Topological Hochschild homology

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24 April 2018

1 Hochschild homology

Good references for Hochschild homology are [Lod13, Chapter 1] and [Wei94, Chapter 9]. Let k be a commutative ring, A a k-algebra, and M an (A, A)-bimodule.

Definition 1.1. The **cyclic bar construction** of A with coefficients in M is the simplicial k-module $B^{cyc}(A; M)$ with n-simplicies $M \otimes_k A^{\otimes_k n}$ and face maps and degeneracies given by the following formulae:

$$d_i(m \otimes a_1 \otimes \cdots \otimes a_n) = \begin{cases} ma_1 \otimes a_2 \otimes \cdots \otimes a_n & i = 0, \\ m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n & 0 < i < n, \\ a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1} & i = n, \end{cases}$$

$$s_i(m \otimes a_1 \otimes \cdots \otimes a_n) = m \otimes a_1 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n$$

Definition 1.2. The **Hochschild homology** of A with coefficients in M are the homotopy groups of the geometric realization of the cyclic bar construction:

$$HH_n^k(A; M) := \pi_n |B^{cyc}(A; M)|.$$

These are also the homology groups of the associated (via the Dold-Kan correspondence) **Hochschild chain complex** $C_\bullet(A; M)$.

$$0 \longleftarrow M \xleftarrow{b} M \otimes_k A \xleftarrow{b} M \otimes_k A \otimes_k A \xleftarrow{b} \cdots$$

where $b = \sum_{i=0}^{n} (-1)^i d_i$.

Example 1.3. When $M = A = k$, the Hochschild complex is

$$k \xleftarrow[0]{} k \xleftarrow[1]{} k \xleftarrow[0]{} k \xleftarrow[1]{} k \qquad \cdots$$

and therefore

$$HH_n(k) = \begin{cases} k & (n = 0), \\ 0 & (n > 0). \end{cases}$$

Example 1.4 ([Lod13, 1.1.6]). The zeroth Hochschild homology of A with coefficients in M is the **module of coinvariants of** M**:**

$$HH_0(A;M) = M_A := {}^{M}/_{\langle ma - am \mid m \in M, a \in A \rangle}$$

When $M = A$, $HH_0(A) = A/[A,A]$, the quotient of A by its commutator submodule. If A is abelian, the commutator submodule is zero so $HH_0(A) = A$.

Example 1.5 ([Wei94, 9.1.2]). Let G be a group and let $A = k[G]$ be the group algebra of G with coefficients in k. Let M be a right G-module, considered as an (A, A)-bimodule with trivial left action. Then the Hochschild homology of A with coefficients in M is the group homology of G with coefficients in M: $HH_*(A; M) = H_*(G; M)$.

Example 1.6 ([Wei94, 9.1.4]). Let n be a positive integer and assume that k is a field of characteristic coprime to n. Let $A = k[x]/\langle x^{n+1}\rangle$. Then $HH_i(A) = A/x^nA$ for all $i \geq 1$.

2 Topological Hochschild homology

The original references for topological Hochschild homology are [B8̈5b, B8̈5c, B8̈5a], although they are hard to find (email me if you'd like a copy). The section on the Loday construction follows [AR05, Section 3].

Fix a symmetric monoidal category of spectra $(\mathbf{Sp}, \wedge, \mathbb{S})$. Denote the monoidal category of commutative ring spectra in **Sp** by $(\mathbf{CommRingSp}, \wedge, \mathbb{S})$, and the monoidal category of R-algebra spectra for a ring spectrum R by $(\mathbf{AlgSp}_R, \wedge_R, \mathbb{S})$.

2.1 The Loday Construction

The **Loday construction** gives a concise description for topological Hochschild homology via

$$THH(R) = R \otimes S^1.$$

where S^1 is a simplicial model of the circle with n-many n-simplicies. A good description of this construction of the spectrum $THH(R)$ can be found in [AR05, Section 3], and we follow it here.

