THE DOLD-KAN CORRESPONDENCE

1. Simplicial sets

We shall now introduce the notion of a *simplicial set*, which will be a presheaf on a suitable category. It turns out that simplicial sets provide a (purely combinatorial) model for the homotopy theory of CW complexes, although we shall not prove this here. We will restrict ourselves to describing their basic properties, and then move on to our ultimate goal, the Dold-Kan correspondence.

1.1. The simplex category.

Definition 1.1. Let Δ be the category of finite (nonempty) ordered sets and order-preserving morphisms. The object $[n]$ will denote the set $\{0, 1, \dots, n\}$ with the usual ordering. Thus Δ is equivalent to the subcategory consisting of the $[n]$. This is called the **simplex category.**

There is a functor from Δ to the category **Top** of topological spaces. Given $[n]$, we send it to the *standard topological n-simplex* Δ_n that consists of points $(t_0, \dots, t_n) \in \mathbb{R}^{n+1}$ such that each $t_i \in [0, 1]$ and $\sum t_i = 1$. Given a morphism $\phi : [m] \to [n]$ of ordered sets, we define $\Delta_m \to \Delta_n$ by sending

$$(t_0, \dots, t_m) \mapsto (u_j), \quad u_j = \sum_{\phi(i)=j} t_i.$$

Here the empty sum is to be regarded as zero.

For instance, an *inclusion* of ordered sets $[n-1] \hookrightarrow [n]$ will embed Δ_{n-1} as a *face* of Δ_n.

1.2. Simplicial sets.

Definition 1.2. A **simplicial set** $X_\bullet$ is a contravariant functor from Δ to the category of sets. In other words, it is a presheaf on the simplex category. A **morphism** of simplicial sets is a natural transformation of functors. The class of simplicial sets thus becomes a category **SSet**.

A **simplicial object** in a category $\mathcal{C}$ is a contravariant functor $\Delta \to \mathcal{C}$.

We have just seen that the category Δ is equivalent to the subcategory consisting of the $[n]$. As a result, a simplicial set $X_\bullet$ is given by specifying sets X_n for each $n \in \mathbb{Z}_{\geq 0}$, together with maps

$$X_n \to X_m$$

for each map $[m] \to [n]$ in Δ. These maps are required to satisfy compatibility conditions (i.e., form a functor). The set X_n is called the set of *n-simplices* of $X_\bullet$.

Example 1.3. Let X be a topological space. Then we define its *singular simplicial set* $\mathrm{Sing} X_\bullet$ as follows. We let $(\mathrm{Sing} X)_n = \hom_{\mathbf{Top}}(\Delta_n, X)$. Using the functoriality of Δ_n discussed above, it is clear that there are maps $(\mathrm{Sing} X)_n \to (\mathrm{Sing} X)_m$ for each $[m] \to [n]$.

Example 1.4. Given $n \in \mathbb{Z}_{\geq 0}$, we define the *standard n-simplex* $\Delta[n]_\bullet$ via

$$\Delta[n]_m = \hom_\Delta([m], [n]).$$

Given a category $\mathcal{C}$, we know that there is a way of generating presheaves on $\mathcal{C}$. For each $X \in \mathcal{C}$, we consider the presheaf h_X defined as $Y \mapsto \hom_{\mathcal{C}}(Y, X)$; the presheaves obtained are the *representable* presheaves. The standard simplices are a special case of that.

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Example 1.5. Let $X_\bullet$ be a simplicial set, and $Y_\bullet \subset X_\bullet$ be a *simplicial subset*, so $Y_n \subset X_n$ for each n and the obvious diagrams commute. Then we can define a *quotient simplicial set set* $(X/Y)_\bullet$, whose n-simplices are X_n/Y_n.

Consider the quotient of $\Delta[1]_\bullet$ modulo the boundary $\Delta[0]_\bullet \sqcup \Delta[0]_\bullet$, imbedded in $\Delta[1]_\bullet$ via the two maps $[0] \rightrightarrows [1]$. This is the *simplicial circle.*

Example 1.6. Arbitrary (small) limits and colimits exist in **SSet**, and are calculated "pointwise"; this is true for any presheaf category.

Finally, we note the universal property of the standard n-simplices.

Proposition 1.7. *Let $X_\bullet$ be a simplicial set. Then there is a natural bijection*

$$X_n \simeq \hom_{\mathbf{SSet}}(\Delta[n]_\bullet, X_\bullet).$$

In other words, mapping from a standard n-simplex into $X_\bullet$ is equivalent to giving an n-simplex of X.

Proof. Immediate from Yoneda's lemma. □

1.3. **Generalities on presheaves.** We are interested in describing functors on the category of simplicial sets. It will be convenient to describe them first on the standard n-simplices $\Delta[n]_\bullet$. In general, this will be sufficient to characterize the functor. In fact, we are going to see that the values on the standard n-simplices (that is, on the simplex category Δ) will be enough, in many cases, to determine a functor out of **SSet**. We shall discuss this in a general context of presheaves on any small category, though.

Let $\mathcal{C}$ be any small category. We shall, most often, take $\mathcal{C}$ to be Δ. Let $\hat{\mathcal{C}}$ be the category of presheaves on $\mathcal{C}$, so for instance $\mathbf{SSet} = \hat{\Delta}$.

