Derived Algebraic Geometry X: Formal Moduli Problems

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Introduction

The following thesis plays a central role in deformation theory:

($*$) If X is a moduli space over a field k of characteristic zero, then a formal neighborhood of any point $x \in X$ is controlled by a differential graded Lie algebra.

This idea was developed in unpublished work of Deligne, Drinfeld, and Feigin, and has powerfully influenced subsequent contributions of Hinich, Kontsevich-Soibelman, Manetti, Pridham, and many others. One of our main goals in this paper is to give a precise formulation (and proof) of ($*$), using the language of higher category theory.

The first step in formulating ($*$) is to decide exactly what we mean by a moduli space. For simplicity, let us work for now over the field $\mathbf{C}$ of complex numbers. We will adopt Grothendieck's "functor of points" philosophy, and identify an algebro-geometric object X (for example, a scheme) with the functor $R \mapsto X(R) = \mathrm{Hom}(\mathrm{Spec}\, R, X)$. This suggests a very general definition:

Definition 0.0.1. A *classical moduli problem* is a functor $X : \mathrm{Ring}_{\mathbf{C}} \to \mathcal{S}\mathrm{et}$, where $\mathrm{Ring}_{\mathbf{C}}$ denotes the category of commutative $\mathbf{C}$-algebras and $\mathcal{S}\mathrm{et}$ denotes the category of sets.

Unfortunately, Definition 0.0.1 is not adequate for the needs of this paper. First of all, Definition 0.0.1 requires that the functor X take values in the category of sets. In many applications, we would like to consider functors X which assign to each commutative ring R some collection of geometric objects parametrized by the affine scheme $\mathrm{Spec}\, R$. In such cases, it is important to keep track of automorphism groups.

Example 0.0.2. For every commutative $\mathbf{C}$-algebra R, let $X(R)$ denote the category of elliptic curves $E \to \mathrm{Spec}\, R$ (morphisms in the category $X(R)$ are given by isomorphisms of elliptic curves). Then F determines a functor from $\mathrm{Ring}_{\mathbf{C}}$ to Gpd, where Gpd denotes the 2-category of *groupoids*. In this case, X determines an underlying set-valued functor, which assigns to each commutative ring R the set $\pi_0 X(R)$ of isomorphism classes of elliptic curves over R. However, the groupoid-valued functor $X : \mathrm{Ring}_{\mathbf{C}} \to \mathrm{Gpd}$ is much better behaved than the set-valued functor $\pi_0 X : \mathrm{Ring}_{\mathbf{C}} \to \mathcal{S}\mathrm{et}$. For example, the functor X satisfies descent (with respect to the flat topology on the category of commutative rings), while the functor $\pi_0 X$ does not: two elliptic curves which are locally isomorphic need not be globally isomorphic.

Because the functor X of Example 0.0.2 is not $\mathcal{S}\mathrm{et}$-valued, it cannot be represented by a scheme. However, it is nevertheless a reasonable geometric object: it is representable by a Deligne-Mumford stack. To accommodate Example 0.0.2, we would like to adjust Definition 0.0.1 to allow groupoid-valued functors.

Variant 0.0.3. Let $\mathcal{C}$ be an ∞-category. A *$\mathcal{C}$-valued classical moduli problem* is a functor $\mathrm{N}(\mathrm{Ring}_{\mathbf{C}}) \to \mathcal{C}$. Here $\mathrm{Ring}_{\mathbf{C}}$ denotes the category of commutative algebras over the field $\mathbf{C}$ of complex numbers.

Remark 0.0.4. We recover Definition 0.0.1 as a special case of Variant 0.0.3, by taking $\mathcal{C}$ to be (the nerve of) the category of sets. In practice, we will be most interested in the special case where $\mathcal{C}$ is the ∞-category $\mathcal{S}$ of spaces.

The next step in formulating ($*$) is to decide what we mean by a formal neighborhood of a point x in a moduli space X. Suppose, for example, that $X = \mathrm{Spec}\, A$ is an affine algebraic variety over the field $\mathbf{C}$ of complex numbers. Then a closed point $x \in X$ is determined by a $\mathbf{C}$-algebra homomorphism $\phi : A \to \mathbf{C}$, which is determined a choice of maximal ideal $\mathfrak{m} = \ker(\phi) \subseteq A$. One can define the *formal completion* of X at the point x to be the functor $X^{\wedge} : \mathrm{Ring} \to \mathcal{S}\mathrm{et}$ given by the formula

$$X^{\wedge}(R) = \{f \in X(R) : f(\mathrm{Spec}\, R) \subseteq \{x\} \subseteq \mathrm{Spec}\, A\}.$$

In other words, $X^{\wedge}(R)$ is the collection of commutative ring homomorphisms $\phi : A \to R$ having the property that ϕ carries each element of $\mathfrak{m}$ to a nilpotent element of R. Since $\mathfrak{m}$ is finitely generated, this is equivalent to the condition that ϕ annihilates $\mathfrak{m}^n$ for some integer $n \gg 0$, so that the image of ϕ is a quotient of A by some $\mathfrak{m}$-primary ideal.

