^n - (X \times_{\mathrm{CW}} Y)^{n-1} = \coprod_{i+j=n} (X^i - X^{i-1}) \times (Y^j - Y^{j-1}) = \coprod_{i+j=n} \left(\coprod_\alpha e^i_\alpha \times \coprod_\beta f^j_\beta \right) = \coprod_{\substack{i+j=n \\ \alpha,\beta}} e^i_\alpha \times f^j_\beta$$

are the products of the open cells e^i_α in X with the open cells f^j_β in Y for all $i, j \geq 0$ with $i + j = n$.



The topology on $X \times_{\mathrm{CW}} Y$, defined to be the topology coherent with the ascending skeletal filtration, is finer than the product topology.

Theorem 4.13. [5, Theorem A.6] *There is a bijective continuous map $X \times_{\mathrm{CW}} Y \to X \times Y$. This map is a homeomorphism if X and Y have countably many cells.*

In all cases relevant for us, $X \times_{\mathrm{CW}} Y$ and $X \times Y$ are homeomorphic.

5. The Homotopy Extension Property

The Homotopy Extension Property will be very important to us. Let X be a space with a subspace $A \subset X$.

Definition 5.1. [1, VII.1] *The pair (X, A) has the Homotopy Extension Property (HEP) if any partial homotopy $A \times I \to Y$ of a map $X \to Y$ into any space Y can be extended to a (full) homotopy of the map.*

Diagrammatically, (X, A) has the HEP if it is always possible to complete the diagram

$$\begin{tikzcd} X \times \{0\} \cup A \times I \arrow[r] \arrow[d, hook, "\mathrm{incl}"'] & Y \\ X \times I \arrow[ur, dotted] & \end{tikzcd}$$

for any space Y and any partial homotopy of a map $X \to Y$.

The pair $(X, \emptyset)$ always has the HEP. A **nondegenerate base point** is a point $x_0 \in X$ such that $(X, \{x_0\})$ has the HEP.

Proposition 5.2. [5, p 14, Ex 0.26] *Let X a space and $A \subset X$ a subspace. The following three conditions are equivalent*

(1) (X, A) has the HEP.
(2) $X \times \{0\} \cup A \times I$ is a retract of $X \times I$.
(3) $X \times \{0\} \cup A \times I$ is a deformation retract of $X \times I$.

Proof. (1) $\iff$ (2): This is a special case of Proposition 2.12.
(3) $\implies$ (2): Clear.
(2) $\implies$ (3): The problem is to turn a retraction $r\colon X \times I \to X \times \{0\} \cup A \times I$ into a deformation retraction $H\colon X \times I \times I \to X \times I$ of the cylinder $X \times I$ onto the cylinder on the inclusion $X \times \{0\} \cup A \times I$. Write the retraction $r\colon X \times I \to X \times \{0\} \cup A \times I \subset X \times I$ as $r(x,t) = (r_1(x,t), r_2(x,t))$ where r_1 is a map into X and r_2 a map into I. Then we can concoct a deformation retraction $H\colon X \times I \times I \to X \times I$ by [2, p 329]

$$H(x,t,s) = (r_1(x,st), (1-s)t + r_2(x,st))$$

We check that $H(x,t,0) = (r_1(x,t), t + r_2(x,0)) = (x, t+0) = (x,t)$, $H(x,t,1) = (r_1(x,t), r_2(x,t)) = r(x,t) \in X \times \{0\} \cup A \times I$, $H(x,0,s) = (x,0)$, and $H(a,t,s) = (r_1(a,ts), (1-s)t + r_2(a,st)) = (a, (1-s)t + st) = (a,t)$ for all $a \in A$ so that H is indeed a deformation retraction of the cylinder $X \times I$ on the identity onto the cylinder on the inclusion, $X \times \{0\} \cup A \times I$. $\square$

5.1. What is the HEP good for?

The next theorem explains what the HEP can do for you.

Theorem 5.3. [5, Proposition 0.17] [5, Proposition 0.18, Ex 0.26] [1, VII.4.5] *Suppose that (X, A) has the HEP.*

(1) If the inclusion map has a homotopy left inverse then A is a retract of X.
(2) If the inclusion map is a homotopy equivalence then A is a deformation retract of X.
(3) If A is contractible then the quotient map $X \to X/A$ is a homotopy equivalence.
(4) The homotopy type of the adjunction space $Y \cup_\varphi X$ only depends on the homotopy class of the attaching map $\varphi\colon A \to Y$ for any space Y and any map $\varphi\colon A \to Y$.