Proposition 1.8. *Any presheaf on $\mathcal{C}$ is canonically the colimit of representable presheaves.*

Proof. Let $F \in \hat{\mathcal{C}}$ be a presheaf on $\mathcal{C}$. For each $X \in \mathcal{C}$, we let h_X be the representable presheaf defined, as above, by $h_X(Y) = \hom_{\mathcal{C}}(Y, X)$. Now form the category $\mathcal{D}$ whose objects are morphisms of presheaves $h_X \to F$, such that the morphisms between $h_X \to F$ and $h_Y \to F$ are given by commutative diagrams

$$\begin{array}{ccc} h_X & \longrightarrow & h_Y \\ & \searrow \quad \swarrow & \\ & F & \end{array}.$$

Note that these commutative diagrams depend on nothing fancy: a morphism $h_X \to h_Y$ is just a map $X \to Y$, in view of Yoneda's lemma. There is a functor $\phi : \mathcal{D} \to \hat{\mathcal{C}}$ sending $h_X \to F$ to h_X. The image of this functor consists of representable presheaves (clear) and, by definition of the category $\mathcal{D}$, there is a map of presheaves

$$\phi(c) \to F, \quad \forall c \in \mathcal{D}$$

that commutes with the diagrams. So there is induced a map

$$\varinjlim_{\mathcal{D}} \phi(c) \to F. \tag{1}$$

This is a map from a colimit of representable presheaves to F. The claim is that it is an isomorphism.

But by the Yoneda lemma, if $X \in \mathcal{C}$ and $\alpha \in F(X)$, then there is a map $h_X \to F$ in $\hat{\mathcal{C}}$ such that the identity in $h_X(X)$ is sent to α in $F(X)$. It follows that we can hit any element in any part of the presheaf F by a representable presheaf. Thus the map (1) is surjective.

Now let $X \in \mathcal{C}$ be a fixed object. We want to show that the map

$$\varinjlim_{\mathcal{D}} \phi(c)(X) \to F(X)$$

is injective. Note that we can calculate direct limits in $\hat{\mathcal{C}}$ "pointwise." Suppose two elements $\alpha_1 \in \phi(c_1)(X)$ and $\alpha_2 \in \phi(c_2)(X)$ are mapped to the same element of $F(X)$. Then c_1, c_2 correspond to maps $h_{Y_1} \to F, h_{Y_2} \to F$ given by elements $\beta_1 \in F(Y_1), \beta_2 \in F(Y_2)$, and α_1, α_2 correspond to maps $f_1 : X \to Y_1, f_2 : X \to Y_2$. The fact that they map to the same thing in $F(X)$ means that $f_1^*(\beta_1) = f_2^*(\beta_2)$, where the star denotes pulling back. Call $\gamma = f_1^*(\beta_1) = f_2^*(\beta_2)$.

We are now going to show that α_1, α_2 are identified in the colimit. To see this, we construct diagrams

$$\begin{array}{ccc} h_X & \xrightarrow{f_1} & h_{Y_1} \\ & {\scriptstyle\gamma}\searrow \quad \swarrow{\scriptstyle\beta_1} & \\ & F & \end{array}$$

and

$$\begin{array}{ccc} h_X & \xrightarrow{f_2} & h_{Y_2} \\ & {\scriptstyle\gamma}\searrow \quad \swarrow{\scriptstyle\beta_2} & \\ & F & \end{array} .$$

The first comes from the map $f_1 : X \to Y_1$ and the map $h_X \to F$ given by γ; the map $h_{Y_1} \to F$, given by β_1, is just the object c_1. The second diagram is similar. The first shows that the object $\alpha_1 \in h_{Y_1}(X)$ of the colimit is identified with the identity of $h_X(X)$ by f_1 in the diagram (where h_X is an element of $\mathcal{D}$ by the map $h_X \to F$ given by γ). Similarly α_2 is identified with this in the colimit, so α_1, α_2 are identified. It follows that (1) is also injective.

The fact that this colimit is "canonical" follows from the fact that if $F \to F'$ is a morphism of presheaves, there is a functor between the categories $\mathcal{D}$ associated to each of them. ☐

To clarify, if F is a presheaf, then we have described a category $\mathcal{D}_F$ and a functor $\mathcal{D}_F \to \mathcal{C}$ such that F is the colimit of $\mathcal{D}_F \to \mathcal{C} \to \hat{\mathcal{C}}$. This association is functorial; if $F \to F'$ is a morphism of presheaves, then there is a functor $\mathcal{D}_F \to \mathcal{D}_{F'}$ that fits into an obvious commutative diagram.

Corollary 1. *Any simplicial set is canonically a colimit of standard n-simplices.*

Proof. This follows from the previous result with $\mathcal{C} = \Delta$. ☐

Warning: Just because every element of $\hat{\mathcal{C}}$ is a colimit of representable presheaves does not mean that every element of $\hat{\mathcal{C}}$ is representable, even if $\mathcal{C}$ is cocomplete. For instance, the empty presheaf (which assigns to each element of the category $\emptyset$) is *never* representable (if $\mathcal{C}$ is not empty).[1] The problem is that the Yoneda embedding does not commute with colimits.

1.4. **Adjunctions.** Let $\mathcal{C}$ be a category, and $\mathcal{D}$ a cocomplete category. We are interested in colimit-preserving functors

$$\overline{\mathbf{F}} : \hat{\mathcal{C}} \to \mathcal{D}.$$

Here, as before, $\hat{\mathcal{C}}$ is the category of presheaves on $\mathcal{C}$. We shall, in this section, write functors out of a presheaf category with a line above them, and functors just defined out of $\mathcal{C}$ without the line. Functors will be in bold.