Definition 0.0.5. Let R be a commutative algebra over the field $\mathbf{C}$ of complex numbers. We will say that R is a *local Artinian* if it is finite dimensional as a $\mathbf{C}$-vector space and has a unique maximal ideal $\mathfrak{m}_R$. The collection of local Artinian $\mathbf{C}$-algebras forms a category, which we will denote by $\mathrm{Ring}_{\mathbf{C}}^{\mathrm{art}}$.

The above analysis shows that if X is an affine algebraic variety over $\mathbf{C}$ containing a point x, then the formal completion $X^{\wedge}$ can be recovered from its values on local Artinian $\mathbf{C}$-algebras. This motivates the following definition:

Definition 0.0.6. Let $\mathcal{C}$ be an ∞-category. A *$\mathcal{C}$-valued classical formal moduli problem* is a functor $\mathrm{N}(\mathrm{Ring}_{\mathbf{C}}^{\mathrm{art}}) \to \mathcal{C}$.

If X is a $\mathcal{S}$et-valued classical moduli problem and we are given a point $\eta \in X(\mathbf{C})$, we can define a $\mathcal{S}$et-valued classical formal moduli problem $X^{\wedge}$ by the formula

$$X^{\wedge}(R) = X(R) \times_{X(R/\mathfrak{m}_R)} \{\eta\}.$$

We will refer to $X^{\wedge}$ as the *completion of X at the point η*. If X is Gpd-valued, the same formula determines a Gpd-valued classical formal moduli problem $X^{\wedge}$ (here we take a homotopy fiber product of the relevant groupoids).

Example 0.0.7. For every commutative $\mathbf{C}$-algebra R, let $X(R)$ denote the groupoid whose objects are smooth proper R-schemes and whose morphisms are isomorphisms of R-schemes. Suppose we are given a point $\eta \in X(\mathbf{C})$, corresponding to smooth and proper algebraic variety Z over $\mathbf{C}$. The formal completion $X^{\wedge}$ assigns to every local Artinian $\mathbf{C}$-algebra R the groupoid $X^{\wedge}(R)$ of *deformations over Z* over R: that is, smooth proper morphisms $f : \overline{Z} \to \operatorname{Spec} R$ which fit into a pullback diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & \overline{Z} \arrow[d] \\ \operatorname{Spec} \mathbf{C} \arrow[r] & \operatorname{Spec} R. \end{tikzcd}$$

Example 0.0.7 is a typical example of the kind of formal moduli problem we would like to study. Let us summarize some well-known facts about the functor $X^{\wedge}$:

(a) The functor $X^{\wedge}$ carries the ring $\mathbf{C}[\epsilon]/(\epsilon^2)$ to the groupoid of first-order deformations of the variety Z. Every first order deformation of Z has an automorphism group which is canonically isomorphic to $\mathrm{H}^0(Z; T_Z)$, where T_Z denotes the tangent bundle of Z.

(b) The collection of isomorphism classes of first order deformations of Z can be canonically identified with the cohomology group $\mathrm{H}^1(Z; T_Z)$.

(c) To every first order deformation η_1 of Z, we can assign an obstruction class $\theta \in \mathrm{H}^2(Z; T_Z)$ which vanishes if and only if η_1 extends to a second-order deformation $\eta_2 \in X^{\wedge}(\mathbf{C}[\epsilon]/(\epsilon^3))$.

Assertion (a) and (b) are very satisfying: they provide a concrete geometric interpretations of certain cohomology groups, and (b) can be given a conceptual proof using the interpretation of H^1 as classifying torsors. By contrast, (c) is often proven by an ad-hoc argument which uses the local triviality of the first order deformation to extend locally, and then realizes the obstruction as a cocycle representing the (possible) inability to globalize this extension. This argument is computational rather than conceptual, and it does not furnish a geometric interpretation of the entire cohomology group $\mathrm{H}^2(Z; T_Z)$.

Let us now sketch an explanation for (c) using the language of spectral algebraic geometry, which does not share these defects. The key observation is that we can enlarge the category on which the functor X is defined. If R is a connective $\mathbb{E}_\infty$-algebra over $\mathbf{C}$, we can define $X(R)$ to be the underlying ∞-groupoid of the ∞-category of spectral schemes which are proper and smooth over R. If R is equipped with an augmentation $\epsilon : R \to \mathbf{C}$, we let $X^\wedge(R)$ denote the fiber product $X(R) \times_{X(\mathbf{C})} \{\eta\}$, which we can think of as a classifying space for *deformations of* Z over $\operatorname{Spec} R$. In the special case where R is a discrete local Artinian $\mathbf{C}$-algebra, we recover the groupoid-valued functor described in Example 0.0.7. However, we can obtain more information by evaluating the functor $X^\wedge$ on $\mathbb{E}_\infty$-algebras over $\mathbf{C}$ which are not discrete. For example, let $\mathbf{C}[\delta]$ denote the square-zero extension $\mathbf{C} \oplus \mathbf{C}[1]$. One can show that there is a canonical bijection $\mathrm{H}^2(Z; T_Z) \simeq \pi_0 X^\wedge(\mathbf{C}[\delta])$. We can regard this as an analogue of (c): it gives a description of cohomology group $\mathrm{H}^2(Z; T_Z)$ as the set of isomorphism classes of first order deformations of Z to the "nonclassical" commutative ring $\mathbf{C}[\delta]$.

