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MULTIPLICATIVE INFINITE LOOP SPACE THEORY

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In [22], Thomason and I gave a synthesis which combined the different existing infinite loop space theories into a single coherent whole. In particular, we proved that, up to equivalence, there is only one sensible way to pass from space level input data to spectrum level output.

In [21], I elaborated the additive theory by showing how to incorporate into it a theory of pairings. This explained how to pass from space level pairing data to pairings of spectra. The input data there was as general as would be likely to find use, and the output, while deduced using one particular infinite loop space machine, automatically applied to all machines by virtue of the uniqueness theorem.

We shall here obtain a comparably complete multiplicative infinite loop space theory. The idea is to start with input data consisting of ring spaces up to all possible higher coherence homotopies and to obtain output consisting of ring spectra with enriched internal structure. Applications of such internal structure abound, both in infinite loop space theory and its applications to geometric topology [5, 14, 15] and in stable homotopy theory [3, 16].

We shall explain the notion of a 'category of ring operators' $\mathscr{J}$ in Section 1 and the notion of a $\mathscr{J}$-space in Section 2. We shall see that this notion includes as special cases both the $(\mathscr{C}, \mathscr{G})$-spaces that were the input of the E_∞ ring theory in [14] and the $\mathscr{F}\wr\mathscr{F}$-spaces that provide the simplest input for a Segal style development of multiplicative infinite loop space theory. On a technical note, we shall define $\mathscr{J}$-spaces with a cofibration condition, but we shall see in Appendix C that the cofibration condition results in little loss of generality. We shall also see that if $\mathscr{J}$ and $\mathscr{K}$ are categories of ring operators which are equivalent in a suitable sense, then the categories of $\mathscr{J}$-spaces and of $\mathscr{K}$-spaces are equivalent. All of this is precisely parallel to the additive theory in [22].

The main applications start with categories with products $\oplus$ and $\otimes$ which satisfy the axioms for a commutative ring up to coherent natural isomorphism. Making as many diagrams as possible commute strictly, one arrives at the notion of a bipermutative category. We shall show in Section 3 that bipermutative categories functorially give rise to $\mathscr{F}\int\mathscr{F}$-spaces, this being a direct application of standard categorical constructions. The use of $\mathscr{F}\int\mathscr{F}$-spaces here substitutes for an incorrect passage from bipermutative categories to E_∞ ring spaces in [14], and the present theory originated with the need to fill this gap. We also take the opportunity to extend and complete various remarks in [14] about such things as units and analogs of monomial matrices in general bipermutative categories.

Thus the input data is well understood. However, in contrast to the additive theory, it is not immediately apparent what the appropriate output should be. In [14], Quinn, Ray, and I introduced E_∞ ring spectra and, more generally, $\mathscr{G}$-ring spectra for appropriate operads $\mathscr{G}$. Here $\mathscr{G}$ might model n-fold loop spaces rather than infinite loop spaces, and need for this extra generality arises naturally in the study of Thom spectra [3, 10] and the splitting of n-fold loop spaces [4]. The notion of a $\mathscr{G}$-ring spectrum (which will be recalled in Section 8) is not very complicated and feeds effortlessly into the study of classifying spaces [14], the calculation of homology operations [5], and applications in stable homotopy theory [3, 16]. I am *convinced that this is definitively the right notion of enriched internal structure on* ring spectra.

Thus we seek to pass from $\mathscr{J}$-spaces to $\mathscr{G}$-spectra. Here $\mathscr{J}$ is taken to be the 'wreath product' $\hat{\mathscr{G}}\int\hat{\mathscr{C}}$, where $(\mathscr{C}, \mathscr{G})$ is a suitable 'operad pair' and $\hat{\mathscr{C}}$ and $\hat{\mathscr{G}}$ are the 'categories of operators' associated to $\mathscr{C}$ and $\mathscr{G}$. The category of ring operators $\mathscr{J}$ maps to $\mathscr{F}\int\mathscr{F}$, hence $\mathscr{F}\int\mathscr{F}$-spaces are $\mathscr{J}$-spaces by pullback, and $(\mathscr{C}, \mathscr{G})$-spaces are those $\mathscr{J}$-spaces defined in terms of strict Cartesian powers of a given space rather than the Cartesian powers up to homotopy allowed in the general definition.

It is at this point that problems and subtleties arise. The action of $\mathscr{G}$ on $\mathscr{G}$-spectra is defined in terms of strict smash products. It is therefore too much to expect that the spectra associated to general $\mathscr{J}$-spaces are $\mathscr{G}$-spectra. One possible solution would be to define a weaker notion than that of a $\mathscr{G}$-spectrum, using smash products up to homotopy, and so formulate a new target category for the output. However, in any such approach, the resulting objects would inevitably be far more complicated than $\mathscr{G}$-spectra, and their use would entail elaborate reworkings of the theories of [3, 5, and 14] to arrive at the desired applications.

It was proven in [14] that the spectra associated to $(\mathscr{C}, \mathscr{G})$-spaces are $\mathscr{G}$-spectra. An alternative approach therefore suggests itself. One might first try to replace general $\mathscr{J}$-spaces by equivalent $(\mathscr{C}, \mathscr{G})$-spaces and then pass to $\mathscr{G}$-spectra. In fact, this can be done, but for intrinsically interesting reasons having to do with the nature of free $\mathscr{J}$-spaces, there seems to be no direct, one step, construction of such a replacement functor.

What turns out to be the case is that there is an intermediate notion of a $(\hat{\mathscr{C}}, \mathscr{G})$-space, for which addition is defined using Cartesian products up to homotopy but multiplication is then superimposed using strict Cartesian products. There is a functorial way to replace general $\mathscr{J}$-spaces X by equivalent $(\hat{\mathscr{C}}, \mathscr{G})$-spaces VX, and the spectra EY associated to $(\hat{\mathscr{C}}, \mathscr{G})$-spaces Y are $\mathscr{G}$-spectra. This allows us to pass from $\mathscr{J}$-spaces to $\mathscr{G}$-spectra and so recover all the applications. Moreover, the zeroth spaces of $\mathscr{G}$-spectra are $(\mathscr{C}, \mathscr{G})$-spaces, there is a functorial way to replace $(\mathscr{C}, \mathscr{G})$-spaces Y by equivalent $(\mathscr{C}, \mathscr{G})$-spaces $V_{\oplus}Y$, and there is a natural $(\mathscr{C}, \mathscr{G})$-map $V_{\oplus}Y \to E_0 Y$ which is a group completion of the underlying additive structure. In sum, $V_{\oplus}VX$ is a $(\mathscr{C}, \mathscr{G})$-space equivalent to the original $\mathscr{J}$-space X, and the group completion $V_{\oplus}VX \to E_0 VX$ preserves the $(\mathscr{C}, \mathscr{G})$-structure.

Effectively then, this allows us to arrive at the simplest possible output at the price of considerable complexity inside the black box which converts input to output. The complexity and shape of the theory were imperiously dictated by the surprising behavior of the input data when reformulated in terms of actions by monads.

We state our results on the passage from input to output in Section 4. Their proofs require certain categorical preliminaries presented in Section 5. These concern the relationship between monads defined on adjoint pairs of ground categories and the construction of monads associated to topological categories which contain discrete subcategories with the same objects. The heart of the theory is in Sections 6 and 7. These give a conceptual and combinatorial analysis of the monads associated to categories of ring operators and their relationship to the underlying purely additive and purely multiplicative monads. The phenomena produced by the interplay between the conceptual categorical framework and the concrete combinatorial structure of free $\mathscr{J}$-spaces strike me as altogether fascinating. It is the nature of these phenomena which dictates the shape of the theory. The passage to $\mathscr{G}$-spectra is completed in Section 8, and several results identifying the output $\mathscr{G}$-spectra associated to certain generic types of input are also presented there.

One possibly unsatisfactory feature of this theory is that, in contrast to the work of [21 and 22], which was based on the premise that all machines are equivalent, these results are bound up with one particular choice of machinery. We return to this point in Appendix D, where we compare our work with that of Segal [24, §5] and Woolfson [31, 32].

I am very grateful to Steinberger for finding the mistakes in [14] and a related mistake in [19]. Corrigenda are given in Appendices A and B, but the basic conclusion is that all applications are correct as originally stated. A sequel with Fiedorowicz [6] will justify the homological applications of [5]. The key idea of focussing on replacement functors is part of his contribution to that paper, and he also noticed that my first draft of this paper actually proved significantly more than it claimed. I am also grateful to Thomason for a number of illuminating conversations.

Contents

1. Categories of ring operators and operad pairs

We begin by constructing categories which will parametrize the ring operations on ring spaces up to higher coherence homotopies. Here categories will be small and topological, with the identity function Ob $\mathscr{C} \to$ Mor $\mathscr{C}$ a cofibration, and Cat will denote the category of small topological categories and continuous functors.

Let N denote the discrete space of based sets $\mathbf{n} = \{0, 1, \dots, n\}$ with basepoint zero. Let $\mathscr{F}$ denote the category with objects $\mathbf{n}$ and morphisms all based functions $\phi : \mathbf{m} \to \mathbf{n}$. Let $\Pi \subset \mathscr{F}$ be the subcategory consisting of those morphisms ϕ such that $\phi^{-1}(j)$ has at most one element for $1 \le j \le n$. Recall from [22, 1.1] that a category of operators is a category $\hat{\mathscr{G}}$ with object space N such that $\hat{\mathscr{G}}$ contains Π and maps to $\mathscr{F}$ via a functor $\varepsilon : \hat{\mathscr{G}} \to \mathscr{F}$ which restricts to the inclusion on Π. Observe that Π and $\mathscr{F}$ are each permutative categories under both wedge sums and smash products of finite based sets. To be precise, $\mathbf{m} \vee \mathbf{n}$ is to be identified with $\mathbf{m} + \mathbf{n}$ in blocks and $\mathbf{m} \wedge \mathbf{n}$ is to be identified with $\mathbf{mn}$ via lexicographic ordering of pairs. By convention, the empty wedge sum is $\mathbf{0}$ and the empty smash product is $\mathbf{1}$. We need notations for certain canonical permutations.

Notations 1.1. Let $\phi : \mathbf{m} \to \mathbf{n}$ and $\psi : \mathbf{n} \to \mathbf{p}$ be morphisms of $\mathscr{F}$. Given nonnegative integers r_i for $1 \le i \le m$, let $s_k = \times_{(\psi\phi)(i) = k} r_i$ and let $\sigma_k(\psi, \phi)$ denote the permutation

$$\mathbf{s}_k = \bigwedge_{\psi\phi(i) = k} \mathbf{r}_i \to \bigwedge_{\psi(j) = k} \; \bigwedge_{\phi(i) = j} \mathbf{r}_i = \mathbf{s}_k.$$

Here the left and right equalities are lexicographic identifications and the arrow permutes the factors $\mathbf{r}_i$ from their order on the left (i increasing) to their order on the right (j increasing and, for fixed j, i increasing). By convention, $\mathbf{s}_k = \mathbf{1}$ and $\sigma_k(\psi, \phi) : \mathbf{1} \to \mathbf{1}$ is the identity if there are no i such that $(\psi\phi)(i) = k$. Let

$$\sigma(\psi, \phi) : (\mathbf{s}_1, \dots, \mathbf{s}_p) \to (\mathbf{s}_1, \dots, \mathbf{s}_p)$$

be the morphism in Π^p with kth coordinate $\sigma_k(\psi, \phi)$. For morphisms $g : \mathbf{m} \to \mathbf{n}$ and $h : \mathbf{n} \to \mathbf{p}$ in a category of operators $\hat{\mathscr{G}}$, define

$$\sigma(h, g) = \sigma(\varepsilon(h), \varepsilon(g)).$$

The following definition specifies the appropriate relationship between an 'additive' category of operators $\mathscr{C}$ and a 'multiplicative' category of operators $\mathscr{G}$ for them to together determine a category of ring operators. Let $\mathscr{C}^0$ be the trivial category with unique object $*$.

Definition 1.2. Let $\mathscr{C}$ and $\mathscr{G}$ be categories of operators. An action λ of $\mathscr{G}$ on $\mathscr{C}$ consists of functors $\lambda(g):\mathscr{C}^m\to\mathscr{C}^n$ for each morphism $g:\mathbf{m}\to\mathbf{n}$ such that the following properties hold, where $\varepsilon(g)=\phi$:

(i) On objects, $\lambda(g)$ is specified by

$$\lambda(g)(\mathbf{r}_1,\dots,\mathbf{r}_m)=(\mathrm{s}_1,\dots,\mathrm{s}_n),\quad \text{where } \mathrm{s}_j=\bigwedge_{\phi(i)=j}\mathbf{r}_i.$$

(ii) On morphisms $(\chi_1,\dots,\chi_m)$ of $\Pi^m\subset\mathscr{C}^m$, $\lambda(g)$ is specified by

$$\lambda(g)(\chi_1,\dots,\chi_m)=(\omega_1,\dots,\omega_n),\quad \text{where } \psi_j=\bigwedge_{\phi(i)=j}\chi_i.$$

(iii) On general morphisms $(c_1,\dots,c_m)$ of $\mathscr{C}^m$, $\lambda(g)$ satisfies

$$\varepsilon(\lambda(g)(c_1,\dots,c_m))=(\zeta_1,\dots,\zeta_n),\quad \text{where } \zeta_j=\bigwedge_{\phi(i)=j}\varepsilon(c_i).$$

(iv) For morphisms $\phi:\mathbf{m}\to\mathbf{n}$ of $\Pi\subset\mathscr{G}$, $\lambda(\phi)$ is specified by

$$\lambda(\phi)(c_1,\dots,c_m)=(c_{\phi^{-1}(1)},\dots,c_{\phi^{-1}(n)}).$$

(v) For $g:\mathbf{m}\to\mathbf{n}$ and $h:\mathbf{n}\to\mathbf{p}$, the morphisms $\sigma(h,g)$ in $\Pi^p\subset\mathscr{C}^p$ specify a natural isomorphism of functors $\lambda(hg)\to\lambda(h)\lambda(g)$.

In (i), the jth coordinate is $\mathbf{1}$ if $\phi^{-1}(j)$ is empty (as holds for all j when $m=0$); in (ii)–(iv), the jth coordinate is the identity morphism of $\mathbf{1}$ if $\phi^{-1}(j)$ is empty.

The substantive point is to specify $\lambda(g)$ on morphisms of $\mathscr{C}^m$ not in Π^m, where g is a morphism of $\mathscr{G}$ not in Π, and to do so in such a way that $\lambda(g)$ is a functor and the naturality condition of (v) is satisfied. It is easy to see that the jth coordinate of $\lambda(g)(c_1,\dots,c_m)$ depends only on those c_i with $\phi(i)=j$ and may thus be written $\lambda_j(g)(\times_{\phi(i)=j}c_i)$; as g varies, these should be thought of as parametrized smash products of the c_i. The following implicit examples are central to the theory.

Examples 1.3. (i) The definition itself specifies an action of Π on any category of operators $\mathscr{C}$.

(ii) The definition itself specifies an action of any category of operators $\mathscr{G}$ on Π.

(ii) Most importantly, via formula (iii), the definition itself specifies an action of any category of operators $\mathscr{G}$ on $\mathscr{F}$.

Of course, the definition is not plucked out of the air. With $\lambda(\mathbf{n})=\mathscr{C}^n$, the $\lambda(g)$ and $\sigma(h,g)$ specify a lax functor (or pseudofunctor) $\lambda:\mathscr{G}\to\mathrm{Cat}$ [28; 29; 21, 3.1]. Street [28] has given two general procedures for replacing lax functors by suitably equivalent actual functors. However, the introduction of any such rectification would serve only to introduce wholly unnecessary and irrelevant complications into the theory. The following special case of a standard categorical construction provides the most convenient domain categories for multiplicative infinite loop space theory. We go into detail in order to fix notations.

Definition 1.4. Let λ be an action of $\mathscr{G}$ on $\mathscr{C}$. Definte the wreath product $\mathscr{G}\int\mathscr{C}$ to be the following category. Its space of objects is $\coprod_{n\geq 0} N^n$, with elements denoted $(n; S)$ where $S=(\mathbf{s}_1, \dots, \mathbf{s}_n)$. Its space of morphisms $(m; R)\to(n; S)$ is

$$\coprod_{\phi\in\mathscr{F}(m,n)} \varepsilon^{-1}(\phi)\times \prod_{1\leq j\leq n} \mathscr{C}\Big(\bigwedge_{\phi(i)=j} \mathbf{r}_i, \mathbf{s}_j\Big), \quad \varepsilon: \mathscr{G}\to\mathscr{F},$$

with typical elements denoted $(g; c)$ where $c=(c_1, \dots, c_n)$. For $(h; d):(n; S)\to(p; T)$, composition is specified by

$$(h; d)(g; c) = (hg; e), \quad \text{where } e = d\circ\lambda(h)(c)\circ\sigma(h,g).$$

More explicitly, with $\varepsilon(g)=\phi$ and $\varepsilon(h)=\psi$, the kth coordinate of e, $1\leq k\leq p$, is the composite

$$\bigwedge_{(\psi\phi)(i)=k} \mathbf{r}_i \xrightarrow{\sigma_k(\psi,\phi)} \bigwedge_{\psi(j)=k}\ \bigwedge_{\phi(i)=j} \mathbf{r}_i \xrightarrow{\lambda_k(h)(\times_{\psi(j)=k} c_j)} \bigwedge_{\psi(j)=k} \mathbf{s}_j \xrightarrow{d_k} \mathbf{t}_k.$$

N^0 has the single object $(0; *)$. The morphisms with target $(0; *)$ are written $(g; *)$ for $g:\mathbf{m}\to\mathbf{0}$. With $c_j:\mathbf{1}\to\mathbf{s}_j$, the general notation applies to morphisms with source $(0; *)$.

It is useful to think of a morphism $(\phi; \chi):(m; R)\to(n; S)$ in $\Pi\int\Pi$ as a map

$$\chi_1\times\cdots\times\chi_n : \mathbf{r}_{\phi^{-1}(1)}\times\cdots\times\mathbf{r}_{\phi^{-1}(n)}\to\mathbf{s}_1\times\cdots\times\mathbf{s}_n,$$

where $r_{\phi^{-1}(j)}=1$ if $\phi^{-1}(j)$ is empty. This makes sense since $\phi^{-1}(j)$ has at most one element.

The following observation will play a crucial role in our theory.

Lemma 1.5. *There are inclusions of categories*

$$\mathscr{C}\subset\Pi\int\mathscr{C}\subset\mathscr{G}\int\mathscr{C}\supset\mathscr{G}\int\Pi\supset\mathscr{G}$$

Proof. The middle two inclusions are obvious. The first is given by $\mathbf{n}\to(1; \mathbf{n})$ on objects and $c\to(1; c)$ on morphisms. The last is given by $\mathbf{n}\to(n; \mathbf{1}^n)$ on objects and $g\to(g; \mathbf{1}^n)$ on morphisms $g:\mathbf{m}\to\mathbf{n}$.

For use in Section 6, we record the following commutation formula for the morphisms of $\mathscr{G}\int\mathscr{C}$.

Lemma 1.6. *Let* $\mathscr{G}$ *act on* $\mathscr{C}$ *and let*

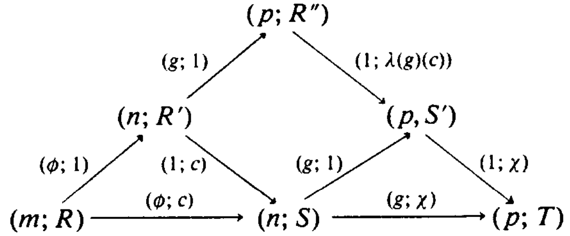
$$(g; \chi):(n; S)\to(p; T) \quad \textit{and} \quad (\phi; c):(m; R)\to(n; S)$$

be morphisms in $\hat{\mathscr{G}}\wr\Pi$ *and* $\Pi\wr\hat{\mathscr{C}}$, *respectively. Let* $\varepsilon(g)=\psi$. *Then the following formula holds in* $\hat{\mathscr{G}}\wr\hat{\mathscr{C}}$:

$$(g;\chi)(\phi;c)=(1;\chi\circ\lambda(g)(c))\circ(g\phi;\sigma(\psi,\phi)).$$

Proof. Inspection of the definitions gives the following commutative diagram, where, for $1\le j\le n$ and $1\le k\le p$,

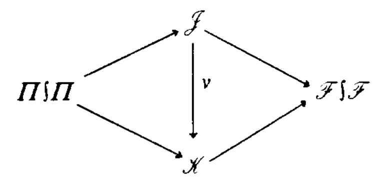
$$\mathbf{r}'_j=\mathbf{r}_{\phi^{-1}(j)},\qquad \mathbf{r}''_k=\bigwedge_{\psi(j)=k}\mathbf{r}_{\phi^{-1}(j)},\qquad \mathbf{s}'_k=\bigwedge_{\psi(j)=k}\mathbf{s}_j.$$



The conclusion follows upon composing the legs of the triangle.

While these categories $\hat{\mathscr{G}}\wr\hat{\mathscr{C}}$ are the ones of real interest, it clarifies ideas and simplifies notations to introduce the following more general concept.

Definition 1.7. A category of ring operators is a category $\mathscr{J}$ with object space $\coprod_{n\ge 0}N^n$ such that $\mathscr{J}$ contain $\Pi\wr\Pi$ and is augmented over $\mathscr{F}\wr\mathscr{F}$ by a functor $\varepsilon:\mathscr{J}\to\mathscr{F}\wr\mathscr{F}$ which restricts to the inclusion on $\Pi\wr\Pi$. A map of categories of ring operators is a functor $\nu:\mathscr{J}\to\mathscr{K}$ such that ν is the identity function on objects and the following diagram commutes:



The map ν is said to be an equivalence if each of its induced maps of morphism spaces is an equivalence.

As in [22], by an equivalence of spaces we understand a weak homotopy equivalence.

We shall add a minor technical condition to the definition in Addendum 2.8 below.

While the definitions above suffice for the work in the next two sections, our later arguments depend on the association of categories of ring operators to operad pairs that is the subject of the rest of this section.

Recall from [12, 1.1] that an operad $\mathscr{C}$ is a sequence of spaces $\mathscr{C}(j)$ such that $\mathscr{C}(0)=\{*\}$, there is a unit element $1\in\mathscr{C}(1)$, there is a right action of the symmetric group Σ_j on $\mathscr{C}(j)$, and there are suitably associative, unital, and equivariant structural maps

$$\gamma:\mathscr{C}(k)\times\mathscr{C}(j_1)\times\cdots\times\mathscr{C}(j_k)\to\mathscr{C}(j_1+\cdots+j_k).$$

As explained in [22, §4], an operad $\mathscr{C}$ determines a category of operators $\hat{\mathscr{C}}$ with

$$\hat{\mathscr{C}}(\mathbf{m},\mathbf{n})=\coprod_{\phi\in\mathscr{F}(\mathbf{m},\mathbf{n})}\prod_{1\le j\le n}\mathscr{C}(|\phi^{-1}(j)|).$$

The composite of $(\phi;c_1,\ldots,c_n)\in\hat{\mathscr{C}}(\mathbf{m},\mathbf{n})$ and $(\psi;d_1,\ldots,d_p)\in\hat{\mathscr{C}}(\mathbf{n},\mathbf{p})$ is

$$\left(\psi\phi;\ \mathop{\times}_{k=1}^{p}\gamma\left(d_k;\ \mathop{\times}_{\psi(j)=k}c_j\right)\sigma_k(\psi,\phi)\right).$$

Note that $\mathbf{0}$ is both an initial and a terminal object of $\hat{\mathscr{C}}$. The notion of an action of one operad on another is specified in [14, VI.1.6]. Since the cited definition contains two misprints and admits some notational simplification, we rephrase it here.

Definition 1.8. An action λ of an operad $\mathscr{G}$ on an operad $\mathscr{C}$ consists of maps

$$\lambda:\mathscr{G}(k)\times\mathscr{C}(j_1)\times\cdots\times\mathscr{C}(j_k)\to\mathscr{C}(j_1\cdots j_k)$$

for $k\ge 0$ and $j_r\ge 0$, interpreted as $\lambda(*)=1\in\mathscr{C}(1)$ when $k=0$, which satisfy the following properties, where

$$g\in\mathscr{G}(k),\quad g_r\in\mathscr{G}(j_r)\qquad\text{for } 1\le r\le k,$$

$$c\in\mathscr{C}(j),\quad c_r\in\mathscr{C}(j_r)\qquad\text{for } 1\le r\le k,$$

$$c_{r,q}\in\mathscr{C}(i_{r,q})\qquad\text{for } 1\le q\le j_r \text{ and } 1\le r\le k.$$

(a) $$\lambda\left(\gamma\left(g;\ \mathop{\times}_{r=1}^{k}g_r\right);\ \mathop{\times}_{r=1}^{k}\mathop{\times}_{q=1}^{j_r}c_{r,q}\right)=\lambda\left(g;\ \mathop{\times}_{r=1}^{k}\lambda\left(g_r;\ \mathop{\times}_{q=1}^{j_r}c_{r,q}\right)\right).$$

(a′) $$\gamma\left(\lambda\left(g;\ \mathop{\times}_{r=1}^{k}c_r\right);\ \mathop{\times}_{Q}\lambda\left(g;\ \mathop{\times}_{r=1}^{k}c_{r,q_r}\right)\right)\nu=\lambda\left(g;\ \mathop{\times}_{r=1}^{k}\gamma\left(c_r;\ \mathop{\times}_{q=1}^{j_r}c_{r,q}\right)\right),$$

where Q runs through the lexicographically ordered set of sequences $(q_1,\ldots,q_k)$ with $1\le q_r\le j_r$ and ν is that permutation of the set of

$$\sum_{Q}\left(\prod_{r=1}^{k}i_{r,q_r}\right)=\prod_{r=1}^{k}\sum_{q=1}^{j_r}i_{r,q_r}$$

letters which corresponds under block sum and lexicographic identifications on the left and right to the natural distributivity isomorphism

$$\bigvee_{Q}\bigwedge_{r=1}^{k}\mathbf{i}_{r,q_r}\to\bigwedge_{r=1}^{k}\bigvee_{q=1}^{j_r}\mathbf{i}_{r,q}.$$

(b) $\lambda(1; c) = c$, where $1 \in \mathscr{G}(1)$ is the unit of $\mathscr{G}$.

(b′) $\lambda(g; 1^k) = 1$, where $1 \in \mathscr{C}(1)$ is the unit of $\mathscr{C}$.

(c) $$\lambda\left(g\sigma; \mathop{\times}_{r=1}^{k} c_r\right) = \lambda\left(g; \mathop{\times}_{r=1}^{k} c_{\sigma^{-1}(r)}\right)\sigma\langle j_1, \dots j_k\rangle,$$

where $\sigma\langle j_1, \dots j_k\rangle$ is that permutation which corresponds under lexicographic identifications to the permutation of smash product factors

$$\sigma : \mathbf{j}_1 \wedge \cdots \wedge \mathbf{j}_k \to \mathbf{j}_{\sigma^{-1}(1)} \wedge \cdots \wedge \mathbf{j}_{\sigma^{-1}(k)}.$$

(c′) $$\lambda\left(g; \mathop{\times}_{r=1}^{k} c_r\tau_r\right) = \lambda\left(g; \mathop{\times}_{r=1}^{k} c_r\right)(\tau_1 \otimes \cdots \otimes \tau_k),$$

where $\tau_1 \otimes \cdots \otimes \tau_k$ is that permutation which corresponds under lexicographic identifications to the smash product

$$\tau_1 \wedge \cdots \wedge \tau_k : \mathbf{j}_1 \wedge \cdots \wedge \mathbf{j}_k \to \mathbf{j}_1 \wedge \cdots \wedge \mathbf{j}_k.$$

The categorical rightness of this combinatorial definition is given by the following result. Let $\phi_j : \mathbf{j} \to \mathbf{1}$ be the morphism in $\mathscr{F}$ which sends i to 1 for $1 \le i \le j$.

Proposition 1.9. *An action λ of an operad $\mathscr{G}$ on an operad $\mathscr{C}$ determines and is determined by an action λ of $\hat{\mathscr{G}}$ on $\hat{\mathscr{C}}$.*

Proof. Given an action λ of $\mathscr{G}$ on $\mathscr{C}$ and given morphisms

$$\left(\phi; \mathop{\times}_{j=1}^{n} g_j\right) : \mathbf{m} \to \mathbf{n} \quad \text{in } \hat{\mathscr{G}}$$

and

$$\left(\chi_i; \mathop{\times}_{u=1}^{r_i} c_{i,u}\right) : \mathbf{r}_i' \to \mathbf{r}_i \quad \text{in } \hat{\mathscr{C}}$$

for $1 \le i \le m$, specify the jth coordinate of the required categorical action by

$$\lambda_j\left(\phi; \mathop{\times}_{j=1}^{n} g_j\right)\left(\mathop{\times}_{\phi(i)=j}\left(\chi_i; \mathop{\times}_{u=1}^{r_i} c_{i,u}\right)\right) = \left(\mathop{\wedge}_{\phi(i)=j} \chi_i; \mathop{\times}_{U} \lambda\left(g_j; \mathop{\times}_{\phi(i)=j} c_{i,u_i}\right)\right),$$

where U runs over the lexicographically ordered set of sequences with ith term u_i satisfying $i \le u_i \le r_i$ for $1 \le i \le |\phi^{-1}(j)|$. Here

$$g_j \in \mathscr{G}(|\phi^{-1}(j)|), \qquad c_{i,u} \in \mathscr{C}(|\chi_i^{-1}(u)|),$$

and the formula makes sense because

$$\mathop{\times}_{\phi(i)=j} |\chi_i^{-1}(u_i)| = \left|\left(\mathop{\wedge}_{\phi(i)=j} \chi_i\right)^{-1}(U)\right|.$$

Each $\lambda(\phi; \times_{j=1}^{n} g_j)$ is a functor by a routine calculation from (a′)–(c′) of the preceding definition. The naturality of Definition 1.2(v) is verified by a similar, but longer, calculation from (a)–(c). Conversely, given an action λ of $\hat{\mathscr{G}}$ on $\hat{\mathscr{C}}$, the formula

$$\lambda(\phi_k; g)\left(\mathop{\times}_{r=1}^{k} (\phi_{j_r}; c_r)\right) = \left(\bigwedge_{r=1}^{k} \phi_{j_r}; \lambda\left(g; \mathop{\times}_{r=1}^{k} c_r\right)\right)$$

defines $\lambda(g; \times_{r=1}^k c_r)$ for $g \in \mathscr{G}(k)$ and $c_r \in \mathscr{C}(j_r)$. This makes sense since $\bigwedge_{r=1}^k \phi_{j_r} = \phi_{j_1 \cdots j_r}$, and the requisite verifications are tedious reversals of those needed for the first part.

There is a trivial operad $\mathscr{P}$ with $\mathscr{P}(0) = \{*\}$, $\mathscr{P}(1) = \{1\}$, and $\mathscr{P}(j)$ empty for $j > 1$; $\hat{\mathscr{P}}$ is precisely Π. There is an operad $\mathscr{N}$ with each $\mathscr{N}(j)$ a point; $\hat{\mathscr{N}}$ is precisely $\mathscr{F}$. As in Examples 1.3, $\mathscr{P}$ acts on any $\mathscr{C}$ and any $\mathscr{G}$ acts on $\mathscr{P}$ and on $\mathscr{N}$. A comparison of Definition 1.7 to [21, 1.4] gives the following relationship between the notions here and the notions used in our theory of pairings. (The latter notions will play no explicit role in this paper.)

