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THE KERVAIRE INVARIANT ONE ELEMENT AND THE DOUBLE TRANSFER†

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THE Kervaire invariant one element $\theta_j \in \pi^s_{2^{j+1}-2}(S^0)$ is shown not to factor through the double transfer unless $j \leqq 4$.

In particular, θ_5 of Barratt–Jones–Mahowald does not factor through the double transfer.

0. INTRODUCTION

The Kervaire invariant one problem has been one of the most fundamental and challenging problems in topology [6, 8, 9, 18, 19, 23]. Of course, the pivotal work was [8], which translated the original geometric problem [18, 19] into the problem of the stable homotopy groups of the sphere:

Is $h_j^2 \in \mathrm{Ext}_{\mathscr{A}}^{2,2^{j+1}}(\mathbb{Z}/2, \mathbb{Z}/2)$ *a permanent cycle in the Adams spectral sequence of the sphere?*

The traditional belief [23, 28, 37] is that, for each j, h_j^2 is a permanent cycle represented by $\theta_j \in \pi^s_{2^{j+1}-2}(S^0)$, which factors through the double transfer $P \wedge P \xrightarrow{\lambda \wedge \lambda} S^0$. Here $\lambda: P \to S^0$ is the Kahn–Priddy map [17], and the double transfer lift of θ_j is forced to have $x_{2^j-1} \otimes x_{2^j-1} \in H_{2^{j+1}-2}(P \wedge P)$ as its stable mod-2 Hurewicz image.

The probability of such a double transfer factorization was primarily supported by the Kahn–Priddy theorem [17] and unpublished calculations of Mark Mahowald. And there was a more general conjecture of Mahowald [28] which would imply that any Kervaire invariant one element factors through the double transfer. Though Singer [38] disproved the naive conjecture for $n = 5$, which states that $\pi^s_*(\wedge^n P) \to E_\infty^{n,n+*}(S^0)$ is onto (where the target is associated with the classical Adams spectral sequence of the sphere [1]), it did not contradict this conjecture of Mahowald, at least on the nose.

Now, the purpose of this paper is to disprove such a belief:

If the Kervaire invariant one element $\theta_j \in \pi^s_{2^{j+1}-2}(S^0)$ *exists and factors through the double transfer* $P \wedge P \to S^0$, *then* $j \leq 4$ (Theorem 3.1).

We will prove this result as follows: In section 1, we show any such a double transfer lift has a BP-Hurewicz image with a gigantic order. In section 2, we study the BP-Adams operation on $BP_{\mathrm{even}}(P \wedge P)$, and show that gigantic order elements in $BP_{\mathrm{even}}(P \wedge P)$ cannot be in the BP-Hurewicz image. And, in section 3, these results are combined to prove Theorem 3.1.

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Now, the aftermath of this result is stated in section 4: In our previous paper [28], some sufficient conditions for an element to factor through the double transfer were given. So we combine our Theorem 3.1 with [28] to get some consequences, and one of the consequence (which can be stated without any technical terminology from [28]) is the following:

θ_j may be represented by a framed hypersurface only if $j \leqq 4$ (Corollary 4.4).

Notation and conventions: H_* stands for the mod-2 homology; $P = \Sigma^\infty \mathbb{R}P^\infty$; $x_i \in H_i(P)$ is the generator; $BP_* = \mathbb{Z}_{(2)}[v_1, v_2, \ldots]$; $\mathscr{A}_* = P(\xi_1, \xi_2, \ldots)$, where $|\xi_n| = 2^n - 1$; $\mathrm{Ext}^{*,*}_{\mathscr{A}_*}(\mathbb{Z}/2, H_*(BP)) \cong \mathrm{Ext}^{*,*}_{E_*}(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2[u_0, u_1, \ldots]$; where $u_i \in \mathrm{Ext}^{1, 2^{i+1}-1}_{E_*}(\mathbb{Z}/2, \mathbb{Z}/2)$ is expressed as $[\xi_{i+1}]$ in the cobar complex, and corresponds to the usual (Hazewinkel [14] or Araki [4], whichever) generator $v_i \in BP_{2^{i+1}-2}$ (resp. p) when $i \geqq 1$ (resp. $i = 0$). $E\langle k\rangle$ is the exterior quotient-Hopf algebra of $\mathscr{A}_*$, generated by $\xi_1, \ldots, \xi_{k+1}$, whose notation is intended to suggest

$$\mathrm{Ext}^{*,*}_{\mathscr{A}_*}(\mathbb{Z}/2, H_*(BP\langle k\rangle)) \cong \mathrm{Ext}^{*,*}_{E\langle k\rangle_*}(\mathbb{Z}/2, \mathbb{Z}/2).$$