Proof. (1) Assume that $r\colon X \to A$ is a map such that $ri \simeq 1_A$. We must change r on A so that it actually fixes points of A. There is a map $X \times \{0\} \cup A \times I \to A$ which on $X \times \{0\}$ is r and on $A \times I$ is a homotopy from ri to the identity of A. Using the HEP we may complete the commutative diagram

$$\begin{tikzcd} X \times \{0\} \cup A \times I \arrow[r, "r \cup ri \simeq 1_A"] \arrow[d, hook] & A \\ X \times I \arrow[ur, dotted, "h"'] & \end{tikzcd}$$

and get a homotopy $h\colon X \times I \to A$. The end-value of this homotopy is a map $h_1\colon X \to A$ such that $h_1 i = 1_A$ (a retract).
(2) Let $i\colon A \to X$ be the inclusion map. The assumption is that there exists a map $r\colon X \to A$ such that $ri \simeq 1_A$ and $ir \simeq 1_X$. By point (1) we can assume that $ri = 1_A$, ie that A is a retract of X. Let

$G\colon X \times I \to X$ be a homotopy with start value $G_0 = 1_X$ and end value $G_1 = ir$. For $a \in A$, $G(a,0) = a$ and $G(a,1) = a$ but we have no control of $G(a,t)$ when $0 < t < 1$. We want to modify G into a deformation retraction, that is a homotopy from 1_X to ir *relative to* A. Since (X,A) has the HEP so does $(X,A)\times(I,\partial I) = (X \times I, A \times I \cup X \times \partial I)$ ((5.8).(3)). Let $H\colon X \times I \times I \to X \times I$ be an extension (a homotopy of homotopies) of the map $X \times I \times \{0\} \cup A \times I \times I \cup X \times \partial I \times I$ given by

$$\begin{aligned} H(x,t,0) &= G(x,t) \\ H(a,t,s) &= G(a,t(1-s)) \text{ for } a \in A \\ H(x,0,s) &= x \\ H(x,1,s) &= G(ir(x),1-s) \end{aligned}$$

Note that H is well-defined since the first line, $H(x,1,0) = G(x,1) = ir(x)$, and the fourth line, $H(x,1,0) = G(ir(x),1) = irir(x) = ir(x)$, yield the same result. The end value of H, $(x,t) \mapsto H(x,t,1)$, is a homotopy rel A of $H(x,0,1) = x$ to $H(x,1,1) = G(ir(x),0) = ir(x)$. This is a homotopy rel A since $H(a,t,1) = G(a,0) = a$ for all $a \in A$.
(3) What we need is a homotopy inverse to the projection map $q\colon X \to X/A$ and this is more or less the same thing as a homotopy $X \times I \to X$ from the identity to a map that collapses A inside A. How can we get such a homotopy? Well, precisely from the HEP! (This could be used as the motivation for HEP.) Let $C\colon A \times I \to A \subset X$ be a contraction of A, a homotopy of the identity map to a constant map. Use the HEP to extend the contraction of A and the identity on X

$$\begin{tikzcd} X \times \{0\} \cup A \times I \arrow[r, "1_X \cup C"] \arrow[d, hook] & X \\ X \times I \arrow[ur, dotted, "h"'] & \end{tikzcd}$$

to a homotopy $h\colon X \times I \to X$ such that h_0 is the identity map of X, h_t sends A to A for all $t \in I$, and h_1 sends A to a point of A. By the universal property of quotient maps (General Topology, 2.81), the homotopy h induces a homotopy $\bar{h}$ and the map h_1 induces a map $\widetilde{h}_1$ such that the diagram

$$\begin{tikzcd} & X \times \{0\} \cup A \times I \arrow[r, "C \cup 1_X"] \arrow[d, hook] & X \arrow[d, equal] & \\ & X \times I \arrow[r, "h"] \arrow[d, "q \times 1_I"'] & X \arrow[d, "q"] & h_0 = 1_X,\ h_1(A) = * \\ \text{quotient map} & X/A \times I \arrow[r, "\bar{h}"] & X/A & \bar{h}_0 = 1_{X/A},\ \bar{h}_1(*) = * \end{tikzcd}$$

commutes. (The product map $q \times 1\colon X \times I \to X/A \times I$ is quotient since I is locally compact Hausdorff (General Topology, 2.87).) Since h_1 takes A to a point, it factors through the quotient space X/A. The lower square considered only at time $t = 1$ can be enlarged to a commutative diagram

$$\begin{tikzcd} X \arrow[r, "h_1"] \arrow[d, "q"'] & X \arrow[d, "q"] \\ X/A \arrow[r, "\bar{h}_1"'] \arrow[ur, "\widetilde{h}_1"] & X/A \end{tikzcd} \qquad \begin{aligned} \widetilde{h}_1 q &= h_1 \simeq h_0 = 1_X \\ q\widetilde{h}_1 &= \bar{h}_1 \simeq \bar{h}_0 = 1_{X/A} \end{aligned}$$

