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Smash products and Γ-spaces

By MANOS LYDAKIS
Fakultät für Mathematik, Universität Bielefeld, Postfach 100131,
33501 *Bielefeld, Germany*
e-mail: manos@math206.mathematik.uni-bielefeld.de

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1. *Introduction*

In this paper we construct a symmetric monoidal smash product of Γ-spaces modelling the smash product of connective spectra. For the corresponding theory of ring-spectra, we refer the reader to [**Sch**].

We give a brief review of Γ-spaces. If n is a non-negative integer, the pointed set n^+ is the set $\{0, \dots, n\}$ with 0 as the basepoint. The category Γ^{op} is the full subcategory of the category of pointed sets, with objects all n^+. The category $\mathcal{GS}$ of Γ-*spaces* is the full subcategory of the category of functors from Γ^{op} to pointed simplicial sets, with objects all F such that $F(0^+) \cong 0^+$. A map of Γ-spaces is a *strict weak equivalence* if it gives a weak equivalence of simplicial sets for every n^+ in Γ^{op}. Segal [**Se**] introduced Γ-spaces and showed that they give rise to a homotopy category equivalent to the homotopy category of connective spectra. Bousfield and Friedlander [**BF**] later provided model category structures for Γ-spaces. We follow the terminology of [**BF**] (only the class of special Γ-spaces, and its subclass of very special Γ-spaces, are considered in [**Se**], where they are called 'Γ-spaces' and 'Γ-spaces A such that $A(\mathbf{1})$ has a homotopy inverse'). Segal proved that the homotopy category of connective spectra is equivalent to the category with objects the very special Γ-spaces and morphisms obtained by inverting the strict weak equivalences of Γ-spaces. Bousfield and Friedlander proved that the category obtained from *all* of $\mathcal{GS}$ by inverting the *stable weak equivalences* (Definition 5·4) is again equivalent to the homotopy category of connective spectra, in such a way that very special Γ-spaces correspond to omega-spectra.

One advantage of the approach of [**BF**] in relating Γ-spaces and spectra, is that $\mathcal{GS}$ has the structure of a closed simplicial model category and this structure is related by a Quillen-pair of functors to a similar one that the category of spectra has. The main results of this paper can be summarized as follows: the smash product of Γ-spaces is compatible with the model category structures of [**BF**], and corresponds to the smash product of spectra under the equivalence of homotopy categories of [**BF**]. We remark however that the proofs of the main results of this paper use neither [**BF**] nor model categories (the main prerequisites for them are some basic facts about bisimplicial sets, although Sections 2–4, and 6, do not use much more simplicial theory than the definition of simplicial objects). The parts of this paper labelled 'remark' or 'example' may have more prerequisites, and arguments there contain, in general, less details than the rest of the paper. On the other hand, the rest of the paper (which contains all the main results) is independent from these parts.

This paper is organized as follows. In Section 2 we define the smash product of Γ-spaces, and prove that it gives $\mathcal{GS}$ the structure of a symmetric monoidal category. In Section 3 we study the filtration of a Γ-space by its *skeleta* (Definition 3·1). The skeleta of a Γ-space are also considered in [**BF**]. We prove that they are the stages of a filtration (i.e. that every skeleton injects in the next one), and describe in Theorem 3·10 how one stage of the filtration is built from the next by attaching representable Γ-spaces.

The results of Section 3 are useful in reducing the proof of whether a certain property is shared by all Γ-spaces, to proving that the representable Γ-spaces have that property. Questions about representable Γ-spaces are usually very easy to answer, because the representable Γ-spaces are very well-known functors. There is one of them for each non-negative integer n, namely the functor Γ^n that takes the pointed set X to the n-fold cartesian product of X with itself. Reduction to representable Γ-spaces can be used to give easy proofs of certain interesting facts, some of which seem to be more well-known than others. An example of the first kind (see Proposition 5·20) is that every Γ-space is a homotopy functor when *extended degreewise* (Definition 2·10). An example of the second kind (see Proposition 5·21) is that, for any Γ-space F and any connected space X, the *assembly map* from $S^1 \wedge F(X)$ to $F(S^1 \wedge X)$ (Definition 2·12) is as connected as the suspension of $S^1 \wedge X \wedge X$ (in fact, one may replace S^1 by any connected space).

Reduction to representable Γ-spaces is used in Sections 4 and 5, to prove that smashing by a *cofibrant* Γ-space (Definition 3·1) preserves cofibrations (see Theorem 4·6, which is actually a little stronger), strict weak equivalences (Theorem 5·1), and stable weak equivalences (Theorem 5·12), and that the smash product of Γ-spaces corresponds to the smash product of spectra if one of the factors is cofibrant (Theorem 5·11). In the last section we prove that our skeleta and our cofibrations agree with those in [**BF**] (this is not needed in the rest of the paper).

There is an interesting subclass of the class of all cofibrant Γ-spaces, consisting of the Q-*cofibrant* Γ-spaces (see [**Sch**], especially Lemma A·3 and the paragraph immediately preceding it). The fact that the cofibrant Γ-spaces are the cofibrant objects in strict and stable model category structures for $\mathcal{GS}$ [**BF**], has its counterpart in the fact that the Q-cofibrant Γ-spaces are also the cofibrant objects in strict and stable model category structures for $\mathcal{GS}$. The strict structure is a special case of a general construction of Quillen [**Q**], and the stable structure is constructed by Schwede [**Sch**]. Being Q-cofibrant is a relatively strong condition on a Γ-space. For example, if the Γ-space is also discrete, then it must be a sum of representable Γ-spaces. On the other hand, this implies that the strict and stable notions of 'Q-fibrant' are weaker than the corresponding notions in [**BF**] and this makes the 'Q-model category structures' ideal for the applications of this smash product in [**Sch**]. We work with the weaker notions of 'cofibrant' in this paper, partly because we obtain more general theorems this way and partly because there are interesting cofibrant Γ-spaces which are not Q-cofibrant (for example, certain Γ-spaces that Segal associates to categories with finite sums; see Example 3·5). There are many more examples of cofibrant Γ-spaces which are not Q-cofibrant (see Example 3·3).

The definition and the basic properties of this smash product have been discovered independently by J. Smith (unpublished). Most of the results of Section 2 have been known to category-theorists for a long time and in greater generality (cf. [**D**]).

2. *Smash products and function objects*

The following conventions will save us a lot of writing. A *space* is a pointed simplicial set. A space is *discrete* if its underlying simplicial set is constant. A map always preserves all the available structure. For example, if X and Y are spaces, then a map $X \to Y$ is the same thing as a pointed simplicial map $X \to Y$.

We choose a smash product functor from $\Gamma^{\mathrm{op}} \times \Gamma^{\mathrm{op}}$ to Γ^{op}, extend it to one for all pointed sets and write $X \wedge Y$ for the smash product of the pointed sets X and Y. Such a functor exists, because Γ^{op} is equivalent to the category of finite pointed sets. We write X^+ for the space obtained from the simplicial set X by attaching a disjoint basepoint. If F is a Γ-space and Y is a space, the Γ-space $F \wedge Y$ takes n^+ to $F(n^+) \wedge Y$. We write $\mathscr{S}_*$ for the category of spaces. We write $\mathscr{C}(A, B)$ for the set of morphisms from A to B in a category $\mathscr{C}$.

2·1. *Definition.* We introduce function objects in $\mathscr{GS}$. The first one is the pointed set $\mathscr{GS}(F, F')$. The second is the space $\hom(F, F')$ that has $\mathscr{GS}(F \wedge (\Delta^q)^+, F')$ as its pointed set of q-simplices. The third function object is the Γ-space $\mathrm{Hom}(F, F')$ that takes m^+ to $\hom(F, F'(m^+ \wedge\ ))$, where we define $F'(m^+ \wedge\ )$ evaluated at n^+ to be $F'(m^+ \wedge n^+)$.

2·2. Theorem. *There exists a functor from $\mathscr{GS} \times \mathscr{GS}$ to $\mathscr{GS}$, whose value at (F, F') we call the smash product $F \wedge F'$ of F and F', and an isomorphism*

$$\mathscr{GS}(F \wedge F', F'') \cong \mathscr{GS}(F, \mathrm{Hom}(F', F''))$$

natural in the Γ-spaces F, F' and F''.

