Lecture Notes in Mathematics

Barry Mazur
William Messing

# Universal Extensions and One Dimensional Crystalline Cohomology

370

Springer

# Lecture Notes in Mathematics

Edited by A. Dold, Heidelberg and B. Eckmann, Zürich

## 370

Barry Mazur
Harvard University, Cambridge, MA/USA
William Messing
Massachusetts Institute of Technology, Cambridge, MA/USA

# Universal Extensions and One Dimensional Crystalline Cohomology

Springer-Verlag
Berlin · Heidelberg · New York 1974

AMS Subject Classifications (1970): F-30, K-05, G-05, L-05

ISBN 3-540-06659-4 Springer-Verlag Berlin · Heidelberg · New York
ISBN 0-387-06659-4 Springer-Verlag New York · Heidelberg · Berlin

Library of Congress Catalog Card Number 73-21377. Printed in Germany.

Offsetdruck: Julius Beltz, Hemsbach/Bergstr.

## INTRODUCTION

The object of this paper is to prove these results announced by Grothendieck.

Theorem 1. If A is an abelian scheme over S its universal extension is crystalline in nature and its Lie algebra is isomorphic to the one-dimensional crystalline cohomology of $A^*$ over S, $R^1 f_{*,\mathrm{crys}}(\mathcal{O}_{A^*,\mathrm{crys}})$.

Theorem 2. If G is a Barsotti-Tate (= p-divisible) group on S , a base such that p is locally nilpotent, then its universal extension is crystalline in nature, and its Lie algebra provides a generalization of the classical Dieudonné module theory for Barsotti-Tate groups.

## UNIVERSAL EXTENSIONS

If A is an abelian variety over a field k , the universal extension of A is defined to be an extension of algebraic groups

$$(*) \qquad 0 \to V(A) \to E(A) \to A \to 0$$

where V(A) is a vector group over k and such that (*) is universal for extensions of A by vector groups.

Rosenlicht [22] defined this notion and showed that any abelian variety A possesses a universal extension. The key to his construction are the isomorphisms

$$\mathrm{Ext}(A,G_a) \cong H^1(A,\mathcal{O}_A) \cong \mathrm{Hom}_k(\underline{\omega}_{A^*},G_a)$$

which gives

$$\mathrm{Ext}^1(A,V) \cong \mathrm{Hom}_k(\underline{\omega}_{A^*},V)$$

where $V$ is an arbitrary vector group over $k$, $A^*$ is the dual abelian variety, with zero section

$$e: \mathrm{Spec}\ k \to A^*$$

and $\underline{\omega}_{A^*} = e^*\Omega^1{}_{A^*/S}$. Taking $V = \underline{\omega}_{A^*}$, the universal extension is the element in $\mathrm{Ext}(A,V)$ corresponding to $1 \in \mathrm{Hom}(\underline{\omega}_{A^*}\ \underline{\omega}_{A^*})$.

In the same paper, Rosenlicht described the relationship between differentials of the $2^{\mathrm{nd}}$ kind and rational cross-sections of the universal extension.

In [27] Weil observed that when working on an abelian variety $A$ over an arbitrary field, consideration of extensions of $A$ by a vector group replaces the study of differentials of the second kind, while consideration of extensions of $A$ by a torus replaces the study of differentials of the third kind. He attributes these ideas (in the classical case) to Severi. Over $\mathbb{C}$, Barsotti in [1 bis] established algebraically the isomorphism

$$\mathrm{Ext}(A,G_a) \cong H^1(\mathcal{O}_A) \cong \frac{\text{differentials of second kind}}{\text{holomorphic differentials} + \text{exact differentials}}$$

(See Serre's [24] and [25] for a beautiful account of these ideas).

Another approach to the universal extension is provided by Tate's definition of generalized Picard varieties [26]. He considers the group of divisors on $A$ not containing the zero $e$, which are algebraically equivalent to zero, modulo the subgroup of principal divisors (f) where $f \equiv 1 \mod \underline{m}_e^2$ ($\underline{m}_e$= maximal ideal at $e$). (See also [15 bis]). Both Tate and Lang ask whether this abstract group carries a natural algebraic structure. This algebraic structure was provided by Murre [18] and also by Oort (unpublished).

Grothendieck, more recently, provided still another viewpoint on the universal extension (by means of the theory of group extensions with integrable connections - which he called $\natural$-extensions). In a letter to Tate, Grothendieck announced that the universal extension over $\mathbb{C}$ is crystalline in nature and conjectured that the same is true over any base. In his Montreal lectures he discussed the relation between the universal extension and "the generalized Dieudonné theory" [13].

A discussion of the crystalline nature of the universal extension and applications to the deformation theory of abelian varieties and Barsotti-Tate groups is given in [16] via the theory of the exponential. Previously Cartier had in [5] solved these problems (at least when the base is a perfect field) for p-divisible formal Lie groups. His approach also yielded the result that the Lie algebra of the universal extension is the Dieudonné module (a result which we generalize below).

We shall treat alongside the theory for abelian varieties the corresponding theory for p-divisible (= Barsotti-Tate) groups. Amusingly enough, we repeat the complicated history sketched above.

Thus, in Chapter I §1 we introduce the universal extension (in a more general context, but) in the spirit of Rosenlicht, and Serre. In Chapter I § 2 we identify the universal extension with something we call Extrig (rigidified extensions) which is modelled on Tate's approach.

In Chapter I §3 we identify Extrig with $\mathrm{Ext}^{\natural}$ ($\natural$-extensions) and thus pass to Grothendieck's.

From $\mathrm{Ext}^{\natural}$ one may establish the crystalline nature in a lengthy, but straightforward way (c.f. Chapter II), and also pass to a hypercohomological interpretation of the universal extension (Chapter I §4) thereby establishing the link with De Rham cohomology.

In Chapter I §5 we mention some connections between the constructions we have dealt with and the Mordell-Weil group of an elliptic curve over $\mathbb{Q}$ .

In chapter II we discuss the crystalline nature of the universal extension, i.e. its relation to "generalized" Dieudonné Theory. The results of §9 and 13 and 15 imply that for $A$, an abelian variety over a perfect field $k$ (char $k = p > 0$), and $G$, its associated $p$-divisible group, there is a canonical isomorphism between the Dieudonné module of $G$ and the crystalline $H^1$ of $A$ . The reduction modulo $p$ of this statement was proven by Oda [ 18 bis].

Throughout this chapter we rely heavily on the work of Berthelot, Grothendieck and Illusie.

We refer the reader to the introduction to Chapter II for more precision on its contents.

## OPEN QUESTIONS

a) Give a comparison of our theory of Dieudonné crystals associated to $p$-divisible formal Lie groups (over $S$) with Cartier's theory.

b) Find a Dieudonné crystal theory for finite, locally-free $p$-groups over $S_0$ (a base of characteristic $p$).

c) Determine whether the functor $G \mapsto \mathbb{D}^*(G)$ on a base $S_0$ , of characteristic $p$ , is fully-faithful.

# CONTENTS

# CHAPTER ONE

## EXPLICIT CONSTRUCTIONS OF UNIVERSAL EXTENSIONS

### §1. GROUP SCHEMES AND THEIR RELATIONS TO VECTOR GROUPS

By *group scheme* over $S$ we shall mean commutative flat separated group scheme locally of finite presentation over $S$.

If $Q$ is any quasi-coherent $\mathcal{O}_S$-module, we may regard $Q$ as a sheaf for the fppf site by the rule:

$$Q(S') = \Gamma(S', \varphi^* Q)$$

where $\varphi : S' \to S$ is the structural morphism. If $L$ is a locally free $\mathcal{O}_S$-module of finite rank, then $L$, regarded as a sheaf for the fppf site over $S$, is representable by a group scheme which is locally isomorphic to a finite product of $G_a$'s. Call such a group scheme $L$ a *vector group over $S$*.

Fix a group scheme $G$ over $S$ and consider the following two universal problems:

*Problem A* (*Universal homormorphism problem*):

*Vector group hull of $G$*:

Find a mapping

$$\alpha : G \to V$$

to a vector group over $S$, which is universal for mappings of $G$, to vector groups, in the following sense:

The induced mapping

$$\alpha : \mathrm{Hom}_{\mathcal{O}_S}(V,M) \to \mathrm{Hom}_S(G,M)$$

is an isomorphism for all vector groups $M$ over $S$. If such a $V$ can be found, call it the vector-group hull of $G$.

Quasi-coherent hull of $G$:

Find a mapping

$$\alpha : G \to Q$$

where $Q$ is a quasi-coherent sheaf, universal for mappings of $G$ to quasi-coherent sheaves over $S$.

Problem B (Universal extension problem):

Assuming $\mathrm{Hom}(G,V) = (0)$ for all vector groups, $V$; find an extension of group schemes over $S$:

$$(\epsilon) \qquad 0 \to V(G) \to E(G) \to G \to 0$$

such that $V(G)$ is a vector group, and such that $(\epsilon)$ is universal for all extensions of $G$ by vector groups over $S$. More precisely, we would like the mapping

$$\mathrm{Hom}_{\mathcal{O}_S}(V(G),M) \to \mathrm{Ext}^1_S(G,M)$$

induced by $(\epsilon)$ to be an isomorphism. If such an $(\epsilon)$ can be found call it the universal extension of $G$. Clearly $(\epsilon)$ and $E(G)$ and $V(G)$ are determined up to canonical isomorphism by their role in problem B, and they are functors on the subcategory of group schemes admitting a solution to problem B.

Examples and discussion: (I. Existence of Solution to Problem A)

(1.1). Suppose $\underline{\mathrm{Hom}}(G,G_a)$ is a locally free $\mathcal{O}_S$-module of finite rank. Set $V = \underline{\mathrm{Hom}}_{\mathcal{O}_S}(\underline{\mathrm{Hom}}(G,G_a),\mathcal{O}_S)$. Then

$$\begin{aligned}\underline{\mathrm{Hom}}(G,M) &= \underline{\mathrm{Hom}}(G,G_a)\otimes_{\mathcal{O}_S}M\\ &= \underline{\mathrm{Hom}}_{\mathcal{O}_S}(V,M)\end{aligned}$$

and consequently $V$ is the vector group hull of $G$.

(1.2). Suppose that the Cartier dual of $G$ is representable by a group scheme. By the <u>Cartier dual</u> we mean the presheaf on Sch/S given by

$$G^* = \underline{\mathrm{Hom}}_{\mathrm{gr}}(G,G_m).$$

Then if

$$\begin{tikzcd} S \arrow[r, hook, "e"] & G^* \end{tikzcd}$$

denotes the zero-section of $G^*/S$, let

$$\begin{tikzcd} S \arrow[r, hook, "e_1"] & G^*_1 = \mathrm{Inf}^1(G^*) \end{tikzcd}$$

denote the first infinitesimal neighborhood of the zero-section.

The commutative diagram

$$\begin{tikzcd} & G^*_1 \arrow[d] \\ S \arrow[ur] \arrow[r] & G^* \end{tikzcd}$$

is a morphism of S-pointed S-schemes.

There is a natural isomorphism of functors on the category Sch/S

$$\underline{\mathrm{Hom}}_{\text{S-pointed S-schemes}}(G^*_1,G_m) \cong e^*\Omega^1_{G^*/S}$$

and we shall use the notation $\underline{\omega}_{G^*}$ to denote the quasi-coherent sheaf over S defined by either side of the above formula. We have a natural isomorphism

$$G^*_1 \cong \underline{\mathrm{Spec}}\ (\mathcal{O}_S \oplus \underline{\omega}_{G^*})\ .$$

Let $\alpha: G \to \underline{\omega}_{G^*}$ denote the composition

$$\alpha: G \to \underline{\mathrm{Hom}}(G^*, G_m) \to \underline{\mathrm{Hom}}_{S\text{-pointed } S\text{-schemes}}(G^*_1, G_m) = \underline{\omega}_{G^*}$$

(1.3) Examples of groups G such that $G^*$ is representable are the following:

a) G finite and locally-free

b) G locally constant for the f.p.q.c. topology [11, $SGA_3$ X 5.3]

c) G of multiplicative type <u>and</u> quasi-isotrivial [11, $SGA_3$ X 5.7]

d) G an abelian scheme (here $G^* = (0)$ since denoting by $\pi: G \to S$ the structural morphism, $\pi_*(\mathcal{O}_G) = \mathcal{O}_S$ universally)

e) $G = Z[T]$ where T is a finite, locally-free S-scheme. (i.e. for variable S' over S, $\Gamma(S', G)$ is the free-Z -module on $[\mathrm{Hom}_S(S', T)]$.

The only example which requires (perhaps) any justification is e). But here $\underline{\mathrm{Hom}}_{gr}\ (G\ , G_m) \simeq \mathrm{Hom}(T, G_m)$ and hence its S'-valued points are simply the units in $\Gamma(T \underset{S}{\times} S', \mathcal{O}_{T \underset{S}{\times} S})$. The representability follows now because we can (locally) choose a finite basis for the $\mathcal{O}_S$-module, $\mathcal{O}_T$, and a unit is a section such that multiplication by it defines an automorphism.

(1.4) Proposition: Let $G$ be any abelian presheaf on $^{Sch}/_S$ such that $G^*$ is representable. The functor on quasi-coherent $\mathcal{O}_S$-modules $M \to \mathrm{Hom}_{gr}(G,M)$ is representable by $\underline{\omega}_{G^*}$ and the homomorphism $\alpha\colon G \to \underline{\omega}_{G^*}$ above is the universal homomorphism from $G$ to quasi-coherent $\mathcal{O}_S$-modules.

Proof: Let us first show that the functor is representable. For $M$ a quasi-coherent $\mathcal{O}_S$-module let $S_M$ be the affine $S$-scheme $\underline{\mathrm{Spec}}(\mathcal{O}_S \oplus M)$, where $\mathcal{O}_S \oplus M$ is made into an algebra by requiring $M^2 = (0)$ (i.e., it is the "dual numbers" on $M$). Denote by $\pi_M$(resp. $\eta_M$) the structural morphism (resp. unit section) of $S_M$ which corresponds to the algebra homomorphisms $\mathcal{O}_S \to \mathcal{O}_S \oplus M$ (resp. $\mathcal{O}_S \oplus M \to \mathcal{O}_S$, $M$ being mapped to zero).

There is an obvious homomorphism $\pi_{M*}(G_{m_{S_M}}) \to G_{m_S}$ which arises functorially because $id_{(\widehat{Sch}/S)} = \eta_M^{*} \circ \pi_M^{*}$ and because there is a map $\pi_{M*} \to \eta_M^{*}$ since $\pi_M \circ \eta_M = id_S$. The kernel of the map is $M$ and using the definition of $G_m$ we see that there is an exact sequence:

$$0 \to M \to \pi_{M*}(G_{m_{S_M}}) \to G_{m_S} \to 0$$

Thus
$$\begin{aligned}
\mathrm{Hom}_{gr}(G,M) &\cong \mathrm{Ker}[\mathrm{Hom}_{gr}(G,\pi_{M*}(G_{m_{S_M}})) \to \mathrm{Hom}_{gr}(G,G_{m_S})] \\
&\cong \mathrm{Ker}[\mathrm{Hom}_{gr}(\pi_M^{*}(G),G_{m_{S_M}}) \to \mathrm{Hom}_{gr}(G,G_{m_S})] \\
&=_{\mathrm{dfn.}} \mathrm{Ker}[\Gamma(S_M,G^*) \to \Gamma(S,G^*)] \\
&\cong \mathrm{Ker}[\Gamma(S_M,G^*_{S_M}) \to \Gamma(S,\eta_{M_*}(G^*))] \\
&\cong \mathrm{Hom}_{\mathcal{O}_{S_M}}(\underline{\omega}_{G^*} \otimes \mathcal{O}_{S_M},M) \quad \text{, by}
\end{aligned}$$

[8,II §4, 3.5] since $G^*$ is representable. Finally, by adjointness we have

$\mathrm{Hom}_{\mathcal{O}_{S_M}}(\underline{\omega}_{G^*} \otimes \mathcal{O}_{S_M}, M) \simeq \mathrm{Hom}_{\mathcal{O}_S}(\underline{\omega}_{G^*}, M)$.

Because all of the above isomorphisms are functorial in the quasi-coherent module $M$, it follows that $\underline{\omega}_{G^*}$ represents the functor $M \mapsto \mathrm{Hom}_{gr}(G,M)$.

To calculate what the universal map $G \to \underline{\omega}_{G^*}$ is, let us first observe that for $M = \underline{\omega}_{G^*}$, $S_M$ is the first infinitesimal neighborhood of the unit section of $G^*, \mathrm{Inf}^1(G^*)$. From the explicit definition of the mapping

$$\mathrm{Hom}_{\mathcal{O}_{\mathrm{Inf}^1(G^*)}}(\underline{\omega}_{G^*} \otimes \mathcal{O}_{\mathrm{Inf}^1(G^*)}, \underline{\omega}_{G^*}) \to \mathrm{Ker}[\Gamma(\mathrm{Inf}^1 G^*, G^*) \to \Gamma(S, G^*)]$$

given in [8, II 4, 3.2] it follows that $\mathrm{id}_{\underline{\omega}_{G^*}} \in \mathrm{Hom}_{\mathcal{O}_S}(\underline{\omega}_{G^*}, \underline{\omega}_{G^*})$ corresponds to the inclusion $\mathrm{Inf}^1(G^*) \hookrightarrow G^*$.

From this point on, the remainder of the proof of the proposition is entirely formal. Recall that $G^* =_{dfn} \mathrm{Hom}_{gr}(G, G_m)$ and hence there is a tautological pairing $G \times G^* \to G_m$ which defines two group homomorphisms $G_{G^*} \to G_{m_{G^*}}$ and $G^*_G \to G_{m_G}$, the knowledge of which allows us to reconstruct the pairing. The homomorphism $G_{G^*} \to G_{m_{G^*}}$ is (by very definition of $G^*$) universal in an obvious sense. Thus the morphism $\mathrm{Inf}^1(G^*) \to G^*$ defines a homomorphism $G_{\mathrm{Inf}^1(G^*)} \to G_{m_{\mathrm{Inf}^1(G^*)}}$ as well as a morphism $\mathrm{Inf}^1 G^*_G \to G_{m_G}$. In particular for any $S$-scheme $T$ and point $\xi \in G(T)$ we obtain a morphism $\mathrm{Inf}^1 G^*_T \to G_{m_T}$ which is simply the restriction of the map $G^*_T \to G_{m_T}$ to $\mathrm{Inf}^1 G^*_T$. This element is $\alpha(\xi)$ and hence the proposition is proved since the two ways of obtaining a map $T \times \mathrm{Inf}^1(G^*) \to T \times G_m$:

a) viewing $G_{\mathrm{Inf}^1(G^*)} \to G_{m_{\mathrm{Inf}^1(G^*)}}$ as giving for $\xi \in G(T)$ a map $\mathrm{Inf}^1(G^*)_T \to G_{m_T}$

b) $\mathrm{Inf}^1 G^* \to G_{m_T}$ as the restriction of $G^*_T \to G_{m_T}$ both come from restricting the map $G \times G^* \to G_m$ to $T \times \mathrm{Inf}^1(G^*) \to G \times G^*$.

(1.5) Corollary: For $G$ an abelian scheme and $M$ quasi-coherent, $\mathrm{Hom}_{gr}(G,M) = (0)$ .

Proof: In this case $G^* = (0)$ by 1.2(d) and hence $\underline{\omega}_{G^*} = (0)$.

(1.6) A given group scheme $G/S$ may have a vector group hull and a quasi-coherent hull which differ. Consider $S = \mathrm{Spec}(\mathbb{Z}_p)$ and $G = \mathbb{Z}/p$. Its vector group hull is zero, whereas its quasi-coherent hull, by the previous proposition, is $\underline{\omega}_{\mu_p}$.

A related issue is the question of commutation with base change. The quasi-coherent hull, constructed by the previous proposition commutes with all base changes, whereas the vector group hull constructed in (1.1) does not.

## 2. EXISTENCE OF SOLUTIONS TO PROBLEM B

(1.7) Suppose that

(a) $\underline{\mathrm{Hom}}(G,G_a) = 0$

(b) $\underline{\mathrm{Ext}}(G,G_a)$ is a locally free $\mathcal{O}_S$-module of finite rank

as sheaves for the Zariski topology over $S$ . Set

$$V(G) = \underline{\mathrm{Hom}}_{\mathcal{O}_S}(\underline{\mathrm{Ext}}(G,G_a),\mathcal{O}_S)$$

Then a universal extension of $G$ exists with the above $V(G)$ as vector group.

This assertion follows easily from the evident

$$\underline{\mathrm{Ext}}(G,M) = \underline{\mathrm{Ext}}(G,G_a) \otimes_{\mathcal{O}_S} M$$

where $M$ is any locally free $\mathcal{O}_S$-module of finite rank.

There are three important cases where hypotheses (a) and (b) hold:

### (1.8) Barsotti-Tate groups over bases $S$ such that $p$ is nilpotent on $S$.

If $G$ is a Barsotti-Tate group (i.e. a p-divisible group) over such an $S$, let $G^*$ denote the Cartier dual of $G$, and let $G(n)$ be the kernel of multiplication by $p^n$. If $n$ is sufficiently large so that $p^n = 0$ on $S$, then $\underline{\omega}_{G^*} = \underline{\omega}_{G^*(n)}$ is locally free of finite rank over $S$ and the argument of ( 16 IV,1) shows that $\underline{\mathrm{Ext}}(G,G_a)$ is $\underline{\mathrm{Hom}}_{\mathcal{O}_S}(\underline{\omega}_{G^*},\mathcal{O}_S)$. Therefore the hypotheses (a) and (b) above hold. The construction given shows more. Namely, there is the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & G(n) \arrow[r] \arrow[d, "\alpha"'] & G \arrow[r, "p^n"] \arrow[d] & G \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \underline{\omega}_{G^*} \arrow[r] & E(G) \arrow[r] & G \arrow[r] & 0
\end{tikzcd}$$

where the vertical map $\alpha$ is the vector group hull of $G(n)$. This construction clearly commutes with *all* base changes.

### (1.9) Abelian schemes over any base $S$.

If $G$ is an abelian scheme over $S$ of dimension $d$, it satisfies the following hypotheses for all $S'/S$:

a) Any morphism of sheaves of *sets* over $S'$

$$\varphi\colon G_{S'} \to Q_{S'}$$

to any quasi-coherent sheaf $Q$ over $S'$ is a *constant* map.

Explicitly, $\varphi$ admits a factorization:

$$\begin{tikzcd}
G_{S'} \arrow[rr] \arrow[dr, "f_{S'}"'] & & Q_{S'} \\
& S' \arrow[ur, "\text{section}"'] &
\end{tikzcd}$$

(b) $\quad f_*\mathcal{O}_{G_{S'}} = \mathcal{O}_{S'}$

(c) $\quad R^1f_*\mathcal{O}_{G_{S'}} = R^1f_*\mathcal{O}_G \otimes \mathcal{O}_{S'}$ is locally free of rank d .

Here is a proof of a):

Lemma: Let $f: A \to S$ be an abelian scheme and M a quasi-coherent $\mathcal{O}_S$-module. Any map $A \to M$ is constant.

Proof: A map $A \to M$ , may be viewed as an element of $\Gamma(A,f^*(M))$ $= \Gamma(S,f_*f^*(M))$. The map $\Gamma(S,M) \to \Gamma(S,f_*f^*M)$ corresponds then to $\eta: S \to M \mapsto \eta \circ f: A \to M$ . Thus to conclude it suffices to show the map $\Gamma(S,M) \to \Gamma(S,f_*f^*M)$ is bijective. Let us form the cartesian square:

$$\begin{tikzcd}
A & A_{S[M]} \arrow[l] \arrow[d, "f_M"] \\
S \arrow[from=u, "f"'] & S[M] \arrow[l]
\end{tikzcd}$$

Then
$$\begin{aligned}
\Gamma(S,\mathcal{O}_S) \oplus \Gamma(S,M) &= \Gamma(S[M],\mathcal{O}_{S[M]}) \xrightarrow{\sim} \Gamma(S[M],f_{M*}(\mathcal{O}_{A_{S[M]}})) \\
&= \Gamma(A_{S[M]},\mathcal{O}_{A_{S[M]}}) = \Gamma(A,\mathcal{O}_A) \oplus \Gamma(A,f^*(M)) \\
&= \Gamma(A,\mathcal{O}_A) \oplus \Gamma(S,f_*f^*M)
\end{aligned}$$
since (b) $\quad f_*(\mathcal{O}_A) = \mathcal{O}_{S'}$ universally.

Let

$$\omega = \underline{\mathrm{Hom}}_{\mathcal{O}_S}(R^1f_*\mathcal{O}_G,\mathcal{O}_S).$$

(1.10) Proposition: If $G$ satisfies the above hypotheses (a), (b), (c), then $G$ possesses a universal extension,

$$0 \to \omega \to E(G) \to G \to 0$$

which is indeed universal for all extensions of $G$ by quasi-coherent sheaves. (We assume $G \to S$ is quasi-compact).

Proof. (After Rosenlicht, and Serre , [22,25])

Let $M$ denote a quasi-coherent sheaf. By our assumptions (notably **a**) the category of extensions, $EXT(G,M)$ is rigid. Thus, the presheaf for the flat topology

$$S' \mapsto \text{Ext}^1(G_{S'},M_{S'})$$

is a sheaf.

We shall show that the composition

$$\lambda\colon \text{Ext}^1(G,M) \to H^1(G,f^*M) \to \Gamma(S,R^1f_*f^*M).$$

is an isomorphism. But by the above remark, we may assume $S$ affine.

$\lambda$ is injective:

For let $E$ be an extension of $G$ by $M$ and assume $\varphi\colon G \to E$ is a section (as sheaves of sets). By subtracting $\varphi(0)$ we may suppose that $\varphi(0) = 0$ . The map $G \times G \to E$ which expresses the obstruction to $\varphi$ being a homomorphism actually maps $G \times G$ into $M$ and brings $0$ to $0$ . After hypothesis (a), one may see that this obstruction is zero.

$\lambda$ is surjective:

Let E be a principal homogeneous space for M over the base G . Since S is affine, E admits a section e lying over the zero-section of G . We now follow Serre's prescription for imposing a group structure on E with zero-section e , which establishes E as a group extension

$$0 \to M \to E \to G \to 0$$

[25, VII, 15] . To follow out this prescription one need only know that the cohomology class in $H^1(G,f^*M)$ representing the principal homogeneous space E is primitive. But $H^1(G,f^*M)$ consists entirely of primitive elements as follows from the Kunneth formula if G is an abelian scheme and [21 bis, III,4.2] in general.

Our plan is to establish the isomorphism

$$\mathrm{Hom}_{\mathcal{O}_S}(\omega,M) \cong \mathrm{Ext}^1(G,M)$$

and, consequently, representability of the functor

$$M \mapsto \mathrm{Ext}^1(G,M)$$

We do this by demonstrating these isomorphisms:

$$\Gamma(S,R^1f_*f^*M) \cong \Gamma(S,R^1f_*\mathcal{O}_G \otimes M) \cong \mathrm{Hom}_{\mathcal{O}_S}(\omega,M)$$

To establish the first isomorphism above, we need a lemma:

(1.11) Lemma: $R^1f_*\mathcal{O}_G \otimes M \to R^1f_*f^*M$ is an isomorphism, for M any quasi-coherent $\mathcal{O}_S$-module.

(N.B. This follows from (c) but the following proof is valid whenever $R^*f_*\mathcal{O}_G$ is a flat $\mathcal{O}_S$-module).

Proof. We shall force the Kunneth theorem (10, $\text{EGA}_{\text{III}}$, 6.7.8) to yield this result, resorting to a technical trick. Let $S[M]$ denote the scheme, affine over $S$, whose underlying space is $S$, and whose structural sheaf is $\mathcal{O}_S \oplus M$, taken to be a ring in the obvious say. Form the diagram,

$$\begin{tikzcd}
& G[M] \arrow[dl, "g_G"'] \arrow[dr] \arrow[dd, "F"] & \\
G \arrow[dr, "f"'] & & S[M] \arrow[dl, "g_S"] \\
& S &
\end{tikzcd}$$

and note that $R^1F_*\left(\mathcal{O}_{G[M]}\right) = R^1f_*\mathcal{O}_G \oplus R^1f_*f^*M$, using that $g_G$ is affine.

But, by Kunneth,

$$R^1F_*\left(\mathcal{O}_{G[M]}\right) = R^1f_*\mathcal{O}_G \otimes (\mathcal{O}_S \oplus M)$$

using that $g_S$ is affine.

The lemma follows, and so does (1.10).

(1.12) If $A$ is an abelian scheme over the base $S$, where $p$ is nilpotent on $S$, let $G$ denote the p-divisible group associated to $A$ over $S$. It is an easy exercise to see the pullback to $G$ of the universal extension of $A$ over $S$ is the universal extension of $G$ over $S$. More explicitly, consider the map

$$\lambda : \omega_{G^*} \longrightarrow \omega_{A^*}$$

which determines the pullback to $G$ of the universal extension of $A$. This map $\lambda$ is easily seen to be the natural isomorphism.

## §2. RIGIDIFICATION OF HOM AND EXT

(2.1) Fix an S-group scheme G and an exact sequence (of fppf sheaves of abelian groups over S)

$$(\varepsilon) \qquad 0 \to G \to E \to F \to 0$$

Let $F_1 = \mathrm{Inf}^1_S(F) \subset F$ denote the first infinitesimal neighborhood of the zero section of F over S . Regard $F_1$ as an S-pointed sheaf.

By definition a <u>rigidification</u> r of the extension $(\varepsilon)$ is a homomorphism of S-pointed S-schemes making the following commutative diagram:

$$\begin{tikzcd} F_1 \arrow[rr, "r"] \arrow[dr] & & E \arrow[dl] \\ & F & \end{tikzcd}$$

A <u>rigidified extension of F by G</u> is a pair consisting in an extension $(\varepsilon)$ together with a rigidification of it , r .

If H is an S-group scheme, an <u>$(\varepsilon)$-rigidified homomorphism</u> from G to H consists in a homomorphism of S-groups

$$\varphi: G \to H$$

together with a rigidification r of the induced (pushout) exact sequence $(\varphi_* \varepsilon)$.

If

$$(\varepsilon) \quad 0 \to G \to E \to F \to 0$$
$$(\varepsilon') \quad 0 \to G \to E' \to F \to 0$$

are two extensions, provided with rigidifications r , r' respectively, then on the Baer sum $(\bar{\varepsilon})$ of $(\varepsilon)$ and $(\varepsilon')$ there is a natural rigidification $\bar{r}$, which we shall call the Baer sum of the rigidifications $r$ and $r'$. This is obtained from the natural rigidification on the external product:

$$\begin{tikzcd}
(\varepsilon\times\varepsilon'): 0 \arrow[r] & G\times G \arrow[r] & E\times E' \arrow[rr] & & F\times F' \arrow[r] & 0 \\
& & (F\times F)_1 \arrow[u, "\bar{r}"] \arrow[rr] & & F_1\times F_1 \arrow[llu, "r\times r'"'] &
\end{tikzcd}$$

Denote by Extrig $(F,G)$ the set of isomorphism classes of rigidified extensions of $F$ by $G$. Denote by $(\varepsilon)$-Homrig$(G,H)$ the set of isomorphism classes of $(\varepsilon)$-rigidified homomorphisms from $G$ to $H$. One checks easily that Baer sum induces an abelian group structure on Extrig$(F,G)$ and on $(\varepsilon)$-Homrig$(G,H)$. Extrig$(F,G)$ is bifunctorial in $F$ and $G$. As for $(\varepsilon)$-Homrig$(G,H)$, it is functorial in $H$, and if $\varphi:G \to G'$ is a homomorphism of $S$-groups, one gets a natural homomorphism

$$(\varphi_*\varepsilon)\text{-Homrig}(G',H) \to (\varepsilon)\text{-Homrig}(G,H).$$

There are two objects of this section:

To express the universal extension of a Barsotti-Tate group (over a base $S$ on which $p$ is locally nilpotent) as a direct limit of $\varepsilon$-Homrig's (2.5.7).

To express the universal extension of an abelian scheme as an Extrig (2.6.7).

(2.2) Let us consider the special case where $(\varepsilon)$ is an exact sequence of finite locally-free groups and where $H = G_m$. Furthermore let us assume that the base scheme, $S$, is affine.

(2.2.1) Proposition There is an exact sequence of abelian groups:

(2.2.2) $$0 \to \Gamma(S,\underline{\omega}_F) \to (\varepsilon)\text{-Homrig}(G,G_m) \to \Gamma(S,G^*) \to 0$$

Proof: The map $(\varepsilon)\text{-Homrig}(G,G_m) \to \Gamma(S,G^*)$ is defined by forgetting the rigidification $r$ of the rigidified homomorphism $(\varphi,r)$. Given $\varphi\colon G \to G_m$, consider the corresponding extension

$(\varepsilon_{\phi})$ $$0 \to G_m \to G_m \overset{G}{\amalg} E \to F \to 0$$

It makes $G_m \overset{G}{\amalg} E$ a principal homogeneous space over $F$ under the group $G_m$. Thus by descent [11,10; S.G.A$_1$ XI 4.3, EGA$_{IV}$ 17.7.3] $G_m \overset{G}{\amalg} E$ is a smooth $F$-scheme. Viewing $F_1$ as an $F$-scheme via the inclusion $F_1 \subset F$ we view $S$ as an $F$-scheme defined by the vanishing of an ideal of square zero: namely $\underline{\omega}_F$. Because $G_m \overset{G}{\amalg} E$ is smooth over $F$, the identity section can be lifted so as to obtain a commutative diagram:

(2.2.3)
$$\begin{tikzcd}
0 \arrow[r] & G_m \arrow[r] & G_m \overset{G}{\amalg} E \arrow[r] & F \arrow[r] & 0 \\
 & & & F_1 \arrow[u, hook] \arrow[ul, dashed] & \\
 & & & S \arrow[u, hook] \arrow[uul] &
\end{tikzcd}$$

This shows the map $(\varepsilon)\text{-Homrig}(G,G_m) \to \Gamma(S,G^*)$ is surjective.