with a diagonal map $\widetilde{h}_1$ – and this is the homotopy inverse to q that we are looking for!
(4) Let $\varphi_0\colon Y \to and\ \varphi_1\colon Y \to be$ two attaching maps. Suppose that $\varphi\colon A \times I \to Y$ is a homotopy from φ_0 to φ_1. We want to show that $Y \cup_{\varphi_0} X$ and $Y \cup_{\varphi_1} X$ are homotopy equivalent. The point is that both $Y \cup_{\varphi_0} X$ and $Y \cup_{\varphi_1} X$ are deformation retracts of $Y \cup_{\varphi} (X \times I)$. We get the deformation retractions of $Y \cup_{\varphi} (X \times I)$

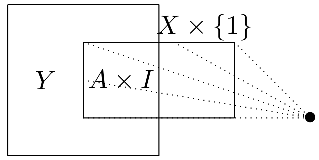


FIGURE 6. A deformation retraction $Y \cup_\varphi (X \times I) \times I \to Y \cup_\varphi (X \times I)$ of $Y \cup_\varphi (X \times I)$ onto $Y \cup_{\varphi_1} X$

onto $Y \cup_{\varphi_0} X$ or $Y \cup_{\varphi_1} X$ from the deformation retractions of Proposition 5.2.(3) of $X \times I$ onto $A \times I \cup X \times \{0\}$ or $A \times I \cup X \times \{1\}$. The idea behind the proof is indicated in Figure 6.

We intend to show that the inclusions

$$Y \cup_{\varphi_0} X = Y \cup_\varphi (X \times \{0\} \cup A \times I) \subset Y \cup_\varphi (X \times I) \supset Y \cup_\varphi (X \times \{1\} \cup A \times I) = Y \cup_{\varphi_1} X$$

are homotopy equivalences. (The equality signs are there because all points of $A \times I$ have been identified to points in Y.) The left inclusion is a homotopy equivalence because the subspace is a deformation retract of the big space. The deformation retraction $\overline{h}$ of $Y \cup_\varphi (X \times I)$ onto $Y \cup_{\varphi_0} X$ is induced by the universal property of adjunction spaces (General Topology, 14.17) as in the diagram

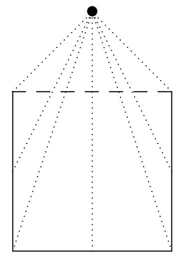
$$\begin{tikzcd}
A \times I \arrow[rrr, "f"] \arrow[ddd, hook, "\mathrm{incl}"'] & & & Y \arrow[ddd] \\
& A \times I \times I \arrow[ul, "\pi_1 \times \pi_2"'] \arrow[r, "f \times 1"] \arrow[d, hook, "\mathrm{incl}"'] & Y \times I \arrow[ur, "\pi_1"'] \arrow[d] & \\
& X \times I \times I \arrow[r] \arrow[dl, "h"'] & Y \cup_\varphi (X \times I) \times I \arrow[dr, dotted, "\overline{h}"] & \\
X \times I \arrow[rrr] & & & Y \cup_\varphi (X \times I)
\end{tikzcd}$$

from a deformation retraction $h\colon X \times I \times I \to X \times I$ of $X \times I$ onto $X \times \{0\} \cup A \times I$ (Proposition 5.2.(3)). Here, the outer square is the push-out diagram for $Y \cup_\varphi (X \times I)$ and the inner square is just this diagram crossed with the unit interval. The homotopy $h\colon X \times I \times I \to X \times I$ starts as the identity map, is constant on the subspace $X \times \{0\} \cup A \times I \subset X \times I$, and ends as a retraction of $X \times I$ onto this subspace. The induced homotopy $\overline{h}\colon Y \cup_\varphi (X \times I) \times I \to Y \cup_\varphi (X \times I)$ starts as the identity map, is constant on the subspace $Y \cup_\varphi (X \times \{0\} \cup A \times I) = Y \cup_{\varphi_0} X$, and ends as a retraction onto this subspace. We conclude that $Y \cup_\varphi (X \times I)$ deformation retracts onto its subspace $Y \cup_{\varphi_0} X$. Similarly, $Y \cup_\varphi (X \times I)$ deformation retracts onto its subspace $Y \cup_{\varphi_1} X$. Thus $Y \cup_{\varphi_0} X$ and $Y \cup_{\varphi_1} X$ are homotopy equivalent spaces. (Note that we proved this by construction a zig-zag $Y \cup_{\varphi_0} X \to Y \cup_\varphi (X \times I) \leftarrow Y \cup_{\varphi_1} X$ of homotopy equivalences, not by constructing a direct homotopy equivalence between the two spaces.) $\square$

5.2. **Are there any pairs of spaces that have the HEP?.** Our work on HEP pairs would be futile if there weren't any pairs that enjoying this property. But we shall next see that pairs with the HEP are uniquitous: It is difficult, but not impossible, to find a pair that does not have the HEP.