Proof. We need the category $\mathscr{GGS}$ of $\Gamma \times \Gamma$-spaces. It is the category of functors from $\Gamma^{\mathrm{op}} \times \Gamma^{\mathrm{op}}$ to $\mathscr{S}_*$ taking $(0^+, 0^+)$ to a point. Given F'' in $\mathscr{GS}$, let RF'' in $\mathscr{GGS}$ take (m^+, n^+) to $F''(m^+ \wedge n^+)$. The *external smash product* $F \tilde{\wedge} F'$ of the Γ-spaces F and F' takes (m^+, n^+) to $F(m^+) \wedge F'(n^+)$. Note that

$$\mathscr{GS}(F, \mathrm{Hom}(F', F'')) \cong \mathscr{GGS}(F \tilde{\wedge} F', RF'').$$

This isomorphism is similar to the one in

$$\mathscr{S}_*(X, \hom_*(Y, Z)) \cong \mathscr{S}_*(X \wedge Y, Z),$$

where the space $\hom_*(X, Y)$ has $\mathscr{S}_*(X \wedge (\Delta^q)^+, Y)$ as its pointed set of q-simplices for X and Y spaces. Thus, if R has a left adjoint $L: \mathscr{GGS} \to \mathscr{GS}$, then we may take $F \wedge F'$ to be $L(F \tilde{\wedge} F')$. But L does exist. Given F''' in $\mathscr{GGS}$ let $LF'''(n^+)$ be the colimit over all $i^+ \wedge j^+ \to n^+$ of $F'''(i^+, j^+)$ (to see that $LF'''(0^+)$ is a point, note that the identity map of 0^+ is terminal among the maps of the form $i^+ \wedge j^+ \to 0^+$).

2·3. Corollary. *There exist isomorphisms*

$$\hom(F \wedge F', F'') \cong \hom(F, \mathrm{Hom}(F', F''))$$

$$\mathrm{Hom}(F \wedge F', F'') \cong \mathrm{Hom}(F, \mathrm{Hom}(F', F''))$$

natural in the Γ-spaces F, F', and F''.

Proof. The first isomorphism is similar to the one in Theorem 2·2. For the second, use the first and the isomorphism $\mathrm{Hom}\,(F', F'')(n^+\wedge\ ) \cong \mathrm{Hom}\,(F', F''(n^+\wedge\ ))$.

2·4. *Remark.* The smash product of the Γ-spaces F and F' is the universal Γ-space F'' with a map of $\Gamma\times\Gamma$-spaces $F\,\widetilde{\wedge}\,F' \to RF''$. We indicate below a similarity between this definition and the definition of the tensor product of abelian groups. We first recall from [**Se**] that abelian groups embed as a full subcategory of Γ-spaces.

Fix an abelian group A. It determines a Γ-space HA, where $HA(n^+) = A \otimes \widetilde{\mathbb{Z}}[n^+]$ and $\widetilde{\mathbb{Z}}[n^+]$ is the reduced free abelian group on the pointed set n^+. Fix another abelian group A'. A map of sets $f\colon A \to A'$ is a group homomorphism if and only if there is a map of Γ-spaces

$$\widetilde{f}\colon HA \to HA'$$

such that $\widetilde{f}_{1^+} = f$. Further, this $\widetilde{f}$ is unique, if it exists.

This observation expresses linearity in terms of Γ-spaces. The following observation does this for bilinearity. Fix a third abelian group A''. A map of sets $g\colon A \wedge A' \to A''$ is bilinear if and only if there is a map of $\Gamma\times\Gamma$-spaces

$$\widetilde{g}\colon HA\,\widetilde{\wedge}\,HA' \to RHA''$$

such that $\widetilde{g}_{(1^+,1^+)} = g$. Further, this $\widetilde{g}$ is unique, if it exists.

2·5. *Definition.* Given n^+ in Γ^{op}, define the Γ-space Γ^n by $\Gamma^n(m^+) = \Gamma^{\mathrm{op}}(n^+, m^+)$.

2·6. LEMMA. *If n is a non-negative integer and F is a Γ-space, then* $\mathrm{hom}\,(\Gamma^n, F)$ *is isomorphic to $F(n^+)$.*

Proof. This follows from the Yoneda lemma, the isomorphism

$$\mathscr{GS}(\Gamma^n \wedge (\Delta^q)^+, F) \cong \mathscr{SET}^{\Gamma^{\mathrm{op}}\times\Delta^{\mathrm{op}}}(\Gamma^n \times \Delta^q, F)$$

(where we wrote F also for the associated functor from $\Gamma^{\mathrm{op}}\times\Delta^{\mathrm{op}}$ to sets) and the fact that $(n^+, [q]\,)$ represents $\Gamma^n \times \Delta^q$.

2·7. *Definition.* The Γ-space $\mathbb{S}$ is the inclusion of Γ^{op} in the category of spaces (we identify a pointed set with its associated discrete space).

2·8. PROPOSITION. *The Γ-spaces $\mathbb{S}$ and Γ^1 are isomorphic.*

2·9. LEMMA. *The above smash product is associative and commutative, up to natural isomorphism and $\mathbb{S}$ acts as a unit, up to natural isomorphism.*

Proof. We claim that $F \wedge \mathbb{S} \cong F$. This follows from Theorem 2·2, Proposition 2·8 and the fact that $\mathrm{Hom}\,(\Gamma^1, F'')$ is isomorphic to F'' (this is essentially a special case of Lemma 2·6).

To check that $F \wedge F' \cong F' \wedge F$, we check that $\mathscr{GS}(F \wedge F', F'') \cong \mathscr{GS}(F' \wedge F, F'')$ for all $F'' \in \mathrm{ob}\,\mathscr{GS}$. This follows from the isomorphisms

$$\mathscr{GGS}(F\,\widetilde{\wedge}\,F', RF'') \cong \mathscr{GGS}(F'\,\widetilde{\wedge}\,F, RF'' \circ T)$$

and $RF'' \cong RF'' \circ T$, where T is the obvious involution of $\Gamma^{\mathrm{op}} \times \Gamma^{\mathrm{op}}$.

Finally, we compare both $(F \wedge F') \wedge F''$ and $F \wedge (F' \wedge F'')$ to a more symmetric Γ-space $F \wedge F' \wedge F''$, where the space $(F \wedge F' \wedge F'')(n^+)$ is defined to be the colimit of $F(i^+) \wedge F'(j^+) \wedge F''(k^+)$ over $i^+ \wedge j^+ \wedge k^+ \to n^+$ (to simplify the exposition, we write as if the smash product of pointed sets was associative instead of associative up to unique natural isomorphism). The isomorphism

$$\mathscr{GS}((F \wedge F') \wedge F'', F''') \cong \mathscr{GS}(F \wedge F' \wedge F'', F''')$$

is obtained by observing that

$$\begin{aligned} \mathscr{GS}((F \wedge F') \wedge F'', F''') &\cong \mathscr{GS}(F \wedge F', \mathrm{Hom}\,(F'', F''')) \\ &\cong \mathscr{GGS}(F \mathbin{\widetilde{\wedge}} F', R\,\mathrm{Hom}\,(F'', F''')) \\ &\cong \mathscr{GGGS}(F \mathbin{\widetilde{\wedge}} F' \mathbin{\widetilde{\wedge}} F'', R'F''') \end{aligned}$$

where $\mathscr{GGGS}$ is the category of functors from $\Gamma^{\mathrm{op}} \times \Gamma^{\mathrm{op}} \times \Gamma^{\mathrm{op}}$ to $\mathscr{S}_*$ taking $(0^+, 0^+, 0^+)$ to 0^+, and $R'F'''$ takes (i^+, j^+, k^+) to $F'''(i^+ \wedge j^+ \wedge k^+)$. The second isomorphism is similar.

2·10. *Definition.* Given a Γ-space F, we obtain an extended functor from spaces to spaces, which we again denote by F, as follows. If X is any pointed set, define $F(X)$ as the colimit of $F(n^+)$ over $n^+ \to X$. If X is any space, define $F(X)$ as the diagonal of the pointed bisimplicial set which, evaluated at $[p]$, yields $F(X_p)$. We say that F is *extended degreewise*.

