The Kervaire Invariant
(joint work with Mike Hill and Doug Ravenel)
Theorem (Hill, H., Ravenel)

If $M$ is a smooth, stably framed manifold of Kervaire invariant one, then the dimension of $M$ is one of

$$2, 6, 14, 30, 62, \text{ or } 126$$
Topology before the 1930's

counting the solutions to $n$ equations in $n$ unknowns

$R^n$ $(S^n)$ $y = f(x)$ $R^n$ $(S^n)$
Definition of the degree of a map

\[ f : M \rightarrow S^n \quad (\text{dim } M = n) \]
Toplogy before the 1930’s  

(Brouwer, Hopf)

**Definition of the degree of a map**

\[ f : M \rightarrow S^n \quad (\text{dim} \ M = n) \]

**Theorem:** Two maps of the same degree are homotopic.
Topology before the 1930’s  (Brouwer, Hopf)

Definition of the degree of a map

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Theorem: Two maps of the same degree are homotopic.

Key tool: Cohomology \( H^*(M) \).
Pontryagin (1930's)

\[ y = f(x) \]

\[ \mathbb{R}^n \quad (\mathbb{S}^n) \]

\[ \mathbb{R}^{n+k} \quad (\mathbb{S}^{n+k}) \]
Pontryagin (1930's)

Problem: Count the solutions to systems of \( n \) equations in \( (n+k) \) unknowns
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Answer: A stably framed manifold of dimension \( k \).
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$y_1, y_2$ 

$M_1 = f^{-1}(y_1)$ 

$M_2 = f^{-1}(y_2)$
Pontryagin (1930’s)

Framed cobordism
Pontryagin (1930’s)

\[ \Omega_k := \{ \text{stably framed } k\text{-manifolds} \}/\text{cobordism} \]

**Theorem:** The above construction gives a bijection

\[ \pi_{n+k}(S^n) \approx \Omega_k \]

where

\[ \pi_{n+k}(S^n) := \{ \text{maps } S^{n+k} \rightarrow S^n \}/\text{homotopy} \]
Pontryagin (1930's)

\[ k=0 \]
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\[ \pi_n(S^n) = \mathbb{Z} \]
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(the degree)

\[ \pi_{n+1}(S^n) = \mathbb{Z}/2 \]
Pontryagin (1930's)

$k=2$
Pontryagin (1930’s)

\[ k=2 \quad \text{genus } M = 0 \quad \Rightarrow \quad M \text{ is a boundary} \]

(since \( S^2 \) bounds a disk and \( \pi_2(\text{GL}_n(\mathbb{R})) = 0 \))
Pontryagin (1930’s)

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(since \( S^2 \) bounds a disk and \( \pi_2(\text{GL}_n(\mathbb{R}))=0 \))

Suppose the genus of \( M \) is greater than 0.
Pontryagin (1930’s)

$k=2$
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\[ k = 2 \]
Pontryagin (1930’s)

$k=2$

choose an embedded arc
Pontryagin (1930's)

\[ k=2 \]

choose an embedded arc

cut the surface open and glue in disks
Pontryagin (1930’s)

\[ k = 2 \]

framed surgery
Pontryagin (1930’s)

Obstruction: \( \varphi : H_1(M; \mathbb{Z}/2) \rightarrow \mathbb{Z}/2 \)
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Argument: Since the dimension of \( H_1(M; \mathbb{Z}/2) \) is even, there is always a non-zero element in the kernel of \( \varphi \), and so surgery can be performed.
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**Obstruction:** \( \varphi : H_1(M; \mathbb{Z}/2) \to \mathbb{Z}/2 \)

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**Conclusion:** \( \Omega_2 = \pi_{n+2}(S^n) = 0 \).
Pontryagin (1930’s)

Error: The function $\varphi$ is not linear:

$$\varphi(x+y) - \varphi(x) - \varphi(y) = \int_M x \, y$$
Pontryagin (1930's)

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The Arf invariant of $\varphi$ gives an isomorphism

$$\Omega_2 = \pi_{n+2}(S^n) = \mathbb{Z}/2$$
Pontryagin (1930’s)
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Question: In which dimensions every stably framed manifold cobordant to a (homotopy) sphere?