We have the standard Yoneda embedding $X \mapsto h_X$ of $\mathcal{C} \to \hat{\mathcal{C}}$. Thus any such functor $\overline{\mathbf{F}} : \hat{\mathcal{C}} \to \mathcal{D}$ determines a functor $\mathbf{F} : \mathcal{C} \to \mathcal{D}$. However, we know that any object of $\hat{\mathcal{C}}$ is a colimit of representable

[1] I learned this from `http://mathoverflow.net/questions/59503/question-on-the-interpretation-of-a-presheaf-category-as-a-co-`

presheaves. So any colimit-preserving functor $\hat{\mathcal{C}} \to \mathcal{D}$ is determined by what it does on $\mathcal{C}$, embedded in $\hat{\mathcal{C}}$ via the Yoneda embedding.

Conversely, let $\mathbf{F} : \mathcal{C} \to \mathcal{D}$ be any functor. We want to extend this to a functor $\overline{\mathbf{F}} : \hat{\mathcal{C}} \to \mathcal{D}$ that preserves colimits. For each presheaf G, we can write it (Theorem 1.8) as a colimit of representable presheaves over some category $\mathcal{D}_G$ and functor $\mathcal{D}_G \to \mathcal{C}$; if $G \to G'$ is a morphism of presheaves, we get a commutative diagram

$$\begin{array}{ccc} \mathcal{D}_G & \longrightarrow & \mathcal{D}_{G'} \\ & \searrow \quad \swarrow & \\ & \mathcal{C} & \end{array} .$$

So we can define

$$\overline{\mathbf{F}}(G) = \varinjlim_{c \in \mathcal{D}_G} \mathbf{F}(c).$$

By functoriality of $\mathcal{D}_G$, this is a functor. This extends $\mathbf{F}$ because for a representable presheaf G, the associated category $\mathcal{D}_G$ has a final object (namely, G itself!). We will see that this functor commutes with colimits. In fact:

Proposition 1.9. *If $\mathcal{D}$ is cocomplete, there is a natural bijection between left adjoints $\overline{\mathbf{F}} : \hat{\mathcal{C}} \to \mathcal{D}$ and functors $\mathbf{F} : \mathcal{C} \to \mathcal{D}$, given by restriction.*

Proof. Given a left adjoint $\overline{\mathbf{F}} : \hat{\mathcal{C}} \to \mathcal{D}$, restriction gives a functor $\mathbf{F} : \mathcal{C} \to \mathcal{D}$, and $\overline{\mathbf{F}}$ is determined from $\mathbf{F}$ as above, because a left adjoint commutes with colimits.

Conversely, we need to show that if $\mathbf{F} : \mathcal{C} \to \mathcal{D}$ is any functor, then the functor $\overline{\mathbf{F}} : \hat{\mathcal{C}} \to \mathcal{D}$ built from it as above is a left adjoint.

So we need to find a right adjoint $\overline{\mathbf{G}} : \mathcal{D} \to \hat{\mathcal{C}}$. We do this by sending $D \in \mathcal{D}$ to the presheaf $X \mapsto \hom_{\mathcal{C}}(\mathbf{F}X, D)$. We need now to see that $\overline{\mathbf{F}}, \overline{\mathbf{G}}$ are indeed adjoints. This follows formally:

$$\begin{aligned} \hom_{\hat{\mathcal{C}}}(\mathbf{F}, \overline{\mathbf{G}}D) &\simeq \varprojlim_{h_X \to F} \hom_{\hat{\mathcal{C}}}(h_X, \overline{\mathbf{G}}D) \\ &\simeq \varprojlim_{h_X \to F} \overline{\mathbf{G}}D(X) \\ &\simeq \varprojlim_{h_X \to F} \hom_{\mathcal{D}}(\mathbf{F}X, D) \\ &\simeq \hom_{\mathcal{D}}(\varinjlim_{h_X \to F} \mathbf{F}X, D) \\ &\simeq \hom_{\mathcal{D}}(\mathbf{F}F, D) \end{aligned}$$

□

From this, we can get a characterization of representable functors on presheaf categories.

Corollary 2. *Any contravariant functor $\overline{\mathbf{F}} : \hat{\mathcal{C}} \to \mathbf{Set}$ that sends colimits to limits is representable.*

Proof. Let $\overline{\mathbf{F}} : \hat{\mathcal{C}} \to \mathbf{Set}^{op}$ be a functor that commutes with colimits. Then, as we have seen, $\overline{\mathbf{F}}$ has an adjoint $\overline{\mathbf{G}} : \mathbf{Set}^{op} \to \hat{\mathcal{C}}$. Let $F = \overline{\mathbf{G}}(*) \in \hat{\mathcal{C}}$, where $*$ is the one-point set. Then we claim that F is a universal object. To see this, consider the chain of equalities for any presheaf F'

$$\begin{aligned} \hom_{\hat{\mathcal{C}}}(F', F) &\simeq \hom_{\hat{\mathcal{C}}}(F', \overline{\mathbf{G}}(*)) \\ &\simeq \hom_{\mathbf{Set}^{op}}(\overline{\mathbf{F}}F', *) \\ &\simeq \hom_{\mathbf{Set}}(*, \overline{\mathbf{F}}F') \\ &\simeq \mathbf{F}F'. \end{aligned}$$