2·11. *Convention.* Consider all functors from spaces to spaces that satisfy the following three conditions. First, they are determined by their behaviour on discrete spaces by using degreewise evaluation and diagonalization, just as in the preceding definition. Second, they commute with filtered colimits. Third, they take one-point spaces to one-point spaces. These functors are precisely the functors from spaces to spaces which are isomorphic to (the degreewise extension of) a Γ-space. In fact, given any two such functors F and F', restriction to Γ^{op} gives a bijection between the degreewise extended maps from F to F' and the maps of Γ-spaces from the restriction of F to the restriction of F'. This allows us to identify any such functor with the degreewise extension of its associated Γ-space and we do this in what follows without any other comment. For example, we identify $\mathbb{S}(X)$ with X, for any space X.

2·12. *Definition.* We define a natural map $F \wedge F' \to F \circ F'$, which we call the *assembly map* (here F denotes the extended functor from spaces to spaces defined in Definition 2·10, so that the composition $F \circ F'$ makes sense). This map is an isomorphism when F' equals some Γ^n (Proposition 2·16).

We first define a map $F(X) \wedge Y \to F(X \wedge Y)$, natural in the spaces X and Y. This is the map that most resembles other assembly maps in the literature and it is the special case $F' = \mathbb{S} \wedge Y$. Using degreewise extension, it suffices to define a map of the form $F(n^+) \wedge m^+ \to F(n^+ \wedge m^+)$. We do this as follows. Given i in m^+ and x in $F(n^+)$, the element $\phi(x \wedge i)$ is defined as $\kappa_*(x)$, where κ denotes the map from n^+ to $n^+ \wedge m^+$ that takes j to $j \wedge i$. There is a similar map $X \wedge F(Y) \to F(X \wedge Y)$, natural in the spaces X and Y. We use the name *special assembly map* to refer to any of these two maps.

To handle the general case, by definition of the smash product, it is enough to specify a natural map $F(n^+) \wedge F'(m^+) \to F(F'(n^+ \wedge m^+))$. This is defined as the composition

$$F(n^+) \wedge F'(m^+) \to F(n^+ \wedge F'(m^+)) \to F(F'(n^+ \wedge m^+)),$$

where the first map is a special assembly map, and the second is given by applying F to a special assembly map.

2·13. *Remark.* We are now able to identify the Γ-spaces that represent 'algebras over the sphere spectrum'. Surprisingly enough, these turn out to be well known.

Let us say that a Γ-space F is a *Gamma-ring*, if there are maps $\mu: F \wedge F \to F$ and $\eta: \mathbb{S} \to F$ satisfying the usual associativity and unit conditions. Then μ corresponds to $\widetilde{\mu}: F \,\widetilde{\wedge}\, F \to RF$, i.e. to a map

$$\widetilde{\mu}: F(X) \wedge F(Y) \to F(X \wedge Y)$$

natural in X and Y in Γ^{op}, which extends degreewise to give a similar map, denoted again by $\widetilde{\mu}$, natural in the spaces X and Y. In fact, the Γ-space F is a Gamma-ring if and only if it is an FSP, as defined by Bökstedt in [**Bö**], under $\widetilde{\mu}$, η, and the special assembly map of Definition 2·12. Further, this defines a full embedding of Gamma-rings in FSPs, and one can show that all connective FSPs are in the image of this embedding, up to stable weak equivalence of FSPs.

2·14. *Convention.* If I is a small category, F a functor from I to pointed sets, i an object of I and $x \in F(i)$, we denote the image of x in colim F by $[i, x]$.

2·15. PROPOSITION. *For any non-negative integers m and n, the Γ-spaces Γ^{mn} and $\Gamma^m \wedge \Gamma^n$ are isomorphic.*

Proof. An isomorphism from Γ^{mn} to $\Gamma^m \wedge \Gamma^n$ is defined by mapping $f: m^+ \wedge n^+ \to k^+$ to $[f, 1 \wedge 1]$, where the notation is as in 2·14. Its inverse takes $[g, \phi \wedge \psi]$ to $g \circ (\phi \wedge \psi)$.

2·16. PROPOSITION. *The assembly map $F \wedge F' \to F \circ F'$ is an isomorphism, whenever F' equals some Γ^n.*

Proof. The case $F = \Gamma^m$ follows from the previous proposition. Note that, for fixed F', the functor $F \circ F'$ preserves all limits and colimits and the functor $F \wedge F'$ preserves all colimits, since it is a left adjoint. The conclusion will follow if we show that any F is an iterated colimit of diagrams involving only Γ-spaces of the form Γ^m for some m. There is a standard trick to write F this way, which we recall below.

We claim that F is isomorphic to the coequalizer of the two obvious maps

$$\bigvee_{l^+ \to m^+} \Gamma^m \wedge F(l^+) \rightrightarrows \bigvee_{m^+} \Gamma^m \wedge F(m^+).$$

To see this, we may assume that F is discrete. Write κ for the canonical map to the above coequalizer from $\bigvee_{m^+} \Gamma^m \wedge F(m^+)$. A map in one direction takes $\kappa(f \wedge x)$ to $f_*(x)$. Its inverse takes $x \in F(m^+)$ to $\kappa(1 \wedge x)$, where 1 denotes the identity map of m^+.

2·17. *Remark.* It follows from the proof of Proposition 2·16 that there is a second description of the value of the Γ-space F on the space X, namely $F(X)$ is naturally isomorphic to the coequalizer of

$$\bigvee_{l^+ \to m^+} \Gamma^m(X) \wedge F(l^+) \rightrightarrows \bigvee_{m^+} \Gamma^m(X) \wedge F(m^+),$$

where the two maps are induced from the corresponding maps in the proof of Proposition 2·16. The above statement is true if we take $\Gamma^m(X)$ to be as defined in Definition 2·10, or if we let $\Gamma^m(X)$ equal the m-fold cartesian product of X with itself (see convention 2·11).

Recall from [**Bor**, Definition 6·1·2] the definition of a symmetric monoidal category.

2·18. Theorem. *The category of* Γ*-spaces is symmetric monoidal with respect to the above smash product.*

Proof. We have already verified in Lemma 2·9 that the smash product of Γ-spaces is associative, commutative, and unital. It remains to show that, given a positive integer N and Γ-spaces $F_1, \ldots, F_N$, certain natural automorphisms of $F_1 \wedge \cdots \wedge F_N$ equal the identity. In fact, every natural automorphism of $F_1 \wedge \cdots \wedge F_N$ equals the identity. The case $N = 2$ should suffice to explain the proof, so we assume $N = 2$ below and we write F and F' instead of F_1 and F_2.

We may assume that F and F' are discrete. Let ϕ be a natural automorphism of $F \wedge F'$. Given a non-negative integer m and $x \in F(m^+)$, write $\hat{x}$ for the map from Γ^m to F that takes $f: m^+ \to k^+$ to $f_* x$. By definition of the smash product, every z in $(F \wedge F')(k^+)$ is of the form $(\hat{x} \wedge \hat{y})[f, 1_{m^+} \wedge 1_{n^+}]$ for some $f: m^+ \wedge n^+ \to k^+$ in Γ^{op}, where the notation is as in Convention 2·14. Thus it suffices to show that, for all non-negative integers m and n, if $F = \Gamma^m$ and $F' = \Gamma^n$ then $\phi = 1$.

Proposition 2·16 implies that $\mathscr{GS}(\Gamma^m \wedge \Gamma^n, \Gamma^m \wedge \Gamma^n) \cong \Gamma^{\mathrm{op}}(m^+ \wedge n^+, m^+ \wedge n^+)$. Let $\phi = f^*$ for some $f: m^+ \wedge n^+ \to m^+ \wedge n^+$ in Γ^{op}. We show that all $i \wedge j$ in $m^+ \wedge n^+$ are fixed under f. Note that this is trivially true if $m = n = 1$, since ϕ is an automorphism.