By definition the kernel of this map consists of pairs $(0,\tau)$ where $\tau$ is a rigidification on the trivial extension:

$$0 \to G_m \to G_m \times F \to F \to 0.$$

But to give a morphism of $S$-pointed $S$-schemes, $F_1 \to G_m \times F$, which projects to the inclusion $F_1 \subset F$, is equivalent to giving a morphism of S-pointed S-scheme, $F_1 \to G_m$, which is the same as giving an element in $\Gamma(S,\underline{\omega}_F)$.

Since it is clear that the map $\Gamma(S,\underline{\omega}_F) \to (\varepsilon)\text{-Homrig}(G,G_m)$ defined by the above is additive, the proof of the proposition is complete.

(2.3) Let $(\varepsilon)\text{-}\underline{\text{Homrig}}\ (G,G_m)$ denote the sheaf associated to the ZARISKI presheaf whose value on an S-scheme $S'$ is $(\varepsilon_{S'})\text{-Homrig}(G_{S'},G_{m_{S'}})$. Then without any hypothesis on the scheme $S$ we have the following corollary:

(2.3.1) *Corollary*: There is an exact sequence of ZARISKI (resp. f.p.p.f.,...) sheaves on S:

$$0 \to \underline{\omega}_F \to (\varepsilon)\text{-}\underline{\text{Homrig}}(G,G_m) \to G^* \to 0$$

In particular $(\varepsilon)\text{-}\underline{\text{Homrig}}(G,G_m)$ is a commutative flat S-group, provided $\underline{\omega}_F$ is finite, locally-free.

(2.4) Let $(\varepsilon)\ 0 \to G \to E \to F \to 0$ be an exact sequence of finite, locally-free S-groups. The next proposition is the basic result relating $(\varepsilon)$-rigidified homomorphisms to the construction given in (1.4) above. It and its analogue for abelian schemes given below in (2.6) are the basic results which will allow us to obtain an explicit description of the universal extension of a Barsotti-Tate group (resp. an abelian scheme).

(2.4.1) *Proposition*. There is a canonical and functorial homomorphism of groups $E^* \to (\varepsilon)\text{-}\underline{\text{Homrig}}(G,G_m)$, which will be explicitly constructed in the proof, rendering the following diagram commutative:

(2.4.2)
$$\begin{tikzcd}
0 \arrow[r] & F^* \arrow[r] \arrow[d, "-\alpha"'] & E^* \arrow[r] \arrow[d] & G^* \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \underline{\omega}_F \arrow[r] & (\epsilon)\text{-}\underline{\mathrm{Homrig}}(G,G_m) \arrow[r] & G^* \arrow[r] & 0
\end{tikzcd}$$

Proof: $(\epsilon)$-$\underline{\mathrm{Homrig}}(G,G_m)$ is the sheaf associated to the presheaf $S' \mapsto (\epsilon_{S'})\text{-Homrig}(G_{S'}, G_{m_{S'}})$. Thus it suffices to construct a mapping on the level of presheaves, and since every "object" occuring in (2.4.2) commutes with base change it suffices to construct the map $\Gamma(S,E^*) \to (\epsilon)\text{-Homrig}(G,G_m)$. Let $\phi : E \to G_m$ be an element in $\Gamma(S,E^*)$. Because we require the right hand square of (2.4.2) to commute we must assign to $\phi$ a pair $(\phi|G,r)$ where r is a rigidification of the extension $(\phi|G)_*(\epsilon)$. That is we must define r, a morphism of pointed S-schemes, which renders the following diagram commutative:

(2.4.3)
$$\begin{tikzcd}
0 \arrow[r] & G \arrow[r] \arrow[d, "\phi|G"'] & E \arrow[r, "\pi"] \arrow[d] \arrow[dl, "\phi"'] & F \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & G_m \arrow[r] & G_m \overset{G}{\amalg} E \arrow[r, "\tilde{\pi}"] & F \arrow[r] & 0 \\
 & & & F_1 \arrow[u, hook] \arrow[ul, dashed, "r"] &
\end{tikzcd}$$

Using $\phi$ we obtain a splitting, $G_m \overset{G}{\amalg} E \xrightarrow{\tilde{\phi}} G_m$, of the lower horizontal line of (2.4.3). Composing the "trivial" rigidification $r_o : F_1 \hookrightarrow F \hookrightarrow G_m \times F$ with $(\tilde{\phi},\tilde{\pi})^{-1} : G_m \times F \to G_m \overset{G}{\amalg} E$ we obtain the desired rigidification r.

It remains to show that the left hand square of (2.4.2) is commutative. Thus let $\psi: F \to G_m$ be given so that the diagram corresponding to (2.4.3) is:

$$\begin{tikzcd}
0 \arrow[r] & G \arrow[r] \arrow[d, "0"'] & E \arrow[r, "\pi"] \arrow[d] \arrow[dl, "\psi\circ\pi"'] & F \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & G_m \arrow[r] & G_m \overset{G}{\underline{\amalg}} E \arrow[r] & F \arrow[r] & 0
\end{tikzcd}$$

Identifying $G_m \overset{G}{\underline{\amalg}} E$ with $G_m \times F$, then $(\widetilde{\psi\circ\pi}, \breve{\pi})$ is identified with the automorphism of $G_m \times F$ taking $(x,f)$ to $(x+\psi(f),f)$. This shows that to $\psi\circ\pi$ the pair $(0,-\psi|F_1)$ is assigned. By definition of $\alpha$ and of the map $\underline{\omega}_F \to (\epsilon)$-$\underline{\text{Homrig}}(G,G_m)$ it follows that the diagram commutes. Finally the fact that the map $E^* \to (\epsilon)$-$\underline{\text{Homrig}}(G,G_m)$, which has been defined above is a homomorphism of groups, follows directly from the definitions.

## (2.5) (The Universal Extension of a Barsotti-Tate group)

Assume that our base $S$ is killed by $p^n$ and fix a Barsotti-Tate group G on S. For any $i \geq 1$ let $(\epsilon_{n,i})$ be the extension:

$$(\epsilon_{n,i}) \qquad 0 \to G(i) \to G(n+i) \xrightarrow{p^i} G(n) \to 0$$

By (2.4.1) we obtain a commutative diagrams:

$$(2.5.1_i) \qquad \begin{tikzcd}
0 \arrow[r] & G^*(n) \arrow[r] \arrow[d, "-\alpha"'] & G^*(n+i) \arrow[r, "p^n"] \arrow[d] & G^*(i) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \underline{\omega}_{G(n)} \arrow[r] & (\epsilon_{n,i})\text{-}\underline{\text{Homrig}}(G(i),G_m) \arrow[r] & G^*(i) \arrow[r] & 0
\end{tikzcd}$$

From the proof of (2.4.1) and the explicit definition of (2.7) it follows that the following diagrams

(2.5.2)
$$\begin{tikzcd}
0 \arrow[r] & G(i+1) \arrow[r] \arrow[d, "p"'] & G(n+i+1) \arrow[r, "p^{i+1}"] \arrow[d, "p"'] & G(n) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & G(i) \arrow[r] & G(n+1) \arrow[r, "p_i"] & G(n) \arrow[r] & 0
\end{tikzcd}$$

give rise to commutative diagrams:

(2.5.3)
$$\begin{tikzcd}[column sep=small]
0 \arrow[rr] & & G^*(n) \arrow[rrr] \arrow[dl, "id"'] \arrow[dd, "-\alpha"] & & & G^*(n+i) \arrow[rrr, "p^i"] \arrow[dl] \arrow[dd] & & & G^*(i) \arrow[rr] \arrow[dl] \arrow[dd] & & 0 \\
0 \arrow[r] & G^*(n) \arrow[rrr] \arrow[dd, "-\alpha"'] & & & G^*(n+i+1) \arrow[rrr, "p^{i+1}"] \arrow[dd] & & & G^*(i+1) \arrow[rrr] \arrow[dd] & & & 0 \\
0 \arrow[rr] & & \underline{\omega}_{G}(n) \arrow[rrr] \arrow[dl, "id"'] & & & (\epsilon_{n,i})\underline{\mathrm{Homrig}} \arrow[rrr] \arrow[dl] & & & G^*(i) \arrow[rr] \arrow[dl] & & 0 \\
0 \arrow[r] & \underline{\omega}_{G}(n) \arrow[rrr] & & & (\epsilon_{n,i+1})\underline{\mathrm{Homrig}} \arrow[rrr] & & & G^*(i+1) \arrow[rrr] & & & 0
\end{tikzcd}$$

Hence passing to the direct limit we find a commutative diagram:

(2.5.4)
$$\begin{tikzcd}
0 \arrow[r] & G^*(n) \arrow[r] \arrow[d, "-\alpha"'] & G^* \arrow[r, "p^n"] \arrow[d] & G^* \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \underline{\omega}_{G}(n) \arrow[r] & \varinjlim(\epsilon_{n,i})\underline{\mathrm{Homrig}} \arrow[r] & G^* \arrow[r] & 0
\end{tikzcd}$$

But we know that pushing out the extension $0 \to G^*(n) \to G^* \xrightarrow{p^n} G^* \to 0$ via $\alpha$ gives the universal extension of $G^*$. Hence there is a canonical isomorphism $E(G^*) \xrightarrow{\sim} \varinjlim(\epsilon_{n,i})\underline{\mathrm{Homrig}}(G(i),G_m)$ which makes the following diagram commute:

(2.5.5)
$$\begin{tikzcd}
0 \arrow[r] & \underline{\omega}_{G}(n) \arrow[r] \arrow[d, "-id"'] & E(G^*) \arrow[r] \arrow[d] & G^* \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \underline{\omega}_{G}(n) \arrow[r] & \varinjlim(\epsilon_{n,i})\text{-}\underline{\mathrm{Homrig}} \arrow[r] & G^* \arrow[r] & 0
\end{tikzcd}$$

Also it follows that the hypotheses that $p^n$ kills $S$ can be replaced by the assumption that $p$ is locally-nilpotent on $S$ .

To be more precise consider the exact sequences:

$$(\epsilon_i) \qquad 0 \to G(i) \to G \xrightarrow{p^i} G \to 0$$

The map of sequence $(\epsilon_{n,i})$ to $(\epsilon_i)$ defines the homomorphism $(\epsilon_i)\text{-}\underline{\mathrm{Homrig}}(G(i),G_m) \to (\epsilon_{n,i})\text{-}\underline{\mathrm{Homrig}}(G(i),G_m)$. If $p^n$ kills $S$ , then this map is an isomorphism because $G_1 = \mathrm{Inf}^1(G) \subseteq G(n)$ [16,II 3.3.16]. Thus the map $(\epsilon_i)\text{-}\underline{\mathrm{Homrig}}(G(i),G_m) \to G^*(i)$ is an epimorphism since this is a local property on $S$ . Also the fact that $G_1$ is affine on $S$ insures that the map $\underline{\omega}_G \to (\epsilon_i)\text{-}\underline{\mathrm{Homrig}}(G(i),G_m)$ is well-defined and that the sequence:

$$0 \to \underline{\omega}_G \to (\epsilon_i)\text{-}\underline{\mathrm{Homrig}}(G(i),G_m) \to G^*(i) \to 0$$

is exact.

Passing to the direct limit we obtain an exact sequence:

$$(2.5.6) \qquad 0 \to \underline{\omega}_G \to \varinjlim (\epsilon_i)\text{-}\underline{\mathrm{Homrig}}(G(i),G_m) \to G^* \to 0$$

Let $0 \to \underline{\omega}_G \to E(G^*) \to G^* \to 0$ be the universal extension of $G^*$ by a vector group. Then there is a unique linear map $\underline{\omega}_G \to \underline{\omega}_G$ giving the extension (2.5.6) by pushing out. By (2.5.5) this map is $-\mathrm{id}$ locally and hence is $-\mathrm{id}$ . Finally because of the functoriality of $\underline{\mathrm{Homrig}}$ discussed in (2.1) we can state:

(2.5.7) <u>Proposition</u>. Let $S$ be a scheme on which $p$ is locally nilpotent. The two contravariant functors from the category of Barsotti-Tate groups to the category of abelian (f.p.p.f) sheaves on $S$ :

a) $G \mapsto E(G^*)$

b) $G \mapsto \varinjlim(\varepsilon_i)\text{-}\underline{\mathrm{Homrig}}(G(i),G_m)$

are canonically isomorphic. Furthermore the natural exact sequence

$$0 \to \underline{\omega}_G \to \varinjlim(\varepsilon_i)\text{-}\underline{\mathrm{Homrig}}(G(i),G_m) \to G^* \to 0$$

is "the" universal extension of $G^*$ by a vector group.

(2.6) (The Universal extension of an abelian scheme)

Let $S$ be a scheme and $A$ an abelian scheme on $S$. Let $0 \to G_m \to E \to A \to 0$ be an extension of $A$ by $G_m$. Then $E$ is representable and the morphism $E \to A$ is smooth, so that if $S$ is affine this extension admits a rigidification. Thus if we denote by $\underline{\mathrm{Extrig}}(A,G_m)$ the ZARISKI sheaf associated to the presheaf $S' \mapsto \mathrm{Extrig}(A_{S'},G_{m_S})$ we find (just as in (2.2.1)) an exact sequence:

$$0 \to \underline{\omega}_A \to \underline{\mathrm{Extrig}}(A,G_m) \to \underline{\mathrm{Ext}}^1(A,G_m) \to 0 \tag{2.6.1}$$

But the dual abelian scheme, $A^*$, exists and is isomorphic to $\underline{\mathrm{Ext}}^1(A,G_m)$ [21,19]. From descent it follows that $\underline{\mathrm{Extrig}}(A,G_m)$ is representable and is a smooth S-group.

(2.6.2) We shall see below that the extension (2.6.1) is the universal extension of $A^*$ by a vector group. Let us begin with a special case where an explicit isomorphism between the universal extension and the extension (2.6.1) can be given. Thus assume $p^n$ is zero on $S$. Recall then that $\underline{\omega}_{A(n)} = \underline{\omega}_A$ and

and that the universal extension of $A^*$ by a vector group is obtained as a "pushout" as in the following diagram:

(2.6.3)
$$\begin{tikzcd}
0 \arrow[r] & A^*(n) \arrow[r] \arrow[d, "\alpha"'] & A^* \arrow[r, "p^n"] \arrow[d] & A^* \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \underline{\omega}_{A(n)} \arrow[r] & \mathcal{E} \arrow[r] & A^* \arrow[r] & 0
\end{tikzcd}$$

Our isomorphism is obtained from a homomorphism $A^* \to \underline{\text{Extrig}}(A,G_m)$ which renders the diagram obtained by replacing $\mathcal{E}$ by $\underline{\text{Extrig}}$ commutative. To define the map it suffices to do so on the level of presheaves, and hence, because everything is compatible with base change, to define a map $\Gamma(S,A^*) \to \text{Extrig}(A,G_m)$.

If $0 \to G_m \to E \to A \to 0$ is an extension we can pull it back via the homomorphism $A \xrightarrow{p^n} A$ and obtain a commutative diagram:

(2.6.4)
$$\begin{tikzcd}
& & & \text{Inf}^1(A(n))=A(n)_1 \arrow[dl, dashed] \arrow[d, hook] & \\
& & \text{Ker} \arrow[r, "\sim"] \arrow[d] & A(n) \arrow[d, hook] & \\
0 \arrow[r] & G_m \arrow[r] \arrow[d, equal] & E \times_A A \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "p^n"] & 0 \\
0 \arrow[r] & G_m \arrow[r] & E \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

The kernel of the map $E \times_A A \xrightarrow{pr_1} E$ is mapped isomorphically under the projection $pr_2 : E \times_A A \to A$ to $A(n)$. This allows us to find a unique arrow $A(n)_1 \to \text{Ker}$ making the diagram commute. Because $p^n$ kills $S$, $\text{Inf}^1(A(n)) = \text{Inf}^1(A) = A_1$ and hence composing this arrow with the inclusion $\text{Ker} \hookrightarrow E \times_A A$ we obtain a rigidified extension of $A$ by $G_m$. This defines the desired homomorphism. It remains to show that the diagram:

(2.6.5)
$$\begin{tikzcd}
0 \arrow[r] & A^*(n) \arrow[r] \arrow[d, "\alpha"'] & A^* \arrow[r, "p^n"] \arrow[d] & A^* \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \underline{\omega}_{A(n)} \arrow[r] & \underline{\text{Extrig}} \arrow[r] & A^* \arrow[r] & 0
\end{tikzcd}$$

is commutative. The right hand square commutes by definition of the morphism $A^* \to \underline{\text{Extrig}}(A,G_m)$. To check the commutativity of the left hand square let the extension

$$(\varepsilon) \qquad 0 \to G_m \to E \to A \to 0$$

represent an element in $A^*(n)$. Then there exists a unique homomorphism $\phi$: $E \times_A A \to G_m$ which splits the extension in the upper row of (2.6.4). It follows from the explicit form of Cartier duality given for example in [18 bis] that the identification of $A^*(n)$ with $A(n)^* = \underline{\text{Hom}}(A(n),G_m)$ makes correspond to $(\varepsilon)$ the homomorphism $\psi$: $A(n) \to G_m$ which is the following composition:

$$(2.6.6) \qquad A(n) \xhookrightarrow{i} E \times_A A \xrightarrow{\phi} G_m$$

Thus going around the left hand square:
$A^*(n) \xrightarrow{\alpha} \underline{\omega}_{A(n)} \to \underline{\text{Extrig}}(A,G_m)$ assigns to $(\varepsilon)$ the trivial extension $G_m \times A$ together with the rigidification whose components are $\psi|A(n)_1$: $A(n)_1 \to G_m$ and the canonical inclusion $A(n)_1 \hookrightarrow A$ .

We must check that this extension is isomorphic to the extension given by the upper row of (2.6.4), via an isomorphism respecting the rigidified structures. The unique isomorphism between these two extensions is given by the map $\tau$: $E \times_A A \to G_m \times A$

whose components are $\emptyset$ and $pr_2$ .

For $x$, an $S'$-valued point of $A_1$, the rigidification on $E \times_A A$ assigns to it the $S'$-valued point $(0,x)$. Certainly the second component of $\tau((0,x))$ is $x$ while the first component is $\emptyset((0,x)) = \emptyset \circ i|A_1(x) = \psi|A_1(x)$. Thus $\tau$ is an isomorphism of rigidified extensions and the diagram (2.6.5) commutes as asserted.

(2.6.7) *Proposition*: Let $S$ be a scheme, $A$ an abelian scheme on $S$ and let $E(A^*)$ denote the universal extension of $A^*$ by a vector group. The canonical morphism $E(A^*) \to \underline{\mathrm{Extrig}}(A,G_m)$ (arising from the definition of the universal extension and the extension (2.6.1)) is an isomorphism, which is functorial in $A$ .

*Proof*: Observe first that both the universal extension and $\underline{\mathrm{Extrig}}(A,G_m)$ are compatible with arbitrary base change. For $E(A^*)$ this follows from the fact that all objects (= group or map between groups) entering into the proof of its existence in (1.9) are compatible with base change. To show the map $E(A^*) \to \underline{\mathrm{Extrig}}(A,G_m)$ is an isomorphism is equivalent to showing that the map $\underline{\omega}_A \to \underline{\omega}_A$ giving rise to it is an isomorphism. This problem is local on $S$ and hence $S$ can be assumed to be affine. Because $A$ is proper and smooth on $S$ (hence of finite presentation on $S$ ) we can assume that $S = \mathrm{Spec}(R)$ where $R$ is a ring of finite type over $Z$ [10, $EGA_{IV}$ 8.9.1, 8.10.5, ... ]. From (2.6) it follows that for any maximal ideal $\underline{m} \subset R$ , the corresponding map $\underline{\omega}_A/\underline{m}^n\underline{\omega}_A \to \underline{\omega}_A/\underline{m}^n\underline{\omega}_A$ is an isomorphism $(n \geq 1)$. Hence the determinant of the corresponding endomorphism of

$\underline{\omega}_A \otimes \hat{R}_{\underline{m}}$ is a unit in $\hat{R}_{\underline{m}}$. This implies that this determinant is actually invertible in $R_{\underline{m}}$. Because this holds for all maximal ideals $\underline{m}$, the endomorphism of $\underline{\omega}_A$ is an automorphism.

To check the functoriality of this isomorphism, consider two abelian schemes A, B on S and a homomorphism $u: A \to B$. The assertion means that the following diagram is commutative:

(2.6.8)
$$\begin{tikzcd}
0 \arrow[rr] & & \underline{\omega}_B \arrow[rr] \arrow[dl, "id"'] \arrow[dd] & & E(B^*) \arrow[rr] \arrow[dl] \arrow[dd] & & B^* \arrow[r] \arrow[dl, "id"'] \arrow[dd, "u^*"] & 0 \\
0 \arrow[r] & \underline{\omega}_B \arrow[rr, crossing over] \arrow[dd] & & \underline{\text{Extrig}}(B,G_m) \arrow[rr, crossing over] & & B^* \arrow[rr, crossing over] & & 0 \\
0 \arrow[rr] & & \underline{\omega}_A \arrow[rr] \arrow[dl, "id"] & & E(A^*) \arrow[rr] \arrow[dl] & & A^* \arrow[r] \arrow[dl, "id"] & 0 \\
0 \arrow[r] & \underline{\omega}_A \arrow[rr] \arrow[from=uu, crossing over] & & \underline{\text{Extrig}}(A,G_m) \arrow[rr] \arrow[from=uu, crossing over] & & A^* \arrow[rr] \arrow[from=uu, crossing over, "u^*"'] & & 0
\end{tikzcd}$$

To check that the two ways of going from $E(B^*)$ to $\underline{\text{Extrig}}(A,G_m)$ coincide, observe that their difference is a map $E(B^*) \to \underline{\omega}_A$ which vanishes on $\underline{\omega}_B$ and hence gives a map $B^* \to \underline{\omega}_A$, necessarily zero by (1.5).

## §3. RIGIDIFIED EXTENSIONS AND $\natural$-EXTENSIONS

Let

$$(*) \qquad 0 \to G_m \to E \to A \to 0$$

be an extension over an affine base $S$ , where $A/S$ is an abelian scheme.

In this section we will show, in detail, how the following two additional structures on $(*)$ are equivalent:

(a) A rigidification of $(*)$

(b) An integrable connection on $E$ regarded as a $G_m$-torseur over $A$ (this connection being required to be compatible with the group structure of the extension $E$).

In this way we shall obtain yet another explicit description of the universal extension of an abelian scheme.

### (3.1) The definitions.

By torseur for $G$ over $S$ we shall mean principal homogeneous space, locally trivial for the étale topology. There are many equivalent ways to define connection and we shall take the definition using the fewest words:

Definition: Let $X$ be an $S$-scheme, $G$ a commutative smooth $S$-group, and $P$ a torseur on $X$ under the group $G_X$. Let $\Delta^1(X) = \Delta^1(X/S)$ denote the first infinitesimal neighborhood of the diagonal map $X \to X \times_S X$ . The two projections

$p_j: X \times_S X \to X$ $(j=1,2)$ induce morphisms $p_j:\Delta^1(X) \to X$ .

A <u>connection</u> $\nabla$ on the $G_X$-torseur $P$ is an isomorphism of $G_{\Delta^1(X)}$-torseurs:

$$\nabla: p_1^*(P) \to p_2^*(P)$$

which restricts to the identity on $X$ . (That is , $\Delta^*(\nabla) = \mathrm{id}_P$).

Given an $\mathcal{O}_X$-module $E$ a connection on $E$ is an $\mathcal{O}_{\Delta^1(X)}$ isomorphism $\nabla:p_1^*(E) \to p_2^*(E)$ restricting to the identity on $X$. Given $(E,\nabla)$, an $\mathcal{O}_X$-module with connection, we may obtain an $\mathcal{O}_S$-linear homomorphism

$$\nabla':E \to E \otimes \Omega^1_{X/S}$$

(satisfying the Leibniz product rule) as follows:

Denote by $j_1,j_2$ the two ring homomorphisms $\mathcal{O}_X \to \mathcal{O}_{\Delta^1(X)}$ corresponding to the two projections $p_1,p_2$. One obtains the corresponding morphisms $j_1(E): E \to p_1^*(E)$, $j_2(E):E \to p_2^*(E)$. Define:

$$\nabla' = \nabla^{-1}\circ j_2(E) - j_1(E).$$

(3.1.2) <u>Examples</u>

a) If $G = G_m$ , then connections on the $G_m$-torseur $P$ are in one-one correspondence with connections on the line bundle, $\mathcal{L}$ , which is associated to $P$ .

b) If $G = G_a$, then $G_a$-torseurs $P$ correspond to extensions $(\epsilon)$ of $\mathcal{O}_X$ by $\mathcal{O}_X$:

$$(\epsilon) \qquad 0 \to \mathcal{O}_X \to E \to \mathcal{O}_X \to 0$$

and connections on $P$ correspond to isomorphisms of extensions $p_1^*(\epsilon) \xrightarrow{\sim} p_2^*(\epsilon)$ which restrict to $\mathrm{id}_\epsilon$ on $X$ .

(3.1.3) The G-torseurs with connection $(P,\nabla)$ are the objects of a category in which the morphisms, $\mathrm{Hom}(P,\nabla),(Q,\bar{\nabla}))$ are precisely those morphisms $\eta:P \to Q$ of G-torseurs such that the following diagram commutes:

$$\begin{tikzcd}
p_1^*(P) \arrow[r, "p_1^*(\eta)"] \arrow[d, "\nabla"'] & p_1^*(Q) \arrow[d, "\bar{\nabla}"] \\
p_2^*(P) \arrow[r, "p_2^*(\eta)"'] & p_2^*(Q)
\end{tikzcd}$$

Such an $\eta:P \to Q$ is said to be <u>horizontal</u> when the connections on $P$ and $Q$ are understood as being given.

(3.1.4) (<u>The curvature of a connection</u>). The curvature tensor will be an element in $\Gamma(X,\Omega^2_{X/S} \otimes \underline{\mathrm{Lie}}(G))$. First we define the curvature of a connection on the trivial bundle $G_X$ and then show that these tensors can be patched together to give a definition for an arbitrary torseur $P$ .

A connection on $G_X$ is simply an automorphism of $G_{\Delta^1(X)}$ which restricts to the identity. It is completely determined by telling what it does to the unit section and hence is determined by giving an arbitrary element $\xi$ in $\mathrm{Ker}(\Gamma(\Delta^1(X),G) \to \Gamma(X,G))$ $= \mathrm{Hom}_{\mathcal{O}_X}(\underline{\omega}_G \otimes \mathcal{O}_X, \Omega^1{}_{X/S}) = \Gamma(X,\Omega^1_{X/S} \otimes \underline{\mathrm{Lie}}(G))$. The image of $\xi$ in $\Gamma(X,\Omega^2_{X/S} \otimes \mathrm{Lie}(G))$ under $d \otimes \mathrm{id} : \Omega^1_{X/S} \otimes \underline{\mathrm{Lie}}\ G \to \Omega^2_{X/S} \otimes \mathrm{Lie}(G)$ **is by definition** the <u>curvature form</u> of the connection.

Now if $P$ is an arbitrary G-torseur on $X$ , endowed with a connection, then after an étale base change $X' \to X$ , by our definition of torseur, P becomes trivial. There is an induced connection on $P_{X'}$. Choosing a trivialization of $P_{X'}$, construct the curvature of the induced connection which lies in $\Gamma(X',\Omega^2_{X'/S} \otimes \underline{\mathrm{Lie}}(G)) = \Gamma(X',f^*\Omega^2_{X/S} \otimes \underline{\mathrm{Lie}}(G))$. (We obtain the above equality because $X' \to X$ is étale.) In order to show that this local construction descends to define a section of $\Omega^2 \otimes \mathrm{Lie}(G)$ over $X$, which will be by definition the <u>curvature</u>, it suffices to show that the curvature of $P_{X'}$ is independent of the choice of trivialization, since then the application of $p_1^*$ and $p_2^*$ to our section in $\Gamma(X',\Omega^1_{X'/S} \otimes \underline{\mathrm{Lie}}(G))$ yields the same section of $\Gamma(X' \times_X X', \Omega^1_{X' \times_X X'} \otimes \underline{\mathrm{Lie}}(G))$ and we can apply descent. To do this take two trivializations

$$\phi: P \to G, \qquad \psi: P \to G$$

and express the comparison $\psi \circ \phi^{-1}$ as an S-morphism

$$g: X \to G .$$

One checks readily that the difference between the two curvatures is given by $d\alpha \in \Gamma(X,\Omega^2_{X/S} \otimes \underline{\mathrm{Lie}}(G))$ where $\alpha = p_2^*(g) - p_1^*(g)$ is interpreted as an element in

$$(*) \qquad \begin{tikzcd} \mathrm{Ker}(\mathrm{Hom}(\Delta^1(X),G) \arrow[r] \arrow[dr, "\sim"] & \mathrm{Hom}(X,G)) \\ & \Gamma(X,\Omega^1_{X/S} \otimes \underline{\mathrm{Lie}}(G)) \end{tikzcd}$$

and $d$ is induced from exterior differentiation

$$d: \Omega^1_{X/S} \to \Omega^2_{X/S} .$$

We must now show

(3.1.5) <u>Lemma</u>: $d\alpha = 0$

<u>Proof</u>: Let the same letter $\pi$ denote the structural morphisms $\pi: G \to S$, $\pi: X \to S$ for no confusion will result.

The element $\alpha$ may be viewed as a homomorphism:

$$\alpha: \underline{\omega}_G \to \pi_*\Omega^1_{X/S}$$

by means of the isomorphism

$$(**) \qquad \Gamma(X, \Omega^1_{X/S} \otimes \underline{\mathrm{Lie}}(G)) \cong \mathrm{Hom}_{\mathcal{O}_S}(\underline{\omega}_G, \pi_*\Omega^1_{X/S})$$

Using the diagram:

$$(3.1.6) \qquad \begin{tikzcd} & G\times G \arrow[r, "\text{subtraction}"] & (G) \\ \Delta^1(X) \arrow[ur, "{(p_1\circ g, p_2\circ g)}"] \arrow[r, "\Delta^1(g)"] & \Delta^1(G) \arrow[u, hook] \arrow[r] & \mathrm{Inf}^1_S(G) \arrow[u, hook] \\ X \arrow[u, hook] \arrow[r, "g"'] & G \arrow[u, hook] \arrow[r] & S \arrow[u, hook] \end{tikzcd}$$

and the isomorphisms (*), (**) above one can see that $\alpha$ is the composition of the two top horizontal arrows in the following diagram:

$$\begin{tikzcd} \underline{\omega}_G \arrow[r] & \pi_*\Omega^1_{G/S} \arrow[r, "dg"] \arrow[d, "d"'] & \pi_*\Omega^1_{X/S} \arrow[d, "d"'] \\ & \pi_*\Omega^2_{G/S} \arrow[r] & \pi_*\Omega^2_{X/S} \end{tikzcd}$$

Since the image of $\underline{\omega}_G$ in $\pi_*\Omega^1_{G/S}$ is killed by $d$, the lemma follows, and our construction of the global curvature of the G-torseur P is concluded.

If the curvature associated to $(P,\nabla)$ is zero we say that the connection $\nabla$ is <u>integrable</u>.

(3.1.7) (<u>The multiplicative de Rham complex</u>)

Consider the map of sheaves for $X_{et}$, the small étale site of X:

$$G_X \xrightarrow{\partial} \Omega^1_{X/S} \otimes \underline{\text{Lie}}\ G$$

$$g \longmapsto \alpha = p_2^*(g) - p_1^*(g)\ .$$

(3.1.5) implies that

$$\Omega^{\bullet}_{X/S}(G) =_{\text{defn}} G_X \xrightarrow{\partial} \Omega^1_{X/S} \otimes \underline{\text{Lie}}\ G \xrightarrow{d} \Omega^2_{X/S} \otimes \underline{\text{Lie}}\ G \cdots \to \Omega^n_{X/S} \otimes \underline{\text{LieG}} \to \cdots$$

may be viewed as a complex of sheaves on $X_{et}$.

If $G = G_a$ we obtain the ordinary de Rham complex

$$\mathcal{O}_X \to \Omega^1_{X/S} \longrightarrow \cdots\cdot$$

If $G = G_m$ , we obtain a complex called the <u>multiplicative</u> de Rham complex:

$$\mathcal{O}_X^* \xrightarrow{d\ \log} \Omega^1_{X/S} \xrightarrow{d} \Omega^2_{X/S} \longrightarrow \cdots$$

(3.1.8) A G-torseur endowed with an integrable connection is what Grothendieck calls a ϟ-torseur. The ϟ-torseurs form a full sub-category of the category introduced in (3.1.3) Denote this category by TORS $^{ϟ}(X,G)$.

(3.1.9) Because G is commutative we can define the contracted product $P \overset{G}{\wedge} Q$ of two G-torseurs. It is by definition the associated sheaf of the presheaf which is the quotient of $P \times Q$ by the action of $G$: $g.(p,q) = (gp, g^{-1}q)$. $P \overset{G}{\wedge} Q$ is made into a $G$-torseur by letting $G$ act on either of the factors. If $P$ and $Q$ are endowed with connections $\nabla_P$ and $\nabla_Q$, then

$$\begin{tikzcd}
p_1^*(P) \overset{G}{\wedge} p_1^*(Q) \arrow[r, "\sim"] & p_2^*(P) \overset{G}{\wedge} p_2^*(Q) \\
p_1^*(P \overset{G}{\wedge} Q) \arrow[u, "\wr"] & p_2^*(P \overset{G}{\wedge} Q) \arrow[u, "\wr"]
\end{tikzcd}$$

$\nabla_P \overset{G}{\wedge} \nabla_Q$:

defines a connection on $P \overset{G}{\wedge} Q$. Furthermore the curvature tensor associated to $\nabla_P \overset{G}{\wedge} \nabla_Q$ is the sum of that associated to $\nabla_P$ and that associated to $\nabla_Q$. In particular, the contracted product of $\natural$-torseurs is a $\natural$-torseur.