□

1.5. **Geometric realization.** We recall that there was a functor $\Delta \to \mathbf{Top}$ that sent $[n]$ to the topological n-simplex Δ_n. The category $\mathbf{Top}$ is cocomplete, so it follows that there is induced a unique colimit-preserving functor

$$\mathbf{SSet} \to \mathbf{Top}$$

that sends the standard n-simplex $\Delta[n]_\bullet$ (i.e., the simplicial set corresponding to $[n]$ under the Yoneda embedding) to Δ_n, with the maps $\Delta_n \to \Delta_m$ associated to $[n] \to [m]$ as discussed earlier. So, in our previous notation, the functor $\Delta \to \mathbf{Top}$ is $\mathbf{F}$, and the extension to $\mathbf{SSet}$ is $\overline{\mathbf{F}}$.

Definition 1.10. This functor is called **geometric realization.** The geometric realization of $X_\bullet$ is denoted $|X|$.

As a left adjoint, geometric realization commutes with colimits. It is a basic fact, which we do not prove, that geometric realization commutes with finite limits if the limits are taken in the category of *compactly generated* spaces.

We can explicitly describe $|X|$. Namely, one forms the *simplex category*, which has objects consisting of all maps

$$\Delta[n]_\bullet \to X_\bullet$$

with morphisms corresponding to maps $\Delta[n]_\bullet \to \Delta[m]_\bullet$ fitting into a commutative diagram. Then we can define

$$|X| = \varinjlim_{\Delta[n]_\bullet \to X_\bullet} \Delta_n.$$

This functor has a right adjoint. In fact, this adjoint is none another than the *singular simplicial set* $\mathrm{Sing} T_\bullet$ for a topological space T! To see this, recall that we computed the adjoint to be

$$\mathbf{G}T = \{[n] \mapsto \hom_{\mathbf{Top}}(\mathbf{F}[n], T)\},$$

and since $\mathbf{F}$ takes $[n]$ to Δ_n, it is easy to see that this is the singular simplicial set.

Proposition 1.11. *The functors* $|\cdot| : \mathbf{SSet} \to \mathbf{Top}, \mathrm{Sing} : \mathbf{Top} \to \mathbf{SSet}$ *form an adjoint pair.*

1.6. **The simplicial identities.** We shall define certain important morphisms in the simplex category Δ and show that they generate the category, modulo certain relations.

Let $n \in \mathbb{Z}_{\geq 0}$. We define

$$d^i : [n-1] \to [n], \quad 0 \leq i \leq n, \quad d^i(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } i \geq j \end{cases}.$$

Here d^i maps the ordered set $[n-1]$ to $[n]$ via an inclusion, but where the element i in $[n]$ is omitted. These are called the *coface maps.* So one is supposed to think of the coface map is being the string $0 \to 1 \to \cdots \to i-1 \to i+1 \to \cdots \to n$ of $n-1$ elements in $[n]$.

Similarly, we define the *codegeneracy maps*

$$s^i : [n] \to [n-1], \quad 0 \leq i \leq n, \quad s^i(j) = \begin{cases} j & \text{if } j < i \\ j-1 & \text{if } i \geq j \end{cases}.$$

The codegeneracy s^i is a surjective map, where the elements $i, i+1$ are mapped to the same element. One is supposed to think of this as the string of n elements $0 \to 1 \to \cdots \to i-1 \to i \to i \to i+1 \to \cdots \to n-1$ of elements of $[n-1]$.

Lemma 1.12 (Cosimplicial identities)**.** *We have:*

$$d^j d^i = d^i d^{j-1}, \ [n-2] \to [n] \quad i < j \tag{2}$$

$$d^i d^i = d^{i+1} d^i, \ [n-2] \to [n] \tag{3}$$

Proof. We can think of the map $d^j d^i : [n-2] \to [n]$ as a map that clearly omits j from the image. Moreover, $d^j(i) = i$ is omitted. Similarly, we see that $d^i d^{j-1}$ omits i and $d^i(j-1) = j$ from the image. Since both maps are injective, the first assertion is clear. The second assertion can be proved similarly. □

We now describe identities involving the codegeneracies. (We follow [1].)

Lemma 1.13 (More cosimplicial identities). *We have:*

$$s^j d^j = s^j d^{j+1} = 1 \tag{4}$$

$$s^j d^i = d^i s^{j-1}, \quad i < j \tag{5}$$

$$s^j d^i = d^{i-1} s^j, \quad i > j+1 \tag{6}$$

$$s^j s^i = s^i s^{j+1}, \quad i \leq j \tag{7}$$

We omit the verification, which is easy.

Let $X_\bullet$ be a simplicial set. There are induced maps

$$d_i : X_n \to X_{n-1}, \quad s_i : X_n \to X_{n+1}$$

for each n, by applying the functor $X_\bullet$ to the d^i, s^i. These are called the *face* and *degeneracy* maps, respectively.