Let $I: 1^+ \to m^+$ and $J: 1^+ \to n^+$ correspond to i and j. It suffices to show that $f \circ (I \wedge J) = I \wedge J$. But

$$
\begin{aligned}
f \circ (I \wedge J) &= (f \circ (I \wedge J))^* 1_{m^+ \wedge n^+} \\
&= (I \wedge J)^* f^* 1_{m^+ \wedge n^+} \\
&= (I \wedge J)^* \phi 1_{m^+ \wedge n^+} \\
&= (I^* \wedge J^*) \phi 1_{m^+ \wedge n^+} \\
&= \phi (I^* \wedge J^*) 1_{m^+ \wedge n^+} \\
&= \phi (I \wedge J) \\
&= I \wedge J,
\end{aligned}
$$

where the fourth equality follows from identifying Γ^{mn} with $\Gamma^m \wedge \Gamma^n$ using Proposition 2·16, the fifth equality follows from naturality and the last equality follows from the last line of the previous paragraph.

2·19. *Remark.* The composition product $(F, F') \mapsto F \circ F'$ is associative and unital (with unit $\mathbb{S}$), up to natural isomorphism, and the assembly map is compatible with associativity and unit isomorphisms. In the language of monoidal categories, the assembly map makes the identity functor of $\mathscr{GS}$ a lax monoidal functor from the monoidal category $(\mathscr{GS}, \circ, \mathbb{S})$ to the monoidal category $(\mathscr{GS}, \wedge, \mathbb{S})$.

3. *The skeleton filtration*

In this section we define a filtration

$$* = F^{(0)} \subset F^{(1)} \subset \cdots \subset \bigcup_{m=0}^{\infty} F^{(m)} = F$$

of a Γ-space F and we prove in Theorem 3·10 that there is a pushout square giving $F^{(m)}$ in terms of $F^{(m-1)}$ and Γ^m.

3·1. *Definition.* Let m be a non-negative integer and F be a Γ-space. The m-skeleton of F is the Γ-space $F^{(m)}$ defined as follows. In case F is discrete, define

$$F^{(m)}(n^+) = \{f_* x |\ x \in F(k^+), f: k^+ \to n^+, k \leqslant m\},$$

and extend this definition degreewise in the general case.

Let Σ_n be the group of automorphisms of n^+ in Γ^{op}. If Σ_n acts freely (off the basepoint) on $F/F^{(n-1)}(n^+)$ for all positive integers n, we say that the Γ-space F is *cofibrant*. A map of Γ-spaces is a *cofibration* provided that it is injective and that its cofiber is cofibrant.

3·2. PROPOSITION. *The Γ-space Γ^n is cofibrant for all n.*

Proof. The elements of $(\Gamma^n)^{(m-1)}(m^+)$ are the non-surjective maps from n^+ to m^+. We want to show that given $x: n^+ \to m^+$ and $\sigma \in \Sigma_m$, if $\sigma \neq 1$ and x is onto then $\sigma \circ x \neq x$. But if $\sigma i \neq i$ and $xj = i$ then $\sigma x j = \sigma i \neq i = xj$.

3·3. *Example.* If f: $F \to F'$ is an injective map of Γ-spaces and F' is cofibrant, then both F and the cofibre of f are cofibrant and f is a cofibration. This is true because, if G is any group, all subobjects and cofibres of free pointed G-sets are free (in the pointed sense). Thus the inclusions $(\Gamma^n)^{(n-1)} \subset \Gamma^n$ and $\Gamma^n \vee \Gamma^m \to \Gamma^{n+m}$ are cofibrations. These are probably the easiest examples of cofibrations that are not Q-cofibrations (since their cofibres are not sums of representables).

3·4. *Example.* Recall the Γ-space $H\mathbb{Z}$ of Remark 2·4 and let e: $n^+ \to \widetilde{\mathbb{Z}}[n^+]$ be the canonical map. Thus $\widetilde{\mathbb{Z}}[n^+] = \mathbb{Z}e(1) + \cdots + \mathbb{Z}e(n)$. Then $H\mathbb{Z}$ is not cofibrant. In fact, the element $e(1) + e(2)$ of $H\mathbb{Z}(2^+)$ is fixed by Σ_2, and does not belong to $H\mathbb{Z}^{(1)}(2^+) = \mathbb{Z}e(1) \bigcup \mathbb{Z}e(2)$. A similar argument shows that HA is not cofibrant, for any non-trivial abelian group A.

3·5. *Example.* We recall certain well-known Γ-spaces associated to finite sums in a category $\mathscr{C}$. We show that, under a mild assumption on $\mathscr{C}$, they are cofibrant (despite a formal similarity to the Γ-spaces of the previous example; see below). We also show that, in all interesting cases, these are not Q-cofibrant.

Given an object n^+ of Γ^{op}, let $\mathscr{P}_n$ be the category of pointed subsets of n^+ and inclusions. Let $\mathscr{C}$ be a small category with a chosen initial object $*$. Then $\mathscr{C}$ determines a Γ-space F whose value at n^+ is the nerve of the following category. Its objects are all functors from $\mathscr{P}_n$ to $\mathscr{C}$ that preserve sums (in the sense that they take a diagram $S' \subset S \supset S''$ in $\mathscr{P}_n$ expressing S as the sum of S' and S'' to a similar diagram in $\mathscr{C}$) and take 0^+ to $*$. Its morphisms are the isomorphisms (of functors from $\mathscr{P}_n$ to $\mathscr{C}$) between its objects. The Γ-spaces F of this type were among the important examples considered in [**Se**]. For example, if $\mathscr{C}$ is the category of finite based sets, then the homotopy groups of $\Omega F(S^1)$ are the stable homotopy groups of spheres, and if $\mathscr{C}$ is the category of finitely generated projective modules over some ring R, then the homotopy groups of $\Omega F(S^1)$ are the algebraic K-theory groups of R. If we assume that finite sums exist in $\mathscr{C}$, then $F(n^+)$ is homotopy equivalent to the product $F(1^+)^n$ and we see a similarity with the Γ-spaces of Example 3·4. In fact, the sum in $\mathscr{C}$ makes $F(1^+)$ an abelian H-space, in particular an abelian monoid in the homotopy category ho $(\mathscr{S}_*)$. The construction HA of Remark 2·4 is possible for any abelian monoid A in a category $\mathscr{D}$ having finite products and a zero object, and it produces a functor $\Gamma^{\mathrm{op}} \to \mathscr{D}$. Finally, F lifts $H(F(1^+))$, in the sense that F and $H(F(1^+))$ are isomorphic as functors $\Gamma^{\mathrm{op}} \to \mathrm{ho}\,(\mathscr{S}_*)$.

We now show that F is cofibrant if and only if the initial object of $\mathscr{C}$ is unique (thus we may always replace $\mathscr{C}$ by an equivalent category whose associated Γ-space is cofibrant). Let σ be a non-trivial element of Σ_n and let C in $F_0(n^+)$ be fixed by σ. Choose a non-trivial cycle for σ, i.e. choose an injection $i\colon \mathbb{Z}/m \to n^+$ such that $m \geqslant 2$ and $\sigma i(a) = i(a+1)$ for all a in $\mathbb{Z}/m$. Let S be the image of i, so that we have a representation of $C(S)$ as the sum of all $C_a = C(\{i(a)\})$, with associated maps $f_a\colon C_a \to C(S)$. Since $\sigma C = C$, all maps f_a are equal. For any object C' of $\mathscr{C}$ and any maps g and g' from C_0 to C', let $h\colon C(S) \to C'$ be the unique map such that $hf_0 = g$ and $hf_a = g'$ for all non-zero elements a of $\mathbb{Z}/m$. Then $g = hf_0 = hf_1 = g'$, i.e. C_0 is an initial object. The conclusion follows since the vertices of $F^{(n-1)}(n^+)$ consist of those functors C such that for some element i of n^+ we have $C(\{i\}) = *$ and since actions on nerves of categories are free if and only if they are free on objects.

