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The universality of the Rezk nerve

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We functorially associate to each relative ∞–category $(\mathcal{R}, \boldsymbol{W})$ a simplicial space $\mathrm{N}^{\mathrm{R}}_{\infty}(\mathcal{R}, \boldsymbol{W})$, called its *Rezk nerve* (a straightforward generalization of Rezk's "classification diagram" construction for relative categories). We prove the following *local* and *global* universal properties of this construction: (i) that the complete Segal space generated by the Rezk nerve $\mathrm{N}^{\mathrm{R}}_{\infty}(\mathcal{R}, \boldsymbol{W})$ is precisely the one corresponding to the localization $\mathcal{R}[\![\boldsymbol{W}^{-1}]\!]$; and (ii) that the Rezk nerve functor defines an equivalence $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{\infty}[\![\boldsymbol{W}_{\mathrm{BK}}^{-1}]\!] \xrightarrow{\sim} \mathcal{C}\mathrm{at}_{\infty}$ from a localization of the ∞–category of relative ∞–categories to the ∞–category of ∞–categories.

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0 Introduction

0.1 The Rezk nerve

A *relative ∞–category* is a pair $(\mathcal{R}, \boldsymbol{W})$ of an ∞–category $\mathcal{R}$ and a subcategory $\boldsymbol{W} \subset \mathcal{R}$ containing all the equivalences, called the subcategory of *weak equivalences*. Freely inverting the weak equivalences, we obtain the *localization* of this relative ∞–category, namely the initial functor

$$\mathcal{R} \to \mathcal{R}[\![\boldsymbol{W}^{-1}]\!]$$

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from $\mathcal{R}$ which sends all maps in $\boldsymbol{W}$ to equivalences. In general, it is extremely difficult to access the localization.[1] To ameliorate this state of affairs, in this paper we provide a novel method of accessing this localization via Rezk's theory of *complete Segal spaces*.

To describe this, let us first recall that the ∞–category $\mathcal{CSS}$ of complete Segal spaces participates in a diagram

$$s\mathcal{S} \underset{\mathrm{U}_{\mathcal{CSS}}}{\overset{\mathrm{L}_{\mathcal{CSS}}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{CSS} \underset{\mathrm{N}_\infty}{\overset{\mathrm{N}_\infty^{-1}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\sim}}}} \mathcal{C}\mathrm{at}_\infty.$$

That is, it sits as a reflective subcategory of the ∞–category $s\mathcal{S}$ of simplicial spaces, and it is equivalent to the ∞–category $\mathcal{C}\mathrm{at}_\infty$ of ∞–categories. In particular, one can contemplate the complete Segal space (or equivalently, the ∞–category) *generated* by an arbitrary simplicial space Y, much as one can contemplate the 1–category generated by an arbitrary simplicial set: this is encoded by the unit

$$Y \xrightarrow{\eta} \mathrm{L}_{\mathcal{CSS}}(Y)$$

of the adjunction (where we omit the inclusion functor $\mathrm{U}_{\mathcal{CSS}}$ for brevity).

Now, given a relative ∞–category $(\mathcal{R}, \boldsymbol{W})$, its *Rezk nerve* is a certain simplicial space

$$\mathrm{N}_\infty^{\mathrm{R}}(\mathcal{R}, \boldsymbol{W}) \in s\mathcal{S}$$

which "wants to be" the complete Segal space

$$\mathrm{N}_\infty(\mathcal{R}[\![\boldsymbol{W}^{-1}]\!]) \in \mathcal{CSS}$$

corresponding to its localization:

- it admits canonical maps
$$\mathrm{N}_\infty(\mathcal{R}) \to \mathrm{N}_\infty^{\mathrm{R}}(\mathcal{R}, \boldsymbol{W}) \to \mathrm{N}_\infty(\mathcal{R}[\![\boldsymbol{W}^{-1}]\!]),$$
and moreover
- its construction manifestly dictates that for any ∞–category $\mathcal{C}$, the restriction map
$$\hom_{s\mathcal{S}}(\mathrm{N}_\infty^{\mathrm{R}}(\mathcal{R}, \boldsymbol{W}), \mathrm{N}_\infty(\mathcal{C})) \to \hom_{s\mathcal{S}}(\mathrm{N}_\infty(\mathcal{R}), \mathrm{N}_\infty(\mathcal{C})) \simeq \hom_{\mathcal{C}\mathrm{at}_\infty}(\mathcal{R}, \mathcal{C})$$

[1] For instance, even in the case that $\mathcal{R}$ is a one-object 1–category and we are only interested in its 1–*categorical* localization, ie the composite $\mathcal{R} \to \mathcal{R}[\![\boldsymbol{W}^{-1}]\!] \to \mathrm{ho}(\mathcal{R}[\![\boldsymbol{W}^{-1}]\!]) \simeq \mathcal{R}[\boldsymbol{W}^{-1}]$ — that is, in the case that we are interested in freely inverting certain elements of a monoid — obtaining a concrete description is nevertheless an intractable (in fact, computationally undecidable) task, closely related to the so-called "word problem" for generators and relations in abstract algebra.

factors through the subspace (ie collection of path components) of those functors $\mathcal{R} \to \mathcal{C}$ sending all maps in $\boldsymbol{W} \subset \mathcal{R}$ to equivalences in $\mathcal{C}$.

Unfortunately, life is not quite so simple: the Rezk nerve is not generally a complete Segal space (or even a Segal space).[2] Nevertheless, the second-best possible thing is true.

Theorem (3.8) *The above maps extend to a commutative diagram*

$$\begin{tikzcd}
\mathrm{N}_\infty(\mathcal{R}) \arrow[r] \arrow[d, "\eta"', "\wr"] & \mathrm{N}^{\mathrm{R}}_\infty(\mathcal{R}, \boldsymbol{W}) \arrow[r] \arrow[d, "\eta"] & \mathrm{N}_\infty(\mathcal{R}[\![\boldsymbol{W}^{-1}]\!]) \arrow[d, "\wr"', "\eta"] \\
\mathrm{L}_{\mathcal{CSS}}(\mathrm{N}_\infty(\mathcal{R})) \arrow[r] & \mathrm{L}_{\mathcal{CSS}}(\mathrm{N}^{\mathrm{R}}_\infty(\mathcal{R}, \boldsymbol{W})) \arrow[r, "\sim"'] & \mathrm{L}_{\mathcal{CSS}}(\mathrm{N}_\infty(\mathcal{R}[\![\boldsymbol{W}^{-1}]\!]))
\end{tikzcd}$$

In other words, the complete Segal space generated by the Rezk nerve of $(\mathcal{R}, \boldsymbol{W})$ is precisely the one corresponding to its localization.

This theorem provides a *local* universal property of the Rezk nerve: it asserts that the composite

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathrm{N}^{\mathrm{R}}_\infty} s\mathcal{S} \xrightarrow{\mathrm{L}_{\mathcal{CSS}}} \mathcal{CSS} \underset{\sim}{\xrightarrow{\mathrm{N}_\infty^{-1}}} \mathcal{C}\mathrm{at}_\infty$$

takes each relative ∞–category $(\mathcal{R}, \boldsymbol{W})$ to its localization $\mathcal{R}[\![\boldsymbol{W}^{-1}]\!]$. However, it says nothing about the effect of this composite on *morphisms* of relative ∞–categories. To this end, we also prove the following:

Theorem (Propositions 3.9 and 3.11) *The above composite is canonically equivalent to the localization functor*

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty.$$

In particular, denoting by $\boldsymbol{W}_{\mathrm{BK}} \subset \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ *the subcategory of maps which it takes to equivalences, the above composite induces an equivalence*

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty[\![\boldsymbol{W}_{\mathrm{BK}}^{-1}]\!] \xrightarrow{\sim} \mathcal{C}\mathrm{at}_\infty.$$

In other words, the Rezk nerve functor does indeed *functorially* compute localizations of relative ∞–categories, and, moreover, the induced "homotopy theory" on the ∞–category $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ of relative ∞–categories — that is, the relative ∞–category

[2] We provide sufficient conditions on $(\mathcal{R}, \boldsymbol{W})$ for its Rezk nerve $\mathrm{N}^{\mathrm{R}}_\infty(\mathcal{R}, \boldsymbol{W})$ to be a (complete) Segal space in [15].

structure $(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty, \boldsymbol{W}_{\mathrm{BK}})$ that results therefrom — gives a presentation of the ∞–category $\mathcal{C}\mathrm{at}_\infty$ of ∞–categories. We therefore deem this result as capturing the *global* universal property of the Rezk nerve.

Remark 0.1 The Rezk nerve functor is a close cousin of Rezk's "classification diagram" functor of [17, Section 3.3]; to emphasize the similarity, we denote the latter functor by

$$\mathrm{RelCat} \xrightarrow{\mathrm{N}^{\mathrm{R}}} ss\mathcal{S}\mathrm{et}$$

and refer to it as the 1*–categorical Rezk nerve*. In fact, as we explain in Remark 3.2, this is essentially just the restriction of the ∞–categorical Rezk nerve functor, in the sense that there is a canonical commutative diagram

$$\begin{tikzcd} \mathrm{RelCat} \arrow[r, "\mathrm{N}^{\mathrm{R}}"] \arrow[d] & s(s\mathcal{S}\mathrm{et}) \arrow[r, "s(|-|)"] & s\mathcal{S} \\ \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \arrow[urr, bend right, "\mathrm{N}^{\mathrm{R}}_\infty"'] & & \end{tikzcd}$$

in $\mathcal{C}\mathrm{at}_\infty$. In Remark 3.10, we use this observation to show that our global universal property of the ∞–categorical Rezk nerve can be seen as a generalization of work of Barwick and Kan.

0.2 Conventions

Though it stands alone, this paper belongs to a series on *model ∞–categories*. These papers share many key ideas; thus, rather than have the same results appear repeatedly in multiple places, we have chosen to liberally cross-reference between them. To this end, we introduce the following "code names":

title	reference	code
Model ∞–categories, I: Some pleasant properties of the ∞–category of simplicial spaces	[11]	S
The universality of the Rezk nerve	n/a	N
On the Grothendieck construction for ∞–categories	[16]	G
Hammocks and fractions in relative ∞–categories	[15]	H
Model ∞–categories, II: Quillen adjunctions	[12]	Q
Model ∞–categories, III: The fundamental theorem	[13]	M

Thus, for instance, to refer to [13, Theorem 1.9], we will simply write Theorem M.1.9. (The letters are meant to be mnemonical: they stand for "simplicial space", "nerve", "Grothendieck", "hammock", "Quillen" and "model", respectively.)

We take quasicategories as our preferred model for ∞–categories, and in general we adhere to the notation and terminology of Lurie [7; 9].[3] In fact, our references to these two works will be frequent enough that it will be convenient for us to adopt Lurie's convention and use the code names T and A for them, respectively.

However, we work invariantly to the greatest possible extent: that is, we primarily work *within the ∞–category of ∞–categories*. Thus, for instance, we will omit all technical uses of the word "essential", eg we will use the term *unique* in situations where one might otherwise say "essentially unique" (ie parametrized by a contractible space). For a full treatment of this philosophy as well as a complete elaboration of our conventions, we refer the interested reader to Appendix S.A. The casual reader should feel free to skip this on a first reading; on the other hand, the careful reader may find it useful to peruse that section before reading the present paper. For the reader's convenience, we also provide a complete index of the notation that is used throughout this sequence of papers in Appendix S.B.

Outline

We now provide a more detailed outline of the contents of this paper.

- In Section 1, we undertake a study of relative ∞–categories and their localizations.
- In Section 2, we briefly review the theory of complete Segal spaces.
- In Section 3, we introduce the Rezk nerve and state its local and global universal properties. We give a proof of the global universal property which relies on the local one, but we defer the proof of the local one to Section 4.
- In Section 4, we prove the local universal property of the Rezk nerve. Though much of the proof is purely formal, at its heart it ultimately relies on some rather delicate model-categorical arguments.

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[3] A notable exception is the term "localization": we refer to the notion described in [7, Definition 5.2.7.2] as a "left localization". (The latter is a special case of the former; see Example 1.13.)

Yusufzai for providing helpful comments on a preliminary draft. It is also our pleasure to thank Katherine de Kleer for writing a Python script verifying the identities for the simplicial homotopies defined therein.[4] We would also like to thank an anonymous referee for a very careful report, as well as for pointing out both a subtle error and a means of fixing it. Lastly, we thank the NSF graduate research fellowship program (grant DGE-1106400) for its financial support during the time that this work was carried out.

1 Relative ∞–categories and their localizations

Given an ∞–category and some chosen subset of its morphisms, we are interested in freely inverting those morphisms. In order to codify these initial data, we introduce the following:

Definition 1.1 A *relative ∞–category* is a pair $(\mathcal{R}, \boldsymbol{W})$ of an ∞–category $\mathcal{R}$ and a subcategory $\boldsymbol{W} \subset \mathcal{R}$, called the subcategory of *weak equivalences*, such that $\boldsymbol{W}$ contains all the equivalences (and in particular, all the objects) in $\mathcal{R}$. These form the evident ∞–category $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$.[5] Weak equivalences will be denoted by the symbol $\xrightarrow{\approx}$. Though we will of course write $\mathcal{R}$ for the ∞–category obtained by forgetting $\boldsymbol{W}$, to ease notation we will also sometimes simply write $\mathcal{R}$ for the pair $(\mathcal{R}, \boldsymbol{W})$. We write $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at} \subset \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ for the full subcategory on those relative ∞–categories $(\mathcal{R}, \boldsymbol{W})$ such that $\mathcal{R} \in \mathcal{C}\mathrm{at} \subset \mathcal{C}\mathrm{at}_\infty$.

Remark 1.2 As we are working invariantly, our Definition 1.1 is not quite a generalization of the 1–category RelCat as given eg in [1, Section 3.1] or [6, Definition 3.1], an object of which is a *strict* category $\mathcal{R} \in \mathrm{Cat}$ (ie a simplicial set satisfying the Segal condition; see subitem S.A(4)(c)) equipped with a wide subcategory $\boldsymbol{W} \subset \mathcal{R}$ (ie one containing all the objects). For emphasis, we will therefore sometimes refer to objects of RelCat as *strict relative categories*.

In addition to being the only meaningful variant in the invariant world, Definition 1.1 allows for a clean and aesthetically appealing definition of localization, namely as a left adjoint (see Definition 1.8). In any case, as we are ultimately only interested in relative ∞–categories because we are interested in their localizations, this requirement is no real loss.

[4] This script is readily available upon request.

[5] To be precise, one can view $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \simeq \mathrm{Fun}^{\text{surj mono}}([1], \mathcal{C}\mathrm{at}_\infty) \subset \mathrm{Fun}([1], \mathcal{C}\mathrm{at}_\infty)$ as the full subcategory on those functors selecting the inclusion of a surjective monomorphism.

Despite these differences, there is an evident functor

$$\mathrm{RelCat} \to \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at},$$

to which we will refer on occasion.

Notation 1.3 In order to disambiguate our notation associated to various relative ∞–categories, we introduce the following conventions:

- When multiple relative ∞–categories are under discussion, we will sometimes decorate them for clarity. For instance, we may write $(\mathcal{R}_1, \boldsymbol{W}_1)$ and $(\mathcal{R}_2, \boldsymbol{W}_2)$ to denote two arbitrary relative ∞–categories, or we may instead write $(\mathcal{I}, \boldsymbol{W}_{\mathcal{I}})$ and $(\mathcal{J}, \boldsymbol{W}_{\mathcal{J}})$.
- Moreover, we will eventually study certain "named" relative ∞–categories; for example, there is a *Barwick–Kan relative structure* on $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ itself (see Definition 1.16). We will always subscript the subcategory of weak equivalences of such a relative ∞–category with (an abbreviation of) its name; for example, we will write $\boldsymbol{W}_{\mathrm{BK}} \subset \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$. We may also merely similarly subscript the ambient ∞–category to denote the relative ∞–category; for example, we will write $(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{\mathrm{BK}} = (\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty, \boldsymbol{W}_{\mathrm{BK}})$.
- Finally, there will occasionally be two different ∞–categories with relative structures of the same name. In such cases, if disambiguation is necessary, we will additionally superscript the subcategory of weak equivalences with the name of the ambient ∞–category. For instance, we would write $\boldsymbol{W}_{\mathrm{BK}}^{\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty} \subset \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ to distinguish it from the subcategory $\boldsymbol{W}_{\mathrm{BK}}^{\mathrm{RelCat}} \subset \mathrm{RelCat}$.

We have the following fundamental source of examples of relative ∞–categories:

Example 1.4 If $\mathcal{R} \to \mathcal{C}$ is any functor of ∞–categories, we can define a relative ∞–category $(\mathcal{R}, \boldsymbol{W})$ by declaring $\boldsymbol{W} \subset \mathcal{R}$ to be the subcategory on those maps that are sent to equivalences in $\mathcal{C}$. Note that $\boldsymbol{W} \subset \mathcal{R}$ will automatically have the two-out-of-three property.

Definition 1.5 In the situation of Example 1.4, we will say that the functor $\mathcal{R} \to \mathcal{C}$ *creates* the subcategory $\boldsymbol{W} \subset \mathcal{R}$.