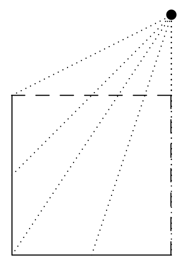
Corollary 5.4. *The pair (D^n, S^{n-1}) has the HEP for all $n \geq 1$. In fact, (CX, X) has the HEP for all spaces X.*

Proof. For instance, for $n=1$, $D^1\times I\subset \mathbf{R}\times I\subset \mathbf{R}^2$ (deformation) retracts onto $D^1\times\{0\}\cup S^0\times I$ by radial projection from $(0,2)$ as indicated in this picture:



In fact, $D^n\times I\subset \mathbf{R}^n\times I\subset\mathbf{R}^{n+1}$ (deformation) retracts onto $D^n\times\{0\}\cup S^{n-1}\times I$ by a radial projection from $(0,\dots,0,2)$.

More generally, for any space X, the pair (CX,X) has the HEP because $CX\times\{0\}\cup X\times I$ is a retract of $CX\times I$. The below picture indicates a retraction $R\colon I\times I\to\{0\}\times I\cup I\times\{0\}$, sending all of $\{1\}\times I$ to the point $(1,0)$.



The map $1_X\times R\colon X\times I\times I\to X\times\{0\}\times I\cup X\times I\times\{0\}$ factors through

$$\begin{tikzcd} X\times I\times I \arrow[r, "1_X\times R"] \arrow[d] & X\times\{0\}\times I\cup X\times I\times\{0\} \arrow[d] \\ \frac{X\times I}{X\times\{1\}}\times I \arrow[r, dotted] & X\times\{0\}\times I\cup\frac{X\times I}{X\times\{1\}}\times\{0\} \end{tikzcd}$$

to give the required retraction $CX\times I\to X\times I\cup CX\times\{0\}$. (Remember that the left vertical map is a quotient map by the Whitehead Theorem from General Topology.) $\square$

Proposition 5.5. *If (X,A) has the HEP and X is Hausdorff, then A is a closed subspace of X.*

Proof. $X\times\{0\}\cup A\times I$ is a closed subspace of $X\times I$ since it is a retract (Proposition 2.14). Now look at X at level $\frac12$ inside the cylinder $X\times I$. $\square$

See [1, VII.1.5] for a necessary condition for an inclusion to have the HEP.

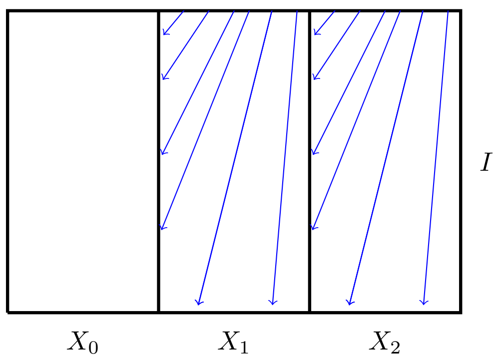
Example 5.6. (A closed subspace that does not have the HEP) (General topology exam, Problem 3). (I,A) where $A=\{0\}\cup\{\frac1n|n=1,2,\dots\}$ does *not* have the HEP since $I\times\{0\}\cup A\times I$ is not a retract of $I\times I$. Indeed, assume that $r\colon I\times I\to I\times\{0\}\cup A\times I$ is a retraction. For each $n\in\mathbf{Z}_+$, the map $t\to(t\times 1)$, $t\in[\frac{1}{n+1},\frac1n]$, is a path in $I\times I$ from $\frac{1}{n+1}\times 1$ to $\frac1n\times 1$ and its image under the retraction, $t\mapsto r(t\times 1)$, $\frac{1}{n+1}\le t\le\frac1n$, is a path in A connecting the same two points. Such a path must pass through all points of $(\frac{1}{n+1},\frac1n)\times\{0\}\subset I\times\{0\}$ because $\pi_1 r([\frac{1}{n+1},\frac1n]\times\{1\})\supset[\frac{1}{n+1},\frac1n]$ by connectedness. Thus there is a point $t_n\in(\frac{1}{n+1},\frac1n)$ such that $r(t_n\times 1)\in(\frac{1}{n+1},\frac1n)\times\{0\}$. This contradicts continuity of r for $t_n\times 1$ converges to 0×1 and $r(t_1\times 1)$ converges to $0\times 0\neq r(0\times 1)$. (A similar, but simpler, argument shows that there is no retraction $r\colon A\times I\to A\times\{0\}\cup\{0\}\times I$ so that 0 is a degenerate base-point of A.)