Let R be a commutative ring spectrum and let U be a finite set. Define

$$R \otimes U := \bigwedge_{u \in U} R;$$

this is the smash product of one copy of R for each element of U. This is again a commutative ring spectrum. It is often convenient to specify the copy of R in the smash product corresponding to $u \in U$ by $\{u\} \otimes R$, so we will write

$$R \otimes U = \bigwedge_{u \in U} R \otimes \{u\}.$$

If $f\colon U \to V$ is a function of finite sets, define $f \otimes R\colon U \otimes R \to V \otimes R$ as follows. For each $v \in V$, define a map

$$\bigwedge_{u \in f^{-1}(v)} R \xrightarrow{f_v} R \otimes \{v\}$$

by iterated multiplication; since R is a commutative ring spectrum, there is no ambiguity in the order of multiplication. If $f^{-1}(v)$ is empty, this is the unit map $\mathbb{S} \to R \otimes \{v\}$. The function $R \otimes f\colon R \otimes U \to R \otimes V$ is the smash product over all $v \in V$ of the maps f_v.

It follows that the construction $U \otimes R$ is functorial in U. We extend it degree wise to simplicial finite sets K to define a simplicial spectrum

$$R \otimes K\colon [q] \mapsto R \otimes K_q$$

with face and degeneracy maps $R \otimes d_i$ and $R \otimes s_j$.

Definition 2.1. The **Loday construction** $\mathcal{L}_K(R)$ of a commutative ring spectrum R with respect to a simplicial finite set K is the geometric realization of $R \otimes K$.

$$\mathcal{L}_K(R) := |R \otimes K|$$

Remark 2.2.

(a) The choice of name **Loday Construction** comes from [HHL+18].

(b) The definition of the Loday construction $K \otimes R$ depends only on the geometric realization of the simplicial set K, up to weak equivalence. See for example [AR05, Lemma 3.8].

(c) For any simplicial spectrum $X_\bullet$, its geometric realization is given by the coend

$$|X_\bullet| := \int^{[n] \in \Delta} (X_n) \wedge |\Delta^n|_+.$$

This is in analogy to the geometric realization of a simplicial set $K_\bullet$, which is given by the coend

$$|K_\bullet| := \int^{[n] \in \Delta} K_n \times \Delta^n.$$

The construction and its properties are detailed in [EKMM07, Chapter X].

(d) The Loday construction describes how the category of ring spectra with the smash product is tensored over simplicial finite sets.

Definition 2.3. The **topological Hochschild homology** of a commutative ring spectrum R is the spectrum

$$\mathrm{THH}(R) = |\mathcal{L}_{S^1}(R)| = |R \otimes S^1|.$$

If we restrict to the category of R-algebra spectra and use instead the smash product over R, then we obtain a relative Loday construction, which we denote $\mathcal{L}_K(R; A)$. In this manner, the category of R-algebra spectra is tensored over simplicial finite sets. To distinguish this from the tensor over simplicial finite sets as a commutative ring specturm, we write this as $A \otimes_R S^1$.

Definition 2.4. The **topological Hochschild homology** of an R-algebra spectrum X is the spectrum

$$\mathrm{THH}^R(A) = |\mathcal{L}_{S^1}(A; R)| = |A \otimes_R S^1|.$$

In this notation, $\mathrm{THH}^{\mathbb{S}}(A)$ is the topological Hochschild homology of A as an $\mathbb{S}$-algebra, i.e. a ring spectrum. We will sometimes drop the superscript if the base ring spectrum is understood.

Remark 2.5. It is also possible to define the topological Hochschild homology $\mathrm{THH}^R(A; M)$ of an R-algebra A with coefficients in an A-module M, but this is not necessary for our purposes. See [EKMM07, IX.1.1].

2.2 Connection to Hochschild homology

Why does this deserve to be called topological Hochschild homology? Let k be a discrete commutative ring and let A be a k-algebra. Assume that A is flat as a k-algebra. Consider

$$\mathrm{THH}^{Hk}(HA) = |HA \otimes_{Hk} S^1|.$$

Since S^1 has q-many q-simplicies, the simplicial spectrum $HA \otimes_{Hk} S^1$ has q-simplicies:

$$\underbrace{HA \wedge_{Hk} HA \wedge_{Hk} \cdots \wedge_{Hk} HA}_{q}$$

The homotopy of this spectrum is

$$\underbrace{A \otimes_k A \otimes_k A \otimes_k \cdots \otimes_k A}_{q}.$$

In fact, the degeneracies and face maps of $HA \otimes_{Hk} S^1$ become the degeneracies and faces of the cyclic bar construction after applying π_* level-wise. In particular,

$$\pi_*(HA \otimes_{Hk} S^1) \cong B^{cyc}(A).$$

Therefore,

$$\pi_* \operatorname{THH}^{Hk}(HA) = \pi_*|HA \otimes_{Hk} S^1| \cong \pi_*|B^{cyc}(A)| = HH^k_*(A).$$