The interpretation of obstructions as elements of $\mathrm{H}^2(X, T_X)$ can now be obtained as follows. The ordinary commutative ring $\mathbf{C}[\epsilon]/(\epsilon^3)$ is a square-zero extension of $\mathbf{C}[\epsilon]/(\epsilon^2)$ by the ideal $\mathbf{C}\,\epsilon^2$, and therefore fits into a pullback diagram of $\mathbb{E}_\infty$-rings

$$\begin{tikzcd} \mathbf{C}[\epsilon]/(\epsilon^3) \arrow[r] \arrow[d] & \mathbf{C}[\epsilon]/(\epsilon^2) \arrow[d] \\ \mathbf{C} \arrow[r] & \mathbf{C}[\delta]. \end{tikzcd}$$

In §IX.9, we saw that this pullback square determines a pullback square of spaces

$$\begin{tikzcd} X(\mathbf{C}[\epsilon]/(\epsilon^3)) \arrow[r] \arrow[d] & X(\mathbf{C}[\epsilon]/(\epsilon^2)) \arrow[d] \\ X(\mathbf{C}) \arrow[r] & X(\mathbf{C}[\delta]), \end{tikzcd}$$

and therefore a fiber sequence of spaces

$$X^\wedge(\mathbf{C}[\epsilon]/(\epsilon^3)) \to X^\wedge(\mathbf{C}[\epsilon]/(\epsilon^2)) \to X^\wedge(\mathbf{C}[\delta]).$$

In particular, every first-order deformation η_1 of Z determines an element of $\pi_0 X^\wedge(\mathbf{C}[\delta]) \simeq \mathrm{H}^2(Z; T_Z)$, which vanishes precisely when η_1 can be lifted to a second order deformation of Z.

The analysis that we have just provided in Example 0.0.7 cannot be carried out for an arbitrary classical formal moduli problem (in the sense of Definition 0.0.6): it depends crucially on the fact that the functor $X^\wedge$ could be defined on $\mathbb{E}_\infty$-rings which are not assumed to be discrete. This motivates another variant of Definition 0.0.1:

Definition 0.0.8. Let $\mathrm{CAlg}^{\mathrm{sm}}_{\mathbf{C}}$ denote the ∞-category of small $\mathbb{E}_\infty$-algebras over $\mathbf{C}$ (see Proposition 1.1.11). A *formal moduli problem* over $\mathbf{C}$ is a functor $X : \mathrm{CAlg}^{\mathrm{sm}}_{\mathbf{C}} \to \mathcal{S}$ which satisfies the following pair of conditions:

(1) The space $X(\mathbf{C})$ is contractible.

(2) For every pullback diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R_0 \arrow[d] \\ R_1 \arrow[r] & R_{01} \end{tikzcd}$$

in $\mathrm{CAlg}^{\mathrm{sm}}_{\mathbf{C}}$ for which the underlying maps $\pi_0 R_0 \to \pi_0 R_{01} \leftarrow \pi_0 R_1$ are surjective, the diagram

$$\begin{tikzcd} X(R) \arrow[r] \arrow[d] & X(R_0) \arrow[d] \\ X(R_1) \arrow[r] & X(R_{01}) \end{tikzcd}$$

is a pullback square.

Remark 0.0.9. Let $\mathrm{CAlg}^{\mathrm{cn}}_{\mathbf{C}}$ denote the ∞-category of connective $\mathbb{E}_\infty$-algebras over the field $\mathbf{C}$ of complex numbers, and let $X : \mathrm{CAlg}^{\mathrm{cn}}_{\mathbf{C}} \to \mathcal{S}$ be a functor. Given a point $x \in X(\mathbf{C})$, we define the *formal completion* of X at the point x to be the functor $X^\wedge : \mathrm{CAlg}^{\mathrm{sm}}_{\mathbf{C}} \to \mathcal{S}$ given by the formula $X^\wedge(R) = X(R) \times_{X(\mathbf{C})} \{x\}$. The functor $X^\wedge$ automatically satisfies condition (1) of Definition 0.0.8. Condition (2) is not automatic, but holds whenever the functor X is defined in a sufficiently "geometric" way. To see this, let us imagine that there exists some ∞-category of geometric objects $\mathcal{C}$ with the following properties:

- (a) To every object $A \in \mathrm{CAlg}^{\mathrm{cn}}_{\mathbf{C}}$ we can assign an object $\operatorname{Spec} A \in \mathcal{C}$, which is contravariantly functorial in A.
- (b) There exists an object $\mathcal{X} \in \mathcal{C}$ which represents X, in the sense that $X(A) \simeq \mathrm{Hom}_{\mathcal{C}}(\operatorname{Spec} A, \mathcal{X})$ for every small $\mathbf{C}$-algebra A.