Example 5.7 (A closed subspace that does not have the HEP)**.** Let C be the quasi-circle [5, Ex 1.3.7]. Collapsing the interval $A=[-1,1]\subset C$ lying on the vertical axis gives a quotient map $C\to S^1$ [6, §28] which is not a homotopy equivalence (since $\pi_1(S^1)=\mathbf{Z}$ and $\pi_1(C)$ is trivial) even though A is contractible. Thus (C,A) does not have the HEP.

Proposition 5.8 (Hereditary properties of the HEP)**.** *Suppose (X,A) has the HEP.*

(1) *(Transitivity) If $X_0 \subset X_1 \subset X_2$ and both pairs (X_2,X_1) and (X_1,X_0) have the HEP, then (X_2,X_0) has the HEP. More generally, if $X = \bigcup X_k$ has the coherent topology with respect to its subspaces $X_0 \subset X_1 \subset \cdots \subset X_{k-1} \subset X_k \subset \cdots$ where each pair of consecutive subspaces has the HEP, then (X,X_0) has the HEP.*
(2) *$Y \times (X,A) = (Y \times X, Y \times A)$ has the HEP for all spaces Y.*
(3) *$(X,A) \times (I,\partial I) = (X \times I, X \times \partial I \cup A \times I)$ has the HEP.*
(4) *$(Y \cup_\varphi X, Y \cup_\varphi A)$ has the HEP for all spaces Y and all maps $\varphi\colon B \to Y$ defined on a closed subspace B of A. In particular, $(Y \cup_\varphi X, Y)$ has the HEP for any attaching map $\varphi\colon A \to Y$. (See Figure 7)*
(5) *The n-cellular extension $(Y \cup_\varphi \coprod D^n, Y)$ of any space Y has the HEP for any attaching map $\varphi\colon \coprod S^{n-1} \to Y$.*

Proof. (1) In the first case, there are retractions $r_2\colon X_2 \times I \to X_1 \times I \cup X_2 \times \{0\}$ and $r_1\colon X_1 \times I \cup X_2 \times \{0\} \to X_0 \times I \cup X_2 \times \{0\}$. Then $r_1 r_2$ is a retraction of $X_2 \times I$ onto $X_0 \times I \cup X_2 \times \{0\}$.



$X_2 \times I$ retracts onto $X_0 \times I \cup X_2 \times \{0\}$
$r_2\colon X_2 \times I \to X_1 \times I \cup X_2 \times \{0\}$
$r_1\colon X_1 \times I \cup X_2 \times \{0\} \to X_0 \times I \cup X_2 \times \{0\}$
$r_2 r_1\colon X_2 \times I \to X_0 \times I \cup X_2 \times \{0\}$

In the general case, there are retractions $r_k\colon X_k \times I \cup X \times \{0\} \to X_{k-1} \times I \cup X \times \{0\}$. There is a well-defined retraction $r_1 r_2 \cdots r_k \cdots \colon X \times I \to X_0 \times I \cup X \times \{0\}$ that on $X_k \times I \cup X \times \{0\}$ is

$$X_k \times I \cup X \times \{0\} \xrightarrow{r_k} X_{k-1} \times I \cup X \times \{0\} \xrightarrow{r_{k-1}} \cdots \xrightarrow{r_2} X_1 \times I \cup X \times \{0\} \xrightarrow{r_1} X_0 \times I \cup X \times \{0\}$$

This retraction $X \times I \to X_0 \times I \cup X \times \{0\}$ is continuous because the product topology on $X \times I$ is coherent with the filtration $X_k \times I$, $k = 0,1,\ldots$. (Verify this claim!)
(2) We use Proposition 5.2. Let $r\colon X \times I \to X \times I$ be a retraction onto $X \times \{0\} \cup A \times I$. Then the product map $1_Y \times r$ is a retraction of $(Y \times X) \times I$ onto $(Y \times X) \times \{0\} \cup (Y \times A) \times I$.
(3) [2, 7.5 p. 330]
(4) We use Proposition 5.2 again. Let $r\colon X \times I \to X \times I$ be a retraction onto $X \times \{0\} \cup A \times I$. The universal property of quotient maps provides a factorization, $\overline{1_{Y\times I} \amalg r}$, of $1_{Y \times I} \amalg r$

$$\begin{tikzcd}
(Y \amalg X) \times I \arrow[r, "1_{Y\times I} \amalg r"] \arrow[d, "q \times 1_I"'] & (Y \amalg X) \times I \arrow[d, "q \times 1_I"] \\
(Y \cup_\varphi X) \times I \arrow[r, "\overline{1_{Y\times I} \amalg r}"'] & (Y \cup_\varphi X) \times I
\end{tikzcd}$$