Lemma 1.10. *An action of $\mathscr{N}$ on $\mathscr{C}$ determines and is determined by a structure of permutative operad on $\mathscr{C}$.*

By [21, 1.5], $\mathscr{C}$ is a permutative operad if and only if $\hat{\mathscr{C}}$ is a permutative category of operators in the sense of [21, 1.3]. The following analog is easily verified by comparison of Definition 1.2 to [21, 1.3].

Lemma 1.11. *An action of $\mathscr{F}$ on a category of operators $\hat{\mathscr{C}}$ determines and is determined by a structure of permutative category of operators on $\hat{\mathscr{C}}$.*

When $\hat{\mathscr{C}}$ comes from an operad $\mathscr{C}$, the two lemmas above have precisely the same content. If $\mathscr{F}$ acts on $\hat{\mathscr{C}}$, $\lambda(\phi_2)$ gives the product $\wedge : \hat{\mathscr{C}} \times \hat{\mathscr{C}} \to \hat{\mathscr{C}}$. If $\hat{\mathscr{C}}$ is permutative, the $\lambda(\phi)$ can be defined as in Definition 1.2(ii) in terms of $\wedge$ on $\hat{\mathscr{C}}$.

2. Actions by categories of ring operators

We first define the notion of a $\mathscr{J}$-space for a category of ring operators $\mathscr{J}$. The essential point is the incorporation of higher coherence homotopies for distributivity, but the subtle point is the correct handling of 0 and 1. We then discuss special cases and compare the categories of $\mathscr{J}$-spaces as $\mathscr{J}$ varies.

Let $\mathscr{U}$ denote the category of compactly generated weak Hausdorff spaces and let $\mathscr{T}$ denote the category of nondegenerately based spaces in $\mathscr{U}$.

Let $\delta_i : \mathbf{n} \to \mathbf{1}$ be the projection in Π specified by $\delta_i(j) = 1$ if $i = j$ and $\delta_i(j) = 0$ otherwise. Say that a morphism $(\phi; \chi) : (m; R) \to (n; S)$ in $\Pi \int \Pi$ is an injection if $\phi : \mathbf{m} \to \mathbf{n}$ is an injection and if $\chi_j : \mathbf{r}_i \to \mathbf{s}_j$ is an injection when $\phi(i) = j$; $\chi_j : \mathbf{1} \to \mathbf{s}_j$ can be any morphism in Π (including 0) when $j \notin \operatorname{Im} \phi$. These are precisely those morphisms of $\Pi \int \Pi$ which admit left inverses. For an injection $(\phi; \chi)$, let $\Sigma(\phi; \chi)$

denote the group of automorphisms $(\sigma;\tau):(n;S)\to(n;S)$ such that $(\sigma;\tau)\mathrm{Im}(\phi;\chi)=\mathrm{Im}(\phi;\chi)$, where $(\phi;\chi)$ and $(\sigma;\tau)$ are interpreted as maps of based sets. Say that a space is aspherical if its homotopy groups are trivial.

Definition 2.1. Let $\mathscr{J}$ be a category of ring operators. A $\mathscr{J}$-space is a functor $X:\mathscr{J}\to\mathscr{U}$, written $(n;S)\mapsto X(n;S)$ on objects, such that the evaluation maps

$$\mathscr{J}((m;R),(n;S))\times X(m;R)\to X(n;S)$$

are continuous and the following properties hold (where we use the same names for morphisms in $\mathscr{J}$ and for their images under X).

(1) $X(0;*)$ is aspherical and contains a nondegenerate basepoint $*$.

(2) $X(1;0)$ is aspherical.

(3) The map $\delta':X(1;\mathbf{s})\to X(1,\mathbf{1})^s$ with coordinates $(1;\delta_v)$ is an equivalence.

(4) The map $\delta'':X(n;S)\to\prod_{j=1}^{n}X(1;\mathbf{s}_j)$ with coordinates $(\delta_j;1)$ is an equivalence.

(5) If $(\phi;\chi):(m;R)\to(n;S)$ is an injection in $\Pi\int\Pi$, then $(\phi;\chi):X(m;R)\to X(n;S)$ is a $\Sigma(\phi,\chi)$-equivariant cofibration.

Let $\mathscr{J}[\mathscr{U}]$ denote the category of $\mathscr{J}$-spaces, its morphisms being the natural transformations under $\mathscr{J}$. A map $X\to X'$ of $\mathscr{J}$-spaces is said to be an equivalence if each $X(n;S)\to X'(n;S)$ is an equivalence.

We shall often have $X(0;*)=\{*\}$. The single basepoint $*$ determines both 0 and 1. Indeed, we have canonical injections $(0;0):(0;*)\to(1;\mathbf{s})$ for $s\geq 0$ and $(0;1^n):(0;*)\to(n;\mathbf{1}^n)$ for $n\geq 1$. Applied to $*$, these yield compatible nondegenerate basepoints $0\in X(1;\mathbf{s})$ and $1\in X(n;\mathbf{1}^n)$. For nontriviality, 0 and 1 must lie in different path components of $X(1;\mathbf{1})$. The role of the cofibration condition will not become apparent until Section 7. Its verification is not difficult in practice, and some of the condition can be arranged by a whiskering construction to be explained in Appendix C.

The definition is analogous to that of a $\hat{\mathscr{C}}$-space for a category of operators $\hat{\mathscr{C}}$ [22, 1.2]; however, in the cofibration condition of the cited definition, Σ_ϕ for $\phi:\mathbf{m}\to\mathbf{n}$ in Π was intended to be the group of permutations $\sigma:\mathbf{n}\to\mathbf{n}$ such that $\sigma\,\mathrm{Im}\,\phi=\mathrm{Im}\,\phi$ (rather than $\sigma\phi=\phi$). With this correction, the following result is an immediate consequence of the definitions. Recall Lemma 1.5.

Lemma 2.2. *Let X be a $\mathscr{J}$-space, where $\mathscr{J}=\hat{\mathscr{G}}\int\hat{\mathscr{C}}$. Define $X_\oplus:\hat{\mathscr{C}}\to\mathscr{T}$ and $X_\otimes:\hat{\mathscr{G}}\to\mathscr{T}$ to be the restrictions of X to the subcategories $\hat{\mathscr{C}}$ and $\hat{\mathscr{G}}$ of $\mathscr{J}$; the basepoints understood are* 0 *and* 1 *respectively. Then $X_\oplus$ is a $\hat{\mathscr{C}}$-space and $X_\otimes$ is a $\hat{\mathscr{G}}$-space.*

Restriction gives a forgetful functor $\mathscr{J}[\mathscr{U}]\to(\Pi\int\Pi)[\mathscr{U}]$, and a $\mathscr{J}$-space is to be thought of as an underlying $\Pi\int\Pi$-space with additional structure. Clearly conditions (1)–(5) refer only to $\Pi\int\Pi$. In turn, a $\Pi\int\Pi$-space is to be thought of as a collection of spaces $X(n;S)$ with all the formal and homotopical properties that would be present if $X(n;S)$ were $Z^{s_1}\times\cdots\times Z^{s_n}$ for a space Z with two basepoints 0 and 1. We make this precise in the following analog of [22, 1.3].

Definition 2.3. Let $\mathscr{T}_e$ denote the category of spaces Z with nondegenerate basepoints 0 and 1. Let $L:(\Pi\int\Pi)[\mathscr{U}]\to\mathscr{T}_e$ be the functor which sends a $\Pi\int\Pi$-space X to the space $X(1;\mathbf{1})$. Let $R:\mathscr{T}_e\to(\Pi\int\Pi)[\mathscr{U}]$ be the functor which sends a space Z to the $\Pi\int\Pi$-space RZ such that $(RZ)(0;*)=\{*\}$ and $(RZ)(n;S)=Z^{s_1}\times\cdots\times Z^{s_n}$ for $n>0$, where $Z^0=\{0\}$; for a morphism $(\phi;\chi):(m;R)\to(n;S)$ in $\Pi\int\Pi$, the induced map

$$(\phi;\chi):Z^{r_1}\times\cdots\times Z^{r_m}\to Z^{s_1}\times\cdots\times Z^{s_n}$$

has (j,v)th coordinate 0 if $\chi_j^{-1}(v)$ is empty, 1 if $\phi^{-1}(j)$ is empty and the morphism $\chi_j:\mathbf{1}\to\mathbf{s}_j$ satisfies $\chi_j(1)=v$, and the (i,u)th domain coordinate if $\phi(i)=j$ and $\chi_j(u)=v$. Observe that L and R are left and right adjoints,

$$\mathscr{T}_e(LX,Z)\cong(\Pi\int\Pi)[\mathscr{U}](X,RZ),$$

since a map $f:X(1,1)\to Z$ extends uniquely to a map $\tilde f:X\to RZ$, the (j,v)th coordinate of $\tilde f(n;S):X(n;S)\to Z^{s_1}\times\cdots\times Z^{s_n}$ being the composite of f and the projection $(\delta_j;\delta_v):X(n;S)\to X(1;1)$. Let $\delta:X\to RLX$ be the unit of the adjunction (that is, $\delta=\tilde 1$).

Note that RS^0 is a sub $\Pi\int\Pi$-space of RZ (if $0\neq 1$), but that $(\phi;\chi)$ need not be basepoint preserving in either possible sense.

The term 'category of ring operators' is justified by the following observation.

Lemma 2.4. *An $\mathscr{F}\int\mathscr{F}$-space with underlying $\Pi\int\Pi$-space RZ determines and is determined by a structure of commutative topological semi-ring on Z.*

Proof. If Z is a commutative topological semi-ring, define

$$(\phi;\chi):Z^{r_1}\times\cdots\times Z^{r_m}\to Z^{s_1}\times\cdots\times Z^{s_n}$$

for morphisms $\phi:\mathbf{m}\to\mathbf{n}$ and $\chi_j:\bigwedge_{\phi(i)=j}\mathbf{r}_i\to\mathbf{s}_j$ in $\mathscr{F}$ as follows. For $y_i=(z_{i,1},\dots,z_{i,r_i})\in Z^{r_i}$, let $(\phi;\chi)(y_1,\dots,y_m)$ have (j,v)th coordinate, $1\le j\le n$ and $1\le v\le s_j$, the element

$$\sum_{\chi_j(U)=v}\ \prod_{\phi(i)=j} z_{i,u_i}$$

where U runs over the lexicographically ordered set of sequences with ith term u_i satisfying $1\le u_i\le r_i$ for $i\in\phi^{-1}(j)$, this set being identified with $\bigwedge_{\phi(i)=j}\mathbf{r}_i-\{0\}$. Here the empty sum is 0 and the empty product is 1. The requisite functoriality is a tedious calculation. Conversely, if RZ is an $\mathscr{F}\int\mathscr{F}$-space, the maps

$$(1;\phi_2):Z\times Z\to Z\quad\text{and}\quad(\phi_2;1):Z\times Z\to Z$$

specify an addition and multiplication with respect to which Z is a commutative topological semi-ring. The essential point is the deduction of distributivity from the commutative diagram

$$\begin{array}{ccccc} & & (2;1,1) & & \\ & \nearrow^{(1;\,\phi_2,\phi_2)} & & \searrow^{(\phi_2;\,1)} & \\ (2;2,2) & & \xrightarrow{(\phi_2;\,\phi_4)} & & (1;1) \\ & \searrow_{(\phi_2;\,1)} & & \nearrow_{(1;\,\phi_4)} & \\ & & (1;4) & & \end{array}$$

in $\mathcal{F}\int\mathcal{F}$ (where $\phi_4 = \phi_2 \wedge \phi_2$). The unit condition for the multiplication comes from the commutative diagrams

$$\begin{array}{ccccc} & & (2;1,1) & & \\ & \nearrow^{(\iota_i;\,1,1)} & & \searrow^{(\phi_2;\,1)} & \\ (1;1) & & = \!=\!=\!= & & (1;1), \end{array}$$

where $\iota_i(1) = i$ for $i = 1$ or $i = 2$, and our conventions on empty smash products and the basepoint 1. The interested reader is invited to fill in complete details, as the exercise provides considerable insight into the working of our definitions.

The preceding result is the special case $(\mathcal{C}, \mathcal{G}) = (\mathcal{L}, \mathcal{L})$ of the following one. We recall the notion of an action of an operad pair on a space which was the essential starting point of the multiplicative infinite loop space theory in [14].

Definition 2.5. A $\mathcal{G}$-space with zero, or $\mathcal{G}_0$-space, is a $\mathcal{G}$-space (Z, ξ) with unit 1 and a second basepoint 0 such that $\xi(g;\ \times_{r=1}^{k} z_r) = 0$ if any $z_r = 0$. A $(\mathcal{C}, \mathcal{G})$-space is a $\mathcal{C}$-space (Z, θ) and a $\mathcal{G}_0$-space (Z, ξ) such that the following diagrams commute:

$$\begin{array}{ccc} \mathcal{G}(k) \times \left(\overset{k}{\underset{r=1}{\times}} \mathcal{C}(j_r) \times Z^{j_r} \right) & \xrightarrow{1 \times \theta^k} & \mathcal{G}(k) \times Z^k \\ \Big\downarrow \xi & & \Big\downarrow \xi \\ \mathcal{C}(j_1 \cdots j_k) \times Z^{j_1 \cdots j_k} & \xrightarrow{\theta} & Z \end{array}$$

Here ξ on the left is specified by

$$\xi\left(g;\ \overset{k}{\underset{r=1}{\times}} \left(c_r;\ \overset{j_r}{\underset{q=1}{\times}} z_{r,q}\right)\right) = \left(\lambda\left(g;\ \overset{k}{\underset{r=1}{\times}} c_r\right);\ \underset{Q}{\times}\, \xi\left(g;\ \overset{k}{\underset{r=1}{\times}} z_{r,q_r}\right)\right)$$

where Q runs over the lexicographically ordered set of sequences $(q_1, \ldots, q_k)$ with $1 \leq q_r \leq j_r$.

Proposition 2.6. *Let $\mathcal{J} = \hat{\mathcal{G}} \int \hat{\mathcal{C}}$, where $(\mathcal{C}, \mathcal{G})$ is an operad pair. A $\mathcal{J}$-space with underlying $\Pi\int\Pi$-space RZ determines and is determined by a structure of $(\mathcal{C}, \mathcal{G})$-space on Z.*

Proof. If Z is a $(\mathscr{C}, \mathscr{G})$-space, define

$$\left((\phi; g); \underset{j=1}{\overset{n}{\times}} (\chi_j; c_j)\right): Z^{r_1} \times \cdots \times Z^{r_m} \to Z^{s_1} \times \cdots \times Z^{s_n}$$

as follows, where $g = (g_1, \dots, g_n)$ and $c_j = (c_{j,1}, \dots, c_{j,s_j})$ with

$$g_j \in \mathscr{G}(|\phi^{-1}(j)|) \quad \text{and} \quad c_{j,v} \in \mathscr{C}(|\chi_j^{-1}(v)|).$$

With the notations of the proof of Lemma 2.4, let the (j, v)th coordinate of the image of $(y_1, \dots, y_m)$ be

$$\theta\left(c_{j,v}; \underset{\chi_j(U)=v}{\times} \xi\left(g_j; \underset{\phi(i)=j}{\times} z_{i,u_i}\right)\right).$$

This is the same formula as in the cited proof but with k-fold products parametrized by $\mathscr{G}(k)$ and k-fold sums parametrized by $\mathscr{C}(k)$. The parametrized distributivity law of the previous definition implies the requisite functoriality by direct computation from the definitons. For the converse, assume RZ is a $\mathscr{J}$-space. The morphisms

$$((\phi_k; g); 1): (k; 1, \dots, 1) \to (1; 1) \quad \text{and} \quad (1; (\phi_k; c)): (1, \mathrm{k}) \to (1; 1)$$

in $\mathscr{J}$ determine action maps

$$\xi: \mathscr{G}(k) \times Z^k \to Z \quad \text{and} \quad \theta: \mathscr{C}(k) \times Z^k \to Z.$$

Definition 1.2(ii) and consideration of the relevant injections shows that $0 \in Z$ is a strict zero for ξ, and functoriality implies the requisite parametrized distributivity law.

The following remarks should be compared to [22, 1.4 and 4.3] and will be followed up in Appendix D.

Remarks 2.7. The notion of an $\mathscr{F}\int\mathscr{F}$-space provides appropriate domain data for a Segal type development of multiplicative infinite loop space theory, and the present notion of a $\mathscr{J}$-space provides the appropriate simultaneous generalization of the notions of $\mathscr{F}\int\mathscr{F}$-space and $(\mathscr{C}, \mathscr{G})$-space. In turn, each of these notions is a natural generalization of the notion of an $(\mathscr{N}, \mathscr{N})$-space, or commutative topological semi-ring.

We can compare categories of $\mathscr{J}$-spaces precisely as we compared categories of $\hat{\mathscr{C}}$-spaces in [22, p. 207]. Thus assume given a map $\nu: \mathscr{J} \to \mathscr{K}$ of categories of ring operators. For a $\mathscr{J}$-space Y, pullback along ν gives a $\mathscr{J}$-space $\nu^* Y$. We associate a $\mathscr{K}$-space $\nu_* X$ to a $\mathscr{J}$-space X as follows, where ν must be an equivalence for proper behavior on the underlying $\Pi\int\Pi$-space level. For $(n; S)$ in $\coprod_{n \geq 0} N^n$, the space

$$\mathscr{K}(n; S) = \coprod_{(m;R)} \mathscr{K}((m; R), (n; S))$$

is a right graph over $\mathscr{J}$. For a $\mathscr{J}$-space X, $\coprod X(n; S)$ is a left graph over $\mathscr{J}$, again denoted X. We have a two-sided bar construction

$$(\nu_* X)(n; S) = B(\mathscr{K}(n; S), \mathscr{J}, X).$$

In [22, p. 208], we collapsed out by a trivial cofibration in order to fix basepoints. Here we prefer to choose the basepoint

$$* \in X(0; *) \subset \mathscr{K}(0; *) \times X(0; *) \subset (\nu_* X)(0, *),$$

the first inclusion being given by the identity morphism of $(0; *)$. Just as in [22, p. 208], $\nu_* X$ is a continuous functor $\mathscr{K} \to \mathscr{U}$, and the following addition to the definition of a category of ring operators ensures that $\nu_* X$ satisfies the cofibration condition of Definition 2.1(5).

Addendum 2.8. We require of a category of ring operators $\mathscr{J}$ that left composition $\mathscr{J}(m; R) \to \mathscr{J}(n; S)$ by an injection $(\phi; \chi)$ in $\Pi \int \Pi$ be a $\Sigma(\phi; \chi)$-cofibration.

This holds trivially when $\mathscr{J} = \hat{\mathscr{G}} \int \hat{\mathscr{C}}$ for an operad pair $(\mathscr{C}, \mathscr{G})$ since here the maps in question are inclusions onto unions of components. Now the proof of the following result is exactly the same as the proof of [22, 1.8].

Theorem 2.9. *Let $\nu : \mathscr{J} \to \mathscr{K}$ be an equivalence of categories of ring operators. For a $\mathscr{J}$-space X, $\nu_* X$ is a $\mathscr{K}$-space and there are natural equivalences of $\mathscr{J}$-spaces*

$$\nu^* \nu_* X \leftarrow 1_* X \rightarrow X,$$

where 1_* is induced by the identity functor of $\mathscr{J}$. For a $\mathscr{K}$-space Y, there is a natural equivalence of $\mathscr{K}$-spaces $\nu_* \nu^* Y \to Y$.

Thus the categories of $\mathscr{J}$-spaces and of $\mathscr{K}$-spaces are essentially equivalent.

3. Bipermutative categories and $\mathscr{F} \int \mathscr{F}$-spaces

A symmetric bimonoidal category is a symmetric monoidal category under operations $\oplus$ and $\otimes$ which satisfy the axioms for a commutative semi-ring up to coherent natural isomorphism. Here coherence has been made precise by Laplaza [18]. A bipermutative category is a symmetric bimonoidal category in which all diagrams commute strictly that might reasonably be expected to do so. Precisely, such a category $\mathscr{A}$ is permutative under $\oplus$ and $\otimes$, with unit objects 0 and 1 and commutativity isomorphisms c and $\bar{c}$. The right distributivity law holds strictly and 0 is a strict zero for $\otimes$. The left distributivity isomorphism

$$a \otimes (b \oplus c) \xrightarrow{\bar{c}} (b \oplus c) \otimes a = (b \otimes a) \oplus (c \otimes a) \xrightarrow{\bar{c} \oplus \bar{c}} (a \otimes b) \oplus (a \otimes c)$$

determined by $\tilde{c}$ is coherent with c in a sense made precise in [14, p. 154]. Clearly it would be unreasonable to ask that both distributive laws hold strictly.

Our concern in this section, which precisely parallels [21, §4], is to discuss the passage from bipermutative categories to $\mathscr{F}\int\mathscr{F}$-spaces via the classifying space functor B. We could use symmetric bimonoidal categories directly but, as a matter of aesthetics, we prefer to eliminate all coherence isomorphisms that can be eliminated. It is shown in [14, VI §3] that symmetric bimonoidal categories can be replaced functorially by equivalent bipermutative categories. However, just as discussed in [29 and 21, §2], we do not want to restrict ourselves to the strict morphisms of symmetric bimonoidal and bipermutative categories used in [14].

Thus define a morphism $F:\mathscr{A}\to\mathscr{B}$ of symmetric bimonoidal categories to be a functor F together with natural transformations (not necessarily isomorphisms)

$$\nu:0\to F(0),\qquad \Phi:Fa\oplus Fb\to F(a\oplus b),$$

$$\tilde{\nu}:1\to F(1),\qquad \tilde{\Phi}:Fa\otimes Fb\to F(a\otimes b),$$

such that the appropriate coherence diagrams commute. (See Lewis [8] and [14; VI §3]; a complete treatment of coherence here is not in the literature but has been obtained by Laplaza and is more or less implicit in Kelly's work in [9]). A morphism of bipermutative categories is a morphism of underlying symmetric bimonoidal categories. These are lax morphisms; strict morphisms would have isomorphisms in the symmetric bimonoidal case and equalities in the bipermutative case.

The association of equivalent bipermutative categories to bimonoidal categories is functorial in both senses. One can associate strict morphisms to strict morphisms by exploiting the freeness of the constructions involved or one can associate lax morphisms to lax morphisms by exploiting the constructed equivalences. Following Thomason, we showed in [21, 4.3] that, by settling for adjunction rather than actual equivalence, one can replace permutative categories and lax morphisms by permutative categories and strict morphisms. The corresponding assertion for bipermutative categories is more delicate if true, and I have not been able to obtain a proof.

We proceed to construct a functor from the category of bipermutative categories and (lax) morphisms to the category of $\mathscr{F}\int\mathscr{F}$-spaces. Just as in [29 and 21, §4], we first pass to lax functors $\mathscr{F}\int\mathscr{F}\to\mathrm{Cat}$ and lax natural transformations between them, then use Street's first construction [28] to rectify these to actual functors $\mathscr{F}\int\mathscr{F}\to\mathrm{Cat}$ and actual transformations, and finally apply the classifying space functor. The necessary categorical definitions and constructions are given in detail in [21, §3] and will not be repeated.

Thus let $\mathscr{A}$ be a bipermutative category. We construct an associated lax functor $A:\mathscr{F}\int\mathscr{F}\to\mathrm{Cat}$ as follows. Let

$$A(n;S)=\mathscr{A}^{s_1}\times\cdots\times\mathscr{A}^{s_n},$$

where $A(0;*)$ is the trivial category $\{*\}$ and $\mathscr{A}^0$ is the trivial category $\{0\}$. For a morphism $(\phi;\chi):(m;R)\to(n;S)$, specify the functor

$$A(\phi;\chi):A(m;R)\to A(n;S)$$

by the following formula on both objects and morphisms:

$$A(\phi;\chi)\left(\mathop{\times}_{i=1}^{m}\mathop{\times}_{u=1}^{r_i} a_{i,u}\right)=\mathop{\times}_{j=1}^{n}\mathop{\times}_{v=1}^{s_j}\bigoplus_{\chi_j(U)=v}\bigotimes_{\phi(i)=j} a_{i,u_i}.$$

The notations and conventions here are precisely the same as in the proof of Lemma 2.4. There we had precise commutativity and distributivity and so obtained a functor $\mathscr{F}\int\mathscr{F}\to\mathscr{U}$ from a commutative semi-ring. Here we have coherence isomorphisms which give lax functoriality. In detail, note first that A takes identity morphisms to identity functors. However, for a second morphism $(\psi;\omega):(n;S)\to(p;T)$,

$$A(\psi\phi;\xi)\left(\mathop{\times}_{i=1}^{m}\mathop{\times}_{u=1}^{r_i} a_{i,u}\right)=\mathop{\times}_{k=1}^{p}\mathop{\times}_{w=1}^{t_k}\bigoplus_{\xi_k(Y)=w}\bigotimes_{(\psi\phi)(i)=k} a_{i,y_i},$$

where $\xi_k=\omega_k\circ(\bigwedge_{\psi(j)=k}\chi_j)\circ\sigma_k(\psi,\phi)$ and Y runs through appropriate sequences regarded as elements of $\bigwedge_{(\psi\phi)(i)=k}\mathbf{r}_i$, while

$$A(\psi;\omega)A(\phi;\chi)\left(\mathop{\times}_{i=1}^{m}\mathop{\times}_{u=1}^{r_i} a_{i,u}\right)=\mathop{\times}_{k=1}^{p}\mathop{\times}_{w=1}^{t_k}\bigoplus_{\omega_k(V)=w}\bigotimes_{\psi(j)=k}\bigoplus_{\chi_j(U)=v_j}\bigotimes_{\phi(i)=j} a_{i,u_i},$$

where U and V run through appropriate sequences regarded as elements of $\bigwedge_{\phi(i)=j}\mathbf{r}_i$ and $\bigwedge_{\psi(j)=k}\mathbf{s}_j$. The commutativity isomorphisms c and $\bar{c}$ in $\mathscr{A}$, together with the strict right distributive law, induce a natural isomorphism

$$\sigma((\psi;\omega),(\phi;\chi)):A(\psi\phi;\xi)\to A(\psi;\omega)A(\phi;\chi).$$

The compatibility diagram in [21, 3.1] for a composable triple of morphisms commutes by coherence, hence these isomorphisms give A a structure of lax functor.

For a morphism $F:\mathscr{A}\to\mathscr{B}$ of bipermutative categories, the functors

$$F^{s_1}\times\cdots\times F^{s_n}:\mathscr{A}^{s_1}\times\cdots\times\mathscr{A}^{s_n}\to\mathscr{B}^{s_1}\times\cdots\times\mathscr{B}^{s_n}$$

and natural transformations

$$B(\phi;\chi)(F^{r_1}\times\cdots\times F^{r_m})\to(F^{s_1}\times\cdots\times F^{s_n})A(\phi;\chi)$$

with (j,v)th coordinate

$$\bigoplus_{\chi_j(U)=v}\bigotimes_{\phi(i)=j}F(a_{i,u_i})\to F\left(\bigoplus_{\chi_j(U)=v}\bigotimes_{\phi(i)=j}a_{i,u_i}\right)$$

determined by the natural transformations given as part of the definition of a morphism together specify a lax natural transformation of lax functors. Note the need for the two unit transformations to handle 0's and 1's. The compatibility diagram in [21, 3.2] required for composites of morphisms commutes by coherence.

By [21, 3.4], we have associated functors $\bar{A}:\mathscr{F}\int\mathscr{F}\to\mathbf{Cat}$ and natural transformations $\tilde{F}: \tilde{A} \to \tilde{B}$. We also write $\tilde{A} = \mathscr{A}$ to emphasize that it is a collection of categories to which the classifying space functor B can be applied. We give $\mathscr{A}(0; *)$ the base object $*$ specified (in terms of the explicit constructions of [21, 3.4] by the identity morphism of $(0; *)$ and the unique object $* \in A(0; *)$. Since $\mathscr{A}(n; S)$ is related to $\mathscr{A}^{s_1} \times \cdots \times \mathscr{A}^{s_n}$ by a pair of adjoint functors, one of which is part of a lax natural transformation $\eta: A \to \tilde{A}$, it is immediately obvious that the homotopy type conditions in (1)–(4) of Definition 2.1 are satisfied by the functor $B\mathscr{A}$. The cofibration condition (5) holds trivially when $\mathscr{A}$ is discrete, since then the maps in question are inclusions of subcomplexes in CW-complexes, and can be deduced in general by inspection of the constructions and use of our standing hypothesis that $\mathrm{Id}: \mathrm{Ob}\,\mathscr{A} \to \mathrm{Mor}\,\mathscr{A}$ is a cofibration. In view of Remarks C.8 below, we omit the details.

Thus we, or rather those category theorists who study coherence, have proven the following basic result.

Theorem 3.1. *There is a functor, written $\mathscr{A} \to B\mathscr{A}$ on objects and $F \to B\tilde{F}$ on morphisms, from the category of bipermutative categories to the category of $\mathscr{F}\wr\mathscr{F}$-spaces. There is also a natural homotopy equivalence $\eta: B\mathscr{A} \to B\mathscr{A}(1; 1)$.*

The following remarks explain the relationship between the construction just given and the analogous constructions for $\mathscr{A}$ regarded just as a permutative category.

Remarks 3.2. Let $\mathscr{A}_{\oplus}$ and $\mathscr{A}_{\otimes}$ denote $\mathscr{A}$ regarded as a permutative category under $\oplus$ and $\otimes$. The corresponding lax functors $A_{\oplus}: \mathscr{F} \to \mathrm{Cat}$ and $A_{\otimes}: \mathscr{F} \to \mathrm{Cat}$ are precisely the restrictions to $\mathscr{F} \subset \Pi\wr\mathscr{F}$ and to $\mathscr{F} \subset \mathscr{F}\wr\Pi$ of $A: \mathscr{F}\wr\mathscr{F} \to \mathrm{Cat}$. Write $\mathscr{F}_{\oplus}$ and $\mathscr{F}_{\otimes}$ to distinguish these two copies of $\mathscr{F}$ contained in $\mathscr{F}\wr\mathscr{F}$. There result natural transformations

$$\zeta_{\oplus}: \tilde{A}_{\oplus} \to (\tilde{A} \mid \mathscr{F}_{\oplus}) \quad \text{and} \quad \zeta_{\otimes}: \tilde{A}_{\otimes} \to (\tilde{A} \mid \mathscr{F}_{\otimes})$$

of functors $\mathscr{F} \to \mathrm{Cat}$ (compare [21, 3.6]). Their component functors (for $\mathbf{n} \in \mathscr{F}$) induce equivalences on passage to classifying spaces since they are compatible with the component functors of the respective lax natural transformations η with domain $A_{\oplus}$ and $A_{\otimes}$.

The following observation about general permutative categories is particularly useful when applied to $\mathscr{A}_{\oplus}$.