Lemma 1.14 (Simplicial identities). *For any simplicial set $X_\bullet$, we have*

$$d_i d_j = d_{j-1} d_i, \quad i < j \tag{8}$$

$$d_i d_i = d_i d_{i+1} \tag{9}$$

$$d_j s_j = d_{j+1} s_j = 1 \tag{10}$$

$$d_i s_j = s_{j-1} d_i, \quad i < j \tag{11}$$

$$d_i s_j = s_j d_{i-1}, \quad i > j+1 \tag{12}$$

$$s_i s_j = s_{j+1} s_i \quad i \leq j \tag{13}$$

Proof. This is now clear from the cosimplicial identities. □

One way to think about this is that "the smaller map can be moved to the inside." For instance, if we have $d_i d_j$ with $i < j$, then we can move the "smaller" map d_i to the inside of the composition. Another thing to keep in mind is that for a simplicial set $X_\bullet$, the degeneracy maps are *injective*; indeed, they have canonical sections, namely the face maps.

2. Simplicial abelian groups

A *simplicial abelian group* $A_\bullet$ is a simplicial object in the category of abelian groups. This means that there are abelian groups $A_n, n \in \mathbb{Z}_{\geq 0}$ and group-homomorphisms $A_n \to A_m$ for each map $[m] \to [n]$ in Δ.

2.1. **Three different complexes.**

Following [1], we are going to define several ways of making a chain complex from a simplicial abelian group. They will all have the same homotopy type, but one of them will be the most convenient for the Dold-Kan correspondence.

Definition 2.1. The **Moore complex** of a simplicial abelian group $A_\bullet$ is the complex A_* which in dimension n is A_n. The boundary map

$$\partial : A_n \to A_{n-1}$$

is the map $\sum_{i=0}^n (-1)^i d_i$, the alternating sum of the face maps. The simplicial identities easily imply that this is in fact a chain complex. Thus $A_\bullet \mapsto A_*$ defines a functor from simplicial abelian groups to chain complexes.

The *singular chain complex* of a topological space X can be obtained by taking the Moore complex of $\mathbb{Z}[\mathrm{Sing} X_\bullet]$, where $\mathbb{Z}[]$ denotes the operation of taking the free abelian group. (Note that applying $\mathbb{Z}$ turns a simplicial set into a simplicial abelian group.)

Recall that if $X_\bullet$ is a simplicial set, then a simplex $x \in X_n$ is called *degenerate* if it is in the image of one of the degeneracy maps (from X_{n-1}).

Proposition 2.2. *Let $A_\bullet$ be a simplicial abelian group. There is a subcomplex $DA_* \subset A_*$ of the Moore complex such that DA_n consists of the sums of degenerate simplices in degree n.*

Proof. We need only check that DA_* is stable under ∂. In particular, we have to check that $\partial(s_i a)$ is a sum of degeneracies for any $a \in A_{n-1}$. Now this is

$$\partial(s_i a) = \sum (-1)^j d_j(s_i a) = \sum_{j \neq i, i+1} d_j s_i a,$$

because the terms $(-1)^i(d_i s_i a - d_{i+1} s_i a) = (-1)^i(a - a) = 0$ vanishes in view of the simplicial identities. Moreover, the simplicial identities show that we can move the d part inside in the rest of the terms of the summation, potentially changing the subscript of the s. So $\partial s_i a$ belongs to DA_{n-1}. ☐

Definition 2.3. Consequently, if $A_\bullet$ is a simplicial abelian group, we can consider the chain complex $(A/DA)_*$. This is functorial in $A_\bullet$, and there is a natural transformation

$$A_* \to (A/DA)_*.$$

Nonetheless, in defining the Dold-Kan correspondence, we shall use a different construction (which we will prove is isomorphic to $(A/DA)_*$).

Definition 2.4. If $A_\bullet$ is a simplicial abelian group, we define the **normalized** complex NA_* as follows. In dimension n, NA_n consists of the subgroup of A_n that is killed by the face maps $d_i, i < n$. The differential

$$NA_n \to NA_{n-1}$$

is given by $(-1)^n d_n$.

It needs to be checked that we indeed have a chain complex. Suppose $a \in NA_n$; we must show that $d_{n-1} d_n a = 0$. But $d_{n-1} d_n = d_{n-1} d_{n-1}$ by the simplicial identities, and we know that d_{n-1} kills a. Thus the verification is clear.

We thus have three different ways of obtaining a complex from $A_\bullet$. By the way we defined the normalized chain complex, we have natural morphisms

$$NA_* \to A_* \to (A/DA)_*.$$

Our goal is to prove:

Theorem 2.5 (Dold-Kan)**.** *The functor $A_\bullet \mapsto NA_*$ defines an equivalence of categories between chain complexes of abelian groups and simplicial abelian groups. Moreover, the three complexes $NA_*, A_*, (A/DA)_*$ are all naturally homotopically equivalent (and the first and the last are even isomorphic).*

2.2. **The functor in the opposite direction.** A priori, the normalized chain complex of a simplicial abelian group $A_\bullet$ looks a lot different from $A_\bullet$, which a priori has much more structure. Nonetheless, we are going to see that it is possible to recover $A_\bullet$ entirely from this chain complex.

A key step in the proof of the Dold-Kan correspondence will be the establishment of the functorial decomposition for any simplicial abelian group $A_\bullet$

$$\bigoplus_{\phi: [n] \twoheadrightarrow [k]} NA_k \simeq A_n. \tag{14}$$

Here the map from a factor NA_k corresponding to some $\phi : [n] \twoheadrightarrow [k]$ to A_n is given by pulling back by ϕ. We will establish this below.