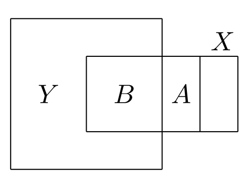
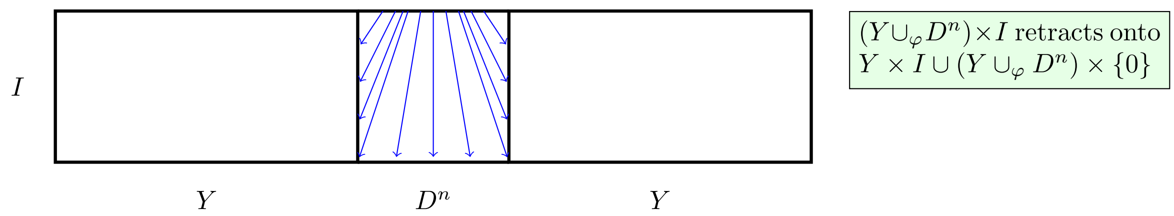


FIGURE 7. The pair $(Y \cup_\varphi X, Y \cup_\varphi A)$

that is a retraction of $Y \cup_\varphi X \times I$ onto $Y \cup_\varphi X \times \{0\} \cup Y \cup_\varphi A \times I$. To prove continuity, note that the left vertical map is a quotient map since I is locally compact Hausdorff (General Topology, 2.87). This shows that $(Y \cup_\varphi X, Y \cup_\varphi A)$ has the HEP. If the attaching map φ is defined on all of A, we have that $(Y \cup_\varphi X, Y \cup_\varphi A) = (Y \cup_\varphi X, Y)$ so this pair has the HEP.



(5) This is a special case of (4) since $(\coprod D^n, \coprod S^{n-1})$ has the HEP (Corollary 5.4).

□

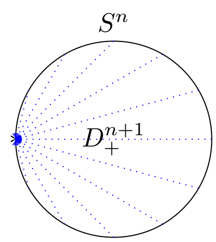
Corollary 5.9. [8, Footnote 32] *Any relative CW-complex (X, A) (Definition 4.6) has the HEP. In particular, any CW-pair (X, A) has the HEP.*

Proof. There is a filtration of X

$$A = X^{-1} \subset X^0 \subset X^1 \subset \cdots \subset \cup X^{n-1} \subset \cup X^n \subset \cdots \subset X$$

where X^n, $n \geq 0$, is obtained from $A \cup X^{n-1}$ by attaching n-cells. Since a cellular extension has the HEP, transitivity (Proposition 5.8.(1)) implies that also (X, A) has the HEP. □

Example 5.10 (S^∞ is contractible)**.** Choose $* = 1$ as the base-point of $\mathbf{R} \supset S^0 \subset S^\infty$. Let D^{n+1}_+ denote the upper half of $S^{n+1} = D^{n+1}_- \cup D^{n+1}_+$. Since the base point $\{*\}$ is a deformation retract of the disc D^{n+1}_+



there is a homotopy $R^n \colon S^n \times [\frac{n}{n+1}, \frac{n+1}{n+2}] \to S^{n+1}$ from the inclusion map of S^n into S^{n+1} to the constant map $S^n \to *$ and this homotopy is relative to the base point $\{*\}$.

Since (S^1, S^0) has the HEP, the partial homotopy $R^0 \colon S^0 \times [0, 1/2] \to S^1$ extends to a homotopy $S^1 \times [0, 1/2] \to S^1$, relative to the base point, from the identity map of S^1 to some map $f_1 \colon S^1 \to S^1$ that sends S^0 to $*$.

Since (S^2, S^1) has the HEP, the homotopy $S^1 \times [1/2, 2/3] \to S^2 \colon (x, t) \to R^1(f_1(x), t)$, which is constant on $S^0 \times [1/2, 2/3]$, combined with the already constructed homotopy $S^1 \times [0, 1/2] \to S^1$ and the identity on $S^2 \times \{0\}$, extends to a homotopy $S^2 \times [0, 2/3] \to S^2$, constant on $S^1 \times [1/2, 2/3]$, from the identity map of S^2 to some map $f_2 \colon S^2 \to S^2$ that sends S^1 to $*$.

Continue like this and get a homotopy $S^\infty \times [0, 1] \to S^\infty$, from the identity to the constant map relative to the base point.