If $X$ is an $S$-group, then it is possible to impose additional structures on a $G_X$-torseur $P$: namely to require that $P$ has the structure of an $S$-group so that we obtain a (central) extension

$$0 \to G \to P \to X \to 0$$

In our context (i.e. given that $P$ is a torseur) the most convenient way to express this is by giving an isomorphism:

$$\beta: \pi_1^*(P) \overset{G}{\wedge} \pi_2^*(P) \xrightarrow{\sim} s^*(P)$$

(where $\pi_1, \pi_2$: $X \times X \to X$ are the projections and $s$: $X \times X \to X$ is the addition law) and requiring the appropriate diagrams, (expressing the associativity and commutativity) to commute.

(3.1.10) By combining the notion of torseur endowed with an integrable connection, with the notion of a group extension of $G$ by $X$ we are led to the following definition (following again Grothendieck's terminology).

Definition. A $\natural$-extension of the smooth group G by the commutative group X is a triple $(P\nabla,\beta)$, where $(P,\nabla)$ is a $\natural$-torseur on X under G, $(P,\beta)$ defines a group structure on P, making it an extension of X by G, and where $\beta:\pi_1^*(P) \overset{G}{\wedge} \pi_2^*(P) \xrightarrow{\sim} s^*(P)$ is a horizontal morphism.

We denote by $\mathrm{EXT}^{\natural}(X,G)$ the category whose objects are the $\natural$-extensions and whose morphisms are the horizontal morphisms between extensions. Because G is commutative, the category of extensions of X by G, EXT(X,G) is endowed with a "composition law" which corresponds to taking the contracted product of the underlying torseurs. Upon passing to the set of isomorphism classes of objects the induced composition law gives the standard group structure to $\mathrm{Ext}^1(X,G)$. From the description of the composition law in terms of contracted product of torseurs it is clear that we can define the "Baer sum" of two $\natural$-extensions and that by passing to isomorphism classes we obtain a group $\mathrm{Ext}^{\natural}(X,G)$.

Let

$$(\epsilon) \qquad 0 \to A \to B \to C \to 0$$

be an exact sequence of finite locally free (commutative) S -groups An $(\epsilon)$-$\natural$ homomorphism $A \to G$ is by definition a pair $(\phi,\nabla)$ where $\phi: A \to G$ is a homomorphism and $\nabla$ is a connection on $G \overset{A}{\amalg} B$ making

$$(\epsilon_{\phi}) \qquad 0 \to G \to G \overset{A}{\amalg} B \to C \to 0$$

a $\natural$-extension of $C$ by $G$ .

The set of $(\epsilon)$-$\natural$ homomorphisms $A \to G$ is made into a group by defining $(\phi,\nabla)+(\phi',\nabla') = (\phi+\phi',\overline{\nabla})$ where $\overline{\nabla}$ defines the structure of $\natural$-extension on the "Baer sum" of $(\epsilon_{\phi})$ and $(\epsilon_{\phi'})$. We shall denote this group by $(\epsilon)$-$\natural$ Hom$(A,G)$.

(3.2) (<u>The isomorphisms</u>)

In this section we shall construct a homomorphism

(3.2.1) $$\mathrm{Ext}^{\natural}(A,G) \to \mathrm{Extrig}(A,G).$$

As a consequence, one then obtains a homomorphism

(3.2.2) $$(\epsilon)\text{-}\natural\,\mathrm{Hom}(A,G) \to (\epsilon)\text{-}\mathrm{Homrig}(A,G).$$

Later we shall prove that over an affine base $S$ (3.2.2) is an isomorphism if $G = G_m$ , and (3.2.1) is an isomorphism if $G = G_m$ and $A$ is an abelian scheme.

Let the $\natural$-extension,

$(\epsilon)$ $$0 \to G \to E \to A \to 0$$

be given.

Denote by $i$ , the inclusion $\mathrm{Inf}^1(A) \hookrightarrow A$ , $\pi$: $\mathrm{Inf}^1(A) \to S$ the structural morphism and by $\tau$: $\mathrm{Inf}^1(A) \to \Delta^1(A)$ the morphism determined by $p_1 \circ \tau = e_A \circ \pi$ , $p_2 \circ \tau = i$ .

Since the $\natural$-structure on $E$ is given by an isomorphism $\nabla$: $p_1^*(E) \xrightarrow{\sim} p_2^*(E)$ , we can "pull back" $\nabla$ via $\tau$ to obtain:

$$\tau^*(\nabla): \pi^* \circ e_A^*(E) \to i^*(E) .$$

Since $E$ is a group $e_A^*(E)$ and hence $\pi^* e_A^*(E)$ is equipped with an obvious choice of section, the unit section. Via $\tau^*(\nabla)$ we transfer this section to obtain a section of $i^*(E)$ and hence by composition with $i^*(E) \to E$, we obtain finally a morphism $\sigma\colon \mathrm{Inf}^1(A) \to E$. It is this $\sigma$, which we shall choose to be the rigidification of the extension $(\varepsilon)$. In order to show that this is legitimate let us verify that $\sigma$ possesses the three properties required of a rigidification:

1) $\sigma$ is a morphism of $S$-schemes

2) the following diagram commutes

$$\begin{tikzcd}
E \arrow[r] & A \\
& \mathrm{Inf}^1(A) \arrow[ul, "\sigma"] \arrow[u, hook]
\end{tikzcd}$$

3) $\sigma$ is a morphism of $S$-pointed schemes.

To check 1): $\mathrm{Inf}^1(A) \xrightarrow{\sigma} E \to S = \mathrm{Inf}^1(A) \xrightarrow{\sigma} E \to A \to S$
$= \mathrm{Inf}^1(A) \to \mathrm{Inf}^1(A) \times_A E \xrightarrow{\mathrm{proj}} E \to A \to S =$
$\mathrm{Inf}^1(A) \to \mathrm{Inf}^1(A) \times_A E \xrightarrow{\mathrm{proj}} \mathrm{Inf}^1(A) \hookrightarrow A \to S = \mathrm{Inf}^1(A) \xrightarrow{\pi} S$.

To check 2) it suffices to observe that $i^*(E) = \mathrm{Inf}^1(A) \times_A E$ and that $\sigma$ is the composition of a section in $\Gamma(\mathrm{Inf}^1(A), i^*(E))$ and the projection $i^*(E) \to E$.

Finally let us check that 3) holds. We are to show that $S \to \mathrm{Inf}^1(A) \xrightarrow{\sigma} E = S \xrightarrow{e} E$. The left hand side can be computed as follows:

$$\begin{aligned}
& S \hookrightarrow \mathrm{Inf}^1(A) \xrightarrow{\sigma} E = S \hookrightarrow \mathrm{Inf}^1(A) \to \mathrm{Inf}^1(A) \times_{\overset{i}{\to} A} E \xrightarrow{\mathrm{proj}} E \\
&= S \hookrightarrow \mathrm{Inf}^1(A) \xrightarrow{(\mathrm{id},\, e\circ\pi)} \mathrm{Inf}^1(A) \times_{\overset{e\circ\pi}{\to} A} E \xrightarrow{\tau^*(\nabla)} \mathrm{Inf}^1(A) \times_{\overset{i}{\to} A} E \xrightarrow{\mathrm{proj}} E \\
&= S \xhookrightarrow{u} \mathrm{Inf}^1 A \times_{\overset{e\circ\pi}{\to} A} E \xrightarrow{\tau^*(\nabla)} \mathrm{Inf}^1(A) \times_{\overset{i}{\to} A} E \xrightarrow{\mathrm{proj}} E
\end{aligned}$$

where the components of $u\colon S \to \mathrm{Inf}^1(A) \underset{\underset{e\circ\pi}{\longrightarrow} A}{\times} E$ are $e_{\mathrm{Inf}^1(A)}$ and $e_E$ .

Thus to conclude (3) it must be shown that $\tau^*(\nabla)$ preserves the second component of this morphism. To do this let us return momentarily to the given $\nabla\colon p_1^*(E) \xrightarrow{\sim} p_2^*(E)$. By composing $e_A\colon S \hookrightarrow A$ with $\Delta\colon A \to \Delta^1(A)$ , $S$ may be viewed as a $\Delta^1(A)$-scheme. Via this both $p_1^*(E)$ and $p_2^*(E)$ have an obvious section $\delta_1$(resp $\delta_2$) with values in the $\Delta^1(A)$-scheme $S$: namely the section with components $S \hookrightarrow \Delta^1(A)$ and $S \xhookrightarrow{e_E} E$ . Under the identification of $\Delta^* p_i^*(E)$ with $E$ , the unit section $S \xhookrightarrow{e_E} E$ is identified with the section just described of $p_1^*(E)$ with values in the $\Delta^1(A)$-scheme $S$ . But by definition of a connection, $\Delta^*(\nabla) = \mathrm{id}_E$ , and hence $\nabla$ must map $\delta_1\colon S \to p_1^*(E)$ into the corresponding section $\delta_2\colon S \to p_2^*(E)$; <u>that is</u> the second component remains $S \xhookrightarrow{e_E} E$ .

Let us now consider the first factor $S \xhookrightarrow{e_{\mathrm{Inf}^1(A)}} \mathrm{Inf}^1(A)$ Because $\tau \circ e_{\mathrm{Inf}^1(A)} = \Delta \circ e_A$ , it follows immediately from the definitions that $\tau^*(\delta_1) = u$ . This implies that $\tau^*(\nabla)\circ u$ has as its second component the unit section $e_E\colon S \hookrightarrow E$ , and completes the proof.

(3.2.3) <u>Proposition</u>: a) If $A$ is an abelian scheme the homomorphism $\mathrm{Ext}^{\natural}(A,G_m) \to \mathrm{Extrig}(A,G_m)$ is an isomorphism.

b) The homomorphism

$(\epsilon)^{\natural}\mathrm{Hom}(A,G_m) \to (\epsilon)\mathrm{Homrig}(A,G_m)$ is an isomorphism if $S$ is affine.

<u>Proof</u>: a) In order to prove $\mathrm{Ext}^{\natural}(A,G_m) \to \mathrm{Extrig}(A,G_m)$ is an isomorphism, let us construct an inverse. Assume given a

rigidified extension

(3.2.4)
$$\begin{tikzcd} 0 \arrow[r] & G_m \arrow[r] & E \arrow[r, "j"] & A \arrow[r] & 0 \\ & & & \mathrm{Inf}^1(A) \arrow[ul, "\sigma"] \arrow[u, hook] & \end{tikzcd}$$

$\sigma$ defines a section of $i^*(E)$, and hence a trivialization $\rho: (e_A \circ \pi)^*(E) \xrightarrow{\sim} i^*(E)$, via $e \mapsto (\mathrm{id}_{\mathrm{Inf}^1(A)}, \sigma)$.

By definition of $\mathrm{Inf}^1(A)$, the map $p_2 - p_1: \Delta^1(A) \to A$ factors through $\mathrm{Inf}^1(A)$. Let us write it as $\Delta^1(A) \xrightarrow{\eta} \mathrm{Inf}^1(A) \xhookrightarrow{i} A$. Thus $\eta^*(\rho): (e_A \circ \pi_{\Delta^1(A)})^*(E) \xrightarrow{\sim} (p_2 - p_1)^*(E)$ is an isomorphism where $\pi_{\Delta^1(A)}: \Delta^1(A) \to S$ is the structural morphism.
Multiplying both source and target of this map by $p_1^*(E)$ and using the fact that $E$ is a group we obtain a diagram where the lower horizontal arrow is defined so as to render it commutative

(3.2.5)
$$\begin{tikzcd} (e_A \circ \pi_{\Delta^1(A)})^*(E) \wedge p_1^*(E) \arrow[r, "\eta^*(\rho) \wedge p_1^*(E)"] \arrow[d, "\wr"'] & (p_2 - p_1)^*(E) \wedge p_1^*(E) \arrow[d, "\wr"] \\ p_1^*(E) \arrow[r, "\nabla'"'] & p_2^*(E) \end{tikzcd}$$

Our inverse mapping is now defined by associating to the rigidified extension above the $\natural$-extension with the same underlying extension and the $\natural$-structure defined by $\nabla'$. To show that the definition makes sense and actually gives an inverse, five statements must be proved:

1) $\Delta^*(\nabla') = \mathrm{id}_E$

2) The map $\mathrm{Extrig}(A, G_m) \to \mathrm{Ext}^{\natural}(A, G_m) \to \mathrm{Extrig}(A, G_m)$ is the identity.

3) The map $\mathrm{Ext}^{\natural}(A,G_m) \to \mathrm{Extrig}(A,G_m)$ is injective

4) $\nabla'$ is integrable

5) The isomorphism $\pi_1^*(E) \overset{G_m}{\wedge} \pi_2^*(E) \xrightarrow{\sim} s^*(E)$ is horizontal.

The proofs we give of the first two statements are entirely formal, while those of the remaining three actually **use the assumptions** that $A$ is an abelian scheme and $G = G_m$ .

1) Since $\Delta^*(\nabla')$ is a morphism over $A$ , it suffices to show that it is the identity when $E$ is viewed as a sheaf on $\mathrm{Sch}/S$ . Since our situation commutes with base change it suffices to show the mapping it induces, $E(S) \to E(S)$, is the identity. Let $\zeta: S \to E$ be given so that $\zeta$ defines morphisms $\zeta_1: S \longrightarrow P_1^*(E)$ and $\zeta_2: S \to p_2^*(E)$. Since $\Delta: A \to \Delta^1(A)$ is a **monomorphism** it suffices to show that $\nabla' \circ \zeta_1 = \zeta_2$ . To check that it is true let us recall the definition of the vertical isomorphisms in the diagram (3.2.5) above.

Let $\alpha,\beta: T \to A$ be given and consider the torseurs $E_\alpha$, $E_\beta$, $E_{\alpha+\beta}$ deduced from $E$ by the corresponding base changes. $E_\alpha \overset{G}{\wedge} E_\beta$ is a sheaf associated to the quotient of $E_\alpha \times_T E_\beta$ by the action of $G$ . Thus if $T'$ is any S-scheme elements of $\Gamma(T', E_\alpha \overset{G}{\wedge} E_\beta)$ are given locally by triples (of S-morphisms) $x: T' \to E$, $y: T' \to E$ , $t': T' \to T$ where $T' \xrightarrow{x} E \to A$ $= T' \to T \xrightarrow{\alpha} A$ , $T' \xrightarrow{y} E \to A = T' \to T \xrightarrow{\beta} A$ . Thus the isomorphism in question is determined by associating to $(t',x,y)$ the pair $(t',x+y) \in \Gamma(T', E_{\alpha+\beta})$.

Return now to diagram (3.2.4). Then $\zeta_1 = (\Delta \circ j \circ \zeta, \zeta)$, $\zeta_2 = (\Delta \circ j \circ \zeta, \zeta)$ and after the above **ex**plication of the vertical isomorphism it is obvious that $\zeta_1$ corresponds to the class of $(\Delta \circ j \circ \zeta, e_E, \zeta)$. On the other hand projection of $(e_A \circ \pi_{\Delta^1(A)})^*(E) = \eta^*((e_A \circ \pi_{\mathrm{Inf}^1(A)})^*(E))$ to $(e_A \circ \pi_{\mathrm{Inf}^1(A)})^*(E)$ assigns to $(\Delta \circ j \circ \zeta, e_E)$ the pair $(\eta \circ \Delta \circ j \circ \zeta, e_E)$ which it transforms via $\rho$ into $(\eta \circ \Delta \circ j \circ \zeta, \sigma \circ \eta \circ \Delta \circ j \circ \zeta)$. Thus $\eta^*(\rho)$ transforms $(\Delta \circ j \circ \zeta, e_E)$ into $(\Delta \circ j \circ \zeta, \sigma \circ \eta \circ \Delta \circ j \circ \zeta)$. Therefore $\eta^*(\rho) \wedge p_1^*(E)$ will transform the class of $(\Delta \circ j \circ \zeta, e_E, \zeta)$ to the class of $(\Delta \circ j \circ \zeta, \sigma \circ \eta \circ \Delta \circ j \circ \zeta, \zeta)$. As $e_A \circ \pi_A = (p_2 - p_1) \circ \Delta = i \circ \eta \circ \Delta$ and also $e_A \circ \pi_A = i \circ e_{\mathrm{Inf}^1(A)} \circ \pi_A$, it follows that $\eta \circ \Delta = e_{\mathrm{Inf}^1(A)} \circ \pi_A$. Hence $\sigma \circ \eta \circ \Delta \circ j \circ \zeta = \sigma \circ e_{\mathrm{Inf}^1(A)} \circ \pi_A \circ j \circ \zeta = e_E \circ \pi_A \circ j \circ \zeta = e_E$. Thus under the isomorphism $(p_2 - p_1)^*(E) \wedge p_1^*(E) \xrightarrow{\sim} p_2^*(E)$ $(\Delta \circ j \circ \zeta, \sigma \circ \eta \circ \Delta \circ j \circ \zeta, \zeta)$ corresponds to $(\Delta \circ j \circ \zeta, \zeta)$ which shows (finally) that $\Delta^*(\nabla') = \mathrm{id}_E$.

2) Let us begin with the rigidified extension

$$\begin{tikzcd}
0 \arrow[r] & G_m \arrow[r] & E \arrow[r] & A \arrow[r] & 0 \\
 & & & \mathrm{Inf}^1(A) \arrow[ul, "\sigma"] \arrow[u] &
\end{tikzcd}$$

We associate a connection $\nabla'$ on $E$ to $\sigma$ and then a rigidification $\sigma'$ is associated to $\Delta'$. It is to be shown that $\sigma' = \sigma$. Using the definition of $\sigma'$ it is the projection onto $E$ of $\tau^*(\nabla')(\mathrm{id}_{\mathrm{Inf}^1(A)}, e_E \circ \pi_{\mathrm{Inf}^1(A)})$. Hence it is the projection onto $E$ of $\nabla'(\tau, e_E \circ \pi_{\mathrm{Inf}^1(A)})$. But as it follows from the definition of $\nabla'$ in terms of the diagram (3.2.5) above this projection is simply the sum:

$$\text{proj. onto } E(\eta^*(\rho)(\tau, e_E \circ \pi_{\mathrm{Inf}^1(A)})) + e_E \circ \pi_{\mathrm{Inf}^1(A)}$$
$$= \text{proj. onto } E(\eta^*(\rho)(\tau, e_E \circ \pi_{\mathrm{Inf}^1(A)}))$$
$$= \text{proj. onto } E(\rho(\eta \circ \tau, e_E \circ \pi_{\mathrm{Inf}^1(A)}))$$

But since $i \circ \eta \circ \tau = (p_2 - p_1) \circ \tau = p_2 \circ \tau - p_1 \circ \tau = i - e_A \circ \pi_{\mathrm{Inf}^1(A)} = i$, and since $i$ is a monomorphism, it follows that $\eta \circ \tau = \mathrm{id}_{\mathrm{Inf}^1(A)}$. This implies, by the very definition of $\rho$, that $\sigma' = \sigma$.

3) To show the map $\mathrm{Ext}^{\natural}(A, G_m) \to \mathrm{Extrig}(A, G_m)$ is injective, we must show that if $\nabla$ defines a $\natural$-structure on the trivial extension

$$0 \to G_m \to G_m \times A \to A \to 0$$

whose associated rigidification, $\sigma$, is trivial, then $\nabla$ is trivial. But $\nabla$ is determined by giving a section of $\Gamma(\mathcal{O}^*_{\Delta^1(A)})$ of the form $1+\omega$, $\omega \in \Gamma(A, \Omega^1_A)$. The corresponding $\rho$ (associated to the rigidification $\sigma$) is, because it is an automorphism of $G_{m_{\mathrm{Inf}^1(A)}}$, determined by a unit in $\Gamma(\mathcal{O}^*_{\mathrm{Inf}^1(A)})$ of the form $1+\omega'$, $\omega' \in \Gamma(S, \underline{\omega}_A)$. One has: $\omega' = \tau^*(\omega)$. But, because $A$ is an abelian scheme, this mapping $\Gamma(A, \Omega^1_{A/S}) \to \Gamma(S, \underline{\omega}_A)$ is an isomorphism.

4) To show the connection $\nabla'$ is integrable we shall use a trick which will be repeated below in showing that $\nabla'$ is compatible with the group structure on $E$. The curvature tensor $c(\nabla')$ is an element of $\Gamma(S, \pi_{A*}(\Omega^2_{A/S}))$. As mentioned in (3.1), $E$ corresponds to a line bundle $\mathcal{L}_E$ and $\nabla'$ to a connection on this line bundle. Thus because $A$ is an abelian scheme, and hence all global 1-forms are closed, the curvature $c(\nabla')$ is

actually independent of the connection on $E$. Notice this allows us to define a morphism $\underline{\mathrm{Ext}}(A,G_m) \to \pi_{A*}(\Omega^2_{A/S})$. Namely if $S'$ is an (absolutely) affine $S$-scheme and $0 \to G_{m_{S'}} \to E' \to A_{S'} \to 0$ is an extension, we can take any structure of rigidified extension on it, then by the above procedure put a connection on $E'$ and hence finally obtain the curvature tensor which lies in $\Gamma(S',\pi_{A_{S'}*}(\Omega^2_{A_{S'}/S'})) = \Gamma(S',\pi_{A*}(\Omega^2_{A/S})_{S'})$. Passing to the associated sheaves gives the morphism $\underline{\mathrm{Ext}}(A,G_m) \to \pi_{A*}(\Omega^2_{A/S})$. Since $\underline{\mathrm{Ext}}(A,G_m)$ is an abelian scheme and $\pi_{A*}(\Omega^2_{A/S})$ is a vector group, this morphism is constant. Clearly the image of the trivial extension is zero and thus the map is identically zero implying that the connection $\nabla'$ is integrable.

5) To show the connection $\nabla'$ is compatible with the group structure let us replace $E$ by the corresponding line bundle $\mathcal{L}_E$. Then we are to show the isomorphism $s^*(\mathcal{L}_E) \xrightarrow{\sim} \pi_1^*(\mathcal{L}_E) \otimes \pi_2^*(\mathcal{L}_E)$ is horizontal. Using this isomorphism the problem can be interpreted as that of showing that two connections on $s^*(\mathcal{L}_E)$ are the same. Taking their "difference" we obtain a section, $\delta(\nabla')$ in $\Gamma(S,\underline{\omega}_{A\times A})$. In order to imitate the trick used in 4) above, we will use the following lemma.

(3.2.6) Lemma: Let $X/S$ be a scheme, $\mathcal{L}_1,\mathcal{L}_2$ line bundles on $X$, $\nabla_1,\nabla_2,\nabla_1',\nabla_2'$ connections on $\mathcal{L}_i$, $\phi$: $\mathcal{L}_1 \to \mathcal{L}_2$ an isomorphism. Let $\delta$ (resp. $\delta'$) denote the "difference" between $\phi^*(\nabla_2)$ and $\nabla_1$ (resp $\phi^*(\nabla_2')$ and $\nabla_1'$). Then we have the following formula $\delta - \delta'$ = "difference" between $\nabla_2$ and $\nabla_2'$ - "difference" between

$\nabla_1$ and $\nabla_1'$ .

Proof: The assertion is local hence we can assume $X = \mathrm{Spec}(B)$ S affine, $\mathscr{L}_1, \mathscr{L}_2$ trivial. Translating then $\nabla_1, \nabla_1', \nabla_2, \nabla_2'$ corresponding to differential forms $\omega_1, \omega_1', \omega_2, \omega_2' \in \Omega_B^1$ and $\phi$ corresponds to the mapping multiplication by a unit $b \in B^*$. Thus $\phi^*(\nabla_2)$ corresponds to $\frac{db}{b} + \omega_2$ so $\phi^*(\nabla_2) - \nabla_1 = \frac{db}{b} + (\omega_2 - \omega_1)$ and analogously $\phi^*(\nabla_2') - \nabla_1' = \frac{db}{b} + (\omega_2' - \omega_1')$. Subtracting we find the result.

In applying the lemma take $\mathscr{L}_1 = s^*(\mathscr{L})$, $\mathscr{L}_2 = \pi_1^*(\mathscr{L}) \otimes \pi_2^*(\mathscr{L})$ and for any two connections $\bar{\nabla}, \bar{\bar{\nabla}}$ on $\mathscr{L}$ let $\nabla_1 = s^*(\bar{\nabla})$, $\nabla_1' = s^*(\bar{\bar{\nabla}})$, $\nabla_2 = \pi_1^*(\bar{\nabla}) \otimes \pi_2^*(\bar{\nabla})$, $\nabla_2' = \pi_1^*(\bar{\bar{\nabla}}) \otimes \pi_2^*(\bar{\bar{\nabla}})$. Then if $\bar{\nabla} - \bar{\bar{\nabla}} = \psi \in \Gamma(A, \Omega_A^1)$ the lemma says that $\delta(\bar{\nabla}) - \delta(\bar{\bar{\nabla}}) = \pi_1^*(\psi) + \pi_2^*(\psi) - s^*(\psi)$. But because A is an abelian scheme $\psi$ is primitive and hence $\delta(\bar{\nabla}) = \delta(\bar{\bar{\nabla}})$.

Because $\delta(\nabla)$ is independent of the connection put on the line bundle $\mathscr{L}$, we can just as in 4) above define a morphism $\underline{\mathrm{Ext}}(A, G_m) \to \underline{\omega}_{A \times A}$. As the trivial connection on the trivial extension is compatible with the group structure, any connection placed on any extension is similarly compatible since the morphism is constantly zero.

b) Assume S is affine and consider the extension of finite locally free-groups:

$$(\mathscr{E}) \qquad 0 \to A \to B \to C \to 0$$

From (2.2.1) there is an exact sequence

$$(3.2.7) \qquad 0 \to \Gamma(S, \underline{\omega}_C) \to (\mathscr{E})\text{-Homrig}(A, G_m) \to \mathrm{Hom}_{S\text{-}gr}(A, G_m) \to 0$$

The indeterminacy in putting a structure of $\natural$-extension on the trivial extension $0 \to G_m \to G_m \times C \to C \to 0$ is given by $\Gamma(S, \underline{\omega}_C)$ since the differential form defining the connection must be primitive (i.e. translation invariant). Thus there is also an exact sequence

$$(3.2.8) \qquad 0 \to \Gamma(S, \underline{\omega}_C) \to (\mathcal{E})\text{-}\natural\mathrm{Hom}(A, G_m) \to \mathrm{Hom}_{S\text{-}gr}(A, G_m)$$

Obviously (3.2.7) receives a map from (3.2.8) which is the identity on $\Gamma(S, \underline{\omega}_C)$ and on $\mathrm{Hom}_{S\text{-}gr}(A, G_m)$, and which is the map (3.2.2) on the middle terms. Hence to conclude that (3.2.2) is an isomorphism it suffices to prove that the map $(\mathcal{E})\text{-}\natural\mathrm{Hom}(A, G_m) \to \mathrm{Hom}_{S\text{-}gr}(A, G_m)$ is surjective.

Let $\phi\colon A \to G_m$ be a homomorphism and consider the corresponding extension

$$(\mathcal{E}_\phi) \qquad 0 \to G_m \to E \to C \to 0 \ , \quad E = G_m \overset{A}{\amalg} B.$$

If the set of structures of $\natural$-extension on $(\mathcal{E}_\phi)$ is not empty it is principal homogeneous under $\Gamma(S, \underline{\omega}_C)$. Replacing $S$ by an arbitrary $S$-scheme $S'$ we see that for variable $S'$ the functor $S' \mapsto$ set of structures of $\natural$-extension on $(\mathcal{E}_\phi)_{S'}$ is formally principal homogeneous under $\underline{\omega}_C$. Since $C$ is finite and locally free $\underline{\mathrm{Ext}}^1(C, G_m) = (0)$ and hence locally for the f.p.p.f. topology, the extension $(\mathcal{E}_\phi)$ is trivial. This implies that we actually have a torseur. By descent it is locally trivial for the Zariski topology and thus because $S$ is affine it is trivial. Hence $(\mathcal{E}_\phi)$ actually admits a structure of $\natural$-extension; which proves $(\mathcal{E})\text{-}\natural\mathrm{Hom}(A, G_m) \to \mathrm{Hom}_{S\text{-}gr}(A, G_m)$ is surjective.

## §4. THE RELATION BETWEEN ONE-DIMENSIONAL DE RHAM COHOMOLOGY AND THE LIE ALGEBRA OF THE UNIVERSAL EXTENSION

Let A be an abelian scheme over S. We shall establish an isomorphism between $H^1_{DR}(A)$ and the lie algebra of Extrig$(A,G_m)$. The most convenient way to do this is to find yet another interpretation of Extrig$(A,G_m)$, this time in terms of differential forms (see the construction of $E^\natural$ below).

### (4.1). The Definition of $E^\natural$.

(4.1.1). Let A/S be an abelian scheme. Its De **Rham** cohomology is quite simple:

a) all the $H^i_{DR}(A)$, $H^q(\Omega^p_A)$ are locally free (and hence their formation commutes with base change).

b) The Hodge-DR spectral sequence degenerates at $E_1$.

c) $H^*_{DR}(A) = \Lambda^* H^1_{DR}(A)$

The first thing we do is give a geometric interpretation to a portion of the long exact sequence of (hyper) cohomology associated to the short exact sequence of complexes

$$\begin{tikzcd}
 & 0 \arrow[d] & \mathcal{O}^* \arrow[d, "d\log"] & \mathcal{O}^* \arrow[d] & \\
0 \arrow[r] & \Omega^1 \arrow[r] \arrow[d] & \Omega^1 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & \Omega^2 \arrow[d] & \Omega^2 \arrow[d] & 0 \arrow[d] & \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

which we abbreviate to: $0 \to \tau_1(\Omega^\cdot) \to \Omega^* \to \mathcal{O}^* \to 0$

Define a functor on S-schemes by:

$S' \mapsto$ the group of isomorphism classes of line bundles on $A_{S'}$ endowed with an integrable connection.

Write $P^{\natural}$ for this functor and $\underline{P}^{\natural}$ for the associated Zariski sheaf. For any $S'$, there is the forgetting map:

$$P^{\natural}(S') \to H^1(\mathcal{O}^*_{A_{S'}})$$

which by passage to the associated sheaves yields ( since $A$ is an abelian scheme) a homomorphism

$$\underline{P}^{\natural} \xrightarrow{\pi} \underline{\mathrm{Pic}}(A)$$

Because global 1-forms on an abelian scheme are closed, and because the map $H^o(\mathcal{O}^*_A) \xrightarrow{d \log} H^o(\Omega^1_A)$ is the zero map, the indeterminacy in putting an integrable connection on the trivial bundle $\mathcal{O}$ is precisely $\Gamma(A,\Omega^1_A) = \Gamma(S,\underline{\omega}_A)$. Passing to the associated sheaves we find the kernel of the map $\pi$ to be $\underline{\omega}_A$. What is the obstruction to putting an integrable connection on a line bundle $\mathcal{L}$ (over $A$)? The obstruction to putting <u>any</u> connection on $\mathcal{L}$ is furnished by the cocycle arising as the logarithmic derivative of the transition function defining $\mathcal{L}$:

$$H^1(\mathcal{O}^*_A) \to H^1(\Omega^1_A) \quad , \quad (f_{ij}) \mapsto \frac{df_{ij}}{f_{ij}}$$

There is an obvious map

$$H^1(\mathcal{O}^*_A) \to \mathbb{H}^2(\tau_1(\Omega^{\bullet}_A))$$

given in terms of Čech cocyles (for some affine open cover $\underline{\mathcal{U}}$ of $A$) by

$$(f_{ij}) \longmapsto ((\frac{df_{ij}}{f_{ij}})),0) \in C^1(\mathcal{U},\Omega^1)\oplus C^o(\mathcal{U},\Omega^2)$$

If this cocycle is a coboundary there are <u>closed</u> 1-forms $\omega_i$

such that $\frac{df_{ij}}{f_{ij}} = \omega_i - \omega_j$ and hence $\mathcal{L}$ will admit an integrable connection. The converse is equally trivial.

(4.1.2) Proposition:

$$P^{\natural}(S) \rightarrow \mathbb{H}^1(\Omega^*_A) .$$

Proof: To any line bundle with integrable connection $(\mathcal{L},\nabla)$ we associate the cohomology class of the Čech cocycle $((f_{ij}),(\omega_i)) \in C^1(\mathcal{O}^*_A) \oplus C^0(\Omega^1_A)$ where $f_{ij}$ are the transition functions and $\omega_i$ is the "connection form" for the induced connection on $\mathcal{L}|\mathcal{U}_i$.

Q.E.D.

Thus we have arrived at the geometrical description of a portion of the above mentioned cohomology sequence:

(4.1.3)

$$\begin{tikzcd}
0 \arrow[r] & H^0(\Omega^1_A) \arrow[r] \arrow[d, equal] & P^{\natural}(S) \arrow[r] \arrow[d, "\wr"'] & \mathrm{Pic}(A) \arrow[r] \arrow[d, equal] & \mathbb{H}^2(\tau_1(\Omega^{\bullet}_A)) \arrow[d, equal] \\
0 \arrow[r] & \mathbb{H}^1(\tau_1(\Omega^{\bullet}_A)) \arrow[r] & \mathbb{H}^1(\Omega^*_A) \arrow[r] & H^1(\mathcal{O}^*_A) \arrow[r] & \mathbb{H}^2(\tau_1(\Omega^{\bullet}_A))
\end{tikzcd}$$

Now we shall consider Lie algebras. For any group functor $G$ on Sch/S, the formation of $\underline{\mathrm{Lie}}(G)$ commutes with taking of the associated Zariski sheaf. Thus to calculate the Lie algebra of $\underline{P}^{\natural}$ it suffices to calculate that of $P^{\natural}$.