Now, let us *assume* that (14) is true. Motivated by this, we shall define a functor from chain complexes to simplicial abelian groups.

Let us now determine how the simplicial maps will play with the decomposition (which we are assuming) $A_n = \bigoplus_{[n]\twoheadrightarrow[k]} NA_k$. Given $f : [m] \to [n]$ and a factor NA_k of A_n (for some epimorphism $\phi : [n] \twoheadrightarrow [k]$), we want to know where f^* takes NA_k into A_m. We can factor the composite $[m] \to [n] \twoheadrightarrow [k]$ as $[m] \stackrel{\psi_1}{\twoheadrightarrow} [m'] \stackrel{\psi_2}{\hookrightarrow} [k]$. It is easy to see that simplicial maps induced by injections in Δ preserve NA. There is a commutative diagram:

$$\begin{array}{ccc} NA_k & \xrightarrow{\psi_2^*} & NA_{m'} \\ \downarrow{\scriptstyle \phi^*} & & \downarrow{\scriptstyle \psi_1^*} \\ A_n & \xrightarrow{f} & A_m \end{array} .$$

Now ψ_1 is an epimorphism. It follows that $\psi_1^* : NA_{m'} \to A_m$ is one of the maps in the canonical decomposition.

It follows that we have a *recipe* for determining where the ϕ-factor NA_k of A_n goes:

(1) Consider the composite $[m] \xrightarrow{f} [n] \xrightarrow{\phi} k$, and factor this as a composite $\psi_2 \circ \psi_1$ with $\psi_1 : [m] \twoheadrightarrow [m']$ an epimorphism and $\psi_2 : [m'] \hookrightarrow [k]$ a monomorphism.
(2) Then NA_k (embedded in A_n via ϕ^*) gets sent to $NA_{m'} \subset A_m$ (embedded in A_m via ψ_1^*).
(3) The map $NA_k \to NA_{m'}$ is given by ψ_2^*.

2.3. **Simplicial abelian groups from chain complexes.** Motivated by this, let us describe the inverse construction. Let C_* be a chain complex (nonnegatively graded, as always). We define a simplicial abelian group $\sigma C_\bullet$ such that

$$\sigma C_n = \bigoplus_{[n]\twoheadrightarrow[k]} C_k.$$

The sum is taken over all surjections $[n] \twoheadrightarrow [k]$. We can make this into a simplicial abelian group using the above "recipe" describing how the canonical decomposition for a simplicial abelian group behaves, but there is a bit of subtlety.

Since the ψ_2^* in the explanation of (3) above does not a priori make sense, let us note that if we restrict to the subcategory $\Delta' \subset \Delta$ consisting of *injective* maps, then the map $[n] \mapsto C_n$ becomes a contravariant functor in a natural way. Indeed, we let the map $C_n \to C_m$ induced by an injection $[m] \hookrightarrow [n]$ be zero unless $m = n - 1$ and we are working with the map d^n, in which case we let the map $C_n \to C_{n-1}$ be the differential. Since C_* is a chain complex, this is indeed a functor. So a chain complex gives an abelian presheaf on the "semi-simplicial" category.

Note that if we started with a simplicial abelian group $A_\bullet$, then if the chain complex NA is made into a contravariant functor $\Delta' \to \mathbf{Ab}$, we have gotten nothing new: we just recover the simplicial structure maps. Indeed, if $\psi : [m] \hookrightarrow [n]$ is an injection, then the map $\psi^* : NA_n \to NA_m$ is zero unless $\psi = d^n$ and $m = n - 1$. Otherwise ψ will contain a d^i for some $i < n$, and the definition of NA completes the proof.

We thus see:

Lemma 2.6. *Let C_* be a chain complex. Then there is a functor from $\Delta' \to \mathbf{Ab}$ sending $[n] \to C_n$ and an injection $[m] \hookrightarrow [n]$ to zero unless $m = n-1$ and the injection is d^n, in which case it is the differential. If $A_\bullet$ is a simplicial abelian group, this construction agrees with the simplicial maps when restricted to NA_*.*

Let us now, finally, show how to make $\sigma C_\bullet$ into a simplicial abelian group. Given some map $[m] \to [n]$ in Δ, we map the individual terms as follows. Let $\phi : [n] \twoheadrightarrow [k]$ be an epimorphism in

Δ, giving a factor $C_k \subset \sigma C_n$. We then map

$$C_k \to \sigma C_m = \bigoplus_{[m] \twoheadrightarrow [l]} C_l$$

as follows. If $[n] \twoheadrightarrow [k]$ is the given surjection, then we have a map $[m] \to [n] \to [k]$, which we can factor as a composite $[m] \twoheadrightarrow [m'] \overset{\psi}{\hookrightarrow} [k]$, of a surjection and an injection. So we send C_k (via ψ^*, which is defined by the functoriality) to $C_{m'}$, imbedded in σC_m as the factor corresponding to the surjection $[m] \twoheadrightarrow [m']$.

Lemma 2.7. *The above construction gives a functor from chain complexes to simplicial abelian groups.*

In fact, the above construction will give a simplicial object from any semi-simplicial object. (A *semi-simplicial object* is a presheaf on the category of finite ordered sets and injective order-preserving maps.)