To verify that $X^\wedge$ satisfies condition (2) of Definition 0.0.8, it suffices to show that when $\phi : R_0 \to R_{01}$ and $\phi' : R_1 \to R_{01}$ are maps of small $\mathbb{E}_\infty$ algebras over $\mathbf{C}$ which induce surjections $\pi_0 R_0 \to \pi_0 R_{01} \leftarrow \pi_0 R_1$, then the diagram

$$\begin{tikzcd}
\operatorname{Spec} R_{01} \arrow[r] \arrow[d] & \operatorname{Spec} R_1 \arrow[d] \\
\operatorname{Spec} R_0 \arrow[r] & \operatorname{Spec}(R_1 \times_{R_{01}} R_0)
\end{tikzcd}$$

is a pushout square in $\mathcal{C}$. This assumption expresses the idea that $\operatorname{Spec}(R_0 \times_{R_{01}} R_1)$ should be obtained by "gluing" $\operatorname{Spec} R_0$ and $\operatorname{Spec} R_1$ together along the common closed subobject $\operatorname{Spec} R_{01}$.

Example 0.0.10. Let $\mathcal{C}$ denote the ∞-category $\mathrm{Stk}_{\mathbf{C}}$ of spectral Deligne-Mumford stacks over $\mathbf{C}$. The construction $\operatorname{Spec}^{\text{ét}} : \mathrm{CAlg}^{\mathrm{cn}}_{\mathbf{C}} \to \mathcal{C}$ satisfies the gluing condition described in Remark 0.0.9 (Corollary IX.6.5). It follows that every spectral Deligne-Mumford stack $\mathfrak{X}$ over $\mathbf{C}$ equipped with a base point $x : \operatorname{Spec}^{\text{ét}} \mathbf{C} \to \mathfrak{X}$ determines a formal moduli problem $X^\wedge : \mathrm{CAlg}^{\mathrm{sm}}_{\mathbf{C}} \to \mathcal{S}$, given by the formula

$$X^\wedge(R) = \mathrm{Map}_{\mathrm{Stk}_{\mathbf{C}}}(\operatorname{Spec}^{\text{ét}} R, \mathfrak{X}) \times_{\mathrm{Map}_{\mathrm{Stk}_{\mathbf{C}}}(\operatorname{Spec}^{\text{ét}} \mathbf{C}, \mathfrak{X})} \{x\}.$$

We refer to $X^\wedge$ as the *formal completion of $\mathfrak{X}$ at the point x*.

Remark 0.0.11. Let $X : \mathrm{CAlg}^{\mathrm{sm}}_{\mathbf{C}} \to \mathcal{S}$ be a formal moduli problem. Then X determines a functor $\overline{X} : \mathrm{hCAlg}^{\mathrm{sm}}_{\mathbf{C}} \to \mathcal{S}\mathrm{et}$ between ordinary categories (here $\mathrm{hCAlg}^{\mathrm{sm}}_{\mathbf{C}}$ denotes the homotopy category of $\mathrm{CAlg}^{\mathrm{sm}}_{\mathbf{C}}$), given by the formula $\overline{X}(A) = \pi_0 X(A)$. It follows from condition (2) of Definition 0.0.8 that if we are given maps of small E_∞-algebras $A \to B \leftarrow A'$ which induce surjections $\pi_0 A \to \pi_0 B \leftarrow \pi_0 A'$, then the induced map

$$\overline{X}(A \times_B A') \to \overline{X}(A) \times_{\overline{X}(B)} \overline{X}(A')$$

is a surjection of sets. There is a substantial literature on set-valued moduli functors of this type; see, for example, [50] and [33].

Warning 0.0.12. If X is a formal moduli problem over $\mathbf{C}$, then X determines a classical formal moduli problem (with values in the ∞-category $\mathcal{S}$) simply by restricting the functor X to the subcategory of $\mathrm{CAlg}^{\mathrm{sm}}_{\mathbf{C}}$ consisting of ordinary local Artinian $\mathbf{C}$-algebras (which are precisely the discrete objects of $\mathrm{CAlg}^{\mathrm{sm}}_{\mathbf{C}}$).

If $\mathfrak{X} = (\mathcal{X}, \mathcal{O})$ is a spectral Deligne-Mumford stack over $\mathbf{C}$ equipped with a point $\eta : \operatorname{Spec} \mathbf{C} \to \mathfrak{X}$ and X is defined as in Example 0.0.10, then the restriction $X_0 = X| \mathrm{N}(\mathrm{Ring}^{\mathrm{art}}_{\mathbf{C}})$ depends only on the pair $(\mathcal{X}, \pi_0 \mathcal{O})$. In particular, the functor X cannot be recovered from X_0.

In general, if we are given a classical formal moduli problem $X_0 : \mathrm{N}(\mathrm{Ring}^{\mathrm{art}}_{\mathbf{C}}) \to \mathcal{S}$, there may or may not exist a formal moduli problem X such that $X_0 = X| \mathrm{N}(\mathrm{Ring}^{\mathrm{art}}_{\mathbf{C}})$. Moreover, if X exists, then it need not be unique. Nevertheless, classical formal moduli problems X_0 which arise naturally are often equipped with a natural extension $X : \mathrm{CAlg}^{\mathrm{sm}}_{\mathbf{C}} \to \mathcal{S}$ (as in our elaboration of Example 0.0.7).