To conclude this example, we show that if $\mathscr{C}$ has finite sums and more than one object, then F is not Q-cofibrant. By lemma A·3 of [**Sch**], the zero-simplices of a Q-cofibrant Γ-space split as a sum of various Γ^n. Thus, if P is the discrete Γ-space given by the zero-simplices of F, it suffices to show that P has no non-trivial maps to Γ^n, for any n. An element x of $\Gamma^n(m^+)$ is trivial, if so are its images under all maps $\Gamma^n \to \Gamma^1$. Thus we may assume that $n = 1$. Fix a map $f\colon P \to \Gamma^1$ and an element C in $P(n^+)$. We show that $f(C) = 0$. Let $m = 2n$. Let p and q be the maps $m^+ \to n^+$ which take all $i \leqslant n$, resp. all $i > n$, to 0 and such that, for $1 \leqslant i \leqslant n$, $p(n+i) = i$, and $q(i) = i$. Because $\mathscr{C}$ has finite sums, there exists D in $P(m^+)$, such that $p_*D = q_*D = C$ (i.e. given a pointed subset S of m^+, the object $D(S)$ is some choice of the sum over $i \in S$ of $D(i)$, with $D(i)$ equal to $C(i)$ if $i \in n^+$ and to $C(i-n)$ if $i \in n^+$; further, this choice is fixed if $S \subset n^+$ (then $D(S) = C(S)$) or if $S \cap n^+ = 0^+$ (then $D(S) = Cp(S)$)). In case $f(D) \in n^+$ we have $f(C) = fp_*(D) = p_*f(D) = 0$. In case $f(D) \notin n^+$ we have $f(C) = fq_*(D) = q_*f(D) = 0$.

3·6. Proposition. *Suppose that D is a pullback square of pointed sets and that F is a functor from pointed sets to pointed sets. Then $F(D)$ is also a pullback square, provided that all four maps of D are injective.*

Proof. Suppose that D is the square below

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ i\downarrow & & \downarrow j \\ Z & \xrightarrow{g} & W \end{array}$$

where we may assume that all maps are inclusions. Choose a retraction u to i and extend it to a retraction v to j. In other words, the equalities $fu = vg$ and $vj = 1$

hold. This can be done because $X = Y \cap Z$. Because every injective map of pointed sets is split, all maps in $F(D)$ are again injective. It remains to show that given y in $F(Y)$ and z in $F(Z)$, if they map to the same element in $F(W)$ then they lift to $F(X)$. Because g_* is injective, it suffices to lift y. But $u_*(z)$ is such a lift, since $f_*u_*(z) = v_*g_*(z) = v_*j_*(y) = y$.

3·7. *Definition.* For any non-negative integers n and m, let $\binom{n}{m}$ denote the set of those injective maps from m^+ to n^+ which are increasing with respect to the usual order.

3·8. PROPOSITION. *For any Γ-space F and any positive integers m and n, there is a pushout square*

$$\begin{array}{ccc} \binom{n}{m}^+ \wedge F^{(m-1)}(m^+) & \subset & \binom{n}{m}^+ \wedge F(m^+) \\ \downarrow & & \downarrow \\ F^{(m-1)}(n^+) & \subset & F^{(m)}(n^+) \end{array}$$

where the map $\binom{n}{m}^+ \wedge F(m^+) \to F^{(m)}(n^+)$ takes $f \wedge x$ to f_*x.

Proof. We may assume that F is discrete. Since the square above is commutative, we obtain a map from the pushout of the truncated square to $F^{(m)}(n^+)$, which is surjective. It remains to show that this map is injective.

Since the map from $\binom{n}{m}^+ \wedge F^{(m-1)}(m^+)$ to $F^{(m-1)}(n^+)$ is onto, it suffices to show that, given $f_*x = g_*y$ in $F^{(m)}(n^+)$ with x and y in $F(m^+)$ and f and g in $\binom{n}{m}$, if $f \wedge x \neq g \wedge y$ then x and y are in $F^{(m-1)}(m^+)$. If $f = g$, then $x = y$ since f_* is injective, so there is nothing to prove in this case. Assume now that $f \neq g$ and let p^+ be their pullback. Since f and g are distinct increasing injections, they have distinct images, and therefore $p < m$. The conclusion now follows from Proposition 3·6.

3·9. *Definition.* We view Γ^m as a Γ-Σ_m-space, that is, a functor from Γ^{op} to Σ_m-spaces taking 0^+ to a point, by using the mapping space action (that is, the image of $f \in \Gamma^m_k$ under the action of $\sigma \in \Sigma_m$ is $f \circ \sigma^{-1}$).

3·10. THEOREM. *For any Γ-space F and any positive integer m, there is a pushout square*

$$\begin{array}{ccc} \partial(\Gamma^m \wedge F(m^+))/\Sigma_m & \subset & (\Gamma^m \wedge F(m^+))/\Sigma_m \\ \downarrow & & \downarrow \\ F^{(m-1)} & \subset & F^{(m)} \end{array}$$

where $\partial(\Gamma^m \wedge F(m^+))$ is defined as

$$\Gamma^m \wedge F^{(m-1)}(m^+) \cup_{(\Gamma^m)^{(m-1)} \wedge F^{(m-1)}(m^+)} (\Gamma^m)^{(m-1)} \wedge F(m^+)$$

and the map $(\Gamma^m \wedge F(m^+))/\Sigma_m \to F^{(m)}$ takes the orbit of $f \wedge x$ to f_*x.

Proof. The cofibre of the inclusion $\partial(\Gamma^m \wedge F(m^+)) \subset \Gamma^m \wedge F(m^+)$ is isomorphic to $\Gamma^m/(\Gamma^m)^{(m-1)} \wedge (F/F^{(m-1)})(m^+)$. Further, the inclusion of $\binom{n}{m}^+$ in $\Gamma^m(n^+)$ induces an isomorphism between $\binom{n}{m}^+$ and the Σ_m-orbits of $\Gamma^m/(\Gamma^m)^{(m-1)}(n^+)$. Finally, the action of Σ_m on $\Gamma^m(n^+)/(\Gamma^m)^{(m-1)}(n^+)$ is free. The conclusion now follows from Proposition 3·8, since a commutative square of pointed sets with horizontal maps injective is a pushout if and only if the induced map on horizontal cofibres is an isomorphism.

3·11. PROPOSITION. *If F is a discrete cofibrant Γ-space then there exists a pointed set S such that $F^{(n)}$ is obtained from $F^{(n-1)}$ by attaching $\Gamma^n \wedge S$ along $(\Gamma^n)^{(n-1)} \wedge S$.*

Proof. Let $S = F/F^{(n-1)}(n^+)/\Sigma_n$. For every non-basepoint element s in S, choose a representative $x(s)$ in $F(n^+)$, so that the orbit $\Sigma_n x(s)$ equals s, where we have denoted the image of $x(s)$ in $F/F^{(n-1)}(n^+)$ again by $x(s)$. Consider the diagram below, where the map $S \to F(n^+)$ is defined by $s \mapsto x(s)$.

$$\begin{array}{ccc} (\Gamma^n)^{(n-1)} \wedge S & \subset & \Gamma^n \wedge S \\ \downarrow & & \downarrow \\ \partial(\Gamma^n \wedge F(n^+))/\Sigma_n & \subset & (\Gamma^n \wedge F(n^+))/\Sigma_n \\ \downarrow & & \downarrow \\ F^{(n-1)} & \subset & F^{(n)} \end{array}$$

The top square is a pullback square with all four maps injective and every element of $(\Gamma^n \wedge F(n^+))/\Sigma_n$ can be lifted either to $\partial(\Gamma^n \wedge F(n^+))/\Sigma_n$ or to $\Gamma^n \wedge S$. It follows that the top square is a pushout, therefore so is the composed square, since the bottom square is a pushout by Theorem 3·10.

4. *Smash products and cofibrations*

In this section we show that the smash product of Γ-spaces behaves well with respect to injective maps and cofibrations.