Remarks 3.3. Let $(\mathscr{A}, \Box, *, c)$ be any permutative category and let $\mathscr{A}_*$ denote the full subcategory of $\mathscr{A}$ with object space $\{*\}$. The morphism space of $\mathscr{A}_*$ is a topological monoid under both composition and $\Box$, and these operations satisfy $(a \Box b)(a' \Box b') = aa' \Box bb'$ and have the identity map of $*$ as common unit. Therefore these operations coincide and are commutative. It follows that $c = 1: * \to *$. Thus $\mathscr{A}_*$ is a sub permutative category of $\mathscr{A}$.

In practice, when $\mathscr{A}$ is bipermutative, $\mathscr{A}_0 \subset \mathscr{A}_\oplus$ is usually trivial and $\mathscr{A}_1 \subset \mathscr{A}_\otimes$ is usually discrete, so that $B\mathscr{A}_1$ is a $K(\pi, 1)$. In this situation, the full analysis of the relationship between $B\mathscr{A}_1$ and the spectrum determined by $B\mathscr{A}$ will depend on the following observations.

Remarks 3.4. Let $\mathscr{A}_1$ be a commutative topological monoid with unit 1 and let $\mathscr{A}_1^+$ be the union of $\mathscr{A}_1$ and a disjoint basepoint 0. Regard $\mathscr{A}_1^+$ as a category with objects 0 and 1 in the obvious way and note that $B(\mathscr{A}_1^+) = (B\mathscr{A}_1)^+$. There is a functor $A_1^+ : \mathscr{F}\int\Pi \to \mathrm{Cat}$ specified on objects by

$$A_1^+(n; S) = (\mathscr{A}_1^+)^{s_1} \times \cdots \times (\mathscr{A}_1^+)^{s_n}$$

and specified on morphisms precisely as was A above. No addition is needed because of the use of Π, and A_1^+ is an actual functor by commutativity. Rectification gives another functor $\tilde{A}_1^+$, also written $\mathscr{A}_1^+$, and the left adjoints $\varepsilon : \tilde{A}_1^+(n; S) \to A_1^+(n; S)$ of [21, 3.4] specify an (actual) natural transformation $\tilde{A}_1^+ \to A_1^+$. Clearly $BA_1^+ = RB\mathscr{A}_1^+$ as an $\mathscr{F}\int\Pi$-space. If $\mathscr{A}_1$ arises from a bipermutative category $\mathscr{A}$, we have an evident inclusion $A_1^+ \to A$ of lax functors $\mathscr{F}\int\Pi \to \mathrm{Cat}$ and thus a natural transformation $\zeta : \tilde{A}_1^+ \to (\tilde{A} \mid \mathscr{F}\int\Pi)$ by [21, 3.6]. We therefore have the diagram

$$RB\mathscr{A}_1^+ \xleftarrow{B\varepsilon} B\tilde{\mathscr{A}}_1^+ \xrightarrow{B\zeta} B(\tilde{\mathscr{A}} \mid \mathscr{F}\int\Pi)$$

of $\mathscr{F}\int\Pi$-spaces, in which the component maps of $B\varepsilon$ are equivalences.

The force of these observations comes from the following remarks on unit morphisms, which show that $\mathscr{A}_1$ naturally generates a sub bipermutative category of $\mathscr{A}$.

Remarks 3.5. (i) Let $\mathscr{E}$ denote the subcategory of $\mathscr{F}$ consisting of those morphisms which are isomorphisms (that is, permutations). It is closed under wedges and smash products (thought of as $\oplus$ and $\otimes$ on permutations) and is thus bipermutative. It is easy to see that any bipermutative category $\mathscr{A}$ admits a strict unit morphism $e : \mathscr{E} \to \mathscr{A}$. Indeed, $\mathscr{E}$ is the free bipermutative category generated by $\mathbf{0}$ and $\mathbf{1}$. (Compare [14, p. 161].)

(ii) Let $\mathscr{A}_1$ be any commutative topological monoid. Then $\mathbf{0}$ and $\mathscr{A}_1$ generate a wreath product bipermutative category $\mathscr{E}\int\mathscr{A}_1$. Its object space is N, there are no morphisms $\mathbf{m} \to \mathbf{n}$ for $m \neq n$, and the monoid $\Sigma_n\int\mathscr{A}_1$ is the space of morphisms $\mathbf{n} \to \mathbf{n}$. Addition and multiplication

$$\oplus : (\Sigma_m\int\mathscr{A}_1) \times (\Sigma_n\int\mathscr{A}_1) \to \Sigma_{m+n}\int\mathscr{A}_1$$

and

$$\otimes : (\Sigma_m\int\mathscr{A}_1) \times (\Sigma_n\int\mathscr{A}_1) \to \Sigma_{mn}\int\mathscr{A}_1$$

are specified in terms of $\oplus$ and $\otimes$ on $\mathscr{E}$ by the formulas

$$\left(\sigma; \mathop{\times}_{i=1}^{m} a_i\right) \oplus \left(\tau; \mathop{\times}_{j=1}^{n} b_j\right) = \left(\sigma \oplus \tau; \mathop{\times}_{i=1}^{m} a_i, \mathop{\times}_{j=1}^{n} b_j\right)$$

and

$$\left(\sigma;\ \overset{m}{\underset{i=1}{\times}} a_i\right)\otimes\left(\tau;\ \overset{n}{\underset{j=1}{\times}} b_j\right)=\left(\sigma\otimes\tau;\ \underset{(i,j)}{\times} a_i b_j\right).$$

(iii) By $e(\sigma; a_1,\dots,a_n)=e(\sigma)\circ(a_1\oplus\cdots\oplus a_m)$ for $\sigma\in\Sigma_m$ and $a_i: \mathbf{1}\to\mathbf{1}$, the unit $e:\mathscr{E}\to\mathscr{A}$ of a bipermutative category $\mathscr{A}$ extends to a strict morphism $e:\mathscr{E}\wr\mathscr{A}_1\to\mathscr{A}$ of bipermutative categories. When $\mathscr{A}$ is the general linear category $\mathscr{GL}A$ or orthogonal category $\mathscr{O}A$ of a commutative ring A, $\mathscr{E}\wr\mathscr{A}_1$ is the appropriate subcategory of monomial matrices.

4. The passage from $\mathscr{J}$-spaces to $\mathscr{G}$-spectra

As the previous sections make clear, $\mathscr{J}$-spaces provide the appropriate input data for multiplicative infinite loop space theory, where $\mathscr{J}=\mathscr{G}\wr\mathscr{C}$ for an operad pair $(\mathscr{C},\mathscr{G})$. The special case of $(\mathscr{C},\mathscr{G})$-spaces, or $\mathscr{J}$-spaces with underlying $\Pi\wr\Pi$-space RZ, is particularly simple. We explain here how our machinery converts $\mathscr{J}$-spaces on the input side to $\mathscr{G}$-spectra and thus, by passage to 0th spaces, to $(\mathscr{C},\mathscr{G})$-spaces on the output side. Proofs are deferred to the following sections. To accomplish this conversion, we shall make heavy use of an intermediate category. The following elaboration of Definition 2.5 sets the stage.

Definition 4.1. Let $\Pi[\mathscr{T}]_e$ denote the category of Π-spaces $Y:\Pi\to\mathscr{T}$ with a second nondegenerate basepoint $1\in Y_1$ which is sent to 0 under the map $Y_1\to Y_0$ induced by $0:\mathbf{1}\to\mathbf{0}$. We define a diagram of adjoint pairs of functors

$$\mathscr{T}_e \underset{L'}{\overset{R'}{\rightleftarrows}} \Pi[\mathscr{T}]_e \underset{L''}{\overset{R''}{\rightleftarrows}} (\Pi\wr\Pi)[\mathscr{U}], \qquad R:\mathscr{T}_e\to(\Pi\wr\Pi)[\mathscr{U}],\quad L:(\Pi\wr\Pi)[\mathscr{U}]\to\mathscr{T}_e$$

Just as in [22, 1.3], which gives details, $L'Y=Y_1$ and $R'Z=\{Z^n\}$ for $Z\in\mathscr{T}_e$. For a $\Pi\wr\Pi$-space X, define $L''X=\{X(1,s)\}=X_\oplus$, as in Lemma 2.2. For $Y\in\Pi[\mathscr{T}]_e$ define $R''Y$ by

$$(R''Y)(0;*)=\{*\}\quad\text{and}\quad(R''Y)(n;S)=Y_{s_1}\times\cdots\times Y_{s_n}\quad\text{for } n>0.$$

For a morphism $(\phi;\chi):(m;R)\to(n;S)$ in $\Pi\wr\Pi$, let

$$(\phi;\chi): Y_{r_1}\times\cdots\times Y_{r_m}\to Y_{s_1}\times\cdots\times Y_{s_n}$$

have jth coordinate the composite of the projection to Y_{r_i} and the map $\chi_j: Y_{r_i}\to Y_{s_j}$ if $\phi(i)=j$; if $\phi^{-1}(j)$ is empty, then χ_j is a map $Y_1\to Y_{s_j}$ and is to be applied to the point $1\in Y_1$. We have

$$\Pi[\mathscr{T}]_e(L''X,Y)\cong(\Pi\wr\Pi)[\mathscr{U}](X,R''Y)$$

since a map $f: L''X \to Y$ extends uniquely to a map $\bar{f}: X \to R''Y$, the jth coordinate of $\bar{f}(n; S): X(n; S) \to (R'Y)(n; S)$ being $f_j \circ (\delta_j; 1)$. Clearly $R = R''R'$ and $L = L'L''$. Write

$$\delta: X \to RLX, \qquad \delta': Y \to R'L'Y, \qquad \delta'': Z \to R''L''Z$$

for the units of the various adjunctions; thus $\delta = R''\delta'L'' \circ \delta''$. By the very definitions of Π-spaces and of $\Pi\int\Pi$-spaces, these maps are all (weak) equivalences.

The following definition gives our intermediate notion.

Definition 4.2. A $(\hat{\mathscr{C}}, \mathscr{G})$-space is a Π-space Y (with $1 \in Y_1$) together with a structure of $\mathscr{J}$-space on $R''Y$. For example, $R'Z$ is a $(\hat{\mathscr{C}}, \mathscr{G})$-space for any $(\mathscr{C}, \mathscr{G})$-space Z.

We shall arrive at a more intrinsic description at the end of Section 6, but the present formulation is well suited to the explanation of what our basic theorems say. The intuition is that the additive, $\hat{\mathscr{C}}$-space, structure on Y is based on use of the products up to homotopy $Y_s \simeq Y_1^s$, but the multiplicative, $\mathscr{G}$-space, structure is then superimposed on actual products $Y_{s_1} \times \cdots \times Y_{s_n}$.

Let us say that an operad $\mathscr{C}$ is Σ-free if Σ_j acts freely on $\mathscr{C}(j)$ for all j. The following result asserts that $\mathscr{J}$-spaces can be replaced functorially by equivalent $(\hat{\mathscr{C}}, \mathscr{G})$-spaces.

Theorem 4.3. *Let $\mathscr{J} = \mathscr{G}\int\mathscr{C}$, where $\mathscr{C}$ and $\mathscr{G}$ are Σ-free operads. There is a functor U from $\mathscr{J}$-spaces to $\mathscr{J}$-spaces and a functor V from $\mathscr{J}$-spaces to $(\hat{\mathscr{C}}, \mathscr{G})$-spaces together with a natural pair of equivalences of $\mathscr{J}$-spaces*

$$X \xleftarrow{\ \varepsilon\ } UX \xrightarrow{\ \delta\ } R''VX.$$

There is also a functor U from $(\hat{\mathscr{C}}, \mathscr{G})$-spaces to $(\hat{\mathscr{C}}, \mathscr{G})$-spaces such that $UR'' = R''U$. When $X = R''Y$ for a $(\hat{\mathscr{C}}, \mathscr{G})$-space Y, the diagram just displayed is obtained by application of R'' to a diagram

$$Y \xleftarrow{\ \varepsilon\ } UY \cong VR''Y$$

of $(\hat{\mathscr{C}}, \mathscr{G})$-spaces in which ε is a natural equivalence.

A second functor, $V_{\oplus}$, which replaces $(\hat{\mathscr{C}}, \mathscr{G})$-spaces by equivalent $(\mathscr{C}, \mathscr{G})$-spaces will appear in Theorems 4.6 and 4.8. It would obscure the structure of the theory to think of the composite $V_{\oplus}V$ as a single replacement functor, although that is a convenient point of view in the homological applications of [6].

To state our results on the passage from $(\hat{\mathscr{C}}, \mathscr{G})$-spaces to $\mathscr{G}$-spectra, we need some recollections. An operad $\mathscr{C}$ is said to be spacewise contractible if each $\mathscr{C}(j)$ is contractible. An E_∞ operad is one which is Σ-free and spacewise contractible. There is a canonical (additive) E_∞ operad, namely the Steiner operad $\mathscr{K}_\infty$. Its jth space consists of appropriate j-tuples of paths of embeddings $R^\infty \to R^\infty$, and it acts naturally on infinite loop spaces. See Steiner [26] and [21, §6]. The operad $\mathscr{K}_\infty$ supplants the ugly little convex bodies partial operad I introduced in [14], and my results about the latter apply more simply to the former. There is also a canonical (multiplicative) E_∞ operad, namely the linear isometries operad $\mathscr{L}$. Its jth space consists of linear isometries $(R^\infty)^j \to R^\infty$, and [14, VII.2.3] specifies an action of $\mathscr{L}$ on $\mathscr{K}_\infty$ (compare [21, 6.7]).

Throughout the rest of this section, we assume given an operad pair $(\mathscr{C}', \mathscr{G}')$. We define a second operad pair

$$(\mathscr{C}, \mathscr{G}) = (\mathscr{C}' \times \mathscr{K}_\infty, \mathscr{G}' \times \mathscr{L}),$$

we set $\mathscr{J} = \hat{\mathscr{G}} \int \hat{\mathscr{C}}$, and we regard $\hat{\mathscr{G}}' \int \hat{\mathscr{C}}'$-spaces as $\mathscr{J}$-spaces by pullback. Ignoring the multiplicative operads for the moment, we have a functor E from $\hat{\mathscr{C}}$-spaces to spectra by [21, §6]. We shall recall our conventions on spectra and the definition of E in Section 8. When $Y = R'Z$ for a $\mathscr{C}$-space Z, EY coincides with the spectrum EZ constructed earlier in [14, VII §3]. A $\Pi \int \hat{\mathscr{C}}$-space X determines an underlying $\hat{\mathscr{C}}$-space $X_\oplus$, by Lemma 2.2, and we define $EX = EX_\oplus$. Remarks 8.3 below will show the inevitability of this definition.

We shall recall the notion of a $\mathscr{G}$-spectrum in Section 8. As pointed out in the introduction, this notion is defined in terms of strict smash products, and it is too much to ask that EX be a $\mathscr{G}$-spectrum when X is a $\mathscr{J}$-space. However, EX is equivalent to EVX, by Theorem 4.3, and we shall prove the following result. By Lemma C.1, the hypothesis $Y_0 = \{0\}$ can be arranged functorially and so results in no loss of generality.

Theorem 4.4. *For $(\hat{\mathscr{C}}, \mathscr{G})$-spaces Y with $Y_0 = \{0\}$, EY is functorially a $\mathscr{G}$-spectrum. If $Y = R'Z$ for a $(\mathscr{C}, \mathscr{G})$-space Z, then $EY = EZ$ as a $\mathscr{G}$-spectrum.*

The last statement asserts the compatibility of the old passage from $(\mathscr{C}, \mathscr{G})$-spaces to $\mathscr{G}$-spectra of [14, VII §4] with the new passage from $(\hat{\mathscr{C}}, \mathscr{G})$-spaces to $\mathscr{G}$-spectra. By [14, VII.2.4], we have the following basic result.

Theorem 4.5. *The zeroth space of a $\mathscr{G}$-spectrum is functorially a $(\mathscr{C}, \mathscr{G})$-space.*

The previous three theorems together show that the composite functor E_0V converts the fuzzy structure of a $\mathscr{J}$-space X on the input side to the precise structure of a $(\mathscr{C}, \mathscr{G})$-space on the output side. The force of the construction comes from the following relationship between the input and the output. We first recall information implicit in our earlier treatment of the purely additive theory.

Theorem 4.6. *There is a functor $U_\oplus$ from $\hat{\mathscr{C}}$-spaces to $\hat{\mathscr{C}}$-spaces and a functor $V_\oplus$ from $\hat{\mathscr{C}}$-spaces to $\mathscr{C}$-spaces together with a natural diagram of maps of $\hat{\mathscr{C}}$-spaces*

$$Y \xleftarrow{\varepsilon} U_\oplus Y \xrightarrow{\delta} R'V_\oplus Y \xrightarrow{R'\gamma} R'E_0Y \tag{$*$}$$

such that ε is an equivalence with natural inverse τ (of Π-spaces), δ is an equivalence, and $\gamma: V_{\oplus}Y \to E_0Y$ is a group completion if $\mathscr{C}'$ is spacewise contractible. Therefore $\iota = \gamma L'(\delta\tau): Y_1 \to E_0Y$ is also a group completion if $\mathscr{C}'$ is spacewise contractible. There is also a functor $U_{\oplus}$ from $\mathscr{C}$-spaces to $\mathscr{C}$-spaces such that $U_{\oplus}R' = R'U_{\oplus}$. When $Y = R'Z$ for a $\mathscr{C}$-space Z, the diagram just displayed is obtained by application of R' to a diagram

$$Z \xleftarrow{\ \varepsilon\ } U_{\oplus}Z \cong V_{\oplus}R'Z \xrightarrow{\ \gamma\ } E_0Z \qquad (**)$$

of $\mathscr{C}$-spaces in which ε is an equivalence with natural inverse τ.

We shall see that the following result is implicit in Definition 4.2. Recall the notation $X_{\otimes}$ from Lemma 2.2 and the notion of a $\mathscr{G}_0$-space from Definition 2.5.

Lemma 4.7. *For $(\hat{\mathscr{C}}, \mathscr{G})$-spaces Y, Y_1 is a $\mathscr{G}$-space whose associated $\hat{\mathscr{G}}$-space is precisely $(R''Y)_{\otimes}$. If $Y_0 = \{0\}$, then Y_1 is a $\mathscr{G}_0$-space.*

The following theorem asserts the compatibility of the given $(\hat{\mathscr{C}}, \hat{\mathscr{G}})$-space structure on Y with the derived $(\mathscr{C}, \mathscr{G})$-space structure on E_0Y.

Theorem 4.8. *If Y is a $(\hat{\mathscr{C}}, \mathscr{G})$-space with $Y_0 = \{0\}$, then $U_{\oplus}Y$ is a $(\hat{\mathscr{C}}, \mathscr{G})$-space, $V_{\oplus}Y$ is a $(\mathscr{C}, \mathscr{G})$-space, $(*)$ is a diagram of maps of $(\hat{\mathscr{C}}, \mathscr{G})$-spaces, and $L\tau$ and therefore also $\iota: Y_1 \to E_0Y$ are maps of $\mathscr{G}_0$-spaces. If $Y = R'Z$ for a $(\mathscr{C}, \mathscr{G})$-space Z, then $(**)$ is a diagram of maps of $(\mathscr{C}, \mathscr{G})$-spaces.*

The last statement was proven in [14, VII §4].

The following relationship between $\mathscr{G}_0$-spaces and $\mathscr{G}$-spectra was pointed out in [14, p. 70]. Let Σ^∞ denote the stabilization functor from spaces to spectra.

Lemma 4.9. *For $\mathscr{G}_0$-spaces Z, $\Sigma^\infty Z$ is functorially a $\mathscr{G}$-spectrum. If E is a $\mathscr{G}$-spectrum and $f: Z \to E_0$ is a map of $\mathscr{G}_0$-spaces, then the adjoint $\tilde{f}: \Sigma^\infty Z \to E$ of f is a map of $\mathscr{G}$-spectra.*

Applied to $\iota: Y_1 \to E_0Y$, this gives the following result.

Corollary 4.10. *For a $(\hat{\mathscr{C}}, \mathscr{G})$-space Y with $Y_0 = \{0\}$, $\tilde{\iota}: \Sigma^\infty Y_1 \to EY$ is a map of* $\mathscr{G}$-spectra.

Of course, our main interest is in the case $Y = VX$ (after modification by Lemma C.1 so as to arrange $Y_0 = \{0\}$). Here we have the following important consequence of the results above.

Corollary 4.11. *For a $\mathscr{J}$-space X, δ and ε are equivalences in the following diagram* *of $\mathscr{G}$-spaces:*

$$X_{\otimes} \xleftarrow{\ \varepsilon\ } (UX)_{\otimes} \xrightarrow{\ \delta\ } (R''VX)_{\otimes}.$$

Here $(R''VX)_{\otimes}$ is the $\mathscr{G}$-space associated to the $\mathscr{G}$-space V_1X, and $\iota: V_1X \to E_0VX$ is a map of $\mathscr{G}$-spaces.

For a $\mathscr{G}$-spectrum E, such as EVX, FE denotes the space of unit components of E_0 and SFE denotes the component of 1. These are $\mathscr{G}$-spaces and thus give rise to spectra when $\mathscr{G}'$ is spacewise contractible. Many of the applications of [14] were based on the use of these "spectra of units". Such applications work more generally in view of the present theory but are otherwise unchanged since the notion of $\mathscr{G}$-spectrum is unchanged.

In particular, the results above apply to the $\mathscr{F}\int\mathscr{F}$-spaces $X = B\mathscr{A}$ associated to bipermutative categories $\mathscr{A}$. Here we take $(\mathscr{C}', \mathscr{G}') = (\mathscr{N}, \mathscr{N})$ and have $(\mathscr{C}, \mathscr{G}) = (\mathscr{K}_\infty, \mathscr{L})$. That is, we regard X as an $\mathscr{L}\int\mathscr{K}_\infty$-space by pullback. We agree to write $E\mathscr{A} = EVB\mathscr{A}$. By Remarks 3.2 and Theorem 4.3, $E\mathscr{A}$ is naturally equivalent to the spectrum $E\mathscr{A}_{\oplus} = EB\mathscr{A}_{\oplus}$ associated to the underlying additive permutative category of $\mathscr{A}$. All but the last statement of the following result are immediate from Remarks 3.6 and naturality.

Proposition 4.12. *The unit e of $E\mathscr{A}$ factors naturally as a composite*

$$S \to E\mathscr{E} \to E(\mathscr{E}\int\mathscr{A}_1) \to E\mathscr{A}$$

of maps of $\mathscr{L}$-spectra. Moreover, $e: S \to E\mathscr{E}$ is an equivalence and $E(\mathscr{E}\int\mathscr{A}_1)$ is equivalent as an $\mathscr{L}$-spectrum to $\Sigma^\infty B\mathscr{A}_1^+$.

The assertion about $E\mathscr{E}$ is a multiplicatively enriched version of the Barratt–Quillen theorem to the effect that the spectrum determined by the category of finite sets and their isomorphisms is the sphere spectrum. The assertion about $E(\mathscr{E}\int\mathscr{A}_1)$ is of particular interest when specialized to categories of monomial matrices.

We shall round out the theory by generalizing some further results proven in [14] in Section 8, but it is high time we turned to the proofs of the results claimed.

5. Generalities on monads

We require some categorical preliminaries (and earlier notations should be forgotten for the moment). The proofs of the results stated in the previous section are all based on use of the general categorical two-sided bar construction $B_*(F, C, X)$ introduced in [12, §9]. Here C is a monad in some ground category $\mathscr{V}$, X is a C-object in $\mathscr{V}$, and $F: \mathscr{V} \to \mathscr{V}'$ is a C-functor with values in some other category $\mathscr{V}'$ (see [12, §2 and §9] for the definitions). $B_*(F, C, X)$ is a simplicial object in $\mathscr{V}'$, its object of q-simplices being FC^qX. When $\mathscr{V}' = \mathscr{U}$, we obtain a space

$$B(F,C,X) = |B_*(F,C,X)|$$

by geometric realization. More generally, $\mathscr{V}'$ might be a category of functors $\Xi \to \mathscr{U}$ for some category Ξ, and then $B(F,C,X)$ will also be a functor $\Xi \to \mathscr{U}$.

We shall first give a general discussion of the behavior of this construction with respect to appropriate changes of ground category. We shall next give a general result, due to Beck [1], on distributivity diagrams and interchange of monads. We shall then give a generic categorical construction of the monads relevant to our theory. Finally, we shall review the material of [22, §5] on monads associated to categories of operators in the light of the new categorical perspective.

Until otherwise specified, we assume given categories $\mathscr{V}$ and $\mathscr{W}$ and an adjoint pair of functors $L: \mathscr{W} \to \mathscr{V}$ and $R: \mathscr{V} \to \mathscr{W}$ such that LR is the identity functor of $\mathscr{V}$. Thus for $Y \in \mathscr{W}$ and $Z \in \mathscr{V}$, we have

$$\mathscr{V}(LY,Z) \cong \mathscr{W}(Y,RZ) \quad \text{and} \quad LRZ = Z.$$

We let $\delta: \mathrm{Id} \to RL$ denote the unit of the adjunction. We then have the following equalities of natural transformations:

$$L\delta = \mathrm{Id}: L \to L \quad \text{and} \quad \delta R = \mathrm{Id}: R \to R.$$

We are interested in the relationship between monad structures on functors

$$C: \mathscr{V} \to \mathscr{V} \quad \text{and} \quad D: \mathscr{W} \to \mathscr{W}.$$

Proposition 5.1. *Let (C,μ,η) be a monad in $\mathscr{V}$.*

(i) *RCL is a monad in $\mathscr{W}$ with unit and product the composites*

$$\mathrm{Id} \xrightarrow{\delta} RL \xrightarrow{R\eta L} RCL \quad \textit{and} \quad RCLRCL = RCCL \xrightarrow{R\mu L} RCL.$$

(ii) *If (F,λ) is a C-functor (in some category $\mathscr{V}'$), then FL is an RCL-functor with action transformation*

$$\lambda L: FLRCL = FCL \to FL.$$

(iii) *If (Z,ξ) is a C-object in $\mathscr{V}$, then RZ is an RCL-object in $\mathscr{W}$ with action map*

$$R\xi: RCLRZ = RCZ \to RZ.$$

For examples, (F,λ) might be (C,μ) and (Z,ξ) might be (CZ',μ) for any $Z' \in \mathscr{V}$. Since $LR = \mathrm{Id}$, we have the equality

$$B_*(F,C,Z) = B_*(FL,RCL,RZ).$$

Because of the asymmetry in our standing assumptions on L and R, LDR need not inherit a monad structure when D is a monad in $\mathscr{W}$.

Proposition 5.2. *Let (D, ν, ζ) be a monad in $\mathscr{W}$, define $C = LDR$, and suppose that (C, μ, η) is a monad in $\mathscr{V}$ such that the following diagrams commute:*

$$\begin{array}{ccc} LR & \xrightarrow{L\zeta R} & LDR \\ \| & & \| \\ Id & \xrightarrow{\eta} & C \end{array} \qquad \text{and} \qquad \begin{array}{ccc} LDDR & \xrightarrow{L\nu R} & LDR \\ {\scriptstyle LD\delta DR}\downarrow & & \| \\ LDRLDR = CC & \xrightarrow{\mu} & C \end{array}$$

Let $\bar{\delta}$ denote the common composite in the diagram

$$\begin{array}{ccc} D & \xrightarrow{D\delta} & DRL \\ {\scriptstyle \delta D}\downarrow & \searrow^{\bar{\delta}} & \downarrow{\scriptstyle \delta DRL} \\ RLD & \xrightarrow{RLD\delta} & RLDRL = RCL \end{array}$$

(i) *$\bar{\delta}$ is a morphism of monads in $\mathscr{W}$.*

(ii) *If (F, λ) is a C-functor, then FL is a D-functor with action*

$$FLD \xrightarrow{FL\bar{\delta}} FLRCL = FCL \xrightarrow{\lambda L} FL;$$

in particular, RCL is a D-functor and $\bar{\delta}: D \to RCL$ is a map of D-functors.

(iii) *If (Z, ξ) is a C-object, then RZ is a D-object with action*

$$DRZ \xrightarrow{\bar{\delta}R} RCLRZ = RCZ \xrightarrow{R\xi} RZ;$$

in particular, for $Y \in \mathscr{W}$, RCLY is a D-object and $\bar{\delta}: DY \to RCLY$ is a map of D-objects.

The proof of (i) is elementary and (ii) and (iii) follow via (ii) and (iii) of the previous result. The hypothesis on C can usually be verified from the structure of D. Precisely, we have the following result.

Proposition 5.3. *Let $C = LDR$ for a monad (D, ν, ζ) in $\mathscr{V}$. Consider*

$$\delta DR = \bar{\delta}R : DR \to RC \quad \text{and} \quad LD\delta = L\bar{\delta} : LD \to CL.$$

If either of these is a natural isomorphism, then C is a monad in $\mathscr{V}$ with unit η and product μ the composites

$$\mathrm{Id} = LR \xrightarrow{L\zeta R} LDR \quad \text{and} \quad LDRLDR \xrightarrow{(LD\delta DR)^{-1}} LDDR \xrightarrow{L\nu R} LDR.$$

Moreover, if (RZ, ψ) is a D-object, then Z is a C-object with action

$$L\psi : CZ = LDRZ \to LRZ = Z,$$

and C-objects in $\mathscr{V}$ may be identified with D-objects of the form RZ in $\mathscr{W}$. In the case when $LD\delta$ is an isomorphism, if (Y,ψ) is a D-object, then LY is a C-object with action

$$CLY \xrightarrow{(LD\delta)^{-1}} LDY \xrightarrow{L\psi} LY$$

and $\delta: Y \to RLY$ is a morphism of D-objects.

Proof. We are defining η and μ by the diagrams of the previous result, but, for the associativity of μ, it would not be enough just to assume that $LD\delta DR$ is an isomorphism. For the second statement, the correspondences $(RZ,\psi) \mapsto (Z, L\psi)$ and $(Z,\xi) \mapsto (RZ, R\xi \circ \bar{\delta}R)$ between actions are mutually inverse.

The essential point is that the hypothesis $DR \cong RC$ or $LD \cong CL$ is a computational property of D which may or may not hold in practice. We shall encounter all possible variants, and the basic structure of our theory is largely determined by the behavior of the monads we shall encounter with respect to the adjunctions of Definition 4.1.

The previous results have the following consequences, special cases of which will lead to the proofs of Theorems 4.3, 4.6, and 4.8.

Corollary 5.4. *If $\bar{\delta}R: DR \to RC$ is an isomorphism in the previous proposition, then*

$$B_*(GR, C, Z) \cong B_*(G, D, RZ)$$

for a C-object Z and D-functor G, hence

$$B_*(F, C, Z) \cong B_*(FL, D, RZ)$$

for a C-functor F. If $L\bar{\delta}: LD \to CL$ is an isomorphism, then

$$B_*(F, C, LY) \cong B_*(FL, D, Y)$$

for a D-object Y, LY being a C-object via $CLY \cong LDY \to LY$.