(4.1.4) Proposition: $H^1_{DR}(A/S)$ is canonically isomorphic to $\underline{\mathrm{Lie}}(\underline{P}^{\natural})$.

Proof: We must examine $\mathrm{Ker}(P^{\natural}(S[\epsilon]) \rightarrow P^{\natural}(S))$ which by (4.1.2) can be regarded as the kernel of

$$\mathbb{H}^1(\Omega^*_{A_{S[\epsilon]}}) \rightarrow \mathbb{H}^1(\Omega^*_A) .$$

But we have a split exact sequence of complexes of sheaves of abelian groups on A:

$$0 \to \Omega^{\cdot}_A \to \Omega^{*}_{A_{S[\epsilon]}} \longrightarrow \Omega^{*}_A \to 0$$

and hence (at least as abelian groups) $\mathbb{H}^1(\Omega^{\cdot}_A) \xrightarrow{\sim} \underline{\mathrm{Lie}}(P^{\natural})(S)$
The fact that the module structures coincide is a straightforward verification. Passing to the associated sheaves we find $H^1_{DR}(A/S) \xrightarrow{\sim} \underline{\mathrm{Lie}}(\underline{P}^{\natural})$ as desired.

(4.1.5) Lemma. $\mathbb{H}^*(\tau_1(\Omega^{\cdot}_A))$ is locally free (and hence commutes with arbitrary base change).

Proof: From the exact sequence $0 \to \tau_1(\Omega^{\cdot}_A) \to \Omega^{\cdot}_A \to \mathcal{O}_A \to 0$, using the local freeness of $H^*_{DR}(A)$ , $H^*(\mathcal{O}_A)$ and the degeneration of Hodge => De Rham, we read the result from the short exact sequences: $0 \to \mathbb{H}^i(\tau_1(\Omega_A)) \to \mathbb{H}^i(\Omega_A) \to H^i(\mathcal{O}_A) \to 0$

Knowing $\mathbb{H}^2(\tau_1(\Omega_A))$ is a locally free module commuting with base change we obtain the exact sequence of Zariski sheaves on Sch/S.

$$0 \to \underline{\omega}_A \to \underline{P}^{\natural} \to \underline{\mathrm{Pic}}(A) \to \mathbb{H}^2(\tau_1(\Omega_A))$$

Let us consider the dual abelian scheme $A^* = \underline{\mathrm{Pic}}^{0}(A)$ and the composite of its inclusions into $\underline{\mathrm{Pic}}(A)$ with the map $\underline{\mathrm{Pic}}(A) \to \mathbb{H}^2(\tau_1(\Omega^{\cdot}_A))$. This composite is zero because there are <u>no</u> non-trivial homomorphisms from a abelian scheme to a (locally-free) quasi-coherent module. Hence the image of $\underline{P}^{\natural}$ in $\underline{\mathrm{Pic}}(A)$ contains $A^*$ and there is an exact sequence

$$0 \to \underline{\omega}_A \to \underline{P}^{\natural} \times_{\underline{\mathrm{Pic}}(A)} A^* \to A^* \to 0$$

(4.1.6) Definition $E^{\natural} = \underline{P}^{\natural} \times_{\underline{\mathrm{Pic}}(A)} A^*$

Thus $E^{\natural}$ is actually a smooth group scheme which is obtained by considering the Zariski sheaf associated to the presheaf assigning to $S'/S$ the set of isomorphism classes of $(\mathcal{L},\nabla)$ where the cohomology class of $\mathcal{L}$ is primitive or equivalently the $G_{m_{A_{S'}}}$ torseur corresponding to $\mathcal{L}$ is an extension of $A_{S'}$ by $G_{m_{S'}}$.

(4.1.7) Proposition: $H^1_{DR}(A/S)$ is canonically isomorphic to $\underline{\mathrm{Lie}}(E^{\natural})$.

Proof: $\underline{\mathrm{Lie}}(\underline{P}^{\natural} \times_{\mathrm{Pic}(A)} A^*) \simeq \underline{\mathrm{Lie}}(\underline{P}^{\natural}) \times_{\underline{\mathrm{Lie}}(\underline{\mathrm{Pic}}(A))} \underline{\mathrm{Lie}}(A^*)$

and as is well known $\underline{\mathrm{Lie}}(A^*) \rightarrow \underline{\mathrm{Lie}}(\underline{\mathrm{Pic}}(A))$ is an isomorphism.

### (4.2) The isomorphism between $\mathrm{Ext}^{\natural}$ and $E^{\natural}$.

For any abelian scheme $A/S$ define a homomorphism,

$$\underline{\mathrm{Ext}}^{\natural}(A,G_m) \underset{\zeta}{\rightarrow} E^{\natural} = \underline{P}^{\natural} \times_{\underline{\mathrm{Pic}}(A)} \underline{\mathrm{Pic}}^{0}(A)$$

as follows: Any element $e$ in $\mathrm{Ext}^{\natural}(A,G_m)$ may be regarded as an isomorphism class of invertible sheaves on $A$ endowed with an integrable connection and with a horizontal isomorphism

$$s^*(G) \overset{\epsilon}{\rightarrow} p^*_1(L) \otimes p^*_2(L)$$

where $p_1,p_2\colon A \times A \rightarrow A$ are the projections and $s = p_1+p_2$ is the sum morphism. By forgetting $\epsilon$, (resp. the connection) we obtain an element of $P^{\natural}$ (resp. $\mathrm{Pic}^{0}(A)$).

(4.2.1) Proposition

The above morphism is an isomorphism,

$$\mathrm{Ext}^{\natural} \xrightarrow[\zeta]{\sim} E^{\natural} .$$

Proof: It is injective. Any two horizontal isomorphisms between line bundles differ by multiplication by a unit in $\Gamma(S,\mathcal{O}_S)$. Thus if there is a horizontal isomorphism, an isomorphism compatible with the $\epsilon$'s is also horizontal.

To show that it is surjective, we shall define a morphism of S-schemes $\eta : A^{*} \to \underline{\omega}_{A\times A}$ which expresses the obstruction to surjectivity of $\zeta$: Let L be in $\mathrm{Ext}(A,G_m)$. Choose any integrable connection $\nabla$ on L . This induces connections on $s^*(L), p_1{}^*(L)$, $p_2^*(L)$, $p_1^*(L) \otimes p_2^*(L)$.

The extension-structure of L gives us an explicit isomorphism,

$$s^*(L) \overset{\epsilon}{\underset{\sim}{\to}} p_1^*(L) \otimes p_2^*(L) .$$

Consider the difference between the connection on $s^*(L)$ and the pullback of the connection on $p_1^*(L)\otimes p_2^*(L)$ via the above morphism. This difference $i(\nabla)$ is a section of $\underline{\omega}_{A\times A}$ . By (3.2.6) $i(\nabla)$ depends only on L and not on the integrable connection $\nabla$ chosen.

We define $\eta(L) = i(\nabla)$. Since $A^*$ is an abelian scheme and $\underline{\omega}_{A\times A}$ is a locally free module, $\eta$ is a constant map. Since $\eta(0) = 0$, $\eta$ is identically zero. It follows that $\epsilon$ is horizontal and $\zeta$ is surjective.

(4.6.3) The sheaf $\underline{P}^{\natural}$ in concrete terms.

Consider the morphism of complexes $\Omega^*_{A/S} \to \mathcal{O}^*_A$ and the corresponding mapping induced on the exact sequence of terms of low degree, from the Leray spectral reference:

$$\begin{tikzcd}
0 \arrow[r] & H^1(\mathcal{O}^*_S) \arrow[r] \arrow[d, "\wr"'] & \mathbb{H}^1(\Omega^*_A) \arrow[r] \arrow[d] & \Gamma(S, R^1 f_* (\Omega^*_A)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^1(\mathcal{O}^*_S) \arrow[r] & H^1(A, \mathcal{O}^*_A) \arrow[r] & \Gamma(S, \underline{\mathrm{Pic}}(A)) \arrow[r] & 0
\end{tikzcd}$$

Consider on the other hand the group $\mathrm{Pic}^{\natural}_e(A) =_{\mathrm{dfn}} \{$isomorphism class of triples $(\mathcal{L}, \alpha, \nabla)$ where $(\mathcal{L}, \alpha)$ is an e-rigidified line bundle on $A$ and $\nabla$ an integrable connection on $\mathcal{L}\}$. Here isomorphisms are to be horizontal and respect the e-rigidification. There is an obvious map $\mathrm{Pic}^{\natural}_e(A) \to P^{\natural}(S)$

$$(\mathcal{L}, \alpha, \nabla) \mapsto (\mathcal{L}, \nabla)$$

If $(\mathcal{L}, \nabla)$, $(\mathcal{L}', \nabla')$ are isomorphic, an isomorphism compatible with the rigidifications can be chosen since to modify an isomorphism we use a global section of $\Gamma(A, \mathcal{O}^*_A) = \Gamma(S, \mathcal{O}^*_S)$ and clearly this will not alter the horizontality. Hence the map is injective. We obviously have a commutative diagram:

$$\begin{tikzcd}
(\mathcal{L}, \alpha, \nabla) \arrow[d, mapsto, ""{name=L}] & \mathrm{Pic}^{\natural}_e(A) \arrow[r] \arrow[d, ""{name=M}] & P^{\natural}(S) \arrow[d] \\
(\mathcal{L}, \alpha) & \mathrm{Pic}_e(A) \arrow[r] & H^1(A, \mathcal{O}^*_A)
\arrow[from=L, to=M, squiggly]
\end{tikzcd}$$

Given $(\mathcal{L}, \nabla)$ in $P^{\natural}(S)$, $\mathcal{L} \otimes f^*e^*(\mathcal{L}^{-1})$ is rigidified and $f^*e^*(\mathcal{L}^{-1})$ can be given the "stupid" connection so that it is in the image of $H^1(S, \mathcal{O}^*_S) \to \mathbb{H}^1(\Omega^*_A)$. Thus the map $\mathrm{Pic}^{\natural}_e(A) \to \underline{P}^{\natural}(S)$ $= H^0(\mathbb{R}^1 f_*(\Omega^*_A))$ is surjective. If $(\mathcal{L}, \alpha, \nabla) \mapsto 0$, then $\mathcal{L} = f^*(\mathcal{L}')$, $\nabla =$ trivial connection, and $\mathcal{O}_S \simeq e^*(\mathcal{L}) = e^*f^*(\mathcal{L}')$ $= \mathcal{L}' \Rightarrow \mathcal{L} \simeq \mathcal{O}_A$, and $\nabla =$ trivial connection, which obviously implies $(\mathcal{L}, \alpha, \nabla) \simeq (\mathcal{O}_A, \text{obv.}, 0)$. Thus the map $\mathrm{Pic}^{\natural}_e(A) \to \underline{P}^{\natural}(S)$

is an isomorphism and we have the desired description of $\underline{P}^\natural(S)$ as {e-rig line bundles + $\nabla$}, a description which is obviously compatible with the description of $\Gamma(S,\underline{Pic}(A))$ as {e-rigidified line bundles.}

Since $E^\natural =_{dfn} P^\natural \times_{\underline{Pic}(A)} \underline{Ext}(A,G_m)$ , it is clear that $E^\natural$ admits the following description, its points with values in S (or for that matter any S-scheme S') consist of isomorphism classes of extensions $0 \to G_m \to E \to A \to 0$ such that E is as $G_{m_A}$-torseur endowed with an integrable connection.

## (4.4) The Universal Extension of an Abelian Scheme in the Analytic Category over $\mathbb{C}$ .

Let A/S be an abelian scheme over S , where S is a scheme locally of finite type over $\mathbb{C}$ . We may view A/S as a family of complex analytic spaces. The theory of Extrig carries over, with no significant change, in the analytic category. One thus obtains the analytic versions and natural maps below:

$$[\underline{Extrig}(A,G_m)]^{an} \longrightarrow \underline{Extrig}(A^{an},G_m^{an})$$

$$[\underline{Extrig}(A,G_a)]^{an} \longrightarrow \underline{Extrig}(A^{an},G_a{}^{an})$$

(4.4.1) Proposition: The morphisms above are isomorphisms.

Proof: This follows for each fibre (over S) by GAGA . Consequently our morphisms are analytic morphisms bijective on underlying pointsets. By consideration of vertical and horizontal tangent vectors one checks that the jacobian criterion is satisfied.

Q.E.D.

As a consequence, the exponential sequence of analytic groups over $\mathbb{C}$

$$0 \rightarrow 2\pi i\mathbb{Z} \rightarrow G_a \xrightarrow{\exp} G_m \rightarrow 0$$

gives rise to the following diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \underline{\omega}_A \arrow[r, "\cong"] \arrow[d] & \underline{\omega}_A \arrow[d] & \\
 & & \underline{\mathrm{Extrig}}(A^{an},G_a) \arrow[r] \arrow[d] & \underline{\mathrm{Extrig}}(A^{an},G_m) \arrow[d] & \\
0 \arrow[r] & R^1f_*\mathbb{Z} \arrow[r] & R^1f_*\mathcal{O}_{A^{an}} \arrow[r] \arrow[d] & \underline{\mathrm{Pic}}^0(A) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

which gives us (using the snake lemma) the following exact sequence:

$$0 \rightarrow H^1(A^{an},\mathbb{Z}) \rightarrow \mathrm{Extrig}(A^{an},G_a) \rightarrow \mathrm{Extrig}(A^{an},G_m) \rightarrow 0$$

over any affine base $S$ .

(4.4.2) <u>Corollary</u>: One has an exact sequence of analytic groups over S:

$$0 \rightarrow R^1f_*\mathbb{Z} \rightarrow \underline{H}^1_{DR}(A^{an}/S) \rightarrow E(A^*)^{an} \rightarrow 0$$

<u>Proof</u>: Note that $\underline{H}_{DR}$ refers to relative de Rham cohomology over the base $S$ . $R^1f_*\mathbb{Z}$ refers to the locally constant sheaf of abelian groups.

The corollary follows from our identifications

$$\underline{\mathrm{Extrig}}(A,G_m) = E(A^*)$$

$$\underline{\mathrm{Extrig}}(A,G_a) = H^1_{DR}(A/S)$$

## §5. FRAGMENTARY COMMENTS CONCERNING NÉRON MODELS AND UNIVERSAL EXTENSIONS

Let $S$ be a connected Dedekind scheme. ( $S$ = Spec $D$ where $D$ is a Dedekind domain). Let $N$ be a Néron model over $S$ . This means that there is a nonempty open $U \subset S$ such that $N/U$ is an abelian scheme, and $N/S$ is the Néron model of $N/U$. Let $N'/U$ denote the dual abelian scheme and let $N'/S$ be its Neron model over $S$ . Define $N^o \subseteq N$ to be the open subgroup scheme all of whose fibres are connected.

The easy part of an unpublished duality theorem of Artin and Mazur asserts

(5.1) Lemma The duality of Abelian schemes

$$\underline{\mathrm{Ext}}^1_U(N_U,G_m) \xrightarrow{\cong} N'_U$$

extends to an isomorphism of functors evaluated on smooth $S$-schemes:

$$\underline{\mathrm{Ext}}^1_S(N^o,G_m) \xrightarrow{\cong} N'$$

We sketch a proof of this lemma by showing that $\underline{\mathrm{Ext}}^1_S(N^o,G_m)$ enjoys the Néron property [11,SGA$_7$IX,1]. To do this one must take $T/S$ a smooth "test" scheme and consider the diagram with exact rows, [11, SGA$_7$VII 1.3.5,1.3.8]:

$$\begin{tikzcd}
0 \arrow[r] & \underline{\mathrm{Ext}}^1_S(N^o,G_m)(T) \arrow[r] \arrow[d, "\alpha"] & \underline{\mathrm{Pic}}_S(N^o)(T) \arrow[r, "\sigma"] \arrow[d, "\beta"] & \underline{\mathrm{Pic}}_S(N^o \times_S N^o)(T) \arrow[d, "\gamma"] \\
0 \arrow[r] & \underline{\mathrm{Ext}}^1_U(N^o_U,G_m)(T_U) \arrow[r] & \underline{\mathrm{Pic}}_U(N^o_U)(T_U) \arrow[r, "\sigma"] & \underline{\mathrm{Pic}}_U(N^o_U \times_U N^o_U)(T_U)
\end{tikzcd}$$

where $\sigma = \mathrm{proj}_1^* + \mathrm{proj}_2^* - \mathrm{sum}^*$

Since $N^o \times_S T$ and $N^o \times_S N^o \times_S T$ are regular schemes and since $N^o \times_S T/T$ $N^o \times_S N^o \times_S T/T$ have connected geometric fibres, β and γ are isomorphisms. Thus α is an isomorphism as well, and the sketch of the proof of (5.1) is concluded.

(5.2) Corollary There is an exact sequence of smooth groups/S:

$$0 \to \underline{\omega}_{N'} \to \underline{\mathrm{Extrig}}_S(N'^{o}, G_m) \to N \to 0$$

Proof: $f_*(\mathcal{O}_{N'^o}) = \mathcal{O}_S$ . Thus there is an exact sequence of Zariski sheaves on the category of smooth S-schemes.

$$0 \to \underline{\omega}_{N'} \to \underline{\mathrm{Extrig}}(N'^{o}, G_m) \to \underline{\mathrm{Ext}}(N'^{o}, G_m) \to 0$$

(c.f. the discussion preceeding (2.6.1))

From the lemma, $N \cong \underline{\mathrm{Ext}}(N'^{o}, G_m)$, and hence $\underline{\mathrm{Extrig}}(N'^{o}, G_m)$ is a smooth group.

Write $\underline{E}(N) = \underline{\mathrm{Extrig}}_S(N'^{o}, G_m)$.

A surprise is that the exact sequence

(5.2.1) $$0 \to \underline{\omega}_{N'} \to E(N) \to N \to 0$$

is not necessarily the universal extension of $N'$. In fact, as L. Breen and M. Raynaud have shown: there are Néron models $N$ which possess no universal extension. A sketch of their elegant argument is included below. Therefore we refer to (5.2.1) as the canonical extension of a Néron model $N$ by a vector group.

It appears to us that this canonical extension deserves systematic study, and indeed the first question one may ask about it is the following, which we pose in purposely vague language:

Find a functorial characterization of the canonical extension (5.2.1) of a Néron model.

It is especially interesting to consider the canonical extension over the base $S = \mathrm{Spec}(\mathbb{Z})$.

Let $M = N(\mathbb{Z}) \cong N(\mathbb{Q})$ denote the Mordell-Weil group of the abelian variety $N_{\mathbb{Q}}$. This is a finitely generated group. Let $M^* = E(N)(\mathbb{Z})$. Since $S$ is affine, (5.2.1) gives the exact sequence

$$(5.2.2) \qquad 0 \to \underline{\omega}_{N'}(\mathbb{Z}) \to M^* \to M \to 0$$

Since $\underline{\omega}_{N'}(\mathbb{Z})$ is a free abelian group whose rank is $\dim N = d$, we see that $M^*$ is a finitely generated abelian group of rank $d + \mathrm{rank}(M)$. What is curious is that $M^*$ has a strong tendency to be free. Explicitly:

(5.3) Theorem: If $p$ divides the order of the torsion subgroup of $M^*$ then either $p = 2$ or $p$ is a prime of bad reduction for $N$.

(5.4) Corollary: If the order of the torsion subgroup of $M$ is relatively prime to

$2 \times$ product of primes of bad reduction of $N$

then $M^*$ is a free abelian group.

Proof of Theorem: Let $x^* \in M^*$ be a nontrivial element of order $p$. Since $E(N)$ is separated it suffices to show $x^*$ is zero, after having base changed to $S = \mathrm{Spec}(\mathbb{Z}_p)$.

By our assumption, $N$ is an abelian scheme over $S$, and $E(N)$ is the universal extension of $N$. Let $N(p)_{/S}$ be the Barsotti-Tate group associated to the abelian scheme $N/S$. Then over $S_\nu = \mathrm{Spec}(\mathbb{Z}/p^\nu)$ for any $\nu$, $E(N)(p)$ is the universal extension of the Barsotti-Tate group $N(p)$. The element $x^* \in M^*$ may be viewed as a section of $E(N)(p)$ and its image, $y$, in $N(p)$ generates a finite flat group $G$ over $S$ of order $p$. Since $p \neq 2$, and since $G$ has a nontrivial rational section, by the classification theory of finite flat groups of order $p$ over $\mathbb{Z}_p$ [20,Theorem 2], $G = \mathbb{Z}/p$.

Let the subscript $\nu \geq 1$ denote restriction to the base $S_\nu = \mathrm{Spec}\ \mathbb{Z}/p^\nu$.

Let $N(p)^{et}$ denote the etale quotient of $N(p)$, and let $E(N(P)^{et})_\nu$ denote the universal extension of $N(p)^{et}_\nu$. We have the diagram

$$\begin{tikzcd}
E(N)(p)_\nu \arrow[r] \arrow[d] & N(p)_\nu \arrow[d] \\
E(N(p)^{et})_\nu \arrow[r] & N(p)^{et}_\nu
\end{tikzcd}$$

Since $G = \mathbb{Z}/p$ the image of $G$ in $N(p)^{et}$ is nonzero. Consequently the image of the section $x^*$ in $N(p)^{et}$ is nonzero. It follows that the image of $x^*$ in $E(N(p)^{et}_\nu)$ is nonzero. But this is a contradiction because the universal extension of an étale p-divisible group over $\hat{S} = \mathrm{Spf}(\mathbb{Z}_p)$ has no nontrivial section of order $p$.

(5.5) As a special case of the above theorem, take an elliptic curve C over $\mathbb{Q}$ whose Mordell-Weil group is a finite group F of odd order relatively prime to the conductor of C .

Since any odd finite group of real points of C is cyclic, F is a cyclic group.

Making a choice of sign of the Néron differential of C enables us to identify $\omega_{N'}(\mathbb{Z}) \cong \mathbb{Z}$ (where N is the Néron model of C) and consequently the exact sequence (5.2.2) becomes

$$(5.5.1) \qquad 0 \to \mathbb{Z} \to M^* \to F \to 0$$

But the theorem implies, under our hypotheses that M* is free, and consequently the exact sequence (5.5.1) becomes:

$$(5.5.2) \qquad 0 \to \mathbb{Z} \xrightarrow{\varphi} \mathbb{Z} \xrightarrow{\pi} F \to 0$$

where $\varphi$ consists in multiplication by the order of F .

Consequently the canonical extension of the Néron model of C determines in this case a *canonical free resolution of the Mordell-Weil group of C* . In particular, choosing a Néron differential of C (there are two possible choices, and to choose one of these two amounts to the same as orienting the real locus of C ) gives (in the case considered above) a *canonical* generator of the Mordell-Weil group, defined to be the image of $1 \in \mathbb{Z}$ under $\pi$ in (5.5.2). (call this the *generator defined by the canonical extension*) It may occur to the reader that the topology of the real locus of C enables one to obtain yet another canonical generator of F: Since F is a finite subgroup of the connected component of the real locus of C , which is a circle (oriented, after a choice of Neron differential), it makes sense to consider that element of $F$, closest to the origin in the circle, where "closest" means in the direction of orientation of the circle. Call this the topologically-defined generator.

Tate has made some computations which abundantly support the opinion that there is no relation at all between the generator defined by the canonical extension and the topologically defined generator.

(5.6) Example of Breen and Raynaud.

The following is taken from a letter of L. Breen.

Let $R$ be a discrete valuation ring with uniformizer $\pi$ and residue field $k$. Let $N_{/R}$ be the Néron model of an elliptic curve. Let $\bar{N}$ denote its fibre at $k$. Suppose one of two special cases

I) $\bar{N} = G_a$

II) $\bar{N} = G_m$

Consider the short exact sequence of Zariski sheaves on the smooth site over $S = \operatorname{Spec} R$,

$$0 \to G_a \xrightarrow[\text{mult. by } \pi]{} G_a \longrightarrow i_*G_a \to 0$$

(Here $i : \operatorname{Spec} k \to S$ is the canonical injection).

This induces the exact sequence

$$0 \to \operatorname{Hom}_S(N, i_*G_a) \longrightarrow \operatorname{Ext}^1(N, G_a) \xrightarrow{\pi} \operatorname{Ext}^1(N, G_a) \to \operatorname{Ext}^1_S(N, i_*G_a)$$

But

$$\operatorname{Hom}_S(N, i_*G_a) = \operatorname{Hom}_k(\bar{N}, G_a)$$
$$\operatorname{Ext}^1_S(N, i_*G_a) = \operatorname{Ext}^1_k(\bar{N}, G_a)$$

and consequently

(*) $\mathrm{Ext}^1(N,G_a) \xrightarrow{\pi} \mathrm{Ext}^1(N,G_a)$ enjoys the following properties in each of our two special cases:

Case I: (*) is not injective

Case II: (*) is surjective

(5.6.1) Corollary: In either case, $\underline{\mathrm{Ext}}^1(N,G_a)$ is not a locally free sheaf of $\mathcal{O}_S$-modules, and there is no universal extension of $N$ by a vector group (over $R$).

Proof: If $\underline{\mathrm{Ext}}^1(N,G_a)$ were locally free then $\mathrm{Ext}^1(N,G_a)$ $= H^0(S,\underline{\mathrm{Ext}}^1(N,G_a))$ would be a free R-module and consequently multiplication by $\pi$ would be injective and not surjective on it. (N.B. $\mathrm{Ext}^1(N,G_a)$ is not zero since the canonical extension is non-trivial). Moreover if there were an extension of $N$ by a vector group $V$, which was universal we would have

$$\mathrm{Hom}(V,G_a) \cong \mathrm{Ext}^1(N,G_a)$$

consider

$$\mathrm{Hom}_S(V,G_a) \xrightarrow{\pi} \mathrm{Hom}_S(V,G_a).$$

Since $V$ is a vector group, $\pi$ is not surjective and is injective, contradicting the situation that obtained in either case I or case II.

# CHAPTER TWO

## UNIVERSAL EXTENSIONS AND CRYSTALS

In this chapter we describe the crystalline nature of the universal extension. More precisely we associate with an abelian scheme (resp. Barsotti-Tate group) $G/S$ a crystal , $\mathbb{E}^*(G)$, on $S$ whose value of $S$, $\mathbb{E}^*(G)_S$ , is the universal extension $E(G^*)$ of $G^*$ by a vector group, By applying the functor *Lie* we then obtain a crystal in locally-free modules, $\mathbb{D}^*(G)$. If $f: G \rightarrow S$ is an abelian scheme then $\mathbb{D}^*(G)$ is nothing but the usual crystalline cohomology, $R^1 f_{crys*}(\mathcal{O}_{(G/S)_{crys}})$. On the other hand when $G$ is a Barsotti-Tate group, $\mathbb{D}^*(G)$ is the *generalized Dieudonné module* associated to $G$ .

One procedure for constructing crystals from the universal extension was given in [16]. Here we shall use a completely different approach allowing us to construct the crystals intrinsically without making use of liftings. Unfortunately, it seems that in order to verify that our crystals have reasonable properties (and in fact that the sheaves constructed are crystals) we must fall back on liftings.

We shall discuss separately the constructions for abelian schemes and for Barsotti-Tate groups. For abelian schemes the construction is straightforward. The procedure for Barsotti-Tate groups is more technical. The reason for the additional complications is the following: For $G$ an abelian scheme our description of $E(G^*)$ uses exclusively the whole group $G$ , while for $G$ a Barsotti-Tate group we use the individual $G(n)$'s as well. But while $G$ is smooth (resp. formally smooth) and hence amenable to standard crystalline techniques, the individual $G(n)$'s are not usually smooth. We assume some familiarity with crystalline theory [2,3].

## §1. THE CRYSTALLINE NATURE OF THE UNIVERSAL EXTENSION OF AN ABELIAN SCHEME

Let $S_o \hookrightarrow S$ be a (locally)-nilpotent immersion defined by an ideal $I$, endowed with (locally) nilpotent divided powers $(\gamma_n)_{n\geq 0}$. Let $A$ and $B$ be abelian schemes on $S$ and $f_o\colon A_o \to B_o$ a homomorphism between their reductions to $S_o$. $f_o$ induces a map on the dual abelian schemes $f_o^*\colon B_o^* \to A_o^*$ and hence a map on the corresponding universal extensions $E(B_o^*) \to E(A_o^*)$. We've shown in chapter I [2.6.7,3.2.3] that this is the map

$$(1.1) \qquad \underline{\mathrm{Ext}}^{\natural}(B_o,G_m) \to \underline{\mathrm{Ext}}^{\natural}(A_o,G_m)$$

induced by $f_o$.

We shall construct a homomorphism $E(B^*) \to E(A^*)$ lifting (1.1). Although this morphism depends on the triple $(A,B,f_o)$ we shall denote it by $E_S^*(f_o)$. From the construction it follows that these homomorphisms enjoy the following properties:

(i) transitivity (= functoriality):

Given $A,B,C$, $\quad A_o \xrightarrow{f_o} B_o \xrightarrow{g_o} C_o$,

$$E_S^*(g_o \circ f_o) = E_S^*(f_o) \circ E_S^*(g_o)$$

(ii) additivity:

Given two homomorphisms $f_o,f_1\colon A_o \to B_o$

$$E_S^*(f_o+f_1) = E_S^*(f_o) + E_S^*(f_1)$$

(iii) functoriality in $S$:

Assume given a commutative diagram

$$\begin{tikzcd} S_o \arrow[r, hook] \arrow[dr, hook] & S \arrow[d, "\phi"] \\ & S' \end{tikzcd}$$

where $\phi$ is a divided power morphism [2,3].

Let $A', B'$ be abelian schemes on $S'$ with $A = \phi^*(A')$, $B = \phi^*(B')$ and let $f_o: A_o \to B_o$ be a homomorphism as above. The following diagram commutes:

$$\begin{tikzcd}
E(B^*) \arrow[r, "E_S^*(f_o)"] \arrow[d, "\wr"'] & E(A^*) \arrow[d, "\wr"] \\
\phi^*(E(B'^*)) \arrow[r, "\phi^*(E_{S'}^*(f_o))"'] & \phi^*(E(A'^*))
\end{tikzcd}$$

(iv) compatibility with liftable maps:

Given a homomorphism $f: A \to B$ with reduction $f_o: A_o \to B_o$,

$$E(f^*) = E_S^*(f_o)$$

(1.2) Remarks

(i) Conditions (i) and (iv) imply $E_S^*(f_o)$ is an isomorphism when $f_o$ is (take $f = id_A$ in (iv))

(ii) Note we do not assert and in general it will not be true that $E_S^*(f_o)$ induces a morphism of extensions.

(1 3) The construction of $E_S^*(f_o)$.

We construct for each flat S-scheme , $T$ , a homomorphism $Ext^{\natural}(B_T, G_m) \to Ext^{\natural}(A_T, G_m)$. It is functorial in $T$ and passing to the associated Zariski sheaves yields a homomorphism between sheaves on the small flat site of $S$ :

$$\underline{Ext}^{\natural}(B, G_m) \to \underline{Ext}^{\natural}(A, G_m) .$$

But because $E(B^*)$ is a flat S-scheme, the map "restriction to the small flat site":

$$\mathrm{Hom}(E(B^*),E(A^*)) \to \mathrm{Hom}_{S_{flat}}(E(B^*),E(A^*))$$

is bijective. Thus we obtain our homomorphism

$$E^*_S(f_o): E(B^*) \to E(A^*) .$$

Because the construction of the map $\mathrm{Ext}^{\natural}(B_T,G_m) \to \mathrm{Ext}^{\natural}(A_T,G_m)$ is functorial in $T$, we shall assume that $T = S$. Consider the following diagram

$$\begin{tikzcd}
A_o \arrow[d, dashed, "f_o"'] \arrow[dd, bend left] & & A \arrow[dd, bend left] \\
B_o \arrow[d] & & B \arrow[d] \\
S_o \arrow[rr, hook] & & S
\end{tikzcd}$$

Recall that if $X$ is any smooth S-scheme, the category of line bundles with integrable connection on $X$ is equivalent to the category of invertible modules on the nilpotent crystalline site of X/S. This equivalence is functorial in the smooth S-scheme $X$.

Also it preserves the algebraic structure inherent in these categories, i.e. it is an equivalence of Picard categories [7]. In particular when we pass to the groups of isomorphism classes of objects, we obtain a canonical isomorphism.

On the other hand since the ideal of the thickening $S_o \hookrightarrow S$ has nilpotent divided powers, there is, for any stack $\mathcal{J}$, an equivalence of categories between $\mathcal{J}$-crystals on $X \times_S S_o/S$ and $\mathcal{J}$-crystals on $X/S$. In particular, with $\mathcal{J}$ = invertible modules, we find invertible modules on $(X_o/S)_{crys} \xrightarrow{\approx}$ invertible modules on $(X/S)_{crys}$. Once again this equivalence is functorial in X and preserves the algebraic structure.

Consider the map

$$H^1(B,\mathcal{O}^*_{B/S_{crys}}) \xrightarrow{\sim} H^1(B_0,\mathcal{O}^*_{(B_0/S)_{crys}}) \xrightarrow{f_0^*} H^1(A_0,\mathcal{O}^*_{(A_0/S)_{crys}}) \xrightarrow{\sim} H^1(A,\mathcal{O}^*_{(A/S)_{crys}})$$

The fact that $f_0$ is a group homomorphism, plus the functoriality (indicated above) applied to the "primitivity maps" $s^*-p_1^*-p_2^*$, shows that our composite maps $\mathrm{Ext}^{\natural}(B,G_m)$ to $\mathrm{Ext}^{\natural}(A,G_m)$. This is the desired homomorphism.