4·1. LEMMA. *Let $f: F \to F'$ be a map of Γ-spaces and n be a positive integer such that f_{n^+} is injective. Then the equality $f(F(n^+)) \bigcap (F')^{(m)}(n^+) = f(F^{(m)}(n^+))$ holds, for all non-negative integers $m \leqslant n$.*

Proof. Fix $x \in F(n^+)$, $k \leqslant m$, $y \in F'_k$, $s: k^+ \to n^+$ and suppose $s_* y = f_{n^+} x$. Write $s = s's''$ with s' injective and s'' surjective. Replacing y by $(s'')_* y$ and s by s', we see that we may assume that s is injective. Choose $r: n^+ \to k^+$ with $rs = 1$. Then $f_{n^+} s_* r_* x = s_* r_* f_{n^+} x = s_* r_* s_* y = s_* y = f_{n^+} x$ and therefore $s_* r_* x = x$ since f_{n^+} is injective, i.e. $f_{n^+} x$ is in $f(F^{(m)}(n^+))$.

4·2. LEMMA. *Let $f: F \to F'$ be a map of Γ-spaces and n be a positive integer such that for all non-negative integers $m \leqslant n$ the map f_{m^+} is injective. Then the map $f^{(m)}_{l^+}$ is injective, for all non-negative integers $m \leqslant n$ and all non-negative integers l.*

Proof. The proof is by induction on m, the case $m = 0$ being trivial. Since $f^{(m-1)}_{l^+}$ is injective by induction, it suffices to show that $f^{(m)}_{l^+}/f^{(m-1)}_{l^+}$ is injective. This follows from Proposition 3·8, since f_{m^+} is injective by hypothesis, $f^{(m-1)}_{m^+}$ is injective by induction and $f(F(m^+)) \bigcap (F')^{(m-1)}(m^+) = f(F^{(m-1)}(m^+))$ by Lemma 4·1.

4·3. PROPOSITION. *For any Γ-space F, smashing with F preserves injective maps.*

Proof. Fix an injective map of Γ-spaces $f: F' \to F''$. Since the smash product $F \wedge F'$ of the Γ-spaces F and F' can be evaluated degreewise in F, we may assume that F is discrete. Similarly, we may assume that F' and F'' are discrete. Since injections of pointed sets are preserved by filtered colimits, it suffices to show that for all non-negative integers m the map $F \wedge f^{(m)}$ is injective. This will be shown by induction on m, the case $m = 0$ being trivial.

Note that a map of cofibration sequences of pointed sets is injective if it is injective on subobjects and quotient objects and that taking Σ_m-orbits preserves injective maps. Applying Theorem 3·10 we see that it suffices to show that

$$F \wedge \Gamma^m/(\Gamma^m)^{(m-1)} \wedge f_{m^+}/f_{m^+}^{(m-1)}$$

is injective. This amounts to showing that $f_{m^+}/f_{m^+}^{(m-1)}$ is injective. Note that f_{m^+} is injective by hypothesis and that $f_{m^+}^{(m-1)}$ is injective, by Lemma 4·2. The conclusion follows because $f(F'(m^+)) \cap (F'')^{(m-1)}(m^+) = f((F')^{(m-1)}(m^+))$, by Lemma 4·1.

4·4. PROPOSITION. *If $F \to F'$ and $\widetilde{F} \to \widetilde{F}'$ are two injections of Γ-spaces, then the canonical map $F \wedge \widetilde{F}' \cup_{F \wedge \widetilde{F}} F' \wedge \widetilde{F} \to F' \wedge \widetilde{F}'$ is injective.*

Proof. Consider the diagram below.

$$\begin{array}{ccc}
F \wedge \widetilde{F}' & = & F \wedge \widetilde{F}' \\
\downarrow & & \downarrow \\
F \wedge \widetilde{F}' \cup_{F \wedge \widetilde{F}} F' \wedge \widetilde{F} & \to & F' \wedge \widetilde{F}' \\
\downarrow & & \downarrow \\
(F'/F) \wedge \widetilde{F} & \to & (F'/F) \wedge \widetilde{F}'
\end{array}$$

It follows from Proposition 4·3 that the top right vertical map is injective, as is the map $F \wedge \widetilde{F} \to F' \wedge \widetilde{F}$ and its cobase change, the top left vertical map. Thus the columns give cofibration sequences of spaces for each n^+ in Γ^{op} and the conclusion follows since the bottom horizontal map is injective, by Proposition 4·3 again.

4·5. LEMMA. *If F and F' are cofibrant Γ-spaces, then so is $F \wedge F'$.*

Proof. We may assume that F and F' are discrete. Assume for the moment that $F = \Gamma^m$.

If $F' = \Gamma^n$, the conclusion follows from Propositions 2·15 and 3·2. It follows from Proposition 4·3 that the map $F \wedge (\Gamma^n)^{(n-1)} \to F \wedge \Gamma^n$ is a cofibration, because it is an injection of Γ-spaces with cofibrant target. Note that cofibrations are closed under cobase change and sequential colimits. The case F' is any discrete cofibrant Γ-space now follows using Proposition 3·11.

This completes the proof in case $F = \Gamma^m$. The proof of the general case proceeds as in the previous paragraph.

4·6. THEOREM. *If $F \to F'$ and $\widetilde{F} \to \widetilde{F}'$ are two cofibrations of Γ-spaces, then the canonical map κ: $F \wedge \widetilde{F}' \cup_{F \wedge \widetilde{F}} F' \wedge \widetilde{F} \to F' \wedge \widetilde{F}'$ is a cofibration.*

Proof. The cofibre of κ is isomorphic to $(F'/F) \wedge (\widetilde{F}'/\widetilde{F})$. The conclusion now follows from Lemma 4·5 and Proposition 4·4.

5. *Smash products and weak equivalences*

There are three main results in this section, Theorems 5·1, 5·11 and 5·12.

5·1. THEOREM. *Smashing with a cofibrant Γ-space preserves strict weak equivalences.*

5·2. PROPOSITION. *For any strict weak equivalence of Γ-spaces $F \to F'$ and any non-negative integer m, the map $F^{(m)} \to (F')^{(m)}$ is a strict weak equivalence.*

Proof. This follows immediately from Proposition 3·8.

Proof of Theorem 5·1 Since the smash product of the Γ-spaces F and F' can be evaluated degreewise in F, we may assume that F is discrete. By Proposition 4·3 it suffices to show that, for all non-negative integers n, smashing with $F^{(n)}$ preserves strict weak equivalences. We prove this by induction on n. The case $n = 0$ is trivial. It follows from Proposition 3·11 that $F^{(n)}$ is obtained from $F^{(n-1)}$ by attaching $\Gamma^n \wedge S$ along $(\Gamma^n)^{(n-1)} \wedge S$, for some pointed set S. The conclusion follows because smashing with $F^{(n-1)}$, $(\Gamma^n)^{(n-1)}$ and Γ^n preserves strict weak equivalences, since the inductive hypothesis applies to the first two and, by Proposition 2·16, smashing with Γ^n is the same as composing with Γ^n and composing with any Γ-space preserves strict weak equivalences. □

Before we state the remaining main results of this section, we define spectra, stable weak equivalences of Γ-spaces and the (naive) smash product of spectra.

5·3. *Definition.* A *spectrum* E consists of a sequence of spaces E_n and a sequence of maps $E^n_{n+1}\colon S^1 \wedge E_n \to E_{n+1}$, for $n = 0, 1, \ldots$, where S^1 is the space $\Delta^1/\partial\Delta^1$. A map $f\colon E \to E'$ of spectra is a sequence of maps $f_n\colon E_n \to E'_n$ such that

$$f_{n+1} \circ E^n_{n+1} = (E')^n_{n+1} \circ (S^1 \wedge f_n).$$

A spectrum determines direct systems

$$\cdots \longrightarrow \pi_m(E_n) \longrightarrow \pi_{m+1}(E_{n+1}) \longrightarrow \cdots$$

and we define the homotopy groups of E by $\pi_n(E) = \operatorname{colim}_k \pi_{n+k}(E_k)$. A map of spectra is called a *weak equivalence* provided that it induces isomorphisms on homotopy groups.

5·4. *Definition.* Define *the spectrum* $F(S)$ *associated to the* Γ*-space* F by setting $F(S)_n = F(S^n)$, where S^{n+1} is defined recursively as $S^1 \wedge S^n$ and where the maps $F(S)^n_{n+1}$ are obtained from the special assembly map. A map f of Γ-spaces is called a *stable weak equivalence* provided that $f(S)$ is a weak equivalence of spectra.