1. THE ADAMS SPECTRAL SEQUENCE OF $BP_*(P \wedge P)$

The main result of this section is to show that any double transfer lift of the Kervaire invariant one element has a gigantic BP_*-order (Proposition 1.2). Therefore, we must face $BP_*(P \wedge P)$, which has a very complicated additive structure. To overcome this difficulty, we use the affirmative solution [31, 35] of the Conner–Floyed conjecture, which allows us to use more tractible $\mathrm{Ext}^{*,*}_{E\langle 2\rangle_*}(\mathbb{Z}/2, H_*(P \wedge P))$, to evaluate the BP_*-order. (Note that the multiplication by $u_0 \in \mathrm{Ext}^{1,1}_{E\langle 2\rangle_*}(\mathbb{Z}/2, \mathbb{Z}/2)$ on $\mathrm{Ext}^{*,*}_{E\langle 2\rangle_*}(\mathbb{Z}/2, H_*(P \wedge P))$ corresponds to the multiplication by p on $BP_*(P \wedge P)$.) We begin with a summary of known results, which are necessary for our approach:

PROPOSITION A. (a) The Adams spectral sequence

$$\mathrm{Ext}^{*,*}_{\mathscr{A}_*}(\mathbb{Z}/2, H_*(BP \wedge P \wedge P)) \Rightarrow BP_*(P \wedge P)$$

collapses.

(b) *As $\mathbb{Z}/2[u_2]$-modules,*

$$\mathrm{Ext}^{*,*}_{E\langle 2\rangle_*}(\mathbb{Z}/2, H_*(P \wedge P)) = \mathrm{Ext}^{*,*}_{E\langle 1\rangle_*}(\mathbb{Z}/2, H_*(P \wedge P)) \otimes \mathbb{Z}/2[u_2].$$

(c) $\mathrm{Ext}^{*,*+2n}_{E\langle 1\rangle_*}(\mathbb{Z}/2, H_*(P \wedge P))$ *is concentrated in the 0th line; more precisely,*

$$\mathrm{Ext}^{*,*+2n}_{E\langle 1\rangle_*}(\mathbb{Z}/2, H_*(P \wedge P)) = \begin{cases} 0 & \text{if } * \geq 1 \\ \oplus_{k+l=n+1} \mathbb{Z}/2\{x_{2k-1} \otimes y_{2l-1}\} & \text{if } * = 0. \end{cases}$$

Proof. (a) This is proved more generally in [15, 16], using the solution of the Conner–Floyd conjecture [31, 35]. Though this particular case would follow from [21].

(b) This is essentially known in [15, (1.2) and Lemma 1.4], but we will give a proof for reader's convenience: It is sufficient to show that the Bockstein long exact sequence of [3]

$$\to \mathrm{Ext}^{s-1,t-7}_{E\langle 2\rangle_*}(\mathbb{Z}/2, H_*(P \wedge P)) \xrightarrow{\cdot u_2} \mathrm{Ext}^{s,t}_{E\langle 2\rangle_*}(\mathbb{Z}/2, H_*(P \wedge P))$$
$$\to \mathrm{Ext}^{s,t}_{E\langle 1\rangle_*}(\mathbb{Z}/2, H_*(P \wedge P)) \to \mathrm{Ext}^{s,t-7}_{E\langle 2\rangle_*}(\mathbb{Z}/2, H_*(P \wedge P))$$

is in fact short exact:

$$0 \to \mathrm{Ext}^{s-1,t-7}_{E\langle 2\rangle_*}(\mathbb{Z}/2, H_*(P \wedge P)) \xrightarrow{\cdot u_2} \mathrm{Ext}^{s,t}_{E\langle 2\rangle_*}(\mathbb{Z}/2, H_*(P \wedge P))$$
$$\to \mathrm{Ext}^{s,t}_{E\langle 1\rangle_*}(\mathbb{Z}/2, H_*(P \wedge P)) \to 0.$$