(1.4) Remark: Given $S_0 \hookrightarrow S$ as above and $A_0$ an abelian scheme on $S_0$, we can define for a flat S-scheme $T$, an abelian group

$$\mathrm{Prim}[H^1\left(A_{0_{T_0}},\mathcal{O}^*_{(A_{0_{T_0}/T})_{crys}}\right)] \subseteq H^1\left(A_{0_{T_0}},\mathcal{O}^*_{(A_{0_{T_0}/T})_{crys}}\right)$$

to be the kernel of $s^*-p_1^*-p_2^*$.
Passing to the associated sheaf for the Zariski topology we obtain a group which is canonically identified with the universal extension of (the dual of) any lifting of $A_0$. This is an example of an "intrinsic" definition of the crystal alluded to above.

(1.5) Now pass to tangent spaces. We've already seen that Lie(E(A*)) is canonically isomorphic to $H^1_{DR}(A/S)$. The general crystalline machine, [2] tells us that this module is $H^1(\mathcal{O}_{(A/S)_{crys}})$. Alternatively, this result can be deduced in the standard way from the fact that the tangent space to $G_m$ is $G_a$:

Consider the commutative diagram

$$\begin{tikzcd} A \arrow[d] & A[\varepsilon] \arrow[l, "\pi"'] \arrow[d] \\ S & S[\varepsilon] \arrow[l] \end{tikzcd}$$

defining $A[\varepsilon]$. Since $A$ is smooth we can assume that $I$ is the zero ideal. The morphism of topoi $(A[\varepsilon]/S[\varepsilon])_{crys} \to (A/S)_{crys}$ induced by $\pi$ is easily understood (because $S[\varepsilon] \to S$ is flat): For $F$ a sheaf on $A[\varepsilon]/S[\varepsilon]$, $\pi_*(F)_{(U \hookrightarrow T, J, \gamma)} = F_{(U[\varepsilon] \hookrightarrow T[\varepsilon], \ldots)}$. Visibly $\pi_*$ is exact. For any $(U \hookrightarrow T, J, \gamma)$ in the crystalline site of $A/S$ we have a split exact sequence of ordinary sheaves (on $T$)

$$0 \to \mathcal{O}_T \to \pi_*(\mathcal{O}^*_{A[\varepsilon]})_{U \hookrightarrow T} \to \mathcal{O}^*_T \to 0 .$$

This tells us we have a split exact sequence

$$0 \to \mathcal{O}_{A_{crys}} \to \pi_*(\mathcal{O}^*_{A[\varepsilon]_{crys}}) \to \mathcal{O}^*_{A_{crys}} \to 0$$

Applying $H^1$ and using the exactness of $\pi_*$ to know $H^1(\pi_*(\mathcal{O}^*_{A[\varepsilon]})) = H^1(\mathcal{O}^*_{A[\varepsilon]})$ we conclude.

(1.6) <u>Remark</u>: In particular we see that the map $H_{DR}(B) \to H_{DR}(A)$ furnished by crystalline cohomology is precisely the map obtained from $E^*_S(f_o)$ by applying the functor <u>Lie</u>.

## §2. STABILITY OF $(\epsilon_{m,n})$-HOM$^{\natural}$

Fix a prime $p$ , In §2-§6 below we shall work with a pair $(S,N)$ where $S$ is a scheme, and $N$ a number such that $p^N \cdot 1_S = 0$ .

Let $G$ be a Barsotti-Tate group on $S$ and

$$(\epsilon_{m,n}) \quad 0 \to G(n) \to G(m+n) \xrightarrow{p^n} G(m) \to 0$$

the doubly indexed family of exact sequences.

We have the push out maps

$$(2.0.1)\quad \begin{tikzcd} (\epsilon_{m,n+1}) \arrow[d] & :0 \arrow[r] & G(n+1) \arrow[r] \arrow[d, "p"'] & G(m+n+1) \arrow[r, "p^{n+1}"] \arrow[d, "p"'] & G(m) \arrow[r] \arrow[d, equal] & 0 \\ (\epsilon_{m,n}) & :0 \arrow[r] & G(n) \arrow[r] & G(m+n) \arrow[r, "p^n"] & G(m) \arrow[r] & 0 \end{tikzcd}$$

and the pullback maps

$$(2.0.2)\quad \begin{tikzcd} (\epsilon_{m+1,n}) \arrow[d] & 0 \arrow[r] & G(n) \arrow[r] \arrow[d, equal] & G(m+n+1) \arrow[r] \arrow[d, "p"] & G(m+1) \arrow[r] \arrow[d, "p"] & 0 \\ (\epsilon_{m,n}) & 0 \arrow[r] & G(n) \arrow[r] & G(m+n) \arrow[r] & G(m) \arrow[r] & 0 \end{tikzcd}$$

(2.1) Lemma (Stability in the second index):

For $n \geq N$ , the maps

(i) $(\epsilon_{N,n})$-$\underline{\mathrm{Hom}}^{\natural}(G(n),G_a) \to (\epsilon_{N,n+1})$-$\underline{\mathrm{Hom}}^{\natural}(G(n+1),G_a)$

and (ii) $\underline{\mathrm{Hom}}(G(n),G_a) \to \underline{\mathrm{Hom}}(G(n+1),G_a)$

are isomorphisms.

Proof: By the five-lemma it suffices to show the maps (ii) are isomorphisms. Consider the sequence $0 \to G(1) \to G(n+1) \xrightarrow{p} G(n) \to 0$. We must show $\mathrm{Hom}(G(n+1),G_a) \to \mathrm{Hom}(G(1),G_a)$ is the zero map.

But because $G(n+1) \xrightarrow{p^n} G(1)$ is an epimorphism, it suffices to note that $\mathrm{Hom}(G(n+1),G_a) \xrightarrow{p^n} \mathrm{Hom}(G(n+1),G_a)$ is zero since $p^n$ kills $S$ .

(2.2) Lemma (Stability in the first index)

$$(\epsilon_{m,n})\text{-}\underline{\mathrm{Hom}}^{\natural}(G(n),G_a) \simeq (\epsilon_{m',n})\text{-}\underline{\mathrm{Hom}}^{\natural}(G(n),G_a) \quad \text{if} \quad m' \geq m \geq N$$

Proof: Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \underline{\omega}_{G(m')} \arrow[r] \arrow[d, "i"'] & (\epsilon_{m',n})\text{-}\underline{\mathrm{Hom}}^{\natural}(G(n),G_a) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(G(n),G_a) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \underline{\omega}_{G(m)} \arrow[r] & (\epsilon_{m,n})\text{-}\underline{\mathrm{Hom}}^{\natural}(G(n),G_a) \arrow[r] & \underline{\mathrm{Hom}}(G(n),G_a) \arrow[r] & 0
\end{tikzcd}$$

and use the fact that $i$ is an isomorphism if $m' \geq m \geq N$ [16,II. 3.3.20]

(2.3) Remark: The analogue of (2.2) remains true when $G_a$ is replaced by any smooth group, and in particular by $G_m$

## §3. EXTENSIONS OF TRUNCATED BARSOTTI-TATE GROUPS BY $G_a$

Assume now that $S$ is affine. The following proposition tells us in particular that $\mathrm{Ext}^1(G,G_a)$ is isomorphic to $\mathrm{Ext}^1(G(N),G_a)$ via the map induced by $G(N) \hookrightarrow G$ and hence that $\mathrm{Ext}^2(G,G_a) = (0)$. Undoubtedly this last fact can be obtained via Breen's method [4] for calculating Ext.

(3.1) Proposition: The coboundary map coming from the sequence

$$0 \to G(N) \to G(2N) \to G(N) \to 0$$
$$\mathrm{Hom}(G(N),G_a) \xrightarrow{\delta} \mathrm{Ext}^1(G(N),G_a)$$

is an isomorphism.

Proof: The proof of (2.1) shows that the map is injective. Surjectivity is equivalent to the assertion that $\mathrm{Ext}^1(G(N),G_a) \to \mathrm{Ext}^1(G(2N),G_a)$ is the zero map. To see that this is true note that the groups in question are by the appendix (functorially) isomorphic to $\mathrm{Ext}^1(\mathcal{L}.^{G(N)*},G_a)$ (resp. $\mathrm{Ext}^1(\mathcal{L}.^{G(2N)*},G_a)$). But by [16, II, 3.3.10] this map is zero.

(3.2) Corollary. The map $\mathrm{Ext}^1(G,G_a) \to \mathrm{Ext}^1(G(N)\ G_a)$ is an isomorphism.

Proof: Consider the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & G(N) \arrow[r] \arrow[d, equal] & G(2N) \arrow[r] \arrow[d, hook] & G(N) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & G(N) \arrow[r] & G \arrow[r, "p^N"] & G \arrow[r] & 0
\end{tikzcd}$$

Since the connecting homomorphism is functorial there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(G(N),G_a) \arrow[r, "\delta", "\cong"'] \arrow[d, equal] & \mathrm{Ext}^1(G,G_a) \arrow[d] \\
\mathrm{Hom}(G(N),G_a) \arrow[r, "\delta", "\cong"'] & \mathrm{Ext}^1(G(N),G_a)
\end{tikzcd}$$

Three sides being isomorphisms, the corollary is established.

## §4. ON THE EXISTENCE OF $\natural$-STRUCTURES

Let $T$ be any scheme, $H$ a commutative group scheme on $T$. Fix an extension of $H$ by a smooth commutative group scheme $L$ (in practice $L = G_m$ or $G_a$).

$$(4.1) \qquad 0 \to L \to E \to H \to 0$$

Given a $\natural$-structure on this extension we can modify it by adding an element of $\Gamma(T, \underline{\omega}_H \otimes \underline{\mathrm{Lie}}(L))$ to obtain a new $\natural$-structure on the extension. Conversely if we have two $\natural$-structures on the extension then their difference is an element of $\Gamma(T, \underline{\omega}_H \otimes \underline{\mathrm{Lie}}(L))$.

We denote by $\mathrm{Hom}^{\nabla}(H,L)$ the subgroup of $\mathrm{Hom}(H,L)$ consisting of the maps $\Phi: H \to L$ with $d\Phi = 0 \in \Gamma(\underline{\omega}_H \otimes \underline{\mathrm{Lie}}(L))$. For an arbitrary $\Phi: H \to L$ the automorphism of the trivial extension

$$0 \to L \to L \times H \to H \to 0$$

corresponding to $\Phi$, transforms the trivial $\natural$-structure into the $\natural$-structure given by $d\Phi$. This discussion explains why the following sequence is exact:

$$(4.2) \quad 0 \to \mathrm{Hom}^{\nabla}(H,L) \to \mathrm{Hom}(H,L) \to \Gamma(\underline{\omega}_H \otimes \underline{\mathrm{Lie}}(L)) \to \mathrm{Ext}^{\natural}(H,L) \to \mathrm{Ext}^1(H,L)$$

We can also pass to sheaves for the flat topology to obtain the sequence

$$(4.3) \quad 0 \to \underline{\mathrm{Hom}}^{\nabla}(H,L) \to \underline{\mathrm{Hom}}(H,L) \to \underline{\omega}_H \otimes \underline{\mathrm{Lie}}(L) \to \underline{\mathrm{Ext}}^{\natural}(H,L) \to \underline{\mathrm{Ext}}^1(H,L)$$

(4.4) Lemma: Assume $T$ is affine and let

$$0 \to L \to E \to H \to 0$$

be an extension which defines the zero section of $\Gamma(T, \underline{\mathrm{Ext}}^1(H,L))$.

Then this extension carries a $\natural$-structure.

Proof: For variable $T'/T$ consider the $\natural$-structures on the restriction of the extension to $T'$. As noted above we obtain in this way a sheaf which is formally principal homogeneous under $\underline{\omega}_H \otimes \underline{\mathrm{Lie}}(L)$ . By assumption, locally this sheaf has sections, and hence the quasi-coherence of $\underline{\omega}_H \otimes \underline{\mathrm{Lie}}(L)$ implies (since $T$ is affine) that it has a global section.

(4.5) Remark: The lemma can be explained "geometrically" as follows: By assumption our extension is a torseur under $\underline{\mathrm{Hom}}(H,L)$. Let $\{U_i\}$ be a cover of $T$ on which it is trivial and $\Phi_{ij} \in \Gamma(U_i \cap U_j, \underline{\mathrm{Hom}}(H,L))$ a corresponding cocycle. Since the cocycle $(\mathbf{d}\Phi_{ij})$ is a coboundary we can find $\natural$-structures $\rho_i$ on the trivial extension over $U_i$ such that $\rho_i - \rho_j = d\Phi_{ij}$. Thus $\Phi_{ij}$ is an isomorphism of $\natural$-extensions over $U_i \cap U_j$ and by gluing we obtain a $\natural$-structure on our original extension.

(4.6) Remark: Let $H$ be finite and locally-free and $L = G_m$ . Since $\underline{\mathrm{Ext}}^1(H,G_m) = (0)$ it follows that (if $T$ is affine) any extension of $H$ by $G_m$ has a $\natural$-structure.

(4.7) The following discussion will be used in the proof of (4.12) below. Let $T$ be a scheme, and $X$ an arbitrary $T$-scheme. Let $T[\varepsilon]$ be the scheme of dual numbers over $T$ , $X[\varepsilon] =_{\mathrm{dfn.}} X \underset{T}{\times} T[\varepsilon], \pi_X : X[\varepsilon] \to X$ the structural map. On $X$ there is an exact sequence of sheaves:

$$0 \to G_a \to \pi_* G_m \to G_m \to 0$$

Corresponding to this sequence there is an "exact sequence" (of Picard categories) [7][15]:

$$0 \to \mathrm{TORS}(X,G_a) \to \mathrm{TORS}(X[\varepsilon],G_m) \to \mathrm{TORS}(X,G_m) \to 0$$

This means that we have an equivalence of categories, compatible with the "addition laws"

$$\text{(4.8)} \qquad \mathrm{TORS}(X,G_a) \xrightarrow{\approx} \begin{array}{l}\text{category of pairs } (P,\Phi) \text{ where } P \text{ is a} \\ G_m\text{-torseur on } X[\varepsilon] \text{ and } \Phi{:}P|X \to G_m \text{ is} \\ \text{an isomorphism of } G_m \text{ torseurs on } X\end{array}$$

This equivalence is functorial in the T-scheme $X$. Let us denote the above category of pairs by $\mathrm{TORS}(\,T[\varepsilon]/T;\ X[\varepsilon],G_m)$. Because a $\natural$-torseur $P$ on $X$ under $G_a$ is the torseur $P$, plus the additional structure of an isomorphism of torseurs $\nabla{:}\pi_1^*(P) \xrightarrow{\sim} \pi_2^*(P)$, satisfying the condition $\Delta^*(\nabla) = \mathrm{id}_P$ (where $\pi_1,\pi_2{:}\ \Delta^1(X) \to X$ are the projections), it follows from the functorial nature of the above equivalence of categories that there is an induced equivalence:

$$\text{(4.9)} \qquad \mathrm{TORS}^\natural(X,G_a) \xrightarrow{\approx} \mathrm{TORS}^\natural(T[\varepsilon]/T\ ;\ X[\varepsilon],\ G_m)$$

where the category on the right has as objects those pairs $(P,\Phi)$ with $P$ a $\natural$-torseur and $\Phi$ a horizontal isomorphism.

Let $G$ be any T-group scheme. Extensions are torseurs $P$, plus isomorphisms

$$s^*(P) \xrightarrow{\sim} p_1^*(P) \wedge p_2^*(P)$$

satisfying the commutative diagram (1.1.4.1) and (1.2.1) of [11, SGA 7, Exposé VII]. Thus the functorial nature of (4.8) implies that it induces an equivalence:

(4.10) $\quad \mathrm{EXT}(G,G_a) \xrightarrow{\approx} \mathrm{EXT}(T[\varepsilon]/T;\ G[\varepsilon],\ G_m)$

Combining (4.9) and (4.10) there is an equivalence of categories

(4.11) $\quad \mathrm{EXT}^{\natural}(G,G_a) \xrightarrow{\approx} \mathrm{EXT}^{\natural}(T[\varepsilon]/T;\ G[\varepsilon],\ G_m)$

(4.12) Proposition: Assume $T$ is affine, $H$ a finite locally-free $T$-group. Any extension of $H$ by $G_a$ admits a $\natural$-structure.

Proof: Fix an extension $E$. View $E$ via (4.10) as an extension of $H[\varepsilon]$ by $G_m$ together with a trivialization, $\Phi$, of the restriction of this extension to $T$. By (4.11), $\natural$-structures on $E$ are the same as $\natural$-structures on $E$ (thought of as an extension of $H[\varepsilon]$ by $G_m$) which satisfy the additional property that $\Phi$ is horizontal.

Via $\Phi$ we transport the trivial $\natural$-structure on $H \times G_m$ to $E|T$ to obtain a $\natural$-structure $\nabla_0$. Because $H$ is finite and locally free we can speak of the torseur (under $\underline{\omega}_{H[\varepsilon]}$) of $\natural$-structures on $E$. Denote it by $\mathcal{J}$ and denote by $\mathcal{J}_0 (= \mathcal{J}/T)$ the torseur under $\underline{\omega}_H$ of $\natural$-structures on $E|T$. Since $T$ is affine, we can choose an isomorphism $\underline{\omega}_{H[\varepsilon]} \xrightarrow{\sim} \mathcal{J}$, whence an induced isomorphism $\underline{\omega}_H \xrightarrow{\sim} \mathcal{J}_0$. Viewing $\nabla_0$ as an element in $\Gamma(\underline{\omega}_H)$, the (obvious) fact that $\Gamma(\underline{\omega}_{H[\varepsilon]}) \to \Gamma(\underline{\omega}_H)$ is surjective shows that $E$ has a $\natural$-structure lifting $\nabla_0$ and completes the proof of the proposition.

## §5. RELATION BETWEEN EXT$^\natural$ AND $\epsilon$-HOM$^\natural$

(5.1) Proposition: Let $n \geq 2N$. The natural homomorphism $(\epsilon_{n,n})\text{-Hom}^\natural(G(n),G_a) \to \text{Ext}^\natural(G(n),G_a)$ is an isomorphism.

Proof: Consider the following commutative diagram:

(5.2)
$$\begin{tikzcd}
0 \arrow[r] & \underline{\omega}_{G(n)} \arrow[r] \arrow[d, equal] & (\epsilon_{n,n})\text{-Hom}^\natural(G(n),G_a) \arrow[r] \arrow[d] & \text{Hom}(G(n),G_a) \arrow[r] \arrow[d, "\delta", "\wr"'] & 0 \\
0 \arrow[r] & \underline{\omega}_{G(n)} \arrow[r] & \text{Ext}^\natural(G(n),G_a) \arrow[r] & \text{Ext}(G(n),G_a) \arrow[r] & 0
\end{tikzcd}$$

Here $\delta$ is the coboundary map which was shown above to be an isomorphism in (3.1). The result will follow once it is shown that $\underline{\omega}_{G(n)} \to \text{Ext}^\natural(G(n),G_a)$ is injective. To do this we must show that the map

(5.4)
$$\text{Hom}(G(n),G_a) \to \underline{\omega}_{G(n)}$$

occuring in (4.2) is the zero map. Consider the sequence

$$0 \to G(n) \to G(2n) \to G(n) \to 0$$

It has been shown in the proof of (2.1) that $\text{Hom}(G(2n),G_a) \to \text{Hom}(G(n),G_a)$ is the zero map, and has been shown in [16,II 3.3.20] that $\underline{\omega}_{G(2n)} \to \underline{\omega}_{G(n)}$ is an isomorphism. Thus (5.4) is the zero map and the proposition is proved.

(5.5) Remark: The proposition probably remains true assuming only $n \geq N$. What must be shown is that (5.4) is the zero map under this weaker assumption. For $N = 1$, it is very easy to show this.

## §6. CRYSTALLINE EXTENSIONS AND $\natural$-EXTENSIONS

(6.1) Here we recall Grothendieck's definition of generalized extensions, and then we specialize the notion to arrive at the definition of crystalline extension.

Crystalline extensions will be used in showing that Lie(E(G*)) is "crystalline in nature."

We shall constantly work with the following structure:

(6.2) Fix a scheme $T$, and $G$ a commutative $T$-group. For each $T$-scheme $T'$ let $\mathcal{T}_{T'}$ = category of $G_{T'}$-torseurs. The usual contracted product of $G_{T'}$-torseurs yields a functor

$$\mathcal{T}_{T'} \times \mathcal{T}_{T'} \to \mathcal{T}_{T'} .$$

This structure is an example of a fibered category $\mathcal{T}$ on $\mathcal{C}$ = Sch/T which is fibered in strictly commutative Picard categories [7,15 ].

If $\mathcal{T}$ is any fibered category in strictly commutative Picard categories over $\mathcal{C}$ (any category), and $H$ any commutative group in $\mathcal{C}$, we may define the notion of $\mathcal{T}$-extension of $H$:

(6.3) Definition: An $\mathcal{T}$-extension of $H$ is an object $P$ of $\mathcal{T}_H$ equipped with an isomorphism $s^*(P) \simeq p_1^*(P) \wedge p_2^*(P)$ such that the analogues of the usual diagrams (expressing the associativity and commutativity of the composition law) are commutative. If products do not exist in $\mathcal{C}$, the definition is modified by requiring that for every pair of points $p_1, p_2 \colon X \to H$ we be given an isomorphism $(p_1+p_2)^*(P) \simeq p_1^*(P) \wedge p_2^*(P)$ satisfying the usual conditions as discussed in [11, SGA$_7$VII].

These extensions form a category EXT$(H,\mathcal{T})$ whose morphisms are the morphism $\Phi: P \to Q$ in $\mathcal{T}_H$ such that the following diagram commutes

$$\begin{tikzcd}
s^*(P) \arrow[r, "\sim"] \arrow[d, "s^*(\Phi)"'] & p_1^*(P) \wedge p_2^*(P) \arrow[d, "p_1^*(\Phi)\wedge p_2^*(\Phi)"] \\
s^*(Q) \arrow[r, "\sim"] & p_1^*(Q) \wedge p_2^*(Q)
\end{tikzcd}$$

The functor $\wedge: \mathcal{T} \times_{\mathcal{C}} \mathcal{T} \to \mathcal{T}$ induces a composition law on the category $\mathrm{EXT}(H,\mathcal{T})$. Passing to isomorphism classes of objects we obtain a commutative group $\mathrm{Ext}(H,\mathcal{T})$. Finally the category $\mathrm{EXT}(H,\mathcal{T})$ varies functorially with $H$ and $\mathcal{T}$.

(6.4) We shall give several examples which illustrate the above.

(6.5) $\mathcal{C} = (\mathrm{Sch}/T)$ , $\mathcal{T}$ = G-torseurs,

$\mathrm{EXT}(H,\mathcal{T})$ is in a natural way equivalent to $\mathrm{EXT}(H,G)$.

(6.6) $\mathcal{C} = (\mathrm{Sch}/T)$ , $G$ a smooth T-group, $\mathcal{T} = \mathcal{G}$-torseurs under $G$. $\mathrm{EXT}(H,\mathcal{T})$ is in a natural way equivalent to $\mathrm{EXT}^{\mathcal{G}}(H,G)$.

(6.7) Let $(T,I,\gamma)$ be a divided power scheme, i.e. $I \subset \mathcal{O}_T$ is endowed with divided powers. Let $(\mathrm{Sch}/T)' = \mathcal{C}$ be the full sub-category of $\mathrm{Sch}/T$ consisting of those $X \to T$ such that the divided powers on $I$ extend to $X$. Fix a smooth commutative T-group $G$(e.g. $G = G_a$ or $G = G_m$). For any $X$ in $(\mathrm{Sch}/T)'$ let $G_X$ be the sheaf of groups on $\mathrm{Crys}(X/T,I,\gamma)$, cf[3], defined by

$$\Gamma((U,T',\delta),G_X) = G(T') = \mathrm{Hom}_T(T',G) \quad .$$

If $f: X' \to X$ is a morphism in $(\mathrm{Sch}/T)'$, then there is an induced map $f^*_{\mathrm{crys}}(G_X) \to G_{X'}$ . This allows us to define the fiber of $\mathcal{J}$ at $X$, $\mathcal{J}_X$, to be $\mathrm{TORS}(\mathrm{Crys}(X/T,I,\gamma),G_X)$, the category of torseurs on the crystalline site of $X$ with structural group $G_X$. The operation $\wedge$ is just the usual contracted product of torseurs. Since morphisms between torseurs are necessarily isomorphism this category admits an alternative description: It is equivalent to the category of crystals in (small Zariski) G-torseurs, i.e. crystals for the stack $(U,T',\delta) \mapsto (G_X)_{(U,T',\delta)}$-torseurs. If $H$ is a group in $(\mathrm{Sch}/T)'$, we denote the category of extensions of $H$ by $\mathcal{J}$ by $\mathrm{EXT}^{\mathrm{crys}/T}(H,G)$ and refer to it as the <u>category of crystalline extensions</u>.

(6.8) <u>Remarks</u>: (i) When $G = G_m$, $\mathrm{TORS}(\mathrm{Crys}(X/T,I,\gamma),G_m)$ is equivalent to the category of invertible modules on $\mathrm{Crys}(X/T,I,\gamma)$.

(ii) Where $G = G_a$, $\mathrm{TORS}(\mathrm{Crys}(X/T,I,\gamma),G_a)$ is equivalent to the category $\mathrm{EXT}_{\mathcal{O}_{X_{\mathrm{crys}}}}(\mathcal{O}_{X_{\mathrm{crys}}},\mathcal{O}_{X_{\mathrm{crys}}})$

(iii) Although the localization allowed in $\mathrm{Crys}(X/T,I,\gamma)$ is quite coarse this will not be bothersome since for the groups $G_m$ and $G_a$ Zariski torseurs are the same as (say) f.p.p.f. torseurs. When we do use $G_m$, the torseurs we'll consider will in fact have sections over closed sub-schemes defined by nilpotent ideals (c.f. §11). Because, previously, "torseur" was used with reference to one of the large sites (ZARISKI, ETALE, F.P.P.F.: for $G_m$ and $G_a$ the notions coincide) we recall how to pass from torseurs on the small site to torseurs on the large one. For simplicity let's work in the Zariski topology. For any scheme $Y$ there are two morphisms of topoi: $p : Y_{ZAR} \to Y_{zar}$, $r: Y_{zar} \to Y_{ZAR}$. The morphism $p$ is defined by $\Gamma(Z,p^*(F)) = \Gamma(Z,g^*(F))$, if $g:Z \to Y$ and $F$ is an ordinary Zariski sheaf on $Y$. The morphism $r$ is defined simply by restricting a sheaf $\mathcal{F}$ on the large Zariski site to the sub-category of opens of $Y$.

The functor

$$P \mapsto p^*(P) \overset{p^*(\mathcal{O}_Y^*)}{\wedge} G_m$$

establishes an equivalence between $G_m$-torseurs on the small and large sites of $Y$ (similarly for $G_a$-torseurs). The functoriality of this equivalence follows from that of the morphisms $p$ in a straightforward manner.

(iv) Given $X/T$, there are functors

$$(6.9) \quad \mathrm{TORS}^{crys/T}(X,G_m) \to \mathrm{TORS}^{\natural}(X,G_m)$$

$$(6.10) \quad \mathrm{TORS}^{crys/T}(X,G_a) \to \mathrm{TORS}^{\natural}(X,G_a)$$

If $X/T$ is smooth and $\mathrm{Crys}(X/T)$ is replaced by the nilpotent crystalline site, then (6.9) is an equivalence of categories [2]. Using the fact that the "standard" connection of $\mathcal{O}_X$ is nilpotent together with the interpretation of an object in $\mathrm{TORS}^{\natural}(X,G_a)$ as a short exact sequence of modules with integrable connection:

$$(6.11) \qquad 0 \to \mathcal{O}_X \to \mathcal{M} \to \mathcal{O}_X \to 0$$

we see that (when $X/T$ is smooth) (6.10) is an equivalence of categories.

(6.9) and (6.10) are functorial in $X$. Furthermore they are compatible with the "composition laws" with which both source and target are endowed. Let $H$ be a T-group such that

$H$, $H\times_T H$ , $H\times_T H\times_T H$ all belong to $(\mathrm{Sch}/T)'$ (e.g. $H/T$ flat, $I$ principal). There are induced functors (compatible with the composition laws):

$$(6.12) \qquad \mathrm{EXT}^{\mathrm{crys}/T}(H,G_m) \rightarrow \mathrm{EXT}^{\not S}(H,G_m)$$

$$(6.13) \qquad \mathrm{EXT}^{\mathrm{crys}/T}(H,G_a) \rightarrow \mathrm{EXT}^{\not S}(H,G_a) \text{ .}$$

If $H/T$ is smooth and we restrict to the nilpotent crystalline site (resp. no restriction) then (6.12) (resp.(6.13)) is an equivalence of categories.

(6.14) We shall need one last example of generalized extensions. Let $(T,I,\gamma)$ be as above and let $T_o = \mathrm{Var}(I)$. Let $\mathcal{C} = (\mathrm{Sch}/T_o)$. Let $G$ be a smooth commutative $T$-group and define $\mathcal{T}$ exactly as in (6.7), i.e. $\mathcal{T}_X$= category of $G_X$-torseurs on $\mathrm{Crys}(X/T,I,\gamma)$ for any $T_o$-scheme $X$ . If $H$ is a group in $\mathcal{C}$ we shall denote the category $\mathrm{EXT}(H,\mathcal{T})$ by $\mathrm{EXT}^{\mathrm{crys}/T}(H/T_o,G)$, and if it is clear that $H$ is a $T_o$-group we shall drop the "$T_o$" from the notation.

(6.15) <u>Remarks</u>: (i) The reason for distinguishing between (6.7) and (6.14) is that a $T_o$-group scheme is never a $T$-group scheme.

(ii) If $T'$ is a closed subscheme of $T_o$ and $\mathcal{C} = (\mathrm{Sch}/T')$, then with $\mathcal{T}$ as in (6.14) there is the category $\mathrm{EXT}^{\mathrm{crys}/T}(H/T',G)$. This category differs from that of (6.14) since (because the ideal of $T'$ in $T$ need not have divided powers) even if $H$ can be lifted to $T$ , the category of crystalline extension of a lifting can be different from this category.

(6.16) Let us indicate the functorial variation of examples (6.7) and (6.14) when $(T,I,\gamma)$ varies. Let $(T',I',\gamma') \to (T,I,\gamma)$ be a divided power morphism. First assume $X$ is a flat T-scheme, $X'$ a flat $T'$-scheme, and assume we are given a commutative diagram

$$\begin{tikzcd}
X' \arrow[r] \arrow[d] & X \arrow[d] \\
(T',I',\gamma') \arrow[r] & (T,I,\gamma)
\end{tikzcd}$$

Let $G$ be a smooth T-group, $G' = G \times_T T'$ . Since crystalline torseurs are crystals the general procedure for taking the inverse image of a crystal [3, IV 1.2; or 16, III, 3.8] permits us to define a functor

(6.17) $$\mathrm{TORS}^{crys/T}(X,G) \to \mathrm{TORS}^{crys/T'}(X',G')$$

This functor varies functorially with $(X',X)$. In particular if $H$ is a flat T-group, $H' = H \times_T T'$, there is an induced functor

(6.18) $$\mathrm{EXT}^{crys/T}(H,G) \to \mathrm{EXT}^{crys/T'}(H',G')$$

If we assume $X$ (resp $X'$) is a $T_o$(resp. $T_o'$) scheme, the map (6.17) is still defined. Furthermore, if $H$ is a $T_o$ group, $H' = H \times_{T_o} T_o'$, there is an induced functor

$$\mathrm{EXT}^{crys/T}(H/T_o,G) \to \mathrm{EXT}^{crys/T'}(H'/T_o',G') .$$

## §7. THE CRYSTALLINE NATURE OF $\mathrm{EXT}^{\natural}(-,G_a)$

Here we let $S$ be a scheme on which $p$ is locally nilpotent. and let $(I,\gamma)$ be a divided power ideal of $\mathcal{O}_S$. Let $G$ be a Barsotti-Tate group on $S$. The inclusions $G(n) \hookrightarrow G(n+1)$ induce functors

$$\mathrm{EXT}^{\mathrm{crys}/S}(G(n+1),G_a) \to \mathrm{EXT}^{\mathrm{crys}/S}(G(n),G_a)$$

$$\mathrm{EXT}^{\natural}(G(n+1),G_a) \longrightarrow \mathrm{EXT}^{\natural}(G(n),G_a)\ .$$

By passage to $\varprojlim$,[9], we obtain from (6.10) a functor

(7.1) $$\varprojlim \mathrm{EXT}^{\mathrm{crys}/S}(G(n),G_a) \to \varprojlim \mathrm{EXT}^{\natural}(G(n),G_a)\ .$$

(7.2) Theorem: The functor (7.1) is an equivalence of categories.

Proof: Note that (7.1) is induced by the functor

(7.3) $$\varprojlim \mathrm{TORS}^{\mathrm{crys}/S}(G(n),G_a) \to \varprojlim \mathrm{TORS}^{\natural}(G(n),G_a).$$

Since the category of crystalline extensions (resp. $\natural$-extensions) is defined as consisting of crystalline (resp. $\natural$) torseurs, $P$, endowed with an isomorphism $s^*(P) \simeq p_1^*(P) \wedge p_2^*(P)$ (satisfying the associativity condition) and since the functor (7.3) is itself functorial with respect to the Barsotti-Tate group, $G$; it suffices to show that (7.3) is an equivalence of categories. Assuming momentarily (7.7) below, we shall show that (7.3) is faithful, full and essentially surjective.

1) faithful. Let $(\Phi_n),(\Psi_n)$ be two morphisms between the object $(P_n)$ and $(Q_n)$ of $\varprojlim \mathrm{TORS}^{\mathrm{crys}/S}(G(n),G_a)$. Assuming $(\Phi_{n_{G(n)} \xrightarrow{id} G(n)}) = (\Psi_{n_{G(n)} \xrightarrow{id} G(n)})$, we must show $(\Phi_n) = (\Psi_n)$.