(See also [EKMM07, IX.1.7])

Here is a handy table to clarify the analogy.

unstable	**stable**
space	spectrum
$\mathbb{Z}$	$\mathbb{S}$
$\mathbb{Z}$-module (abelian group)	$\mathbb{S}$-module (spectrum)
$\mathbb{Z}$-algebra (ring)	$\mathbb{S}$-algebra (ring spectrum)
k-algebra	R-algebra spectrum
tensor product $\otimes_k$	smash product $\wedge_R$
cyclic bar construction $B^{cyc}(A)$	Loday construction $R \otimes S^1$
realization of the cyclic bar construction $\lvert B^{cyc}(A)\rvert$	topological Hochschild spectrum $\operatorname{THH}(R)$
Hochschild homology $HH_n(A) = \pi_n \lvert B^{cyc}(A)\rvert$	$\pi_n \operatorname{THH}(R)$

Indeed, computing topological Hochschild homology often comes down to computing Hochschild homology.

Theorem 2.6 ([EKMM07, IX.1.11])**.** *Let E and R be commutative ring spectra, and A a commutative R-algebra. If $E_*(A)$ is flat over $E_*(R)$, then there is a spectral sequence of $E_*(R)$-modules*

$$E^2_{i,j} \cong HH_j^{E^*(R)}(E_*(A))_i \implies E_{i+j}(\operatorname{THH}^R(A))$$

Proof sketch [MS93, Proposition 3.1] (c.f. [EKMM07, Theorem X.2.9]). Given a simplicial spectrum $X_\bullet$, there is a simplicial filtration on $|X_\bullet|$ and a spectral sequence (called the **skeleton spectral sequence**)

$$E^1_{i,j} = E_i(X_j) \implies E_{i+j}(|X_\bullet|)$$

When $X_\bullet$ is the simplicial spectrum $A \otimes S^1$ whose realization is $\operatorname{THH}(A)$, then

$$E^1_*(X_j) = E_*(A^{\wedge_R (j+1)}) \cong (E_*A)^{\otimes_{E_*R}(j+1)},$$

this last isomorphism by the flatness assumption. The homology of this complex computes the ordinary Hochschild homology of E_*A, so we have

$$E^2_{i,j} = HH_j^{E_*R}(E_*A)_i,$$

where j is the homological degree and i is the simplicial degree of the graded E_*R-algebra E_*A. □

Bökstedt uses this spectral sequence in [B85c] when $R = \mathbb{S}$, and $E = A = H\mathbb{F}_p$ to find the homology of the spectrum $THH(H\mathbb{F}_p)$, which is then used to compute the homotopy type of this spectrum. With these choices of R, E and A, we have

$$E^2_{i,j} \cong HH_j^{(H\mathbb{F}_p)_*\mathbb{S}}((H\mathbb{F}_p)_*(H\mathbb{F}_p))_i \implies (H\mathbb{F}_p)_{i+j}\, THH^{\mathbb{S}}(H\mathbb{F}_p).$$

This can be simplified: notice that

$$(H\mathbb{F}_p)_*\mathbb{S} = \pi_*(H\mathbb{F}_p \wedge \mathbb{S}) = \pi_*(H\mathbb{F}_p) = \mathbb{F}_p \text{ in degree } 0$$

Also, notice that $(H\mathbb{F}_p)^*(H\mathbb{F}_p) = [H\mathbb{F}_p, H\mathbb{F}_p]$ is the set of all cohomology operations mod p, or the Steenrod algebra. So $(H\mathbb{F}_p)_*(H\mathbb{F}_p)$ is the dual Steenrod algebra, which we write $\mathcal{A}_p$. So the E^2-page of this spectral sequence simplifies to computing

$$E^2_{i,j} = HH_j^{\mathbb{F}_p}(\mathcal{A}_p)_i \implies (H\mathbb{F}_p)_{i+j}\, THH^{\mathbb{S}}(H\mathbb{F}_p).$$

Theorem 2.7 ([B85c, Theorem 1.1a]). *As rings,* $THH_*(H\mathbb{F}_p) \cong \mathbb{F}_p[\sigma]$ *where* σ *is in degree* 2.