Corollary 5.5. *Under the hypotheses of Proposition 5.2, there is a natural diagram*

$$Y_* \xleftarrow{\varepsilon_*} B_*(D, D, Y) \xrightarrow{\delta_*} RB_*(CL, D, Y)$$

of simplicial D-objects, where Y is a D-object and Y_* denotes the constant simplicial D-object which is Y in each degree. If $\bar{\delta}R: DR \to RC$ is an isomorphism and $Y = RZ$ for a C-object Z, then this diagram is obtained by application of R to a natural diagram

$$Z_* \xleftarrow{\varepsilon_*} B_*(C, C, Z) \cong B_*(CL, D, RZ)$$

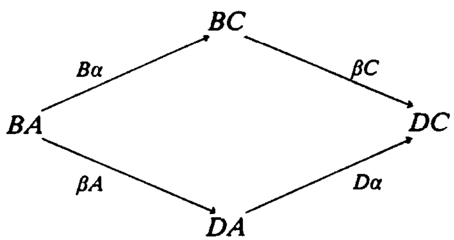
of simplicial C-objects. Moreover, the ε_* are simplicial homotopy equivalences (in the respective ground categories $\mathscr{W}$ and $\mathscr{V}$) with natural homotopy inverses τ_*.

Proof. The assertions about ε_* are general nonsense [8, 9.8]. The map δ_* is $B_*(\bar{\delta}, 1, 1)$, and we have used the evident relation

$$B_*(RCL, D, Y) = RB_*(CL, D, Y).$$

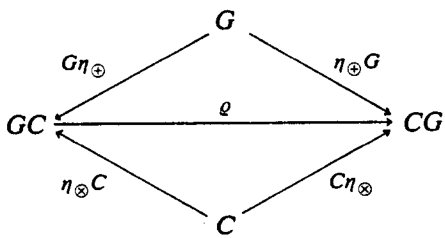
The rest is immediate modulo a little diagram chasing for the assertion that δ_* is a map of simplicial D-objects.

We are particularly interested in the relationship between 'additive' and 'multiplicative' monads defined on the same ground category, and we shall have three different applications of the following result of Beck [1]. Given a monad G in $\mathscr{V}$, let $G[\mathscr{V}]$ denote the category of G-objects in $\mathscr{V}$. Recall that the composite $\beta\alpha$ of natural transformations $\alpha : A \to C$ and $\beta : B \to D$ such the range category of A and C is the domain category of B and D is defined to be the common composite in the diagram

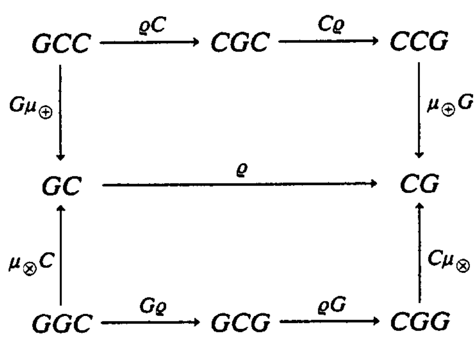


Proposition 5.6. *Let $(C, \mu_\oplus, \eta_\oplus)$ and $(G, \mu_\otimes, \eta_\otimes)$ be monads in the same category $\mathscr{V}$. Then the following distributivity data relating C and G are equivalent.*

(i) *A natural transformation $\varrho : GC \to CG$ such that the following diagrams of functors commute:*



and



(ii) *A natural transformation $\mu : CGCG \to CG$ such that the following properties hold:*

(a) *(CG, μ, η) is a monad in $\mathscr{V}$, where $\eta = \eta_\oplus \eta_\otimes : \mathrm{Id} \to CG$.*

(b) *$C\eta_\otimes : C \to CG$ and $\eta_\oplus G : C \to CG$ are morphisms of monads.*

(c) *The composite*

$$CG \xrightarrow{C\eta_\otimes \eta_\oplus G} CGCG \xrightarrow{\mu} CG$$

is the identity.

(iii) *A structure of monad in $G[\mathscr{V}]$ on C; that is, for a G-action $GX \to X$, a functorial induced G-action $GCX \to CX$ (and thus $GCCX \to CCX$ by iteration) such that $\eta_\oplus : X \to CX$ and $\mu_\oplus : CCX \to CX$ are morphisms of G-objects in $\mathscr{V}$.*

When given such data, the category of CG-objects in $\mathscr{V}$ is isomorphic to the category of C-objects in $G[\mathscr{V}]$.

Proof. We outline the argument. The verifications of the claims to follow are lengthy, but straightforward, diagram chases [1, p. 120–128]. Given ϱ as in (i), define μ to be the composite

$$CGCG \xrightarrow{C\varrho G} CCGG \xrightarrow{\mu_\oplus \mu_\otimes} CG.$$

Given μ as in (ii), define ϱ to be the composite

$$GC \xrightarrow{\eta_\oplus GC\eta_\otimes} CGCG \xrightarrow{\mu} CG.$$

These are inverse correspondences between the data of (i) and the data of (ii). Given ϱ as in (i) and given $(X, \xi) \in \mathscr{G}[\mathscr{V}]$, the composite

$$GCX \xrightarrow{\varrho} CGX \xrightarrow{C\xi} CX$$

specifies an action of G on CX as required in (iii). Given the data of (iii), let ϱ be the composite

$$GC \xrightarrow{GC\eta_\otimes} GCG \longrightarrow CG,$$

where, for $X \in \mathscr{V}$, the second arrow is the action of G on CGX induced by the action $\mu_\otimes$ of G on GX. These are inverse correspondences between the data of (i) and the data of (iii). Suppose given these equivalent data. Then a CG-object (X, ψ) in $\mathscr{V}$ determines a C-object (X, ξ, θ) in $G[\mathscr{V}]$ by letting ξ and θ be the pullback actions

$$GX \xrightarrow{\eta_\oplus G} CGX \xrightarrow{\psi} X \quad \text{and} \quad CX \xrightarrow{C\eta_\otimes} CGX \xrightarrow{\psi} X,$$

and a C-object (X, ξ, θ) in $G[\mathscr{V}]$ determines a CG-object (X, ψ) in $\mathscr{V}$ by letting ψ be the composite

$$CGX \xrightarrow{C\xi} CX \xrightarrow{\theta} X.$$

These correspondences are inverse isomorphisms of categories.

The following generalization of [22, 5.1] gives a generic categorical source for the monads we need.

Construction 5.7. Let $\mathscr{G}$ be a topological category and let Ξ be a discrete subcategory with the same objects. Let $[\Xi, \mathscr{U}]$ denote the category of functors $\Xi \to \mathscr{U}$ and natural transformations between them. We construct a monad (D, ν, ζ) in $[\Xi, \mathscr{U}]$ such that a continuous functor $\mathscr{G} \to \mathscr{U}$ is the same thing as a D-object in $[\Xi, \mathscr{U}]$. For an object $n \in \Xi$, $(DX)(n)$ is the coend

$$\int_{\Xi} \mathscr{G}(m,n) \times X(m) = \coprod_m \mathscr{G}(m,n) \times X(m)/(\sim),$$

where the equivalence relation is specified by

$$(\phi\psi, x) \sim (\phi, \psi x)$$

for morphisms $\phi : m \to n$ in $\mathscr{G}$ and $\psi : k \to m$ in Ξ and points $x \in X(k)$. Technically, this description of the coend is not valid without point-set topological assumptions designed to ensure that the quotient space lies in $\mathscr{U}$; we tacitly assume such hypotheses. Composition on the left by maps $n \to p$ in Ξ gives maps $(DX)(n) \to (DX)(p)$ such that DX is a functor $\Xi \to \mathscr{G}$. The identity maps and composition of the category $\mathscr{G}$ induce maps

$$\zeta : X(n) \to (DX)(n) \quad \text{and} \quad \nu : (DDX)(n) \to (DX)(n),$$

and these specify natural transformations of functors $\Xi \to \mathscr{U}$. The monad identities are inherited from the category axioms. If $X : \Xi \to \mathscr{U}$ extends to a continuous functor $\mathscr{G} \to \mathscr{U}$, then the evaluation maps $\mathscr{G}(m,n) \times X(m) \to X(n)$ induce maps $\xi : (DX)(n) \to X(n)$ which specify an action of D on X. Conversely, if $\xi : DX \to X$ satisfies the identities required for an action of D on X, then the composites

$$\mathscr{G}(m,n) \times X(m) \to (DX)(n) \xrightarrow{\xi} X(n)$$

are the evaluation maps of a continuous functor $\mathscr{G} \to \mathscr{U}$.

To illustrate the combination of this construction with the theory at the beginning of the section, suppose given an operad $\mathscr{C}$ with associated category of operators $\hat{\mathscr{C}}$. We agree to let $[\Pi, \mathscr{T}]$ denote the category of functors $Y : \Pi \to \mathscr{T}$ which satisfy the cofibration condition required of Π-spaces. Since 0 is an initial object of Π, the basepoints of all Y_n are determined by that of Y_0. Then our construction specializes to give a monad $(\hat{C}, \hat{\mu}, \hat{\eta})$ in $[\Pi, \mathscr{T}]$. The cofibration condition ensures that each $\hat{C}_n Y$ lies in $\mathscr{T}$, and it is easy to see that this condition is also satisfied by $\hat{C}Y$.

On the other hand, [12, §2] gave a more combinatorial specification of a monad (C, μ, η) in $\mathscr{T}$. As we shall need the details later, we recall the definition of the spaces CZ.

Construction 5.8. Let $\Lambda \subset \Pi$ be the subcategory of injections. Thus Λ is generated by permutations and by the degeneracy operators $\sigma_q : \mathbf{r} \to \mathbf{1} \to \mathbf{r}$, $1 \le q \le r$, specified by $\sigma_q(i) = i$ for $i < q$ and $\sigma_q(i) = i+1$ for $i \ge q$. Via their permutations and the degeneracy operators $\sigma_q : \mathscr{C}(r) \to \mathscr{C}(r-1)$ specified by

$$c\sigma_q = \gamma(c;\, 1^{q-1} \times * \times 1^{r-q}), \quad \text{where } 1 \in \mathscr{C}(1) \text{ and } * \in \mathscr{C}(0),$$

operads determine contravariant functors $\Lambda \to \mathscr{U}$. Then

$$CZ = \int_{\Lambda} \mathscr{C}(r) \times Z^r = \coprod_r \mathscr{C}(r) \times_{\Sigma_r} Z^r/(\sim),$$

where the equivalence relation is generated by

$$(c, z_1, \ldots, z_r) \sim (c\sigma_q, z_1, \ldots, z_{q-1}, z_{q+1}, \ldots, z_r) \quad \text{if } z_q = 0.$$

Analysis of these monads requires two pieces of information, which are given in [22, 5.7 and 5.6] and are verified by direct inspection of the constructions. The first asserts that the monad C is entirely determined by the monad $\hat{C}$ and the adjunction (L', R') relating $\mathscr{T}$ to $[\Pi, \mathscr{T}]$.

Lemma 5.9. *$L\hat{C}R' = C$ and $\delta\hat{C}R' : \hat{C}R' \to R'C$ is a natural isomorphism. Moreover, the unit and product of C are those specified in Proposition* 5.3.

This means that $\hat{C}_n R'Z \cong (CZ)^n$. Note that $L'\hat{C} \not\cong CL'$. This result entitles us to all the rest of the conclusions of Propositions 5.2 and 5.3 and their corollaries. The second piece of information is an invariance statement. It is a consequence of the way the spaces $\hat{C}_n Y$ are built up by successive cofibrations.

Lemma 5.10. *Assume that $\mathscr{C}$ is Σ-free. If Y and Y' are in $[\Pi, \mathscr{T}]$ and $f : Y \to Y'$ is a natural transformation such that each $f_n : Y_n \to Y'_n$ is an equivalence, then each $\hat{C}_n f : \hat{C}_n Y \to \hat{C}_n Y'$ is also an equivalence.*

The conclusion of the lemma is necessary for the validity of the following result and thus for the utility of the monad $\hat{C}$.

Corollary 5.11. *If $\mathscr{C}$ is Σ-free, then $\hat{C}$ restricts to a monad in the category $\Pi[\mathscr{T}]$ of Π-spaces.*

Proof. It must be shown that each $\delta' : \hat{C}_n Y \to (\hat{C}_1 Y)^n$ is an equivalence when each $\delta' : Y_n \to Y_1^n$ is an equivalence. This is immediate by specialization of the diagram defining δ in Proposition 5.2 to $D = \hat{C}$. Indeed, its top and bottom are equivalences by application of the previous lemma to $\delta' : Y \to RLY'$, its right side is an isomorphism by Lemma 5.9, and its left side is given by the maps in question.

6. The monads associated to categories of ring operators

Assume given an operad pair $(\mathscr{C}, \mathscr{G})$ and let $\mathscr{J} = \hat{\mathscr{G}} \int \hat{\mathscr{C}}$. We agree to let $[\Pi \int \Pi, \mathscr{U}]$ denote the category of functors $X : \Pi \int \Pi \to \mathscr{U}$ which come with a nondegenerate basepoint $* \in X(0; *)$ and which satisfy the cofibration condition required of $\Pi \int \Pi$-spaces in Definition 2.1(5). Then Construction 5.7 specializes to give a monad $(\bar{J}, \bar{\mu}, \bar{\eta})$ in $[\Pi \int \Pi, \mathscr{U}]$ associated to $\mathscr{J}$. The cofibration condition ensures that each $\bar{J}X(n; S)$ lies in $\mathscr{U}$ (by the combinatorial analysis in the next section). Since the action of $\Pi \int \Pi$ on $\bar{J}X$ is induced by composition $\Pi \int \Pi \times \mathscr{J} \to \mathscr{J}$, it is not hard to check that $\bar{J}X$ also satisfies the cofibration condition.

As in Definition 4.1, we have adjunctions (L'', R'') relating $[\Pi, \mathscr{T}]_e$ to $[\Pi \int \Pi, \mathscr{U}]$ and (L', R') relating $\mathscr{T}_e$ to $[\Pi, \mathscr{T}]$ and a composite adjunction (L, R). Here $[\Pi, \mathscr{T}]_e$ denotes the category of functors $Y : \Pi \to \mathscr{T}$ in $[\Pi, \mathscr{T}]$ which come with a nondegenerate basepoint $1 \in Y_1$ which maps to $0 \in Y_0$ and which satisfy the cofibration condition above Lemma 2.2.

We aim to understand the relationship of $\bar{J}$ to these adjunctions and to the various additive and multiplicative monads in sight. In particular, we shall prove Theorem 4.3 (and Lemma 4.7) and shall lay the groundwork for the proofs of Theorems 4.4 and 4.8. Formal proofs are given in this section. Proofs which depend on detailed analysis of the construction of $\bar{J}$ are deferred until the following section, and we begin with statements of results of the latter type. The following result is crucial.

Theorem 6.1. *Define $\hat{J} = L''\bar{J}R''$ and $J = L'\hat{J}R' = L\bar{J}R$. Then $\delta''\bar{J}R'' : \bar{J}R'' \to R''\hat{J}$ is a natural isomorphism. In general, the remaining natural transformations $L''\bar{J} \to \hat{J}L''$, $\hat{J}R' \to R'J$, and $L'\hat{J} \to JL'$ are not isomorphisms.*

The first statement means that $\bar{J}R''Y(n; S) \cong \prod_{j=1}^{n} \hat{J}_{s_j} Y$. It entitles us to all of the conclusions of Propositions 5.2 and 5.3 and their corollaries. In particular, $\hat{J}$ is a monad and $\hat{J}$-spaces are the same things as $(\hat{\mathscr{C}}, \mathscr{G})$-spaces (modulo homotopy type conditions to be discussed shortly). This is analogous to Lemma 5.9, the essential difference being that there we had a preassigned monad C on hand, whereas here $\hat{J}$ is a new construction. We shall obtain a precise description of $\hat{J}$ in Corollary 7.3.

The second statement dictates the central role played by $(\hat{\mathscr{C}}, \mathscr{G})$-spaces in our theory. In fact, $\hat{J}_n R'Z$ is not even equivalent to $(JZ)^n$ in general. Therefore, $\bar{J}RZ$ is not equivalent to RJZ. It is this fact which obstructs the direct replacement of $\mathscr{J}$-spaces by equivalent $(\mathscr{C}, \mathscr{G})$-spaces. We shall explain the relationship of the functor J to the notion of a $(\mathscr{C}, \mathscr{G})$-space in Proposition 6.11.

The following invariance result is analogous to Lemma 5.10, but the way in which the spaces $\bar{J}X(n; S)$ are built up from successive cofibrations is far more elaborate.

Theorem 6.2. *Assume that $\mathscr{C}$ and $\mathscr{G}$ are Σ-free. If X and X' are in $[\Pi \int \Pi, \mathscr{U}]$ and $f : X \to X'$ is a natural transformation such that each $f(n; S) : X(n; S) \to X'(n; S)$ is* *an equivalence, then each $Jf(n;S):JX(n;S)\to JX'(n;S)$ is also an equivalence. The analogous assertion for $\hat{J}$ is also valid.*

Again, by specialization of the diagram defining $\bar{\delta}$ in Proposition 5.2 to $D=J$ and the adjunction (L'',R''), the following result is an immediate consequence of the preceding theorems. Let us say that $X\in[\Pi\int\Pi,\mathscr{U}]$ is a semi $\Pi\int\Pi$-space if $\delta'':X\to R''L''X$, but not necessarily $\delta:X\to RLX$, is an equivalence.

Corollary 6.3. *If $\mathscr{C}$ and $\mathscr{G}$ are Σ-free, then J restricts to a monad in the category of semi $\Pi\int\Pi$-spaces. That is, $\delta'':JX\to R''L''JX$ is an equivalence if $\delta'':X\to R''L''X$ is an equivalence.*

We do not claim, and it is not true, that JX is a $\Pi\int\Pi$-space if X is a $\Pi\int\Pi$-space. Nor is it true that $\hat{J}Y$ is a Π-space if Y is a Π-space.

We need one more preliminary to prove Theorem 4.3, namely the following analog of [12, 12.2]. The proof is essentially the same as that of the cited result and will therefore be omitted.

Theorem 6.4. *For simplicial objects X in $[\Pi\int\Pi,\mathscr{U}]$, there is a natural isomorphism $\nu:|JX|\to J|X|$ in $[\Pi\int\Pi,\mathscr{U}]$ such that the following diagrams commute:*

$$
\begin{array}{ccc}
 & & |JX| \\
 & {\scriptstyle |\bar\eta|}\nearrow & \downarrow{\scriptstyle \nu} \\
|X| & & \\
 & {\scriptstyle \bar\eta}\searrow & \\
 & & J|X|
\end{array}
\quad \textit{and} \quad
\begin{array}{ccc}
|JJX| & \xrightarrow{J\nu\cdot\nu} & JJ|X| \\
{\scriptstyle |\bar\mu|}\downarrow & & \downarrow{\scriptstyle \bar\mu} \\
|JX| & \xrightarrow{\nu} & J|X|
\end{array}
$$

If (X,ξ) is a simplical J-space, then $(|X|,|\xi|\nu^{-1})$ is a J-space. The analogous assertions for $\hat{J}$ are also valid.

We can now define the functors needed to prove Theorem 4.3.

Definition 6.5. For a $\mathscr{J}$-space (or J-space) X, define

$$UX=B(J,J,X)\quad\text{and}\quad VX=B(\hat{J}L'',J,X).$$

For a $(\mathscr{C},\mathscr{G})$-space (or $\hat{J}$-space) Y, define

$$UY=B(\hat{J},\hat{J},Y)\cong VR''Y.$$

Here UX is a $\mathscr{J}$-space and VX and UY are $(\mathscr{C},\mathscr{G})$-spaces (the following paragraph implying the requisite homotopy type conditions irrespective of the fact that the functors J and $\hat{J}$ fail to preserve $\Pi\int\Pi$-spaces and Π-spaces).

We recall that geometric realization preserves products and (weak) equivalences and carries simplicial homotopies to homotopies (see [12, §11], [13, A.4], and [2, App. 4.8]). In particular, $|Y|$ is a Π-space if Y is a simplicial Π-space, $|R''Y| \cong R''|Y|$, and $|X|$ is a (semi) $\Pi\int\Pi$-space if X is a simplicial (semi) $\Pi\int\Pi$-space. Theorem 4.3 is now an immediate consequence of Corollaries 5.4 and 5.5. In the latter result, δ_* is an equivalence since $\bar{\delta}'': \bar{J}X \to R''\hat{J}L''X$ is an equivalence for any $\Pi\int\Pi$-space X.

The additive and multiplicative specializations of J play special roles, and here Theorem 6.1 can be improved. Let $\bar{C}$ denote the monad in $[\Pi\int\Pi, \mathscr{U}]$ associated to $\Pi\int\mathscr{C}$ and recall the inclusion of $\hat{\mathscr{C}}$ in $\Pi\int\mathscr{C}$ from Lemma 1.5.

Proposition 6.6. *$L''\bar{C}R'' = \hat{C}$ and both $\delta''\bar{C}R'' : \bar{C}R'' \to R''\hat{C}$ and $L''\bar{C}\delta'' : L''\bar{C} \to \hat{C}L''$ are natural isomorphisms. Therefore $L\bar{C}R = C$ and $\delta\bar{C}R : \bar{C}R \to RC$ is a natural isomorphism.*

Of course, $L\bar{C} \not\cong CL$ since $L'\hat{C} \not\cong CL'$. As in Lemma 2.2, $L''X = X_{\oplus}$ is the $\hat{\mathscr{C}}$-space associated to a $\Pi\int\mathscr{C}$-space X; its $\hat{C}$-action is just the composite $\hat{C}(X_{\oplus}) \cong (\hat{C}X)_{\oplus} \to X_{\oplus}$.

The following observations should help clarify the multiplicative specialization and will lead to the proof of Lemma 4.7.

Remark 6.7. For $Z \in \mathscr{T}_e$, we have two different derived functors $\Pi \to \mathscr{T}$ with nth space Z^n. The first, $R'Z$, has degeneracies (i.e. actions by injections in Π) defined with respect to 0. The second, which we denote by $R'_{\otimes}Z$, has degeneracies defined with respect to 1. Recall the notation $X_{\otimes}$ from Lemma 2.2. For $Y \in [\Pi, \mathscr{T}]_e$ we have $(R''Y)_{\otimes} = R'_{\otimes}Y_1 : \Pi \to \mathscr{T}$; indeed both of these functors have nth space Y_1^n and have degeneracies defined with respect to 1.

If Y is a $(\hat{\mathscr{C}}, \mathscr{G})$-space, then $(R''Y)_{\otimes}$ is a $\hat{\mathscr{G}}$-space by Lemma 2.2. By application of Proposition 5.3 and Lemma 5.9 to $(L', R'_{\otimes})$ and $\hat{G}$, this structure determines and is determined by a structure of $\mathscr{G}$-space on Y_1. This proves all but the last statement of Lemma 4.7. For that we need a slight variant of Construction 5.8.

Construction 6.8. For $Z \in \mathscr{T}_e$, construct GZ by use of the operad $\mathscr{G}$ and basepoint 1 and define G_0Z to be the quotient space of GZ obtained by identifying all points any of whose coordinates in Z are 0. Observe that G_0 gives a monad in $\mathscr{T}_e$ such that a G_0-space is the same thing as a G-space with zero.

Now let $\bar{G}$ denote the monad in $[\Pi\int\Pi, \mathscr{U}]$ associated to $\hat{\mathscr{G}}\int\Pi$.

Proposition 6.9. *Define $L''\bar{G}R'' = \tilde{G}$. Then $L'\tilde{G}R' = G_0$ on $\mathscr{T}_e$ and $L'\tilde{G}\delta' : L'\tilde{G} \to G_0L'$ is a natural isomorphism on the full subcategory of those $Y \in [\Pi, \mathscr{T}]_e$ such that $Y_0 = \{0\}$. Thus, for such $Y, \tilde{G}_1Y \cong G_0Y_1$ and therefore*

$$(\bar{G}R''Y)_{\otimes} \cong (R''\tilde{G}Y)_{\otimes} = R'_{\otimes}\tilde{G}_1Y \cong R'_{\otimes}G_0Y_1.$$

Of course, $\bar{G}R'' \cong R''\bar{G}$ by Theorem 6.1. However, neither $L''\bar{G} \cong \tilde{G}L''$ nor $\tilde{G}R' \cong R'G_0$. The last statement implies the last statement of Lemma 4.7. Propositions 5.2 and 5.3 and their corollaries apply both to (L'', R'') and $\bar{G}$ and to (L', R') and $\tilde{G}$ restricted to objects Y with $Y_0 = \{0\}$. In particular, $\tilde{G}$ is a monad in $[\Pi, \mathscr{T}]_e$ such that a $\tilde{G}$-space is the same thing as a $\bar{G}$-space of the form $R''Y$. Here $\tilde{G}$ must not be confused with the monad $\hat{G}$ in $[\Pi, \mathscr{T}]$. The former relates to the 'additive' inclusion of Π in $\hat{\mathscr{G}}\int\Pi$. The latter relates to the 'multiplicative' inclusion of $\hat{\mathscr{G}}$ in $\hat{\mathscr{G}}\int\Pi$.

Remarks 6.10. When $Y_0 \neq \{0\}$, $\tilde{G}_1 Y \not\cong G_0 Y_1$. It is for this and related reasons that the hypothesis $Y_0 = \{0\}$ occurs in Section 4. One can generalize the previous result to all Y by replacing (L', R') by an adjunction relating $[\Pi, \mathscr{T}]_e$ to the category $\mathscr{T}_2$ of retraction diagrams

$$Z_0 \xrightarrow{\iota} Z \xrightarrow{\varrho} Z_0, \quad \varrho\iota = \mathrm{id},$$

with $Z_0 \in \mathscr{T}$, $Z \in \mathscr{T}_e$, $\iota(0) = 0$, $\varrho(0) = 0 = \varrho(1)$, and ι a cofibration. Taking $Z_0 = Y_0$ and $Z = Y_1$, we see that any $Y \in [\Pi, \mathscr{T}]_e$ determines a retraction diagram $L_2'Y$, and it is easy to check that $R' : \mathscr{T}_e \to [\Pi, \mathscr{T}]_e$ extends to a right adjoint $R_2' : \mathscr{T}_2 \to [\Pi, \mathscr{T}]_e$ to L_2'. One can generalize G_0 to a functor $G_0 : \mathscr{T}_2 \to \mathscr{T}_2$ such that $L_2'\tilde{G}Y = G_0 L_2' Y$ for all $Y \in [\Pi, \mathscr{T}]_e$.

So far in this section, we have concentrated on the relationship between monads defined on different ground categories: spaces, Π-spaces, and $\Pi\int\Pi$-spaces. To recapitulate, we have functors

$$\begin{array}{ll} \bar{J},\ \bar{C}, \text{ and } \bar{G} & \text{on } [\Pi\int\Pi, \mathscr{U}], \\ \tilde{J},\ \tilde{C}, \text{ and } \tilde{G} & \text{on } [\Pi, \mathscr{T}]_e, \\ J,\ C, \text{ and } G_0 & \text{on } \mathscr{T}_e. \end{array}$$

We agree to ignore $\hat{G}$, which plays no further role in this section. The second row is obtained from the first via L'' ? R''. The third row is obtained from the second via L' ? R' and is therefore also obtained from the first via L ? R. All of these functors except J are monads.

In the rest of this section, we are concerned with the relationship between the functors on a given row. In the notion of a $\mathscr{J}$-space, the additive and multiplicative products are intertwined. The central feature of the earlier theory of $(\mathscr{C}, \mathscr{G})$-spaces was a reinterpretation in which the multiplicative products were subsumed in the structure of the ground category. A similar reformulation of the notion of a $(\hat{\mathscr{C}}, \mathscr{G})$-space will be central to the proofs of Theorems 4.4 and 4.8, and we shall also give such a reformulation of the notion of a $\mathscr{J}$-space.

We first reconsider the notion of a $(\mathscr{C}, \mathscr{G})$-space. Looking at Definition 2.5, we see that a $(\mathscr{C}, \mathscr{G})$-space is a C-space and a G_0-space such that a certain parametrized distributivity diagram commutes. According to [14, VI §1], the left-hand maps ξ in that diagram induce an action of $\mathscr{G}$ on CZ such that C restricts to a monad in the category $G_0[\mathscr{T}_e]$ of G_0-spaces, and a $(\mathscr{C}, \mathscr{G})$-space is just a C-object in $G_0[\mathscr{T}_e]$. That is, the distributivity diagram asserts that $\theta: CZ \to Z$ is a map of G_0-spaces.

Proposition 5.6 leads to an alternative proof of this that both places the functor J in perspective and suggests the proper generalization to $(\mathscr{C}, \mathscr{G})$-spaces and $\mathscr{J}$-spaces.

Proposition 6.11. *For $Z \in \mathscr{T}_e$, there are natural maps*

$$\varrho': G_0CZ \to JZ \quad \text{and} \quad \varrho'': JZ \to CG_0Z$$

whose composite $\varrho: G_0C \to CG_0$ satisfies the conditions of (i) *of Proposition* 5.6. *Therefore C is a monad in $G_0[\mathscr{T}_e]$, CG_0 is a monad in $\mathscr{T}_e$, and a CG_0-space determines and is determined by a C-object in the category of G_0-spaces, that is to say, by a $(\mathscr{C}, \mathscr{G})$-space.*

Proof. We shall be a bit schematic in the definitions. We have

$$G_0CZ = \coprod_{(m,R)} \mathscr{G}(m) \times \mathscr{C}(r_1) \times Z^{r_1} \times \cdots \times \mathscr{C}(r_m) \times Z^{r_m}/(\sim),$$

$$JZ = \coprod_{(m,R)} \mathscr{G}(m) \times \mathscr{C}(R,1) \times Z^{r_1} \times \cdots \times Z^{r_m}/(\sim),$$

$$CG_0Z = \coprod_{(n,S)} \mathscr{C}(n) \times \mathscr{G}(s_1) \times Z^{s_1} \times \cdots \times \mathscr{G}(s_n) \times Z^{s_n}/(\sim).$$

Here $R = (r_1, \dots, r_m)$, $S = (s_1, \dots, s_n)$, and $\mathscr{C}(R, 1)$ is specified in Notations 7.1 below; it is of the form $\coprod \mathscr{C}(|\chi^{-1}(1)|)$, where the union is taken over certain morphisms $\chi: \wedge_{i=1}^m \mathbf{r}_i \to \mathbf{1}$ in $\mathscr{T}$, and all we really need to know is that $\chi = \phi_{r_1 \cdots r_m}$, with $|\chi^{-1}(1)| = r_1 \cdots r_m$, is allowed. The equivalence relation for JZ is specified in Corollary 7.3. The map ϱ' is obtained by passage to quotients from

$$\varrho'(g, c_1, y_1, \dots, c_m, y_m) = (g, \lambda(g; c_1, \dots, c_m), y_1, \dots, y_m),$$

where $g \in \mathscr{G}(m)$, $c_i \in \mathscr{C}(r_i)$, and $y_i \in Z^{r_i}$. The map ϱ'' is obtained by passage to quotients from

$$\varrho''(g, c, y_1, \dots, y_m) = \left(c, \bigtimes_{\chi(U)=1} (g, Y_U)\right),$$

where $g \in \mathscr{G}(m)$, $c \in \mathscr{C}(|\chi^{-1}(1)|)$, $y_i = (y_{i,1}, \dots, y_{i,r_i}) \in Z^{r_i}$, U runs through the sequences $(u_1, \dots, u_m)$ with $1 \le u_i \le r_i$ and $\chi(U) = 1$, and $y_U = (y_{1,u_1}, \dots, y_{m,u_m})$. Thus

$$\varrho(g, c_1, y_1, \dots, c_m, y_m) = \left(\lambda(g; c_1, \dots, c_m), \bigtimes_U (g, y_U)\right).$$

The diagrams of Proposition 5.6 commute by the formulas required of λ in Definition 1.8; compare [14, p. 145].