Fix an $n$ and let $(U\hookrightarrow T,J,\delta)$ be an object of the crystalline site $\mathrm{Crys}(G(n)/S,I,\gamma)$. Obviously it is permissible to assume $T$ is affine. By (7.7) below we can find for $m$ sufficiently large a commutative diagram

(7.4)
$$\begin{tikzcd}
U \arrow[r, hook] \arrow[d, phantom, "\cap"] & T \arrow[ddl, dashed, "f"] \\
G(n) \arrow[d, hook] & \\
G(n+m) &
\end{tikzcd}$$

Let us use a vertical bar "|" to denote restriction (or more properly inverse image). By hypothesis there are commutative diagrams:

(7.5)
$$\begin{tikzcd}
P_n \arrow[r, dash, "\sim"] \arrow[d, "\Phi_n"'] & P_{n+m}|G(n) \arrow[d, "\Phi_{n+m}|G(n)"] \\
Q_n \arrow[r, dash, "\sim"] & Q_{n+m}|G(n)
\end{tikzcd}$$

$$\begin{tikzcd}
P_n \arrow[r, dash, "\sim"] \arrow[d, "\Psi_n"'] & P_{n+m}|G(n) \arrow[d, "\Psi_{n+m}|G(n)"] \\
Q_n \arrow[r, dash, "\sim"] & Q_{n+m}|G(n)
\end{tikzcd}$$

But by definition of the inverse image of a crystal [3,IV 1.2, or 16, III, 3.8] we have

$$(\Phi_{n+m}|G(n))_{U\hookrightarrow T} = f^*(\Phi_{n+m_{G(n+m)\xrightarrow{id}G(n+m)}})$$

and similarly for $(\Psi_{n+m}|G(n))_{U\hookrightarrow T}$. Hence the commutativity of the diagrams (7.5) allow us to conclude $(\Phi_n)_{U\hookrightarrow T} = (\Psi_n)_{U\hookrightarrow T}$ .

2) <u>full</u>: Here it will be convenient to denote the image of an object $(P_n)$(resp. on arrow $(\Phi_n)$) of $\underleftarrow{\mathrm{Lim}}\ \mathrm{TORS}^{\mathrm{crys}/S}(G(n),G_a)$ under (7.3) by $(\overline{P}_n)$(resp.$(\overline{\Phi}_n)$). Let $(\sigma_n): (\overline{P}_n) \to (\overline{Q}_n)$ be a morphism in $\underleftarrow{\mathrm{Lim}}^{♄}\mathrm{TORS}\ (G(n),G_a)$. We must show that there is a morphism $(\Phi_n): (P_n) \to (Q_n)$ in $\underleftarrow{\mathrm{Lim}}\ \mathrm{TORS}^{\mathrm{crys}/S}(G(n),G_a)$ with $(\overline{\Phi}_n) = (\sigma_n)$ . Just as in the proof of faithfulness above, we fix an $n$ and an object $(U \hookrightarrow T,J,\delta)$ of $\mathrm{Crys}(G(n)/S,I,\gamma)$. Using diagram (7.4) we define $\Phi_{n_{U\hookrightarrow T}}$ to be the map obtained via transport of structure using the isomorphisms $P_n \simeq P_{n+m}|G(n)$ and $Q_n \simeq Q_{n+m}|G(n)$ from $f^*(\sigma_{n+m})$. It must be shown that this definition is independent of the choice of $f:T \to G(n+m)$, a lifting of $U \subseteq G(n) \hookrightarrow G(n+m)$. Let $f_1,f_2$ be two liftings. By definition of the divided power neighborhood [3,I 4.32] of $\Delta: G(n+m) \to G(n+m) \times_S G(n+m)$, there is a map $\hat{f}: T \to D(2)_{G(n+m)/S}$ with $p_1 \circ \hat{f} = f_1$ , $p_2 \circ \hat{f} = f_2$ . Augmenting $m$ if necessary we can assume that $\Omega^1_{G(n+m)/S}$ is locally-free of finite rank [16,II 3.3.20]. Since a ♄-torseur under $G_a$ can be interpreted as an exact sequence of modules with integrable connection, it follows from [3,II,4.3.4,4.3.10 ] that $p_1^*(\sigma_{n+m})$ is identifiable with $p_2^*(\sigma_{n+m})$, once we identify $p_i^*(P_{n+m})$ with $P_{n+m_{G(n+m)\hookrightarrow D(2)_{G(n+m)/S}}}$ (and similarly for $p_i^*(Q_{n+m})$). Hence we can identify $f_1^*(\sigma_{n+m})$ and $f_2^*(\sigma_{n+m})$ with $\hat{f}^*(p_1^*(\sigma_{n+m})) = \hat{f}^*(p_2^*(\sigma_{n+m}))$. This shows our definition of $\Phi'_{n_{U\hookrightarrow T}}$ is independent of the choice of lifting and completes the proof that (7.3) is full.

3) <u>essentially surjective</u>: The proof here is quite similar to the proof of fullness above. Given an object $(P'_n)$ in $\underleftarrow{\mathrm{Lim}}\ \mathrm{TORS}^{♄}(G(n),G_a)$, we obtain $(P_n)$ in $\underleftarrow{\mathrm{Lim}}\ \mathrm{TORS}^{\mathrm{crys}/S}(G(n),G_a)$

by defining $P_{n_{U\hookrightarrow T}}$ to be $f^*(P'_{n+m})$, where $f$ is <u>any</u> morphism making (7.4) commutative. The fact that this definition makes sense and yields an object $(P_n)$ such that $(\bar{P}_n) \cong (P'_n)$ follows by again invoking the above cited results in Berthelot's thesis [3,II 4.3.4, 4.3.10].

(7.6) <u>Remark</u>: The proof of faithfulness is valid if $G_a$ is replaced by any smooth commutative S-group. For fullness and essential surjectivity the interpretation of $G_a$-torseurs as extensions of $\mathcal{O}$ by $\mathcal{O}$ (and hence as modules with additional structure) was necessary in order to apply the results in [2,3]. But if we modify the target by replacing $\mathrm{TORS}^{\text{ϟ}}(G(n),G_m)$ by $\mathrm{TORS}^{\mathrm{nil}\ -\text{ϟ}}(G(n),G_m)$ (i.e. the category of line bundles endowed with a nilpotent integrable connection) or if we modify the source by using the nilpotent crystalline site (s) , then the above proof carries over to yield equivalences

(7.6.1) $$\varprojlim \mathrm{EXT}^{\mathrm{crys}/S}(G(n),G_m) \xrightarrow{\approx} \varprojlim \mathrm{EXT}^{\mathrm{nil}\ -\text{ϟ}}(G(n),G_m)$$

(7.6.2) $$\varprojlim \mathrm{EXT}^{\mathrm{Nil\text{-}crys}/S}(G(n),G_m) \xrightarrow{\approx} \varprojlim \mathrm{EXT}^{\text{ϟ}}(G(n),G_m)$$

In the course of the above proof, use was made of:

(7.7) <u>Lemma</u>: Let $G$ be a Barsotti-Tate group on $S$ . $G$ is formally smooth for nilimmersions (i.e. if $X$ is an (absolutely) affine scheme over $S$, and $X_o$ is a closed sub-scheme defined by an ideal in which every element is nilpotent, then any morphism $X_o \to G$ can be lifted to $X$ ).

Proof: Let $(X,X_o)$ be as in the above explication. Write $X = \mathrm{Spec}(A)$, $X_o = \mathrm{Spec}(A/I)$. For $\lambda \in L =$ (set of finite subsets of $I$), let $I_\lambda$ be the finitely generated sub-ideal of $I$ generated by $\lambda$, and let $X_\lambda = \mathrm{Spec}(A/I_\lambda)$. Since $X_o$ is affine, the map $X_o \to G$ factors through some $G(n)$. Because $G(n)$ is locally of finite presentation over $S$ and $X_o = \varprojlim X_\lambda$, it follows from [10,$\mathrm{EGA}_{IV}$8.13.1] that $X_o \to G(n)$ can be lifted to $X_\lambda \hookrightarrow G(n)$ (for some $\lambda$). But $X_\lambda \hookrightarrow X$ is a nilpotent immersion. The result now follows since Barsotti-Tate groups are formally smooth [16,II,3.3.13].

(7.8) Corollary (of 7.2):The category $\varprojlim \mathrm{EXT}^{crys/S}(G(n),G_a)$ is rigid.

Proof: By (7.2) this category is equivalent to $\varprojlim \mathrm{EXT}^{\natural}(G(n),G_a)$. The automorphism group of the zero object ($G(n)\times G_a$, trivial connection) of this category consists of compatible families of homomorphisms $g_n\colon G(n) \to G_a$ with $dg_n = 0$. But $(g_n) \in \varprojlim \mathrm{Hom}(G(n),G_a) = \mathrm{Hom}(G,G_a) = (0)$ and hence each $g_n$ is zero.

(7.9) Let us denote by $\mathrm{EXT}^{crys/S}(G,G_a)$ the category $\varprojlim \mathrm{EXT}^{crys/S}(G(n),G_a)$. Similarly we write $\mathrm{EXT}^{\natural}(G,G_a)$(resp. $\mathrm{TORS}^{\natural}(G,G_a)$, $\mathrm{EXTRIG}(G,G_a)$,... for the categories $\varprojlim \mathrm{EXT}^{\natural}(G(n),G_a)$(resp. $\varprojlim \mathrm{TORS}^{\natural}(G(n),G_a)$, $\varprojlim \mathrm{EXTRIG}(G(n),G_a)$,...). Finally we write $\mathrm{Ext}^{crys/S}(G,G_a)$, etc. for the abelian group of isomorphism classes of objects of $\mathrm{EXT}^{crys/S}(G,G_a)$, etc. Observe that the action of $\Gamma(S,\mathcal{O}_S)$ on $G_a$ gives $\mathrm{Ext}^{crys/S}(G,G_a)$ a module structure.

Having introduced all this notation we can state the following immediate consequence of (7.8):

(7.10) <u>Corollary</u>: The (small) Zariski presheaf on $S$ $U \mapsto \mathrm{Ext}^{\mathrm{crys}/U}(G|U,G_a)$ is a sheaf of $\mathcal{O}_S$-modules.

(7.11) Let us denote this $\mathcal{O}_S$-module by $\underline{\mathrm{Ext}}^{\mathrm{crys}/S}(G,G_a)$. The following proposition tells us that $\underline{\mathrm{Ext}}^{\mathrm{crys}/S}(G,G_a)$ is canonically isomorphic with $\underline{\mathrm{Lie}}(E(G^*))$, the tangent space of the universal extension of the Cartier dual of $G$.

(7.12) <u>Proposition</u>: Assume $S$ is affine and $p^N$ kills $S$. The natural map $\mathrm{Ext}^{\mathrm{crys}/S}(G,G_a) \to \mathrm{Ext}^{\natural}(G(n),G_a)$ is an isomorphism provided $n \geq 2N$.

<u>Proof</u>: By (7.2) we may replace the source by $\mathrm{Ext}^{\natural}(G,G_a)$. Let $(P_i)$ represent an element in $\mathrm{Ext}^{\natural}(G,G_a)$. To demonstrate injectivity we must show that

$$\begin{gathered}[P_n \cong \text{trivial } \natural\text{-extension of } G(n) \text{ by } G_a] \\ \Downarrow \\ [(P_i) \cong \text{trivial } \natural\text{-extension of } G \text{ by } G_a]\end{gathered}$$

Let $0_i$ denote the trivial $\natural$-extension of $G(i)$ by $G_a$. We are to produce for each $i \geq n$ an isomorphism $\theta_i\colon 0_i \xrightarrow{\simeq} P_i$ such that these form a compatible family.

Let $(\bar{P}_i)$ be the object of $\mathrm{EXT}(G,G_a)$ obtained by forgetting the $\natural$-structure on each $P_i$. Since our definition of $\mathrm{EXT}(G,G_a)$ as $\varprojlim \mathrm{EXT}(G(i),G_a)$ coincides with the usual definition as the category of extensions of fppf sheaves, it follows from (3.2) that there is a unique isomorphism $(\tau_i):(P_i)\xrightarrow{\sim}(\overline{O}_i)$. By hypothesis there is an isomorphism $\sigma\colon P_n\xrightarrow{\sim}O_n$. But $\sigma\circ\tau_n^{-1}$ is (by the proof of (5.1), where it is shown that (5.4) is the zero map) a <u>horizontal</u> automorphism of $O_n$. Hence $\tau_n$ is actually an isomorphism between $P_n$ and $O_n$ (and not only between the underlying extensions). It remains to explain why each $\tau_i$ is horizontal. Using $\tau_i$ we obtain, via transport of structure, a $\natural$ structure on $\overline{O}_i$. This corresponds to an element $\eta_i$ of $\Gamma(S,\underline{\omega}_{G(i)})$. By hypothesis $\eta_n = 0$ and since for $i \geq n$ the maps $\underline{\omega}_{G(i)} \to \underline{\omega}_{G(n)}$ are isomorphisms it follows that each $\eta_i = 0$. Thus for $i \geq n$, $\tau_i$ is horizontal and injectivity is established.

Let $R$ be a $\natural$-extension of $G(n)$ by $G_a$. To prove surjectivity we must establish the existence of an object $(P_i)$ in $\mathrm{EXT}^{\natural}(G,G_a)$ with $P_n \simeq R$. By (3.2), there is an object $(\overline{P}_i)$ in $\mathrm{EXT}(G,G_a)$ with $\overline{P}_n \simeq \overline{R}$, $\overline{R}$ being the underlying extension of $R$. Choosing an isomorphism $\phi$ between $\overline{P}_n$ and $\overline{R}$, we endow, via transport of structure, $\overline{P}_n$ with a $\natural$-structure so that $\phi$ becomes a <u>horizontal</u> isomorphism. We must endow each $\overline{P}_i (i \geq n)$ with a $\natural$-structure so that the given maps $\overline{P}_n \simeq \overline{P}_i|G(n)$ are horizontal. Via transport of structure we put a $\natural$-structure on $\overline{P}_i|G(n)$. Since $S$ is affine, (4.12) tells us that $\overline{P}_i$ has at least one $\natural$-structure. But the set of $\natural$-structures on $\overline{P}_i$ (resp. $\overline{P}_i|G(n)$) is principal homogeneous under $\Gamma(S,\underline{\omega}_{G(i)})$ (resp. $\Gamma(S,\underline{\omega}_{G(n)})$). Surjectivity now follows since the map $\Gamma(S,\underline{\omega}_{G(i)}) \to \Gamma(S,\underline{\omega}_{G(n)})$ is onto.

(7.13) Corollary: Let $p$ be locally nilpotent on $S$, $(I,\gamma)$ be a divided power ideal in $\mathcal{O}_S$, $G$ a Barsotti-Tate group on $S$. There is a (functorial in G) exact sequence

$$0 \to \underline{\omega}_G \to \underline{\mathrm{Ext}}^{\mathrm{crys}/S}(G,G_a) \to \underline{\mathrm{Ext}}(G,G_a) \to 0 \tag{7.14}$$

which is canonically identified with the sequence obtained from the universal extension of $G^*$ by taking tangent spaces. In particular $\underline{\mathrm{Ext}}^{\mathrm{crys}/S}(G,G_a)$ is a locally-free (of finite type) $\mathcal{O}_S$-module.

Proof: This follows immediately from (8.7), (3.2), (5.1) and (7.12).

(N.B.) The reader can check that our forward reference to (8.7) does not involve any logical circularity.

## §8. PASSAGE TO LIE ALGEBRAS

To apply the results of §2-§7 to the universal extension we must relate $\mathrm{Homrig}(-,G_m)$ to $\mathrm{Homrig}(-,G_a)$ and $\mathrm{Hom}^{\natural}(-,G_m)$ to $\mathrm{Hom}^{\natural}(-,G_a)$.

Consider as usual an exact sequence of finite locally free S-groups

$$(\varepsilon) \qquad 0 \to A \to B \to C \to 0$$

giving rise to the sequence

$$(8.1) \qquad 0 \to \underline{\omega}_C \to (\varepsilon)\text{-}\underline{\mathrm{Homrig}}(A,G_m) \to A^* \to 0$$

For affine $S$ , the sequence of $S$-valued points is exact. Thus the snake lemma together with a previously noted fact (passage to Lie algebra commutes with passage to associated Zariski sheaf) tells us that the corresponding sequence

$$(8.2) \qquad 0 \to \underline{\omega}_C \to \underline{\mathrm{Lie}}((\varepsilon)\text{-}\underline{\mathrm{Homrig}}(A,G_m)) \to \underline{\mathrm{Lie}}(A^*) \to 0$$

is also exact.

If we replace $G_m$ by $G_a$ we have the **analogue** of (8.1):

$$(8.3) \qquad 0 \to \underline{\omega}_C \to (\varepsilon)\text{-}\underline{\mathrm{Homrig}}(A,G_a) \to \underline{\mathrm{Hom}}(A,G_a) \to 0$$

Let $\pi\colon S[\varepsilon] \to S$ be the structural map so that there is an exact sequence on $S$

$$0 \to G_a \to \pi_*(G_m) \to G_m \to 0$$

Let $\phi : A \to G_a$ be a homomorphism and $\sigma$ be a rigidification on the resulting extension

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\Phi"'] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & G_a \arrow[r] & E \arrow[r] & C \arrow[r] & 0 \\
& & & \mathrm{Inf}(C) \arrow[u, hook] \arrow[ul, dashed, "\sigma"] &
\end{tikzcd}$$

Applying $\pi^*$ to the whole diagram and "pushing out" along the map $\pi^*(G_a) \to \pi^*(\pi_* G_m) \to G_m$, we obtain an element of $\underline{\mathrm{Lie}}((\epsilon)\text{-}\underline{\mathrm{Homrig}}(A,G_m))$. This prodedure defines a homomorphism from the extension (8.3) to (8.2), which is an isomorphism on end-groups. Hence

$$\mathrm{Lie}((\epsilon)\text{-}\underline{\mathrm{Homrig}}(A,G_m)) \cong (\epsilon)\text{-}\underline{\mathrm{Homrig}}(A,G_a)$$

(8.4) Remark: The above discussion is valid also when "Homrig" is replaced by "$\mathrm{Hom}^{ϟ}$", and hence $\underline{\mathrm{Lie}}((\epsilon)\text{-}\underline{\mathrm{Hom}}^{ϟ}(A,G_m)) \cong (\epsilon)\text{-}\underline{\mathrm{Hom}}^{ϟ}(A,G_a)$.

(8.5) Let $S$ be a scheme with $p^N \cdot 1_S = 0$ and let $G$ be a Barsotti-Tate group on $S$. The universal extension of $G^*$ by a vector group is

$$0 \to \underline{\omega}_{G(N)} \to \varinjlim (\epsilon_{N,n})\text{-}\underline{\mathrm{Hom}}^{ϟ}(G(n),G_m) \to G^* \to 0 \tag{8.6}$$

Because "$\varinjlim$" is exact and $\underline{\mathrm{Lie}}$ is defined as a kernel, it follows from the preceeding discussion that

$$\begin{aligned}
&\underline{\mathrm{Lie}}(\varinjlim(\epsilon_{N,n})\text{-}\underline{\mathrm{Hom}}^{ϟ}(G(n),G_m)) \\
&= \varinjlim \underline{\mathrm{Lie}}((\epsilon_{N,n})\text{-}\underline{\mathrm{Hom}}^{ϟ}((G(n),G_m)) \\
&= \varinjlim (\epsilon_{N,n})\text{-}\underline{\mathrm{Hom}}^{ϟ}(G(n),G_a) \\
&= (\epsilon_{N,N})\text{-}\underline{\mathrm{Hom}}^{ϟ}(G(n),G_a) \qquad \text{(by 2.1))}
\end{aligned}$$

Summarizing, we state

(8.7) *Proposition*: If $p^N$ kills $S$ and $n \geq N$, then the tangent space $\underline{\mathrm{Lie}}(E(G^*))$ is $(\epsilon_{N,n})$-$\underline{\mathrm{Hom}}^{\text{Ш}}(G(n),G_a)$.

## §9. THE CRYSTALLINE NATURE OF THE LIE ALGEBRA OF THE UNIVERSAL EXTENSION

Fix a scheme S on which p is locally nilpotent and let $G \in B.T.(S)$ , the category of Barsotti-Tate groups on S . Let us explain how to endow $\underline{\text{Lie}}(E(G))$ with a crystalline structure. More precisely we'll define a contravariant functor

$$\mathbb{D}^*: \text{B.T.}(S)^\bullet \to (\text{Crystals in locally-free modules on } S).$$

Let U be open in S and let $U \to (T,I,\gamma)$ be a divided power thickening and assume p is locally nilpotent on T . Let G be a Barsotti-Tate group over S and let G (again) denote its restriction to U .Let G' be any lifting of G to T . Using the abuse of notation indicated in (6.14), we know $\text{EXT}^{\text{crys}/T}(G,G_a) \cong \text{EXT}^{\text{crys}/T}(G',G_a)$ since reduction module a divided power ideal induces a functorial equivalence between crystals (of any species whatsoever) on $G'(n)/T$ and crystals on $G(n)/T$ . As a consequence of the work of Grothendieck and Illusie [13, 14 ] we know that, locally on T , we can find such a G'. If H' is a Barsotti-Tate group on T and $H = H' \times_T U$ then a homomorphism $u: G \to H$ induces a map $\underline{\text{Ext}}^{\text{crys}/T}(H,G_a) \to \underline{\text{Ext}}^{\text{crys}/T}(G,G_a)$ . Thus we obtain a map $f_u: \underline{\text{Lie}}(E(H'^*)) \to \underline{\text{Lie}}(E(G'^*))$. If u is an isomorphism, then $f_u$ is an isomorphism. In particular, it follows that whenever G' and G" are liftings of G to a divided power neighborhood, $\underline{\text{Lie}}(E(G'^*))$ and $\underline{\text{Lie}}(E(G''^*))$ are canonically isomorphic.

Let $V \hookrightarrow (T',I',\gamma) \xrightarrow{\Phi} U \hookrightarrow (T,I,\gamma)$ be a morphism in the crystalline site of S . If G' is a lifting of G to T then $\Phi^*(G')$ is a lifting of $G|V$ to $T'$. Thus we obtain a commutative diagram of isomorphisms

(9.1)
$$\begin{tikzcd}
\Phi^*(\underline{\mathrm{Lie}}(E(G^{*}))) \arrow[r, "\sim"] \arrow[d, "\wr"'] & \underline{\mathrm{Lie}}(E(\Phi^*(G'^*))) \arrow[d, "\wr"'] \\
\Phi^*(\underline{\mathrm{Ext}}^{crys/T}(G,G_a)) \arrow[r, "\sim"] & \underline{\mathrm{Ext}}^{crys/T'}(G|V,G_a)
\end{tikzcd}$$

Thus the functor $\mathbb{D}^*$ can be explicity defined by.

(9.2)
$$\mathbb{D}^*(G)_{U \hookrightarrow (T,I,\gamma)} = \underline{\mathrm{Ext}}^{crys/T}(G,G_a)$$

(9.3) <u>Remark</u>: The above definition of $\mathbb{D}^*$ is intrinsic, i.e. it is defined entirely in terms of $S$ (without using liftings of Barsotti-Tate groups). Liftings are used to show that $\mathbb{D}^*(G)_{U \hookrightarrow (T,I,\gamma)}$ is locally-free and to show that $\mathbb{D}^*(G)$ is a crystal rather than just a sheaf on the crystalline site.

## §10. A DEFORMATIONAL DUALITY THEOREM FOR BARSOTTI-TATE GROUPS: AN EASY CONSEQUENCE OF THE THEORY OF ILLUSIE

Let $S$ be an affine scheme on which $p^N$ is zero. Let $S_o \hookrightarrow S$ be a closed subscheme defined by the vanishing of an ideal $I \subset \mathcal{O}_S$ with $I^{k+1} = (0)$. Let $G$ be a Barsotti-Tate group on $S$. Denote by $G(S/S_o)$ the kernel of $G(S) \to G(S_o)$ and denote by $EXT(S/S_o; G, G_m)$ the category of extensions of $G$ by $G_m$ trivialized over $S_o$. We write $Ext(S/S_o;G,G_m)$ for the group of isomorphism classes of objects of $EXT(S/S_o;G,G_m)$.

In [16, appendix, 2.5] under the additional assumptions

1) $S = Spec(R)$, $R$ an artin local ring
2) $S_o = Spec(k)$, $k$ = residue field of $R$, $k$ perfect
3) $G = \mathbb{Q}_p/\mathbb{Z}_p$

it was proved that there is a canonical isomorphism

(10.1) $$\mu(R) \simeq Ext_S(\mathbb{Q}_p/\mathbb{Z}_p, G_m)$$

Since $\mu = G^*$ is a formal group and since $S_o = Spec(k)$, and $k$ is a field; $\mu(S) = G^*(S/S_o)$. On the other hand the fact that $k$ is perfect implies $Ext(S/S_o, G, G_m) \simeq Ext(G, G_m)$. Thus the isomorphism can be written as

$$G^*(S/S_o) \simeq Ext(S/S_o;G,G_m)$$

Making extensive use of L. Illusie's deformation theory [14, VII], we prove the following generalization:

(10.2) <u>Deformational duality Theorem</u>: If $S$, $S_o$, $G$ satisfy the initial conditions above then there is a canonical (functorial) isomorphism

$$G^*(S/S_o) \xrightarrow{\approx} \mathrm{Ext}(S/S_o;G,G_m).$$

We shall give two constructions of a map $G^*(S/S_o) \to \mathrm{EXT}(S/S_o;G,G_m)$.

(10.3) Let $\Phi\colon G(n) \to G_m$ be an element of $G^*(S)$. By pushing out the Kummer sequence we obtain an extension of $G$ by $G_m$

$$\begin{tikzcd}
0 \arrow[r] & G(n) \arrow[r] \arrow[d, "\Phi"'] & G \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & G_m \arrow[r] & E \arrow[r] & G \arrow[r] & 0
\end{tikzcd}$$

If $\Phi \in G^*(S/S_o)$, then the restriction of the extension $E$ to $S_o$ has a canonical trivialization.

(10.4) Let us write $\mathrm{TORS}(S,T_p(G^*))$ for the category $\varprojlim \mathrm{TORS}(S,G^*(n))$ (i.e. the category whose objects are compatible families of torseurs, $P(n)$ a torseur under $G^*(n)$, where the transition morphism $G^*(n+m) \to G^*(n)$ is $p^m$). Similarly we write $\mathrm{TORS}\ (S/S_o,T_p(G^*))$ for the category of torseurs under $T_p(G^*)$ equipped with a trivialization over $S_o$.

Because the $G(n)$'s are finite and locally-free $\underline{\mathrm{Ext}}^1(G(n),G_m) = (0)$ and hence $\mathrm{TORS}(S,G^*(n)) \xrightarrow{\approx} \mathrm{EXT}(G(n),G_m)$. Explicitly an equivalence is given as follows:

Given a $G^*(n)$ torseur P we twist $G_m \times G(n)$ by the torseur-P. In down to earth terms this means we take the sheaf-theoretic quotient of $P \times (G_m \times G(n))$ by the action of $G^*(n)$ given by

$$(p,g,x) + \Phi = (p-\Phi\ ,\ g-\Phi(x),x)$$

where $p \in P(S')$, $g \in G_m(S')$, $x \in G(n)(S')$, $\Phi: G(n)_{S'} \to G_{m_{S'}}$, $S'$ an S-scheme.

A quasi-inverse to this functor is given by assigning to an extension

$$0 \to G_m \xrightarrow{i} E \xrightarrow{\pi} G(n) \to 0$$

the $G^*(n)$-torseur of splittings of this extension, i.e. the torseur P with

$$P(S') = \{\sigma: E \to G_m \mid \sigma \circ i = id_{G_{m_{S'}}}\}$$

where $\sigma + \Phi =_{\text{def}} \sigma + \Phi \circ \pi$, for $\Phi: G(n)_{S'} \to G_{m_{S'}}$. Since $TORS(S,T_p(G^*)) \overset{\approx}{=} \underleftarrow{\text{Lim}}\ EXT(G(n),G_m) \overset{\approx}{=} EXT(G,G_m)$ we define a map $G^*(S) \to EXT(G,G_m)$ by composing the above equivalence with the map $G^*(S) \xrightarrow{\alpha} TORS(S,T_p(G^*))$ whose definition is as follows: if $g^* \in G^*(S)$, let $\alpha(g^*)$ be the family $(P(n))$ where $P(n)$ is the $G^*(n)$-torseur $(p^n)^{-1}(g^*)$ arising from the exact sequence

$$0 \to G^*(n) \to G^* \xrightarrow{p^n} G^* \to 0$$

Clearly this induces a map $G^*(S/S_o) \longrightarrow TORS(S/S_o,T_p(G))$.

**Remark:** The fact that the two definitions in (10.3) and (10.4) are equivalent is a trivial exercise in the use of the definition of the Cartier dual. For the proof of (10.2) it is more convenient to work with (10.4) while for the eventual application to the construction of crystals (10.3) is more convenient.

(10.5) Let us observe that the category $EXT(S/S_o;G,G_m)$ is rigid. For if we identify an automorphism of the trivial extension

$$0 \to G_m \to G_m \times G \to G \to 0$$

with an element $f$, of $\mathrm{Hom}(G,G_m)$ then to say this automorphism defines a map in $\mathrm{EXT}(S/S_o ; G,G_m)$ is equivalent to saying $f|S_o = 0$. But from [16,II 3.3.17 + proof of 3.3.21] we know this implies $f = 0$.

(10.6) Let us prove the map $G^*(S/S_o) \to \mathrm{Tors}(S/S_o, T_p(G^*))$ is injective. Given $g^* \in G^*(S/S_o)$, to say the corresponding torseur $P(g^*)$ is trivial means that there is a sequence of elements $(g_n)$, $g_n \in G^*(S)$ such that

1) $p \cdot g_{n+1} = g_n$
2) $p^n g_n = g^*$
3) $g_n | S_o = 0$ for all $n$

But $G^*(S/S_o) \subseteq \Gamma(S, \mathrm{Inf}^k(G^*)) \subseteq G(Nk)$ [16,II 3.3.16]. Hence $p^{Nk}$ kills each $g_n$. It follows that $g^* = 0$.

(10.7) The proof of the surjectivity of the map $G^*(S/S_o) \to \mathrm{Tors}(S/S_o, T_p(G^*))$ seems to be more difficult. Since this is an assertion about any Barsotti-Tate group, we shall drop the "*".

(10.8) Let $P_o$ be a torseur under $T_p(G_o)(G_o = G \underset{S}{\times} S_o)$. Denote by $D(P_o)$(resp. $D(P_o(n))$) the set of isomorphism classes of deformations of $P_o$ (resp. $P_o(n)$) to a $T_p(G)$ (resp. $G(n)$) torseur on $S$.

(10.9) Proposition (using Illusie):

(i) For each $n$, $D(P_o(n)) \neq \emptyset$

(ii) For $n \geq n' \geq N$, $D(P_o(n)) \to D(P_o(n'))$ is surjective

(iii) $D(P_o) \to \varprojlim D(P_o(n))$ is onto and hence by (i) and (ii), $D(P_o) \neq \emptyset$.

(iv) For $n \geq n' \geq kN$, the map $D(P_o(n)) \to D(P_o(n'))$ is bijective.

(v) If $n \geq kN$, the map $D(P_o) \to D(P_o(n))$ is bijective.

Proof: (i) By using an induction on $k$, we can assume $k = 1$. Then from the theory of deformations of torseurs [14,VII:2.4.4, 2.4.4.1, 4.1.1.3] we know that the obstruction to lifting $P_o(n)$ lies in $H^2(S, \ell_G^\vee \overset{L}{\otimes} I)$. Using the notation of [16,II 3.3.9], $\ell_G^\vee \overset{L}{\otimes} I$ "is" the complex $L_o^\vee \otimes I \to L_{-1}^\vee \otimes I$, a complex of quasi-coherent sheaves on $S$. Since $S$ is affine the $H^2$ is zero and $P_o(n)$ can be lifted.

(ii) Once again using induction on $k$, leads us to the case $k = 1$.

From [14,VII 2.4.4, 2.4.4.1, 4.1.1.3] we know that $D(P_o(n))$ is principal homogeneous under $H^1(S, \ell_G^\vee \overset{L}{\otimes} I)$. Since $n, n' \geq N$, it follows from [16,II 3.3.6,3.3.20] that this $H^1$ is $\Gamma(\underline{n}_{G(n)}^\vee \otimes I)$(resp $\Gamma(\underline{n}_{G(n')}^\vee \otimes I)$). But by [16,II 3.3.4,3.3.7, 3.3.16] we know the projection $G(2n) \to G(n)$ (resp. $G(2n) \to G(n')$) induces an isomorphism $\underline{n}_{G(n)} \to \underline{n}_{G(2n)}$ (resp. $\underline{n}_{G(n')} \to \underline{n}_{G(2n)}$) From the functorial nature of the co-Lie complex follows a commutative diagram

$$\begin{tikzcd}
\underline{n}_{G(n')} \arrow[rr] \arrow[dr, "\sim"'] & & n_{G(n)} \arrow[dl, "\sim"] \\
& \underline{n}_{G(2n)} &
\end{tikzcd}$$

Thus the map $\underline{n}^{\vee}_{G(n)} \otimes I \to \underline{n}^{\vee}_{G(n')} \otimes I$ is an isomorphism and (ii) follows now from the fact that $D(P_o(n) \neq \emptyset$ .