He also computes the topological Hochschild homology of the integers.

Theorem 2.8 ([B85c, Theorem 1.1b]).

$$THH_n(\mathbb{Z}) = \begin{cases} \mathbb{Z} & (n = 0), \\ 0 & (n = 2i, i > 0), \\ \mathbb{Z}/i\mathbb{Z} & (n = 2i - 1). \end{cases}$$

3 The trace map

The trace map is a way to extract information about K-theory, which is hard to compute, via topological Hochschild homology, which is significantly easier to compute. This frequently loses information, but the trace map factors through **topological cyclic homology**, which is much closer to K-theory.

3.1 Hochschild Homology of an Exact Category

Let **C** be an exact category.

Definition 3.1 ([McC94, Section 2])**.** The **additive cyclic nerve** of **C** is the simplicial abelian group $N^{cyc}_\bullet(\mathbf{C})$ with N-simplicies

$$N^{cyc}_n(\mathbf{C}) = \bigoplus_{(c_0,c_1,\ldots,c_n)} \mathrm{Hom}_{\mathbf{C}}(c_1,c_0) \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbf{C}}(c_2,c_1) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbf{C}}(c_n,c_{n-1}) \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbf{C}}(c_0,c_n)$$

where the sum runs over all $(n+1)$-tuples $(c_0, c_1, \ldots, c_n)$ of objects in **C**. The face maps and degeneracies are

$$d_i(\alpha_0 \otimes \alpha_1 \otimes \cdots \otimes \alpha_n) = \begin{cases} \alpha_0 \otimes \alpha_1 \otimes \cdots \otimes \alpha_i\alpha_{i+1} \otimes \cdots \otimes \alpha_n & 0 \leq i < n \\ \alpha_n\alpha_0 \otimes \alpha - 1 \otimes \cdots \otimes \alpha_n & i = n \end{cases}$$

$$s_i(\alpha_0 \otimes \cdots \otimes \alpha_n) = \begin{cases} \alpha_0 \otimes \cdots \otimes \alpha_i \otimes \mathrm{id}_{c_{i+1}} \otimes \alpha_{i+1} \otimes \cdots \otimes \alpha_n & 0 \leq i < n \\ \alpha_0 \otimes \cdots \otimes \alpha_n \otimes \mathrm{id}_{c_0} i = n \end{cases}$$

Definition 3.2. The **Hochschild homology** groups of an exact category **C** are the homotopy groups of the geometric realization of the additive cyclic nerve.

$$HH_n(\mathbf{C}) := \pi_n |N^{cyc}_\bullet(\mathbf{C})|$$

This is an appropriate generalization of Hochschild homology, insofar as it agrees with the Hochschild homology of a ring when **C** is the category of finitely generated projective modules over that ring.

Theorem 3.3 ([McC94, Corollary 2.4.4])**.** *When* A *is a commutative ring,*

$$HH_n(A) = HH_n(\mathbf{Proj}^{fg}(A)),$$

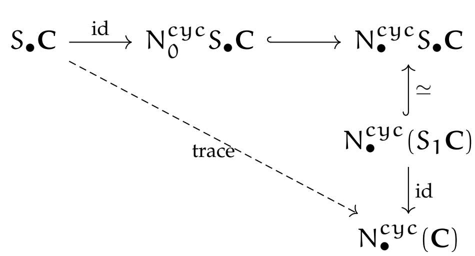
where $\mathbf{Proj}^{fg}(A)$ *is the category of finitely generated projective* A*-modules.*

We may likewise define the topological Hochschild homology of any category enriched in spectra. This is done in [BM12, Section 10].

3.2 The trace map

This section follows [McC94] and [KM00].

For an exact category **C**, we may construct the K-theory space of **C** via the Waldhausen $S_\bullet$ construction: $K(\mathbf{C}) = \Omega|S_\bullet\mathbf{C}|$. The trace map is defined via a map from $S_\bullet\mathbf{C}$ to $N^{cyc}(\mathbf{C})$ by inverting a weak equivalence as in the diagram below.