5·5. *Example.* Recall from Lemma 2·6 and Remark 2·8 that the maps of Γ-spaces from $\mathbb{S}$ to F are given by the vertices of $F(1^+)$. Now let F be as in Example 3·5 with $\mathscr{C} = \Gamma^{\mathrm{op}}$. Thus $F(1^+)$ may be identified with the nerve of the isomorphisms of Γ^{op}. The map $\mathbb{S} \to F$ determined by the object 1^+ of Γ^{op} is a stable weak equivalence. This is essentially the version of the Barratt–Priddy–Quillen–Segal theorem proved in [**Se**].

5·6. *Remark.* The analogue of Proposition 5·2 for stable weak equivalences is false. A counterexample for $m = 1$ and $F' = *$ is provided by setting $F(n^+)$ equal to $n^+ \wedge n^+$.

5·7. *Remark.* We are now able to describe $\boldsymbol{\Sigma}^\infty$ and $\boldsymbol{\Omega}^\infty$ functors. The functor $\boldsymbol{\Sigma}^\infty$ associates to a space X the Γ-space $\Sigma^\infty X$, that takes n^+ to $n^+ \wedge X$. The functor $\boldsymbol{\Omega}^\infty$ associates to a Γ-space F the space $F(1^+)$. Then $\boldsymbol{\Sigma}^\infty$ is left adjoint to $\boldsymbol{\Omega}^\infty$ and takes weak equivalences of spaces to stable (in fact, strict) weak equivalences of Γ-spaces, when restricted to cofibrant objects. This is no restriction at all, because (recall that 'space' means 'pointed simplicial set' in this paper) all spaces are cofibrant. We add it for the sake of symmetry, because in order for the functor $\boldsymbol{\Omega}^\infty$ to take stable weak equivalences to weak equivalences of spaces we have to restrict it to *stably fibrant* objects (see [**BF**]), or at least to *stably* Q*-fibrant* objects (see [**Sch**]). These classes of Γ-spaces are contained in the class of very special Γ-spaces and for very special Γ-spaces the notions of stable and strict weak equivalence coincide (see sections 4 and 5 of [**BF**]).

5·8. *Definition.* Let t be the natural isomorphism that interchanges the second and third factors in the smash product of spaces $X \wedge Y \wedge Z \wedge W$.

5·9. *Definition.* The *smash product* $E \wedge E'$ of the spectra E and E' is the spectrum given by $(E \wedge E')_{2n} = E_n \wedge E'_n$, $(E \wedge E')_{2n+1} = S^1 \wedge (E \wedge E')_{2n}$, $(E \wedge E')^{2n}_{2n+1} = 1$ and $(E \wedge E')^{2n+1}_{2n+2} = (E^n_{n+1} \wedge (E')^n_{n+1}) \circ t$.

5·10. PROPOSITION. *Smashing with a spectrum preserves weak equivalences.*

Proof. This follows essentially because π_n commutes with sequential colimits of spaces (recall that 'space' means 'pointed simplicial set' and $\pi_n(X)$ is the set of pointed homotopy classes of maps from S^n to the singular complex of the realization of X).

5·11. THEOREM. *There is a map of spectra $F(S) \wedge F'(S) \to (F \wedge F')(S)$, natural in the Γ-spaces F and F', which is a weak equivalence if one of the factors is cofibrant.*

5·12. THEOREM. *Smashing with a cofibrant Γ-space preserves stable weak equivalences.*

Proof. This follows immediately from Theorem 5·11 and Proposition 5·10.

5·13. *Definition.* An *S^2-spectrum* E consists of a sequence of spaces E_{2n} and a sequence of maps $E^{2n}_{2n+2}: S^2 \wedge E_{2n} \to E_{2n+2}$, for $n = 0, 1, \ldots$. We define maps, homotopy groups and weak equivalences of such objects, so that forgetting the odd terms of a spectrum gives a functor $E \mapsto E^*$ that preserves weak equivalences. A Γ-space F determines an S^2-spectrum $F(S_t)$, where $F(S_t)_{2n} = F(S^{2n})$. The structural maps are given by the composition

$$S^1 \wedge S^1 \wedge F(S^n \wedge S^n) \to F(S^1 \wedge S^1 \wedge S^n \wedge S^n) \xrightarrow{t_*} F(S^1 \wedge S^n \wedge S^1 \wedge S^n),$$

where the first map is the special assembly map.

5·14. PROPOSITION. *The S^2-spectra $F(S)^*$ and $F(S_t)$ are naturally isomorphic.*

Proof. Define the spectrum S by $S = \mathbb{S}(S)$, and the S^2-spectrum S_t by $S_t = \mathbb{S}(S_t)$. Note that the conclusion of the proposition is true in the special case $F = \mathbb{S}$, i.e. S^* and S_t are isomorphic. The general case follows because $F(S)^*$, respectively $F(S_t)$, is obtained from a functorial construction which, to a Γ-space F and an S^2-spectrum E, associates the S^2-spectrum $F(E)$, by letting $E = S^*$, respectively $E = S_t$. □

The only reason we consider $F(S_t)$, and, in fact, the only reason we consider S^2-spectra, is to be able to write the map in the statement of Theorem 5·11 as a composition of simpler maps. One of these simpler maps is given by Proposition 5·14 and another by Proposition 5·15.

5·15. PROPOSITION. *There is a non-trivial map $(F(S) \wedge F'(S))^* \to (F \wedge F')(S_t)$, natural in the Γ-spaces F and F'.*

Proof. Recall that there is a natural map $F(m^+) \wedge F'(n^+) \to (F \wedge F')(m^+ \wedge n^+)$ (in fact, $F \wedge F'$ is essentially defined by saying that it is universal with this property). This map extends degreewise to a natural map $F(X) \wedge F'(Y) \to (F \wedge F')(X \wedge Y)$, where X and Y are spaces. The map we want is obtained from this map by evaluating on spheres.

5·16. Lemma. *If F and F' are Γ-spaces and F is cofibrant, then the map*

$$(F(S) \wedge F'(S))^* \to (F \wedge F')(S_t)$$

is a weak equivalence.

Proof of Theorem 5·11. Note that the forgetful functor $E \mapsto E^*$ from spectra to S^2-spectra has a left adjoint L, such that $(LE)_{2n} = E_{2n}$, $(LE)_{2n+1} = S^1 \wedge E_{2n}$, $(LE)^{2n}_{2n+1} = 1$ and $(LE)^{2n+1}_{2n+2} = E^{2n}_{2n+2}$. The conclusion now follows from Proposition 5·14, Lemma 5·16 and the fact, for any spectra E and E', the equality $E \wedge E' = L(E \wedge E')^*$ holds.

5·17. *Definition.* We say that a Γ-space F is $o(n)$-connected provided that for any simply-connected space X the space $F(X)$ is as connected as $X^{\wedge n}$. We say that a Γ×Γ-space F is $o(n)$-connected provided that so is its restriction to the diagonal. We say that a map of Γ-spaces, resp. Γ×Γ-spaces, is $o(n)$-connected provided that so is its (pointwise) homotopy cofibre.

5·18. Lemma. *The map of Γ×Γ-spaces given by*

$$F(X) \wedge F'(Y) \to (F \wedge F')(X \wedge Y)$$

is $o(3)$-connected, for any Γ-spaces F and F' with F cofibrant.

Proof of Lemma 5·16. By Lemma 5·18, there is a constant c such that the map $F(S^n) \wedge F'(S^n) \to (F \wedge F')(S^{2n})$ is $(3n + c)$-connected for $n > 1$.

