Boletín de la Sociedad Matemática Mexicana Vol. 37, 1992

FINITE LOCALIZATIONS

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This short note is a response to the articles [7] of Doug Ravenel and [3] of Mark Mahowald and Hal Sadofsky. I give cleaner and more general constructions of the "telescopic" or "finite localization," which they write L'_n. I prefer to call this the *finite* $E(n)$-localization and write L_n^f for it, because, as I shall show, it can be characterized in exactly the same terms as the Bousfield localization, but with the addition of a finiteness assumption. If B is a finite $E(n-1)$-acyclic spectrum with a v_n-self-map $\phi : B \longrightarrow \Sigma^{-q} B$ [2], then $L_n^f B$ is the mapping telescope of B; so L_n^f is a generalization of this construction in that it can be applied to any spectrum X.

By the same method, a finite localization $L_{\mathcal{A}}^f$ can be defined for any set $\mathcal{A}$ of homotopy types of finite spectra. Of particular interest is the case in which $\mathcal{A}$ is the set of finite E-acyclic spectra for some spectrum E, and in this case we will write L_E^f for the corresponding finite localization. The construction of this localization is simpler than that of the Bousfield homology localization—one can work entirely in the homotopy category, and a countable telescope suffices for the construction. It turns out to be easy to show that $L_{\mathcal{A}}^f$ is always "smashing" (i.e., the natural map $X \longrightarrow X \wedge L_{\mathcal{A}}^f S^0$ is an equivalence) and coincides with Bousfield localization with respect to the spectrum $L_{\mathcal{A}}^f S^0$.

For any spectrum E, there is a canonical map $L_E^f X \longrightarrow L_E X$. The "telescope conjecture" for E (advertised for $E = E(n)$ by Ravenel in [4]) is the assertion that this map is an equivalence. It is equivalent to require that any E-acyclic spectrum has an exhaustive filtration whose associated quotients are wedges of *finite* E-acyclic spectra. This structural feature is exactly what Bousfield checks for K-theory, and by virtue of Ravenel's computation [5] we now know that it fails for $E(2)$ at all primes. It would be be very interesting to have invariants vanishing on finite $E(n)$-acyclics and compatible with wedges and cofibrations, but not vanishing on all $E(n)$-acyclics.

1. Finitely $\mathcal{A}$-local spectra

Recall Bousfield's definitions from [1]; here E is any spectrum.

Definition (1). 1. A spectrum W is *E-local* iff $[T, W] = 0$ for every E-acyclic spectrum T.

2. A map $X \longrightarrow Y$ is an *E-equivalence* iff $E_* f$ is an isomorphism.

We have also the variant of this condition, which is clearly implied by it:

2.′ Any map from X to an E-local W extends uniquely to a map from Y to W.

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Bousfield proves:

THEOREM (2). [1] *For any spectra E and X, there is an E-equivalence from X to an E-local spectrum.*

It is easy to see that this map is initial among maps from X to E-local spectra, and terminal among E-equivalences out of X. Either property shows that the map is unique up to canonical equivalence, and it is written $X \longrightarrow L_E X$. Its existence shows that Condition 2.′ for a map f implies that the map is an E-equivalence, since the map $Z \longrightarrow *$ from the mapping cone of f to a point, being initial among maps from Z to E-local spectra, must be E-localization and in particular an E-equivalence.

We now modify these definitions slightly, by testing only against *finite* E-acyclic spectra. In fact, the same methods work for any set $\mathcal{A}$ of homotopy types of finite spectra.

Definition (3). 1. A spectrum W is *finitely $\mathcal{A}$-local* iff $[\Sigma^n A, W] = 0$ for every $A \in \mathcal{A}$ and every $n \in \mathbf{Z}$.

2. A spectrum Z is *finitely $\mathcal{A}$-acyclic* iff $[Z, W] = 0$ for every finitely $\mathcal{A}$-local spectrum W.

3. A map $f : X \longrightarrow Y$ is a *finite $\mathcal{A}$-equivalence* iff its mapping cone is finitely $\mathcal{A}$-acyclic.