Proof. In other words, we need to show that if we have a composite $[m] \to [n] \to [p]$, then the corresponding map $\sigma C_p \to \sigma C_m$ induced by the composite $[p] \to [m]$ is the composite of the maps $\sigma C_p \to \sigma C_n \to \sigma C_m$. So consider a factor of σC_p corresponding to a surjection $[p] \twoheadrightarrow [p']$. Now we can draw a commutative diagram in the simplex category Δ:

$$\begin{array}{ccccc} [m] & \longrightarrow & [n] & \longrightarrow & [p] \\ \downarrow & & \downarrow & & \downarrow \\ [m'] & \hookrightarrow & [n'] & \hookrightarrow & [p'] \end{array}$$

A close look at this will establish the claim, since C_* is a functor from the category of finite ordered sets and injective, order-preserving maps. ☐

As a result, we have (finally!) constructed our functor σ from chain complexes to simplicial abelian groups. Note that there is a natural transformation

$$(\sigma N A_*)_\bullet \to A_\bullet$$

for any simplicial abelian group $A_\bullet$. On the n-simplices, this is the map

$$\bigoplus_{\phi:[n] \twoheadrightarrow [k]} NA_k \to A_n$$

where the factor corresponding to ϕ is mapped to A_n by pulling back by ϕ. This is the map discussed above. It is immediate from the definition that this is a simplicial map. The crux of the proof of the Dold-Kan correspondence is that this is an isomorphism.

2.4. **The canonical splitting.** We have just defined the functor σ from chain complexes to simplicial abelian groups, and the natural transformation $\sigma(NA_*)_\bullet \to A_\bullet$ for any simplicial abelian group $A_\bullet$. We want to show that this is a quasi-inverse to N, that is, the above natural transformation is an isomorphism. Thus we need to show:

Proposition 2.8 (One half of Dold-Kan)**.** *For a simplicial abelian group $A_\bullet$, we have for each n, an isomorphism of abelian groups*

$$\bigoplus_{\phi:[n] \twoheadrightarrow [k]} NA_k \simeq A_n.$$

Here the map is given by sending a summand NA_k to A_n via the pull-back by the term $\phi : [n] \twoheadrightarrow [k]$. Alternatively, the morphism of simplicial abelian groups

$$\sigma(NA_*)_\bullet \to A_\bullet$$

is an isomorphism.

This is going to take some work, and we are going to need first a simpler splitting that will, incidentally, show that NA_* and $(A/DA)_*$ are isomorphic. We are going to prove the above result by induction, using:

Lemma 2.9. *Let $A_\bullet$ be a simplicial abelian group. Then the map*

$$NA_n \oplus DA_n \to A_n$$

is an isomorphism.

So we have a canonical splitting of each term of a simplicial abelian group. This splitting is into the degenerate simplices (or rather, their linear combinations) and the ones almost all of whose faces are zero.

Proof. Following [1], we shall prove this by induction. Namely, for each $k < n$, we define $N_k A_n = \bigcap_0^k \ker d_k$ and $D_k A_n$ to be the group generated by the images of $s_j(A_{n-1}), j \leq k$. So these are partial versions of the NA_n, DA_n. The claim is that there is a natural splitting

$$N_k A_n \oplus D_k A_n = A_n.$$

When $k = n-1$, the result will be proved (note that $D_{n-1}A_n$ is the group generated by degenerate simplices because the degeneracies $s_i : A_{n-1} \to A_n$ only go up to $n-1$).

When $k = 0$, the splitting is

$$\ker d_0 \oplus \mathrm{im} s_0 = A_n.$$

We can see this as follows. We have maps

$$A_{n-1} \underset{s_0}{\overset{d_0}{\leftrightarrows}} A_n \ .$$

Here s_0 is a *split injection*, with d_0 being a section. But in general, whenever $i : A \to B$ is a split injection with section $q : B \to A$, then B splits as $\ker q \oplus \mathrm{im} i$.

Now let us suppose we have established the splitting $A_n = N_{k-1}A_n \oplus D_{k-1}A_n$. We need to establish it for k. For this we will write some exact sequences.

a) We have a split exact sequence:

$$0 \to A_{n-1}/D_{k-1}A_{n-1} \overset{s_k}{\to} A_n/D_{k-1}A_n \to A_n/D_k A_n \to 0. \tag{15}$$

Indeed, exactness of this sequence will be clear once we show that is well-defined. But if $j < k$, then $s_k s_j = s_j s_{k-1}$, so s_k sends $D_{k-1}A_{n-1}$ into $D_{k-1}A_n$. The splitting is given by d_k.

b) Similarly, we have a split exact sequence (where the simplicial identities show that $s_k(N_{k-1}A_{n-1}) \subset N_{k-1}A_n$)

$$0 \to N_{k-1}A_{n-1} \overset{s_k}{\to} N_{k-1}A_n \to N_k A_n \to 0. \tag{16}$$

This is perhaps less obvious. This is equivalent to the claim that the map

$$N_k A_n \oplus N_{k-1}A_{n-1} \overset{(i,s_k)}{\to} N_{k-1}A_n$$

is an isomorphism. (Here i denotes the inclusion.)

We first claim that it is surjective. Indeed, if $a \in N_{k-1}A_n$, then $a - s_k d_k a$ lies in fact in $N_k A_n$. This is because d_k is a section of the split injection s_k, and because $d_j(a - s_k d_k a)$ for $j < k$ by using the simplicial identities to move d_j to the inside. Conversely, to see that it is injective, it suffices to note that if $s_k b \in N_k A_n$ for $b \in N_{k-1}A_{n-1}$, then $b = 0$; but $b = d_k(s_k b) = 0$ by definition of $N_k A_n$.