Observe the role played in the definition of ϱ'' by the diagonal maps implicit in the notion of distributivity. On the $\mathscr{J}$-space level, these are incorporated into the construction of the $JX(n; S)$. One therefore has the following more conceptually satisfying result, which works for any categories of operators $\hat{\mathscr{C}}$ and $\hat{\mathscr{G}}$ such that $\hat{\mathscr{G}}$ acts on $\hat{\mathscr{C}}$. Observe that the inclusions of $\Pi \wr \hat{\mathscr{C}}$ and $\hat{\mathscr{G}} \wr \Pi$ in $\mathscr{J}$ induce morphisms $\bar{C} \to J$ and $\bar{G} \to J$ of monads in $[\Pi \wr \Pi, \mathscr{U}]$.

Theorem 6.12. *For $X \in [\Pi \wr \Pi, \mathscr{U}]$, there is a natural transformation*

$$\bar{\varrho} : \bar{G}\bar{C}X \to \bar{C}\bar{G}X$$

of functors $\Pi \wr \Pi \to \mathscr{U}$ such that $\bar{\varrho}$ is natural in X and satisfies the conditions of (i) *of Proposition 5.6. Therefore $\bar{C}$ is a monad in $\bar{G}[\Pi \wr \Pi, \mathscr{U}]$, $\bar{G}\bar{C}$ is a monad in $[\Pi \wr \Pi, \mathscr{U}]$, and a $\bar{C}\bar{G}$-space determines and is determined by a $\bar{C}$-object in the category of $\bar{G}$-spaces. Moreover, the diagram*

$$\begin{array}{ccccc}
\bar{G}\bar{C}X & & \xrightarrow{\quad\bar{\varrho}\quad} & & \bar{C}\bar{G}X \\
\downarrow & & & & \downarrow \\
JJX & \xrightarrow{\bar{\mu}} & JX & \xleftarrow{\bar{\mu}} & JJX
\end{array}$$

commutes, and the composite

$$\bar{C}\bar{G} \longrightarrow JJ \xrightarrow{\bar{\mu}} J$$

is an isomorphism of monads in $[\Pi \wr \Pi, \mathscr{U}]$.

Proof. The diagram dictates the definition of $\bar{\varrho}$. With the notations of Lemma 1.6, its (p, T)th map is obtained by passage to quotients from the maps

$$\begin{array}{c}
(\hat{\mathscr{G}} \wr \Pi)((n; S), (p; T)) \times (\Pi \wr \hat{\mathscr{C}})((m; R), (n; S)) \times X(m; R) \\
\Big\downarrow {\scriptstyle \bar{\varrho}} \\
(\Pi \wr \hat{\mathscr{C}})((p; R''), (p; T)) \times (\hat{\mathscr{G}} \wr \Pi)((m; R), (p; R'')) \times X(m; R)
\end{array}$$

specified by

$$\bar{\varrho}((g; \chi), (\phi; c), x) = ((1; \chi \circ \lambda(g)(c)), (g\phi; \sigma(\psi, \phi)), x).$$

The diagram is then just a reinterpretation of the cited lemma. If $(g; \chi) = (1; 1)$, then the right side is

$$((1; c), (\phi; 1), x) \sim ((\phi; c), (1; 1), x).$$

If $(\phi; c) = (1; 1)$, then the right side is

$$((1; \chi), (g; 1), x) \sim ((1; 1), (g; \chi), x).$$

These give the diagrams involving $\eta_{\oplus}$ and $\eta_{\otimes}$ of Proposition 5.6(i), and the diagrams involving $\mu_{\oplus}$ and $\mu_{\otimes}$ are verified by formal calculations based on the definition of composition in $\mathscr{G}\int\Pi$ and $\Pi\int\mathscr{C}$, the definition of $\bar{\varrho}$, the naturality of $\sigma(\psi,\phi)$, the functoriality of $\lambda(g)$, and the equivalence relation used to define $\bar{C}\bar{G}X$; compare Definitions 1.2 and 1.4. Routine diagram chases show that the specified composite $\bar{C}\bar{G}\to J$ is a morphism of monads. Since $\mathscr{G}\int\Pi$ and $\Pi\int\mathscr{C}$ generate $\mathscr{J}$ under composition, the component maps $(\bar{C}\bar{G}X)(n;\mathbf{S})\to JX(n;\mathbf{S})$ are obviously surjective. In view of the amalgamations over $\Pi\int\Pi$, they are easily checked to be homeomorphisms. A more conceptual proof that $\bar{C}\bar{G}\cong J$ is also possible. Clearly J-spaces are $\bar{C}\bar{G}$-spaces by pullback. Conversely, suppose that X is a $\bar{C}\bar{G}$-space (in $[\Pi\int\Pi,\mathscr{U}]$). For a typical morphism $(g;c):(m;\mathbf{R})\to(n;\mathbf{S})$ in $\mathscr{J}$, we have the factorization

$$(g;c)=(1;c)(g;1^n):(m;\mathbf{R})\to(n;\mathbf{R}')\to(n;\mathbf{S}),$$

where $\mathbf{r}'_j=\bigwedge_{\phi(i)=j}\mathbf{r}_i$ if $\varepsilon(g)=\phi$. Since a $\bar{C}\bar{G}$-space is a $\bar{C}$-space and a $\bar{G}$-space, we can use this factorization to define $(g;c):X(m;\mathbf{R})\to X(n;\mathbf{S})$. Direct calculation shows that this specifies X as a functor $\mathscr{J}\to\mathscr{U}$ and that we may identify J-spaces and $\bar{C}\bar{G}$-spaces. For $X\in[\Pi\int\Pi,\mathscr{U}]$, JX is the free J-space generated by X, and there results a J-map $JX\to\bar{C}\bar{G}X$. The given composite $\bar{C}\bar{G}X\to JX$ is the $\bar{C}\bar{G}$-map obtained by the freeness of $\bar{C}\bar{G}X$. The respective composites are identity maps by freeness.

Thus $\mathscr{J}$-spaces are $\bar{C}$-spaces and $\bar{G}$-spaces such that the additive action $\bar{C}X\to X$ is a map of $\bar{G}$-spaces. It is the analog for $(\hat{\mathscr{C}},\mathscr{G})$-spaces that we really need, and the following result is a purely formal consequence of Propositions 5.3 and 5.6 and Theorems 6.1 and 6.12.

Corollary 6.13. *For $Y\in[\Pi,\mathscr{T}]_e$, define $\hat{\varrho}:\tilde{G}\hat{C}Y\to\hat{C}\tilde{G}Y$ by commutativity of the diagram*

$$\begin{array}{ccc}
\tilde{G}\hat{C}Y=L''\bar{G}R''L''\bar{C}R''Y & \xrightarrow{(L''\bar{G}\delta''\bar{C}R'')^{-1}} & L''\bar{G}\bar{C}R''Y \\
\downarrow{\scriptstyle\hat{\varrho}} & & \downarrow{\scriptstyle L''\bar{\varrho}R''} \\
\hat{C}\tilde{G}Y=L''\bar{C}R''L''\bar{G}R''Y & \xleftarrow{L''\bar{C}\delta''\bar{G}R''} & L''\bar{C}\bar{G}R''Y
\end{array}$$

Then $\hat{\varrho}$ satisfies the conditions of (i) *of Proposition* 5.6. *Therefore $\hat{C}$ is a monad in $\tilde{G}[\Pi,\mathscr{T}]_e$, $\hat{C}\tilde{G}$ is a monad in $[\Pi,\mathscr{T}]_e$, and a $\hat{C}\tilde{G}$-space determines and is determined by a $\hat{C}$-object in the category of $\tilde{G}$-spaces. Moreover, the natural composite*

$$\hat{C}\tilde{G}\longrightarrow\hat{J}\hat{J}\xrightarrow{\hat{\mu}}\hat{J}$$

is an isomorphism of monads in $[\Pi,\mathscr{T}]_e$.

Thus $(\hat{\mathscr{C}},\mathscr{G})$-spaces are $\hat{C}$-spaces and $\tilde{G}$-spaces such that $\hat{C}Y\to Y$ is a map of $\tilde{G}$-spaces.

Proposition 6.14. *The relationship between the previous result and Proposition* 6.11 *is given by the following commutative diagram:*

$$
\begin{array}{ccccc}
G_0CZ = L'\tilde{G}R'L'\hat{C}R'Z & & \xrightarrow{\quad (L'\tilde{G}\delta'\hat{C}R')^{-1} \quad} & & L'\tilde{G}\hat{C}R'Z \\
\downarrow{\scriptstyle \varrho'} & & & & \downarrow{\scriptstyle L'\hat{\varrho}R'} \\
JZ = L'\hat{J}R'Z & \xleftarrow{\ L'\hat{\mu}R'\ } & L'\hat{J}\hat{J}R'Z & \longleftarrow & L'\hat{C}\tilde{G}R'Z \\
\downarrow{\scriptstyle \varrho''} & & & & \downarrow{\scriptstyle L'\hat{C}\delta'\tilde{G}R'} \\
CG_0Z & & \Longleftrightarrow\!=\!=\!= & & L'\hat{C}R'L'\tilde{G}R'Z
\end{array}
$$

Therefore $\hat{C}R'Z \cong R'CZ$ *as* $\tilde{G}$*-spaces for* G_0*-spaces* Z *and* $C = L'\hat{C}R'$ *as monads in* $G_0[\mathscr{T}_e]$.

Proof. The second statement follows by simple diagram chases from the first. The first requires explicit calculation. We omit the details since we could just as well define ϱ' and ϱ'' by the diagram and deduce that the composite ϱ satisfies the conditions of Proposition 5.6(i).

7. Combinatorial analysis of the functor JX

This section is technical and is devoted to the detailed analysis of JX required to prove Theorems 6.1 and 6.2 and Propositions 6.6 and 6.9. By construction 5.7, Definition 1.4, and the specification of $\hat{\mathscr{C}}$ and $\hat{\mathscr{G}}$ in terms of $\mathscr{C}$ and $\mathscr{G}$ (above Definition 1.8), we find that

$$JX(n;S) = \coprod_{(m;R)} \coprod_{(\phi;\chi)} \prod_{j=1}^{n} \left[\mathscr{G}(|\phi^{-1}(j)|) \times \prod_{v=1}^{s_j} \mathscr{C}(|\chi_j^{-1}(v)|) \right] \times X(m;R)/(\sim),$$

where $(\phi;\chi)$ runs through $(\mathscr{F}\wr\mathscr{F})((m;R),(n;S))$.

The morphisms of $\mathscr{F}$ were discussed in [22, p. 217]. A morphism $\phi:\mathbf{m}\to\mathbf{n}$ is a projection if it is a surjection in Π; it is effective if $\phi^{-1}(0)=\{0\}$; it is ordered if $\phi(i)<\phi(i')$ implies $i<i'$. Any ϕ factors as $\varepsilon\pi$ with π a projection and ε effective, uniquely up to permutation of the target of π. If $\varepsilon:\mathbf{m}\to\mathbf{n}$ is effective, then $\varepsilon\tau$ is ordered for some permutation τ of $\mathbf{m}$. An ordered effective morphism $\phi:\mathbf{m}\to\mathbf{n}$ is uniquely of the form $\Phi_{m_1}\vee\cdots\vee\Phi_{m_n}$, where $m_j=|\phi^{-1}(j)|$ and $m_1+\cdots+m_n=m$. Such ϕ determine and are determined by partitions $M=(m_1,\ldots,m_n)$ of m, and such partitions M determine partitionings $R=(R_1,\ldots,R_n)$, where R_j is the jth block subsequence of R, with m_j entries.

To proceed further, we need some notations; in reading them, m and R should at first be thought of as m_j and R_j. For an operad $\mathscr{C}$, write $c\omega\in\mathscr{C}(q)$ for the action of an injection $\omega : \mathbf{q} \to \mathbf{r}$ on an element $c \in \mathscr{C}(r)$. Recall the notion of an injection in $\Pi \int \Pi$ from Section 2.

Notations 7.1. (i) Consider a sequence $R = (\mathbf{r}_1, \ldots, \mathbf{r}_m)$ and a morphism $\chi : \bigwedge_{i=1}^m \mathbf{r}_i \to \mathbf{s}$ in $\mathscr{F}$. Say that χ is R-effective if for each i, $1 \le i \le m$, and each u, $1 \le u \le r_i$, there is a sequence $U = (u_1, \ldots, u_m)$ such that $u_i = u$, $1 \le u_j \le r_j$ for $j \ne i$, and $\chi(U) \ne 0$. By convention, any $\chi : \mathbf{1} \to \mathbf{s}$ is R-effective when $m = 0$ and R is the empty sequence $*$. Let $\mathscr{E}(R, s)$ denote the set of R-effective morphisms $\bigwedge_{i=1}^m \mathbf{r}_i \to \mathbf{s}$ and define

$$\mathscr{C}(R,s) = \coprod_{\chi \in \mathscr{E}(R,s)} \prod_{v=1}^{s} \mathscr{C}(|\chi^{-1}(v)|).$$

If $R = *$ or if all $\mathbf{r}_i = 0$, let $\mathscr{C}(R,s)^+ = \mathscr{C}(R,s)$. Otherwise let $\mathscr{C}(R,s)^+ = \mathscr{C}(R,s) \amalg \mathscr{C}(0)^s$, where the point $\mathscr{C}(0)^s$ is thought of as indexed on the morphism $0 : \bigwedge_{i=1}^m \mathbf{r}_i \to \mathbf{s}$.

(ii) For an injection $(\psi; \omega) : (k; Q) \to (m; R)$ in $\Pi \int \Pi$ and for $\chi : \bigwedge_{i=1}^m \mathbf{r}_i \to \mathbf{s}$ in $\mathscr{F}$, define $\bar{\chi} : \bigwedge_{h=1}^k \mathbf{q}_h \to \mathbf{s}$ to be the composite

$$\bigwedge_{h=1}^{k} \mathbf{q}_h \xrightarrow{\sigma(\phi_m, \psi)} \bigwedge_{i=1}^{m} \mathbf{q}_{\psi^{-1}(i)} \xrightarrow{\omega_i} \bigwedge_{i=1}^{m} \mathbf{r}_i \xrightarrow{\chi} \mathbf{s};$$

here $\mathbf{q}_{\psi^{-1}(i)} = \mathbf{1}$ if $\psi^{-1}(i)$ is empty and then ω_i may be $0 : \mathbf{1} \to \mathbf{r}_i$. If $\bar{\chi} \ne 0$, then the first two maps of the composite restrict to injections

$$\lambda_v(\psi; \omega; \chi) : \bar{\chi}^{-1}(v) \to \chi^{-1}(v) \quad \text{for } 1 \le v \le s;$$

if $\bar{\chi} = 0$, let $\lambda_v(\psi; \omega; \chi) = 0 : \mathbf{0} \to \chi^{-1}(v)$. Define a map

$$(\psi; \omega) : \mathscr{G}(m) \times \mathscr{C}(R,s)^+ \to \mathscr{G}(k) \times \mathscr{C}(Q,s)^+$$

by requiring $(\psi; \omega)$ to map the χth component to the $\bar{\chi}$th component by the rule

$$\left(g, \mathop{\times}_{v=1}^{s} c_v\right)(\psi; \omega) = \left(g\psi; \mathop{\times}_{v=1}^{s} c_v \lambda_v(\psi; \omega; \chi)\right),$$

where $c_v \in \mathscr{C}(|\chi^{-1}(v)|)$ and thus $c_v \lambda_v(\psi; \omega; \chi) \in \mathscr{C}(|\bar{\chi}^{-1}(v)|)$.

(iii) For an object $(m; R)$ of $\Pi \int \Pi$ and a partition $M = (m_1, \ldots, m_n)$ of m with derived partitioning $(R_1, \ldots, R_n)$ of R, define

$$\mathscr{J}(M,R,S) = \prod_{j=1}^{n} \mathscr{G}(m_j) \times \mathscr{C}(R_j, s_j)$$

and

$$\mathscr{J}(M,R,S)^+ = \prod_{j=1}^{n} \mathscr{G}(m_j) \times \mathscr{C}(R_j, s_j)^+.$$

For injections $(\psi_j; \omega_j) : (k_j; Q_j) \to (m_j; R_j)$ in $\Pi \int \Pi$, $1 \le j \le n$, define $(\psi; \omega) : (k; Q) \to (m; R)$ by

$$\psi = \psi_1 \vee \cdots \vee \psi_n \quad \text{and} \quad \omega = (\omega_1, \ldots, \omega_n).$$

Also, let $R[j]$ be obtained from R by replacing all entries of R_j by 0 and define

$$\zeta_j = (1; 1^{m_1 + \cdots + m_{j-1}} \times 0^{m_j} \times 1^{m_{j+1} + \cdots + m_n}) : (m; R) \to (m; R[j]).$$

With these notations, we have the following description of JX.

Proposition 7.2. *$JX(0;*)=X(0;*)$. For $n>0$ and $S=(\mathbf{s}_1,\dots,\mathbf{s}_n)$,*

$$JX(n,S)=\coprod_{(M,R)}\mathscr{J}(M,R,S)^+\times X(m;R)/(\sim),$$

where the equivalence is given by relations of the form

$$\left(\gamma\left(\mathop{\times}_{j=1}^{n}(\psi_j,\omega_j)\right),x\right)\sim(\gamma,(\psi,\omega)x)$$

for injections $(\psi_j,\omega_j):(k_j;Q_j)\to(m_j;R_j)$ and for $x\in X(k;Q)$ and $\gamma\in\mathscr{J}(M,R,S)^+$ and of the form

$$(\gamma,x)\sim(\gamma,\zeta_j x)$$

for $x\in X(m;R)$ and for any $\gamma\in\mathscr{J}(M,R,S)^+$ with j-th coordinate indexed on $0:\bigwedge_{i=1}^m\mathbf{r}_{j,i}\to\mathbf{s}_j$; here γ is also viewed as an element of $\mathscr{J}(M,R[j],S)^+$ with j-th coordinate indexed on $0:\mathbf{0}\to\mathbf{s}_j$.

Proof. We claim first that if $\chi':\bigwedge_{i=1}^m\mathbf{r}_i'\to\mathbf{s}$ is not R'-effective, then $\chi'=\chi\circ\bigwedge_{i=1}^m\omega_i$, where $\omega_i:\mathbf{r}_i'\to\mathbf{r}_i$ is a projection and χ is R-effective. To see this, suppose $\chi(U)=0$ for all U with ith term u. Let $\nu_j=1:\mathbf{r}_j\to\mathbf{r}_j$ for $j\neq i$, let $\nu_i:\mathbf{r}_i\to\mathbf{r}_i-\mathbf{1}$ be the projection which sends u to 0 and is ordered otherwise, let $\sigma_u:\mathbf{r}_i-\mathbf{1}\to\mathbf{r}_i$ be the ordered injection which misses u, and let $\chi_1=\chi'(1^{i-1}\wedge\sigma_u\wedge 1^{n-i})$. Visibly $\chi'=\chi_1\circ\bigwedge_{i=1}^m\nu_i$, and our claim follows by inductive application of such decompositions. We claim next that, in our original description of $JX(n;S)$, we may restrict attention to those components indexed on morphisms $(\phi;\chi):(m;R)\to(n;S)$ such that $\phi=\phi_{m_1}\vee\cdots\vee\phi_{m_n}$ for some partition M and $\chi=(\chi_1,\dots,\chi_n)$ where each χ_j is R_j-effective. Indeed, if $(\phi';\chi'):(m';R')\to(n;S)$ is not of this form, then it admits a factorization $(\phi';\chi')=(\phi;\chi)(\psi;\omega)$ where $(\phi;\chi)$ is of this form and where ψ and each coordinate of ω is a projection. Moreover, we then have $|\phi^{-1}(j)|=|(\phi')^{-1}(j)|$ and $|\chi_j^{-1}(\upsilon)|=|(\chi_j')^{-1}(\upsilon)|$ for $1\le j\le n$ and $1\le\upsilon\le s_j$. It follows easily that any morphism $(c';g'):(m';R')\to(n;S)$ in $\mathscr{J}$ which augments to $(\phi';\chi')$ in $\mathscr{F}\wr\mathscr{F}$ factors as $(c;g)(\psi;\omega)$ for some morphism $(c;g)$ which augments to $(\phi;\chi)$. In fact, up to permutations, we may take $(c;g)=(c';g')$ as elements of

$$\prod_{j=1}^{n}\left(\mathscr{G}(|\phi^{-1}(j)|)\times\prod_{\upsilon=1}^{s_j}\mathscr{C}(|\chi_j^{-1}(\upsilon)|)\right).$$

The equivalence relation used to define $JX(n;S)$ restricts accordingly. However, to handle degeneracies, it is convenient to allow $\chi_j=0$. If $(\phi;\chi)$ is of this less restrictive form and if $(\phi;\chi)=(\phi';\chi')(\psi;\omega)$, then ψ is an injection and, for each j, either $\chi_j=0$ or each ω_i with $\phi'(i)=j$ is an injection. Further, if $\chi_j=0$ and if $\chi[j]$ is obtained from χ by replacing χ_j by $0:\mathbf{0}\to\mathbf{s}_j$, then $(\phi;\chi)=(\phi;\chi[j])\zeta_j$. With these indications, the remaining details are straightforward.

Specialization gives the following description of $\hat{J}Y$.

Corollary 7.3.

$$\hat{J}_sY = \coprod_{(m;R)} \mathscr{G}(m) \times \mathscr{C}(R,s)^+ \times Y_{r_1} \times \cdots \times Y_{r_m}/(\sim),$$

where the equivalence is given by relations of the form

$$((g;c)(\psi;\omega),(y_1,\ldots,y_k)) \sim ((g;c),(\psi;\omega)(y_1,\ldots,y_k))$$

for injections $(\psi;\omega):(k;Q)\to(m;R)$ *and of the form*

$$((g;*^s),(y_1,\ldots,y_m)) \sim ((g;*^s),(\bar{y}_1,\ldots,\bar{y}_m)),$$

where $y_i \in Y_{r_i}$, $\bar{y}_i \in Y_0$ *is its image with respect to* $0:\mathbf{r}_i \to \mathbf{0}$, *and* $*^s \in \mathscr{C}(0)^s$ *is viewed as an element of both* $\mathscr{C}(R,s)^+$ *and* $\mathscr{C}(\mathbf{0}^m,s)^+$. *In particular, JZ is obtained by setting* $s=1$ *and replacing* Y_r *by* Z^r.

Visibly, the decompositions of the $\mathscr{J}(M,R,S)^+$ as n-fold products are respected by the equivalence relation. Therefore $(JR''Y)(n;S) \cong \prod_{j=1}^n \hat{J}_{s_j}Y$ and $JR'' \cong R''\hat{J}$. Since $\bar{J}X(1;\mathrm{s})$ depends on all $X(m,R)$ and not just the $X(1;\mathrm{s})$, $L''\bar{J} \not\cong \hat{J}L''$. Similarly, $\hat{J}_1Y$ depends on all Y_s and not just Y_1 (or Y_1 and Y_0), hence $L'\hat{J} \not\cong JL'$ (and $L_2'\hat{J} \not\cong JL_2'$). Manifestly, $\hat{J}_sR'Z$ is a very different construction from $(JZ)^s$, hence $\hat{J}R' \not\cong R'J$. These observations prove Theorem 6.1.

Some of these distinctions disappear when $\mathscr{C}$ or $\mathscr{G}$ is the trivial operad $\mathscr{P}$. Recall that $\mathscr{P}(0)=\{*\}$, $\mathscr{P}(1)=\{1\}$, and $\mathscr{P}(m)$ is empty for $m>1$. If $\mathscr{G}=\mathscr{P}$, so that $\bar{J}=\bar{C}$, then the components with $m=1$ in the construction of $\bar{C}X(1;\mathrm{s})$ yield exactly $\hat{C}_sL''X$. The component with $m=0$ is $\mathscr{G}(0)\times\mathscr{C}(*,s)\times X(0;*)$. Since $X(0;*)$ is a retract of $X(1;1)$ via $(0;1):(0;*)\to(1,1)$ and $(0;*):(1;1)\to(0;*)$, we have the relations

$$((1;c)(0;*),(0;1)x) \sim ((*;c),x) \quad \text{for } c\in\mathscr{C}(*,s) \text{ and } x\in X(0;*).$$

These show that this component is unnecessary to the construction. Therefore $L''\bar{C}=\hat{C}L''$ and thus $L''\bar{C}R''=\hat{C}$. This proves Proposition 6.6.

Suppose next that $\mathscr{C}=\mathscr{P}$, so that $\bar{J}=\bar{G}$ and $\hat{J}=\tilde{G}$, and consider $\tilde{G}_sY$ for $s=0$ and $s=1$. Clearly $\mathscr{P}(R,0)^+$ is a point indexed on $\chi=0$, only those R with $\mathbf{r}_i=0$ contribute, and therefore $\tilde{G}_0Y=GY_0$ constructed with respect to the basepoint 0. Less obviously, $\mathscr{P}(R,1)^+$ is a point indexed on $\chi=0$ unless $R=*$ or $R=\mathbf{1}^m$, when $\mathscr{P}(R,1)^+$ consists of a point indexed on $\chi=0$ and a point indexed on $\chi=1$. The components indexed on $\chi=1$ give rise to $GY_1/(\sim)$ where GY_1 is constructed with respect to the basepoint 1 and the equivalence relation is generated by

$$[g;y_1,\ldots,y_r] \sim [g\sigma_q,\iota\varrho y_1,\ldots,\iota\varrho y_{q-1},\iota\varrho y_{q+1},\ldots,\iota\varrho y_r] \quad \text{if } y_q=0,$$

where $\iota:Y_0\to Y_1$ and $\varrho:Y_1\to Y_0$ are determined by $0:\mathbf{0}\to\mathbf{1}$ and $0:\mathbf{1}\to\mathbf{0}$. The components indexed on $\chi=0$ account for the image of GY_0 in $GY_1/(\sim)$. When $Y_0=\{0\}$, these reduce to $\tilde{G}_0Y=\{0\}$ and $\tilde{G}_1Y=G_0Y_1$, and this proves Proposition 6.9. For general Y, these calculations dictate the construction of $G_0:\mathscr{T}_2\to\mathscr{T}_2$ required to validate Remarks 6.10.

It only remains to prove Theorem 6.2. For this purpose, it is convenient to separate the injections in $\Pi\int\Pi$ into permutations and degeneracy operators. Thus define $\Sigma(m;R)$ to be the group of automorphisms of $(m;R)$ in $\Pi\int\Pi$. Notations 7.1(ii) specify a right action of $\Sigma(m;R)$ on $\mathscr{G}(m)\times\mathscr{C}(R,s)$, and we have the following observation.

Lemma 7.4. *If $\mathscr{C}$ and $\mathscr{G}$ are Σ-free, then the action of $\Sigma(m;R)$ on $\mathscr{G}(m)\times\mathscr{C}(R,s)$ is free.*

Proof. If

$$\left(g;\ \mathop{\times}_{\upsilon=1}^{s} c_\upsilon\right)(\psi;\omega)=\left(g\psi;\ \mathop{\times}_{\upsilon=1}^{s} c_\upsilon\lambda_\upsilon(\psi;\omega;\chi)\right)=\left(g;\ \mathop{\times}_{\upsilon=1}^{s} c_\upsilon\right),$$

where $g\in\mathscr{G}(m)$ and $c_\upsilon\in\mathscr{C}(|\chi^{-1}(\upsilon)|)$, then $g\psi=g$ and thus $\psi=1$. Also, we clearly must have $\chi=\bar{\chi}$ and $\lambda_\upsilon(\psi;\omega;\chi)=1$ for all υ. Thus

$$\bigwedge_{i=1}^{m}\omega_i:\bigwedge_{i=1}^{m}\mathbf{r}_i\to\bigwedge_{i=1}^{m}\mathbf{r}_i$$

must restrict to the identity $\chi^{-1}(\upsilon)\to\chi^{-1}(\upsilon)$ for each υ. Since χ is R-effective, this implies that each $\omega_i=1$.

For an object $(m;R)$ of $\Pi\int\Pi$ and a partition M of m, define

$$\Sigma(M,R)=\prod_{j=1}^{n}\Sigma(m_j;R_j)\subset\Sigma(m;R),$$

where the inclusion is given by Notations 7.1(iii). If $\mathscr{C}$ and $\mathscr{G}$ are Σ-free, then $\Sigma(M,R)$ acts freely on $\mathscr{J}(M,R,S)$.

We may now write

$$JX(n;S)=\coprod_{(M,R)}\mathscr{J}(M,R,S)\times_{\Sigma(M,R)}X(m;R)/(\sim).$$

Here the equivalence relation is defined with respect to the appropriate degeneracy operators, all of which correspond to insertions of basepoints (0 or 1) when $X=RZ$ for $Z\in\mathscr{T}_e$. The components of $\mathscr{J}(M,R,S)^+$ not in $\mathscr{J}(M,R,S)$ are unnecessary in view of the second kind of relation specified in Proposition 7.2. Their role is to allow evaluation of those degeneracy operators $(\psi;\omega)$ such that some of the components ω_i with domain **1** are zero.

To prove Theorem 6.2, we shall describe how $JX(n;S)$ is built up inductively by pushouts and unions. We need another set of notations. Observe that $\mathscr{C}(R,s)$ is empty unless $R=*$ or all $r_i=0$ or all $r_i>0$ and define $|R|=0$ in the first two cases and $|R|=r_1+\cdots+r_m-(m-1)$ in the last case; here $m-1$ is subtracted in order to make $|R|=1$ if all $r_i=1$. Recall the σ_q from Construction 5.8.

Notations 7.5. (i) Write $|M|=m$ and $|M,R|=|R_1|+\cdots+|R_n|$. Define $J_mX(n;S)$ to be the image in $JX(n;S)$ of those components indexed on (M,R) with $|M|=m$. Then $JX(n;S)$ is a quotient of $\amalg_k J_kX(n;S)$; filter it by letting $F_mJX(n;S)$ be the image of those components indexed on $k\le m$. In turn, filter each $J_mX(n;S)$ by letting $F_rJ_mX(n;S)$ be the image of those components indexed on (M,R) with $|M,R|\le r$.