(Of course, this corresponds to the fact that $BP_*(P \wedge P)$ is v_2-torsion free.) This follows from (i) $\mathrm{Ext}^{0,t}_{E\langle 2\rangle_*}(\mathbb{Z}/2, H_*(P \wedge P)) \to \mathrm{Ext}^{0,t}_{E\langle 1\rangle_*}(\mathbb{Z}/2, H_*(P \wedge P))$ is an isomorphism. (When $t = 2n$, the target is described explicitly in (c)), (ii) $\mathrm{Ext}^{*,*}_{E\langle 1\rangle_*}(\mathbb{Z}/2, H_*(P \wedge P))$ is generated by $\mathrm{Ext}^{0,*}_{E\langle 1\rangle_*}(\mathbb{Z}/2, H_*(P \wedge P))$ over $\mathbb{Z}/2[u_0, u_1] \cong \mathrm{Ext}^{*,*}_{E\langle 1\rangle_*}(\mathbb{Z}/2, H_*(S^0))$, and (iii) The map $\mathrm{Ext}^{s,t}_{E\langle 2\rangle_*}(\mathbb{Z}/2, H_*(P \wedge P)) \to \mathrm{Ext}^{s,t}_{E\langle 1\rangle_*}(\mathbb{Z}/2, H_*(P \wedge P))$ is a $\mathbb{Z}/2[u_0, u_1]$-module map.

Actually, (i) is an easy calculation, (ii) is well-known [3, 10], and (iii) is trivial from the construction of the Bockstein spectral sequence [3].

(c) The first claim follows from the $E\langle 1\rangle_*$-comodule isomorphism:

$$H_*(P \wedge P) \cong \Sigma^2 H_*(P) \oplus F$$

where F is a cofree $E\langle 1\rangle_*$-comodule. The second claim follows from the reduced $E\langle 1\rangle_*$-coaction

$$\begin{aligned} H_*(P \wedge P) &\to \overline{E\langle 1\rangle_*} \otimes H_*(P \wedge P) \\ x_{2k} \otimes x_{2l} &\mapsto [\xi_1] \otimes (x_{2k-1} \otimes x_{2l} + x_{2k} \otimes x_{2l-1}) \\ &\qquad + [\xi_2] \otimes (x_{2k-3} \otimes x_{2l} + x_{2k} \otimes x_{2l-3}) \\ x_{2k-1} \otimes x_{2l-1} &\mapsto 0 \end{aligned}$$

where $\overline{E\langle 1\rangle_*}$ is the positive dimensional part of $E\langle 1\rangle_*$. □

To make use of this, we need a formula for the u_0-action on $\mathrm{Ext}^{*,*+\mathrm{even}}_{E\langle 2\rangle_*}(\mathbb{Z}/2, H_*(P \wedge P))$ and this is the content of the following lemma.

LEMMA 1.1. *The $\mathbb{Z}/2[u_0, u_1, u_2]$-module structure on the even total degree part of $\mathrm{Ext}^{*,*}_{E\langle 2\rangle_*}(\mathbb{Z}/2, H_*(P \wedge P))$ is given by the $\mathbb{Z}/2[u_2]$-module isomorphism*

$$\mathrm{Ext}^{*,*+\mathrm{even}}_{E\langle 2\rangle_*}(\mathbb{Z}/2, H_*(P \wedge P)) \cong \oplus_{k,l} \mathbb{Z}/2\{x_{2k-1} \otimes x_{2l-1}\} \otimes \mathbb{Z}/2[u_2]$$

and the actions

$$\begin{aligned} u_0(x_{2k-1} \otimes x_{2l-1}) &= u_2(x_{2k-5} \otimes x_{2l-3} + x_{2k-3} \otimes x_{2l-5}) \\ u_1(x_{2k-1} \otimes x_{2l-1}) &= -u_2(x_{2k-5} \otimes x_{2l-1} + x_{2k-3} \otimes x_{2l-3} + x_{2k-1} \otimes x_{2l-5}). \end{aligned}$$

Proof. The $\mathbb{Z}/2[u_2]$-module isomorphism is an immediate consequence of Proposition A (b) and (c). To study the actions by u_0 and u_1, we begin by noting the relations

$$u_0(x_{2k-1} \otimes x_{2l-1}) + u_1(x_{2k-3} \otimes x_{2l-1}) + u_2(x_{2k-7} \otimes x_{2l-1}) = 0 \qquad (1_{k,l})$$

$$u_0(x_{2k-1} \otimes x_{2l-1}) + u_1(x_{2k-1} \otimes x_{2l-3}) + u_2(x_{2k-1} \otimes x_{2l-7}) = 0. \qquad (2_{k,l})$$