(iii) Let $(\zeta_n) \in \varprojlim D(P_o(n))$ and choose for each $n$ a representative $P_n$ of $\zeta_n$ . Then $P_n \overset{G(n)}{\wedge} G(n-1) \simeq P_{n-1}$ and we can sucessively choose these isomorphisms so that $(P_n)$ is a "torseur" under $T_p(G)$ which lifts $P_o$ . The remainder of (iii) is clear.

(iv) For $k = 1$ , the assertion follows from the fact noted in (ii) that $\underline{n}_{G(n)} \simeq \underline{n}_{G(n')}$ . Let us filter $S$ by the closed sub-schemes defined by powers of $I$ : $S_o \subseteq S_1 \subseteq \ldots \subseteq S_{k-1} \subseteq S_k = S$ . By induction on $k$ , we can assume (iv) true for the pair $S_o \hookrightarrow S_{k-1}$ . Let $P(n)$, $P'(n)$ be two deformations of $P_o(n)$ such that the induced $G(n')$-torseurs $P(n'), P'(n')$ are isomorphic deformations of $P_o(n')$. We are to prove that $P(n) \simeq P(n')$. Let $u(n'): P(n') \xrightarrow{\sim} P'(n')$ be an isomorphism of deformations. By the induction hypothesis we can find an isomorphism $v(n): P_{k-1}(n) \to P'_{k-1}(n)$ (where the subscript "k-1" indicates restriction to $S_{k-1}$).

$v(n')$ and $u_{k-1}(n')$ are two isomorphisms between the deformations $P_{k-1}(n')$ and $P'_{k-1}(n')$. Their "difference" is thus an element of $G_{k-1}(n')(S_{k-1}/S_o)$. But from [16, II, 3.3.16] we know $\mathrm{Inf}^{k-1}(G_{k-1}) = \mathrm{Inf}^{k-1}(G_{k-1}((k-1)N)$. Thus multiplication by $p^{(k-1)N}$

kills this difference. Since $n' \geq kN$, this tells us that $v(N) = u_{k-1}(N)$. Observe that via $v(n)$, $P'(n)$ becomes a deformation of $P_{k-1}(n)$, while $P(n)$ is via $id_{P_{k-1}(n)}$ a deformation of $P_{k-1}(n)$. The equality $v(N) = u_{k-1}(N)$ says precisely that $P(N)$ and $P'(N)$ are, via $u(N)$, isomorphic as deformations of $P_{k-1}(N)$. Thus we may apply the result known to be true for the case $k = 1$, to the pair $S_{k-1} \hookrightarrow S$ and the integers $n, N$ (after all, $S_{k-1} \hookrightarrow S$ is a first order thickening). Thus there is an isomorphism $v': P(n) \to P'(n)$ which lifts $v$. This completes the proof.

(v) Let $P, P' \in D(P_o)$ and assume $P(kN) \simeq P'(kN)$. We are to show $P$ is isomorphic to $P'$. From (iv) we know that for $n \geq kN$, $P(n)$ is isomorphic to $P'(n)$. For any $n$ and any $i$ let $\Phi$ and $\psi$ be two isomorphisms between $P(n+kN+i)$ and $P'(n+kN+i)$. Their "difference" is an element of $G(n+kN+i)(S/S_o)$. As noted already in the proof of (iv), this group is killed under multiplication by $p^{kN}$. Thus $\Phi$ and $\psi$ induce the same isomorphism between $P(n)$ and $P'(n)$ : call it $\sigma_n$. It is clear that the $\sigma_n$'s fit together to give an isomorphism between $P$ and $P'$. This completes the proof of the proposition.

(10.10) To complete the proof of (10.2) we must establish surjectivity. From 10.9 (v), it suffices to establish surjectivity for the composite map

(10.11) $$G(S/S_o) \to \mathrm{Tors}(S/S_o, T_p(G)) \to \mathrm{Tors}(S/S_o, G(kN))$$

(10.12) <u>Lemma</u>. Let $S$ be a scheme on which $p$ is locally nilpotent, G be a Barsotti-Tate group on S, and P be a torseur on S under G . Then, P is formally smooth.

Proof: We must show that there is an arrow rendering the following diagram commutative (where X is affine and $X_0$ is defined by the vanishing of an ideal of square-zero).

$$\begin{tikzcd}
P \arrow[d] & X_0 \arrow[l] \arrow[d, hook] \\
S & X \arrow[l] \arrow[ul, dashed]
\end{tikzcd}$$

By making the base change X → S, we can assume S is affine (hence killed by a power of p ). We are given a section of P over $S_0$ and our problem is to lift it. Since G is formally smooth, and P is a G-torseur, it suffices to show that P is trivial (i.e. has a section). Since S is affine, [11, SGA$_4$VI (5.2)] tells us that $H^1(S,G) \simeq \varinjlim H^1(S,G(n))$. Hence we can assume that for some n , P' is a G(n)-torseur on S which has a section over $S_0$ and that $P' \overset{G(n)}{\wedge} G \simeq P$ . Viewing P' as a deformation of the trivial $G_0(n)$-torseur on $S_0$ it defines an element in $\mathrm{Ext}^1(\ell_{G_0(n)},I)$. From [16, II 3.3.9] we know that if n , m are taken sufficiently large, the map $\mathrm{Ext}^1(\ell_{G_0(n)},I) \to \mathrm{Ext}^1(\ell_{G_0(n+m)},I)$ is zero. This tells us in particular that $P' \overset{G(n)}{\wedge} G(n+m)$ is a trivial torseur. Hence P has a section.

(10.13) We consider the exact sequence

$$0 \to G(n) \to G \xrightarrow{p^n} G \to 0$$

where n is an integer $\geq N$ . The functor $\mathrm{TORS}(S/S_0, G(n)) \to \mathrm{TORS}(S/S_0, G)$ induces an equivalence of categories between $TORS(S/S_o,G(n))$ and the category of pairs $(Q,s)$, where $(Q,\mathfrak{s})$ is an object of $TORS(S/S_o,G)$ and $s$ is a section of $Q \overset{G \xrightarrow{p^n}}{\wedge} G$ such that $s|S_o = \mathfrak{s} \overset{G_o \xrightarrow{p^n}}{\wedge} G_o$ ($\mathfrak{s}$ being an element in $\Gamma(S_o,Q)$). This follows immediately from a momentary perusal of the proof of the corresponding fact when $S_o$ is supressed [9,III,3.2.1]. The point is that the quasi-inverse functor is given by $((Q,\mathfrak{s}),s) \longmapsto \pi^{-1}(s)$, where $\pi$ is the obvious map $Q \to Q \overset{G \xrightarrow{pn}}{\wedge} G$.

(10.14) It is now standard [12, p. 17-18] that from the exact sequence

$$0 \to G(n) \to G \xrightarrow{p^n} G \to 0$$

we obtain a long exact sequence:

(10.15) $0 \to G(n)(S/S_o) \to G(S/S_o) \to G(S/S_o) \xrightarrow{\partial} Tors(S/S_o,G(n)) \to Tors(S/S_o,G)$

where $\partial$ is the map (10.11).

From this sequence the surjectivity of $\partial$ follows immediately since (10.12) tells us in particular that the map $Tors(S/S_o,G(n)) \to Tors(S/S_o,G)$ is the zero map. Hence (10.2) has been proved.

(10.16) <u>Corollary</u>: Assume the extension

$$0 \to G_{m_{S_o}} \to E_o \to G_o \to 0$$

arises from pushing out along $g_o \in \Gamma(S_o,G_o^*)$. The set of isomorphism classes of extensions lifting $E_o$ is in bijective correspondance with $\{g \in \Gamma(S,G^*) | g \text{ lifts } g_o\}$.

Proof: One checks immediately that the set of extensions lifting $E_o$ is principal homogeneous under $\mathrm{Ext}(S/S_o;G,G_m)$, and hence the assertion follows immediately from (10.2).

(10.17) It is quite simple to globalize the above result. Let $S$ be a scheme on which $p$ is locally nilpotent and let $G$ be a Barsotti-Tate group on $S$. Since $G$ is locally of finite presentation we know

$$G(S/S_{red}) = \bigcup_{\substack{S_\lambda \text{ defined by a} \\ \text{nilpotent ideal}}} G(S/S_\lambda)$$

whenever $S$ is affine. By abuse of notation we shall continue to write this even if $S$ is not affine. On the other hand if $S_o \subset S_1 \subset S$ and $S$ is an infinitesimal neighborhood of $S_o$, then there is a natural functor $\mathrm{EXT}(S/S_1;G,G_m) \to \mathrm{EXT}(S/S_o;G, G_m)$ which is easily seen to be fully-faithful. By abuse of notation we shall write $\mathrm{EXT}(S/S_{red};G,G_m)$ for the category $\varinjlim \mathrm{EXT}(S/S\ ;G,G_m)$ where the limit is taken over the index set of sub-schemes $S_\lambda$ defined by a nilpotent ideal. Notational consistency dictates that we further abuse notation by writing $\mathrm{Ext}(S/S_{red};G,G_m) = \varinjlim \mathrm{Ext}(S/S_\lambda;G,G_m)$. It is easy to show that we are guilty of a genuine abuse of notation for even if $S$ consists of one point and is of characteristic $p$, there are extensions of $\mathbb{Q}_p/\mathbb{Z}_p$ by $G_m$ which split over $S_{red}$ but do not split where pulled back via a nilpotent immersion.

We've defined above a homomorphism of presheaves on (Sch/S)

(10.18) $T \mapsto G^*(T/T_{red}) \to T \mapsto \mathrm{Ext}(T/T_{red};G_T,G_m)$.

Furthermore (10.2) tells us that this is an isomorphism whenever T is affine.

(10.19) If F is an abelian presheaf on Sch/S we denote by $\overline{F}$ the presheaf on Sch/S defined by $T \mapsto \cup F(T/T_\lambda)$, $T_\lambda$ running through subschemes of T defined by the vanishing of a nilpotent ideal. As an exception, if G is a Barsotti-Tate group on S , "$\overline{G}$" will be used to denote the formal Lie group associated to G. Passing, in (10.18) to associated sheaves for the Zariski topology on Sch/S we obtain an isomorphism

$$(10.20) \qquad \overline{G^*} \xrightarrow{\sim} \underline{\overline{\mathrm{Ext}}}\ (G, G_m)$$

where $\overline{\mathrm{Ext}}$ is the presheaf $T \mapsto \mathrm{Ext}(T/T_{red}; G_T, G_m)$. (N.B. Since $G^*$ is ind-representable by affine schemes (relative to S) sheafification for the Zariski topology gives us an f.p.p.f. sheaf whose sections over an arbitrary S-scheme T can be explicitly described: $\overline{G^*}(T) = \{x \in G^*(T) \mid$ x restricted to any affine open U of T, dies when further restricted to a closed sub-scheme $U_o \subseteq U$ defined by a nilpotent ideal$\}$).

## §11. THE CRYSTALLINE NATURE OF THE FORMAL COMPLETION OF THE UNIVERSAL EXTENSION

Let $S_0 \hookrightarrow (S,I,\gamma)$ be a nilpotent immersion defined by a divided power ideal $I$. Let $G_0$ be a Barsotti-Tate group on $S_0$. We wish to assign to $G_0$ a formal group $\overline{E}^*(G_0)_{S_0} \hookrightarrow S$ which will be canonically isomorphic to the formal group associated to $E(G^*)$, $\overline{E(G^*)}$, wherever $G$ is a lifting of $G_0$ to $S$. We shall give an explicit description of the points of this functor with values in a flat S-scheme $S'$.

(11.1) Let $S_0' = S' \times_S S_0$ and let $G_0' = G_0 \times_{S_0} S_0'$. As explained in (6.14) we can consider the category $\mathrm{EXT}^{\mathrm{crys}/S'}(G_0',G_m)$ $=_{\mathrm{dfn}} \varprojlim\limits_{\mathrm{Lim}} \mathrm{EXT}^{\mathrm{crys}/S'}(G_0'(n),G_m)$.

For any closed subscheme $\overline{S_0'} \hookrightarrow S_0'$ defined by a nilpotent ideal, we have the notion of a crystalline extension of $G_0' \times_{S_0'} \overline{S_0'}$ by $G_m$ (relative to $S'$) as given in (6.15). This allows us to speak of the category whose objects are pairs $(P,\eta)$ where $P$ is an object of $\mathrm{EXT}^{\mathrm{crys}/S'}(G_0',G_m)$ and $\eta$ is a trivialization of the underlying $\natural$-extension of $P$ restricted to $\overline{S_0'}$. When $\overline{S_0'}$ is allowed to vary we obtain a direct system of categories and taking the direct limit we obtain a category which we denote by $\mathrm{EXT}^{\mathrm{crys}/S'}(S_0'/S_0'\,{}_{\mathrm{red}};G_0',G_m)$. We write $\overline{\mathrm{Ext}}^{\mathrm{crys}/S'}(G_0',G_m)$ for the group of isomorphism classes of objects of **this category.**

(11.2) Let $G'$ be a Barsotti-Tate group on $S'$ which lifts $G_0'$. For any closed sub-scheme $\overline{S'}$, of $S'$ which is defined by a nilpotent ideal there is the category of $\overline{S'}$-trivialized

$\natural$-extensions of G' by $G_m$ . Passing to the limit over such closed sub-schemes and then taking isomorphism classes of objects we obtain a group $\overline{\mathrm{Ext}}^{\natural}(G',G_m)$.

(11.3) Proposition: The natural functor

$$\overline{\mathrm{EXT}}^{crys/S'}(G',G_m) \to \overline{\mathrm{EXT}}^{\natural}(G',G_m)$$

is an equivalence of categories.

Proof: The fact that the functor is fully-faithful is proved exactly as was done in the proof of (7.2). In fact it follows immediately from (7.6.1) since $\mathrm{EXT}^{nil-\natural}(G',G_m)$ is a full sub-category of $\mathrm{EXT}^{\natural}(G',G_m)$.

Let E be an object in $\overline{\mathrm{EXT}}^{\natural}(G',G_m)$. Since E becomes the trivial $\natural$-extension when we pass to a closed sub-scheme $\overline{S'} \hookrightarrow S'$ defined by a nilpotent ideal, if we view E as a family of line bundles with integrable connection, $\mathcal{L}_n$ on $G'(n)$, each of these line bundles becomes trivial on $\overline{S'}$. Fix an n and let D be a nilpotent S' derivation of $\mathcal{O}_{G'(n)}$ to itself. For $N >> 0$ $\nabla(D)^N(\mathcal{L}_n) \subset$ (ideal of $\overline{S'}$ in S'). $\mathcal{L}_n$ (since $\mathcal{L}_n|G'(n) \times_{S'} \overline{S'} \simeq (\mathcal{O}$, standard connection). Since the ideal of $\overline{S'}$ in S' is nilpotent, $\nabla(D)$ is a nilpotent endomorphism of $\mathcal{L}_n$ . Thus the connection on each $\mathcal{L}_n$ is nilpotent [3,II,4.3.6] (N.B. Berthelot defines this notion only when $\Omega^1$ is locally-free of finite rank so a more correct assertion would be for $n >> 0$ the connection on each $\mathcal{L}_n$ is nilpotent). Thus our $\natural$-extension E is isomorphic to a crystalline extension and the proof is complete.

(11.4) Corollary: The natural functor

$$\underline{\mathrm{EXT}}^{\mathrm{crys}/S'}(G_o',G_m) \to \underline{\mathrm{EXT}}^{\natural}(G',G_m)$$

is an equivalence of categories.

Proof. Since the closed sub-schemes of $S_o'$ defined by a nilpotent ideal define by composition with $S_o' \hookrightarrow S'$ a co-final system of closed sub-schemes of $S'$ (defined by a nilpotent ideal), and since the ideal of $S_o'$ in $S'$ has divided powers, (11.4) follows immediately from (11.3) plus the usual equivalence $\mathrm{EXT}^{\mathrm{crys}/S'}(G_o',G_m) \approx \mathrm{EXT}^{\mathrm{crys}/S'}(G',G_m)$.

(11.5) Proposition: Let $S$ be affine. There is a natural exact sequence

$$(11.6) \qquad 0 \to \underline{\omega}_G(S/S_{\mathrm{red}}) \to \overline{\mathrm{Ext}}^{\natural}(G,G_m) \to \overline{\mathrm{Ext}}(G,G_m) \to 0$$

Proof: Given $\tau \in \underline{\omega}_G(S/S_{\mathrm{red}})$ let $\tau'$ denote the $\natural$-structure on $G_m \times G$ defined by $\tau$. Assume $(G_m \times G, \tau')$ is isomorphic to the trivial $\natural$-extension $(G_m \times G$, trivial) via an isomorphism $\Phi$ which reduces to $\mathrm{id}_{G_m \times G}$ modulo some nilpotent ideal. Then $\Phi$ is necessarily equal to $\mathrm{id}_{G_m \times G}$ and hence $\tau$ must be $0$.

Let $E$ be a trivialized $\natural$-extension whose underlying extension is isomorphic to $G_m \times G$ via an isomorphism, $\Phi$, respecting the trivializations (all trivializations over some $S_o \hookrightarrow S$ defined by a nilpotent ideal). Using $\Phi$ let us equip $G_m \times G$ with a $\natural$-structure, $\tau'$, by transport of structure. Since

$\tau'$ comes from a unique $\tau \in \Gamma(S,\underline{\omega}_G)$ and since the restriction of $\Phi$ to $S_0$ is compatible with trivializations it follows that $\tau \in \Gamma(S/S_{red},\underline{\omega}_G)$ and exactness at $\overline{\mathrm{Ext}}^{\natural}(G,G_m)$ has been established.

It remains to check the surjectivity of $\overline{\mathrm{Ext}}^{\natural}(G,G_m) \to \overline{\mathrm{Ext}}(G,G_m)$. Let $E$ be an extension of $G$ by $G_m$, and $\Phi_0$ a trivialization of $E \underset{S}{\times} S_0$. From (4.4) it follows that each of the induced extensions

$$0 \rightarrow G_m \rightarrow E \underset{G}{\times} G(n) \rightarrow G(n) \rightarrow 0$$

has a $\natural$-structure. Since for $n$ large the maps $\Gamma(S,\underline{\omega}_{G(n+1)}) \to \Gamma(S,\underline{\omega}_{G(n)})$ are onto it follows that $E$ itself carries at least one $\natural$-structure, $\rho$. The "difference" between $\rho_0$ and the $\natural$-structures on $E_0$ obtained via $\Phi$ is an element of $\Gamma(S_0,\underline{\omega}_{G_0})$. Since the map $\Gamma(S,\underline{\omega}_G) \rightarrow \Gamma(S_0,\underline{\omega}_{G_0})$ is onto we can modify $\rho$ to obtain a new $\natural$-structure on $E$ so that $\Phi_0$ is horizontal. This completes the proof.

(11.7) Corollary: Assume $p$ is locally nilpotent on $S$, $G$ a Barsotti-Tate group on $S$. Sheafifying the sequence (11.6) we obtain an exact sequence

(11.8) $$0 \to \overline{\underline{\omega}}_G \to \underline{\mathrm{Ext}}^{\natural}(G,G_m) \to \underline{\mathrm{Ext}}(G,G_m) \to 0 ,$$

This sequence is canonically isomorphic to the exact sequence of formal groups obtained by completing the universal extension of $G^*$ along the identity section:

(11.9) $$0 \to \overline{\underline{\omega}}_G \to \overline{E(G^*)} \to \overline{G^*} \to 0 .$$

Proof: The exactness of (11.9) is proved in [16,IV (1.2.1)]. From (4.6) we know that for $S$ affine $\Gamma(S,E(G^*))$ is equal to $\varinjlim(\epsilon_n)\text{—}\mathrm{Hom}^{\natural}(G(n),G_m)$ where $(\epsilon_n)$ is the exact sequence

$$(\epsilon_n) \qquad 0 \to G(n) \to G \xrightarrow{p^n} G \to 0$$

and where $(\epsilon_n)\text{-}\mathrm{Hom}^{\natural}(G(n),G_m)$ is the group whose elements are pairs $(\phi : G(n) \to G_m,\ \rho$ a $\natural$-structure on the extension $\phi_*(\epsilon_n))$. Thus using (10.3) we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \bar{\omega}_G \arrow[r] \arrow[d, equal] & \overline{\underrightarrow{\lim}(\epsilon_n)\text{-}\underline{\mathrm{Hom}}^{\natural}(G(n),G_m)} \arrow[r] \arrow[d] & \bar{G}^* \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bar{\omega}_G \arrow[r] & \underline{\mathrm{Ext}}^{\natural}(G,G_m) \arrow[r] & \underline{\mathrm{Ext}}(G,G_m) \arrow[r] & 0
\end{tikzcd}$$

The corollary now follows from (10.20) and the five lemma.

(11.10) Let $S_0 \to (S,I,\gamma)$ be as in the beginning of this section. Assume given two Barsotti-Tate groups $G,H$ on $S$ and a homomorphism $u_0 : G_0 \to H_0$ between their restrictions to $S_0$. We shall associate to $u_0$ a homomorphism $v : \overline{E(H^*)} \to \overline{E(G^*)}$ which lifts $\overline{E(u_0^*)}$.

If $T$ is flat over $S$, the isomorphism (11.4): $\overline{\underline{\mathrm{Ext}}}^{\natural}(G_T,G_m) \simeq \overline{\underline{\mathrm{Ext}}}^{\mathrm{crys}/T}(G_{0_{T}},G_m)$, together with the corresponding isomorphism with $H$ replacing $G$, gives us an arrow $v_T$ rendering the following diagram commutative:

$$\begin{tikzcd}
\overline{\underline{\mathrm{Ext}}}^{\mathrm{crys}/T}(G_{0_{T_0}},G_m) & \overline{\underline{\mathrm{Ext}}}^{\natural}(G_T,G_m) \arrow[l, dash, "\sim"'] \\
\overline{\underline{\mathrm{Ext}}}^{\mathrm{crys}/T}(H_{0_{T_0}},G_m) \arrow[u, "\overline{\underline{\mathrm{Ext}}}^{\mathrm{crys}/T}(u_{0_T},G_m)"] & \overline{\underline{\mathrm{Ext}}}^{\natural}(H_T,G_m) \arrow[l, dash, "\sim"'] \arrow[u, "v_T"']
\end{tikzcd}$$

Sheafifying and using (11.7) we find for $T$ flat over $S$ a morphism $\overline{E(H^*)}(T) \to \overline{E(G^*)}(T)$.

The existence of the homomorphism $\overline{E(H^*)} \to \overline{E(G^*)}$ now follows since $\overline{E(H^*)} = \varinjlim \mathrm{Inf}^k\left(\overline{E(H^*)}\right)$ and each $\mathrm{Inf}^k$ is flat over $S$.

(11.11) It follows immediately from (11.10) that if $G$ and $H$ are two liftings of the Barsotti-Tate group $G_o$ on $S_o$, then $\overline{E(G)}$ is canonically isomorphic to $\overline{E(H)}$ Exactly as in (7.17), (7.18), the functor $\overline{\mathbb{E}}^*$ is explicitly defined by setting for $S'$ an $S$-scheme

$$\Gamma(S', \overline{\mathbb{E}}^*(G_o)_{S_o \hookrightarrow (S,I,\gamma)}) = \Gamma(S', \underline{\mathrm{Ext}}^{\mathrm{crys}/S}(G_o, G_m))$$

where $\underline{\mathrm{Ext}}^{\mathrm{crys}/S}(G_o, G_m)$ denotes the prolongation to $(\mathrm{Sch}/S)$ of the sheaf on the small flat site of $S$ associated to the presheaf:

$$T \mapsto \underline{\mathrm{Ext}}^{\mathrm{crys}/T}(G_{o_{T_o}}, G_m)$$

(11.12) Remarks:

(i) In order to know that $\overline{\mathbb{E}}^*(G_o)_{S_o \hookrightarrow S}$ is a formal group we have made use of a lifting $G$ of $G_o$. In order to know that if

$$\begin{tikzcd} S_o \arrow[r, hook] \arrow[dr, hook] & (S', J, \gamma') \arrow[d, "f"] \\ & (S, I, \gamma) \end{tikzcd}$$

is a commutative diagram where $f$ is a divided power morphism, then $f^*(\overline{\mathbb{E}}^*(G_o)_{S_o \hookrightarrow S}) \to \overline{\mathbb{E}}^*(G_o)_{S_o \hookrightarrow S'}$ is an isomorphism; we make use of a lifting of $G_o$. (If we don't assume the existence of a lifting then there doesn't appear to be any standard terminology which describes what $\overline{\mathbb{E}}^*(G_o)$ is).

(ii) $\overline{\mathbb{E}}^*(G_0)$ is a crystal relative to a crystalline site which sits in-between the nilpotent site and the full Berthelot site: objects are divided power thickenings $S_0 \hookrightarrow (S,I,\delta)$ where $I$ is a nilpotent ideal, *but the divided powers are not necessarily nilpotent*. The reason for this was alluded to in (10.16).

(11.13) Let us check that $\underline{\mathrm{Lie}}(\mathbb{E}^*(G_0))$ is canonically isomorphic to $\mathbb{D}^*(G_0)$ on their common domain of definition: Let $S_0 \hookrightarrow (S,I,\gamma)$ be a divided power thickening of $S_0$ by a nilpotent ideal. Assume $S$ is affine. We shall define a map

$$\mathrm{Ext}^{\mathrm{crys}/S}(G_0,G_a) \to \mathrm{Ker}[\overline{\mathrm{Ext}}^{\mathrm{crys}/S[\epsilon]}(G_{0_{S[\epsilon]}} G_m) \to \overline{\mathrm{Ext}}^{\mathrm{crys}/S}(G_0,G_m)],$$

For any $S_0$-scheme $X$, there is a commutative diagram

$$\begin{tikzcd} X[\epsilon] \arrow[r] \arrow[d] & S[\epsilon] \arrow[d, "\pi"] \\ X \arrow[r] & S \end{tikzcd}$$

which gives rise to a morphism of topoi

$$\pi\colon (X_{[\epsilon]}/S_{[\epsilon]})_{\mathrm{crys}} \longrightarrow (X/S)_{\mathrm{crys}} .$$

Using the definition of $\pi$ [3,III,2.2.3] one checks easily that for any object $(U \hookrightarrow T,J,\delta)$ of the crystalline site of $X$, $\pi^{-1}(U \hookrightarrow T) = U[\epsilon] \hookrightarrow T[\epsilon]$. Thus $\pi_*(G_m)_{U \hookrightarrow T} = G_{m_{T[\epsilon]}}$ and there is an exact sequence of sheaves of groups in $(X/S)_{\mathrm{crys}}$

$$0 \to G_a \to \pi_*(G_m) \to G_m \to 0$$

Thus we obtain an equivalence of categories

$$(11.14)\ \mathrm{TORS}^{crys/S}(X,G_a) \approxeq \mathrm{Ker}[\mathrm{TORS}^{crys/S[\epsilon]}(X[\epsilon],G_m) \to \mathrm{TORS}^{crys/S}(X,G_m)].$$

This equivalence if functorial in the $S_o$-scheme $X$ and hence we obtain

$$(11.15)\ \mathrm{EXT}^{crys/S}(G_o,G_a) \simeq \mathrm{Ker}[\mathrm{Ext}^{crys/S[\epsilon]}(G_o[\epsilon],G_m) \to \mathrm{EXT}^{crys/S}(G_o,G_m)].$$

This permits us to define the map

$$(11.16)\ \mathrm{Ext}^{crys/S}(G_o,G_a) \rightarrowtail \mathrm{Ker}(\overline{\mathrm{Ext}}^{crys/S[\epsilon]}(G_o[\epsilon],G_m) \to \overline{\mathrm{Ext}}^{crys/S}(G_o,G_m)).$$

Before we prove the bijectivity of this map, let us note that the category $\overline{\mathrm{EXT}}^{crys/S}(G_o,G_m)$ is rigid. This follows immediately from (11.4) (and hence we use once again the fact that Barsotti-Tate groups can be lifted).

Let $P,Q$ be representatives of elements of $\mathrm{Ext}^{crys/S}(G_o,G_a)$. To say they define the same element in Ker is equivalent to asserting that there is an isomorphism of crystalline extensions

$$\rho\colon P \overset{G_a}{\wedge} \pi_*(G_m) \simeq Q \overset{G_a}{\wedge} \pi_*(G_m)$$

such that $\rho \wedge G_m$ induces the identity automorphism of the $\natural$-extension $G_m \times G_o$ (once we identify $P \overset{G_a}{\wedge} G_m$ and $Q \overset{G_a}{\wedge} G_m$ with $G_m \times G_o$). But using the rigidity of $\overline{\mathrm{EXT}}^{crys/S}(G_o,G_m)$ noted above, it follows that $\rho \wedge G_m$ is actually the identity automorphism of the <u>crystalline</u> extension $G_m \times G_o$. It now follows from (11.15) that $P \simeq Q$.

On the other hand the surjectivity of (11.16) is clear since a crystalline extension, $P$, of $G_o[\epsilon]$ by $G_m$ trivialized as $\natural$-extension over some closed subscheme $T \subseteq S[\epsilon]$, and which is trivialized over $S$ as crystalline extension (in a compatible fashion over $S \cap T \subset S[\varepsilon]$) defines a crystalline extension of $G_o$ by $G_a$, Q, which is isomorphic to P as a crystalline extension (an isomorphism certainly compatible with the trivialization over $S \cap T$).

## §12. THE CRYSTALLINE NATURE OF THE UNIVERSAL EXTENSION (ON THE NILPOTENT CRYSTALLINE SITE)

In this section we shall show that the universal extension of a Barsotti-Tate group can be extended to a crystal on the nilpotent crystalline site.

Let $S_o$ be a scheme, $(S,I,\gamma)$ a nilpotent divided power thickening. Fix a Barsotti Tate group $G_o$ on $S_o$. Following the procedure(s) used in previous sections we shall define for $S'$ a flat S-scheme, a group $\mathbb{E}(G_o)_{S_o \hookrightarrow S}(S')$, such that sheafification gives us the value of our crystal on $(S,I,\gamma)$.

(12.1) Consider the category whose objects are triples:

(12.2) (i) an element $g_o \in \Gamma(S_o, G_o^*)$

(ii) a nilpotent crystalline extension of $G_o$ by $G_m$ (relative to S), $E \in \mathrm{EXT}^{\mathrm{nil\ crys}/S}(G_o, G_m)$

(iii) an isomorphism $\rho$ between the extension $P_{g_o}$, associated to $g_o$, and the ordinary extension underlying $E$.

Morphisms between $(g_o,E,\rho)$ and $(g_o',E',\rho')$ are defined only if $g_o = g_o'$ and then a morphism is a morphism of crystalline extensions $E \rightarrow E'$ which is compatible with $\rho$ and $\rho'$.

(12.3) Definition: Let $\mathbb{E}^*(G_o)_{S_o \hookrightarrow S}(S)$ = group of isomorphism classes of objects of the above category.

(12.4) Let $G$ be a lifting of $G_o$ to $S$ which we assume to be affine. We construct a map $\Gamma(S,E(G^*)) \rightarrow \mathbb{E}^*(G_o)_{S_o \hookrightarrow S}(S)$ by interpreting an element of $\Gamma(S,E(G^*))$ as an element, $\Phi$, of $\varinjlim(\varepsilon_n)\text{-}\mathrm{Hom}^{\natural}(G(n),G_n)$ (as in the proof of (11.7)) and assigning to $\Phi$ the isomorphism class of the triple:

(i) $g_o$ = restriction to $S_o$ of the element of $\Gamma(S,G^*)$ which is the image of $\Phi$ under $E(G^*) \to G^*$

(ii) $E$, the object of $\mathrm{EXT}^{\text{Nil-crys}/S}(G_o,G_m)$ corresponding to $\Phi$ via the equivalence (7.6.2) plus the equivalence $\mathrm{EXT}^{\text{Nil-crys}/S}(G_o,G_m) \approx \mathrm{EXT}^{\text{Nil-crys}/S}(G,G_m)$

(iii) the canonical isomorphism $P_{g_o} \simeq E$ (i.e. the identity map).

(12.5) <u>Proposition</u>: The map defined in (12.4) is an isomorphism.

<u>Proof</u>: To show injectivity let $\Phi \in \Gamma(S,E(G^*))$ be given, let $g$ = image of $\Phi$ under $E(G^*) \to G^*$. Assume the triple defined by $\Phi$ is isomorphic to the triple (0, trivial crystalline extension, identity), i.e. there is a map $E \simeq G_o \times G_m$ of crystalline extensions and the map on underlying extensions is the identity. Since we can interpret the crystalline extensions $E$ and $G_o \times G_m$ as $\natural$-extensions of $G$ by $G_m$, it follows from (10.3) that $g = 0$. Hence $\Phi$ is given by an element of $\Gamma(S,\underline{\omega}_G)$. The rigidity of the category $\mathrm{EXT}(S/S_o;G,G_m)$ insures that the isomorphism $E \simeq G_o \times G_m$, when interpreted as a map of $\natural$-extension of $G$ by $G_m$, is the identity. This forces the element of $\Gamma(S,\underline{\omega}_G)$, and hence $\Phi$, to be zero.

To prove surjectivity, let $(g_o,E,\rho)$ be a triple. We interpret $E$ (as explained in 12.4(ii)) as an object of

$EXT^{\natural}(G,G_m)$ whose underlying structure of extension we denote by $E'$. From (10.16) the pair $(E',\rho)$ determines an element $g$, of $\Gamma'(S,G^*)$ which lifts $g_o$ . Let $\nabla$ be the $\natural$-structure on $P_g$ obtained via transport of structure from $E$ using the isomorphism $P_g \simeq E'$. If $\Phi = (g,\nabla)$ then, by construction, the image of $\Phi$ is the class of the triple $(g_o,E,\rho)$.