Theorem 0.0.13. *Let* Moduli *denote the full subcategory of* $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{sm}}_{\mathbf{C}}, \mathcal{S})$ *spanned by the formal moduli problems, and let* $\mathrm{Lie}^{\mathrm{dg}}_{\mathbf{C}}$ *denote the category of differential graded Lie algebras over* $\mathbf{C}$ *(see §2.1). Then there is a functor*

$$\theta : \mathrm{N}(\mathrm{Lie}^{\mathrm{dg}}_{\mathbf{C}}) \to \mathrm{Moduli}$$

with the following universal property: for any ∞-category $\mathcal{C}$, composition with θ induces a fully faithful embedding $\mathrm{Fun}(\mathrm{Moduli}, \mathcal{C}) \to \mathrm{Fun}(\mathrm{N}(\mathrm{Lie}^{\mathrm{dg}}_{\mathbf{C}}), \mathcal{C})$*, whose essential image is the collection of all functors* $F : \mathrm{N}(\mathrm{Lie}^{\mathrm{dg}}_{\mathbf{C}}) \to \mathcal{C}$ *which carry quasi-isomorphisms of differential graded Lie algebras to equivalences in* $\mathcal{C}$*.*

Remark 0.0.14. An equivalent version of Theorem 0.0.13 has been proven by Pridham; we refer the reader to [54] for details.

Remark 0.0.15. Let W be the collection of all quasi-isomorphisms in the category $\mathrm{Lie}^{\mathrm{dg}}_{\mathbf{C}}$, and let $\mathrm{Lie}^{\mathrm{dg}}_{\mathbf{C}}[W^{-1}]$ denote the ∞-category obtained from $\mathrm{N}(\mathrm{Lie}^{\mathrm{dg}}_{\mathbf{C}})$ by formally inverting the morphisms in W. Theorem 0.0.13 asserts that there is an equivalence of ∞-categories $\mathrm{Lie}^{\mathrm{dg}}_{\mathbf{C}}[W^{-1}] \simeq \mathrm{Moduli}$. In particular, every differential graded Lie algebra over $\mathbf{C}$ determines a formal moduli problem, and two differential graded Lie algebras $\mathfrak{g}_*$ and $\mathfrak{g}'_*$ determine equivalent formal moduli problems if and only if they can be joined by a chain of quasi-isomorphisms.

Theorem 0.0.13 articulates a sense in which the theories of commutative algebras and Lie algebras are closely related. In concrete terms, this relationship is controlled by the *Chevalley-Eilenberg* functor, which associates to a differential graded Lie algebra $\mathfrak{g}_*$ a cochain complex of vector spaces $C^*(\mathfrak{g}_*)$. The cohomology of this cochain complex is the *Lie algebra cohomology* of the Lie algebra $\mathfrak{g}_*$, and is endowed with a commutative multiplication. In fact, this multiplication is defined at the level of cochains: the construction $\mathfrak{g}_* \mapsto C^*(\mathfrak{g}_*)$ determines a functor C^* from the (opposite of) the category $\mathrm{Lie}^{\mathrm{dg}}_{\mathbf{C}}$ of differential graded Lie algebras over $\mathbf{C}$ to the category $\mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{C}}$ of commutative differential graded algebras over $\mathbf{C}$. This functor carries quasi-isomorphisms to quasi-isomorphisms, and therefore induces a functor between ∞-categories

$$\phi : \mathrm{Lie}^{\mathrm{dg}}_{\mathbf{C}}[W^{-1}]^{op} \to \mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{C}}[W'^{-1}],$$

where W is the collection of quasi-isomorphisms in $\mathrm{Lie}^{\mathrm{dg}}_{\mathbf{C}}$ (as in Remark 0.0.15) and W' is the collection of quasi-isomorphisms in $\mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{C}}$ (here the ∞-category $\mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{C}}[W'^{-1}]$ can be identified $\mathrm{CAlg}_{\mathbf{C}}$ of $\mathbb{E}_\infty$-algebras over $\mathbf{C}$: see Proposition A.7.1.4.11). Every differential graded Lie algebra $\mathfrak{g}_*$ admits a canonical map $\mathfrak{g}_* \to 0$, so that its Chevalley-Eilenberg complex is equipped with an augmentation $C^*(\mathfrak{g}_*) \to C^*(0) \simeq \mathbf{C}$. We may therefore refine ϕ to a functor $\mathrm{Lie}^{\mathrm{dg}}_{\mathbf{C}}[W^{-1}]^{op} \to \mathrm{CAlg}^{\mathrm{aug}}_{\mathbf{C}}$ taking values in the ∞-category $\mathrm{CAlg}^{\mathrm{aug}}_{\mathbf{C}}$ of *augmented* $\mathbb{E}_\infty$-algebras over $\mathbf{C}$. We will see that this functor admits a left adjoint $\mathfrak{D} : \mathrm{CAlg}^{\mathrm{aug}}_{\mathbf{C}} \to \mathrm{Lie}^{\mathrm{dg}}_{\mathbf{C}}[W^{-1}]^{op}$ (Theorem 2.3.1). The functor $\theta : \mathrm{N}(\mathrm{Lie}^{\mathrm{dg}}_{\mathbf{C}}) \to \mathrm{Moduli}$ appearing in the statement of Theorem 0.0.13 can then be defined by the formula

$$\theta(\mathfrak{g}_*)(R) = \mathrm{Map}_{\mathrm{Lie}^{\mathrm{dg}}_{\mathbf{C}}[W^{-1}]}(\mathfrak{D}(R), \mathfrak{g}_*).$$