THEOREM (4). *For any set $\mathcal{A}$ of finite spectra, and any spectrum X, there is a finite $\mathcal{A}$-equivalence from X to a finitely $\mathcal{A}$-local spectrum.*

The same considerations as above show that this map is initial among maps from X to finitely $\mathcal{A}$-local spectra, and terminal among finite $\mathcal{A}$-equivalences out of X. Again, either property shows that the map is unique up to canonical equivalence. We write $\eta : X \longrightarrow L^f_{\mathcal{A}} X$ for it, and call it the *finite $\mathcal{A}$-localization*.

Suppose $\mathcal{A}$ is the set of finite E-acyclic spectra, for some spectrum E, and write L^f_E for $L^f_{\mathcal{A}}$. Since any E-local spectrum is finitely $\mathcal{A}$-local, or since any finite $\mathcal{A}$-equivalence is an E-equivalence, we get a unique factorization

$$\begin{array}{ccc} & & L^f_E X \\ & \nearrow & \\ X & & \downarrow \\ & \searrow & \\ & & L_E X \end{array} .$$

We will say that the *telescope conjecture* holds for E if $L^f_E X \longrightarrow L_E X$ is an equivalence for all X.

The proof of Theorem (4) is extremely simple. Let $X_0 = X$, form the wedge $W_0 = \vee_f A$ over the set of all homotopy classes of maps $f : A \to X$ from all members of $\mathcal{A}$, and let X_1 be the mapping cone of the evident map. Continue the process to form a diagram

$$\begin{array}{ccccccccc} X & = & X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \cdots \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & W_0 & & W_1 & & W_2 & & \end{array} .$$

Form the mapping telescope X_∞. We claim that the map $X \longrightarrow X_\infty$ is a finite $\mathcal{A}$-localization. To see that the map is a finite $\mathcal{A}$-equivalence, let W be finitely $\mathcal{A}$-local. Since

$$[\bigvee A, W] = \prod [A, W] = 0,$$

we get an inverse system of isomorphisms

$$[X, W] \xleftarrow{\cong} [X_0, W] \xleftarrow{\cong} [X_1, W] \xleftarrow{\cong} \cdots,$$

so the Milnor sequence implies that

$$[X, W] \xleftarrow{\cong} [X_\infty, W]$$

as needed. On the other hand, let A be any member of $\mathcal{A}$, and $f : A \longrightarrow X_\infty$. f compresses through some X_n, and is then killed in X_{n+1}—so $f = 0$. This completes the proof.

2. Some properties of finite localizations

If we form the tower of fibers of the maps from X to the stages X_n in the construction of the finite $\mathcal{A}$-localization, we obtain a sequence

$$\begin{array}{ccccccccc} * & = & T_0 & \longrightarrow & T_1 & \longrightarrow & T_2 & \longrightarrow & \cdots \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & W_1 & & W_1 & & W_2 & & \end{array}$$

in which the fibers are the same (up to suspension) as those in the sequence of X_n's. (The octahedral axiom [8] is used here.) The telescopes of the three sequences fit into a cofibration sequence. (Checking this is one place where a return to some underlying category of spectra seems essential; a map between cofibration sequences cannot generally be extended to a "3×3" diagram.) $T_\infty \longrightarrow X$ might thus be called the finite $\mathcal{A}$-acyclicization of X if such a thing were pronounceable. If X is finitely $\mathcal{A}$-acyclic, then $T_\infty \xrightarrow{\simeq} X$:

PROPOSITION (5) *A spectrum is finitely $\mathcal{A}$-acyclic if and only if it is equivalent to the telescope of a sequence of cofibrations whose quotients are wedges of elements of $\mathcal{A}$.*

Thus the telescope conjecture for E is equivalent to the assertion that any E-acyclic spectrum can be expressed as such a telescope.