(ii) If $r_i>1$, let R_i^- be the sequence obtained from R by replacing r_i by r_i-1 and let

$$\sigma_{i,q}=(1;1^{i-1}\times\sigma_q\times 1^{m-i}):(m;R_i^-)\to(m;R).$$

If M is a partition of m with $m_j>0$ and if all entries of R_j are 1, let

$$\lambda_j=(1;1^{m_1+\cdots+m_{j-1}}\times 0^{m_j}\times 1^{m_{j+1}+\cdots+m_n}):(m;R[j])\to(m;R);$$

these satisfy $\zeta_j\lambda_j=1$, with ζ_j as in Notations 7.1(iii). For $|M,R|\ge 1$, define

$$DX(M,R)=\left[\bigcup_{r_i>1,\,1\le q\le r_i}\sigma_{i,q}X(m;R_i^-)\right]\cup\left[\bigcup_{|R_j|=1}\lambda_jX(m;R[j])\right]\subset X(m;R).$$

(iii) Let $\hat{R}_i$ be obtained from R_i by deleting the ith entry $\mathbf{r}_i$. If $r_i=0$ and $\varepsilon=0$ or if $r_i=1$ and $\varepsilon=0$ or 1, define

$$\tau_{i,\varepsilon}=(\sigma_i;1^{i-1}\times\bar{\varepsilon}\times 1^{m-i}):(m-1;\hat{R}_i)\to(m;R),$$

where $\bar{\varepsilon}:\mathbf{1}\to\mathbf{r}_i$ is the morphism 0 if $\varepsilon=0$ or 1 if $\varepsilon=1$. Define

$$CX(m;R)=\bigcup_{0\le\varepsilon\le r_i\le 1}\tau_{i,\varepsilon}X(m-1,\hat{R}_i)\subset X(m;R)$$

and, for $|M,R|\ge 1$,

$$BX(M,R)=CX(m;R)\cap DX(M,R)\subset X(m;R).$$

(iv) Define $KX(n;S)$ to be the image in $JX(n;S)$ of

$$\coprod_{(M,R)}\mathscr{J}(M,R,S)\times_{\Sigma(M,R)}CX(m;R)$$

and define

$$K_mX(n;S)=KX(n;S)\cap J_mX(n;S)$$

and

$$F_rK_mX(n;S)=KX(n;S)\cap F_rJ_mX(n;S).$$

Inspection of the permutations in $\Sigma(M,R)$ makes clear that $DX(m,R)$, $CX(m;R)$, and $BX(M,R)$ are invariant subspaces of $X(m;R)$. With these notations, we have the following inductive descriptions of

$$J_mX(n;S)=\bigcup_{r\ge 0}F_rJ_mX(n;S)\quad\text{and}\quad JX(n;S)=\bigcup_{m\ge 0}F_mJX(n;S).$$

The essential points are that all injections in $\Pi\wr\Pi$ are accounted for as composites of permutations and the specified degeneracy operators and that each of these degeneracy operators raises filtration by exactly one.

Lemma 7.6. *Let $X \in [\Pi \wr \Pi, \mathscr{U}]$, so that X satisfies the cofibration conditions required of a $\Pi \wr \Pi$-space. Then the following three diagrams are pushouts in which the vertical arrows are cofibrations.*

$$\text{(i)}\quad \begin{array}{ccc} \coprod_{|M|=m,\ |M,R|=r} \mathscr{J}(M,R,S) \times_{\Sigma(M,R)} DX(M,R) & \xrightarrow{\nu} & F_{r-1}J_mX(n;S) \\ \cap & & \downarrow \\ \coprod_{|M|=m,\ |M,R|=r} \mathscr{J}(M,R,S) \times_{\Sigma(M,R)} X(m,R) & \longrightarrow & F_rJ_mX(n;S) \end{array}$$

where, for $\gamma \in \mathscr{J}(M,R,S)$,

$$\nu(\gamma, \sigma_{i,q}x) = [\gamma\sigma_{i,q}, x] \quad \textit{and} \quad \nu(\gamma, \lambda_j x) = [\gamma\lambda_j, x].$$

$$\text{(ii)}\quad \begin{array}{ccc} \coprod_{|M|=m,\ |M,R|=r} \mathscr{J}(M,R,S) \times_{\Sigma(M,R)} BX(M,R) & \xrightarrow{\nu} & F_{r-1}K_mX(n;S) \\ \cap & & \downarrow \\ \coprod_{|M|=m,\ |M,R|=r} \mathscr{J}(M,R,S) \times_{\Sigma(M,R)} CX(m,R) & \longrightarrow & F_rK_mX(n;S) \end{array}$$

where ν is the restriction of the corresponding map of (i).

$$\text{(iii)}\quad \begin{array}{ccc} K_mX(n;S) & \xrightarrow{\mu} & F_{m-1}JX(n;S) \\ \cap & & \downarrow \\ J_mX(n;S) & \longrightarrow & F_mJX(n;S) \end{array}$$

where, for $\gamma \in \mathscr{J}(M,R,S)$ and i in the j-th block of m_j numbers in $\{1, \ldots, m\}$,

$$\mu[\gamma, \tau_{i,\varepsilon}x] = [\gamma\tau_{i,\varepsilon}, x] \quad \textit{if } \varepsilon = r_i = 0 \textit{ or } \varepsilon = r_i = 1$$

and

$$\mu[\gamma, \tau_{i,0}x] = [\gamma\tau_{i,0}, \zeta_j x] \quad \textit{if } \varepsilon = 0 \textit{ and } r_i = 1.$$

Proof. The very last definition makes sense because $\gamma\tau_{i,0}$ is indexed on $(\phi; \chi)$ with $\chi_j = 0$; we have used both relations of Proposition 7.2 here, and we interpret ζ_j as the identity when $m_j = 1$ and the jth subsequence of $\hat{R}_i$ is empty. We must first check that ν and μ are well defined; the need for such a verification should have been pointed out in the simpler analog [22, 5.5]. Let $\pi_q : \mathbf{r} \to \mathbf{r-1}$, $1 \le q \le r$, be the projection which sends q to 0, i to i if $i<q$, and i to $i-1$ if $i>q$. For $r_i > 1$, let

$$\pi_{i,q} = (1; 1^{i-1} \times \pi_q \times 1^{m-i}) : (m; R) \to (m; R_i^-).$$

For any r_i, let

$$\varrho_i = (\pi_i; 1^{m-1}) : (m; R) \to (m-1; \hat{R}_i).$$

These morphisms and the ζ_j and the degeneracy operators $\sigma_{i,q}$, λ_j, and $\tau_{i,\varepsilon}$ satisfy commutation relations somewhat similar to those satisfied by the faces and degeneracies of simplicial objects. In particular, $\pi_{i,q}\sigma_{i,q}=1$, $\zeta_j\lambda_j=1$, and $\varrho_i\tau_{i,\varepsilon}=1$. To see that ν in (i) is well defined, suppose for example that $\sigma_{h,p}x=\sigma_{i,q}y$. If $h\neq i$, the $\pi_{h,p}$ and $\sigma_{h,p}$ commute with the $\pi_{i,q}$ and $\sigma_{i,q}$ and we have

$$x=\pi_{h,p}\sigma_{h,p}x=\pi_{h,p}\sigma_{i,q}y=\sigma_{i,q}\pi_{h,p}y$$

and thus

$$y=\pi_{i,q}\sigma_{i,q}y=\pi_{i,q}\sigma_{h,p}x=\pi_{i,q}\sigma_{h,p}\sigma_{i,q}\pi_{h,p}y=\sigma_{h,p}\pi_{h,p}y.$$

Therefore

$$\begin{aligned}[]
[\gamma\sigma_{h,p},x]&=[\gamma\sigma_{h,p},\sigma_{i,q}\pi_{h,p}y]=[\gamma\sigma_{h,p}\sigma_{i,q},\pi_{h,p}y]\\
&=[\gamma\sigma_{i,q}\sigma_{h,p},\pi_{h,p}y]=[\gamma\sigma_{i,q},\sigma_{h,p}\pi_{h,p}y]=[\gamma\sigma_{i,q},y].
\end{aligned}$$

The verification when $h=i$ proceeds similarly, as do the verifications when $\lambda_j x=\lambda_k y$ or when $\lambda_j x=\sigma_{i,q}y$. To see that ν in (ii) is well defined, suppose $\sigma_{i,q}x=\tau_{h,\varepsilon}y$. Then

$$x=\pi_{i,q}\sigma_{i,q}x=\pi_{i,q}\tau_{h,\varepsilon}y=\begin{cases}\tau_{h,\varepsilon}\pi_{i,q}y & \text{if } i<h,\\ \tau_{h,\varepsilon}\pi_{i-1,q}y & \text{if } i>h.\end{cases}$$

The case $h=i$ cannot occur, and these equalities show that $[\gamma\sigma_{i,q},x]\in F_{r-1}K_mX(n;S)$. A similar verification applies when $\lambda_j x=\tau_{h,\varepsilon}y$ unless h is in the jth block of m_j numbers in $\{1,\dots,m\}$, when

$$x=\zeta_j\lambda_j x=\zeta_j\tau_{h,\varepsilon}y=\tau_{h,0}\zeta_j y$$

(ζ_j on the right being the identity if $m_j=1$). To see that μ in (iii) is well defined, we must check (for example) that if $\delta=r_h$ and $\varepsilon=r_i$, then

$$[\beta;\tau_{h,\delta}x]=[\gamma,\tau_{i,\varepsilon}y]\quad\text{in } K_mX(n;S)$$

implies

$$[\beta\tau_{h,\delta},x]=[\gamma\tau_{i,\varepsilon},y]\quad\text{in } F_{m-1}JX(n;S).$$

By inspection of the equivalence relation used to define $J_mX(n;S)$, we may assume that $\beta=\gamma$ and $\tau_{h,\delta}x=\tau_{i,\varepsilon}y$. If $h=i$, we must have $\delta=\varepsilon$ (so as to land in the same $X(m;R)$), and then application of ϱ_i shows that $x=y$. In the remaining cases, easy calculations give

$$x=\begin{cases}\tau_{i-1,\varepsilon}\varrho_h y & \text{if } h<i,\\ \tau_{i,\varepsilon}\varrho_{h-1}y & \text{if } h>i\end{cases}\quad\text{and}\quad y=\begin{cases}\tau_{h,\delta}\varrho_h y & \text{if } h<i,\\ \tau_{h-1,\delta}\varrho_{h-1}y & \text{if } h>i,\end{cases}$$

and then $[\gamma\tau_{h,\delta},x]=[\gamma\tau_{i,\varepsilon},y]$ just as in our first verification. Examination of the injections in $\Pi\wr\Pi$ and their factorizations in terms of permutations and degeneracy operators demonstrates that our diagrams are pushouts. The left vertical arrows in (i) and (ii) are verified to be cofibrations by use of our assumptions on X and a result of Boardman and Vogt [2, App. 2.7]. By the inclusion of the diagram of (i) in that of

(ii) and inductive use of a result of Lewis [11, 2.5], the left vertical arrow of (iii) is also a cofibration.

The proof of Theorem 6.2 is now fairly straightforward. One first verifies by tedious inductive analyses of degeneracy spaces that if each $f(n;S): X(n;S) \to X'(n;S)$ is an equivalence, then so are the restrictions

$$BX(M;R) \to BX'(M;R), \qquad CX(m;R) \to CX'(m;R),$$
$$DX(M,R) \to DX'(M,R)$$

of each $f(m;R)$. One then uses Lemma 7.4 and covering space arguments to show that the $f(m;R)$ induce equivalences on the spaces in the left columns of diagrams (i) and (ii) because they induce equivalences on the corresponding spaces before passage to orbits. One then proceeds inductively, using that, in the presence of cofibrations, pushouts of (weak) equivalences are (weak) equivalences (e.g. by [2, App. 4.8]).

8. The construction and identification of $\mathscr{G}$-spectra

This section is devoted to the proofs of results claimed in Section 4 and to some additional results on the evaluation of the $\mathscr{G}$-spectra associated to certain generic types of input data. We must first recall the notions of $\mathscr{G}$-prespectra and $\mathscr{G}$-spectra.

Consider finite dimensional sub inner product spaces V of R^∞. Let SV denote the one-point compactification of V, with $S\{0\} = S^0$. For a based space Z, let

$$\Sigma^V Z = Z \wedge SV \quad \text{and} \quad \Omega^V Z = F(SV, Z),$$

where $F(Z', Z)$ is the function space of based maps $Z' \to Z$.

Recall from [21, §5] that a prespectrum D consists of based spaces DV for $V \subset R^\infty$ and based maps $\sigma: \Sigma^W DV \to D(V+W)$ for V orthogonal to W such that the adjoints $\tilde{\sigma}: DV \to \Omega^W D(V+W)$ are inclusions and certain obvious unity (for $W = \{0\}$) and associativity conditions hold. D is said to be a spectrum if each $\tilde{\sigma}$ is a homeomorphism, and a prespectrum D determines an associated spectrum LD via

$$(LD)(V) = \bigcup_{V \perp W} \Omega^W D(V+W).$$

The following is a slightly schematic version of the definition of a $\mathscr{G}$-prespectrum given by Quinn, Ray, and myself [14, III.1.1]. Write $\{SV\}$ for the sphere prespectrum and S for the sphere spectrum $\Sigma^\infty S^0 = L\{SV\}$.

Definition 8.1. Let $\mathscr{G}$ be an operad and assume given a morphism of operads $\varepsilon: \mathscr{G} \to \mathscr{L}$; regard elements of $\mathscr{G}(j)$ as linear isometries $(R^\infty)^j \to R^\infty$ via ε. A $\mathscr{G}$-prespectrum is a prespectrum D together with a morphism $e: \{SV\} \to D$ of prespectra and maps

$$\xi_j(g): DV_1 \wedge \cdots \wedge DV_j \to Dg(V_1 \oplus \cdots \oplus V_j)$$

for $j \geq 0$, $g \in \mathscr{G}(j)$, and $V_i \subset R^\infty$ such that the following properties hold; by definition, $\xi_0(*)$ is the zeroth map $e: S^0 \to D_0$.

(a) If $g \in \mathscr{G}(k)$ and $h_r \in \mathscr{G}(j_r)$ for $1 \leq r \leq k$, then

$$\xi_{j_1+\cdots+j_k}(\gamma(g; h_1, \ldots, h_k)) = \xi_k(g) \circ (\xi_{j_1}(h_1) \wedge \cdots \wedge \xi_{j_k}(h_k)).$$

(b) $\xi_1(1)$ is the identity map, where $1 \in \mathscr{G}(1)$ is the unit of $\mathscr{G}$.

(c) If $g \in \mathscr{G}(j)$ and $\tau \in \Sigma_j$, then $\xi_j(g\tau) = \xi_j(g) \circ \tau$.

(d) ξ_j is continuous in g in an appropriate sense.

(e) If $g \in \mathscr{G}(j)$, $x_i \in DV_i$, and $w_i \in W_i$, where $V_i \perp W_i$ for $1 \leq i \leq j$, then

$$\xi_j(g)(\sigma(x_1 \wedge w_1) \wedge \cdots \wedge \sigma(x_j \wedge w_j))$$
$$= \sigma(\xi_j(g)(x_1 \wedge \cdots \wedge x_j) \wedge g(w_1 \oplus \cdots \oplus w_j)).$$

D is a $\mathscr{G}$-spectrum if it is a spectrum, and we again write e for the induced map $S \to D$. A morphism $f: D \to D'$ of $\mathscr{G}$-spectra is a morphism of prespectra such that $f \circ e = e'$ and

$$f \circ \xi_j(g) = \xi_j'(g) \circ (f \wedge \cdots \wedge f) \quad \text{for } g \in \mathscr{G}(j).$$

In particular, $\{SV\}$ is a $\mathscr{G}$-prespectrum and $e: \{SV\} \to D$ is a map of $\mathscr{G}$-prespectra for any $\mathscr{G}$-prespectrum D, and similarly for $\mathscr{G}$-spectra.

Conditions (a)–(c) are just obvious analogs of the associativity, unity, and equivariance conditions required of an action of an operad on a space, and (e) is the obvious commutation with structural maps condition. These conditions are spelled out diagrammatically in [14, p. 67]. Condition (d) is spelled out in [3]; the condition in [14, p. 67] is too weak for the applications in [3] although adequate for the applications in [14]. Since the precise change is technical and could not be motivated here, we prefer not to go into detail.

We recall the following observation from [14, p. 71].

Lemma 8.2. *For $\mathscr{G}$-prespectra D, LD is functorially a $\mathscr{G}$-spectrum.*

As in Section 4, we assume given $(\mathscr{C}', \mathscr{G}')$ and we set

$$(\mathscr{C}, \mathscr{G}) = (\mathscr{C}' \times \mathscr{K}_\infty, \mathscr{G}' \times \mathscr{L}).$$

We wish to prove that spectrum EY associated to $(\mathscr{C}, \mathscr{G})$-space Y is a $\mathscr{G}$-spectrum, and we must first recall the construction of EY (which only depends on Y as a $\mathscr{C}$-space). Let $\mathscr{K}_V$ be the Steiner operad [26; 21 §6] associated to the inner product space V and let $\mathscr{C}_V = \mathscr{C}' \times \mathscr{K}_V$. By [21, 6.9], we have a morphism of monads $\alpha_V: K_V \to \Omega^V \Sigma^V$. Its adjoint $\beta_V: \Sigma^V K_V \to \Sigma^V$ is an action of K_V on Σ^V. By pullback along the projection $\pi: C_V \to K_V$, C_V acts on Σ^V. By Proposition 5.2(ii) and Lemma 5.9, it follows that $\hat{C}_V$ acts on $\Sigma^V L'$. We define

$$(DY)(V) = B(\Sigma^V L', \hat{C}_V, Y).$$

We have inclusions of operads $\sigma:\mathscr{K}_V\to\mathscr{K}_{V+W}$ when $V\perp W$, and these induce $\sigma:\mathscr{C}_V\to\mathscr{C}_{V+W}$. Since suspension commutes with geometric realization [8, 12.1], we obtain maps

$$\begin{array}{ccc}\Sigma^W(DY)(V) & \cong & B(\Sigma^{V+W}L',\hat{C}_V,Y)\\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle B(1,\sigma,1)}\\ (DY)(V+W) & = & B(\Sigma^{V+W}L',\hat{C}_{V+W},Y).\end{array}$$

This specifies a prespectrum DY, and $EY=LDY$. See [21, §6] for an alternative viewpoint and more details.

Remarks 8.3. If $Y=R'Z$ for a $\mathscr{C}$-space Z, then Corollary 5.4 gives

$$(DY)(V)=B(\Sigma^VL',Y)\cong B(\Sigma^V,C_V,Z)\equiv(DZ)(V).$$

If $Y=X_\oplus$ for a $\Pi\int\mathscr{C}$-space X, then Corollary 5.4 gives

$$(DY)(V)=B(\Sigma^VL',\hat{C}_V,Y)\cong B(\Sigma^VL,\bar{C}_V,X)\equiv(DX)(V),$$

where Σ^VL is a $\bar{C}$-functor by Propositions 5.2 and 6.6. Thus the constructions DY and EY for $\hat{\mathscr{C}}$-spaces Y specialize to the analogous constructions DZ and EZ for $\mathscr{C}$-spaces Z and DX and EX for $\Pi\int\mathscr{C}$-spaces X. These observations on bar constructions apply equally well with Σ^V replaced by any other C_V-functor F; here V may be R^∞ itself, when $\mathscr{C}_V=\mathscr{C}$.

Now let Y be a $(\hat{\mathscr{C}},\mathscr{G})$-space such that $Y_0=\{0\}$. We prove that DY is a $\mathscr{G}$-prespectrum. By passage to spectra and use of the previous remarks, this will prove Theorem 4.4. In fact, the proof is virtually the same as that given for $(\mathscr{C},\mathscr{G})$-spaces in [14, p. 188–191]. Via the basepoints 0 and 1 in Y_1 and the natural inclusion of Y_1 in D_0Y, we have a map $S^0\to D_0Y$. This determines $e:\{SV\}\to DY$. Since $Y_0=\{0\}$, all $\hat{C}_0^qY=\{0\}$. Thus all $L'\hat{C}^qY=\hat{C}_1^qY$ are $\mathscr{G}_0$-spaces by Lemma 4.7. As in the cited earlier proof, these actions pull back to finite dimensional inner product spaces and combine with the maps g on suspension coordinates to induce maps

$$\Sigma^{V_1}L'\hat{C}^q_{V_1}Y\wedge\cdots\wedge\Sigma^{V_j}L'\hat{C}^q_{V_j}X\to\Sigma^WC^q_WX,$$

where $W=g(V_1\oplus\cdots\oplus V_j)$. These maps together specify a map of simplicial spaces. On passage to realization and use of the commutation of realizations with products, one obtains the required maps $\xi_j(g)$. The remaining verifications are entirely straightforward and no different from the case $Y=R'Z$.

The following definition provides the functors required for the proof of Theorems 4.6 and 4.8.

Definition 8.4. For a $\hat{\mathscr{C}}$-space Y, define

$$U_{\oplus}Y = B(\hat{C}, \hat{C}, Y) \quad \text{and} \quad V_{\oplus}Y = B(CL', \hat{C}, Y).$$

For a $\mathscr{C}$-space Z, define

$$U_{\oplus}Z = B(C, C, Z) \cong V_{\oplus}R'Z.$$

Here $U_{\oplus}Y$ is a $\hat{C}$-space and $V_{\oplus}Y$ and $U_{\oplus}Z$ are C-spaces.

Precisely as in the argument following Definition 6.5, Corollaries 5.4 and 5.5 couple with standard facts about realization to give equivalences ε and δ of $\hat{C}$-spaces in the diagram

$$Y \xleftarrow{\ \varepsilon\ } U_{\oplus}Y \xrightarrow{\ \delta\ } R'V_{\oplus}Y \xrightarrow{\ R'\gamma\ } R'E_0Y. \qquad (*)$$

We have a composite map $\alpha_\infty \pi : C \to Q$ of monads in $\mathscr{T}$, where $QZ = \operatorname{colim} \Omega^W \Sigma^W Z$. The map $\gamma : V_{\oplus}Y \to E_0 Y$ is the composite

$$B(CL', \hat{C}, Y) \xrightarrow{B(\alpha_\infty\pi, 1, 1)} B(QL', \hat{C}, Y) \xrightarrow{\ \gamma^\infty\ } \operatorname{colim} \Omega^W B(\Sigma^W L', \hat{C}_W, Y).$$

Here γ^∞ is obtained by passage to colimits from the natural comparison $|\Omega^W X| \to \Omega^W |X|$ for a simplicial space X; γ^∞ is an equivalence by [12, 12.3] and a C-map by [12, 12.4]. When $\mathscr{C}'$ is spacewise contractible, $B(\alpha_\infty\pi, 1, 1)$ is a group completion by [13, 2.3] and [14, VI.2.7(iv)] and is a C-map since it is the realization of a simplicial C-map. This proves Theorem 4.6.

Now assume again that Y is a $(\hat{\mathscr{C}}, \mathscr{G})$-space with $Y_0 = \{0\}$. To prove Theorem 4.8, we change ground categories from $\mathscr{T}$ and $\Pi[\mathscr{T}]$ to $G_0[\mathscr{T}_e]$ and the category $\tilde{G}[\Pi, \mathscr{T}]_e$ of $\tilde{G}$-spaces with 0th space $\{0\}$. While $\tilde{G}Y$ will not satisfy the homotopy type condition required of Π-spaces when Y does, this is of no concern since the monads G_0 and $\tilde{G}$ are only used to keep track of underlying multiplicative structures. By Proposition 6.9, we have $L'\tilde{G} \cong G_0 L'$. Therefore, by application of Propositions 5.2 and 5.3 to G_0 and $\tilde{G}$, L' and R' restrict to an adjoint pair of functors relating $G_0[\mathscr{T}_e]$ and $\tilde{G}[\Pi, \mathscr{T}]_e$. We take these categories as $\mathscr{V}$ and $\mathscr{W}$, respectively, in the discussion at the start of Section 5. By Proposition 6.11 and Corollary 6.13, C restricts to a monad in $\mathscr{V}$ and $\hat{C}$ restricts to a monad in $\mathscr{W}$. By Proposition 6.14, $C = L'\hat{C}R'$ as monads in $\mathscr{V}$ and $\hat{C}R' \cong R'C$ as functors $\mathscr{V} \to \mathscr{W}$. Now Corollaries 5.4 and 5.5 apply once more to give that ε and δ in $(*)$ are $\tilde{G}$-maps as well as $\hat{C}$-maps and that the natural inverse τ of ε is a $\tilde{G}$-map. Since $\alpha_\infty\pi$ is a morphism of monads in $G_0[\mathscr{T}_e]$, by [14, VII.2.4], $B(\alpha_\infty\pi, 1, 1)$ above is a G_0-map, and γ^∞ is a G_0-map by an easy explicit computation (these arguments being no different from the case of $(\mathscr{C}, \mathscr{G})$-spaces). This proves Theorem 4.8.

There are several important generic types of $(\hat{\mathscr{C}}, \mathscr{G})$-spaces for which the associated $\mathscr{G}$-spectrum can readily be identified. We first identify $E\hat{C}Y$ for a Π-space Y. $\hat{C}Y$ is the colimit over $V \subset R^\infty$ of the $\hat{C}_V Y$, and we construct a slight modification $D'\hat{C}Y$ of the associated prespectrum $D\hat{C}Y$. Thus let

$$(D'\hat{C}Y)(V) = B(\Sigma^V L', \hat{C}_V, \hat{C}_V Y) \subset B(\Sigma^V L', \hat{C}_V, \hat{C}Y) = (D\hat{C}Y)(V)$$

and specify structural maps σ as before. By a glance at the relevant colimits, we see that

$$LD'\hat{C}Y = LD\hat{C}Y = \hat{E}CY.$$

By general nonsense [12, 9.9], there are natural deformation retractions

$$B(\Sigma^V L', \hat{C}_V, \hat{C}_V Y) \to \Sigma^V L'Y = \Sigma^V Y_1,$$

and these specify a homotopy equivalence $\zeta : D'\hat{C}Y \to \{\Sigma^V Y_1\}$ of prespectra. If Y is also a $\tilde{G}$-space with $Y_0 = \{0\}$, then Y_1 is a G_0-space and $\hat{C}Y$ is a $(\hat{\mathscr{C}}, \mathscr{G})$-space, hence $D'\hat{C}Y$ and $\{\Sigma^V Y_1\}$ are $\mathscr{G}$-prespectra (the first by an argument just like the one for DY above), and ζ is easily checked to be a morphism of $\mathscr{G}$-spectra. On passage to spectra via L, this proves the following result.

Proposition 8.4. *For Π-spaces Y, there is a natural equivalence of spectra*

$$\zeta : E\hat{C}Y \to \Sigma^\infty Y_1.$$

If Y is a $\tilde{G}$-space with $Y_0 = \{0\}$, then ζ is a map of $\mathscr{G}$-spectra.

In fact, ζ is naturally a deformation retraction with inverse inclusion the adjoint $\Sigma^\infty Y_1 \to E\hat{C}Y$ of the evident composite

$$Y_1 \xrightarrow{\eta} \hat{C}_1 Y \xrightarrow{\iota} E_0\hat{C}Y.$$

Of course, $\iota\eta$ is a map of G_0-spaces and its adjoint is a map of $\mathscr{G}$-spectra when Y is a $\tilde{G}$-space with $Y_0 = \{0\}$.

We next recall from [14] the identification of EF_0, where F_0 is the zeroth space of a spectrum F; here F_0 is a $\mathscr{K}_\infty$-space and thus a $\mathscr{C}$-space by pullback. The evaluation maps $\Sigma^V F_0 \cong \Sigma^V \Omega^V FV \to FV$ induce maps

$$\omega : B(\Sigma^V, C_V, F_0) \to FV$$

which specify a map of prespectra (see [14, VII.3.2]). If F is a $\mathscr{G}$-spectrum, then ω is a map of $\mathscr{G}$-prespectra by [14, VII.4.3]. On passage to spectra, the composite

$$F_0 \xrightarrow{\iota} EF_0 \xrightarrow{\omega_0} F_0$$

is the identity. Since F_0 is grouplike, ι is an equivalence if $\mathscr{C}'$ is spacewise contractible. This implies the following result.

Proposition 8.5. *For spectra F, there is a natural map of spectra*

$$\omega : EF_0 \to F.$$

If $\mathscr{C}'$ is spacewise contractible, then ω_0 is an equivalence and thus ω is an equivalence if F is connective. If F is a $\mathscr{G}$-spectrum, then ω is a map of $\mathscr{G}$-spectra.

This is closely related to the specialization of the previous result to spaces (or G_0-spaces) Z. Indeed, QZ is the zeroth space of $\Sigma^\infty Z$, and the following diagram commutes:

$$\begin{array}{ccc} ECZ & \xrightarrow{E\alpha_\infty \pi} & EQZ \\ & {\scriptstyle \zeta} \searrow \quad \swarrow {\scriptstyle \omega} & \\ & \Sigma^\infty Z & \end{array}$$

Armed with this understanding of the relationship between the infinite loop space machinery and suspension spectra, we turn to the last unfinished piece of business from Section 4, namely the proof of the last statement of Proposition 4.12.