In more abstract terms, the relationship between commutative algebras and Lie algebras suggested by Theorem 0.0.13 is an avatar of *Koszul duality*. More specifically, Theorem 0.0.13 reflects the fact that the commutative operad is Koszul dual to the Lie operad (see [29]). This indicates that should be many other versions of Theorem 0.0.13, where we replace commutative and Lie algebras by algebras over some other pair of Koszul dual operads. For example, the Koszul self-duality of the $\mathbb{E}_n$-operads (see [17]) suggests an analogue of Theorem 0.0.13 in the setting of "noncommutative" derived algebraic geometry, which we also prove (see Theorems 3.0.4 and 4.0.8).

Let us now outline the contents of this paper. In §1, we will introduce the general notion of a *deformation theory*: a functor of ∞-categories $\mathfrak{D} : \Upsilon^{op} \to \Xi$ satisfying a suitable list of axioms (see Definitions 1.3.1 and 1.3.9). We will then prove an abstract version of Theorem 0.0.13: every deformation theory $\mathfrak{D}$ determines an equivalence $\Xi \simeq \mathrm{Moduli}^{\Upsilon}$, where $\mathrm{Moduli}^{\Upsilon}$ is a suitably defined ∞-category of formal moduli problems

(Theorem 1.3.12). This result is not very difficult in itself: it can be regarded as a distillation of the purely formal ingredients needed for the proof of results like Theorem 0.0.13. In practice, the hard part is to construct the functor $\mathfrak{D}$ and to prove that it satisfies the axioms of Definitions 1.3.1 and 1.3.9. We will give a detailed treatment of three special cases:

(a) In §2, we treat the case where Υ is the ∞-category $\mathrm{CAlg}_k^{\mathrm{aug}}$ of augmented $\mathbb{E}_\infty$-algebras over a field k of characteristic zero, and use Theorem 1.3.12 to prove a version of Theorem 0.0.13 (Theorem 2.0.2).

(b) In §3, we treat the case where Υ is the ∞-category $\mathrm{Alg}_k^{\mathrm{aug}}$ of augmented $\mathbb{E}_1$-algebras over a field k (of arbitrary characteristic), and use Theorem 1.3.12 to prove a noncommutative analogue of Theorem 0.0.13 (Theorem 3.0.4).

(c) In §4, we treat the case where Υ is the ∞-category $\mathrm{Alg}_k^{(n),\mathrm{aug}}$ of augmented $\mathbb{E}_n$-algebras over a field k (again of arbitrary characteristic), and use Theorem 1.3.12 to prove a more general noncommutative analogue of Theorem 0.0.13 (Theorem 4.0.8).

In each case, the relevant deformation functor $\mathfrak{D}$ is given by some variant of Koszul duality, and our main result gives an algebraic model for the ∞-category of formal moduli problems $\mathrm{Moduli}^{\Upsilon}$. In §5, we will use these results to study some concrete examples of formal moduli problems which arise naturally in deformation theory.

Remark 0.0.16. The notion that differential graded Lie algebras should play an important role in the description of moduli spaces goes back to Quillen's work on rational homotopy theory ([73]), and was developed further in unpublished work of Deligne, Drinfeld, and Feigin. Many other mathematicians have subsequently taken up these ideas: see, for example, the book of Kontsevich and Soibelman ([33]).

Remark 0.0.17. The subject of deformation theory has a voluminous literature, some of which has substantial overlap with the material discussed in this paper. Though we have tried to provide relevant references in the body of the text, there are undoubtedly many sins of omission for which we apologize in advance.

Notation and Terminology

We will use the language of ∞-categories freely throughout this paper. We refer the reader to [40] for a general introduction to the theory, and to [41] for a development of the theory of structured ring spectra from the ∞-categorical point of view. For convenience, we will adopt the following reference conventions:

(T) We will indicate references to [40] using the letter T.

(A) We will indicate references to [41] using the letter A.

(V) We will indicate references to [42] using the Roman numeral V.

(VII) We will indicate references to [43] using the Roman numeral VII.

($VIII$) We will indicate references to [44] using the Roman numeral VIII.

(IX) We will indicate references to [45] using the Roman numeral IX.

For example, Theorem T.6.1.0.6 refers to Theorem 6.1.0.6 of [40].

If $\mathcal{C}$ is an ∞-category, we let $\mathcal{C}^{\simeq}$ denote the largest Kan complex contained in $\mathcal{C}$: that is, the ∞-category obtained from $\mathcal{C}$ by discarding all non-invertible morphisms.