Actually, these follow immediately from the reduced $E\langle 2\rangle_*$ coaction formulas on the elements $x_{2k} \otimes x_{2l-1}$ and $x_{2k-1} \otimes x_{2l}$:

$$\begin{aligned} &H_*(P \wedge P) \to \overline{E\langle 2\rangle_*} \otimes H_*(P \wedge P) \\ &x_{2k} \otimes x_{2l-1} \mapsto [\xi_1] \otimes x_{2k-1} \otimes x_{2l-1} + [\xi_2] \otimes x_{2k-3} \otimes x_{2l-1} + [\xi_3] \otimes x_{2k-7} \otimes x_{2l-1} \\ &x_{2k-1} \otimes x_{2l} \mapsto [\xi_1] \otimes x_{2k-1} \otimes x_{2l-1} + [\xi_2] \otimes x_{2k-1} \otimes x_{2l-3} + [\xi_3] \otimes x_{2k-1} \otimes x_{2l-7}, \end{aligned}$$

where $\overline{E\langle 2\rangle_*}$ is the positive dimensional part of $E\langle 2\rangle_*$. This is because the reduced coaction is the first coboundary in the cobar complex to calculate the Ext-group (see [34] A1], for example). One immediate consequence of these relations is that we only have to show the formula of u_0-action; for the u_1-action formula would follow immediately from the u_0-action formula and either one of these relations.

To prove the u_0-action formula, we form the difference $(1_{k,l}) - (2_{k-1,l+1})$, from which we obtain

$$u_0 \otimes x_{2k-1} \otimes x_{2l-1} = u_0 \otimes x_{2k-3} \otimes x_{2l+1} + u_2(x_{2k-3} \otimes x_{2l-5} - x_{2k-7} \otimes x_{2l-1}). \quad (3_k)$$

Then the u_0-action formula is proved by the mathematical induction on k: When $k = 1$, (3_1) indicates $u_0 \otimes x_1 \otimes x_{2l-1} = 0$, which is exactly what the u_0-action formula tells us for this case. Suppose we have proved the u_0-action formula for $k-1$ (so we know $u_0 \otimes x_{2k-3} \otimes x_{2l+1}$). Then, by (3_k), we get

$$\begin{aligned} u_0 \otimes x_{2k-1} \otimes x_{2l-1} &= u_0 \otimes x_{2k-3} \otimes x_{2l+1} + u_2(x_{2k-3} \otimes x_{2l-5} - x_{2k-7} \otimes x_{2l-1}) \\ &= u_2(x_{2k-7} \otimes x_{2l-1} + x_{2k-5} \otimes x_{2l-3}) \\ &\quad + u_2(x_{2k-3} \otimes x_{2l-5} - x_{2k-7} \otimes x_{2l-1}) \\ &= u_2(x_{2k-5} \otimes x_{2l-3} + x_{2k-3} \otimes x_{2l-5}) \end{aligned}$$

which is nothing but the u_0-action formula for k. □

Finally, we are ready to prove the main result of this section.

PROPOSITION 1.2. *The order of any element* $\Theta \in BP_{2^{j+1}-2}(P \wedge P)$, *which hits* $x_{2^j-1} \otimes x_{2^j-1} \in H_{2^{j+1}-2}(P \wedge P)$ *by the Thom reduction, is a multiple of*

$$2^{[(2^j-1)/4]+1} = \begin{cases} 2^{(2^{j-2})} & \text{if } j \geqq 2 \\ 2 & \text{if } j = 1. \end{cases}$$

Proof. Suppose Θ is detected as $\Omega \in \mathrm{Ext}_{E_*}^{*,*}(\mathbb{Z}/2, H_*(P \wedge P))$. Then, by Proposition A(a) and the fact that the multiplication by 2 corresponds to the multiplication by u_0 in the E_2-term, it suffices to show

$$u_0^{[(2^j-1)/4]}\Omega \neq 0 \in \mathrm{Ext}_{E_*}^{*,*}(\mathbb{Z}/2, H_*(P \wedge P)).$$

To show this, we use the natural map

$$\mathrm{Ext}_{E_*}^{*,*}(\mathbb{Z}/2, H_*(P \wedge P)) \to \mathrm{Ext}_{E\langle 2\rangle_*}^{*,*}(\mathbb{Z}/2, H_*(P \wedge P)).$$