We will make heavy use of the following construction:

Notation 1.6 Given any $(\mathcal{R}_1, \boldsymbol{W}_1), (\mathcal{R}_2, \boldsymbol{W}_2) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$, we define

$$(\mathrm{Fun}(\mathcal{R}_1, \mathcal{R}_2)^{\mathcal{R}\mathrm{el}}, \mathrm{Fun}(\mathcal{R}_1, \mathcal{R}_2)^{\boldsymbol{W}}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$$

by setting

$$\mathrm{Fun}(\mathcal{R}_1, \mathcal{R}_2)^{\mathcal{R}\mathrm{el}} \subset \mathrm{Fun}(\mathcal{R}_1, \mathcal{R}_2)$$

to be the full subcategory on those functors which send $\boldsymbol{W}_1 \subset \mathcal{R}_1$ into $\boldsymbol{W}_2 \subset \mathcal{R}_2$, and setting

$$\mathrm{Fun}(\mathcal{R}_1, \mathcal{R}_2)^{\boldsymbol{W}} \subset \mathrm{Fun}(\mathcal{R}_1, \mathcal{R}_2)^{\mathcal{R}\mathrm{el}}$$

to be the (generally nonfull) subcategory on the natural weak equivalences.[6] It is not hard to see that this defines an internal hom bifunctor for $(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty, \times)$.

It will be useful to have the following terminology:

Definition 1.7 If $\mathcal{C}$ is any ∞–category, we call $(\mathcal{C}, \mathcal{C}^{\simeq})$ the associated *minimal relative ∞–category* and we call $(\mathcal{C}, \mathcal{C})$ the associated *maximal relative ∞–category*. These define fully faithful inclusions

$$\begin{tikzcd} \mathcal{C}\mathrm{at}_\infty \arrow[r, bend left, "\min"] \arrow[r, bend right, "\max"'] & \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \arrow[l] \end{tikzcd}$$

which are respectively left and right adjoint to the forgetful functor $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathrm{U}_{\mathcal{R}\mathrm{el}}} \mathcal{C}\mathrm{at}_\infty$ sending $(\mathcal{R}, \boldsymbol{W})$ to $\mathcal{R}$. For $[n] \in \boldsymbol{\Delta} \subset \mathcal{C}\mathrm{at}_\infty$, we will use the abbreviation $[n]_{\boldsymbol{W}} = \max([n])$, since these relative categories will appear quite often; correspondingly, we will also make the implicit identification $[n] = \min([n])$.

We now come to our central object of interest.

Definition 1.8 The functor min: $\mathcal{C}\mathrm{at}_\infty \to \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ also admits a left adjoint

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathscr{L}} \mathcal{C}\mathrm{at}_\infty,$$

which we refer to as the *localization* functor on relative ∞–categories. For a relative ∞–category $(\mathcal{R}, \boldsymbol{W}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$, we will often write $\mathcal{R}[\![\boldsymbol{W}^{-1}]\!] = \mathscr{L}(\mathcal{R}, \boldsymbol{W})$; we only

[6] If we consider $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \subset \mathrm{Fun}([1], \mathcal{C}\mathrm{at}_\infty)$, then $\mathrm{Fun}(\mathcal{R}_1, \mathcal{R}_2)^{\mathcal{R}\mathrm{el}}$ is simply the ∞–category of natural transformations.

write $\mathscr{L}$ since the notation $(-)[\![(-)^{-1}]\!]$ is a bit unwieldy. Explicitly, its value on $(\mathcal{R}, \boldsymbol{W}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ can be obtained as the pushout

$$\mathcal{R}[\![\boldsymbol{W}^{-1}]\!] \simeq \operatorname{colim}\left(\begin{tikzcd} \boldsymbol{W} \arrow[r] \arrow[d] & \mathcal{R} \\ \boldsymbol{W}^{\mathrm{gpd}} & \end{tikzcd} \right)$$

in $\mathcal{C}\mathrm{at}_\infty$ (and the functor itself can be obtained by applying this construction in families).

Remark 1.9 Using model categories, one can of course compute the pushout in $\mathcal{C}\mathrm{at}_\infty$ of Definition 1.8 by working in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$ (which is left proper), for instance after presenting the map $\boldsymbol{W} \to \boldsymbol{W}^{\mathrm{gpd}}$ using the derived unit of the Quillen adjunction $\mathrm{id}\colon s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}} \rightleftarrows s\mathcal{S}\mathrm{et}_{\mathrm{KQ}} :\!\mathrm{id}$, ie after taking a fibrant replacement via a cofibration in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ of a quasicategory presenting $\boldsymbol{W}$. However, note that this derived unit can be quite difficult to describe in practice, and, moreover, the resulting pushout will generally still be very far from being a quasicategory. Equally inexplicitly, one can also obtain a quasicategory presenting $\mathcal{R}[\![\boldsymbol{W}^{-1}]\!]$ by computing a fibrant replacement in the marked model structure of Proposition T.3.1.3.7 (ie in the specialization of the model structure given there to the case where the base is the terminal object $\mathrm{pt}_{s\mathcal{S}\mathrm{et}}$).

Remark 1.10 We will also use the term "localization" to refer to the canonical map $\mathcal{R} \to \mathcal{R}[\![\boldsymbol{W}^{-1}]\!]$ in $\mathcal{C}\mathrm{at}_\infty$ satisfying the universal property that for any $\mathcal{C} \in \mathcal{C}\mathrm{at}_\infty$, the restriction

$$\hom_{\mathcal{C}\mathrm{at}_\infty}(\mathcal{R}[\![\boldsymbol{W}^{-1}]\!], \mathcal{C}) \to \hom_{\mathcal{C}\mathrm{at}_\infty}(\mathcal{R}, \mathcal{C})$$

defines an equivalence onto the subspace

$$\hom_{\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty}((\mathcal{R}, \boldsymbol{W}), \min(\mathcal{C})) \subset \hom_{\mathcal{C}\mathrm{at}_\infty}(\mathcal{R}, \mathcal{C})$$

of those functors which take $\boldsymbol{W}$ into $\mathcal{C}^{\simeq}$.[7] Thus, by definition, the map $\mathcal{R} \to \mathcal{R}[\![\boldsymbol{W}^{-1}]\!]$ is an epimorphism in $\mathcal{C}\mathrm{at}_\infty$.

Example 1.11 The localization of a minimal relative ∞–category $\min(\mathcal{C}) = (\mathcal{C}, \mathcal{C}^{\simeq})$ is simply the identity functor $\mathcal{C} \xrightarrow{\sim} \mathcal{C}$.

[7]This map can be obtained either by applying $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathscr{L}} \mathcal{C}\mathrm{at}_\infty$ to the counit $\min(\mathcal{R}) \to (\mathcal{R}, \boldsymbol{W})$ of the adjunction $\min \dashv \mathrm{U}_{\mathcal{R}\mathrm{el}}$, or by applying $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathrm{U}_{\mathcal{R}\mathrm{el}}} \mathcal{C}\mathrm{at}_\infty$ to the unit $(\mathcal{R}, \boldsymbol{W}) \to \min(\mathcal{R}[\![\boldsymbol{W}^{-1}]\!])$ of the adjunction $\mathscr{L} \dashv \min$.

Example 1.12 The localization of a maximal relative ∞–category $\max(\mathcal{C}) = (\mathcal{C}, \mathcal{C})$ is the groupoid completion functor $\mathcal{C} \to \mathcal{C}^{\mathrm{gpd}}$ (ie the component at $\mathcal{C}$ of the unit of the adjunction $(-)^{\mathrm{gpd}} \colon \mathcal{C}\mathrm{at}_\infty \rightleftarrows \mathcal{S} : \mathrm{U}_{\mathcal{S}}$).

Example 1.13 Given a left localization adjunction $\mathrm{L} \colon \mathcal{C} \rightleftarrows \mathrm{L}\mathcal{C} : \mathrm{U}$ (ie an adjunction with fully faithful right adjoint), if we define $\boldsymbol{W} \subset \mathcal{C}$ to be created by $\mathcal{C} \xrightarrow{\mathrm{L}} \mathrm{L}\mathcal{C}$, then the localization of $(\mathcal{C}, \boldsymbol{W})$ is precisely $\mathcal{C} \xrightarrow{\mathrm{L}} \mathrm{L}\mathcal{C}$: that is, the functor $\mathcal{C} \xrightarrow{\mathrm{L}} \mathrm{L}\mathcal{C}$ induces an equivalence $\mathcal{C}[\![\boldsymbol{W}^{-1}]\!] \xrightarrow{\sim} \mathrm{L}\mathcal{C}$, which is in fact inverse to the composite $\mathrm{L}\mathcal{C} \xrightarrow{\mathrm{U}} \mathcal{C} \to \mathcal{C}[\![\boldsymbol{W}^{-1}]\!]$. This follows from Proposition T.5.2.7.12, or alternatively from Lemma 1.24 (see Remark 1.25). Of course, a dual statement holds for right localization adjunctions.

For an arbitrary relative ∞–category $(\mathcal{R}, \boldsymbol{W})$, note that the localization map $\mathcal{R} \to \mathcal{R}[\![\boldsymbol{W}^{-1}]\!]$ might *not* create the subcategory $\boldsymbol{W} \subset \mathcal{R}$: there might be strictly more maps in $\mathcal{R}$ which are sent to equivalences in $\mathcal{R}[\![\boldsymbol{W}^{-1}]\!]$. This leads us to the following notion:

Definition 1.14 A relative ∞–category $(\mathcal{R}, \boldsymbol{W})$ is called *saturated* if the localization map $\mathcal{R} \to \mathcal{R}[\![\boldsymbol{W}^{-1}]\!]$ creates the subcategory $\boldsymbol{W} \subset \mathcal{R}$.

Remark 1.15 If a relative ∞–category $(\mathcal{R}, \boldsymbol{W}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ has its subcategory of weak equivalences $\boldsymbol{W} \subset \mathcal{R}$ created by *any* functor $\mathcal{R} \to \mathcal{C}$, then $(\mathcal{R}, \boldsymbol{W})$ will automatically be saturated. To see this, consider the induced factorization

$$\begin{tikzcd}
\mathcal{R} \arrow[rr] \arrow[dr] & & \mathcal{C} \\
& \mathcal{R}[\![\boldsymbol{W}^{-1}]\!] \arrow[ur, dashed] &
\end{tikzcd}$$

in $\mathcal{C}\mathrm{at}_\infty$. This implies that any morphism in $\mathcal{R}$ which is sent to an equivalence in $\mathcal{R}[\![\boldsymbol{W}^{-1}]\!]$ must also be sent to an equivalence in $\mathcal{C}$ (by the functoriality of inverse morphisms), so that by definition it lies in the subcategory $\boldsymbol{W} \subset \mathcal{R}$.

Now, we will be using relative ∞–categories as "presentations of ∞–categories", namely of their localizations. However, a map of relative ∞–categories may induce an equivalence on localizations without itself being an equivalence in $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$. This leads us to the following notion:

Definition 1.16 We define the subcategory $\boldsymbol{W}_{\mathrm{BK}} \subset \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ of *Barwick–Kan weak equivalences* to be created by the localization functor $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathscr{L}} \mathcal{C}\mathrm{at}_\infty$. We denote the resulting relative ∞–category by $(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{\mathrm{BK}} = (\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty, \boldsymbol{W}_{\mathrm{BK}}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$.

The following result then justifies our usage of relative ∞–categories as "presentations of ∞–categories":

Proposition 1.17 *The functors in the left localization adjunction* $\mathscr{L}\colon \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \rightleftarrows \mathcal{C}\mathrm{at}_\infty :\!\min$ *induce inverse equivalences*

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty[\![\boldsymbol{W}_{\mathrm{BK}}^{-1}]\!] \simeq \mathcal{C}\mathrm{at}_\infty$$

in $\mathcal{C}\mathrm{at}_\infty$.

Proof This is a special case of Example 1.13. □

We have the following strengthening of Remark 1.10:

Proposition 1.18 *For any* $(\mathcal{R}, \boldsymbol{W}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ *and any* $\mathcal{C} \in \mathcal{C}\mathrm{at}_\infty$, *the restriction*

$$\mathrm{Fun}(\mathcal{R}[\![\boldsymbol{W}^{-1}]\!], \mathcal{C}) \to \mathrm{Fun}(\mathcal{R}, \mathcal{C})$$

along the localization functor $\mathcal{R} \to \mathcal{R}[\![\boldsymbol{W}^{-1}]\!]$ *defines an equivalence onto the full subcategory of* $\mathrm{Fun}(\mathcal{R}, \mathcal{C})$ *spanned by those functors which take* $\boldsymbol{W}$ *into* $\mathcal{C}^{\simeq}$.

Proof We begin by observing that this functor is a monomorphism in $\mathcal{C}\mathrm{at}_\infty$: this is because we have a pullback diagram

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{R}[\![\boldsymbol{W}^{-1}]\!], \mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Fun}(\boldsymbol{W}^{\mathrm{gpd}}, \mathcal{C}) \arrow[d] \\
\mathrm{Fun}(\mathcal{R}, \mathcal{C}) \arrow[r] & \mathrm{Fun}(\boldsymbol{W}, \mathcal{C})
\end{tikzcd}$$

in $\mathcal{C}\mathrm{at}_\infty$ in which the right arrow is clearly a monomorphism, and monomorphisms are closed under pullback. So, in particular, this functor is the inclusion of a subcategory. Then, to see that it is full, suppose we are given two functors $\mathcal{R}[\![\boldsymbol{W}^{-1}]\!] \rightrightarrows \mathcal{C}$, considered as objects of $\mathrm{Fun}(\mathcal{R}[\![\boldsymbol{W}^{-1}]\!], \mathcal{C})$. A natural transformation between their images in $\mathrm{Fun}(\mathcal{R}, \mathcal{C})$ is given by a functor $[1] \times \mathcal{R} \to \mathcal{C}$ which restricts to the two composites $\mathcal{R} \to \mathcal{R}[\![\boldsymbol{W}^{-1}]\!] \rightrightarrows \mathcal{C}$ on the two objects $0, 1 \in [1]$. Since we already know that $\mathrm{Fun}(\mathcal{R}[\![\boldsymbol{W}^{-1}]\!], \mathcal{C}) \subset \mathrm{Fun}(\mathcal{R}, \mathcal{C})$ is the inclusion of a subcategory, it suffices to obtain an extension

$$\begin{tikzcd}
{[1] \times \mathcal{R}} \arrow[r] \arrow[d] & \mathcal{C} \\
{[1] \times \mathcal{R}[\![\boldsymbol{W}^{-1}]\!]} \arrow[ur, dashed] &
\end{tikzcd}$$

in $\mathcal{C}\mathrm{at}_\infty$. For this, consider the diagram

$$\begin{tikzcd}
& \{0,1\} \times \boldsymbol{W} \arrow[rr] \arrow[dl] \arrow[dd] & & \{0,1\} \times \boldsymbol{W}^{\mathrm{gpd}} \arrow[dl] \arrow[dd] & \\
\{0,1\} \times \mathcal{R} \arrow[rr] \arrow[dd] & & \{0,1\} \times \mathcal{R}[\![\boldsymbol{W}^{-1}]\!] \arrow[rr] \arrow[dd] & & \mathcal{C} \\
& {[1]} \times \boldsymbol{W} \arrow[rr] \arrow[dl] & & {[1]} \times \boldsymbol{W}^{\mathrm{gpd}} \arrow[dl] & \\
{[1]} \times \mathcal{R} \arrow[rr] \arrow[uurrrr, bend left=15] & & {[1]} \times \mathcal{R}[\![\boldsymbol{W}^{-1}]\!] & &
\end{tikzcd}$$

in $\mathcal{C}\mathrm{at}_\infty$ containing and extending the above data. The bottom square is a pushout since the functor $[1] \times -\colon \mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty$ is a left adjoint, and the back square is a pushout by Lemma 1.20. Together, these observations guarantee the desired extension. □

Remark 1.19 Proposition 1.18 implies that Definition 1.8 agrees with Definition A.1.3.4.1.

We now make an easy observation regarding the localization functor, which is necessary for the argument of Proposition 1.18 but will also be useful in its own right.

Lemma 1.20 *The localization functor* $\mathscr{L}\colon \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty$ *commutes with finite products.*

For the proof of Lemma 1.20, it will be convenient to have the following notion:

Definition 1.21 Let $(\mathcal{C}, \otimes)$ be a closed symmetric monoidal ∞–category with internal hom bifunctor

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{\underline{\hom}_{\mathcal{C}}(-,-)} \mathcal{C}.$$

A collection of objects I of $\mathcal{C}$ is called an *exponential ideal* if we have $\underline{\hom}_{\mathcal{C}}(Y, Z) \in I$ for any $Y \in \mathcal{C}$ and any $Z \in I$. We will use this same terminology to refer to a full subcategory $\mathcal{D} \subset \mathcal{C}$ whose objects form an exponential ideal.