COROLLARY (6). *The class of finitely $\mathcal{A}$-acyclic spectra is the smallest class of spectra which is closed under cofibers and all wedges and which contains $\mathcal{A}$.*

Remark (7). These closure conditions on a class $\mathcal{C}$ imply that $\mathcal{C}$ is also closed under retracts. For, it is closed under formation of mapping telescopes (since these are cofibers of maps between wedges), and C is the telescope of the sequence

$$\begin{array}{ccccccccc} C\vee D & \longrightarrow & C\vee\Sigma C & \longrightarrow & C\vee\Sigma^2 D & \longrightarrow & C\vee\Sigma^3 C & \longrightarrow & \cdots \\ \uparrow\alpha & & \uparrow\beta & & \uparrow\alpha & & \uparrow\beta & & \\ C\vee D & & \Sigma C\vee\Sigma D & & \Sigma^2 C\vee\Sigma^2 D & & \Sigma^3 C\vee\Sigma^3 D & & \end{array}$$

where $\alpha = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $\beta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

COROLLARY (8). *If Z is a finitely $\mathcal{A}$-acyclic spectrum and X is any spectrum, then $X\wedge Z$ is again finitely $\mathcal{A}$-acyclic.*

PROPOSITION (9). *For any spectrum X, the natural map*

$$X \cong X\wedge S^0 \xrightarrow{1\wedge\eta} X\wedge L^f_{\mathcal{A}}S^0$$

is a finite $\mathcal{A}$-localization.

Proof. We show that this map satisfies the defining conditions. To see that the target is finitely $\mathcal{A}$-local, let $A\in\mathcal{A}$ and $f: A\longrightarrow X\wedge L^f_{\mathcal{A}}S^0$ a map. There is a finite subspectrum X_α of X, and a stage S^0_n in the directed system constructing $L^f_{\mathcal{A}}S^0$, such that f compresses through $X_\alpha\wedge S^0_n$. Form the adjoint map $A\wedge DX_\alpha\longrightarrow S^0_n$. The evident induction over skelata of DX_α shows that the composite $A\wedge DX_\alpha\longrightarrow S^0_{n+k}$ is null for some k, and hence that $f=0$.

To see that the map is a finite $\mathcal{A}$-equivalence, notice that by Corollary (8), each map in the inverse system

$$[X\wedge S^0, W] = [X\wedge S^0_0, W]\longleftarrow[X\wedge S^0_1, W]\longleftarrow\cdots$$

is bijective. The Milnor sequence then implies that

$$[X,W]\xleftarrow{\cong}[X\wedge L^f_{\mathcal{A}}S^0, W].$$

COROLLARY (10). *The telescope conjecture holds for E if and only if E is smashing (i.e., the map $X\cong X\wedge S^0\longrightarrow X\wedge L_E S^0$ is an equivalence) and the natural map $L^f_E S^0\longrightarrow L_E S^0$ is an equivalence.*

COROLLARY (11). *Finite $\mathcal{A}$-localization is Bousfield localization with respect to the spectrum $L^f_{\mathcal{A}}S^0$.*

Proof. Take for X the spectrum $R = L^f_{\mathcal{A}}S^0$. Since R is finitely $\mathcal{A}$-local, the map

$$R\cong R\wedge S^0\xrightarrow{1\wedge\eta} R\wedge R$$

is an equivalence. Its inverse gives R the structure of a ring-spectrum for which the multiplication map is an equivalence. Standard arguments ([7], proof of 3.7) show that if R is any such ring-spectrum and any spectrum X, the map

$$X \cong X \wedge S^0 \longrightarrow X \wedge R$$

is an R-localization, and this proves the corollary.

3. The examples

We recall a basic fact about the Morava K-theories $K(n)$.

THEOREM (12). [4] *Any $K(n)$-acyclic finite complex is $K(n-1)$-acyclic.*

It follows that the finite localization with respect to $K(n)$ coincides with the finite localization with respect to the spectrum

$$K(\leq n) = K(0) \vee K(1) \vee \cdots \vee K(n).$$

Write L_n^f for this localization functor; this coincides with the finite localization with respect to the spectrum $E(n)$. The spectrum $K(n)$ is not smashing unless $n = 0$, but a deep theorem of Hopkins and Ravenel (see [6]) asserts that $K(\leq n)$ *is* smashing. Write L_n for the Bousfield localization with respect to $K(\leq n)$. The traditional telescope conjecture is the telescope conjecture for this theory $K(\leq n)$. Bousfield [1] used computations of Mark Mahowald and the author to verify it for $n = 1$, and observed that it gave a filtration of any $K(\leq 1)$-acyclic spectrum whose quotients are wedges of certain finite $K(\leq n)$-acyclic spectra; see Corollary (16) below.