This gives a map from the q-simplicies of $S_\bullet \mathbf{C}$ to the $(q-1)$-simplicies of $N^{cyc}_\bullet(\mathbf{C})$. Upon taking realizations, this gives a map of spaces

$$K(\mathbf{C}) = \Omega|S_\bullet \mathbf{C}| \to |N^{cyc}_\bullet(\mathbf{C})|$$

and taking homotopy groups, this gives a homomorphism

$$K_n(\mathbf{C}) \to HH_n(\mathbf{C}).$$

When $\mathbf{C} = \mathbf{Proj}^{fg}(A)$ is the category of finitely generated projective A-modules, then this homomorphism is

$$K_n(A) \to HH_n(A).$$

A similar construction is made for topological Hochschild homology in [BM12, Section 10].

3.3 Dennis trace

For any ring R, we construct the K-theory space of R via the plus construction: $K(R) = B\,GL(R)^+$. The Hurewicz map gives a homomorphism from the homotopy of this space to its homology.

$$K_n(R) = \pi_n(B\,GL(R)^+) \xrightarrow{h} H_n(B\,GL(R)^+;\mathbb{Z})$$

By properties of the plus-construction, the homology of $B\,GL(R)^+$ and the homology of $B\,GL(R)$ agree. Moreover, for any group G, the homology of BG is the same as the homology of G. So we may extend the Hurewicz homomorphism's codomain to be the group homology $H_n(GL(R);\mathbb{Z})$.

$$K_n(R) = \pi_n(B\,GL(R)^+) \xrightarrow{h} H_n(B\,GL(R)^+;\mathbb{Z}) \cong H_n(GL(R);\mathbb{Z}) \tag{1}$$

By [Wei94, Example 9.1.2], there is a homomorphism from the group homology of G to the Hochschild homology of $\mathbb{Z}[G]$, for any ring $\mathbb{Z}$.

$$H_n(G;\mathbb{Z}) \to HH_n(\mathbb{Z}[G])$$

In the case $G = GL_m(R)$, there is an inclusion $\mathbb{Z}[GL_m(R)] \to M_m(R)$, where $M_m(R)$ is the ring of $m \times m$ matrices with coefficients in R. By Morita invariance of Hochschild homology, this last term is isomorphic to $HH_n(R)$, and the isomorphism is induced by the trace map $\mathrm{tr}\colon M_m(R) \to R$. The composite yields a homomorphism:

$$H_n(GL_m(R);\mathbb{Z}) \to HH_n(\mathbb{Z}[GL_m(R)]) \to HH_n(M_m(R)) \xrightarrow[\cong]{\mathrm{tr}_*} HH_n(R).$$

Now taking the colimit over all m, these together yield a homomorphism

$$H_n(GL(R);\mathbb{Z}) \cong \operatorname*{colim}_m H_n(GL_m(R);\mathbb{Z}) \to HH_n(R) \tag{2}$$

The composite of (2) and (1) is referred to as the **Dennis Trace Map**, after Keith Dennis.

3.4 An example

Example 3.4. We can try to find the K-theory of $\mathbb{Z}$ using the Dennis trace map. We have

$$\mathrm{HH}_n(\mathbb{Z}) = \begin{cases} \mathbb{Z} & (n = 0), \\ 0 & (n > 0). \end{cases}$$

So this can't give us any information about the K-theory of $\mathbb{Z}$, except $K_0(\mathbb{Z}) = 0$, which we already knew. This shortcoming is resolved by lifting the trace map to topological Hochschild homology in the next section.

Theorem 3.5 ([B8̈5a, Theorem 1.1]). *The trace map* $\pi_{2k-1} K(\mathbb{Z}) \to \pi_{2k-1} \mathrm{THH}(\mathbb{Z})$ *is surjective for all positive integers* k.

Example 3.6. In Example 3.4, we tried to use the Dennis trace map to learn about the K-theory of the integers, but this failed because $\mathrm{HH}_n(\mathbb{Z}) = 0$ for $n \geq 0$. Let's try again with the trace map $K(\mathbb{Z}) \to \mathrm{THH}(\mathbb{Z})$. We have

$$\mathrm{THH}_n(\mathbb{Z}) = \begin{cases} \mathbb{Z} & (n = 0), \\ 0 & (n = 2j, j > 0), \\ \mathbb{Z}/_j & (n = 2j - 1). \end{cases}$$

And moreover, the trace map $K_{2i-1}(\mathbb{Z}) \to \mathrm{THH}_{2i-1}(\mathbb{Z})$ is surjective, so we learn that $K_{2i-1}(\mathbb{Z})$ is nontrivial for each i. This is a whole lot more than we learned with the Dennis trace!

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