The following straightforward result explains why we are interested in exponential ideals:

Lemma 1.22 *Suppose that* $(\mathcal{C}, \otimes)$ *is a closed symmetric monoidal* ∞*–category, and let* $\mathrm{L}\colon \mathcal{C} \rightleftarrows \mathrm{L}\mathcal{C} :\mathrm{U}$ *be a left localization with unit map* $\mathrm{id}_{\mathcal{C}} \xrightarrow{\eta} \mathrm{L}$ *in* $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ *(where we implicitly consider* $\mathrm{L}\mathcal{C} \subset \mathcal{C}$*). Then, the full subcategory* $\mathrm{L}\mathcal{C} \subset \mathcal{C}$ *is an exponential ideal if and only if the natural map* $\mathrm{L}(\eta \otimes \eta)$ *is an equivalence in* $\mathrm{Fun}(\mathcal{C} \times \mathcal{C}, \mathcal{C})$ *(ie we have*

$$\mathrm{L}(Y \otimes Z) \xrightarrow{\sim} \mathrm{L}(\mathrm{L}(Y) \otimes \mathrm{L}(Z))$$

in $\mathrm{L}\mathcal{C}$ *for all* $Y, Z \in \mathcal{C}$*). In particular, if* $\mathrm{L}\mathcal{C}$ *is closed under the monoidal structure, then* $\mathrm{L}\mathcal{C} \subset \mathcal{C}$ *is an exponential ideal if and only if*

$$\mathrm{L}(Y \otimes Z) \simeq \mathrm{L}(Y) \otimes \mathrm{L}(Z)$$

in $\mathrm{L}\mathcal{C}$ *for all* $Y, Z \in \mathcal{C}$*.*

Proof Suppose that $\mathrm{L}\mathcal{C} \subset \mathcal{C}$ is an exponential ideal. Then, for any $Y, Z \in \mathcal{C}$ and any test object $W \in \mathrm{L}\mathcal{C}$, we have the string of natural equivalences

$$\begin{aligned}
\hom_{\mathcal{C}}(\mathrm{L}(Y \otimes Z), W) &\simeq \hom_{\mathcal{C}}(Y \otimes Z, W) \simeq \hom_{\mathcal{C}}(Y, \underline{\hom}_{\mathcal{C}}(Z, W)) \\
&\simeq \hom_{\mathcal{C}}(\mathrm{L}(Y), \underline{\hom}_{\mathcal{C}}(Z, W)) \simeq \hom_{\mathcal{C}}(\mathrm{L}(Y) \otimes Z, W) \\
&\simeq \hom_{\mathcal{C}}(Z \otimes \mathrm{L}(Y), W) \simeq \hom_{\mathcal{C}}(Z, \underline{\hom}_{\mathcal{C}}(\mathrm{L}(Y), W)) \\
&\simeq \hom_{\mathcal{C}}(\mathrm{L}(Z), \underline{\hom}_{\mathcal{C}}(\mathrm{L}(Y), W)) \simeq \hom_{\mathcal{C}}(\mathrm{L}(Z) \otimes \mathrm{L}(Y), W) \\
&\simeq \hom_{\mathcal{C}}(\mathrm{L}(Y) \otimes \mathrm{L}(Z), W) \simeq \hom_{\mathcal{C}}(\mathrm{L}(\mathrm{L}(Y) \otimes \mathrm{L}(Z)), W).
\end{aligned}$$

Hence, we have an equivalence $\mathrm{L}(Y \otimes Z) \simeq \mathrm{L}(\mathrm{L}(Y) \otimes \mathrm{L}(Z))$ by the Yoneda lemma applied to the ∞–category $\mathrm{L}\mathcal{C}$ (and it is straightforward to check that this equivalence is indeed induced by the specified map). So $\mathrm{L}(\eta \otimes \eta)$ is an equivalence in $\mathrm{Fun}(\mathcal{C} \times \mathcal{C}, \mathcal{C})$, as desired.

On the other hand, suppose that $\mathrm{L}(Y \otimes Z) \xrightarrow{\sim} \mathrm{L}(\mathrm{L}(Y) \otimes \mathrm{L}(Z))$ for all $Y, Z \in \mathcal{C}$. Then, we have the string of natural equivalences

$$\begin{aligned}
&\hom_{\mathcal{C}}(Y, \underline{\hom}_{\mathcal{C}}(Z, W)) \\
&\quad\simeq \hom_{\mathcal{C}}(Y \otimes Z, W) \simeq \hom_{\mathcal{C}}(\mathrm{L}(Y \otimes Z), W) \\
&\quad\simeq \hom_{\mathcal{C}}(\mathrm{L}(\mathrm{L}(Y) \otimes \mathrm{L}(Z)), W) \simeq \hom_{\mathcal{C}}(\mathrm{L}(Y) \otimes \mathrm{L}(Z), W) \\
&\quad\simeq \hom_{\mathcal{C}}(\mathrm{L}(\mathrm{L}(Y)) \otimes \mathrm{L}(Z), W) \simeq \hom_{\mathcal{C}}(\mathrm{L}(\mathrm{L}(\mathrm{L}(Y)) \otimes \mathrm{L}(Z)), W) \\
&\quad\simeq \hom_{\mathcal{C}}(\mathrm{L}(\mathrm{L}(Y) \otimes Z), W) \simeq \hom_{\mathcal{C}}(\mathrm{L}(Y) \otimes Z, W) \simeq \hom_{\mathcal{C}}(\mathrm{L}(Y), \underline{\hom}_{\mathcal{C}}(Z, W)).
\end{aligned}$$

Hence, for any map $Y \to Y'$ in $\mathcal{C}$ which localizes to an equivalence $\mathrm{L}(Y) \xrightarrow{\sim} \mathrm{L}(Y')$ in $\mathrm{L}\mathcal{C} \subset \mathcal{C}$, we obtain an equivalence $\hom_{\mathcal{C}}(Y, \underline{\hom}_{\mathcal{C}}(Z, W)) \xleftarrow{\sim} \hom_{\mathcal{C}}(Y', \underline{\hom}_{\mathcal{C}}(Z, W))$.

It follows that the object $\underline{\hom}_{\mathcal{C}}(Z, W) \in \mathcal{C}$ is local with respect to the left localization, ie that in fact $\underline{\hom}_{\mathcal{C}}(Z, W) \in \mathrm{L}\mathcal{C} \subset \mathcal{C}$. So $\mathrm{L}\mathcal{C} \subset \mathcal{C}$ is an exponential ideal. □

With Lemma 1.22 in hand, we now proceed to prove Lemma 1.20.

Proof of Lemma 1.20 The right adjoint $\min\colon \mathcal{C}\mathrm{at}_\infty \to \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ induces an equivalence onto the full subcategory of minimal relative ∞–categories. It is easy to see that this is an exponential ideal in $(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty, \times)$, and so the result follows from Lemma 1.22. □

The following useful construction relies on Lemma 1.20:

Remark 1.23 Let $(\mathcal{R}_1, \boldsymbol{W}_1), (\mathcal{R}_2, \boldsymbol{W}_2) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$. Then the identity map

$$(\mathrm{Fun}(\mathcal{R}_1, \mathcal{R}_2)^{\mathcal{R}\mathrm{el}}, \mathrm{Fun}(\mathcal{R}_1, \mathcal{R}_2)^{\boldsymbol{W}}) \to (\mathrm{Fun}(\mathcal{R}_1, \mathcal{R}_2)^{\mathcal{R}\mathrm{el}}, \mathrm{Fun}(\mathcal{R}_1, \mathcal{R}_2)^{\boldsymbol{W}})$$

is adjoint to an evaluation map

$$(\mathcal{R}_1, \boldsymbol{W}_1) \times (\mathrm{Fun}(\mathcal{R}_1, \mathcal{R}_2)^{\mathcal{R}\mathrm{el}}, \mathrm{Fun}(\mathcal{R}_1, \mathcal{R}_2)^{\boldsymbol{W}}) \to (\mathcal{R}_2, \boldsymbol{W}_2).$$

By Lemma 1.20, applying the localization functor $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathscr{L}} \mathcal{C}\mathrm{at}_\infty$ yields a map

$$\mathcal{R}_1[\![\boldsymbol{W}_1^{-1}]\!] \times \mathrm{Fun}(\mathcal{R}_1, \mathcal{R}_2)^{\mathcal{R}\mathrm{el}}[\![(\mathrm{Fun}(\mathcal{R}_1, \mathcal{R}_2)^{\boldsymbol{W}})^{-1}]\!] \to \mathcal{R}_2[\![\boldsymbol{W}_2^{-1}]\!],$$

which is itself adjoint to a canonical map

$$\mathrm{Fun}(\mathcal{R}_1, \mathcal{R}_2)^{\mathcal{R}\mathrm{el}}[\![(\mathrm{Fun}(\mathcal{R}_1, \mathcal{R}_2)^{\boldsymbol{W}})^{-1}]\!] \to \mathrm{Fun}(\mathcal{R}_1[\![\boldsymbol{W}_1^{-1}]\!], \mathcal{R}_2[\![\boldsymbol{W}_2^{-1}]\!]).$$

In particular, precomposing with the localization map for the internal hom object yields a canonical map

$$\mathrm{Fun}(\mathcal{R}_1, \mathcal{R}_2)^{\mathcal{R}\mathrm{el}} \to \mathrm{Fun}(\mathcal{R}_1[\![\boldsymbol{W}_1^{-1}]\!], \mathcal{R}_2[\![\boldsymbol{W}_2^{-1}]\!]).$$

Lemma 1.20 also allows us to prove the following result, which will be useful later and which gives a sense of the interplay between relative ∞–categories and their localizations:

Lemma 1.24 *Given any $(\mathcal{R}_1 \boldsymbol{W}_1), (\mathcal{R}_2, \boldsymbol{W}_2) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ and any pair of maps $\mathcal{R}_1 \rightrightarrows \mathcal{R}_2$ in $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$, a natural weak equivalence between them induces an equivalence between their induced functors $\mathcal{R}_1[\![\boldsymbol{W}_1^{-1}]\!] \rightrightarrows \mathcal{R}_2[\![\boldsymbol{W}_2^{-1}]\!]$ in $\mathcal{C}\mathrm{at}_\infty$.*

Proof A natural weak equivalence corresponds to a map $[1]_{\boldsymbol{W}} \times \mathcal{R}_1 \to \mathcal{R}_2$ in $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$. By Lemma 1.20 (and Example 1.12), this gives rise to a map $[1]^{\mathrm{gpd}} \times \mathcal{R}_1[\![\boldsymbol{W}_1^{-1}]\!] \to \mathcal{R}_2[\![\boldsymbol{W}_2^{-1}]\!]$ in $\mathcal{C}\mathrm{at}_\infty$, which precisely selects the desired equivalence. □

Remark 1.25 Lemma 1.24 allows for a simple proof of Proposition T.5.2.7.12, that a left localization is in particular a free localization. Indeed, given a left localization adjunction $\mathrm{L}\colon \mathcal{C} \rightleftarrows \mathrm{L}\mathcal{C} :\!\mathrm{U}$, write $\boldsymbol{W} \subset \mathcal{C}$ for the subcategory created by the functor $\mathrm{L}\colon \mathcal{C} \to \mathrm{L}\mathcal{C}$. Then, this adjunction gives rise to a pair of maps $(\mathcal{C}, \boldsymbol{W}) \xrightarrow{\mathrm{L}} \min(\mathrm{L}\mathcal{C})$ and $\min(\mathrm{L}\mathcal{C}) \xrightarrow{\mathrm{U}} (\mathcal{C}, \boldsymbol{W})$ in $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$. Moreover, the composite

$$\min(\mathrm{L}\mathcal{C}) \xrightarrow{\mathrm{U}} (\mathcal{C}, \boldsymbol{W}) \xrightarrow{\mathrm{L}} \min(\mathrm{L}\mathcal{C})$$

is an equivalence, while the composite

$$(\mathcal{C}, \boldsymbol{W}) \xrightarrow{\mathrm{L}} \min(\mathrm{L}\mathcal{C}) \xrightarrow{\mathrm{U}} (\mathcal{C}, \boldsymbol{W})$$

is connected to $\mathrm{id}_{(\mathcal{C},\boldsymbol{W})}$ by the unit of the adjunction, which is a componentwise weak equivalence (since, for any $Y \in \mathcal{C}$, applying $\mathcal{C} \xrightarrow{\mathrm{L}} \mathrm{L}\mathcal{C}$ to the map $Y \to \mathrm{L}(Y)$ gives an equivalence $\mathrm{L}(Y) \xrightarrow{\sim} \mathrm{L}(\mathrm{L}(Y))$). Hence, it follows that these functors induce inverse equivalences $\mathcal{C}[\![\boldsymbol{W}^{-1}]\!] \simeq \mathrm{L}\mathcal{C}$. (From here, one can obtain the actual statement of Proposition T.5.2.7.12 by appealing to Proposition 1.18.)

Lemma 1.24 also has the following special case, which will be useful to us:

Lemma 1.26 *Given any $\mathcal{C}, \mathcal{D} \in \mathcal{C}\mathrm{at}_\infty$ and any pair of maps $\mathcal{C} \rightrightarrows \mathcal{D}$, a natural transformation between them induces an equivalence between the induced maps $\mathcal{C}^{\mathrm{gpd}} \rightrightarrows \mathcal{D}^{\mathrm{gpd}}$ in $\mathcal{S}$.*

Proof In light of Example 1.12, this follows from applying Lemma 1.24 in the special case that $(\mathcal{R}_1, \boldsymbol{W}_1) = \max(\mathcal{C})$ and $(\mathcal{R}_2, \boldsymbol{W}_2) = \max(\mathcal{D})$. □

Remark 1.27 Lemma 1.26 can also be seen as following from applying Lemma 1.22 to the left localization $(-)^{\mathrm{gpd}}\colon \mathcal{C}\mathrm{at}_\infty \rightleftarrows \mathcal{S} :\!\mathrm{U}_{\mathcal{S}}$. Namely, since the full subcategory $\mathcal{S} \subset \mathcal{C}\mathrm{at}_\infty$ is an exponential ideal for $(\mathcal{C}\mathrm{at}_\infty, \times)$, the left adjoint $(-)^{\mathrm{gpd}}\colon \mathcal{C}\mathrm{at}_\infty \to \mathcal{S}$ commutes with finite products, and hence a natural transformation $[1] \times \mathcal{C} \to \mathcal{D}$ gives rise to a map $([1] \times \mathcal{C})^{\mathrm{gpd}} \simeq [1]^{\mathrm{gpd}} \times \mathcal{C}^{\mathrm{gpd}} \to \mathcal{D}^{\mathrm{gpd}}$ which selects the desired equivalence in $\hom_{\mathcal{S}}(\mathcal{C}^{\mathrm{gpd}}, \mathcal{D}^{\mathrm{gpd}})$.

In turn, Lemma 1.26 has the following useful further special case:

Corollary 1.28 *An adjunction $F\colon \mathcal{C} \rightleftarrows \mathcal{D} :G$ induces inverse equivalences*

$$F^{\mathrm{gpd}}\colon \mathcal{C}^{\mathrm{gpd}} \xrightarrow{\sim} \mathcal{D}^{\mathrm{gpd}} \quad \text{and} \quad \mathcal{C}^{\mathrm{gpd}} \xleftarrow{\sim} \mathcal{D}^{\mathrm{gpd}} :G^{\mathrm{gpd}}$$

in $\mathcal{S}$.

Proof The adjunction $F \dashv G$ has unit and counit natural transformations $\mathrm{id}_{\mathcal{C}} \to G \circ F$ and $F \circ G \to \mathrm{id}_{\mathcal{D}}$, and so the claim follows from Lemma 1.26. □

We note the following interaction between taking localizations and taking homotopy categories:

Remark 1.29 The composite left adjoint

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{(\mathrm{ho}(-),\mathrm{ho}(-))} \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at} \xrightarrow{(-)[(-)^{-1}]} \mathcal{C}\mathrm{at}$$

coincides with the composite left adjoint

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{(-)[\![(-)^{-1}]\!]} \mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathrm{ho}} \mathcal{C}\mathrm{at},$$

since they share a right adjoint

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \hookleftarrow \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at} \xleftarrow{\min} \mathcal{C}\mathrm{at}.$$

Hence, for any $(\mathcal{R}, \boldsymbol{W}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ we have a natural equivalence

$$\mathrm{ho}(\mathcal{R}[\![\boldsymbol{W}^{-1}]\!]) \xrightarrow{\sim} \mathrm{ho}(\mathcal{R})[\mathrm{ho}(\boldsymbol{W})^{-1}]$$

in $\mathcal{C}\mathrm{at} \subset \mathcal{C}\mathrm{at}_\infty$.

We end this section with the following observation (which partly echoes Example S.2.11).

Remark 1.30 Suppose $(\mathcal{R}, \boldsymbol{W})$ is a relative ∞–category. Then $(\mathrm{ho}(\mathcal{R}), \mathrm{ho}(\boldsymbol{W}))$ is a relative category (so is in particular a relative ∞–category). However, its localization $\mathrm{ho}(\mathcal{R})[\![\mathrm{ho}(\boldsymbol{W})^{-1}]\!]$ need not recover $\mathcal{R}[\![\boldsymbol{W}^{-1}]\!]$. This is for the same reason as always for such facts, namely that we lose coherence data when we pass from $\mathcal{R}$ to $\mathrm{ho}(\mathcal{R})$. (Commutative diagrams in $\mathrm{ho}(\mathcal{R})$ need not come from commutative diagrams in $\mathcal{R}$, and when they do they might do so in multiple, inequivalent ways.) An explicit counterexample is provided by the minimal relative ∞–category $(\mathcal{R}, \boldsymbol{W}) = (\mathcal{R}, \mathcal{R}^{\simeq})$: then

$$\mathrm{ho}(\boldsymbol{W}) \simeq \mathrm{ho}(\mathcal{R}^{\simeq}) \simeq \mathrm{ho}(\mathcal{R})^{\simeq} \subset \mathrm{ho}(\mathcal{R})$$

since the equivalences in $\mathcal{R}$ are created by $\mathcal{R} \to \mathrm{ho}(\mathcal{R})$, and hence $\mathrm{ho}(\mathcal{R})[\![\mathrm{ho}(\boldsymbol{W})^{-1}]\!] \simeq \mathrm{ho}(\mathcal{R})$ (while of course $\mathcal{R}[\![\boldsymbol{W}^{-1}]\!] \simeq \mathcal{R}$). One might therefore refer to the ∞–category $\mathrm{ho}(\mathcal{R})[\![\mathrm{ho}(\boldsymbol{W})^{-1}]\!]$ as an "exotic enrichment" of the homotopy category $\mathrm{ho}(\mathcal{R}[\![\boldsymbol{W}^{-1}]\!])$.

2 Complete Segal spaces

We now give an extremely brief review of the theory of complete Segal spaces. This section exists more or less solely to fix notation; we refer the reader seeking a more thorough discussion either to the original paper [17] (which uses model categories) or to [8, Section 1] (which uses ∞–categories).