Question: In which dimensions would Pontryagin’s construction have worked?
1956: Milnor gave an example of a manifold of dimension 7, homeomorphic but not diffeomorphic to the 7-sphere.

1961: Milnor introduced a generalization of Pontryagin’s “surgery” maneuver, and initiated a scheme for studying differentiable structures on manifolds in other dimensions.
Topology circa 1960: Kervaire's example
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Topology circa 1960: Kervaire's example

\[ \partial N \equiv S^9 \]
Topology circa 1960: Kervaire's example

\[ X = N/\partial N \]

(a triangulable manifold)
Topology circa 1960: Kervaire’s example

1960: Kervaire defined for certain manifolds $M$ of dimension $(4k+2)$

$$\varphi : H^{2k+1}(M) \to \mathbb{Z}/2$$

satisfying

$$\varphi(x+y) - \varphi(x) - \varphi(y) = \int_M x \cdot y$$
Topology circa 1960: Kervaire’s example

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He set

$$\Phi(M) = \text{Arf}(\varphi)$$
Topology circa 1960: Kervaire's example

Theorem (Kervaire): If $M$ is a smooth, stably framed manifold of dimensions 10 or 18, then

$$\Phi(M) = 0.$$
Topology circa 1960: Kervaire's example

**Theorem (Kervaire):** If $M$ is a smooth, stably framed manifold of dimensions 10 or 18, then

$$\Phi(M) = 0.$$  

**Theorem (Kervaire):** If $X$ can be smoothed then

$$\Phi(X) = 1.$$
Topology circa 1960: Kervaire’s example

Theorem (Kervaire): If $M$ is a smooth, stably framed manifold of dimensions 10 or 18, then

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Theorem (Kervaire): If $X$ can be smoothed then

$$\Phi(X) = 1.$$ 

Corollary (Kervaire): The triangulable manifold $X$ has no smooth structure.
The number $\Phi(M)$ is called the Kervaire invariant of $M$. 
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**Question:** In which dimensions can there exist a (stably framed) manifold of Kervaire invariant one?
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The Kervaire invariant problem
Topology circa 1960: Kervaire's example

\[ X^{4k+2} = N/\partial N \]

\[ \partial N \equiv S^{4k+1} \]

\[ S^{2k+1} \]

\[ TS^{2k+1} \]

\[ N^{4k+2} \]

\[ S^{2k+1} \]
Question: In which dimensions can $X^{4k+2}$ be given a smooth structure? In which dimensions is $\partial N^{4k+1}$ diffeomorphic to the sphere $S^{4k+1}$?
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Remark: If $X^{4k+2}$ can be given a smooth structure, it then becomes a stably framed manifold which is not cobordant to a homotopy sphere.
Kervaire and Milnor (1958, 1963)

Exotic spheres
Kervaire and Milnor (1958, 1963)

**Exotic spheres**

**Definition:** The group $\Theta_n$ is the group of homotopy $n$-spheres, up to h-cobordism.
Exotic spheres

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The group structure is connected sum.
Kervaire and Milnor (1958, 1963)

\[ \text{dim} > 4 \implies \]

\[ h\text{-cobordism} = \text{diffeomorphism} \ (\text{Smale}) \]

and (Smale, Stallings, Zeeman)

\[ \text{homotopy sphere} \iff \text{topological sphere} \]
Kervaire and Milnor (1958, 1963)

\[ \text{dim} > 4 \Rightarrow \]

\[ h\text{-cobordism} = \text{diffeomorphism} \ (\text{Smale}) \]

and (Smale, Stallings, Zeeman)

\[ \text{homotopy sphere} \Leftrightarrow \text{topological sphere}. \]

\[ \mathcal{H}_n = \text{the group of diffeomorphism classes of smooth structures on the } n\text{-sphere}. \]
Kervaire and Milnor (1958, 1963)