We will say that a map of simplicial sets $f : S \to T$ is *left cofinal* if, for every right fibration $X \to T$, the induced map of simplicial sets $\mathrm{Fun}_T(T, X) \to \mathrm{Fun}_T(S, X)$ is a homotopy equivalence of Kan complexes (in [40], we referred to a map with this property as *cofinal*). We will say that f is *right cofinal* if the induced map $S^{op} \to T^{op}$ is left cofinal: that is, if f induces a homotopy equivalence $\mathrm{Fun}_T(T, X) \to \mathrm{Fun}_T(S, X)$ for every *left* fibration $X \to T$. If S and T are ∞-categories, then f is left cofinal if and only if for every object $t \in T$, the fiber product $S \times_T T_{t/}$ is weakly contractible (Theorem T.4.1.3.1).

Throughout this paper, we will generally use the letter k to denote a field (sometimes assumed to be of characteristic zero). We let Mod_k denote the ∞-category of k-module spectra (more concretely, one can think of the objects of Mod_k as given by chain complexes of vector spaces over k: see Remark 2.1.1). For each $M \in \mathrm{Mod}_k$, the homotopy groups $\pi_* M$ constitute a graded vector space over k. We will say that M is *locally finite* if each homotopy group $\pi_n M$ is finite-dimensional as a vector space over k.

For $0 \leq n \leq \infty$, we let $\mathrm{Alg}_k^{(n)}$ denote the ∞-category of $\mathbb{E}_n$-algebras over k (see Definition A.7.1.3.5). In the special case $n = 1$, we will denote $\mathrm{Alg}_k^{(n)}$ by Alg_k; in the special case $n = \infty$ we will denote $\mathrm{Alg}_k^{(n)}$ by CAlg_k. If $A \in \mathrm{Alg}_k^{(n)}$, then an *augmentation* on A is a map of $\mathbb{E}_n$-algebras $\epsilon : A \to k$. We let $\mathrm{Alg}_k^{(n),\mathrm{aug}} = (\mathrm{Alg}_k^{(n)})_{/k}$ denote the ∞-category of augmented $\mathbb{E}_n$-algebras over k (when $n = 1$ we denote this ∞-category by $\mathrm{Alg}_k^{\mathrm{aug}}$, and when $n = \infty$ we denote it by $\mathrm{CAlg}_k^{\mathrm{aug}}$). If $A \in \mathrm{Alg}_k^{(n),\mathrm{aug}}$, then we will refer to the fiber of the augmentation $\epsilon : A \to k$ as the *augmentation ideal* of A, and often denote it by $\mathfrak{m}_A$. Then $\mathfrak{m}_A$ has the structure of a nonunital $\mathbb{E}_n$-algebra over k. The construction $A \mapsto \mathfrak{m}_A$ determines an equivalence from the ∞-category $\mathrm{Alg}_k^{(n),\mathrm{aug}}$ to the ∞-category of nonunital $\mathbb{E}_n$-algebras over k (Proposition A.5.2.3.15).

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1 Deformation Theories: Axiomatic Approach

Our goal in this paper is to prove several variants of Theorem 0.0.13, which supply algebraic descriptions of various ∞-categories of formal moduli problems. Here is a basic prototype for the kind of result we would like to obtain:

$(*)$ Let Υ be an ∞-category of algebraic objects of some sort, let $\Upsilon^{\mathrm{sm}} \subseteq \Upsilon$ be a full subcategory spanned by those objects which are small (i.e., Artinian), and let $\mathrm{Moduli}^{\Upsilon} \subseteq \mathrm{Fun}(\Upsilon^{\mathrm{sm}}, \mathcal{S})$ be the ∞-category of functors $X : \Upsilon^{\mathrm{sm}} \to \mathcal{S}$ which satisfy a suitable gluing condition (as in Definition 0.0.8). Then there is an equivalence of ∞-categories $\mathrm{Moduli}^{\Upsilon} \simeq \Xi$, where Ξ is some other ∞-category of algebraic objects.

Our goal in this section is to flesh out assertion $(*)$. We begin in §1.1 by introducing the notion of a *deformation context* (Definition 1.1.3). A deformation context is a presentable ∞-category Υ equipped with some additional data (namely, a collection of spectrum objects $E_\alpha \in \mathrm{Stab}(\Upsilon)$). Using this additional data, we will explain how to define the full subcategory $\Upsilon^{\mathrm{sm}} \subseteq \Upsilon$ of *small* objects of Υ (Definition 1.1.8), and the full subcategory $\mathrm{Moduli}^{\Upsilon} \subseteq \mathrm{Fun}(\Upsilon^{\mathrm{sm}}, \mathcal{S})$ of *formal moduli problems* (Definition 1.1.14). Our definitions are very general and therefore suitable for a wide variety of applications. Nevertheless, they are sufficiently powerful to ensure that there is a reasonable *differential* theory of theory of formal moduli problems. In §1.2 we will explain how to associate to every formal moduli problem X a collection of spectra $X(E_\alpha)$, which we call the *tangent complex(es)* of X. The construction is functorial: every map between formal moduli problems $u : X \to Y$ can be differentiated to obtain maps of spectra $X(E_\alpha) \to Y(E_\alpha)$. Moreover, if each of these maps is a homotopy equivalence, then u is an equivalence (Proposition 1.2.10).