Let us write $\mathbf{\Delta} \xrightarrow{[\bullet]} \mathcal{C}\mathrm{at}$ for the standard cosimplicial (strict) category. Then, recall that the *nerve* of $\mathcal{C} \in \mathcal{C}\mathrm{at}$ is by definition the simplicial set $\mathrm{N}(\mathcal{C})_\bullet = \mathrm{hom}^{\mathrm{lw}}_{\mathcal{C}\mathrm{at}}([\bullet], \mathcal{C})$. This defines a fully faithful embedding $\mathrm{N}\colon \mathcal{C}\mathrm{at} \to s\mathcal{S}\mathrm{et}$, with image those simplicial sets which admit *unique* lifts for the inner horn inclusions $\{\Lambda^n_i \to \Delta^n\}_{0<i<n\geq 0}$. In fact, this functor is a right adjoint.

The situation with ∞–categories is completely analogous.

Definition 2.1 The (∞–categorical) *nerve* of an ∞–category $\mathcal{C}$ is the simplicial space

$$\mathrm{N}_\infty(\mathcal{C})_\bullet = \mathrm{hom}^{\mathrm{lw}}_{\mathcal{C}\mathrm{at}_\infty}([\bullet], \mathcal{C}),$$

ie the composite

$$\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{[\bullet]^{\mathrm{op}}} (\mathcal{C}\mathrm{at}_\infty)^{\mathrm{op}} \xrightarrow{\mathrm{hom}_{\mathcal{C}\mathrm{at}_\infty}(-,\mathcal{C})} \mathcal{S}.$$

This defines a fully faithful embedding $\mathrm{N}_\infty\colon \mathcal{C}\mathrm{at}_\infty \hookrightarrow s\mathcal{S}$, with image the full subcategory $\mathcal{CSS} \subset s\mathcal{S}$ of *complete Segal spaces*, ie those simplicial spaces satisfying the *Segal condition* and the *completeness condition*. This inclusion fits into a left localization adjunction $\mathrm{L}_{\mathcal{CSS}}\colon s\mathcal{S} \rightleftarrows \mathcal{CSS} :\mathrm{U}_{\mathcal{CSS}}$. Hence, we obtain an equivalence

$$\mathcal{C}\mathrm{at}_\infty \xrightarrow[\sim]{\mathrm{N}_\infty} \mathcal{CSS},$$

whose inverse

$$\mathcal{CSS} \xrightarrow[\sim]{\mathrm{N}_\infty^{-1}} \mathcal{C}\mathrm{at}_\infty$$

takes an object $Y_\bullet \in \mathcal{CSS}$ to the coend

$$\int^{[n]\in\mathbf{\Delta}} Y_n \times [n]$$

in $\mathcal{C}\mathrm{at}_\infty$. (These claims respectively follow from Proposition A.A.7.10, [5, Theorem 4.12], [17, Theorem 7.2] and [5, Theorem 4.12] again.) This equivalence identifies the subcategory $\mathcal{S} \subset \mathcal{C}\mathrm{at}_\infty$ with the subcategory of *constant* simplicial spaces (which are automatically complete Segal spaces).

Remark 2.2 Complete Segal spaces provide an extremely efficient way of computing the hom spaces in an ∞–category: if $x, y \in \mathcal{C}$, then there is a natural equivalence

$$\hom_{\mathcal{C}}(x,y) \simeq \lim \left(\begin{tikzcd} & \mathrm{N}_\infty(\mathcal{C})_1 \arrow[d, "(s,t)"] \\ \mathrm{pt}_{\mathcal{S}} \arrow[r, "(x,y)"'] & \mathrm{N}_\infty(\mathcal{C})_0 \times \mathrm{N}_\infty(\mathcal{C})_0 \end{tikzcd} \right)$$

in $\mathcal{S}$, where we use the notation $s = \delta_1$ and $t = \delta_0$ to emphasize the roles that these two face maps play in this theory. (Note that $\mathrm{N}_\infty(\mathcal{C})_0 = \hom_{\mathcal{C}\mathrm{at}_\infty}([0], \mathcal{C}) \simeq \mathcal{C}^{\simeq}$ is simply the maximal subgroupoid of $\mathcal{C}$, while $\mathrm{N}_\infty(\mathcal{C})_1 = \hom_{\mathcal{C}\mathrm{at}_\infty}([1], \mathcal{C}) \simeq \mathrm{Fun}([1], \mathcal{C})^{\simeq}$ is the space of morphisms in $\mathcal{C}$.)

Remark 2.3 There is a canonical involution $\mathbf{\Delta} \xrightarrow{\sim} \mathbf{\Delta}$ in $\mathcal{C}\mathrm{at}$, which is the identity on objects but acts on morphisms by "reversing the coordinates": a map $[m] \xrightarrow{\varphi} [n]$ is taken to the map

$$[m] \xrightarrow{i \mapsto (n - \varphi(m-i))} [n].$$

Taking opposites, this induces an involution $\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{\sim} \mathbf{\Delta}^{\mathrm{op}}$, which in turn induces an involution of $s\mathcal{S} = \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{S})$ by precomposition. Unwinding the definitions, we see that this involution $s\mathcal{S} \xrightarrow{\sim} s\mathcal{S}$ restricts to an involution $\mathcal{CSS} \xrightarrow{\sim} \mathcal{CSS}$ which corresponds to the involution $(-)^{\mathrm{op}} \colon \mathcal{C}\mathrm{at}_\infty \xrightarrow{\sim} \mathcal{C}\mathrm{at}_\infty$.

For future use, we record the following observation:

Proposition 2.4 *The diagram*

$$\begin{tikzcd} s\mathcal{S} \arrow[r, shift left, "\mathrm{L}_{\mathcal{CSS}}"] \arrow[r, phantom, "\perp"] \arrow[dr, "|-|"'] & \mathcal{CSS} \arrow[l, shift left, "\mathrm{U}_{\mathcal{CSS}}"] \arrow[r, shift left, "\mathrm{N}_\infty^{-1}"] \arrow[r, phantom, "\sim"] & \mathcal{C}\mathrm{at}_\infty \arrow[l, shift left, "\mathrm{N}_\infty"] \arrow[dl, "(-)^{\mathrm{gpd}}"] \\ & \mathcal{S} & \end{tikzcd}$$

commutes: that is,

- *geometric realization of complete Segal spaces models groupoid completion of ∞–categories, and*
- *for any $Y \in s\mathcal{S}$, the localization map $Y \to \mathrm{L}_{\mathcal{CSS}}(Y)$ becomes an equivalence upon geometric realization.*

Proof For the first claim, note that the functor $(-)^{\mathrm{gpd}}\colon \mathcal{C}\mathrm{at}_\infty \to \mathcal{S}$ is a left localization, and the composite

$$\mathcal{S} \xhookrightarrow{\mathrm{U}_{\mathcal{S}}} \mathcal{C}\mathrm{at}_\infty \xrightarrow[\sim]{\mathrm{N}_\infty} \mathcal{CSS} \xhookrightarrow{\mathrm{U}_{\mathcal{CSS}}} s\mathcal{S}$$

agrees with the functor $\mathrm{const}\colon \mathcal{S} \to s\mathcal{S}$. Hence, the equivalence

$$|{-}| \circ \mathrm{U}_{\mathcal{CSS}} \circ \mathrm{N}_\infty \simeq (-)^{\mathrm{gpd}}$$

in $\mathrm{Fun}(\mathcal{C}\mathrm{at}_\infty, \mathcal{S})$ follows from the uniqueness of left adjoints.

For the second claim, note that the reflective inclusion $\mathrm{const}\colon \mathcal{S} \hookrightarrow s\mathcal{S}$ factors through the reflective inclusion $\mathrm{U}_{\mathcal{CSS}}\colon \mathcal{CSS} \hookrightarrow s\mathcal{S}$. Hence, the factorization $\mathcal{S} \hookrightarrow \mathcal{CSS}$ is also a reflective inclusion. The equivalence

$$|{-}| \simeq |{-}| \circ \mathrm{U}_{\mathcal{CSS}} \circ \mathrm{L}_{\mathcal{CSS}}$$

in $\mathrm{Fun}(s\mathcal{S}, \mathcal{S})$ now also follows from the uniqueness of left adjoints. □

Remark 2.5 We may interpret Proposition 2.4 as saying that, while a simplicial space $Y \in s\mathcal{S}$ can be thought of as generating an ∞–category (namely the one corresponding to $\mathrm{L}_{\mathcal{CSS}}(Y) \in \mathcal{CSS}$), we can already directly extract its groupoid completion from Y itself. This is analogous to the fact that an arbitrary simplicial set can be thought of as generating a quasicategory via fibrant replacement in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$, and the replacement map lies in $\boldsymbol{W}_{\mathrm{Joyal}} \subset \boldsymbol{W}_{\mathrm{KQ}}$ (ie it induces an equivalence on geometric realizations).

Remark 2.6 Given a strict category $\mathcal{C} \in \mathcal{C}\mathrm{at}$, the maps

$$\hom_{\mathcal{C}\mathrm{at}}([n], \mathcal{C}) \to \hom_{\mathcal{C}\mathrm{at}_\infty}([n], \mathcal{C})$$

from hom sets to hom spaces collect into a map

$$\mathrm{N}(\mathcal{C}) \to \mathrm{N}_\infty(\mathcal{C})$$

in $s\mathcal{S}$; in turn, these maps assemble into a natural transformation $\mathrm{N} \to \mathrm{N}_\infty$ in $\mathrm{Fun}(\mathcal{C}\mathrm{at}, s\mathcal{S})$. This map will be an equivalence in $s\mathcal{S}$ if and only if $\mathcal{C}$ is *gaunt*: while the nerve $\mathrm{N}(\mathcal{C}) \in s\mathcal{S}\mathrm{et} \subset s\mathcal{S}$ is always a Segal space, it only satisfies the completeness condition when every isomorphism in $\mathcal{C}$ is actually an identity map.[8] However, by [17, Remark 7.8], the above map induces an equivalence

$$\mathrm{L}_{\mathcal{CSS}}(\mathrm{N}(\mathcal{C})) \xrightarrow{\sim} \mathrm{L}_{\mathcal{CSS}}(\mathrm{N}_\infty(\mathcal{C})) \simeq \mathrm{N}_\infty(\mathcal{C})$$

[8] The Segal condition in $s\mathcal{S}\mathrm{et}$ can be equivalently checked in $s\mathcal{S}$ since the inclusion $s\mathcal{S}\mathrm{et} \subset s\mathcal{S}$ is a right adjoint.

in $\mathcal{CSS} \subset s\mathcal{S}$. In particular, it therefore follows from Proposition 2.4 that it also induces an equivalence

$$|\mathrm{N}(\mathcal{C})| \xrightarrow{\sim} |\mathrm{N}_\infty(\mathcal{C})|$$

in $\mathcal{S}$.

3 The Rezk nerve

Recall that the *localization* of a relative ∞–category $(\mathcal{R}, \boldsymbol{W})$ is the initial ∞–category $\mathcal{R}[\![\boldsymbol{W}^{-1}]\!]$ equipped with a functor from $\mathcal{R}$ which sends the subcategory $\boldsymbol{W} \subset \mathcal{R}$ of weak equivalences to equivalences. Meanwhile, given an arbitrary ∞–category $\mathcal{C}$, observe that the n^{th} space of its nerve can be considered as

$$\mathrm{N}_\infty(\mathcal{C})_n = \hom_{\mathcal{C}\mathrm{at}_\infty}([n], \mathcal{C}) \simeq \mathrm{Fun}([n], \mathcal{C})^{\simeq} \subset \mathrm{Fun}([n], \mathcal{C}),$$

the subcategory of $\mathrm{Fun}([n], \mathcal{C})$ whose morphisms are the natural *equivalences*. Combining these two facts, one is led to suspect that the n^{th} space of the nerve $\mathrm{N}_\infty(\mathcal{R}[\![\boldsymbol{W}^{-1}]\!])_\bullet$ should somehow contain the subcategory

$$\mathrm{Fun}([n], \mathcal{R})^{\boldsymbol{W}} \subset \mathrm{Fun}([n], \mathcal{R})$$

of $\mathrm{Fun}([n], \mathcal{R})$ whose morphisms are the natural *weak equivalences*. Of course, this will not generally form a space, but will instead be an ∞–category. On the other hand, there is a universal choice for a space admitting a map from this ∞–category, namely its groupoid completion. We are thus naturally led to make the following construction, a direct generalization of the "classification diagram" construction for relative categories defined in [17, Section 3.3]:

Definition 3.1 Given a relative ∞–category $(\mathcal{R}, \boldsymbol{W})$, its ($\infty$–categorical) *Rezk prenerve* is the simplicial ∞–category

$$\mathrm{preN}^{\mathrm{R}}_\infty(\mathcal{R}, \boldsymbol{W})_\bullet = \mathrm{Fun}^{\mathrm{lw}}([\bullet], \mathcal{R})^{\boldsymbol{W}},$$

ie the composite

$$\boldsymbol{\Delta}^{\mathrm{op}} \xrightarrow{[\bullet]^{\mathrm{op}}} (\mathcal{C}\mathrm{at}_\infty)^{\mathrm{op}} \xrightarrow{\min^{\mathrm{op}}} (\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)^{\mathrm{op}} \xrightarrow{\mathrm{Fun}(-,\mathcal{R})^{\boldsymbol{W}}} \mathcal{C}\mathrm{at}_\infty.$$

This defines a functor

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathrm{preN}^{\mathrm{R}}_\infty} s\mathcal{C}\mathrm{at}_\infty.$$

Then, the (∞–categorical) *Rezk nerve* functor

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathrm{N}^{\mathrm{R}}_\infty} s\mathcal{S}$$

is given by the composite

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathrm{preN}^{\mathrm{R}}_\infty} s\mathcal{C}\mathrm{at}_\infty \xrightarrow{s(-)^{\mathrm{gpd}}} s\mathcal{S}.$$

Remark 3.2 Recall that Rezk's "classification diagram" construction [17, Section 3.3], which we will denote by

$$\mathrm{RelCat} \xrightarrow{\mathrm{N}^{\mathrm{R}}} s(s\mathcal{S}\mathrm{et})$$

and refer to as the 1–*categorical Rezk nerve* functor, is given by the formula

$$\mathrm{N}^{\mathrm{R}}(\mathcal{R}, \boldsymbol{W})_\bullet = \mathrm{N}(\mathrm{Fun}^{\mathrm{lw}}([\bullet], \mathcal{R})^{\boldsymbol{W}}).$$

Of course, we would like to think of this as a simplicial space using the model category $s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}}$. Indeed, combining Proposition 2.4 and Remark 2.6, we obtain a canonical commutative diagram

$$\begin{tikzcd}
\mathrm{RelCat} \arrow[r, "\mathrm{N}^{\mathrm{R}}"] \arrow[d] & s(s\mathcal{S}\mathrm{et}) \arrow[r, "s(|-|)"] & s\mathcal{S} \\
\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \arrow[urr, bend right, "\mathrm{N}^{\mathrm{R}}_\infty"'] & &
\end{tikzcd}$$

in $\mathcal{C}\mathrm{at}_\infty$; in fact, this even refines to a canonical commutative diagram

$$\begin{tikzcd}
\mathrm{RelCat} \arrow[r, "\mathrm{N}^{\mathrm{R}}"] \arrow[d, hook] & s(s\mathcal{S}\mathrm{et}) \arrow[r] & s\mathcal{C}\mathrm{at}_\infty \arrow[r, "s(-)^{\mathrm{gpd}}"] & s\mathcal{S} \\
\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \arrow[urr, "\mathrm{preN}^{\mathrm{R}}_\infty"'] \arrow[urrr, bend right, "\mathrm{N}^{\mathrm{R}}_\infty"'] & & &
\end{tikzcd}$$

in $\mathcal{C}\mathrm{at}_\infty$ (in which the functor $s(s\mathcal{S}\mathrm{et}) \to s\mathcal{C}\mathrm{at}_\infty$ is obtained by applying $s(-) = \mathrm{Fun}(\boldsymbol{\Delta}^{\mathrm{op}}, -)$ to the localization $s\mathcal{S}\mathrm{et} \to s\mathcal{S}\mathrm{et}[\![\boldsymbol{W}_{\mathrm{Joyal}}^{-1}]\!] \simeq \mathcal{C}\mathrm{at}_\infty$). Thus, at least as far as homotopical content is concerned, the ∞–categorical Rezk nerve functor strictly generalizes its 1–categorical counterpart.

Remark 3.3 In turn, the 1–categorical Rezk nerve functor of Remark 3.2 suggests a similar model-dependent definition of a Rezk nerve functor for "marked quasicategories" (once again landing in $ss\mathcal{S}\mathrm{et}$). In fact, as the first step in the proof of Lemma 4.3, we will show that this construction is a model-categorical presentation

- of the ∞–categorical Rezk nerve when considered in $s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}}$, and in fact
- of the ∞–categorical Rezk prenerve when considered in $s(s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}})_{\mathrm{Reedy}}$.