**Theorem:** The order of $\Theta_{4m-1}$ is given by

$$|\Theta_{4m-1}| = a_m |\pi_{4m-1+n} S^n| \ 2^{2m-4} \ (2^{2m-1}-1) \ B_m/m$$

with

- $B_m = m^{th}$ Bernoulli number
- $a_m = 1$ if $m$ is even, $2$ if $m$ is odd
Kervaire and Milnor (1958, 1963)

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They were unable to settle a factor of 2 in the order of $\Theta_{4m+1}$ and $\Theta_{4m+2}$ (Kervaire's $\partial N$ and $X$).
Kervaire and Milnor (1958, 1963)

Question: What are the orders of the groups \( \Theta_{4m+1} \) and \( \Theta_{4m+2} \)?
Methods of homotopy theory

1966: The state of the art
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Theorem (Kervaire ’60): If the dimension of $M$ is 10 or 18 then $\Theta(M) = 0$
1966: The state of the art

**Theorem (Kervaire ‘60):** If the dimension of $M$ is 10 or 18 then $\Theta(M) = 0$

$\Theta(M)$ can be 1 for $M = S^1 \times S^1, S^3 \times S^3, S^7 \times S^7$ (dimensions 2, 6, 14).
1966: The state of the art

Theorem (Kervaire ‘60): If the dimension of $M$ is 10 or 18 then $\Theta(M) = 0$

$\Theta(M)$ can be 1 for $M = S^1 \times S^1$, $S^3 \times S^3$, $S^7 \times S^7$ (dimensions 2, 6, 14).

Kervaire and Milnor speculated that $\Theta(M)$ can be non-zero only in these three dimensions.
Methods of homotopy theory

Theorem (Brown–Peterson ‘65, ’66): If $M$ is a stably framed manifold of dimension $(8k+2)$ then the Kervaire invariant of $M$ is zero.
Methods of homotopy theory

Theorem (Brown-Peterson '65, '66): If $M$ is a stably framed manifold of dimension $(8k+2)$ then the Kervaire invariant of $M$ is zero.

(This extends Kervaire's sequence of 10 and 18.)
Methods of homotopy theory

Theorem (Browder ’69): If $\Phi(M)$ is non-zero then the dimension of $M$ is of the form $2^{j+1} - 2$. 
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In this dimension a manifold $M$ with Kervaire invariant 1 exists if and only if there is an element $\vartheta_j$ in $\pi_{2j+1-2} S^0$ represented at the $E_2$-term of the classical Adams spectral sequence by $h_j^2$. 

Methods of homotopy theory

Theorem (Barratt, Mahowald, Tangora, Jones ’70–’84):

The elements $\vartheta_j$ exist for $j \leq 5$. 
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(the overwhelming belief was that all $\vartheta_j$ exist)
Four Questions
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Question: In which dimensions every stably framed manifold cobordant to a (homotopy) sphere?
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Question: For which $j$ does $\vartheta_j$ exist?
Four Questions

**Question:** In which dimensions every stably framed manifold cobordant to a (homotopy) sphere?

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Four Questions

Question: In which dimensions every stably framed manifold cobordant to a (homotopy) sphere?

Question: In which dimensions can there exist a (stably framed) manifold of Kervaire invariant one?

Answer: In all dimensions except possibly

2, 6, 14, 30, 62, and 126.
Four Questions

Question: What are $|\Theta_{4m+1}|$ and $|\Theta_{4m+2}|$?
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Answer: Unless $(4m+2)$ is of the form $2, 6, 14, 30, 62, \text{ or } 126,$ the group $\Theta_{4m+1}$ is twice as large as it might have been, while $\Theta_{4m+2}$ is half as large as it might have been.
Four Questions

Question: What are $|\Theta_{4m+1}|$ and $|\Theta_{4m+2}|$?