Figure 8 shows the beginning of a homotopy $S^\infty \times I \to S^\infty$ rel $*$ between the identity map and the constant map. It is continuous because the area where it is constant (indicated by the dotted lines) gets larger and larger as we approach $S^\infty \times \{1\}$.

In the above example we actually proved the following:

Proposition 5.11. *Let X be a CW-complex with skeleta X^n, $n \geq 0$, and base point $* \in X^0$. If all the inclusions $X^n \hookrightarrow X^{n+1}$ are homotopic* rel $*$ *to the constant map $*$, then the identity map of X is homotopic* rel $*$ *to the constant map $*$ and X is contractible.*

Exercise 5.12. The Dunce cap [3] is the quotient of the of the 2-simplex by the identifications indicated in Fig 9. Show that the Dunce cap is contractible, in fact, homotopy equivalent to D^2.

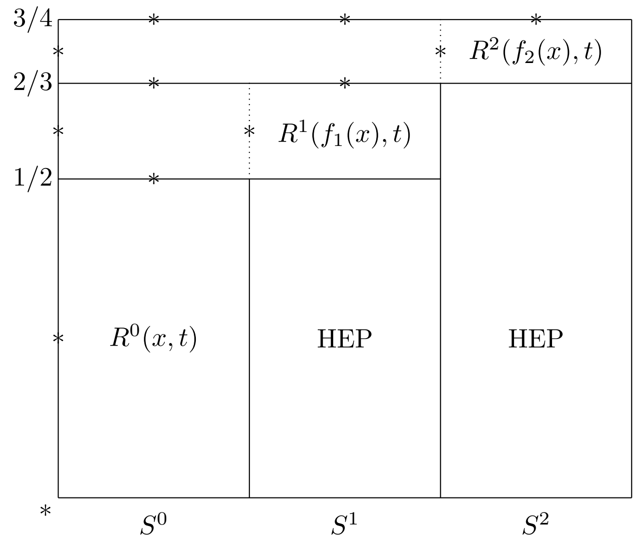


FIGURE 8. S^∞ is contractible

Example 5.13. The unreduced suspension SX and the reduced suspension $\Sigma X = SX/\{x_0\}\times I$ are homotopy equivalent for all CW-complexes X based at a 0-cell $\{x_0\}$.

Example 5.14. [4, 21.21] (Homotopic maps have homotopy equivalent mapping cones). Let $f\colon X \to Y$ be any map. Consider the mapping cone $C_f = Y \cup_f CX$ of f. Since the pair (CX, X) has the HEP (5.4) we know that

- (C_f, Y) has the HEP (Proposition 5.8.(4))
- The homotopy type of C_f only depends on the homotopy class of f (Theorem 5.3.(4))

In Example 2.8 we claimed that the squaring map $2\colon S^1 \to S^1$ is not homotopic to the constant map $0\colon S^1 \to S^1$. To prove this it suffices to show that the mapping cones $C_2 = S^1 \cup_2 D^2 = \mathbf{R}P^2$ and $C_0 = S^1 \cup_0 D^2 = S^1 \vee S^2$ are not homotopy equivalent.

The complex projective plane $\mathbf{C}P^2 = S^2 \cup_\varphi D^4$ is obtained by attaching a 4-cell to the 2-sphere along the Hopf map $S^3 \to S^2$ (4.5). If the attaching map is nullhomotopic then $\mathbf{C}P^2$ is homotopy equivalent to $S^2 \cup_* D^4 = S^2 \vee S^4$.

We shall later develop methods to show that $\mathbf{R}P^2$, $S^1 \vee S^2$ and $\mathbf{C}P^2$, $S^2 \vee S^4$ are not homotopy equivalent. This will show that the squaring map $2\colon S^1 \to S^1$ and the Hopf map $S^3 \to S^2$ are not nullhomotopic.

Example 5.15 (The homotopy type of the quotient space X/A)**.** If (X, A) has the HEP so does the pair $(X \cup CA, CA)$ obtained by attaching X to CA (Proposition 5.8.(4)). Since the cone CA on A is contractible,

$$X \cup CA \to X \cup CA/CA = X/A$$

is a homotopy equivalence (Proposition 5.3.(3)) between the cone on the iclusion of A into X and the quotient space X/A.