Remark 3.4 We have the following slight reformulation of Definition 3.1: in view of Proposition 2.4, the Rezk nerve functor can also be described as a composite

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathrm{preN}^{\mathrm{R}}_\infty} s\mathcal{C}\mathrm{at}_\infty \simeq s\mathcal{CSS} \xhookrightarrow{s(\mathrm{U}_{\mathcal{CSS}})} s(s\mathcal{S}) \xrightarrow{s(|-|)} s\mathcal{S}.$$

Note that the composite functor $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \to s(s\mathcal{S})$ is a right adjoint, whose left adjoint is the left Kan extension

$$\begin{tikzcd}
\mathbf{\Delta} \times \mathbf{\Delta} \arrow[rr, "\mathrm{m}\times\mathrm{m}=(([m],[n])\mapsto[m]\times[n]_W)"] \arrow[d, "\text{よ}"'] & & \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \\
s(s\mathcal{S}) \arrow[urr, dashed, "\text{よ}_!(\mathrm{m}\times\mathrm{m})"'] & &
\end{tikzcd}$$

along the Yoneda embedding, where we write m×m for the upper "min × max" functor for brevity. On the other hand, the functor $s(|-|)\colon s(s\mathcal{S}) \to s\mathcal{S}$ is a left adjoint. Hence, as the Rezk nerve functor is the composite of a right adjoint followed by a left adjoint, understanding its behavior in general is a rather difficult task. (In fact, it follows that $\mathrm{preN}^{\mathrm{R}}_\infty\colon \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \to s\mathcal{C}\mathrm{at}_\infty$ is also a right adjoint, while $s(-)^{\mathrm{gpd}}\colon s\mathcal{C}\mathrm{at}_\infty \to s\mathcal{S}$ is of course also a left adjoint.)

We have the following identifications of the Rezk nerves of minimal and maximal relative ∞–categories: in both of these extremal cases, the Rezk nerve does indeed compute the localization.

Proposition 3.5 *The Rezk nerve functor acts on the full subcategories of $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ spanned by the minimal and maximal relative ∞–categories (both of which can be identified with $\mathcal{C}\mathrm{at}_\infty$) according to the canonical commutative diagram*

$$\begin{tikzcd}
\mathcal{C}\mathrm{at}_\infty \arrow[r, hook, "\min"] \arrow[d, "\mathrm{N}_\infty"', "\wr"] & \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \arrow[d, "\mathrm{N}^{\mathrm{R}}_\infty"] & \mathcal{C}\mathrm{at}_\infty \arrow[l, hook', "\max"'] \arrow[d, "(-)^{\mathrm{gpd}}"] \\
\mathcal{CSS} \arrow[r, hook, "\mathrm{U}_{\mathcal{CSS}}"'] & s\mathcal{S} & \mathcal{S} \arrow[l, hook', "\mathrm{const}"]
\end{tikzcd}$$

in $\mathcal{C}\mathrm{at}_\infty$.

Proof To see that the left square commutes, given any $\mathcal{C} \in \mathcal{C}\mathrm{at}_\infty$ we compute that

$$\mathrm{preN}^{\mathrm{R}}_\infty(\min(\mathcal{C}))_n = \mathrm{Fun}([n], \min(\mathcal{C}))^{\boldsymbol{W}} \simeq \mathrm{Fun}([n], \mathcal{C})^{\simeq} \simeq \hom_{\mathcal{C}\mathrm{at}_\infty}([n], \mathcal{C}) = \mathrm{N}_\infty(\mathcal{C})_n$$

(in a way compatible with the evident simplicial structure maps on both sides), ie we even have a canonical equivalence

$$\mathrm{preN}^{\mathrm{R}}_\infty(\min(\mathcal{C}))_\bullet \simeq \mathrm{N}_\infty(\mathcal{C})_\bullet$$

in $s\mathcal{C}\mathrm{at}_\infty$. As $s(-)^{\mathrm{gpd}} \colon s\mathcal{C}\mathrm{at}_\infty \rightleftarrows s\mathcal{S} : s(\mathrm{U}_{\mathcal{S}})$ is a left localization adjunction, it follows that we also have a canonical equivalence

$$\mathrm{N}^{\mathrm{R}}_\infty(\min(\mathcal{C}))_\bullet \simeq \mathrm{N}_\infty(\mathcal{C})_\bullet$$

in $s\mathcal{S}$.

To see that the right square commutes, given any $\mathcal{C} \in \mathcal{C}\mathrm{at}_\infty$ we first compute that

$$\mathrm{preN}^{\mathrm{R}}_\infty(\max(\mathcal{C}))_n = \mathrm{Fun}([n], \max(\mathcal{C}))^{\boldsymbol{W}} \simeq \mathrm{Fun}([n], \mathcal{C}).$$

Moreover, note that every face-then-degeneracy composite

$$\mathrm{Fun}([n], \mathcal{C}) \xrightarrow{\delta_i} \mathrm{Fun}([n-1], \mathcal{C}) \xrightarrow{\sigma_j} \mathrm{Fun}([n], \mathcal{C})$$

admits a natural transformation either to or from $\mathrm{id}_{\mathrm{Fun}([n],\mathcal{C})}$ (depending on i and j).[9] By Lemma 1.26, it follows that all the structure maps of $\mathrm{N}^{\mathrm{R}}_\infty(\max(\mathcal{C})) \in s\mathcal{S}$ are equivalences, and hence (since $\boldsymbol{\Delta}^{\mathrm{op}}$ is sifted so in particular $(\boldsymbol{\Delta}^{\mathrm{op}})^{\mathrm{gpd}} \simeq \mathrm{pt}_{\mathcal{S}}$) it follows that this simplicial space is constant. The commutativity of the right square now follows from the computation

$$\mathrm{N}^{\mathrm{R}}_\infty(\max(\mathcal{C}))_0 = (\mathrm{Fun}([0], \max(\mathcal{C}))^{\boldsymbol{W}})^{\mathrm{gpd}} \simeq \mathcal{C}^{\mathrm{gpd}},$$

which gives rise to a canonical equivalence $\mathrm{N}^{\mathrm{R}}_\infty(\max(\mathcal{C}))_\bullet \simeq \mathrm{const}(\mathcal{C}^{\mathrm{gpd}}) \simeq \mathrm{N}_\infty(\mathcal{C}^{\mathrm{gpd}})_\bullet$ in $s\mathcal{S}$. □

Now, recall that *any* relative ∞–category $(\mathcal{R}, \boldsymbol{W})$ admits a natural map $\min(\mathcal{R}) = (\mathcal{R}, \mathcal{R}^{\simeq}) \to (\mathcal{R}, \boldsymbol{W})$ (namely the unit of the adjunction $\min \dashv \mathrm{U}_{\mathcal{R}\mathrm{el}}$). Hence, by Proposition 3.5 we obtain a natural map

$$\mathrm{N}_\infty(\mathcal{R}) \to \mathrm{N}^{\mathrm{R}}_\infty(\mathcal{R}, \boldsymbol{W})$$

[9] We refer the reader to Lemma H.4.5 for a more general statement (whose proof of course does not rely on the present discussion in any way).

in $s\mathcal{S}$.[10] This immediately suggests the following two questions:

Question 3.6 When does this map in $s\mathcal{S}$ (or equivalently, its target) actually lie in the full subcategory $\mathcal{CSS} \subset s\mathcal{S}$?

Question 3.7 In light of the composite adjunction

$$\begin{tikzcd} s\mathcal{S} \arrow[r, shift left, "\mathrm{L}_{\mathcal{CSS}}"] \arrow[r, phantom, "\perp"] & \mathcal{CSS} \arrow[l, shift left, "\mathrm{U}_{\mathcal{CSS}}"] \arrow[r, shift left, "\mathrm{N}_\infty^{-1}"] \arrow[r, phantom, "\sim"] & \mathcal{C}\mathrm{at}_\infty, \arrow[l, shift left, "\mathrm{N}_\infty"] \end{tikzcd}$$

what is the ∞–categorical significance of this map?

We give a partial answer to Question 3.6 in [15] (see the *calculus theorem* (Theorem H.6.1)). Meanwhile, the essence of the present paper consists in the following complete answer to Question 3.7, the *local universal property of the Rezk nerve*:

Theorem 3.8 *For any* $(\mathcal{R}, \boldsymbol{W}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ *and any* $\mathcal{C} \in \mathcal{C}\mathrm{at}_\infty$*, we have a commutative square*

$$\begin{tikzcd} \hom_{\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty}((\mathcal{R}, \boldsymbol{W}), \min(\mathcal{C})) \arrow[r, hook] \arrow[d, "\wr"'] & \hom_{\mathcal{C}\mathrm{at}_\infty}(\mathcal{R}, \mathcal{C}) \arrow[d, "\wr"] \\ \hom_{s\mathcal{S}}(\mathrm{N}^{\mathrm{R}}_\infty(\mathcal{R}, \boldsymbol{W}), \mathrm{N}_\infty(\mathcal{C})) \arrow[r] & \hom_{\mathcal{CSS}}(\mathrm{N}_\infty(\mathcal{R}), \mathrm{N}_\infty(\mathcal{C})). \end{tikzcd}$$

In other words, the natural map

$$\mathrm{N}_\infty(\mathcal{R}) \simeq \mathrm{L}_{\mathcal{CSS}}(\mathrm{N}_\infty(\mathcal{R})) \to \mathrm{L}_{\mathcal{CSS}}(\mathrm{N}^{\mathrm{R}}_\infty(\mathcal{R}, \boldsymbol{W}))$$

in $\mathcal{CSS}$ *corresponds to the localization map* $\mathcal{R} \to \mathcal{R}[\![\boldsymbol{W}^{-1}]\!]$ *in* $\mathcal{C}\mathrm{at}_\infty$.

We will give a proof of Theorem 3.8 in Section 4.

Using Theorem 3.8 as input, we now prove the first half of the *global* universal property of the Rezk nerve.

Proposition 3.9 *The composite functor*

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathrm{N}^{\mathrm{R}}_\infty} s\mathcal{S} \xrightarrow{\mathrm{L}_{\mathcal{CSS}}} \mathcal{CSS} \underset{\sim}{\xrightarrow{\mathrm{N}_\infty^{-1}}} \mathcal{C}\mathrm{at}_\infty$$

is canonically equivalent in $\mathrm{Fun}(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty, \mathcal{C}\mathrm{at}_\infty)$ *to the localization functor*

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathscr{L}} \mathcal{C}\mathrm{at}_\infty.$$

[10] This can also be obtained from the levelwise inclusion $\hom^{\mathrm{lw}}_{\mathcal{C}\mathrm{at}_\infty}([\bullet], \mathcal{R}) \simeq (\mathrm{Fun}^{\mathrm{lw}}([\bullet], \mathcal{R})^{\boldsymbol{W}})^{\simeq} \hookrightarrow \mathrm{Fun}^{\mathrm{lw}}([\bullet], \mathcal{R})^{\boldsymbol{W}}$ of maximal subgroupoids.

Proof Consider the commutative triangle

$$\begin{tikzcd} & \min \circ \mathrm{U}_{\mathcal{R}\mathrm{el}} \arrow[dl] \arrow[dr] & \\ \mathrm{id}_{\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty} \arrow[rr] & & \min \circ \mathrm{N}_\infty^{-1} \circ \mathrm{L}_{\mathcal{CSS}} \circ \mathrm{N}_\infty^{\mathrm{R}} \end{tikzcd}$$

in $\mathrm{Fun}(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty, \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)$. Postcomposing with the functor $\mathscr{L}$ yields a commutative triangle

$$\begin{tikzcd} & \mathrm{U}_{\mathcal{R}\mathrm{el}} \arrow[dl] \arrow[dr] & \\ \mathscr{L} \arrow[rr] & & \mathrm{N}_\infty^{-1} \circ \mathrm{L}_{\mathcal{CSS}} \circ \mathrm{N}_\infty^{\mathrm{R}} \end{tikzcd}$$

in $\mathrm{Fun}(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty, \mathcal{C}\mathrm{at}_\infty)$. By Theorem 3.8, the horizontal morphism in this commutative triangle is an equivalence. □

Remark 3.10 Proposition 3.9 can be seen as a generalization of work of Barwick and Kan. To see this, consider the composite pair of Quillen adjunctions

$$s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}} \rightleftarrows ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}} \rightleftarrows \mathrm{RelCat}_{\mathrm{BK}},$$

where

- the first is the left Bousfield localization which defines the Rezk model structure (see [17, Theorem 7.2]) and presents the adjunction $\mathrm{L}_{\mathcal{CSS}}\colon s\mathcal{S} \rightleftarrows \mathcal{CSS} :\!\mathrm{U}_{\mathcal{CSS}}$, and
- the second is the Quillen equivalence which defines the Barwick–Kan model structure (see [1, Theorem 6.1]).

As the latter is constructed using the lifting theorem for cofibrantly generated model categories, its right adjoint preserves all weak equivalences by definition. Moreover, Barwick and Kan provide a natural weak equivalence in $s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}}$ (and hence also in $ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}}$) from the Rezk nerve functor to the right adjoint of their Quillen equivalence (see [1, Lemma 5.4]).

Now, consider the commutative triangle

$$\begin{tikzcd} s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}} \arrow[dr, "\mathrm{id}_{ss\mathcal{S}\mathrm{et}}"'] & & \mathrm{RelCat}_{\mathrm{triv}} \arrow[ll, "\mathrm{N}^{\mathrm{R}}"'] \arrow[dl] \\ & ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}} & \end{tikzcd}$$

in $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}$ (in which we take RelCat with the *trivial* model structure since we are interested in relative categories themselves here). Applying the localization functor

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at} \hookrightarrow \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathscr{L}} \mathcal{C}\mathrm{at}_\infty,$$

this yields a commutative triangle

$$\begin{tikzcd} s\mathcal{S} & & \mathrm{RelCat} \arrow[ll, "s(|-|)\circ \mathrm{N}^{\mathrm{R}}"'] \arrow[dl, "\mathrm{N}_\infty \circ \mathscr{L}"] \\ & \mathcal{CSS} \arrow[from=ul, "\mathrm{L}_{\mathcal{CSS}}"'] & \end{tikzcd}$$

in $\mathcal{C}\mathrm{at}_\infty$, in which

- the upper map coincides with the composite

$$\mathrm{RelCat} \to \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at} \hookrightarrow \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathrm{N}^{\mathrm{R}}_\infty} s\mathcal{S}$$

  by Remark 3.2, and
- the map $\mathrm{RelCat} \to \mathcal{CSS}$ can be identified as indicated since, by what we have just seen, it is equivalent to the projection

$$\mathrm{RelCat} \to \mathrm{RelCat}[\![\boldsymbol{W}_{\mathrm{BK}}^{-1}]\!] \simeq \mathcal{C}\mathrm{at}_\infty$$

  to the underlying ∞–category (which is indeed given by localization).

It follows that we obtain a commutative diagram

$$\begin{tikzcd} \mathrm{RelCat} \arrow[r] \arrow[d, "\mathscr{L}"'] & \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \arrow[r, "\mathrm{N}^{\mathrm{R}}_\infty"] & s\mathcal{S} \arrow[d, "\mathrm{L}_{\mathcal{CSS}}"] \\ \mathcal{C}\mathrm{at}_\infty \arrow[rr, "\sim", "\mathrm{N}_\infty"'] & & \mathcal{CSS} \end{tikzcd}$$

in $\mathcal{C}\mathrm{at}_\infty$, which is precisely the restriction of the assertion of Proposition 3.9 to the category RelCat, as claimed.

We now prove the second half of the global universal property of the Rezk nerve.

Proposition 3.11 *The composite functor*

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathrm{N}^{\mathrm{R}}_\infty} s\mathcal{S} \xrightarrow{\mathrm{L}_{\mathcal{CSS}}} \mathcal{CSS} \simeq \mathcal{C}\mathrm{at}_\infty$$

induces an equivalence

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty[\![\boldsymbol{W}_{\mathrm{BK}}^{-1}]\!] \xrightarrow{\sim} \mathcal{C}\mathrm{at}_\infty.$$

In the proof of Proposition 3.11, it will be convenient to have the following terminology:

Definition 3.12 We define the subcategory $\boldsymbol{W}_{\mathrm{Rezk}} \subset ss\mathcal{S}$ of *Rezk weak equivalences* to be created by the composite

$$s(s\mathcal{S}) \xrightarrow{s(|-|)} s\mathcal{S} \xrightarrow{\mathrm{L}_{\mathcal{CSS}}} \mathcal{CSS} \simeq \mathcal{C}\mathrm{at}_\infty.$$

(This name is meant to be suggestive of Rezk's "complete Segal space" model structure on the category $ss\mathcal{S}\mathrm{et}$ of bisimplicial *sets*.) We denote the resulting relative ∞–category by $ss\mathcal{S}_{\mathrm{Rezk}} = (ss\mathcal{S}, \boldsymbol{W}_{\mathrm{Rezk}}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$. Since left localizations are in particular free localizations (recall Example 1.13), this composite left adjoint induces an equivalence

$$ss\mathcal{S}[\![\boldsymbol{W}_{\mathrm{Rezk}}^{-1}]\!] \xrightarrow{\sim} \mathcal{C}\mathrm{at}_\infty$$

in $\mathcal{C}\mathrm{at}_\infty$.

Proof of Proposition 3.11 Recalling Remark 3.4, we have a composite adjunction

$$\begin{tikzcd} ss\mathcal{S} \arrow[r, shift left, "\text{よ}_!(\mathrm{m}\times\mathrm{m})"] \arrow[r, phantom, "\perp"] & \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \arrow[l, shift left, "\mathrm{preN}^{\mathrm{R}}_\infty"] \arrow[r, shift left, "\mathscr{L}"] \arrow[r, phantom, "\perp"] & \mathcal{C}\mathrm{at}_\infty. \arrow[l, shift left, hook, "\min"] \end{tikzcd}$$

Moreover, it follows from Proposition 3.5 that the right adjoint of this composite adjunction is precisely that of the composite adjunction

$$\begin{tikzcd} s(s\mathcal{S}) \arrow[r, shift left, "s(|-|)"] \arrow[r, phantom, "\perp"] & s\mathcal{S} \arrow[l, shift left, hook, "s(\mathrm{const})"] \arrow[r, shift left, "\mathrm{L}_{\mathcal{CSS}}"] \arrow[r, phantom, "\perp"] & \mathcal{CSS} \arrow[l, shift left, hook, "\mathrm{U}_{\mathcal{CSS}}"] \arrow[r, shift left, "\mathrm{N}_\infty^{-1}"] \arrow[r, phantom, "\sim"] & \mathcal{C}\mathrm{at}_\infty \arrow[l, shift left, "\mathrm{N}_\infty"] \end{tikzcd}$$

whose left adjoint defines $\boldsymbol{W}_{\mathrm{Rezk}} \subset ss\mathcal{S}$, and hence in particular it follows that the right adjoint of our original composite adjunction defines a weak equivalence

$$ss\mathcal{S}_{\mathrm{Rezk}} \xleftarrow[\approx]{\min\circ\,\mathrm{preN}^{\mathrm{R}}_\infty} \min(\mathcal{C}\mathrm{at}_\infty)$$

in $(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{\mathrm{BK}}$.