(12.6) Corollary: Sheafifying the map

$$\Gamma(S,E(G^*)) \to \mathbb{E}^*(G_o)_{S_o \hookrightarrow S}(S)$$

we obtain an isomorphism (of sheaves of groups on the small flat site of $S$)

$$E(G^*) \to \mathbb{E}^*(G_o)_{S_o \hookrightarrow S}$$

(12.7) Let $G_1,G_2$ be two liftings of $G_o$ to $S$ . Just as in (11.10), (11.11) there is a canonical isomorphism $E(G_1^*) \simeq E(G_2^*)$. **In fact more generally we can state**

(12.8) Corollary: There is a functor

$$B.T.(S_o)^o \to \text{Crystals in groups on the nilpotent site of } S_o$$

given by $G_o \mapsto \mathbb{E}^*(G_o)$ (where $\mathbb{E}^*(G_o)_{S_o \hookrightarrow S}$ has been explicitly defined via (12.4)).

(12.9) We now wish to show that "completing along the identity element" the crystal $\mathbb{E}^*(G_o)$ gives us a crystal in formal groups which is canonically isomorphic to the crystal $\overline{\mathbb{E}}^*(G_o)$ (of §11), when the latter is restricted to the nilpotent crystalline site.

Let $S_o \hookrightarrow (S,I,\gamma)$ be a thickening of the nilpotent site of $S_o$. In order to show the formal groups on S $\overline{\left(\mathbb{E}^*(G_o)_{S_o \hookrightarrow S}\right)}$ and $\overline{\mathbb{E}^*}(G_o)_{S_o \hookrightarrow S}$ are isomorphic, it suffices to show that their values on flat S-schemes are functorially isomorphic. Thus by localization it suffices to treat the case when S is affine. Since $\mathbb{E}^*(G_o)_{S_o \hookrightarrow S}$ is ind-representable by affine groups, it follows that

$$\Gamma\left(S,\overline{\left(\mathbb{E}^*(G_o)_{S_o \hookrightarrow S}\right)}\right) = \text{group of classes of}$$

$$\text{triples } (g_o,E,\rho)$$

such that for some nilpotent immersion $T \hookrightarrow S_o$, the inverse image to T of the $\natural$-extension underlying E becomes isomorphic via $\rho_T$ to the trivial $\natural$-extension of $G_{o_T}$ by $G_m$.

To check that this description is correct we use the fact that $\mathbb{E}^*(G_o)_{S_o \xrightarrow{id} S_o} = i^*(\mathbb{E}^*(G_o)_{S_o \hookrightarrow S})$, i: $S_o \hookrightarrow S$ being the inclusion, and the fact that the crystalline extensions of $G_o$ by $G_m$ (relative to $S_o$) are simply the $\natural$-extensions.

Consider now the map

$$(g_o,E,\rho) \mapsto \text{class of } (E,\rho|T) \text{ in } \overline{\mathrm{Ext}}^{crys/S}(G_o,G_m)$$

The injectivity of this map follows from the injectivity of map $G_o^*(S_o/T) \to \mathrm{Ext}(S_o/T;G_o,G_m)$. For if $(g_o',E',\rho')$ is a second triple and $(E',\rho'/T) \simeq (E,\rho/T)$, then there is an isomorphism of crystalline extensions $\eta: E \to E'$ such that $\eta|T \circ \rho|T = \rho'|T$. But $\eta\circ\rho$ and $p'$ are then equal by (10.5).

N.B. We view E as an object of $\mathrm{EXT}^{crys/S}(G_o,G_m)$ using (11.3). The surjectivity of the map follows immediately from the assertion of surjectivity implicit in (10.3).

## §13. RELATION BETWEEN THE UNIVERSAL EXTENSION CRYSTAL OF AN ABELIAN VARIETY AND THAT OF ITS ASSOCIATED BARSOTTI-TATE GROUP

We shall now show that our construction of the crystals (of various sorts) associated to a Barsotti-Tate group is compatible with our earlier construction of the crystals associated to an abelian scheme.

Let $S_o$ be a scheme (with $p$ locally nilpotent), $A_o/S_o$ an abelian scheme, $G_o = \varinjlim A_o(n)$ the associated Barsotti-Tate group. Fix a nilpotent divided power thickening $S_o \hookrightarrow (S,I,\gamma)$ and assume $S_o$ is affine.

(13.1) Lemma: Let the triple $(g_o,E,\rho)$ define an element of $\Gamma(S,E^*(G_o)_{S_o \hookrightarrow S})$. Then up to isomorphism there is a unique crystalline extension $E'$ in $EXT^{nil\text{-}crys/S}(A_o,G_m)$ such that there is an isomorphism $\rho'$ between the extension of $A_o$ by $G_m$ defined by $g_o$ and the extension underlying $E'$, such that $(g_o,E'|G_o, \rho'|G_o)$ is isomorphic to $(g_o,E,\rho)$.
(N.B. $\rho'$ is necessarily unique).

Proof: Let $A/S$ be any abelian scheme lifting $A_o$, let $G$ be the associated Barsotti-Tate group. Corresponding to the triple $(g_o,E,\rho)$, there is a pair $g \in \Gamma(S,G^*)$, $\nabla$ a $\natural$-structure on the extension

$$(13.2) \qquad 0 \to G_m \to \mathcal{E} \to G \to 0$$

obtained by pushing out the "Kummer sequence" along $g$. This $\natural$-structure defines a rigidification on (13.2). But

(13.2) is obtained by restricting to $G$ an extension

$$0 \to G_m \to \mathcal{E}' \to A \to 0$$

Since $\mathrm{Inf}^1(G) = \mathrm{Inf}^1(A)$, this extension has a canonical rigidification, i.e. a canonical $\natural$-structure. It follows immediately from [I,(3.2.3)] that this $\natural$-structure extends the given $\natural$-structure on (13.2). Via the equivalence of categories $\mathrm{EXT}^{\natural}(A,G_m) \approx \mathrm{EXT}^{\text{nil-crys}/S}(A_0,G_m)$ $\mathcal{E}'$ defines an object $E'$ of $\mathrm{EXT}^{\text{nil-crys}/S}(A_0,G_m)$ such that $E' \underset{A_0}{\times} G_0 \simeq E$, and $E'$ clearly satisfies the conditions with $\rho' =$ "id".

Let $E''$ be a second object of $\mathrm{EXT}^{\text{nil crys}/S}(A_0,G_m)$ which satisfies the conditions, i.e. so that there is a $\rho''$. By hypothesis there is an isomorphism $\Phi : E'|G_0 \xrightarrow{\sim} E''|G_0$ of crystalline extensions such that the following diagram commutes

$$\begin{tikzcd}
E'|G_0 \arrow[rr, "\Phi", "\sim"'] & & E''|G_0 \\
& P_{g_0} \arrow[ul, "\rho'|G_0"] \arrow[ur, "\rho''|G_0"'] &
\end{tikzcd}$$

We must show that $E'$ and $E''$ are isomorphic crystalline extensions. Corresponding to $E''$ is a $\natural$-extension $\mathcal{E}''$ of $A$ by $G_m$. Since $\Phi$ is a map of crystalline extensions there is a map $\mathbb{F} : \mathcal{E}'|G \to \mathcal{E}''|G$ which lifts $\Phi$. As the extension underlying the $\natural$-extension $\mathcal{E}''-\mathcal{E}'$ is trivialized over $S_0$, this extension is obtained via pushing out a "Kummer sequence" along an element, $g'$, of $\Gamma(S,G^*)$, such that $g'|S_0 = 0$ [19,(19.1)]. But $g$ and $g+g'$ are two sections in $\Gamma(S,G^*)$ lifting $g_0$

with the corresponding extensions, yielding via $F|G_o$ isomorphic deformations of $P_{g_o}$. Hence from (10.16) it follows that $g' = 0$ and hence that the extensions underlying $\mathcal{E}'$ and $\mathcal{E}''$ are isomorphic via a unique isomorphism $\tau$. By the rigidity of $EXT(A_o,G_m)$, $\tau|S_o \circ \rho = \rho''$, and hence by the rigidity of the category of deformations of $P_{g_o}$, $\tau|G = F$. Since $Inf^1(A) \subseteq G$ $\tau$ induces an isomorphism of the rigidified extensions $\mathcal{E}'$ and $\mathcal{E}''$. But from [I,(3.2.3)] we know this means $\tau$ is an isomorphism of $\natural$-extensions. Via the equivalence $EXT^{\natural}(A,G_m) \cong EXT^{nil\ crys/S}(A_o,G_m)$, we see $\tau$ induces an isomorphism between $E'$ and $E''$. This completes the proof.

(13.3) <u>Remark:</u> Although we have used a lifting in the proof of (13.1) the result is clearly independent of any such choice.

(13.4) Let A and B be abelian schemes on S, G, H the corresponding Barsotti-Tate groups. Assume $u_o: A_o \to B_o$ is a homomorphism inducing $\tilde{u}_o: G_o \to H_o$. In §1 (resp. §12)) there is associated a homomorphism $E(B^*) \to E(A^*)$ (resp. $E(H^*) \to E(G^*)$). It is an immediate consequence of (13.1) that the following diagram commutes:

(13.5)
$$\begin{tikzcd}
E(H^*) \arrow[r] \arrow[d, hook] & E(G^*) \arrow[d, hook] \\
E(B^*) \arrow[r] & E(A^*)
\end{tikzcd}$$

Passing to tangent spaces we find that the map

$$\mathbb{D}^*(H_0)_{S_0 \hookrightarrow S} \longrightarrow \mathbb{D}^*(G_0)_{S_0 \hookrightarrow S}$$

coincides with the map $H^1(B, \mathcal{O}_{B_{crys}}) \to H^1(A, \mathcal{O}_{A_{crys}})$ induced (from $u_0$) by crystalline cohomology.

## §14. GROTHENDIECK'S DUALITY FORMULA FOR THE LIE COMPLEX

Let $S$ be a scheme, $G$ a finite, locally-free (commutative) $S$-group. In the course of the proof given below we shall recall a construction of the co-Lie complex, $\mathcal{L}^G_\cdot$, associated to $G$. Let $M$ be a quasi-coherent $\mathcal{O}_S$-module. From [14,VII,1.1] we know it is entirely harmless to identify $\mathcal{L}^G_\cdot$ and $M$ with the corresponding objects that they define on the flat site of $S$. With this understanding the formula is:

$$(14.1) \qquad R\,\underline{\mathrm{Hom}}_{\mathcal{O}_S}(\mathcal{L}^G_\cdot, M) \simeq \tau_{\leq 1} R\,\underline{\mathrm{Hom}}_{Z}(G^*, M)$$

This isomorphism is functorial in both arguments and when $S$ is affine there is a similar isomorphism with "Hom" replacing "$\underline{\mathrm{Hom}}$".

Taking $M = \mathcal{O}_S$ we find a formula for the Lie complex:

$$(14.2) \qquad \mathcal{L}^{\vee G}_\cdot \simeq \tau_{\leq 1} R\,\underline{\mathrm{Hom}}_{\mathbb{Z}}(G^*, G_a)$$

If $S$ is affine applying $H^1$ (to the formula involving $R\,\mathrm{Hom}$) yields

$$(14.3) \qquad \mathrm{Ext}^1(\mathcal{L}^G_\cdot, M) \simeq \mathrm{Ext}^1_{\mathbb{Z}}(G^*, M)$$

(a formula used above in (3.1))

If instead we took $H^0$ the formula becomes

$$(14.4) \qquad \mathrm{Hom}_{\mathcal{O}_S}(\underline{\omega}_G, M) \simeq \mathrm{Hom}(G^*, M)$$

Proof (Grothendieck): From [11, $SGA_4$ VII,3.5] we know there is a partial resolution of $G$.

$$(14.5) \qquad L_2 \to L_1 \to L_o \xrightarrow{\varepsilon} G$$

Each $L_i$ is a sum of sheaves of the form $\mathbb{Z}[T_i]$ where $T_i$ is a finite product of copies of $G$, $L_o$ is simply $\mathbb{Z}[G]$. This resolution is functorial in $G$. From (I,(1.3)) it follows that $G^i =_{dfn.} L_i^*$ is a smooth commutative group scheme. Because $\underline{\mathrm{Ext}}^i(\mathbb{Z}[T],G_m) = R^i f_{T_*}(G_{m_T}) = 0$ ($f_T$: $T \to S$ being the structural map for a finite locally-free S-scheme), the complex

$$G^\bullet = G^o \to G^1 \to G^2$$

has

$$(14.6) \qquad \begin{cases} H^o(G^\bullet) = G^* \\ H^1(G^\bullet) = \underline{\mathrm{Ext}}^1(G,G_m) = (0) \quad \text{since } G \text{ is finite,} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{locally-free} \end{cases}$$

Thus if $\overline{G} = \mathrm{Ker}(G^1 \to G^2)$ we obtain an exact sequence

$$(14.7) \qquad 0 \to G^* \to G^o \to \overline{G} \to 0$$

It follows from [8,II,5.22] that $\overline{G}$ is a smooth S-group. We define the co-Lie complex of $G^*$ by:

$$(14.8) \qquad \ell_\bullet^{G^*} =_{dfn} \quad \underline{\omega}_{\overline{G}} \to \underline{\omega}_{G^o}$$

(where $\underline{\omega}_{\overline{G}}$ is placed in degree -1)

In (I,(1.2)) we've defined a map

$$L_i \to \underline{\omega}_{G^i}$$

Applying $\underline{\mathrm{Hom}}(\ ,M)$ (resp. $\mathrm{Hom}(\ ,M)$) we obtain a morphism of complexes

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(\underline{\omega}_{G^0},M) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(\underline{\omega}_{G^1},M) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(\underline{\omega}_{G^2},M) \arrow[d] \\
\underline{\mathrm{Hom}}(L_0,M) \arrow[r] & \underline{\mathrm{Hom}}(L_1,M) \arrow[r] & \underline{\mathrm{Hom}}(L_2,M)
\end{tikzcd} \tag{14.9}$$

(I,(1.4)) tells us that (14.9) is an isomorphism of complexes.

Observe that $\mathrm{Ext}^i(\mathbb{Z}[T],M) = R^i f_{T*}(M_T) = (0)$ for $i > 0$ since the map $f_T$ is affine and $M$ is quasi-coherent. Furthermore if $S$ is affine, $\mathrm{Ext}^i(\mathbb{Z}[T],M) = (0)$ for $i > 0$. Since each $\underline{\omega}_{G^i}$ is locally-free, it is also true that $\underline{\mathrm{Ext}}^j(\underline{\omega}_{G^i},M) = (0)$ for $j > 0$ (resp. $\mathrm{Ext}^j(\underline{\omega}_{G^i},M) = 0$ if $S$ is affine).

Since $L$ is a partial resolution of $G$, the complex $\underline{\mathrm{Hom}}(L_0,M) \to \underline{\mathrm{Hom}}(L_1,M) \to \underline{\mathrm{Hom}}(L_2,M)$ has $H^0 = \underline{\mathrm{Hom}}(G,M)$, $H^1 = \underline{\mathrm{Ext}}^1(G,M)$ (resp. without underlining if $S$ is affine). In fact "killing" the $H^2$ of this complex we obtain the complex $\tau_{\leq 1} R\,\underline{\mathrm{Hom}}_{\mathbb{Z}}(G,M)$. On the other hand by applying $\tau_{\leq 1}(\quad)$ to (14.9) we obtain

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(\underline{\omega}_{G^0},M) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(\underline{\omega}_{G},M) \\
\tau_{\leq 1}(R\,\underline{\mathrm{Hom}}_{Z}(G,M)) &
\end{tikzcd}$$

Since the source of this arrow is $R\,\underline{\mathrm{Hom}}(\ell^{G^*},M)$ (14.1) is established.

## §15. COMPARISON WITH CLASSICAL DIEUDONNÉ THEORY

(15.1) Denote by $\mathbb{W}_n = \mathrm{Spec}(\mathbb{Z}[w_o,\dots,w_{n-1}])$ the group scheme of Witt vectors of length $n$ and by $\Phi_n : \mathbb{W}_n \to (\mathbb{G}_a)^n$ the homomorphism given by ghost components. Let $T : \mathbb{W}_n \to \mathbb{W}_{n+m}$ be the homomorphism defined on S-valued points (S any scheme) by

$$T(w_o,\dots,w_{n-1}) = (0,\dots,0,w_o,\dots,w_{n-1})$$

and let $R : \mathbb{W}_{n+1} \to \mathbb{W}_n$ be the homomorphism defined on S-valued points by

$$R(w_o,\dots,w_n) = (w_o,\dots,w_{n-1})$$

Using the mappings $T$, the $\mathbb{W}_n$'s form an inductive system and we denote by $\underset{\rightarrow}{\mathbb{W}}$ the direct limit.

(15.2) Let $k$ be a perfect field of characteristic $p$. Classically [18 bis, 3.12], one defines the Dieudonné module of a unipotent p-divisible group, $G$, as

$$D^*(G) = \mathrm{Hom}_{k\text{-}gr}(G, \underset{\rightarrow}{\mathbb{W}}_k)$$

This definition can be extended to a toroidal p-divisible group, $G$, by setting

$$D^*(G) = D^*(G^*)^{\vee}$$

In (9.2) we defined for $G$ a Barsotti-Tate group over an arbitrary base $S$ (with $p$ locally nilpotent) a crystal on $S$ in locally-free modules, $\mathbb{D}^*(G)$. The category of crystals in locally-free modules on $S_o = \mathrm{Spec}(k)$ (relative to $\mathbb{Z}_p$) is equivalent to the category of free $W(k)$-modules. Explicitly the equivalence is given by

$$M \mapsto \varprojlim M_{W_n(k)}$$

where $M$ is a crystal and $M_{W_n(k)}$ denotes its value on the thickening $S_o \hookrightarrow S_n = \mathrm{Spec}(W_n(k))$

Regarding $\mathbb{D}^*(G)$ as a free $W(k)$-module we can ask about its relation to $D^*(G)$. The answer is provided by the following theorem of Grothendieck.

(15.3) Theorem: There is a canonical isomorphism of functors $D^* \longrightarrow \mathbb{D}^*$ (which will be explicitly constructed below).

(15.4) Because of the decomposition of the category of p-divisible groups/k into the product of the category of toroidal p-divisible groups and the category of unipotent p-divisible groups, it suffices to consider only unipotent groups.

The key to proving (15.3) is Grothendieck's observation that, over $\mathbb{Z}$, the extension

$$(15.5) \qquad 0 \to G_a \to \underrightarrow{\mathbb{W}} \xrightarrow{V} \underrightarrow{\mathbb{W}} \to 0$$

is endowed with a canonical structure of $\natural$-extensions.

To see this one first considers the extensions

$$(15.6_n) \qquad 0 \to \mathbb{G}_a \xrightarrow{T} \mathbb{W}_{n+1} \xrightarrow{R} \mathbb{W}_n \longrightarrow 0 .$$

Let $s: \mathbb{W}_n \longrightarrow \mathbb{W}_{n+1}$ be the set-theoretic section given by:

$$s(w_o,\ldots,w_{n-1}) = (w_o,\ldots,w_{n-1},0)$$

The section $s$ determines a trivialization of the $\mathbb{G}_{a_{\mathbb{W}_n}}$-torseur $\mathbb{W}_{n+1}$. Using this trivialization, endow $\mathbb{W}_{n+1}$ with a structure of $\natural$-torseur, $\nabla_o$ .

We modify $\nabla_0$ by defining a new $\natural$-structure

$$(15.7_n) \qquad \nabla_n = \nabla_0 - \omega_n$$

where

$$(15.8_n) \qquad \omega_n = W_0^{p^n-1}dW_0+\dots+W_{n-1}^{p-1}dW_{n-1} \in \Gamma(\mathbb{Z},\Omega^1_{\mathbb{W}_n})$$

As will be shown in (15.10) $\nabla_n$ makes the extension $(15.6_n)$ into a $\natural$-extension. From the explicit construction of $\nabla_n$ it is immediate that the following is compatible with $\natural$-structures

$$(15.9) \qquad \begin{tikzcd} 0 \arrow[r] & \mathbb{G}_a \arrow[r, "T"] \arrow[d, equal] & \mathbb{W}_{n+1} \arrow[r, "R"] \arrow[d, "T"] & \mathbb{W}_n \arrow[r] \arrow[d, "T"] & 0 \\ 0 \arrow[r] & \mathbb{G}_a \arrow[r, "T"] & \mathbb{W}_{n+2} \arrow[r, "R"] & \mathbb{W}_{n+1} \arrow[r] & 0 \end{tikzcd}$$

Passing to the limit we obtain $\nabla_\infty$, the desired structure of $\natural$-extension on (15.5).

Let us stop here to check

(15.10) Proposition:

(i) Let $t:(\mathbb{G}_a)^n \to (\mathbb{G}_a)^{n+1}$ be the map $(x_0,\dots,x_{n-1}) \mapsto (x_0,\dots,x_{n-1},0)$. View $t$ as a splitting of the extension

$$0 \to \mathbb{G}_a \to \mathbb{G}_a^{n+1} \to \mathbb{G}_a^n \to 0$$

and endow this extension with its trivial $\natural$-structure.

Over $\mathbb{Z}[\frac{1}{p}]$ the diagram

$$(15.11) \qquad \begin{tikzcd} 0 \arrow[r] & \mathbb{G}_a \arrow[r, "T"] \arrow[d, "\begin{matrix} x \\ \downarrow \\ p^n x \end{matrix}"'] & \mathbb{W}_{n+1} \arrow[r, "R"] \arrow[d, "\Phi_{n+1}", "\wr"'] & \mathbb{W}_n \arrow[r] \arrow[d, "\Phi_n", "\wr"'] & 0 \\ 0 \arrow[r] & \mathbb{G}_a \arrow[r] & \mathbb{G}_a^{n+1} \arrow[r] & \mathbb{G}_a^n \arrow[r] & 0 \end{tikzcd}$$

allows us to transport the just-described $\natural$-structure on the lower row to $(15.6_n)$.

Assertion: This $\natural$-structure coincides with $\nabla_n$ .

(ii) $\nabla_n$ makes $(15.6_n)$ into a $\natural$-extension.

Proof: (ii) is an immediate consequence of (i) since the obstruction to the isomorphism

$$s^*(\mathbb{W}_{n+1}) \xrightarrow{\sim} p_1^*(\mathbb{W}_{n+1}) \wedge p_2^*(\mathbb{W}_{n+1})$$

being horizontal is an element of the free abelian group $\Gamma(\Omega^1_{\mathbb{W}_n \times \mathbb{W}_n})$ which dies when we tensor this group with $\mathbb{Z}[\frac{1}{p}]$.

Let $t' = \Phi_{n+1}^{-1} \circ t \circ \Phi_n$ be the splitting obtained by transport of structure. It suffices to show

(15.12) $$d(t'-s) = -\omega_n$$

But $t'-s\ (w_o,\dots,w_{n-1}) = (0,\dots,0,w_n)$
where $p^n w_n + p^{n-1} w_{n-1}^p + \dots + w_o^{p^n} = 0$ .

That is $w_n = \frac{-1}{p^n}(w_o^{p^n} + \dots + p^{n-1} w_{n-1}^p)$

Thus $d(t'-s) = -(W_o^{p^n-1} dW_o + \dots + W_{n-1}^{p-1} dW_{n-1}) = -\omega_n$ .

(15.13) Remark: It follows from (15.10) (i) that the rigidification on $(15.6_n)$ associated to $\nabla_n$ is the restriction of $s$ to $\mathrm{Inf}^1(\mathbb{W}_n)$.

(15.14) We can now define the map $D^*(G) \to \mathbb{D}^*(G)$. For each $n$ interpret the restriction to $W_n(k)$ of the $\natural$-extension (15.5) as being an object in $\mathrm{EXT}^{crys/W_n(k)}(\underrightarrow{\mathbb{W}}_k, \mathbb{G}_{a,W_n(k)})$. Pulling back this object by a homomorphism $\varphi : G \to \underset{\rightarrow}{\mathbb{W}}_{,k}$ gives us an object in $\mathrm{EXT}^{\mathrm{crys}/W_n(k)}(G,\mathbb{G}_a)$. These objects, for variable $n$, piece together and we obtain the desired map

$$\begin{tikzcd}
D^*(G) \arrow[r] \arrow[d, equal] & \mathbb{D}^*(G) \arrow[d, equal] \\
\mathrm{Hom}_{k\text{-gr}}(G,\underset{\rightarrow}{\mathbb{W}}) \arrow[r] & \varprojlim \underline{\mathrm{Ext}}^{\mathrm{crys}/W_n(k)}(G,\mathbb{G}_a)\ .
\end{tikzcd}$$

Proof of (15.3): By Nakayama's lemma it suffices to prove that the reduction modulo $p$ of this map is injective since the source and target are free $W(k)$-modules of the same rank (= height $(G)$).

Let $\varphi : G \to \underset{\rightarrow}{\mathbb{W}}_{k}$ be such that $\varphi^*(\nabla_\infty)$ is the trivial structure of $\natural$-extension. We are to show that $\varphi$ admits a factorization as $\varphi = G \xrightarrow{P} G \xrightarrow{\widetilde{\varphi}} \underset{\rightarrow}{\mathbb{W}}_{,k}$ .

By assumption there is a unique arrow $\psi : G \to W$ which makes the following diagram commutative.

$$(15.15)\qquad \begin{tikzcd}
0 \arrow[r] & \mathbb{G}_a \arrow[r] \arrow[d, equal] & \underset{\rightarrow}{\mathbb{W}} \times_{\underset{\rightarrow}{\mathbb{W}}} G \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, "\varphi"] \arrow[dl, "\psi"'] & 0 \\
0 \arrow[r] & \mathbb{G}_a \arrow[r] & \underset{\rightarrow}{\mathbb{W}} \arrow[r, "V"] & \underset{\rightarrow}{\mathbb{W}} \arrow[r] & 0
\end{tikzcd}$$

We want to show that $\psi$ can be written as $\psi = G \xrightarrow{\tau} \underset{\rightarrow}{\mathbb{W}}_{,k} \xrightarrow{F} \underset{\rightarrow}{\mathbb{W}}_{,k}$ , for some $\tau$ . To show this it suffices to show $\psi|\mathrm{Ker}(F_G) = 0$ . Continue to denote this restricted map by $\psi$ . For $n \gg 0$ (15.15) induces a diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{G}_a \arrow[r] & \mathbb{W}_{n+1} \times_{\mathbb{W}_n} \mathrm{Ker}(F_G) \arrow[r] \arrow[d] & \mathrm{Ker}(F_G) \arrow[r] \arrow[d, "\varphi"] \arrow[dl, "\psi"'] & 0 \\
0 \arrow[r] & \mathbb{G}_a \arrow[r, "T"] & \mathbb{W}_{n+1} \arrow[r, "R"] & \mathbb{W}_n \arrow[r] & 0
\end{tikzcd}$$

From our assumption that $\varphi^*(\nabla_\infty)$ = trivial structure of $\natural$-extension and (15.13) it follows that

$$(15.16) \qquad s \circ R \circ \psi \Big| \mathrm{Inf}^1(\mathrm{Ker}(F_G)) = \psi | \mathrm{Inf}^1(\mathrm{Ker}(F_G)).$$

The following lemma shows that this implies $\psi = 0$ . and completes the proof of (15.3).

(15.17) <u>Lemma</u>: Let H be a finite commutative k-group satisfying

a) $F_H = 0$
b) $V_H$ is nilpotent

Let $\psi : H \to \mathbb{W}_{n+1}$ have components $\psi_0, \ldots, \psi_n$ and assume $\psi_n | \mathrm{Inf}^1(H) = 0$ . Then $\psi = 0$ .

<u>Proof</u>. We use induction on the index of nilpotency of $V_H$ . If $V_H = 0$, then $\psi$ factors through $\mathbb{G}_a \xrightarrow{T} \mathbb{W}_{n+1}$ and we may view $\psi$ as a homomorphism $H \xrightarrow{\psi} \mathbb{G}_a$ . The assumption $\psi | \mathrm{Inf}^1(H) = 0$ implies $\underline{\mathrm{Lie}}(\psi) = 0$ and the result follows from [8,II, §7, 4.3(b)].

Assume the result for groups killed by $V^m$ and that $V^{m+1}$ kills H . Consider the exact sequence (which defines K):

$$0 \to K \to H \xrightarrow{V^m} H^{(p^{-m})} .$$

Because $V^{m+1}$ kills H , $V_{H/K} = 0$ . The induction assumption tells us that $\psi$ factors as

$$\begin{tikzcd}
H \arrow[r] \arrow[d, "\psi"'] & H/K \arrow[dl, "\tilde{\psi}"] \\
\mathbb{W}_{n+1} &
\end{tikzcd}$$

To conclude we must show $\underline{\mathrm{Lie}}(\tilde{\psi}) = 0$ . Because $F_H = 0$ , $V_{K^*}$, $V_{H^*}$ , $V_{(H/K)^*}$ are all zero. The exact sequence

$$0 \to (H/K)^* \to H^* \to K^* \to 0$$

gives rise to an exact sequence of Dieudonne modules

$$\begin{tikzcd}
0 \arrow[r] & D^*(K^*) \arrow[r] \arrow[d, equal] & D^*(H^*) \arrow[r] \arrow[d, equal] & D^*((H/K)^*) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \underline{\mathrm{Lie}}(K) \arrow[r] & \underline{\mathrm{Lie}}(H) \arrow[r] & \underline{\mathrm{Lie}}(H/K) \arrow[r] & 0
\end{tikzcd}$$

But $\psi|\mathrm{Inf}^1(H) = 0$ implies $\underline{\mathrm{Lie}}(\psi) = 0$ and since $\underline{\mathrm{Lie}}(H)$ maps onto $\underline{\mathrm{Lie}}(H/K)$ it follows that $\underline{\mathrm{Lie}}(\tilde{\psi}) = 0$ .

## Bibliography

1. Artin, M. and Mazur,B., Autoduality of the Jacobian (to appear).

1 bis. Barsotti.I., Factor sets and differentials on Abelian varieties,Trans. A.M.S. vol 84, no. 1, pp. 85-108 (1957).

2. Berthelot, P., Cohomologie p-cristalline des schémas,... C.R. Acad. Sc. Paris, t. 269 pp 297-300, 357-360, 397-400, August, September 1969; t. 272 pp. 42-45, 141-144, 254-257.

3. Berthelot, P., Cohomologie cristalline des schémas, Thèse Paris 1972 (to appear).

4. Breen, L., Extensions of abelian sheaves and Eilenberg-MacLane algebras, Inventiones Math., Vol. 9 pp. 15-44 (1969)

5. Cartier, P., Relèvements des Groupes Formels Commutatifs, Séminaire Bourbaki, No. 359 (June 1969).

6. Deligne, P., Théorie de Hodge III, preprint from I.H.E.S. (Bures-sur-Yvette, France) (1972).

7. Deligne, P., Exposé XVIII in $SGA_4$ (c.f. 11 below).

8. Demazure, M., and Gabriel, P., Groupes Algébriques (Tome I) Mason and Cie , Paris, 1970.

9. Giraud, J. Cohomologie non abelienne, Springer-Verlag (1971).

10. Grothendieck, A. and Dieudonné, J., Eléments de Géométrie Algébrique, Publ. Math. IHES, nos. 4,8,11,17,20,24,28,32.

11. Grothendieck, A. et al., Séminaire de Géométrie Algébriques: $SGA_1$; $SGA_3$, $SGA_4$, $SGA_7$ published as Springer Lecture Notes nos. 224; 151-153; 269-70, 305; 289.

12. Grothendieck.A., Catégories Cofibrees Additive et Complexe Cotangent Relatif, Springer Lecture Notes no. 79 (1968).

13. Grothendieck, A., Groupes de Barsotti-Tate et cristaux, Actes du Congres intern. math., 1970, 1, pp. 431-436, Gauthier-Villars, Paris.

14. Illusie, L , Complexe Cotangent et Deformations, Springer Lecture Notes nos. 239 and 283 (1971, 1972).

15. Illusie, L. Mimeographed notes on Picard Stacks and Deformation Theory, MIT, 1970-71 (unpublished).

15 bis. Lang, S., Abelian Varieties, Interscience, New York, 1959.

16. Messing, W., The Crystals associated to Barsotti-Tate groups: with applications to Abelian schemes, Springer Lecture Notes, no. 264 (1972).

17. Messing, W., The Universal Extension of an Abelian variety by a vector group, Symposia Mathematica Vol XI (to appear).

18. Murre, J.P., On Generalized Picard varieties, Math. Ann., 145, 1962 pp. 334-353.

18.bis. Oda, T., The First De Rham cohomology group and Dieudonné modules, Ann. Sci. E.N.S., 4$^e$ serie, t.2. (1969) pp. 63-135.

19. Oort, F., Commutative Group Schemes, Springer Lecture Notes no. 15(1966)

20. Oort, F. and Tate, J., Group schemes of prime order, Ann. Sci. ENS 4$^e$ serie, t.3 (1970) pp 1-21.

21. Raynaud, M., Exposé XIII in Schémas abéliens (Séminaire Faculte d'Orsay 1967-68)(to appear?).

21.bis. Raynaud, M., Faisceaux amples sur les schémas en groupes et les espaces homogènes, Springer Lecture Notes no. 119 (1970).

22. Rosenlicht, M. Extensions of vector groups by abelian varieties, Amer. J. Math., 80, 1958, pp. 685-714.

22.bis. Rosenlicht, M., Group varieties and Differential Forms Proceeding Int. Cong. Math., Amsterdam 1954, North Holland (1956).

23. Serre, J.P., Géométrie algébrique et géométrie analytique, Ann. Inst. Fourier, 1955-56, pp. 1-42.

24. Serre, J.P. Exposé XI in Séminaire Chevalley, 1958-59, Varieties de Picard Secretariat Mathematique Paris (1960).

25. Serre, J.P., Groupes algébriques et corps de classes, Hermann Paris (1959).

26. Tate,J., WC-groups over p-adic fields Séminaire Bourbaki expose 156 (Dec. 1957).

27. Weil, A., Variétiés abéliennes, Colloque d'algèbre et theorie des nombres, Paris, 1949.