Proposition 8.6. *For commutative topological monoids $\mathscr{A}_1$, there is a natural equivalence of $\mathscr{L}$-spectra between $E(\mathscr{E}\int\mathscr{A}_1)$ and $\Sigma^\infty(B\mathscr{A}_1^+)$.*

Proof. Abbreviate $\mathscr{A}=\mathscr{E}\int\mathscr{A}_1$. The problem here is that there is no $\mathscr{L}_0$-map $B\mathscr{A}_1^+\to E_0\mathscr{A}$ in sight. By Remarks 3.4, we do have a diagram

$$RB\mathscr{A}_1^+ \xleftarrow{\simeq} B\mathscr{A}_1^+ \longrightarrow B(\mathscr{A} \mid \mathscr{F}\textstyle\int \Pi)$$

of $\mathscr{F}\int\Pi$-spaces and thus, by pullback, of $\mathscr{L}\int\Pi$-spaces. We apply the functor V of Theorem 4.3 to convert this to a diagram of $(\Pi, \mathscr{L})$-spaces (which works since $\mathscr{P}$ and $\mathscr{L}$ are Σ-free). Similarly, regard the $\mathscr{F}\int\mathscr{F}$-space $B\tilde{\mathscr{A}}$ as an $\mathscr{L}\int\mathscr{K}_\infty$-space by pullback and convert it to a $(\mathscr{K}_\infty, \mathscr{L})$-space via V. Of course, $VB\tilde{\mathscr{A}}$ is a $(\Pi, \mathscr{L})$-space by restriction. Since the construction V is functorial on operad pairs, we now have a diagram

$$\begin{array}{ccccccc} R'B\mathscr{A}_1^+ & \xleftarrow{\varepsilon} & UR'B\mathscr{A}_1^+ \cong VRB\mathscr{A}_1^+ & \xleftarrow{\simeq} & VB\tilde{\mathscr{A}}_1^+ \\ & & & & \downarrow \\ & & & & VB(\tilde{\mathscr{A}} \mid \mathscr{F}\textstyle\int\Pi) \\ & & & & \downarrow \\ & & & & VB\tilde{\mathscr{A}} \end{array}$$

of $(\Pi;\mathscr{L})$-spaces, where we have exploited the second statement of Theorem 4.3. Without changing notation, we apply Lemma C.1 to arrange that all zeroth spaces are $\{0\}$. By Lemma 4.7, we then obtain $\mathscr{L}_0$-spaces by passage to first spaces. By Theorem 4.8, we have an $\mathscr{L}_0$-map $(VB\tilde{\mathscr{A}})_1\to E_0VB\tilde{\mathscr{A}}$. Omitting intermediate maps, this yields a diagram

$$B\mathscr{A}_1^+ \xleftarrow{\simeq} (VB\tilde{\mathscr{A}}_1^+)_1 \longrightarrow E_0VB\tilde{\mathscr{A}}$$

of $\mathscr{L}_0$-maps. Applying Σ^∞ and using its freeness, there results a diagram

$$\Sigma^\infty B\mathscr{A}_1^+ \xleftarrow{\simeq} \Sigma^\infty(VB\tilde{\mathscr{A}}_1^+)_1 \longrightarrow EVB\tilde{\mathscr{A}}$$

of $\mathcal{I}$-spectra. It suffices to prove that the second map here is an equivalence, and for this we are free to ignore the multiplicative structure. As a spectrum,

$$EVB\tilde{\mathcal{A}} \simeq E(B\tilde{\mathcal{A}})_{\oplus} \simeq EB\tilde{\mathcal{A}}_{\oplus}$$

by Theorem 4.3 and Remarks 3.2. Write $\mathcal{A}$ for $\mathcal{A}_{\oplus}$ henceforward, regarding it just as an additive permutative category. Under these equivalences and the equivalence $(VB\tilde{\mathcal{A}}_1^+)_1 \simeq B\mathcal{A}_1^+$, the map in question becomes the map $\Sigma^\infty B\mathcal{A}_1^+ \to EB\mathcal{A}$ adjoint to the composite of the map $B\mathcal{A}_1^+ \to B\mathcal{A}$ induced by the inclusion of $\mathcal{A}_1^+$ in $\mathcal{A}$ and the natural group completion $B\mathcal{A} \to E_0B\tilde{\mathcal{A}}$. Here $EB\tilde{\mathcal{A}}$ is constructed from the $\mathcal{F}$-space $B\tilde{\mathcal{A}}$ (of [21, §4]). On the other hand, as explained in [13, §4 and 14, VI §4], there is a certain E_∞ operad $\mathcal{D}$ which acts naturally on $B\mathcal{A}$ for a permutative category $\mathcal{A}$. By the uniqueness theorem for the passage from permutative categories to spectra [17], there is an equivalence $EB\tilde{\mathcal{A}} \simeq EB\mathcal{A}$ which is compatible with the group completions from $B\mathcal{A}$ to the respective zeroth spaces. However, as in [14, VII.5.8], we have

$$D(B\mathcal{A}_1^+) = \coprod_{j\geq 0} \mathcal{D}(j) \times_{\Sigma_j} B\mathcal{A}_1^j = \coprod_{j\geq 0} B(\Sigma_j \int \mathcal{A}_1) = B\mathcal{A}$$

as $\mathcal{D}$-spaces. Taking $\mathcal{C}' = \mathcal{D}$ in the general (additive) theory, the projection $\mathcal{C} \to \mathcal{D}$ induces an equivalence $\hat{C}Y \to \hat{D}Y$ of $\mathcal{C}$-spaces for any Π-space Y (since $\mathcal{D}$ is an E_∞ operad). Therefore

$$EB\mathcal{A} = ED(B\mathcal{A}_1^+) \simeq EC(B\mathcal{A}_1^+) \simeq \Sigma^\infty B\mathcal{A}_1^+$$

by Proposition 8.4, and the conclusion follows.

There is one further generic identification result in [14], and it generalizes directly to the present context.

Remarks 8.7. In [14, VII §5], it is proven that if X is a $(\mathcal{C}, \mathcal{G})$-space with $\pi_0 X$ the nonnegative integers Z^+ and if M is a nontrivial multiplicative submonoid of Z^+, then, under a homological stability hypothesis, there is a natural equivalence of 'multiplicative' infinite loop spaces

$$(SFEX)[M^{-1}] = (E_0X)_1[M^{-1}] \simeq E_0(X_M)_1.$$

Here X_M denotes the union of those components of X in M; this is a $\mathcal{G}$-space, and $E_0(X_M)$ is the zeroth space of the associated spectrum. The subscripts 1 refer to components of 1, and the claim is that the localization away from M of the 1-component of the zeroth space of the spectrum EX obtained from the additive $\mathcal{C}$-space X is equivalent via the zeroth map of a map of 'multiplicative' spectra to $E_0(X_M)_1$. The result remains true, with precisely the same proof, with X replaced by a $(\hat{\mathcal{C}}, \mathcal{G})$-space Y with $Y_0 = \{0\}$ and $\pi_0 Y_1 = Z^+$ and with X_M replaced by $(Y_1)_M$. The case $Y = VB\mathcal{A}$ for appropriate bipermutative categories $\mathcal{A}$ is of particular interest.

Appendix A. Corrigenda to the theory of E_∞ ring spaces

I shall explain just where the error occurs in the passage from bipermutative categories to E_∞ ring spaces proposed in [14] and will point out how the results above provide substitutes for details based on that passage. I will also take the opportunity to give other corrigenda and addenda to [14]. Errata and addenda to [12] and [13] are given in [5, p. 485–490], and a minor error in [5] is corrected in [20, p. 635]. Those homological calculations of [5] which start from bipermutative categories will be justified in [6].

Scholium A.1. Since the classifying space of a permutative category is a $\mathscr{P}$-space for a certain categorical operad $\mathscr{P}$ and there are evident maps

$$\lambda : \mathscr{P}(k) \times \mathscr{P}(j_1) \times \cdots \times \mathscr{P}(j_k) \to \mathscr{P}(j_1 \cdots j_k),$$

it seems intuitively obvious that $(\mathscr{P}, \mathscr{P})$ is an operad pair and that the two actions of $\mathscr{P}$ on the classfying space of a bipermutative category give it a structure of $(\mathscr{P}, \mathscr{P})$-space. It is these and related 'obvious' assertions in [14] that are false. They implicitly demand both the left and right distributive laws to hold strictly. On the category level, the assertions hold up to natural isomorphism; on the space level, they hold up to homotopy. This is inadequate for multiplicative infinite loop space theory.

The mistake shows up most clearly in [14, VI.2.3], where it is claimed that the operad $\mathscr{M}$ with each $\mathscr{M}_j = \Sigma_j$ acts on itself. In fact, the equivariance formulas required of action maps λ yield an incompatible overdetermination. An $\mathscr{M}$-space is a topological monoid, and an $(\mathscr{M}, \mathscr{M})$-space would have to be a topological semi-ring with noncommutative addition. If there were such a theory, there would be such objects (namely free ones) with no commutation relations and satisfying cancellation laws. But this is absurd. Distributivity would give

$$(x+y)(z+w) = (x+y)z + (x+y)w = xz + yz + xw + yw;$$
$$(x+y)(z+w) = x(z+w) + y(z+w) = xz + xw + yz + yw.$$

Cancellation and the case $z = w = 1$ would give $y + x = x + y$.

Thus all references to the operad pairs $(\mathscr{M}, \mathscr{M})$ and $(\mathscr{P}, \mathscr{P})$ in [14] are nonsense. However, as illustrated by our proof of Proposition 8.6, use of $\mathscr{P}$ alone is entirely correct and often convenient.

As a matter of speculation, there may well be an alternative generalization of the theory of [14] which does allow use of $(\mathscr{P}, \mathscr{P})$. It may be that the maps λ specify a 'lax' action of $\mathscr{P}$ on $\mathscr{P}$ and that $(\mathscr{P}, \mathscr{P})$ acts on $B\mathscr{A}$ in a 'lax' sense. The appropriate lax weakening of the distributivity data of Proposition 5.6 has been developed and exploited by Kelly [9]. With such a definitional framework, one might never have to use Cartesian products up to homotopy, but I have not pursued the idea.

We list the resultant changes, and other corrections and addenda, to [14]. I would like to thank Bruner, Fiedorowicz, Lewis, McClure, E. Miller, Steinberger, and Weibel for spotting errors. The essential points are that the theory of spectra outlined in Chapter II is made rigorous and given an equivariant generalization in [3], that the present theory fully compensates for the errors in Chapter VI, that the previous section streamlines as well as generalizes the theory of Chapter VII, and that Chapter IX is supplanted by [21]. The chapters devoted to applications, namely I, III, IV, V, and VIII, require no significant changes. In retrospect, the time was ripe for the applications when [14] was written, but the theory was premature: if the mistake discussed above had been noticed then, there would have been little chance for a satisfactory repair.

Corrigenda A.2. (Page references are to [14].)

(1) p. 11, 13, 27, 73: The closed image condition on the relevant inclusions is unnecessary in the definitions of $\mathscr{I}$-functors, $\mathscr{I}_*$-functors, prespectra, and $\mathscr{I}_*$-prefunctors.

(2) p. 27: In the interest of simplicity, the use of isometries should be eliminated from the definition of prespectra; by II.1.10 (p. 32), there is no significant loss of information.

(3) p. 29–31: The notations Σ^∞, Ω^∞, and Q_∞ are awkward. In this and other more recent papers, I use the notations $\{\Sigma^V X\}$, L, and Σ^∞ for these notions (respectively), reserving Ω^∞ for the zeroth space functor from spectra to spaces.

(4) p. 32–35: This approach to the stable category requires use of prespectra defined without an inclusion condition. Details are given by Lewis [10], and he and I have generalized this material to a development of stable categories of G-spectra for compact Lie groups G in [3].

(5) p. 39: The promised $\lim^1 = 0$ discussion is given by McClure [23, 3].

(6) p. 41–42: Details, minor corrections, and careful consideration of the $\lim^1$ terms implicit in this comparison of Whitehead prespectra to spectra are also given by McClure [23, 3].

(7) p. 67–68: The continuity condition IV.1.1(d) will be strengthened in [3]; in line with (2), condition IV.1.1(f) should be deleted.

(8) p. 89: The reference to Brumfiel should be [24].

(9) p. 99.: The last part of Lemma 2.4 should be interpreted as saying that the equivalence (λ, q) gives $B(SF; E_T)$ an infinite loop structure.

(10) p. 136: Friedlander [7] has proven the infinite loop complex Adams conjecture. Together with the uniqueness theorem of [18], this completes the odd primary infinite loop analysis of classifying spaces studied in V §7.

(11) p. 144: Misprints in VI.1.6(a) and (a′) are corrected in Definition 1.8 here.

(12) p. 148–150: Lemmas 2.3 and 2.6 and Remarks 2.7(i) are incorrect: $\mathscr{M}$ does not act on $\mathscr{M}$ and $\mathscr{D}$ does not act on $\mathscr{D}$; however, Remarks 2.7(ii)–(iv) remain correct and useful.

(13) p. 158–160: The diagram of Lemma 4.3 commutes only up to natural isomorphism and Proposition 4.4 fails; Remarks 4.5 are correct, but Remarks 3.3 here

provide a better way of looking at the roles of $\mathscr{A}_0$ and $\mathscr{A}_1$.

(14) p. 161, 162, 167: The references to $(\mathscr{Q}, \mathscr{Q})$ must be deleted; Remarks 3.4 and 3.5 here provide generalized substitutes, and Propositions 4.12 and 8.6 show that these substitutes have the desired topological implications.

(15) p. 169–180: The Steiner operads [26] greatly simplify this theory.

(16) p. 181–183: Elimination of isometries as in (2) eliminates the need for restrictions on X here.

(17) p. 187: As pointed out in [22, A.3], Ω is really the shift desuspension in this proof; this makes no difference to the result.

(18) p. 193, 199: Proposition 4.12 here provides a generalized substitute for the first of these usages of $(\mathscr{Q}, \mathscr{Q})$ and combines with Remarks 8.7 to substitute for the second.

(19) p. 204: The present theory substitutes for this mention of $(\mathscr{Q}, \mathscr{Q})$.

(20) p. 205: While the assertion that the plus construction associated to a perfect commutator subgroup of π_1 always yields a simple space is obviously nonsense, the simplicity is immediately apparent in the relevant applications.

(21) p. 208: The diagram is not commutative but is homotopy commutative, as will be shown in [6]; this suffices for the applications.

(22) p. 236–237: $\mathscr{Q}$ must be eliminated from this discussion and thus DS^0 must be interpreted as $B\mathscr{E}$. In view of Proposition 4.12 and Remarks 8.7, the infinite loop diagrams on the cited pages can be reconstructed from the present theory with only a few changes of notation.

(23) p. 244–256: This theory of pairings contains an error (in IX.1.4) and is entirely superceded by the much sharper and more general treatment of [21].

Appendix B. Corrigenda to the theory of A_∞ ring spaces

In [19], I studied A_∞ ring spaces. These are $(\mathscr{C}, \mathscr{G})$-spaces X, where $\mathscr{C}$ is a spacewise contractible (or E_∞) operad and $\mathscr{G}$ is a spacewise contractible non-Σ operad (i.e. operad without permutations). These notions are specified by Definition 1.8, with condition (c) deleted, and Definition 2.5. The present theory generalizes [19, §1–3]. However, $\pi_0 X$ was assumed to be a ring in the substantive parts of [19] and, under this hypothesis, there is no loss of generality in restricting attention to actual $(\mathscr{C}, \mathscr{G})$-spaces.

I used $(\mathscr{Q}, \mathscr{Q})$ in [19] in an attempt to arrange the hypothesis $\mathscr{C}(1) = \{1\}$ (and to get around problems now obviated by use of the Steiner operad). I also made a related combinatorial error, and the details to follow would simplify if one actually could arrange $\mathscr{C}(1) = \{1\}$. The error occurs in the construction of non-Σ operads $\mathscr{H}_n$ acting on the space $M_n X$ of $n \times n$ matrices with entries in a $(\mathscr{C}, \mathscr{G})$-space X.

Scholium B.1. I described the combinatorial details as 'perfectly straightforward' on page 257 of [19]. It serves me right that I got them wrong on page 258. The error begins on line 9; there there are n^{k+1} rather than $n^{m-1}\varepsilon_r$'s; the misprint is not important, but the fact that there are n^{m+1} and not n^{m-1} indices r such that $\varepsilon_r = 1$ is. In fact, with the line of argument there, one would have to define a different $t(j_1, \ldots, j_k)$ for each matrix position (r, s). Then b in (8) would depend on (r, s) and, to allow for this, c must also depend on (r, s).

Steinberger found this mistake and also found a quite simple solution, which he sketched in [25]. We give some details.

Definitions B.2. Let $(\mathscr{C}, \mathscr{G})$ be an operad pair, with $\mathscr{G}$ non-Σ. Fix $n \geq 1$. Define

$$\mathscr{H}_n(j) = M_n\mathscr{C}(n^{j-1}) \times \mathscr{G}(j) \quad \text{for } j > 0.$$

Define $\mathscr{H}_n(0) = \mathscr{C}(1)^n \times \mathscr{G}(0)$, where $\mathscr{C}(1)^n$ is thought of as the space of matrices with diagonal entries in $\mathscr{C}(1)$ and with all remaining entries the point $* \in \mathscr{C}(0)$. Give $\mathscr{H}_n(0)$ the basepoint $* = (1^n, *)$, where $1 \in \mathscr{C}(1)$ is the unit. Let $1 = (J_n, 1) \in \mathscr{H}_n(1)$, where $J_n \in M_n\mathscr{C}(1)$ is the matrix all of whose entries are 1. For $1 \leq r \leq n$, $1 \leq s \leq n$, and $j \geq 1$, let $T(r, s, j)$ be the set of sequences $U = (u_0, \ldots, u_j)$ such that $u_0 = r$, $1 \leq u_i \leq n$, and $u_j = s$. Let $T(r, s, 0)$ be empty if $r \neq s$ and contain just the sequence (r) if $r = s$. Define maps

$$\gamma : \mathscr{H}_n(k) \times \mathscr{H}_n(j_1) \times \cdots \times \mathscr{H}_n(j_k) \to \mathscr{H}_n(j), \quad j = j_1 + \cdots + j_k,$$

by

$$\gamma\left((c; g); \mathop{\times}_{i=1}^{k} (c_i; g_i)\right) = \left(d\zeta; \gamma\left(g; \mathop{\times}_{i=1}^{k} g_i\right)\right).$$

Here $d \in M_n\mathscr{C}(n^{j-1})$ has (r,s)th entry

$$d(r,s) = \gamma\left(c(r,s); \mathop{\times}_{V \in T(r,s,k)} \lambda\left(g; \mathop{\times}_{i=1}^{k} c_i(v_{i-1}, v_i)\right)\right)$$

and $\zeta \in \Sigma_{n^{j-1}}$ is the permutation described as follows. Via an obvious splicing of sequences construction, we have a bijection

$$T(r,s,j) \leftrightarrow \coprod_{V \in T(r,s,k)} T(v_0, v_1, j_1) \times \cdots \times T(v_{k-1}, v_k, j_k);$$

ζ converts the lexicographic ordering on the left to the ordering via the lexicographic ordering of $T(r, s, k)$ and, for fixed V, the lexicographic ordering of the product of k ordered sets on the right. If $k = 0$, γ is to be interpreted as the identity map of $\mathscr{H}_n(0)$; the interpretation when any of the j_i are zero is forced.

We now redefine the notion of a non-Σ operad by letting the 0th space be a based space rather than a point; for an A_∞ operad, we require spacewise contractibility. Then [19, 4.1] takes the following corrected form.

Theorem B.3. *The $\mathscr{H}_n$ are non-Σ operads and are A_∞ operads if $\mathscr{C}$ and $\mathscr{G}$ are space-*

wise contractible. for a $(\mathscr{C},\mathscr{G})$*-space* X*, define*

$$\psi_j : \mathscr{K}_n(j) \times (M_n X)^j \to M_n X$$

by

$$\psi_j(c,g;x_1,\ldots,x_j)(r,s) = \theta_{n^{j-1}}\left(c(r,s); \underset{U \in T(r,s,j)}{\times} \xi_j\left(g; \overset{j}{\underset{i=1}{\times}} x_i(u_{i-1},u_i)\right)\right);$$

when $j=0$*, the definition is to be interpreted as*

$$\psi_0(c,*,I_n)(r,s) = \begin{cases} \theta_1(c(r,r),1) & \text{if } r=s, \\ 0 & \text{if } r \neq s. \end{cases}$$

Then the ψ_j *specify an action of* $\mathscr{K}_n$ *on* $M_n X$.

Proof. The formula for ψ_j is just (1) of [19, p. 256] but with $c(r,s)$ in place of c on the right. Except that steps (4)–(9) can now be omitted, with a concomitant simplification of notation, the details are as given in [19, p. 257–260]. Dare I say that they are perfectly straightforward?

The following is a direct consequence of the definitions.

Corollary B.4. *The maps* $\gamma : \mathscr{K}_n(j) \times \mathscr{K}_n(0)^j \to \mathscr{K}_n(0)$ *specify an action of* $\mathscr{K}_n$ *on the space* $\mathscr{K}_n(0)$ *such that the map* $\psi_0 : \mathscr{K}_n(0) \to M_n X$ *is an* $\mathscr{K}_n$*-map. If* $n=1$ *and* $\mathscr{C}(1)$ *is connected,* ψ_0 *takes values in the component of* $1 \in X$.

To complete our repairs, we also need the following result (stated in passing in [19, 6.4]).

Proposition B.5. *There are maps* $\tau_{m,n} : \mathscr{K}_{m+n} \to \mathscr{K}_m \times \mathscr{K}_n$ *of non-*Σ *operads such that the block sum of matrices map* $\oplus : M_m X \times M_n X \to M_{m+n} X$ *is an* $\mathscr{K}_{m+n}$*-map.*

Proof. Define $\tau : \mathscr{K}_{n+1}(j) \to \mathscr{K}_n(j)$ by $\tau(c,g) = (d,g)$, where $d(r,s) = \gamma(c(r,s); t_{nj})$; here t_{nj} is the degeneracy operator specified in [19, p. 266]. As in the proof on that page (more perfectly straightforward combinatorics), these maps specify a morphism $\mathscr{K}_{n+1} \to \mathscr{K}_n$ of non-Σ operads related to the standard inclusion $\nu_n : M_n X \to M_{n+1} X$, $\nu_n(x) = x \oplus I_1$. By symmetry, there is a morphism $\tau' : \mathscr{K}_{n+1} \to \mathscr{K}_n$ related to the inclusion ν_n', $\nu_n'(x) = I_1 \oplus x$. The components of $\tau_{m,n}$ in $\mathscr{K}_m$ and $\mathscr{K}_n$ are m-fold and n-fold iterates of τ and τ', respectively. That $\oplus$ is an $\mathscr{K}_{m+n}$-map uses more perfectly straightforward combinatorics, along the lines of [19, p. 267].

With these results on hand, we can record the changes required to recover the conclusions asserted in [19].

Corrigenda B.6. (Page references are to [19].)

(1) p. 257–260: Theorem 4.1 must be replaced by Theorem B.3 here.

(2) p. 263–264: Proposition 5.2 is correct, but the proof needs easy changes to account for our corrected definition of the $\mathscr{K}_n$.

(3) p. 266–269: Theorem 6.1 is correct, but I know of no interesting cases where its hypothesis $\mathscr{C}(1)=\{1\}$ holds.

(4) p. 273–275: With $\mathscr{C}(1)\neq\{1\}$, ν_n is not an $\mathscr{H}_{n+1}$-map. However, the results above give the following diagram of $\mathscr{H}_{n+1}$-maps relating the spaces of unit components FM_nX and $FM_{n+1}X$:

$$FM_nX \xleftarrow{\pi_1} FM_nX\times\mathscr{H}_1(0) \xrightarrow{1\times\psi_0} FM_nX\times FX \xrightarrow{\oplus} FM_{n+1}X.$$

Here π_1 is an equivalence since $\mathscr{H}_1(0)$ is contractible. We need only replace ν_n by $\oplus\circ(1\times\psi_0)$ to retrieve the passage to classifying spaces and telescopes discussed on these pages.

(5) p. 279–281: The space F_nX of monomial matrices is a sub $\mathscr{H}_n$-space of FM_nX, and the following diagram of $\mathscr{H}_{n+1}$-spaces includes in the diagram of (4):

$$F_nX \xleftarrow{\pi} F_nX\times\mathscr{H}_1(0) \xrightarrow{1\times\psi_0} F_nX\times FX \xrightarrow{\oplus} F_{n+1}X.$$

However, with $\mathscr{C}(1)\neq\{1\}$, the homeomorphism $\alpha_n: F_nX\to\Sigma_n\int FX$ of p. 280 is not an $\mathscr{H}_n$-map. One can use the contractibility of $\mathscr{C}(1)$ to show that α_n is an sh $\mathscr{H}_n$-map in the sense of Lada [5] and that the diagram above maps into the diagram

$$\Sigma_n\int FX \xleftarrow{\pi} (\Sigma_n\int FX)\times\mathscr{H}_1(0) \xrightarrow{1\times\psi_0} (\Sigma_n\int FX)\times FX \xrightarrow{\oplus} \Sigma_{n+1}\int FX$$

with the appropriate conditions on higher homotopies to allow delooping of the right hand square; see [5, p. 246]. This gives one way of recovering the results of [19, §7], but the details are unpleasant.

Steinberger [25] has proven [19, Conjecture 12.5], which asserts that my definition of the algebraic K-theory of spaces agrees with Waldhausen's [30].

Steiner [27] has shown that the algebraic K-theory space KX of a $(\mathscr{C},\mathscr{G})$-space X is actually an infinite loop space (something I was not at all sure would be true) and that Steinberger's equivalence is one of infinite loop spaces. With his approach, one can entirely dispense with the operads $\mathscr{H}_n$.

Igusa [33] has recently shown that, in the interesting cases, M_nX inherits a structure of A_∞ ring space from X. He has also made the first concrete calculational application of A_∞ ring space theory.

Appendix C. Whiskerings of $\mathscr{J}$-spaces

Let $\mathscr{J}=\hat{\mathscr{G}}\int\hat{\mathscr{C}}$ for an operad pair $(\mathscr{C},\mathscr{G})$. We first show how to arrange $Y_0=\{0\}$ for $(\hat{\mathscr{C}},\mathscr{G})$-spaces Y and then show how to arrange part of the cofibration condition required of $\mathscr{J}$-spaces X in Definition 2.1(5). In Remark C.8, we give an alternative way of arranging this condition in the cases of greatest interest.

Lemma C.1. *Let Y be a $(\hat{\mathscr{C}},\mathscr{G})$-space and define $\bar{Y}$ by letting $\bar{Y}_s=Y_s/Y_0$, where Y_0 is* *embedded in Y_s via $0:\mathbf{0}\to\mathbf{s}$. Then $\tilde{Y}$ is a $(\hat{\mathscr{C}},\mathscr{G})$-space with $\tilde{Y}_0=\{0\}$, and the projection $Y\to\tilde{Y}$ is a natural equivalence of $(\hat{\mathscr{C}},\mathscr{G})$-spaces.*

Proof. Since $\mathbf{0}$ is an initial object of Π, it is easy to see that $\tilde{Y}$ inherits a structure of Π-space from Y. We must check that $R^n\tilde{Y}$ inherits a structure of $\mathscr{J}$-space. For a morphism $(g;c):(m;R)\to(n;S)$ in $\mathscr{J}$, we must see that the following diagram can be completed:

$$\begin{array}{ccc} Y_{r_1}\times\cdots\times Y_{r_m} & \xrightarrow{(g;c)} & Y_{s_1}\times\cdots\times Y_{s_n} \\ \downarrow & & \downarrow \\ \tilde{Y}_{r_1}\times\cdots\times \tilde{Y}_{r_m} & \overset{(g;c)}{\dashrightarrow} & \tilde{Y}_{s_1}\times\cdots\times \tilde{Y}_{s_n} \end{array}$$

Consider any morphism $(1;\chi):(m;P)\to(m;R)$ such that, for $1\le i\le m$, either $\mathbf{p}_i=\mathbf{r}_i$ and $\chi_i=1$ or $\mathbf{p}_i=0$. We construct a square

$$\begin{array}{ccc} (m;P) & \overset{(g;b)}{\dashrightarrow} & (n;Q) \\ {\scriptstyle(1;\chi)}\downarrow & & \downarrow{\scriptstyle(1;\omega)} \\ (m;R) & \xrightarrow{(g;c)} & (n;S) \end{array}$$

Thus let $\mathbf{q}_j=\mathbf{s}_j$, $\omega_j=1$, and $b_j=c_j$ if $\chi_i=1$ for all i such that $\phi(i)=j$, where $\phi=\varepsilon(g)$, and let $\mathbf{q}_j=0$, $\omega_j=0$, and $b_j=0$ if $\chi_i=0$ for any such i. A moment's reflection shows that these squares imply the existence of the dotted arrows $(g;c)$ and these clearly specify $\tilde{Y}$ as a functor $\mathscr{J}\to\mathscr{U}$.

Remarks C.2. One might think that it would be equally simple to replace $\mathscr{J}$-spaces X by $\mathscr{J}$-spaces $\tilde{X}$ with $\tilde{X}(0;*)=\{*\}$ by letting $\tilde{X}(n;S)=X(n;S)/X(0;*)$, where $X(0,*)$ is embedded in $X(n;S)$ via $(0;0^n):(0;*)\to(n;S)$. In fact, $\tilde{X}$ is not even a $\Pi\int\Pi$-space, let alone a $\mathscr{J}$-space, since $(0;*)$ is not an initial object of $\Pi\int\Pi$. If $(\phi;\chi):(m;R)\to(n;S)$ is a morphism with $\phi^{-1}(j)$ empty and $\chi_j\neq0$, then $(\phi;\chi)(0;0^m)\neq(0;0^n)$. Note that $B\mathscr{A}(0;*)\neq\{*\}$ for the $\mathscr{F}\int\mathscr{F}$-spaces constructed from bipermutative categories in Section 3.

It would be asking too much to try to arrange the entire cofibration condition of Definition 1.2(5) by growing whiskers. If Z is a commutative topological semi-ring and we try to attach whiskers at 0 and at 1 to replace these by nondegenerate basepoints, then we quickly see that there is no reasonable way to define addition on the whisker attached at 1.

We explain briefly what can be achieved. Regard $I=[0,1]$ as a commutative topological semi-ring with 0 and 1 as zero and identity elements by letting

$$s+t=\max(s,t)\quad\text{and}\quad st=\min(s,t).$$

Then RI is an $\mathscr{F}\int\mathscr{F}$-space by Lemma 2.3 and thus a $\mathscr{J}$-space by pullback. For a functor $X:\mathscr{J}\to\mathscr{U}$, we have a product functor $I\times X:\mathscr{J}\to\mathscr{U}$ with

$$(I\times X)(n;S)=I^{s_1}\times\cdots\times I^{s_n}\times X(n;S).$$

Definition C.3. Define $WX(n;S)$ to be the subspace of $(I\times X)(n;S)$ which consists of those points (t,x),

$$t=\mathop{\times}_{j=1}^{n}\ \mathop{\times}_{\upsilon=1}^{s_j} t_{j,\upsilon},$$

such that $x\in\operatorname{Im}(1;\omega)$, where $\omega_j:\mathbf{q}_j\to\mathbf{s}_j$ is that ordered injection such that $\upsilon\in\operatorname{Im}\omega_j$ if and only if $t_{j,\upsilon}=1$. Define $\pi:WX(n;S)\to X(n;S)$ by $\pi(t,x)=x$. Observe that $X(n;S)$ embeds in $WX(n;S)$ as the subspace of points (t,x) with all $t_{j,\upsilon}=1$ and that $X(n;S)$ is thereby a deformation retract of $WX(n;S)$.

Lemma C.4. *WX is a subfunctor of $I\times X:\mathscr{J}\to\mathscr{U}$.*

Proof. Consider $(g;c):(m;R)\to(n;S)$ in $\mathscr{J}$ with $\varepsilon(g)=\phi$ and $\varepsilon(c)=\chi$. Let $(d,x)\in WX(m;R)$. We must check that $(g;c)(d;x)\in WX(n;S)$. We have

$$(g;c)(d,x)=(t,(g;c)(x))\quad\text{where}\quad t_{j,\upsilon}=\max_{\chi_j(U)=\upsilon}\ \min_{\phi(i)=j} d_{i,u_i}.$$

In particular, $t_{j,\upsilon}=0$ if $\upsilon\notin\operatorname{Im}\chi_j$ and $t_{j,\upsilon}=1$ if $j\notin\operatorname{Im}\phi$ and $\upsilon\in\operatorname{Im}\chi_j$. Let $\nu_i:\mathbf{p}_i\to\mathbf{r}_i$ and $\omega_j:\mathbf{q}_j\to\mathbf{s}_j$ be those ordered injections such that $u\in\operatorname{Im}\nu_i$ if and only if $d_{i,u}=1$ and $\upsilon\in\operatorname{Im}\omega_j$ if and only if $t_{j,\upsilon}=1$. We are given that $x\in\operatorname{Im}(1;\nu)$ and must verify that $(g;c)(x)\in\operatorname{Im}(1;\omega)$. By easy separate checks of the cases $j\in\operatorname{Im}\phi$ and $j\notin\operatorname{Im}\phi$, the latter separated into the cases $\chi_j=0$ and $\chi_j\neq 0$, we see that there is a morphism ζ_j in $\mathscr{F}$ such that the following square commutes:

$$\begin{array}{ccc}
\bigwedge_{\phi(i)=j}\mathbf{p}_i & \xrightarrow{\ \zeta_j\ } & \mathbf{q}_j\\
{\scriptstyle\wedge\nu_i}\big\downarrow & & \big\downarrow{\scriptstyle\omega_j}\\
\bigwedge_{\phi(i)=j}\mathbf{r}_i & \xrightarrow{\ \chi_j\ } & \mathbf{s}_j
\end{array}$$

An elaboration, as in [22, B.3], shows that there exists b such that

$$(1;\omega)(g;b)=(g;c)(1;\nu),$$

and this implies the conclusion.