Next, we claim that the right adjoint $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathrm{preN}^{\mathrm{R}}_\infty} ss\mathcal{S}$ is a relative functor. To see this, first note that given any $(\mathcal{R}, \boldsymbol{W}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$, we obtain a counit map

$$(\mathcal{R}, \boldsymbol{W}) \xrightarrow{\approx} \min(\mathcal{R}[\![\boldsymbol{W}^{-1}]\!])$$

in $(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{\mathrm{BK}}$ from the adjunction $\mathscr{L} \dashv \min$. Theorem 3.8 and Proposition 3.5 then together imply that applying the functor $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathrm{preN}^{\mathrm{R}}_\infty} ss\mathcal{S}$ to this map yields a weak equivalence

$$\mathrm{preN}^{\mathrm{R}}_\infty(\mathcal{R}, \boldsymbol{W}) \xrightarrow{\approx} \mathrm{preN}^{\mathrm{R}}_\infty(\min(\mathcal{R}[\![\boldsymbol{W}^{-1}]\!])) \simeq \mathrm{const}^{\mathrm{lw}}(\mathrm{N}_\infty(\mathcal{R}[\![\boldsymbol{W}^{-1}]\!]))$$

in $ss\mathcal{S}_{\mathrm{Rezk}}$. Hence, any weak equivalence $(\mathcal{R}_1, \boldsymbol{W}_1) \xrightarrow{\approx} (\mathcal{R}_2, \boldsymbol{W}_2)$ in $(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{\mathrm{BK}}$ induces a commutative diagram

$$\begin{tikzcd} \mathrm{preN}^{\mathrm{R}}_\infty(\mathcal{R}_1, \boldsymbol{W}_1) \arrow[r] \arrow[d, "\wr"'] & \mathrm{preN}^{\mathrm{R}}_\infty(\mathcal{R}_2, \boldsymbol{W}_2) \arrow[d, "\wr"] \\ \mathrm{const}^{\mathrm{lw}}(\mathrm{N}_\infty(\mathcal{R}_1[\![\boldsymbol{W}_1^{-1}]\!])) \arrow[r, "\sim"'] & \mathrm{const}^{\mathrm{lw}}(\mathrm{N}_\infty(\mathcal{R}_2[\![\boldsymbol{W}_2^{-1}]\!])) \end{tikzcd}$$

in $ss\mathcal{S}_{\mathrm{Rezk}}$, and then the top arrow in this square is also in $\boldsymbol{W}_{\mathrm{Rezk}} \subset ss\mathcal{S}$ since it has the two-out-of-three property. So this does indeed define a relative functor

$$(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{\mathrm{BK}} \xrightarrow{\mathrm{preN}^{\mathrm{R}}_\infty} ss\mathcal{S}_{\mathrm{Rezk}}.$$

From here, it follows that the right adjoints of our original composite adjunction form a commutative diagram

$$\begin{tikzcd} ss\mathcal{S}_{\mathrm{Rezk}} & & \min(\mathcal{C}\mathrm{at}_\infty) \arrow[ll, "\mathrm{preN}^{\mathrm{R}}_\infty \circ \min"', "\approx"] \arrow[dl, "\min", "\approx"'] \\ & (\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{\mathrm{BK}} \arrow[ul, "\mathrm{preN}^{\mathrm{R}}_\infty"] & \end{tikzcd}$$

in $(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{\mathrm{BK}}$, and so the entire diagram lies in $\boldsymbol{W}_{\mathrm{BK}} \subset \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ since it has the two-out-of-three property. Hence, we obtain a commutative diagram

$$\begin{tikzcd} ss\mathcal{S}_{\mathrm{Rezk}} \arrow[r, "s(|-|)"] \arrow[rr, bend left, "\approx"] & s\mathcal{S} \arrow[r, "\mathrm{L}_{\mathcal{CSS}}"] & \mathcal{CSS} \simeq \mathcal{C}\mathrm{at}_\infty \\ (\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{\mathrm{BK}} \arrow[u, "\mathrm{preN}^{\mathrm{R}}_\infty", "\wr"'] \arrow[ur, "\mathrm{N}^{\mathrm{R}}_\infty"'] & & \end{tikzcd}$$

in $(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{\mathrm{BK}}$, which proves the claim. □

Remark 3.13 It does not appear possible to give a completely hands-off proof of Proposition 3.11, ie one not relying on Theorem 3.8 (or perhaps even one that would prove Theorem 3.8 as a formal consequence). More specifically, adjunctions of underlying ∞–categories do not necessarily play well with relative ∞–category structures, even if one of the adjoints is a relative functor: one must have some control over the behavior of *both* adjoints.

For instance, the geometric realization functor $s\mathcal{S} \xrightarrow{|-|} \mathcal{S}$ and its restriction to the subcategory $s\mathcal{S}\mathrm{et} \subset s\mathcal{S}$ create subcategories of weak equivalences which define the *Kan–Quillen* *relative ∞–category* structures $(s\mathcal{S}, \boldsymbol{W}_{\mathrm{KQ}}^{s\mathcal{S}}), (s\mathcal{S}\mathrm{et}, \boldsymbol{W}_{\mathrm{KQ}}^{s\mathcal{S}\mathrm{et}}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ (which underlie their respective Kan–Quillen model structures (see Section S.4)). Moreover, these relative ∞–categories give rise to a diagram

$$\begin{tikzcd} s\mathcal{S} \arrow[rr, shift left, "s(\pi_0)"] \arrow[d] \arrow[rr, phantom, "\perp"] & & s\mathcal{S}\mathrm{et} \arrow[ll, hook', shift left, "s(\mathrm{disc})"] \arrow[d] \\ s\mathcal{S}[\![(\boldsymbol{W}_{\mathrm{KQ}}^{s\mathcal{S}})^{-1}]\!] \arrow[dr, "\sim"'] & & s\mathcal{S}\mathrm{et}[\![(\boldsymbol{W}_{\mathrm{KQ}}^{s\mathcal{S}\mathrm{et}})^{-1}]\!] \arrow[dl, "\sim"] \\ & \mathcal{S} & \end{tikzcd}$$

in which the right adjoint commutes with the respective localization functors: in other words, it induces a weak equivalence

$$(s\mathcal{S}_{\mathrm{KQ}}, \boldsymbol{W}_{\mathrm{KQ}}^{s\mathcal{S}}) \xleftarrow{\approx} (s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}, \boldsymbol{W}_{\mathrm{KQ}}^{s\mathcal{S}\mathrm{et}})$$

in $(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{\mathrm{BK}}$. Nevertheless, the left adjoint is clearly very far from also defining a weak equivalence in $(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{\mathrm{BK}}$.

Remark 3.14 Taken together, Propositions 3.9 and 3.11 imply that in fact the adjunction

$$\begin{tikzcd} ss\mathcal{S} \arrow[r, shift left, "よ_!(\mathrm{m}\times\mathrm{m})"] \arrow[r, phantom, "\perp"] & \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \arrow[l, shift left, "\mathrm{preN}_\infty^{\mathrm{R}}"] \end{tikzcd}$$

has

- that both adjoints are relative functors (with respect to their respective Rezk and Barwick–Kan relative structures), and
- that the unit and counit are both natural weak equivalences.

This can be seen as follows.

First of all, recall that in the proof of Proposition 3.11, we already saw that the right adjoint is a relative functor. On the other hand, the left adjoint is a relative functor because the composite left adjoint

$$ss\mathcal{S} \xrightarrow{よ_!(\mathrm{m}\times\mathrm{m})} \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathscr{L}} \mathcal{C}\mathrm{at}_\infty$$

agrees with the left adjoint

$$ss\mathcal{S} \xrightarrow{s(|-|)} s\mathcal{S} \xrightarrow{\mathrm{L}_{\mathcal{CSS}}} \mathcal{CSS} \underset{\sim}{\xrightarrow{\mathrm{N}_\infty^{-1}}} \mathcal{C}\mathrm{at}_\infty$$

(since we have seen in the proof of Proposition 3.11 that they share a right adjoint), and so in fact the subcategory $\boldsymbol{W}_{\mathrm{Rezk}} \subset s\mathcal{sS}$ is created by pulling back the subcategory $\boldsymbol{W}_{\mathrm{BK}} \subset \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$.

Next, we can see that the counit map

$$よ_!(\mathrm{m}{\times}\mathrm{m})(\mathrm{preN}^{\mathrm{R}}_\infty(\mathcal{R}, \boldsymbol{W})) \to (\mathcal{R}, \boldsymbol{W})$$

is a weak equivalence in $(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{\mathrm{BK}}$ as follows. Applying the functor $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathscr{L}} \mathcal{C}\mathrm{at}_\infty$, we obtain a map

$$\mathscr{L}\bigl(よ_!(\mathrm{m}{\times}\mathrm{m})(\mathrm{preN}^{\mathrm{R}}_\infty(\mathcal{R}, \boldsymbol{W}))\bigr) \to \mathcal{R}[\![\boldsymbol{W}^{-1}]\!]$$

in $\mathcal{C}\mathrm{at}_\infty$. Then, again appealing to the fact that these composite left adjoints $s\mathcal{sS} \to \mathcal{C}\mathrm{at}_\infty$ agree, we can reidentify the source as

$$\begin{aligned}\mathscr{L}\bigl(よ_!(\mathrm{m}{\times}\mathrm{m})(\mathrm{preN}^{\mathrm{R}}_\infty(\mathcal{R}, \boldsymbol{W}))\bigr) &\simeq \mathrm{N}_\infty^{-1}\bigl(\mathrm{L}_{\mathcal{CSS}}\bigl(s(|{-}|)(\mathrm{preN}^{\mathrm{R}}_\infty(\mathcal{R}, \boldsymbol{W}))\bigr)\bigr) \\ &\simeq \mathrm{N}_\infty^{-1}\bigl(\mathrm{L}_{\mathcal{CSS}}(\mathrm{N}^{\mathrm{R}}_\infty(\mathcal{R}, \boldsymbol{W}))\bigr).\end{aligned}$$

So, we can reidentify this map as

$$\mathrm{N}_\infty^{-1}\bigl(\mathrm{L}_{\mathcal{CSS}}(\mathrm{N}^{\mathrm{R}}_\infty(\mathcal{R}, \boldsymbol{W}))\bigr) \to \mathcal{R}[\![\boldsymbol{W}^{-1}]\!],$$

which is an equivalence by Theorem 3.8. So the counit map is indeed a weak equivalence in $(\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty)_{\mathrm{BK}}$, ie the counit is a natural weak equivalence.

Finally, we can see that the unit map

$$\mathrm{preN}^{\mathrm{R}}_\infty(よ_!(\mathrm{m}{\times}\mathrm{m})(Y)) \to Y$$

is a weak equivalence in $s\mathcal{sS}_{\mathrm{Rezk}}$ as follows. Applying the composite left adjoint

$$s\mathcal{sS} \xrightarrow{\mathrm{N}_\infty^{-1} \circ \mathrm{L}_{\mathcal{CSS}} \circ s(|{-}|)} \mathcal{C}\mathrm{at}_\infty$$

and appealing to Proposition 3.9, we obtain a map

$$\mathscr{L}(よ_!(\mathrm{m}{\times}\mathrm{m})(Y)) \to \mathrm{N}_\infty^{-1}\bigl(\mathrm{L}_{\mathcal{CSS}}(s(|{-}|)(Y))\bigr)$$

in $\mathcal{C}\mathrm{at}_\infty$, and the same equivalence of composite left adjoints $s\mathcal{sS} \to \mathcal{C}\mathrm{at}_\infty$ implies that this is an equivalence. So the unit map is indeed a weak equivalence in $s\mathcal{sS}_{\mathrm{Rezk}}$, ie the unit is a natural weak equivalence.

4 The proof of Theorem 3.8

Let $(\mathcal{R}, \boldsymbol{W})$ be an arbitrary relative ∞–category. In this section, we show that as a simplicial space, its Rezk nerve $\mathrm{N}^{\mathrm{R}}_\infty(\mathcal{R}, \boldsymbol{W})$ enjoys the desired universal property for mapping *into* complete Segal spaces: for any $\mathcal{C} \in \mathcal{C}\mathrm{at}_\infty$, we have a commutative diagram

$$\begin{tikzcd}
\hom_{\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty}((\mathcal{R}, \boldsymbol{W}), \min(\mathcal{C})) \arrow[r, hook] \arrow[d, "\wr"'] & \hom_{\mathcal{C}\mathrm{at}_\infty}(\mathcal{R}, \mathcal{C}) \arrow[d, "\wr"] \\
\hom_{s\mathcal{S}}(\mathrm{N}^{\mathrm{R}}_\infty(\mathcal{R}, \boldsymbol{W}), \mathrm{N}_\infty(\mathcal{C})) \arrow[r] & \hom_{\mathcal{CSS}}(\mathrm{N}_\infty(\mathcal{R}), \mathrm{N}_\infty(\mathcal{C}))
\end{tikzcd}$$

in $\mathcal{S}$, as asserted in Theorem 3.8.

Most of the proof is reasonably straightforward, and we can give it immediately. But there will be one technical result (Lemma 4.3) that is necessary for the proof which will occupy us for the remainder of the section.

Proof of Theorem 3.8 By definition, the localization $\mathcal{R}[\![\boldsymbol{W}^{-1}]\!] \in \mathcal{C}\mathrm{at}_\infty$ is given as the pushout

$$\begin{tikzcd}
\boldsymbol{W} \arrow[r] \arrow[d] & \mathcal{R} \arrow[d] \\
\boldsymbol{W}^{\mathrm{gpd}} \arrow[r] & \mathcal{R}[\![\boldsymbol{W}^{-1}]\!]
\end{tikzcd}$$

in $\mathcal{C}\mathrm{at}_\infty$; under the equivalence $\mathrm{N}_\infty\colon \mathcal{C}\mathrm{at}_\infty \xrightarrow{\sim} \mathcal{CSS}$, this corresponds to a pushout diagram

$$\begin{tikzcd}
\mathrm{N}_\infty(\boldsymbol{W}) \arrow[r] \arrow[d] & \mathrm{N}_\infty(\mathcal{R}) \arrow[d] \\
\mathrm{N}_\infty(\boldsymbol{W}^{\mathrm{gpd}}) \arrow[r] & \mathrm{N}_\infty(\mathcal{R}[\![\boldsymbol{W}^{-1}]\!])
\end{tikzcd}$$

in $\mathcal{CSS} \subset s\mathcal{S}$. On the other hand, there is an evident commutative diagram

$$\begin{tikzcd}
(\boldsymbol{W}, \boldsymbol{W}^{\simeq}) \arrow[r] \arrow[d] & (\mathcal{R}, \mathcal{R}^{\simeq}) \arrow[dr] & \\
(\boldsymbol{W}, \boldsymbol{W}) \arrow[rr] \arrow[drr] & & (\mathcal{R}, \boldsymbol{W}) \arrow[d] \\
& & (\mathcal{R}[\![\boldsymbol{W}^{-1}]\!], \mathcal{R}[\![\boldsymbol{W}^{-1}]\!]^{\simeq})
\end{tikzcd}$$

in $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$. Applying the functor $\mathrm{N}^{\mathrm{R}}_\infty \colon \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty \to s\mathcal{S}$ and taking the pushout of the upper left span, in light of Proposition 3.5 we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{N}_\infty(\boldsymbol{W}) \arrow[r] \arrow[d] & \mathrm{N}_\infty(\mathcal{R}) \arrow[d] \arrow[dr] & \\
\mathrm{N}_\infty(\boldsymbol{W}^{\mathrm{gpd}}) \arrow[r] \arrow[drr] & \mathrm{p.o.}^{s\mathcal{S}} \arrow[r] \arrow[dr] & \mathrm{N}^{\mathrm{R}}_\infty(\mathcal{R}, \boldsymbol{W}) \arrow[d] \\
& & \mathrm{N}_\infty(\mathcal{R}[\![\boldsymbol{W}^{-1}]\!])
\end{tikzcd}$$

in $s\mathcal{S}$,

- where $\mathrm{p.o.}^{s\mathcal{S}}$ denotes the pushout in $s\mathcal{S}$ of the upper left span, and
- which contains as a subdiagram the above pushout square in $\mathcal{CSS} \subset s\mathcal{S}$ (namely the upper left span along with the object $\mathrm{N}_\infty(\mathcal{R}[\![\boldsymbol{W}^{-1}]\!])$).

Our goal is to prove that the induced map

$$\mathrm{L}_{\mathcal{CSS}}(\mathrm{N}^{\mathrm{R}}_\infty(\mathcal{R}, \boldsymbol{W})) \to \mathrm{L}_{\mathcal{CSS}}(\mathrm{N}_\infty(\mathcal{R}[\![\boldsymbol{W}^{-1}]\!])) \simeq \mathrm{N}_\infty(\mathcal{R}[\![\boldsymbol{W}^{-1}]\!])$$

is an equivalence in $\mathcal{CSS} \subset s\mathcal{S}$.