By the assumption and Proposition A(b) and (c), Θ goes to

$$x_{2^j-1} \otimes x_{2^j-1} \in \mathrm{Ext}_{E\langle 2\rangle_*}^{0,2^{j+1}-2}(\mathbb{Z}/2, H_*(P \wedge P))$$

and so we are reduced to showing

$$u_0^{[(2^j-1)/4]} x_{2^j-1} \otimes x_{2^j-1} \neq 0 \in \mathrm{Ext}_{E\langle 2\rangle_*}^{*,*+\text{even}}(\mathbb{Z}/2, H_*(P \wedge P)).$$

But, this is an immediate consequence of Lemma 1.1, which says

$$\begin{aligned} u_0^{[(2^j-1)/4]} x_{2^j-1} \otimes x_{2^j-1} = u_2^{[(2^j-1)/4]}(x_3 \otimes x_{2^{j-1}+1} \\ + \text{sum of terms of the form } x_{2k-1} \otimes x_{2l-1} \\ \text{with } 2k-1 \geq 7). \end{aligned}$$

□

Remark 1.3. It is not difficult to read off the presentation of $BP\langle 2\rangle_{2*}(P \wedge P)$ with generators and relations, from Theorem 1.4 of [12]. But, it does not look possible that the above Proposition 1.2 follows easily from this result.

2. ADAMS OPERATIONS ON $BP_{2*}(P \wedge P)$

The main result of this section (Proposition 2.2) gives an upper bound for the BP-order of elements in the (even degree) BP-Hurewicz image of $P \wedge P$. For this, we use the BP-Adams operation ψ^3 [5, 32]. To determine the ψ^3-action on $BP_{\text{even}}(P \wedge P)$, we begin by summarizing the necessary known results.

LEMMA B. (a) *$\psi^3 : BP_{2k-1}(P) \to BP_{2k-1}(P)$ acts as the multiplication by 3^k.*
(b) *ψ^3 is a map of ring spectrum, and so commutes with the pairings: The diagram*

$$\begin{array}{ccc} BP_*(P) \otimes BP_*(P) & \xrightarrow[\cong]{\wedge} & BP_{\text{even}}(P \wedge P) \\ \downarrow{\scriptstyle \psi^3 \otimes \psi^3} & & \downarrow{\scriptstyle \psi^3} \\ BP_*(P) \otimes BP_*(P) & \xrightarrow[\cong]{\wedge} & BP_{\text{even}}(P \wedge P) \end{array}$$

commutes.

Proof. (a) This is well-known; see for instance [13].
(b) For the first claim, see [5] or [39]. The fact that $BP_{\text{even}}(P \wedge P)$ is isomorphic to the tensor product is an immediate consequence of the Künneth formula of [21]. □

As an immediate consequence of Lemma B, we get the following corollary.

COROLLARY 2.1. *The action of the Adams operation*

$$\psi^3 : BP_{2n-2}(P \wedge P) \to BP_{2n-2}(P \wedge P)$$

is given by multiplication by 3^n.

Just as in the case of the original Adams operations in K-theory [2, 25], we resort to the usual result in the elementary number theory and this is the content of the following.

LEMMA C. *Write $n = 2^r m$, with m odd. Then*

$$v_2(3^n - 1) = \begin{cases} r + 2 & \text{if } r \geqq 1 \\ 1 & \text{if } r = 0. \end{cases}$$

Proof. This is well-known and quite easy to show: The key is to prove

$$(1 + 2)^{2^r} \equiv 1 + 2^{r+2} \pmod{2^{r+3}}$$

for all $r \geqq 1$, by the mathematical induction on r. □

Now, Corollary 2.1 and Lemma C immediately implies the following proposition.

PROPOSITION 2.2. *The order of any element in*

$$\operatorname{Ker}(\psi^3 - 1)|_{BP_{2^{j+1}-2}(P \wedge P)}$$

divides 2^{j+2}.

3. MAIN THEOREM AND ITS PROOF

Now, we are ready to prove the main result of this paper.

THEOREM 3.1. *If the Kervaire invariant one element* $\theta_j \in \pi^s_{2^{j+1}-2}(S^0)$ *exists and factors through the double transfer* $P \wedge P \to S^0$, *then* $j \leq 4$.

COROLLARY 3.2. $\theta_5 \in \pi^s_{62}(S^0)$ *of Barratt–Jones–Mahowald* [7] *does not factor through the double transfer.*

Proof of Theorem 3.1. Suppose θ_j exists and lifts to $\tilde{\theta}_j \in \pi^s_{2^{j+1}-2}(P \wedge P)$; let Θ_j be the BP-Hurewicz image of $\tilde{\theta}_j$ and n_j be the order of Θ_j.