Suppose in addition that the inclusion map $A \hookrightarrow X$ is homotopic to the constant map $0\colon A \to X$, ie. A is contractible in X. Then there are homotopy equivalences

$$X/A \simeq X \cup CA = X \cup_i CA = C_i \simeq C_0 = X \vee SA$$

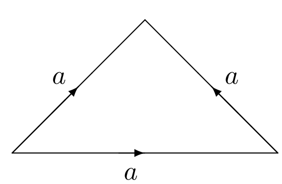


FIGURE 9. The dunce cap

as the inclusion map and the contanst map have homotopy equivalent mapping cones by Example 5.14. For instance, $S^n/S^i \simeq S^n \vee S^{i+1}$ for all $i \leq 0 < n$. (The inclusion $S^i \to S^n$, $0 < i < n$, is nullhomotopic since it factors through the contractible space $S^n - * = \mathbf{R}^n$.) See [5, Exmp 0.8] for an illustration of $S^2/S^0 \simeq S^2 \cup CS^0 \simeq S^2 \vee S^1$.

Example 5.16. (HEP for mapping cylinders.) Let $f\colon X \to Y$ be a map. We apply 5.8 in connection with the mapping cylinder $M_f = Y \cup_f (X \times I)$.

$$(I, \partial I) \text{ has the HEP} \overset{5.8.(2)}{\Longrightarrow} (X \times I, X \times \partial I) \text{ has the HEP} \overset{5.8.(4)}{\Longrightarrow} (M_f, X \cup Y) \text{ has the HEP}$$

$$(I, \{0\}) \text{ has the HEP} \overset{5.8.(2)}{\Longrightarrow} (X \times I, X \times \{0\}) \text{ has the HEP} \overset{5.8.(4)}{\Longrightarrow} (M_f, Y) \text{ has the HEP}$$

The fact that $(M_f, X \cup Y)$ has the HEP implies that also (M_f, X) has the HEP (simply take a constant homotopy on Y). See 5.17 below for another application.

Example 5.17. (HEP for subspaces with mapping cylinder neighborhoods [5, Example 0.15]) For another application of 5.8, suppose that the subspace $A \subseteq X$ has a mapping cylinder neighborhood. This means that A has a closed neighborhood N containing a subspace B (thought of as the boundary of N) such that $N - B$ is an open neighborhood of A and $(N, A \cup B)$ is homeomorphic to $(M_f, A \cup B)$ for some map $f\colon B \to A$. Then (X, A) has the HEP. To see this, let $h\colon X \times \{0\} \cup A \times I \to Y$ by a partial homotopy of a map $X \to Y$. Extend it to a partial homotopy on $X \times \{0\} \cup (A \cup B) \times I$ by using the constant homotopy on $B \times I$. Since $(N, A \cup B)$ has the HEP, we can extend further to a partial homotopy defined on $X \times \{0\} \cup N \times I$. Finally, extend to $X \times I$ by using a constant homotopy on $X - (N - B) \times I$. In this way we get extensions

$$\begin{tikzcd}
X \times \{0\} \cup A \times I \arrow[r, "h"] \arrow[d, hook] & Y \\
X \times \{0\} \cup (A \cup B) \times I \arrow[ur] \arrow[d, hook] & \\
X \times \{0\} \cup N \times I \arrow[uur] \arrow[d, hook] & \\
X \times I \arrow[uuur] &
\end{tikzcd}$$

The final map is continuous since it restricts to continuous maps on the closed subspaces $X - (N - B) \times I$ and $N \times I$ with union $X \times I$.

For instance, the subspace ABC $\subseteq \mathbf{R}^2$ consisting of the three thin letters in the figure on [5, p. 1] is a subspace with a mapping cylinder neighborhood, namely the three thick letters. Thus $(\mathbf{R}^2, \text{ABC})$ has the HEP.

References

[1] Glen E. Bredon, *Topology and geometry*, Graduate Texts in Mathematics, vol. 139, Springer-Verlag, New York, 1993. MR 94d:55001
[2] James Dugundji, *Topology*, Allyn and Bacon Inc., Boston, Mass., 1966. MR 33 #1824
[3] George K. Francis, *A topological picturebook*, Springer-Verlag, New York, 1987. MR 88a:57002
[4] Marvin J. Greenberg and John R. Harper, *Algebraic topology*, Benjamin/Cummings Publishing Co. Inc. Advanced Book Program, Reading, Mass., 1981, A first course. MR 83b:55001
[5] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR 2002k:55001
[6] Jesper M. Møller, *General topology*, `http://www.math.ku.dk/~moller/e03/3gt/notes/gtnotes.pdf`.
[7] James R. Munkres, *Topology. Second edition*, Prentice-Hall Inc., Englewood Cliffs, N.J., 2000. MR 57 #4063
[8] J. H. C. Whitehead, *Combinatorial homotopy. I*, Bull. Amer. Math. Soc. **55** (1949), 213–245. MR 0030759 (11,48b)

Matematisk Institut, Universitetsparken 5, DK–2100 København
E-mail address: `moller@math.ku.dk`
URL: `http://www.math.ku.dk/~moller`