For notational convenience, let us simply write

$$\begin{tikzcd}
(s\mathcal{S})^{\mathrm{op}} \arrow[r, "よ_{(s\mathcal{S})^{\mathrm{op}}}"] \arrow[rr, dashed, bend right, "よ_{\mathcal{CSS}^{\mathrm{op}}}"'] & \mathrm{Fun}(s\mathcal{S}, \mathcal{S}) \arrow[r, "-\circ \mathrm{U}_{\mathcal{CSS}}"] & \mathrm{Fun}(\mathcal{CSS}, \mathcal{S})
\end{tikzcd}$$

for the restricted contravariant Yoneda functor, so that for any $Y \in s\mathcal{S}$ we have

$$よ_{\mathcal{CSS}^{\mathrm{op}}}(Y) = \hom_{s\mathcal{S}}(Y, \mathrm{U}_{\mathcal{CSS}}(-)) \simeq \hom_{\mathcal{CSS}}(\mathrm{L}_{\mathcal{CSS}}(Y), -)$$

in $\mathrm{Fun}(\mathcal{CSS}, \mathcal{S})$. Then, by Yoneda's lemma, our aforestated goal is equivalent to proving that the map

$$\mathrm{N}^{\mathrm{R}}_\infty(\mathcal{R}, \boldsymbol{W}) \to \mathrm{N}_\infty(\mathcal{R}[\![\boldsymbol{W}^{-1}]\!])$$

in $s\mathcal{S}$ induces an equivalence

$$よ_{\mathcal{CSS}^{\mathrm{op}}}(\mathrm{N}^{\mathrm{R}}_\infty(\mathcal{R}, \boldsymbol{W})) \leftarrow よ_{\mathcal{CSS}^{\mathrm{op}}}(\mathrm{N}_\infty(\mathcal{R}[\![\boldsymbol{W}^{-1}]\!]))$$

in $\mathrm{Fun}(\mathcal{CSS}, \mathcal{S})$. Moreover, as the functor $s\mathcal{S} \xrightarrow{\mathrm{L}_{\mathcal{CSS}}} \mathcal{CSS}$ commutes with pushouts (being a left adjoint), it follows that the map

$$\mathrm{p.o.}^{s\mathcal{S}} \to \mathrm{N}_\infty(\mathcal{R}[\![\boldsymbol{W}^{-1}]\!])$$

in $s\mathcal{S}$ induces an equivalence

$$\mathrm{L}_{\mathcal{CSS}}(\text{p.o.}^{s\mathcal{S}}) \xrightarrow{\sim} \mathrm{L}_{\mathcal{CSS}}(\mathrm{N}_\infty(\mathcal{R}[\![\boldsymbol{W}^{-1}]\!])) \simeq \mathrm{N}_\infty(\mathcal{R}[\![\boldsymbol{W}^{-1}]\!])$$

in $\mathcal{CSS} \subset s\mathcal{S}$, and so the above diagram in $s\mathcal{S}$ gives rise to a retraction diagram

$$\begin{tikzcd}
\text{ょ}_{\mathcal{CSS}^{\mathrm{op}}}(\text{p.o.}^{s\mathcal{S}}) & \text{ょ}_{\mathcal{CSS}^{\mathrm{op}}}(\mathrm{N}^{\mathrm{R}}_\infty(\mathcal{R}, \boldsymbol{W})) \arrow[l] \\
 & \text{ょ}_{\mathcal{CSS}^{\mathrm{op}}}(\mathrm{N}_\infty(\mathcal{R}[\![\boldsymbol{W}^{-1}]\!])) \arrow[u] \arrow[ul, "\sim"]
\end{tikzcd}$$

in $\mathrm{Fun}(\mathcal{CSS}, \mathcal{S})$ into which this map which we must show to be an equivalence fits, and which it therefore suffices to show is in fact a diagram of equivalences.

Now, observe that $\mathcal{CSS}$ is complete and hence in particular is cotensored over $\mathcal{S}$, and observe moreover that the functor

$$(s\mathcal{S})^{\mathrm{op}} \xrightarrow{\text{ょ}_{\mathcal{CSS}^{\mathrm{op}}}} \mathrm{Fun}(\mathcal{CSS}, \mathcal{S})$$

factors through the contravariant Yoneda embedding and hence takes values in functors which commute with cotensors. So, by Lemma 4.1, it suffices to show that after postcomposition with $\mathcal{S} \xrightarrow{\pi_0} \mathcal{S}\mathrm{et}$, the above retraction diagram in $\mathrm{Fun}(\mathcal{CSS}, \mathcal{S})$ becomes a diagram of natural isomorphisms in $\mathrm{Fun}(\mathcal{CSS}, \mathcal{S}\mathrm{et})$. Hence, it suffices to show that the induced map

$$\big(\pi_0 \circ \text{ょ}_{\mathcal{CSS}^{\mathrm{op}}}(\mathrm{N}^{\mathrm{R}}_\infty(\mathcal{R}, \boldsymbol{W}))\big) \to (\pi_0 \circ \text{ょ}_{\mathcal{CSS}^{\mathrm{op}}}(\text{p.o.}^{s\mathcal{S}}))$$

is a natural monomorphism in $\mathrm{Fun}(\mathcal{CSS}, \mathcal{S}\mathrm{et})$. This follows from the stronger statement that the composite

$$\big(\pi_0 \circ \text{ょ}_{\mathcal{CSS}^{\mathrm{op}}}(\mathrm{N}^{\mathrm{R}}_\infty(\mathcal{R}, \boldsymbol{W}))\big) \to (\pi_0 \circ \text{ょ}_{\mathcal{CSS}^{\mathrm{op}}}(\text{p.o.}^{s\mathcal{S}})) \to \big(\pi_0 \circ \text{ょ}_{\mathcal{CSS}^{\mathrm{op}}}(\mathrm{N}_\infty(\mathcal{R}))\big)$$

is a natural monomorphism in $\mathrm{Fun}(\mathcal{CSS}, \mathcal{S}\mathrm{et})$, which in turn follows from Lemma 4.3. □

We needed the following easy result in the proof of Theorem 3.8:

Lemma 4.1 *Let $\mathcal{C}$ be an ∞–category admitting a cotensoring*

$$\mathcal{S}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{-\pitchfork-} \mathcal{C},$$

and suppose we are given two space-valued functors $F, G \in \mathrm{Fun}(\mathcal{C}, \mathcal{S})$ that commute with cotensors. Then, a natural transformation $F \to G$ is a natural equivalence in $\mathrm{Fun}(\mathcal{C}, \mathcal{S})$ *if and only if its postcomposition $\pi_0 F \to \pi_0 G$ with $\mathcal{S} \xrightarrow{\pi_0} \mathcal{S}\mathrm{et}$ is a natural isomorphism in* $\mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$.

Proof The "only if" direction is clear. So, suppose we are given a natural transformation $F \to G$ in $\mathrm{Fun}(\mathcal{C}, \mathcal{S})$ such that the induced natural transformation $\pi_0 F \to \pi_0 G$ is a natural equivalence in $\mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$. Since equivalences in $\mathrm{Fun}(\mathcal{C}, \mathcal{S})$ are determined componentwise, it suffices to show that for any $Y \in \mathcal{C}$, the map $F(Y) \to G(Y)$ is an equivalence in $\mathcal{S}$. In turn, since equivalences in $\mathcal{S}$ are created in $\mathrm{ho}(\mathcal{S})$, by Yoneda's lemma it suffices to show that for any $Z \in \mathcal{S}$, the induced map $[Z, F(Y)]_{\mathcal{S}} \to [Z, G(Y)]_{\mathcal{S}}$ is an isomorphism in $\mathcal{S}\mathrm{et}$. But since $\mathcal{C}$ admits cotensors, then we can reidentify this map via the canonical commutative square

$$\begin{tikzcd}
\pi_0(F(Z \pitchfork Y)) \arrow[r, "\cong"] \arrow[d, "\wr"'] & \pi_0(G(Z \pitchfork Y)) \arrow[d, "\wr"] \\
{[Z, F(Y)]_{\mathcal{S}}} \arrow[r] & {[Z, G(Y)]_{\mathcal{S}}}
\end{tikzcd}$$

in $\mathcal{S}\mathrm{et}$, in which the top arrow is an isomorphism by the assumption that $\pi_0 F \to \pi_0 G$ is a natural isomorphism and the vertical arrows are isomorphisms by the assumption that F and G commute with cotensors. This proves the claim. $\square$

Before moving on to Lemma 4.3, it will be convenient to have the following bit of terminology:

Definition 4.2 A morphism in a model category $\mathcal{M}$ is called a *homotopy epimorphism* if it presents an epimorphism in the underlying ∞–category $\mathcal{M}[\![\boldsymbol{W}^{-1}]\!]$.

We now proceed to the technical heart of the proof of Theorem 3.8. We warn the reader that our proof of the following result is (perhaps unexpectedly, and certainly unsatisfyingly) complicated.

Lemma 4.3 *The map* $\mathrm{N}_\infty(\mathcal{R}) \to \mathrm{L}_{\mathcal{CSS}}(\mathrm{N}^{\mathrm{R}}_\infty(\mathcal{R}, \boldsymbol{W}))$ *is an epimorphism in* $\mathcal{CSS}$.

Proof Our proof will proceed using model categories — primarily $ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}}$ and $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$, but also a number of others auxiliarily — and will also use the language of marked simplicial sets (see eg Section T.3.1).

We begin by recalling the two Quillen equivalences between $ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}}$ and $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$ given in [5].

(1) Let us write $\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \xrightarrow{\mathrm{pr}_2} \mathbf{\Delta}^{\mathrm{op}}$ for the second projection map and $\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{i_2} \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}$ for the functor $\mathrm{const}([0]^\circ) \times \mathrm{id}_{\mathbf{\Delta}^{\mathrm{op}}}$. Pullbacks along these two functors induce the Quillen equivalence

$$\mathrm{pr}_2^*\colon s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}} \rightleftarrows ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}} :i_2^*$$

of [5, Theorem 4.11].

(2) Let us write $(\Delta^i)^{\mathrm{gpd}} \in s\mathcal{S}\mathrm{et}$ for the nerve of the strict (ie objects-preserving) groupoid completion of $[i] \in \mathcal{C}\mathrm{at}$, and let us write $t_!\colon ss\mathcal{S}\mathrm{et} \to s\mathcal{S}\mathrm{et}$ for the left Kan extension

$$\begin{tikzcd} \mathbf{\Delta} \times \mathbf{\Delta} \arrow[rr, "{([n],[i]) \mapsto \Delta^n \times (\Delta^i)^{\mathrm{gpd}}}"] \arrow[d] & & s\mathcal{S}\mathrm{et} \\ ss\mathcal{S}\mathrm{et} \arrow[urr, dashed] & & \end{tikzcd}$$

along the (1–categorical) Yoneda embedding. This has a right adjoint $t^!\colon s\mathcal{S}\mathrm{et} \to ss\mathcal{S}\mathrm{et}$ given by

$$t^!(Y) = \{\{\hom_{s\mathcal{S}\mathrm{et}}(\Delta^n \times (\Delta^i)^{\mathrm{gpd}}, Y)\}_{i \geq 0}\}_{n \geq 0},$$

and together these fit into the Quillen equivalence

$$t_!\colon ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}} \rightleftarrows s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}} :t^!$$

of [5, Theorem 4.12].

Now, suppose that $\mathtt{R} \in s\mathcal{S}\mathrm{et}^f_{\mathrm{Joyal}}$ is a quasicategory presenting $\mathcal{R} \in \mathcal{C}\mathrm{at}_\infty$, and let $(\mathtt{R}, \mathtt{W}) \in s\mathcal{S}\mathrm{et}^+$ be the marked simplicial set obtained by marking precisely those edges of $\mathtt{R}$ which present maps in $\boldsymbol{W} \subset \mathcal{R}$. For any $n \geq 0$, the ∞–category $\mathrm{Fun}([n], \mathcal{R})$ is presented by the object

$$\underline{\hom}_{s\mathcal{S}\mathrm{et}}(\Delta^n, \mathtt{R}) = \{\hom_{s\mathcal{S}\mathrm{et}}(\Delta^n \times \Delta^i, \mathtt{R})\}_{i \geq 0} \in s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}},$$

and hence its subcategory

$$\mathrm{Fun}([n], \mathcal{R})^{\boldsymbol{W}} \subset \mathrm{Fun}([n], \mathcal{R})$$

is presented by the object

$$\{\hom_{s\mathcal{S}\mathrm{et}^+}((\Delta^n)^\flat \times (\Delta^i)^\sharp, (\mathtt{R}, \mathtt{W}))\}_{i \geq 0} \in s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}.$$

These constructions are contravariantly functorial in $[n] \in \mathbf{\Delta}$, and hence we obtain that the Rezk prenerve

$$\mathrm{preN}^{\mathrm{R}}_{\infty}(\mathcal{R}, \boldsymbol{W}) = \mathrm{Fun}^{\mathrm{lw}}([\bullet], \mathcal{R})^{\boldsymbol{W}} \in s\mathcal{C}\mathrm{at}_{\infty}$$

is presented by the object

$$\{\{\hom_{s\mathcal{S}\mathrm{et}^+}((\Delta^n)^\flat \times (\Delta^i)^\sharp, (\mathtt{R}, \mathtt{W}))\}_{i\geq 0}\}_{n\geq 0} \in s(s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}})_{\mathrm{Reedy}}.$$

From here, we observe that the Quillen adjunction

$$\mathrm{id}_{ss\mathcal{S}\mathrm{et}} \colon s(s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}})_{\mathrm{Reedy}} \rightleftarrows s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}} :\mathrm{id}_{ss\mathcal{S}\mathrm{et}}$$

presents the left localization adjunction $s((-)^{\mathrm{gpd}}) \colon s\mathcal{C}\mathrm{at}_{\infty} \rightleftarrows s\mathcal{S} : s(\mathrm{U}_{\mathcal{S}})$; as all objects of $s(s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}})_{\mathrm{Reedy}}$ are cofibrant, it follows that when considered as an object of $s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}}$, this same bisimplicial set presents $\mathrm{N}^{\mathrm{R}}_{\infty}(\mathcal{R}, \boldsymbol{W}) \in s\mathcal{S}$. Moreover, in light of the left Bousfield localization

$$\mathrm{id}_{ss\mathcal{S}\mathrm{et}} \colon s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}} \rightleftarrows ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}} :\mathrm{id}_{ss\mathcal{S}\mathrm{et}}$$

presenting the left localization adjunction $\mathrm{L}_{\mathcal{CSS}} \colon s\mathcal{S} \rightleftarrows \mathcal{CSS} : \mathrm{U}_{\mathcal{CSS}}$, when considered as an object of $ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}}$, this same bisimplicial set presents the Rezk nerve

$$\mathrm{N}^{\mathrm{R}}_{\infty}(\mathcal{R}, \boldsymbol{W}) = (\mathrm{Fun}^{\mathrm{lw}}([\bullet], \mathcal{R})^{\boldsymbol{W}})^{\mathrm{gpd}} \in \mathcal{CSS}.$$

We will denote this bisimplicial set by $\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W}) \in ss\mathcal{S}\mathrm{et}$.[11] In particular, note that we have a natural isomorphism $\mathrm{N}^{\mathrm{R}}(\mathtt{R}^\natural) \cong t^!(\mathtt{R})$ in $ss\mathcal{S}\mathrm{et}$, and hence we see that the right Quillen equivalence

$$t^! \colon s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}} \to ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}}$$

presents the equivalence $\mathrm{N}_{\infty} \colon \mathcal{C}\mathrm{at}_{\infty} \xrightarrow{\sim} \mathcal{CSS}$ of ∞–categories.

Now, the natural map

$$\mathtt{R}^\natural \to (\mathtt{R}, \mathtt{W})$$

in $s\mathcal{S}\mathrm{et}^+$ induces a map

$$\mathrm{N}^{\mathrm{R}}(\mathtt{R}^\natural) \to \mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W})$$

in $ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}}$, which, by what we have seen, presents the map

$$\mathrm{N}_{\infty}(\mathcal{R}) \to \mathrm{L}_{\mathcal{CSS}}(\mathrm{N}^{\mathrm{R}}_{\infty}(\mathcal{R}, \boldsymbol{W}))$$

[11] When $(\mathtt{R}, \mathtt{W}) \in s\mathcal{S}\mathrm{et}^+$ is the "marked nerve" of a relative 1–category, this recovers the 1–categorical Rezk nerve of Remark 3.2 (as an object of $ss\mathcal{S}\mathrm{et}$), and so there is no ambiguity in the notation.

in $\mathcal{CSS}$. So, to prove that this latter map is an epimorphism in $\mathcal{CSS}$, it suffices to prove that the former map is a homotopy epimorphism in $ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}}$. However, note that there is a natural isomorphism $t_!(\mathrm{pr}_2^*(\mathtt{R})) \xrightarrow{\cong} \mathtt{R}$ in $s\mathcal{S}\mathrm{et}$, which is in particular a weak equivalence in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$; via the Quillen equivalence of item (2), this corresponds to a weak equivalence $\mathrm{pr}_2^*(\mathtt{R}) \xrightarrow{\approx} t^!(\mathtt{R})$ in $ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}}$. So, it also suffices to show that the composite map

$$\mathrm{pr}_2^*(\mathtt{R}) \xrightarrow{\approx} t^!(\mathtt{R}) \cong \mathrm{N}^{\mathrm{R}}(\mathtt{R}^{\natural}) \to \mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W})$$

is a homotopy epimorphism in $ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}}$.