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the group $\Theta_{4m+1}$ is twice as large as it might have been, while $\Theta_{4m+2}$ is half as large as it might have been.

Kervaire's $X^{4k+2}$ has no smooth structure, and $\partial N^{4k+2}$ is an exotic sphere.
Four Questions

Question: For which $j$ does $\theta_j$ exist?
Four Questions

Question: For which $j$ does $\theta_j$ exist?

Answer: For $j = 1, 2, 3, 4, 5$ and possibly $6$. 
A finite set of things
A finite set of things

The universe was created in 6 days.

Day 1: $\vartheta_1$  Day 2: $\vartheta_2$  Day 3: $\vartheta_3$
Day 4: $\vartheta_4$  Day 5: $\vartheta_5$  Day 6: $\vartheta_6$
A finite set of things

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Day 4: $\vartheta_4$  Day 5: $\vartheta_5$  Day 6: $\vartheta_6$

Birthdays.  Hill: almost 30  Ravenel: 62

Hopkins+Browder: 126
A finite set of things

Possible connections (Bökstedt and others)

\[ E_6 \leftrightarrow \vartheta_4 \]
\[ E_7 \leftrightarrow \vartheta_5 \]
\[ E_8 \leftrightarrow \vartheta_6 \]
Outline of the proof

Cohomology theory: $\Omega$

(like in the study of the degree)
Outline of the proof

Cohomology theory: $\Omega$

(like in the study of the degree)

General properties:

contravariant functor $\mathcal{X} \mapsto \Omega^\ast(\mathcal{X})$

suspension isomorphism $\Omega^\ast + n(\Sigma^n \mathcal{X}) \cong \Omega^\ast(\mathcal{X})$
Outline of the proof

The $\Omega$ degree:

$$\Omega^n(S^{n+2^{j+1}-2}) \xrightarrow{\theta_j} \Omega^n(S^n)$$

$$\Omega^{2-2^{j+1}}(pt) \xleftarrow{} \Omega^0(pt)$$
Outline of the proof

The \( \Omega \) degree:

\[
\Omega^n(S^{n+2^j+1-2}) \xrightarrow{\vartheta_j} \Omega^n(S^n) \xleftarrow{\vartheta_j} \Omega^0(pt) \ni 1
\]
Outline of the proof

Detection Theorem: If $\vartheta_j$ exists then $\Omega^*(\vartheta_j)$ is a non-zero element of $\Omega^{2^{-2^{j+1}}}(pt)$. 
Outline of the proof

Detection Theorem: If $\vartheta_j$ exists then $\Omega^*(\vartheta_j)$ is a non-zero element of $\Omega^{2-2^{j+1}}(pt)$.

Periodicity Theorem: The cohomology theory $\Omega$ is periodic: for any $X$

$$\Omega^{*+256}(X) \approx \Omega^*(X).$$
Outline of the proof

Detection Theorem: If $\emptyset_j$ exists then $\Omega^*(\emptyset_j)$ is a non-zero element of $\Omega^{2-2^{j+1}}(pt)$.

Periodicity Theorem: The cohomology theory $\Omega$ is periodic: for any $X$

$$\Omega^{*+256}(X) \approx \Omega^*(X).$$

Gap Theorem: The groups $\Omega^*(pt)$ are zero for

$$0 < * < 4$$
Outline of the proof

Detection Theorem: Inventory a table of long-known computations.
Outline of the proof

**Detection Theorem:** Inventory a table of long-known computations.

**Periodicity and Gap Theorems:** The slice tower -- new variation on the Postnikov tower in equivariant homotopy theory.
Outline of the proof

Detection Theorem: Inventory a table of long-known computations.

Periodicity and Gap Theorems: The slice tower -- new variation on the Postnikov tower in equivariant homotopy theory.

A large class of naturally constructed cohomology theories satisfy **periodicity and gap** theorems (the gap is always the same, the period varies). We choose $\Omega$ to be one with the smallest period but large enough to satisfy the Detection Theorem.
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