Clearly $\pi:WX\to X$ is a natural transformation of functors $\mathscr{J}\to\mathscr{U}$ and a space-wise equivalence. Also, $WX(0;*)=X(0;*)$. Thus if X satisfies (1)–(4) of Definition 2.1, then so does WX. As to cofibration conditions, we have the following assertions.

Lemma C.5. *If* $(1;\chi):(n;R)\to(n;S)$ *is such that each* $\chi_j:\mathbf{r}_j\to\mathbf{s}_j$ *is an injection, then* $(1;\chi):WX(n;R)\to WX(n;S)$ *is a* $\Sigma(1;\chi)$*-equivariant cofibration.*

Proof. Taking $(g;c)=(1;\chi)$ in the previous proof, we see that each ζ_j is an isomorphism since $p_j=q_j$ by inspection and the diagram gives that ζ_j is an injection. From this we see that

$$\operatorname{Im}(1;\chi)=\{(t,x)\mid t_{j,v}=0 \text{ if } v\notin \operatorname{Im}\chi_j\}\subset WX(n;S).$$

Indeed, if $x\in\operatorname{Im}(1;\omega)$, then $x\in\operatorname{Im}(1;\chi)$ since $(1;\omega)(1;\zeta)=(1;\chi)(1;v)$. The conclusion follows from this description as in [22, B.4].

Remark C.6. Any injection $\Pi\int\Pi$ is a composite of one of the form $(1;\chi)$ as in the previous lemma and one of the form $(\phi;\lambda)$, where $\mathbf{r}_j=\mathbf{s}_j$ and λ_j is the identity for all $j\in\operatorname{Im}\phi$ (the $\lambda_j:\mathbf{1}\to\mathbf{s}_j$ being arbitrary when $j\notin\operatorname{Im}\phi$). Suppose that all such $(\phi;\lambda):X(m;R)\to X(n;S)$ are $\Sigma(\phi;\lambda)$-cofibrations. By a modification of the proof that products of cofibrations are cofibrations to account for the extra condition on the X-coordinate of elements of WX, one can show that the $(\phi;\lambda)$ are also $\Sigma(\phi;\lambda)$-cofibrations for WX and thus that WX satisfies the cofibration condition of Definition 2.1(5).

Remark C.7. For a based space Z, let WZ be formed by attaching a whisker to the basepoint. For a functor Y from Π to based spaces with $Y_s\simeq Y_1^s$, let WY be constructed as in [22, App. B] (but using the sum rather than the product to specify $R'I$ as an $\mathscr{F}$-space) and then let $\bar{W}Y$ be constructed from WY as in Lemma C.1. Then $\bar{W}Y$ is a Π-space with $\bar{W}_0Y=\{0\}$. Given second base points $1\in Z$ and $1\in Y_1$, we find by inspection that $R'WZ=WR'Z$ and $R''WY=WR''Y$. Thus, by Lemmas C.4 and C.1, if RZ or $R''Y$ extends to a functor $\mathscr{J}\to\mathscr{U}$, then so does RWZ or $R''\bar{W}Y$. (While $1\in WZ$ and $1\in\bar{W}_1Y$ need not be nondegenerate, this condition plays no real role in the theory of Section 8.) Thus whiskering of $(\mathscr{C},\mathscr{G})$-spaces Z or $(\hat{\mathscr{C}},\mathscr{G})$-spaces Y without cofibration conditions gives them all of the properties needed for the passage to $\mathscr{G}$-spectra.

Remark C.8. There is an alternative procedure for arranging our cofibration condition that applies when $\mathscr{C}$ and $\mathscr{G}$ are both spacewise contractible. The replacement of $\mathscr{J}$-spaces by $\mathscr{F}\int\mathscr{F}$-spaces of Theorem 2.9 does not depend on cofibration conditions. Because $\mathscr{F}\int\mathscr{F}$ is discrete, application of the geometric realization of the total singular complex functor replaces functors $\mathscr{F}\int\mathscr{F}\to\mathscr{U}$ by equivalent ones for which injections in $\Pi\int\Pi$ induce inclusions of subcomplexes in CW-complexes (with the appropriate equivariance).

Thus the cofibration condition results in no real loss of generality.

Appendix D. Comparison with the theory of Segal and Woolfson

In [24, §5], Segal hinted at an alternative approach to multiplicative infinite loop space theory, and Woolfson supplied some details in [31]. We give a precise comparison of Woolfson's input to our input and a rough comparison of his output to our output.

Woolfson's input is based on use of a different wreath product of $\mathscr{F}$ with itself than our $\mathscr{F}\wr\mathscr{F}$. The following recollections will lead to a precise comparison.

For a permutative category $\mathscr{A}$, let $\mathscr{A}$ also denote the associated lax functor $\mathscr{F}\to\mathrm{Cat}$ with $\mathbf{n}\mapsto\mathscr{A}^n$ ([29, §4 or 21, §4]) and let $\bar{\mathscr{A}}:\mathscr{F}\to\mathrm{Cat}$ be the actual functor of [17, Const. 10]; we assume familiarity with the cited construction and we write $\pi_{s,t}=i^{-1}_{(s,t)}$ in it. The following pair of results was proven in [21, App.]. The relevant categorical definitions are given in [21, 3.1–3.3].

Proposition D.1. *There are lax natural transformations $\delta:\bar{\mathscr{A}}\to\mathscr{A}$ and $\nu:\mathscr{A}\to\bar{\mathscr{A}}$ such that $\delta\nu=\mathrm{Id}:\mathscr{A}\to\mathscr{A}$ and there is a natural homotopy $\xi:\mathrm{Id}\to\nu\delta$ of lax natural transformations of functors $\bar{\mathscr{A}}\to\bar{\mathscr{A}}$ such that each $\xi(n)$ is a natural isomorphism of functors $\bar{\mathscr{A}}_n\to\bar{\mathscr{A}}_n$.*

Corollary D.2. *There is a natural transformation $\bar{\mathscr{A}}\to\bar{\bar{\mathscr{A}}}$ of functors $\mathscr{F}\to\mathrm{Cat}$ such that each $\bar{\mathscr{A}}_n\to\bar{\bar{\mathscr{A}}}_n$ induces an equivalence on passage to classifying spaces.*

Here $\bar{\bar{\mathscr{A}}}$ is Street's first construction on the lax functor $\mathscr{A}$.

There is a wreath product construction which associates a category $\mathscr{G}\wr\mathscr{B}$ to a lax functor $\mathscr{B}:\mathscr{G}\to\mathrm{Cat}$, a functor $\mathscr{G}\wr b:\mathscr{G}\wr\mathscr{B}\to\mathscr{G}\wr\mathscr{B}'$ to a lax natural transformation $b:\mathscr{B}\to\mathscr{B}'$, and a natural transformation $\mathscr{G}\wr b\to\mathscr{G}\wr b'$ to a natural homotopy $\beta:b\to b'$. See for example Thomason [29; §3]; the last clause is not stated there but is easily verified (and was in Thomason's thesis). Thus the proposition has the following consequence.

Corollary D.3. *There are functors $\iota=\mathscr{F}\wr\nu:\mathscr{F}\wr\mathscr{A}\to\mathscr{F}\wr\bar{\mathscr{A}}$ and $\varrho=\mathscr{F}\wr\delta:\mathscr{F}\wr\bar{\mathscr{A}}\to\mathscr{F}\wr\mathscr{A}$ such that $\varrho\iota=\mathrm{Id}$ and there is a natural isomorphism $\xi:\mathrm{Id}\to\iota\varrho$.*

Thus $\mathscr{F}\wr\mathscr{A}$ is equivalent to and a retract of $\mathscr{F}\wr\bar{\mathscr{A}}$. Now take $\mathscr{A}=\mathscr{F}$ regarded as a permutative category under smash product. Then $\mathscr{F}\wr\bar{\mathscr{F}}$ is the category Woolfson denotes $\Gamma\wr\Gamma$ (except that he omits the detailed specification of $\bar{\mathscr{F}}$).

Definition D.4. An $\mathscr{F}\wr\bar{\mathscr{F}}$-space is a functor $X:\mathscr{F}\wr\bar{\mathscr{F}}\to\mathscr{U}$ such that the restriction of X to $\mathscr{F}\wr\mathscr{F}$ is an $\mathscr{F}\wr\mathscr{F}$-space.

Scholium D.5. This differs from Woolfson's notion of a 'hyper-Γ-space' [29, 2.1] in two respects. First, he requires $X(0;*)=S^0$ and requires the functor X to take values in the category of based spaces. As the details of Section 2 make clear, these conventions are incompatible with the unit axioms for rings. Second, he requires no cofibration conditions. In view of the arguments of the previous appendix, the added conditions cause no loss of generality. They have the usual convenience of allowing use of the standard, unthickened, product-preserving geometric realization.

Woolfson sketched a construction associating $\mathscr{F}\int\tilde{\mathscr{F}}$-spaces to certain bipermutative categories. I failed to understand what was intended and was convinced that the idea was completely wrong when I wrote [20, p. 632]. By way of penance, and because the construction is of definite interest, I shall present a detailed treatment. This should also give a feel for the relationship between $\mathscr{F}\int\tilde{\mathscr{F}}$ and $\mathscr{F}\int\mathscr{F}$, and we must first establish notations describing the former. Its objects are of the form $(n;N)$ where $n\geq 0$ and $N=\langle N_s,\pi_{s,t}\rangle$ is an object of $\tilde{\mathscr{F}}_n$. Here s runs through the based subsets of $\mathbf{n}$, (s,t) runs through the pairs of subsets with $s\cap t=\{0\}$, N_s is an object of $\mathscr{F}$ with $N_0=1$, and $\{\pi_{s,t}\}$ is a unital, associative, and commutative system of isomorphisms $\pi_{s,t}:N_s\wedge N_t\to N_{s\cup t}$ (as in [17, Const. 10] with $\pi_{s,t}=i^{-1}_{(s,t)}$). The morphisms are pairs $(\phi;\mu):(m;M)\to(n;N)$ with $\phi:\mathbf{m}\to\mathbf{n}$ in $\mathscr{F}$ and $\mu:\phi_*M\to N$ in $\tilde{\mathscr{F}}_n$, where $\phi_*:\tilde{\mathscr{F}}_m\to\tilde{\mathscr{F}}_n$ denotes the functor induced by ϕ. Composition is specified by $(\psi;\upsilon)(\phi;\mu)=(\psi\circ\phi;\nu\circ\psi_*\mu)$.

Construction D.6. Let $\mathscr{A}$ be a bipermutative category. We construct a functor $\tilde{\mathscr{A}}:\mathscr{F}\int\tilde{\mathscr{F}}\to\mathrm{Cat}$ such that $\mathscr{B}\tilde{\mathscr{A}}$ is an $\mathscr{F}\int\tilde{\mathscr{F}}$-space.

Step 1. Construction of the $(n;N)$-th category $\tilde{\mathscr{A}}(n;N)$. Recall that $\oplus$ on $\mathscr{A}$ gives a functor $\mathscr{A}_\oplus:\mathscr{F}\to\mathrm{Cat}$ with nth category $\mathscr{A}_n$. The objects of $\tilde{\mathscr{A}}(n;N)$ are systems $\langle B_s,\Pi_{s,t}\rangle$, where $B_s=\langle B_s(\sigma),\pi_s(\sigma,\sigma')\rangle$ is an object of $\tilde{\mathscr{A}}_{N_s}$ and $\{\Pi_{s,t}\}$ is a 'unital, associative, and distributive system of isomorphisms $\Pi_{s,t}:B_s\otimes B_t\to B_{s\cup t}$'. Here σ runs through the based subsets of N_s, (σ,σ') runs through the pairs with $\sigma\cap\sigma'=\{0\}$, $B_s(\sigma)$ is an object of $\mathscr{A}$ with $B_s(0)=0$, and $\{\pi_s(\sigma,\sigma')\}$ is a unital, associative, and commutative system of isomorphisms

$$\pi_s(\sigma,\sigma'):B_s(\sigma)\oplus B_s(\sigma')\to B_s(\sigma\cup\sigma').$$

We require $B_0=1\in\mathscr{A}=\tilde{\mathscr{A}}_1$. Each $\Pi_{s,t}$ is itself a system of isomorphisms

$$\Pi_{s,t}(\sigma,\tau):B_s(\sigma)\otimes B_t(\tau)\to B_{s\cup t}(\pi_{s,t}(\sigma\wedge\tau)),$$

where

$$\pi_{s,t}:\sigma\wedge\tau\subset N_s\wedge N_t\to N_{s\cup t},$$

and we require the following diagrams to commute:

$$\begin{array}{ccc} B_s(\sigma)\otimes B_0(1) & & \\ \Big\| & \searrow{\scriptstyle \Pi_{s,0}(\sigma,1)} & \\ B_s(\sigma)\otimes 1=B_s(\sigma) & & \end{array}\qquad \begin{array}{ccc} B_s(\sigma)\otimes B_t(\tau) & \xrightarrow{\Pi_{s,t}(\sigma,\tau)} & B_{s\cup t}(\pi_{s,t}(\sigma\wedge\tau)) \\ \Big\downarrow{\scriptstyle \tilde{c}} & & \Big\| \\ B_t(\tau)\otimes B_s(\sigma) & \xrightarrow{\Pi_{t,s}(\tau,\sigma)} & B_{s\cup t}(\pi_{t,s}(\tau\wedge\sigma)) \end{array}$$

$$\begin{array}{ccc}
B_r(\varrho)\otimes B_s(\sigma)\otimes B_t(\tau) & \xrightarrow{1\otimes\Pi_{s,t}(\sigma,\tau)} & B_r(\varrho)\otimes B_{s\cup t}(\pi_{s,t}(\sigma\wedge\tau)) \\
 & & \Big\downarrow{\scriptstyle \Pi_{r,s\cup t}(\varrho,\,\pi_{s,t}(\sigma\wedge\tau))} \\
{\scriptstyle \Pi_{r,s}(\varrho,\sigma)\otimes 1}\Big\downarrow & & B_{r\cup s\cup t}(\pi_{r,s\cup t}(\varrho,\pi_{s,t}(\sigma\wedge\tau))) \\
 & & \Big\| \\
B_{r\cup s}(\pi_{r,s}(\varrho\wedge\sigma))\otimes B_t(\tau) & \xrightarrow{\Pi_{r\cup s,t}(\pi_{r,s}(\varrho\wedge\sigma),\,\tau)} & B_{r\cup s\cup t}(\pi_{r\cup s,t}(\varrho\wedge\sigma),\tau))
\end{array}$$

$$\begin{array}{ccc}
(B_s(\sigma)\otimes B_s(\sigma'))\otimes B_t(\tau) & = & (B_s(\sigma)\otimes B_t(\tau))\oplus(B_s(\sigma')\otimes B_t(\tau)) \\
{\scriptstyle \pi_s(\sigma,\sigma')\otimes 1}\Big\downarrow & & \Big\downarrow{\scriptstyle \Pi_{s,t}(\sigma,\tau)\otimes\Pi_{s,t}(\sigma',\tau)} \\
B_s(\sigma\cup\sigma')\otimes B_t(\tau) & & B_{s\cup t}(\pi_{s,t}(\sigma\wedge\tau))\oplus B_{s\cup t}(\pi_{s,t}(\sigma'\wedge\tau)) \\
{\scriptstyle \Pi_{s,t}(\sigma\cup\sigma',\tau)}\Big\downarrow & & \Big\downarrow{\scriptstyle \pi_{s\cup t}(\pi_{s,t}(\sigma\wedge\tau),\,\pi_{s,t}(\sigma'\wedge\tau))} \\
B_{s\cup t}(\pi_{s,t}((\sigma\cup\sigma')\wedge\tau)) & = & B_{s\cup t}(\pi_{s,t}(\sigma\wedge\tau)\cup\pi_{s,t}(\sigma'\wedge\tau))
\end{array}$$

The morphisms $\langle\chi_s\rangle:\langle B_s\rangle\to\langle B_s'\rangle$ in $\mathscr{A}(n;N)$ are systems of morphisms $\chi_s:B_s\to B_s'$ in $\mathscr{A}_{N_s}$ such that the following diagrams commute:

$$\begin{array}{ccc}
B_s(\sigma)\otimes B_t(\tau) & \xrightarrow{\Pi_{s,t}(\sigma,\tau)} & B_{s\cup t}(\pi_{s,t}(\sigma\wedge\tau)) \\
{\scriptstyle \chi_s\otimes\chi_t}\Big\downarrow & & \Big\downarrow{\scriptstyle \chi_{s\cup t}} \\
B_s'(\sigma)\otimes B_t'(\tau) & \xrightarrow{\Pi_{s,t}'(\sigma,\tau)} & B_{s\cup t}'(\pi_{s,t}(\sigma\wedge\tau))
\end{array}$$

Composition is inherited from composition in the $\mathscr{A}_{N_s}$.

Step 2. Construction of functors $(\phi;\mu)_*:\mathscr{A}(m;M)\to\mathscr{A}(n;N)$. By abuse, write

$$\phi^{-1}(s)=\{0\}\cup\phi^{-1}(s-\{0\})\quad\text{for } s\subset\mathbf{n}.$$

If $M=\langle M_r,\pi_{r,r'}\rangle$ and $N=\langle N_s,\pi_{s,s'}\rangle$, then

$$\phi_*M=\langle M_{\phi^{-1}(s)},\pi_{\phi^{-1}(s),\phi^{-1}(s')}\rangle\quad\text{and}\quad\mu=\langle\mu_s\rangle,$$

$\mu_s:M_{\phi^{-1}(s)}\to N_s$. Write

$$\mu_s^{-1}(\sigma)=\{0\}\cup\mu_s^{-1}(\sigma-\{0\})\quad\text{for }\sigma\subset N_s.$$

For $\langle A_r,\Pi_{r,r'}\rangle\in\mathscr{A}(m;M)$, define

$$(\phi;\mu)_*\langle A_r,\Pi_{r,r'}\rangle=\langle B_s,\Pi_{s,s'}\rangle,$$

where

$$B_s(\sigma)=A_{\phi^{-1}(s)}(\mu_s^{-1}(\sigma)),\qquad \pi_s(\sigma,\sigma')=\pi_{\phi^{-1}(s)}(\mu_s^{-1}(\sigma),\mu_s^{-1}(\sigma')),$$

and

$$\Pi_{s,s'}(\sigma,\sigma')=\Pi_{\phi^{-1}(s),\phi^{-1}(s')}(\mu^{-1}(\sigma),\mu^{-1}(\sigma')).$$

Similarly, on morphisms, $(\phi;\mu)_*\langle\lambda_r\rangle=\langle\chi_s\rangle$ where $\chi_s=\lambda_{\phi^{-1}(s)}$. Functoriality is easily checked. Note that, as in Sections 2 and 3, there is no choice of base objects for which all $(\phi;\mu)_*$ are based functors.

Step 3. Verification of proper behavior on $\mathscr{F}\wr\mathscr{F}\subset\mathscr{F}\wr\tilde{\mathscr{F}}$. We use the notations of Definition 2.1 and check conditions (1)–(4) there, leaving the cofibration condition (5) to the reader. Clearly $\tilde{\mathscr{A}}(0;*)$ is the trivial category, and $\tilde{\mathscr{A}}(1;s)=\tilde{\mathscr{A}}_s$ in view of the identification of $\tilde{\mathscr{F}}_1$ with $\tilde{\mathscr{F}}$. By [17, Const. 10], $\tilde{\mathscr{A}}(1;0)$ is trivial and $\delta':\tilde{\mathscr{A}}(1;\mathbf{s})\to\tilde{\mathscr{A}}(1;\mathbf{1})^s$ is an equivalence. We must show that $\delta'':\tilde{\mathscr{A}}(n;S)\to\prod_{j=1}^n\tilde{\mathscr{A}}(1;\mathbf{s}_j)$ is an equivalence; here we write $S=(\mathbf{s}_1,\dots,\mathbf{s}_n)$ but think of $N=\nu S\in\tilde{\mathscr{F}}$. Note that an equivalence of the same sort for general N will follow formally in view of the equivalence relating $\mathscr{F}\wr\mathscr{F}$ and $\mathscr{F}\wr\tilde{\mathscr{F}}$. Define $\nu'':\prod_{j=1}^n\tilde{\mathscr{A}}(1;\mathbf{s}_j)\to\tilde{\mathscr{A}}(n;\mathrm{S})$ as follows. If $s=\{0,i_1,\cdots,i_q\}$ with $0<i_1<\cdots<i_q\le n$, then $N_s=\mathbf{s}_{i_1}\wedge\cdots\wedge\mathbf{s}_{i_q}$. The nonzero elements of $\sigma\subset N_s$ are of the form $a_1\wedge\cdots\wedge a_q$ with $1\le a_r\le s_{i_r}$ and are ordered lexicographically. For $B_j\in\tilde{\mathscr{A}}_{s_j}$ and for $s\subset\mathbf{n}$ and $\sigma\subset N_s$, write $B_j(a)=B_j(\{0,a\})$ for $1\le a\le s_j$ and set

$$B_s(\sigma)=\bigoplus_{a_1\wedge\cdots\wedge a_q}B_{i_1}(a_1)\otimes\cdots\otimes B_{i_q}(a_q).$$

Starting from $\pi_s(\sigma,\sigma')=1$ if $\sigma=\{0,a\}$ and $\sigma'=\{0,a'\}$ with $a<a'$, the diagrams in Step 1 of [17, Const. 10] dictate specification of the $\pi_s(\sigma,\sigma')$. Starting from $\Pi_{s,t}(\sigma,\tau)=1$ if $s=\{0,i\}$ and $t=\{0,j\}$ with $i<j$ and if σ and τ are singletons, the diagrams in Step 1 here then dictate specification of the $\Pi_{s,t}(\sigma,\tau)$. This specifies the functor ν'' on objects, and its specification on morphisms is similar. The composite $\delta''\nu''$ is $\prod_{j=1}^n\nu'\delta'$ (because we have used only the $B_j(a)$, ignoring the given $B_j(\sigma)$ for other $\sigma\subset\mathbf{s}_j$). It is not hard to construct a natural isomorphism $\xi'':\mathrm{Id}\to\nu''\delta''$, its form being dictated by our specification of objects (as in the definition of ξ in [17, Const. 10]).

Let us also write $\mathscr{A}$ for the lax functor $\mathscr{F}\wr\mathscr{F}\to\mathrm{Cat}$ with $(n;S)\mapsto\mathscr{A}^{s_1}\times\cdots\times\mathscr{A}^{s_n}$ specified (and denoted A) in Section 3 and let us continue to write $\tilde{\mathscr{A}}$ for the restriction of $\tilde{\mathscr{A}}$ to $\mathscr{F}\wr\mathscr{F}$. Then an elaboration of the last step of the construction gives the following bipermutative analog of Proposition D.1.

Proposition D.7. *There are lax natural transformations* $\delta:\tilde{\mathscr{A}}\to\mathscr{A}$ *and* $\nu:\mathscr{A}\to\tilde{\mathscr{A}}$ *such that* $\delta\nu=\mathrm{Id}:\mathscr{A}\to\mathscr{A}$ *and there is a natural homotopy* $\xi:\mathrm{Id}\to\nu\delta$ *of lax natural transformations of functors* $\tilde{\mathscr{A}}\to\tilde{\mathscr{A}}$ *such that each* $\xi(n;S)$ *is a natural isomorphism of functors* $\tilde{\mathscr{A}}(n;S)\to\tilde{\mathscr{A}}(n;S)$.

Again, via [21, 3.4], this has the following immediate consequence.

Corollary D.8. *There is a natural transformation* $\mathscr{A}\to\tilde{\mathscr{A}}$ *of functors* $\mathscr{F}\wr\mathscr{F}\to\mathrm{Cat}$ *such that each* $\mathscr{A}(n;S)\to\tilde{\mathscr{A}}(n;S)$ *induces an equivalence upon passage to classifying spaces.*

For what it is worth, we record the bipermutative analog of Corollary D.3.

Corollary D.9. *There are functors* $\iota : (\mathscr{F}\int\mathscr{F})\int\mathscr{A} \to (\mathscr{F}\int\mathscr{F})\int\bar{\mathscr{A}}$ *and* $\varrho : (\mathscr{F}\int\mathscr{F})\int\bar{\mathscr{A}} \to (\mathscr{F}\int\mathscr{F})\int\mathscr{A}$ *such that* $\varrho\iota = \mathrm{Id}$ *and there is a natural isomorphism* $\xi : \mathrm{Id} \to \iota\varrho$.

Although $\bar{\mathscr{A}}$ is unnecessary to the theory of this paper, I do see possible applications of it. While it is rather complicated to make precise, it is a much more economical construction than $\tilde{\mathscr{A}}$. As a consequence, however, it is only functorial on strict morphisms of bipermutative categories.

Summarizing, we see that Woolfson's hyper Γ-spaces are our $\mathscr{F}\int\bar{\mathscr{F}}$-spaces and are thus $\mathscr{F}\int\mathscr{F}$-spaces by neglect of extraneous combinatorial structure. His passage from categorical to space level input accepts less general morphisms but is otherwise usable interchangeably with ours.

To explain the relationship between Woolfson's output and ours, observe that we can still construct a category $\mathscr{F}\int\mathscr{S}$ from a (large) symmetric monoidal category $\mathscr{S}$ since $\mathscr{S}$ determines a lax functor $\mathscr{F} \to \mathrm{Cat}$ with $\mathbf{n} \to \mathscr{S}^n$ (see [29, 3.1.2 and 4.1.2]). Via an elaboration of [17, Const. 10], we can associate an actual functor $\bar{\mathscr{S}} : \mathscr{F} \to \mathrm{Cat}$ to this lax functor and so define $\mathscr{F}\int\bar{\mathscr{S}}$. The relationship between $\mathscr{F}\int\mathscr{S}$ and $\mathscr{F}\int\bar{\mathscr{S}}$ is as in Corollary D.3.

Let $\mathscr{I}_*$ be the category of finite dimensional real inner product spaces and linear isometric isomorphisms. (Woolfson uses vector spaces and linear maps, but this makes no real difference.) Then $\mathscr{I}_*$ is symmetric monoidal under $\oplus$ and 1-point compactification gives a morphism $S : \mathscr{I}_* \to \mathscr{T}$ of symmetric monoidal categories.

Recall from [14, p. 73] that an $\mathscr{I}_*$-prefunctor (T, ω, e) is a continuous functor $T : \mathscr{I}_* \to \mathscr{T}$ together with natural transformations $\omega : \wedge \circ (T \times T) \to T \circ \oplus$ and $e : S \to T$ such that ω is commutative and associative and ω and e satisfy some obvious compatibility properties. We define

$$\sigma : TV \wedge SW \xrightarrow{1 \wedge e} TV \wedge TW \xrightarrow{\omega} T(V \oplus W)$$

and require its adjoint to be an inclusion. By restriction to sub inner product spaces of R^∞, T determines an $\mathscr{L}$-prespectrum. The requisite maps $\xi_j(g)$ for $g \in \mathscr{L}(j)$ are just the composites

$$TV_1 \wedge \cdots \wedge TV_j \xrightarrow{\omega} T(V_1 \oplus \cdots \oplus V_j) \xrightarrow{Tg} Tg(V_1 \oplus \cdots \oplus V_j).$$

(Compare Definition 8.1 and [14, p. 73].) Thus $\mathscr{I}_*$-prefunctors are natural precursors of $\mathscr{L}$-spectra and the two notions have essentially the same level of complexity. Of course, our theory manufactures $\mathscr{L}$-spectra from $\mathscr{F}\int\mathscr{F}$-spaces (although these manufactured $\mathscr{L}$-spectra do not arise from $\mathscr{I}_*$prefunctors).

$\mathscr{I}_*$-prefunctors determine functors $\mathscr{F}\int\mathscr{I}_* \to \mathscr{T}$. Indeed, for an object $(n; V) \in \mathscr{F}\int\mathscr{I}_*$, $V = (V_1, \ldots, V_n)$, one defines

$$T(n; V) = TV_1 \times \cdots \times TV_n.$$

For a morphism $(\phi; f):(m; U)\to(n; V)$, $\phi:\mathbf{m}\to\mathbf{n}$ in $\mathscr{F}$ and $f=(f_1,\ldots,f_n)$ with $f_j:\bigoplus_{\phi(i)=j}U_i\to V_j$ in $\mathscr{I}_*$, one defines $(\phi; f):T(m; U)\to T(n; V)$ to be the composite

$$\prod_{i=1}^{m} TU_i\to\prod_{j=1}^{n}\Big(\bigwedge_{\phi(i)=j} TU_i\Big)\xrightarrow{\prod_{j=1}^{n}\omega}\prod_{j=1}^{n}T\Big(\bigoplus_{\phi(i)=j}U_i\Big)$$
$$\xrightarrow{\prod_{j=1}^{n}Tf_j}\prod_{j=1}^{n}TV_j.$$

Moreover, the maps σ above induce maps

$$\sigma: T(n; V)\times S(n; W)\to T(n; V\oplus W) \qquad (*)$$

with appropriate naturality and associativity properties. A similar, but more complicated, construction associates functors $\mathscr{F}\wr\bar{\mathscr{I}}_*\to\mathscr{T}$ to $\mathscr{I}_*$-prefunctors. In both cases, of course, the spaces comprising the functor are given by actual Cartesian products of the TV.

Woolfson's theory takes 'hyperspectra' as output. Modulo details, these are functors $\mathscr{F}\wr\bar{\mathscr{I}}_*\to\mathscr{T}$ such that $T(n; V)$ is homotopy equivalent rather than equal to $\prod_{j=1}^{n}T(1; V_j)$ together with structural maps σ as in $(*)$ and a bit of extra basepoint data designed to enable one to see smash products as opposed to just Cartesian products. That is, hyperspectra are essentially $\mathscr{I}_*$-prefunctors up to homotopy. They lead to a weakened, up to homotopy, notion of an $\mathscr{L}$-spectrum. The main idea of Woolfson's work is to construct functorial extensions of $\mathscr{F}\wr\bar{\mathscr{I}}$-spaces to functors $\mathscr{F}\wr\bar{\mathscr{T}}\to\mathscr{U}$ whose restrictions to $\mathscr{F}\wr\bar{\mathscr{I}}_*$ are hyperspectra. While various choices of detail are possible, it is intrinsic to the conception that smash products up to homotopy appear in the output since products up to homotopy appear in the input. Since our theory converts the same input to actual $\mathscr{L}$-spectra and since there are no known naturally occurring examples of hyperspectra which do not arise from $\mathscr{I}_*$-prefunctors, I see no present applications for the more complicated notion and have not pursued the details required for a precise comparison of machines.

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