5·19. Proposition. *For any Γ-space F there is a natural strict weak equivalence $F^c \to F$ with F^c cofibrant.*

Proof. We may assume that the Γ-space F is discrete. Let $\mathscr{C}(n^+)$ be the following category. Its objects are the pairs (f, x) such that $f\colon m^+ \to n^+$ and $x \in F(m^+)$. There is one morphism from (f, x) to (g, y) for each h in Γ^{op} such that $gh = f$ and $h_*(x) = y$. Define $F'(n^+)$ as the nerve of $\mathscr{C}(n^+)$. There is a map $F'(n^+) \to F(n^+)$ taking (f, x) to $f_*(x)$, and is the disjoint union of projections of nerves of categories to their terminal objects, in particular a weak equivalence. Note that

$$F'_q \cong \coprod_{(k_0,\ldots,k_q)} \Gamma^{k_q} \times \Gamma^{\mathrm{op}}(k^+_{q-1}, k^+_q) \times \cdots \times \Gamma^{\mathrm{op}}(k^+_0, k^+_1) \times F(k^+_0).$$

The required Γ-space F^c is given by a pointed version of this. Let

$$F^c_q = \bigvee_{(k_0,\ldots,k_q)} \Gamma^{k_q} \wedge \Gamma^{\mathrm{op}}(k^+_{q-1}, k^+_q) \wedge \cdots \wedge \Gamma^{\mathrm{op}}(k^+_0, k^+_1) \wedge F(k^+_0)$$

so that we have a canonical map $F' \to F^c$. Then it is still true that there is a map $F^c \to F$ and that, for fixed n^+, there is a section $F(n^+) \to F^c(n^+)$, as well as a map $F^c(n^+) \wedge (\Delta^1)^+ \to F^c(n^+)$ which is a homotopy between the identity and the composition $F^c(n^+) \to F(n^+) \to F^c(n^+)$. All these maps are compatible with the canonical map $F' \to F^c$ and exist essentially because Γ^{op} is a pointed category and F is a pointed functor.

Proof of Lemma 5·18. Suppose first that F equals Γ^n and that F' equals Γ^m. The map in the statement of the lemma corresponds, under the isomorphism of Proposition 2·15 between $\Gamma^n \wedge \Gamma^m$ and Γ^{nm}, to the map ψ that takes (f, g) to $f \wedge g$. Define a filtration

$$* = F_0\Gamma^n \subset F_1\Gamma^n \subset \cdots \subset F_n\Gamma^n = \Gamma^n$$

of Γ^n by letting $F_k\Gamma^n(m^+)$ consists of those $f\colon n^+ \to m^+$ such that the cardinality of $f^{-1}(0)$ is at least $(n+1-k)$. Then $F_k\Gamma^n/F_{k-1}\Gamma^n(X)$ is isomorphic to $\binom{n}{k}^+ \wedge X^{\wedge k}$. In particular, the map $F_k\Gamma^n \subset \Gamma^n$ is $o(k+1)$-connected and $F_1\Gamma^n$ is isomorphic to $n \wedge \Gamma^1$. The restriction of ψ to $F_1\Gamma^n(X) \wedge F_1\Gamma^m(Y)$ is an isomorphism onto $F_1\Gamma^{nm}(X \wedge Y)$. The conclusion follows from this, together with the fact that $F_1\Gamma^k$ and Γ^k are $o(1)$-connected (for any k).

Suppose now that F' is discrete. We prove that the lemma is true in this case, by proving that it is true for all $(F')^{(m)}$. As usual, the proof will be by induction on m, the case $m = 0$ being trivial. The inductive step follows from the previous paragraph and Proposition 3·11.

In case F' is any cofibrant Γ-space, the lemma is true because diagonalization preserves connectivity. The complete special case $F = \Gamma^n$ now follows from Theorem 5·1 and Propositions 3·2 and 5·19. The rest of the proof is similar, i.e. the discrete case is done by induction on skeleta and then the general case of an arbitrary cofibrant Γ-space F follows because diagonalization preserves connectivity. □

We conclude this section by proving certain interesting statements about Γ-spaces.

5·20. PROPOSITION. *Any Γ-space F preserves connectivity, i.e. if f is a k-connected map of spaces, then so is $F(f)$. In particular, F is a homotopy functor, i.e. it preserves weak equivalences of spaces.*

Proof. This follows from Propositions 3·11 and 5·19.

5·21. PROPOSITION. *If X and Y are connected spaces and F is a Γ-space, the map $F(X) \wedge Y \to F(X \wedge Y)$ is as connected as the suspension of $X^{\wedge 2} \wedge Y$.*

Proof. The proof is similar to, but easier than, the proof of Lemma 5·18.

5·22. PROPOSITION. *If X, Y are connected spaces and F, F' are Γ-spaces with F cofibrant, then the map $F(X) \wedge F'(Y) \to (F \wedge F')(X \wedge Y)$ is as connected as the suspension of $(X^{\wedge 2} \wedge Y) \vee (X \wedge Y^{\wedge 2})$.*

Proof. The proof is the same as the proof of Lemma 5·18.

5·23. PROPOSITION. *The assembly-map $F \wedge F' \to F \circ F'$ is a stable weak equivalence, whenever F or F' is cofibrant.*

Proof. Given the previous proposition, it suffices to show that the associated map $f\colon F(X) \wedge F'(Y) \to (F \circ F')(X \wedge Y)$ is highly connected, if so are X and Y. This follows from the definition of f, together with Propositions 5·20 and 5·21.

6. *Cofibrations and strict cofibrations*

In this section we show that our definitions of the skeleta and the cofibrations of Γ-spaces are equivalent to those found in [**BF**, pp. 89, 91], which we now recall.

6·1. *Definition.* Let m be a non-negative integer. The functor $sk_m: \mathscr{GS} \to \mathscr{GS}$ is defined as follows. If F is a Γ-space and n is a non-negative integer, define $(sk_m F)(n^+)$ to be the colimit over all $k^+ \to n^+$ with $k \leqslant m$ of $F(k^+)$.

There is a map $sk_m F \to F$, given for $f\colon k^+ \to n^+$ by $f_*\colon F(k^+) \to F(n^+)$. A map of Γ-spaces $f: F \to F'$ is called a *strict cofibration* provided that for all positive integers n the map

$$g_n\colon F(n^+) \cup_{(sk_{n-1}F)(n^+)} (sk_{n-1}F')(n^+) \to F'(n^+)$$

is injective and the action of Σ_n is free off the image of g_n.

6·2. Proposition. *For any Γ-space F and any non-negative integer m, the Γ-spaces $F^{(m)}$ and $sk_m F$ are isomorphic.*

Proof. The map $sk_m F \to F$ induces a surjective map $sk_m F \to F^{(m)}$, which we show is also injective. Fix a non-negative integer n, as well as maps $f: k^+ \to n^+$ and $g: l^+ \to n^+$ with $k, l \leqslant m$. We have to show that if $f_* x = g_* y$ in $F(n^+)$, then $[f, x] = [g, y]$ in $(sk_m F)(n^+)$, where the notation is as in 2·14. By arguing as in the proof of Lemma 4·1, we may assume that f and g are injective. By Proposition 3·6, if p^+ is the pullback of f and g, then x and y can be lifted to z in $F(p^+)$. If $h: p^+ \to n^+$ is the associated canonical map, both $[f, x]$ and $[g, y]$ equal $[h, z]$.

6·3. Lemma. *Strict cofibrations of Γ-spaces are injective.*

Proof. Let $f: F \to F'$ be a strict cofibration of Γ-spaces. We prove by induction on n that f_{m^+} is injective for all $m \leqslant n$. The case $n = 0$ is trivial.

By Lemma 4·2 and the inductive hypothesis, $F^{(n-1)}(n^+)$ injects into $(F')^{(n-1)}(n^+)$. Therefore $F(n^+)$ injects into $F(n^+) \cup_{F^{(n-1)}(n^+)} (F')^{(n-1)}(n^+)$, which in turn injects into $F'(n^+)$ by the definition of strict cofibration and Proposition 6·2.

6·4. Proposition. *A map of Γ-spaces $f: F \to F'$ is a cofibration if and only if it is a strict cofibration.*

Proof. For any injection of Γ-spaces $f: F \to F'$ and any positive integer n, the map g_n of Definition 6·1 is isomorphic over $F'(n^+)$ to the inclusion

$$f(F(n^+)) \cup (F')^{(n-1)}(n^+) \subset F'(n^+).$$

This follows from Proposition 6·2 and the fact that, by Lemma 4·1,

$$f(F(n^+)) \cap (F')^{(n-1)}(n^+) = f(F^{(n-1)}(n^+)).$$

The conclusion now follows from Lemma 6·3.

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