We now explain how the nilpotence theorem can be used to identify certain L_n^f localizations with more explicit telescopes.

THEOREM (13). [2] 1. *If B is a $K(n-1)$-acyclic finite complex then for some q there is a map $\phi : B \longrightarrow \Sigma^{-q} B$ inducing an isomorphism in $K(n)$ and nilpotent maps in $K(m)$ for all $m \neq n$. (Such a map is called a v_n-self map.)*

2. *If A and B are $K(n-1)$-acyclic finite complexes with v_n-self-maps ψ and ϕ, and $f : A \longrightarrow B$ is any map, then there are positive integers i and j for which the diagram*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow \psi^i & & \downarrow \phi^j \\ \Sigma^{-r} A & \xrightarrow{\Sigma^{-r} f} & \Sigma^{-r} B \end{array}$$

commutes.

Given any self-map $\phi : B \longrightarrow \Sigma^{-q} B$, let $\mathrm{tel}(\phi)$ be the corresponding mapping telescope of the sequence

$$\begin{array}{cccc} B & \xrightarrow{\phi} & \Sigma^{-q} B & \xrightarrow{\Sigma^{-q}\phi} \cdots \\ \uparrow & & \uparrow & \\ C & & \Sigma^{-q} C & \end{array},$$

in which C is the desuspension of the mapping cone of ϕ. Note that C is $K(n)$-acyclic.

PROPOSITION (14). *If B is a $K(n-1)$-acyclic finite complex, with v_n-self-map $\phi : B \longrightarrow \Sigma^{-q}B$, then the map $B \longrightarrow \mathrm{tel}(\phi)$ is a finite $K(n)$-localization.*

Proof. First, the map is a finite $K(n)$-equivalence: Let W be finitely $K(n)$-local. Since C is $K(n)$-acyclic, each map in the sequence

$$[B,W] \overset{[\phi,1]}{\longleftarrow} [\Sigma^{-q}B,W] \overset{[\Sigma^{-q}\phi,1]}{\longleftarrow} \cdots$$

is an isomorphism. The Milnor sequence shows that $[B,W] \longleftarrow [\mathrm{tel}(\phi),W]$ is bijective.

Next, $\mathrm{tel}(\phi)$ is finitely $K(n)$-local: Let A be any finite $K(n)$-acyclic spectrum, and $f : A \longrightarrow \mathrm{tel}(\phi)$ any map. f compresses through a map $g : A \longrightarrow \Sigma^{-kq}B$ for some k. The trivial map is a v_n-self-map of A, so for some j the diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & \Sigma^{-nq}B \\ \downarrow * & & \downarrow \Sigma^{-nq}\phi^j \\ \Sigma^{-jq}A & \xrightarrow{\Sigma^{-jq}g} & \Sigma^{-(n+j)q}B \end{array}$$

commutes. Thus $f = 0$.

Finally, we note that there is a criterion for a spectrum W to be finitely $K(n)$-local in terms of homotopy with suitable coefficients:

PROPOSITION (15). *Let A be any finite spectrum with $K(n+1)_*A \neq 0$ and $K(n)_*A = 0$. A spectrum W is finitely $K(n)$-local if and only if*
(i) *$\pi_*(W;\mathbf{Z}/l) = 0$ for all primes $l \neq p$, and*
(ii) *$[\Sigma^k A, W] = 0$ for all $k \in \mathbf{Z}$.*

Proof. Let $\mathcal{A}$ be the set consisting of A and $S^0 \cup_l e^1$ for all $l \neq p$. The set of finite finitely $\mathcal{A}$-acyclic spectra is closed under cofibers and retracts (by Remark (7)), so it must be the class of finite $K(m)$-acyclics for some m, by the main theorem of [2]. It also consists entirely of $K(n)$-acyclics (since its generators are $K(n)$-acyclic), but contains the $K(n+1)$-nonacyclic A. Hence $m = n$.

COROLLARY (16). *Let A be as in the proposition. Then any finitely $K(n)$-acyclic spectrum is the telescope of a sequence of spectra having wedges of suspensions of A and mod l-Moore spaces for l prime to p as cofibers.*

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