For this, let us also recall the "usual" geometric realization functor $ss\mathcal{S}\mathrm{et} \to s\mathcal{S}\mathrm{et}$ (a homotopy colimit functor with respect to $s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}}$): this is the left Kan extension

$$\begin{tikzcd} \mathbf{\Delta} \times \mathbf{\Delta} \arrow[rr, "{([n],[i]) \mapsto \Delta^n \times \Delta^i}"] \arrow[d] & & s\mathcal{S}\mathrm{et} \\ ss\mathcal{S}\mathrm{et} \arrow[urr, dashed] & & \end{tikzcd}$$

along the (1–categorical) Yoneda embedding, but by [2, Chapter IV, Exercise 1.6] this is (naturally isomorphic to) the functor $\mathrm{diag}^*\colon ss\mathcal{S}\mathrm{et} \to s\mathcal{S}\mathrm{et}$, where $\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{\mathrm{diag}} \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}$ denotes the diagonal functor. Now, the evident morphisms $\Delta^n \times \Delta^i \to \Delta^n \times (\Delta^i)^{\mathrm{gpd}}$ in $s\mathcal{S}\mathrm{et}$ induce a natural transformation $\mathrm{diag}^* \to t_!$ in $\mathrm{Fun}(ss\mathcal{S}\mathrm{et}, s\mathcal{S}\mathrm{et})$. Moreover, it is not hard to see that upon precomposition with $s\mathcal{S}\mathrm{et} \xrightarrow{\mathrm{pr}_2^*} ss\mathcal{S}\mathrm{et}$, this induces the identity natural transformation from $\mathrm{id}_{s\mathcal{S}\mathrm{et}}$ to itself in $\mathrm{Fun}(s\mathcal{S}\mathrm{et}, s\mathcal{S}\mathrm{et})$ (up to isomorphism). Applying these observations to the above composite map in $ss\mathcal{S}\mathrm{et}$, we obtain a commutative square

$$\begin{tikzcd} \mathrm{diag}^*(\mathrm{pr}_2^*(\mathtt{R})) \arrow[r, "\alpha"] \arrow[d, "\wr\|"'] & \mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W})) \arrow[d, "\beta"] \\ t_!(\mathrm{pr}_2^*(\mathtt{R})) \arrow[r, "\gamma"'] & t_!(\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W})) \end{tikzcd}$$

in $s\mathcal{S}\mathrm{et}$, where both objects on the left are (compatibly) isomorphic to $\mathtt{R}$ itself. Since $t_!\colon ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}} \to s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$ is a left Quillen equivalence and all objects of $ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}}$ are cofibrant, it suffices to show that the map γ is a homotopy epimorphism in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$. For this, it suffices to prove that when considered in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$, the map α is a weak equivalence and the map β is a homotopy epimorphism. This, finally, is what we will show.

We begin with the second assertion, that the map

$$\mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W})) \xrightarrow{\beta} t_!(\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W}))$$

is a homotopy epimorphism in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$. In fact, we will show that the natural transformation $\mathrm{diag}^* \to t_!$ in $\mathrm{Fun}(ss\mathcal{S}\mathrm{et}, s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}})$ is a componentwise homotopy epimorphism. Just for the duration of this subproof, let us "reverse" our simplicial coordinates, so that the one we have been denoting by "i" will be the *outer* coordinate while the one we have been denoting by "n" will be the *inner* coordinate. Now, observe that we can rewrite these two functors as

$$\mathrm{diag}^* \cong \int^{[i]\in\mathbf{\Delta}} (-)_i \times \Delta^i \colon s(s\mathcal{S}\mathrm{et}) \to s\mathcal{S}\mathrm{et}$$

and

$$t_! \cong \int^{[i]\in\mathbf{\Delta}} (-)_i \times (\Delta^i)^{\mathrm{gpd}} \colon s(s\mathcal{S}\mathrm{et}) \to s\mathcal{S}\mathrm{et},$$

under which identifications our natural transformation $\mathrm{diag}^* \to t_!$ is induced by the evident map $\Delta^\bullet \to (\Delta^\bullet)^{\mathrm{gpd}}$ in $c(s\mathcal{S}\mathrm{et})$. Moreover, by Proposition T.A.2.9.26, we obtain a left Quillen bifunctor

$$\int^{[i]\in\mathbf{\Delta}} (-)_i \times (-)^i \colon s(s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}})_{\mathrm{Reedy}} \times c(s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}})_{\mathrm{Reedy}} \to s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$$

(since $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$ is cartesian, ie the product bifunctor is left Quillen).[12] As every object of $s(s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}})_{\mathrm{Reedy}}$ is cofibrant, for any object

$$Y_\bullet \in s(s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}})_{\mathrm{Reedy}}$$

the above left Quillen bifunctor induces a left Quillen functor

$$\int^{[i]\in\mathbf{\Delta}} Y_i \times (-)^i \colon c(s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}})_{\mathrm{Reedy}} \to s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}.$$

Moreover, the cofibrant objects of $c(s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}})_{\mathrm{Reedy}}$ are exactly those of $c(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}}$ (since the cofibrations in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$ are exactly those of $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$), and so in particular the objects $\Delta^\bullet, (\Delta^\bullet)^{\mathrm{gpd}} \in c(s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}})_{\mathrm{Reedy}}$ are cofibrant by [3, Corollary 15.9.10].

Now, epimorphisms (being determined by a colimit condition) are preserved by left adjoint functors of ∞–categories. Moreover, by [14, Theorem 2.1], a left Quillen functor between model categories induces a left adjoint functor between ∞–categories, which is presented (in $\mathrm{RelCat}_{\mathrm{BK}}$) by the restriction of the left Quillen functor to the

[12] Since we have flipped our simplicial coordinates, this model structure $s(s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}})_{\mathrm{Reedy}}$ is *different* from the model structure $s(s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}})_{\mathrm{Reedy}}$ that appeared earlier (with respect to the fixed copy of the underlying category $ss\mathcal{S}\mathrm{et}$ in which we have been working).

subcategory of cofibrant objects. So, it suffices to show that the map $\Delta^\bullet \to (\Delta^\bullet)^{\mathrm{gpd}}$ is a homotopy epimorphism in $c(s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}})_{\mathrm{Reedy}}$.

For this, observe that the model category $c(s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}})_{\mathrm{Reedy}}$ presents the ∞–category $c\mathcal{C}\mathrm{at}_\infty$. Since epimorphisms in $c\mathcal{C}\mathrm{at}_\infty = \mathrm{Fun}(\mathbf{\Delta}, \mathcal{C}\mathrm{at}_\infty)$ are determined componentwise, it suffices to show that each $\Delta^i \to (\Delta^i)^{\mathrm{gpd}}$ is a homotopy epimorphism in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$. But this is clear: this map in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$ presents the terminal map

$$[i] \to [i]^{\mathrm{gpd}} \simeq \mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty}$$

in $\mathcal{C}\mathrm{at}_\infty$, which on an arbitrary ∞–category $\mathcal{C}$ corepresents the inclusion

$$\mathcal{C}^{\simeq} \hookrightarrow \hom_{\mathcal{C}\mathrm{at}_\infty}([i], \mathcal{C})$$

of the subspace of length-i sequences of composable *equivalences* (inside of the space of arbitrary length-i sequences of composable morphisms). Thus, the natural transformation $\mathrm{diag}^* \to t_!$ in $\mathrm{Fun}(ss\mathcal{S}\mathrm{et}, s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}})$ is indeed a componentwise homotopy epimorphism, and so in particular we obtain that the map β (which is its component at the object $\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W}) \in ss\mathcal{S}\mathrm{et}$) is a homotopy epimorphism, as claimed.

So, it only remains to show that the map

$$\mathtt{R} \cong \mathrm{diag}^*(\mathrm{pr}_2^*(\mathtt{R})) \xrightarrow{\alpha} \mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W}))$$

is a weak equivalence in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$. Unwinding the definitions, we see that via the evident cosimplicial object

$$\mathbf{\Delta} \xrightarrow{(\Delta^\bullet)^\flat \times (\Delta^\bullet)^\sharp} s\mathcal{S}\mathrm{et}^+,$$

we obtain a canonical isomorphism

$$\mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W})) \cong \hom^{\mathrm{lw}}_{s\mathcal{S}\mathrm{et}^+}((\Delta^\bullet)^\flat \times (\Delta^\bullet)^\sharp, (\mathtt{R}, \mathtt{W})).$$

Moreover, via the canonical isomorphisms

$$\mathtt{R} \cong \hom^{\mathrm{lw}}_{s\mathcal{S}\mathrm{et}}(\Delta^\bullet, \mathtt{R}) \cong \hom^{\mathrm{lw}}_{s\mathcal{S}\mathrm{et}^+}((\Delta^\bullet)^\flat, \mathtt{R}^\flat) \cong \hom^{\mathrm{lw}}_{s\mathcal{S}\mathrm{et}^+}((\Delta^\bullet)^\flat, (\mathtt{R}, \mathtt{W})),$$

this map α is corepresented by the collection of first projection maps

$$(\Delta^n)^\flat \times (\Delta^n)^\sharp \to (\Delta^n)^\flat,$$

which assemble to a natural transformation in $\mathrm{Fun}(\mathbf{\Delta}, s\mathcal{S}\mathrm{et}^+)$. On the other hand, the collection of diagonal maps

$$(\Delta^n)^\flat \to (\Delta^n)^\flat \times (\Delta^n)^\sharp$$

(or, more precisely, the unique maps in $s\mathcal{S}\mathrm{et}^+$ which recover the diagonal maps in $s\mathcal{S}\mathrm{et}$ under the forgetful functor $s\mathcal{S}\mathrm{et}^+ \to s\mathcal{S}\mathrm{et}$) also assemble into a natural transformation in $\mathrm{Fun}(\mathbf{\Delta}, s\mathcal{S}\mathrm{et}^+)$, which likewise corepresents a map

$$\mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W})) \xrightarrow{\rho} \mathtt{R}$$

in $s\mathcal{S}\mathrm{et}$. Clearly, the composite

$$\mathtt{R} \xrightarrow{\alpha} \mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W})) \xrightarrow{\rho} \mathtt{R}$$

is the identity map, since this is true of the composite

$$(\Delta^n)^\flat \to (\Delta^n)^\flat \times (\Delta^n)^\sharp \to (\Delta^n)^\flat$$

of the diagonal map followed by the first projection. On the other hand, we will show that the composite

$$\mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W})) \xrightarrow{\rho} \mathtt{R} \xrightarrow{\alpha} \mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W}))$$

is connected to $\mathrm{id}_{\mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R},\mathtt{W}))}$ by the zigzag of simplicial homotopies illustrated in Figure 1, whose components (ie whose values on the vertices of (the source copies of) $\mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W}))$) are all degenerate edges of (the target copy of) $\mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W}))$. Postcomposing with an arbitrary fibrant replacement

$$\mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W})) \xrightarrow{\approx} \mathbb{R}\big(\mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W}))\big) \twoheadrightarrow \mathrm{pt}_{s\mathcal{S}\mathrm{et}}$$

in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$, we obtain a composite

$$\begin{aligned}\Lambda^2_2 \to \underline{\mathrm{hom}}_{s\mathcal{S}\mathrm{et}}\big(\mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W})), \mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W}))\big) \\ \to \underline{\mathrm{hom}}_{s\mathcal{S}\mathrm{et}}\big(\mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W})), \mathbb{R}\big(\mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W}))\big)\big)\end{aligned}$$

in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$, which, by [4, Chapter 5, Theorem C] — and [4, Proposition 4.8] (and the fact that $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$ is cartesian) — presents a zigzag of natural equivalences in $\mathcal{C}\mathrm{at}_\infty$ between the functors presented by the maps $\mathrm{id}_{\mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R},\mathtt{W}))}$ and $\alpha\rho$ in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$. In turn, this zigzag (along with the natural equivalence in $\mathcal{C}\mathrm{at}_\infty$ presented by the identification $\rho\alpha = \mathrm{id}_{\mathtt{R}}$) witnesses the fact that the maps α and ρ in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$ present inverse equivalences in $\mathcal{C}\mathrm{at}_\infty$, from which we conclude that in particular the map α is indeed a weak equivalence in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$.

Now, all three of η, H_1, and H_2 will be corepresented by maps between the various objects $(\Delta^n)^\flat \times (\Delta^n)^\sharp \in s\mathcal{S}\mathrm{et}^+$; in turn, all of these maps will be obtained by applying

$$\begin{tikzcd}
\mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R},\mathtt{W})) \arrow[d, "\Delta^{\{0\}}\times\mathrm{id}"'] \arrow[ddrr, "\mathrm{id}_{\mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R},\mathtt{W}))}"] & & \\
\Delta^1\times\mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R},\mathtt{W})) \arrow[drr, dashed, "H_1"] & & \\
\mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R},\mathtt{W})) \arrow[u, "\Delta^{\{1\}}\times\mathrm{id}"] \arrow[d, "\Delta^{\{1\}}\times\mathrm{id}"'] \arrow[rr, dashed, "\eta"] & & \mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R},\mathtt{W})) \\
\Delta^1\times\mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R},\mathtt{W})) \arrow[urr, dashed, "H_2"] & & \\
\mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R},\mathtt{W})) \arrow[u, "\Delta^{\{0\}}\times\mathrm{id}"] \arrow[uurr, "\alpha\rho"'] & &
\end{tikzcd}$$

Figure 1: The zigzag of simplicial homotopies in $s\mathcal{S}\mathrm{et}$ in the proof of Lemma 4.3.

the evident "marked nerve" functor $\mathrm{N}^+\colon \mathtt{RelCat}\to s\mathcal{S}\mathrm{et}^+$ to maps between the various objects $[n]\times[n]_{\boldsymbol{W}}\in\mathtt{RelCat}$.

We begin by defining the map $\mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R},\mathtt{W}))\xrightarrow{\eta}\mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R},\mathtt{W}))$: this is corepresented by the marked nerves of the maps

$$[n]\times[n]_{\boldsymbol{W}}\xrightarrow{\eta^n}[n]\times[n]_{\boldsymbol{W}}$$

in $\mathtt{RelCat}$ given by

$$\eta^n(i,j)=\begin{cases}(i,i) & \text{if } i\geq j,\\ (i,j) & \text{if } i<j.\end{cases}$$

It is easy to verify that this does indeed define a map in $\mathtt{RelCat}$, and moreover that assembling these maps for all $n\geq 0$ yields an endomorphism of the object $[\bullet]\times[\bullet]_{\boldsymbol{W}}\in c\mathtt{RelCat}$.

In order to define the simplicial homotopies H_1 and H_2, we first recall a combinatorial reformation of the definition of a simplicial homotopy (see eg [10, Definitions 5.1]):

for any $Y, Z \in s\mathcal{S}\mathrm{et}$ and any $f, g \in \hom_{s\mathcal{S}\mathrm{et}}(Y, Z)$, a simplicial homotopy

$$\begin{tikzcd}
Y \arrow[d, "\Delta^{\{0\}}\times\mathrm{id}"'] \arrow[dr, "g"] & \\
\Delta^1 \times Y \arrow[r, dashed, "h"] & Z \\
Y \arrow[u, "\Delta^{\{1\}}\times\mathrm{id}"] \arrow[ur, "f"'] &
\end{tikzcd}$$

is equivalently given by a family of maps

$$\{h_{i,n} \in \hom_{\mathcal{S}\mathrm{et}}(Y_n, Z_{n+1})\}_{0\leq i\leq n\geq 0}$$

which satisfy the identities

$$\delta_0 h_{0,n} = f_n, \qquad \delta_{n+1} h_{n,n} = g_n, \qquad \delta_i h_{j,n} = \begin{cases} h_{j-1,n-1}\delta_i & \text{if } i < j, \\ \delta_i h_{i-1,n} & \text{if } i = j \neq 0, \\ h_{j,n-1}\delta_{i-1} & \text{if } i > j+1, \end{cases}$$

and

$$\sigma_i h_{j,n} = \begin{cases} h_{j+1,n+1}\sigma_i & \text{if } i \leq j, \\ h_{j,n+1}\sigma_{i-1} & \text{if } i > j. \end{cases}$$

So, for $\varepsilon \in \{1, 2\}$, we will define the simplicial homotopies

$$\Delta^1 \times \mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W})) \xrightarrow{H_\varepsilon} \mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R}, \mathtt{W}))$$

to be corepresented by the marked nerves of families of maps

$$\{H_\varepsilon^{i,n} \in \hom_{\mathrm{RelCat}}([n+1]\times[n+1]_{\boldsymbol{W}}, [n]\times[n]_{\boldsymbol{W}})\}_{0\leq i\leq n\geq 0}$$

satisfying the opposites of the identities given above (with the first two "boundary condition" identities being dictated by their respective sources and targets). Namely, we define

$$H_1^{i,n}(j,k) = \begin{cases} (j,k) & \text{if } 0 \leq j, k \leq i, \\ (j-1, j-1) & \text{if } j > i \text{ and } j \geq k, \\ (j, k-1) & \text{if } k > i \geq j, \\ (j-1, k-1) & \text{if } k > j > i, \end{cases}$$

and

$$H_2^{i,n}(j,k) = \begin{cases} (j,j) & \text{if } j \le i, \\ (j-1,j-1) & \text{if } j > i \text{ and } j \ge k, \\ (j-1,k-1) & \text{if } k > j > i. \end{cases}$$

It is a straightforward (but lengthy) process to verify that

- these satisfy the opposites of the identities given above,
- they restrict along their boundaries to the various maps

  $$\mathrm{id}_{\mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R},\mathtt{W}))}, \eta, \alpha\rho \in \hom_{s\mathcal{S}\mathrm{et}}\big(\mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R},\mathtt{W})), \mathrm{diag}^*(\mathrm{N}^{\mathrm{R}}(\mathtt{R},\mathtt{W}))\big)$$

  as indicated in Figure 1, and
- their values on vertices are all degenerate edges,

as claimed. This completes the proof. □

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