Then, as is well-known [23, 33, 38], the mod-2 Hurewicz image of $\tilde{\theta}_j$ is $x_{2^j-1} \otimes x_{2^j-1}$. So Proposition 1.2 forces

$$2^{(2^{j-2})} \mid n_j \quad \text{if } j \geqq 2.$$

On the other hand, as $\Theta_j \in \operatorname{Ker}(\psi^3 - 1)|_{BP_{2^{j+1}-2}(P \wedge P)}$, Proposition 2.2 implies

$$n_j \mid 2^{j+2}.$$

Combining these two, we get $2^{j-2} \leqq j + 2$ when $j \geqq 2$. But, this happens only when $j \leqq 4$. □

4. AFTERMATH

In our previous paper [28], we studied some sufficient condition for the double transfer lift. To state it, we recall the fundamental concept of [28].

Definition 4.1. Suppose X is a space. Then $\alpha \in \pi^s_n(X_+)$ is called G.F. (= Geometrically Flasque) if α has a framed bordism representative $f: M^n \to X$ such that

$$\Sigma M^n \simeq \Sigma N \bigvee S^{n+1}$$

where N is the $n - 1$ skeleton of M^n.

Remark 4.2. Of course, if α is in the image of

$$\pi_n(\Omega\Sigma(X_+)) \to \pi_n(Q(X_+)) \simeq \pi^s_n(X_+)$$

then it is *G.F.* But, usually the set of *G.F.* elements is much larger than this image. For instance, when X is a point (i.e. the case of the framed bordism groups $\pi^s_n(S^0)$) any element $\alpha \in \pi^s_n(S^0)$ is *G.F.*, since Kervaire–Milnor [19] showed that a framed bordism representative of α can be taken either by a homotopy sphere or the Kervaire manifold.

The following is the main result of [28]:

THEOREM 1 (Minami [28]). *Consider the composite*

$$\pi^s_*(SO_+) \to \pi^s_*(SO) \to \pi^s_*(S^0)_{(2)}$$

where the first map is induced by sending the disjoint basepoint to a basepoint in SO and the second map is induced by the G. Whitehead J-map $J: SO \to SG = Q_1 S^0 \simeq Q_0 S^0$. *This is surjective in the 2-primary part by the Kahn–Priddy theorem. Suppose* $\alpha \in \pi^s_*(S^0)_{(2)}$ *has a G.F. lift* $\tilde{\alpha} \in \pi^s_n(SO_+)$, *then it factors through the double transfer, unless it is Hopf invariant one or (possibly) the generator of the image J in* $\pi^s_{15}(S^0)$.

Therefore, Theorem 3.1 immediately implies the following corollary.

COROLLARY 4.3. *Under the situation of Theorem* 1 *of* [28], *such a G.F. lift of* θ_j *may exist only if* $j \leqq 4$.

From the definition, it is easy to see that such a *G.F.* lift exists for those with a framed hypersurface representative. Therefore, we immediately get the following corollary.

COROLLARY 4.4. θ_j *may be represented by a framed hypersurface only if* $j \leqq 4$.

The first such an example was given by [36], where Adams's μ_{8k+1}, μ_{8k+2} are shown not to be represented by a framed hypersurface when $k \geq 1$. But these elements factor through the double transfer, unlike θ_5. [28].

We also get the following as a pushout of Theorem 2 of [28] and Theorem 3.1.

COROLLARY 4.5. *Suppose* $\theta_j \in \pi^s_{2^{j+1}-2}(S^0)$ *exists and there is a G.F. lift* $\tilde{\theta}_j \in \pi^s_{2^{j+1}-2}$ $(STOP_+)$ *under the composite*

$$\pi^s_{2^{j+1}-2}(STOP_+) \to \pi^s_{2^{j+1}-2}(STOP) \to \pi^s_{2^{j+1}-2}(SG) \to \pi^s_{2^{j+1}-2}(S^0)$$

where the first and the third map are defined as before and the second map is induced by the usual infinite loop map $STOP \to SG$ (*see for example* [22]). *Then* $j \leqq 4$.

Remark 4.6. (1) The method used in the present paper would be applied to some other situations in our future papers [29, 30].

(2) Our Theorem 3.1 might have reminded you of the doomsday conjecture, which was disproved by [24]. We will try to revive a variant of it in [30].

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