Now we are going to fit the exact sequences (15) and (16) into a diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_{k-1}A_{n-1} & \longrightarrow & N_{k-1}A_n & \xrightarrow{a\mapsto a-s_k d_k a} & N_k A_n & \longrightarrow & 0 \\ & & \downarrow \simeq & & \downarrow \simeq & & \downarrow & & \\ 0 & \longrightarrow & A_{n-1}/D_{k-1}A_{n-1} & \longrightarrow & A_n/D_{k-1}A_n & \longrightarrow & A_n/D_k A_n & \longrightarrow & 0 \end{array}$$

It is clear that this diagram commutes. The first square consists of the natural inclusions and projections, so it is obvious. For the second square, the extra term $s_k d_k a$ does not affect things modulo $D_k A_n$, so it commutes as well. Since both rows are exact and the first two columns are isomorphisms by the inductive hypothesis, so is the third. □

Corollary 3. *The map*

$$NA_* \to (A/DA)_*$$

is an isomorphism of chain complexes.

This is why we added the sign to the definition of the differential in constructing NA_*.

2.5. **The proof of Theorem 2.8.** We have a natural map

$$\Phi_n : \bigoplus_{\phi:[n]\twoheadrightarrow[k]} NA_k \to A_n,$$

which we need to prove is an isomorphism. This is a map of simplicial abelian groups.

Let us first show that $\Phi_n : (\sigma NA_*)_n \to A_n$ is surjective. By induction on n, we may assume that $\Phi_m : (\sigma NA_*)_m \to A_m$ is surjective for smaller $m < n$. Now A_n splits as the sum of NA_n and DA_n. Clearly NA_n is in the image of Φ_n (from the factor NA_n). But by the inductive hypothesis, everything in A_{n-1} is in the image of Φ_{n-1}, and taking degeneracies now shows that anything in DA_n is in the image of Φ_n. Thus Φ_n is surjective.

Let us now show that Φ_n is injective. Suppose a family $(a_\phi) \in \bigoplus_{\phi:[n]\twoheadrightarrow[k]} NA_k$ is mapped to zero under this map; we must show that each a_ϕ is zero. By assumption, we have

$$\sum_{\phi:[n]\twoheadrightarrow[k]} \phi^* a_\phi = 0 \in A_n.$$

Suppose some a_ϕ is nonzero. Note that $a_{1:[n]\to[n]}$ is zero by the canonical splitting, since that is the only term that might not be in DA_n.

We shall now define an *ordering* on the surjections $[n] \twoheadrightarrow [k]$. Say that $\phi_1 \leq \phi_2$ if $\phi_1(a) \leq \phi_2(a)$ for each $a \in [n]$. We can assume that ϕ is chosen minimal with respect to this (partial) ordering such that $a_\phi \neq 0$. Now choose a section $\psi : [k] \hookrightarrow [n]$ which is maximal in that ψ is not a section of any $\phi' > \phi$. If we think of ϕ as determining a partition of $[n]$ into k subsets, then we have ψ sending $i \in [k]$ to the last element of the ith subset of $[n]$. Then ψ is a section of ϕ, and of no other $\phi' < \phi$.

If we apply ψ^* to the equation $(a_\phi) = 0$, we find that

$$\Phi_k((\psi^* a_\phi)) = 0$$

which implies by the inductive hypothesis (as $k < n$) that ψ^* pulls back $(a_\phi) \in (\sigma NA_*)_n$ to zero. But the component of the identity $[k] \to [k]$ of this pull-back is just a_ϕ, from the choice of ψ. This means that $a_\phi = 0$.

2.6. **Completion of the proof of Dold-Kan.** We thus have defined a functor N from simplicial abelian groups to chain complexes. We have defined a functor σ in the opposite direction. We have, moreover, seen that the simplicial abelian group associated to NA_* for $A_\bullet$ a simplicial abelian group is just $A_\bullet$ itself, in view of the canonical decomposition of a simplicial abelian group. It suffices now, at least, to prove that the normalized chain complex associated to $\sigma C_\bullet$ is just C_*, for any chain complex C_*.

So we need to compute $N(\sigma C_\bullet)$. In degree n, this consists of elements of

$$\bigoplus_{[n] \twoheadrightarrow [k]} C_k$$

that are killed by the $d_i, i < n$. The claim is that this consists precisely of C_n under the identity $[n] \twoheadrightarrow [n]$! We can see this because we can show that $C_n \subset N(\sigma C)_n$ by direct computation; if $i < n$, then the map $d^i : [n-1] \to [n] \twoheadrightarrow [n]$ pulls C_n down to C_{n-1} via the functor $\Delta' \to \mathbf{Ab}$ induced; however, this functor induces zero on coface maps that are not the highest index. Conversely, we must show that $N(\sigma C_\bullet)_n \subset C_n$. To do this, we have to show that $C_n \subset N(\sigma C_\bullet)_n$; but we know that $N(\sigma C)_n$ is a complement to the degeneracies. However, the $C_k, k < n$ occurring in the expression for $(\sigma C)_n$ are all (clearly) degeneracies. Thus our assertion is clear.

References

[1] Paul Goerss and J. F. Jardine. *Simplicial Homotopy Theory.* Birkhauser, 1999.