In §1.3, we will formulate a general version of $(*)$. For this, we will introduce the notion of a *deformation theory*. A deformation theory is a functor $\mathfrak{D} : \Upsilon^{op} \to \Xi$ satisfying a collection of axioms (see Definitions 1.3.1 and 1.3.9). Our main result (Theorem 1.3.12) can then be stated as follows: if $\mathfrak{D} : \Upsilon^{op} \to \Xi$ is a deformation theory, then $\mathfrak{D}$ determines an equivalence of ∞-categories $\Xi \simeq \mathrm{Moduli}^{\Upsilon}$. The proof of this result will be given in §1.5, using an ∞-categorical variant of Quillen's small object argument which we review in §1.4.

Our work in this section should be regarded as providing a sort of formal outline for proving results like Theorem 0.0.13. In practice, the main difficulty is not in proving Theorem 1.3.12 but in verifying its hypotheses: that is, in constructing a functor $\mathfrak{D} : \Upsilon^{op} \rightarrow \Xi$ which satisfies the axioms listed in Definitions 1.3.1 and 1.3.9. The later sections of this paper are devoted to carrying this out in special cases (we will treat the case of commutative algebras in §2, associative algebras in §3, and $\mathbb{E}_n$-algebras in §4).

1.1 Formal Moduli Problems

In this section, we introduce a general axiomatic paradigm for the study of deformation theory. Let us begin by outlining the basic idea. We are ultimately interesting in studying some class of algebro-geometric objects (such as schemes, or algebraic stacks, or their spectral analogues). Using the functor of points philosophy, we will identify these geometric objects with functors $X : \Upsilon \rightarrow \mathcal{S}$, where Υ denotes some ∞-category of test objects. The main example of interest (which we will study in detail in §2) is the case where Υ to be the ∞-category $\mathrm{CAlg}^{\mathrm{aug}}_k = (\mathrm{CAlg}_k)_{/k}$ of augmented $\mathbb{E}_\infty$-algebras over a field k of characteristic zero. In any case, we will always assume that Υ contains a final object $*$; we can then define a *point* of a functor $X : \Upsilon \rightarrow \mathcal{S}$ to be a point of the space $X(*)$. Our goal is to introduce some techniques for studying a *formal neighborhood* of X around a chosen point $\eta \in X(*)$. This formal neighborhood should encode information about the homotopy fiber products $X(A) \times_{X(*)} \{\eta\}$ for every object $A \in \Upsilon$ which is sufficiently "close" to the final object $*$. In order to make this idea precise, we need to introduce some terminology.

Notation 1.1.1. If Υ is a presentable ∞-category, we let Υ_* denote the ∞-category of pointed objects of Υ (Definition T.7.2.2.1) and $\mathrm{Stab}(\Upsilon)$ the stabilization of Υ (Definition A.1.4.4.1). Then $\mathrm{Stab}(\Upsilon)$ can be described as a homotopy limit of the tower of ∞-categories

$$\cdots \rightarrow \Upsilon_* \xrightarrow{\Omega} \Upsilon_* \xrightarrow{\Omega} \Upsilon_* \rightarrow \cdots$$

In particular, we have forgetful functors $\Omega^{\infty-n}_* : \mathrm{Stab}(\Upsilon) \rightarrow \Upsilon_*$ for every integer $n \in \mathbf{Z}$. We let $\Omega^{\infty-n} : \mathrm{Stab}(\Upsilon) \rightarrow \Upsilon$ denote the composition of $\Omega^{\infty-n}_*$ with the forgetful functor $\Upsilon_* \rightarrow \Upsilon$.

Remark 1.1.2. We can describe $\mathrm{Stab}(\Upsilon)$ explicitly as the ∞-category of strongly excisive functors from $\mathcal{S}^{\mathrm{fin}}_*$ to Υ, where $\mathcal{S}^{\mathrm{fin}}_*$ is the ∞-category of pointed finite spaces (see Corollary A.1.4.4.14; we will return to this description of $\mathrm{Stab}(\Upsilon)$ in §1.2).

Definition 1.1.3. A *deformation context* is a pair $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$, where Υ is a presentable ∞-category and $\{E_\alpha\}_{\alpha \in T}$ is a set of objects of the stabilization $\mathrm{Stab}(\Upsilon)$.

Example 1.1.4. Let k be an $\mathbb{E}_\infty$-ring, and let $\Upsilon = \mathrm{CAlg}^{\mathrm{aug}}_k = (\mathrm{CAlg}_k)_{/k}$ denote the ∞-category of augmented $\mathbb{E}_\infty$-algebras over k. Using Theorem A.7.3.5.14, we can identify $\mathrm{Stab}(\Upsilon)$ with the ∞-category Mod_k of k-module spectra. Let $E \in \mathrm{Stab}(\Upsilon)$ be the object which corresponds to $k \in \mathrm{Mod}_k$ under this identification, so that for every integer n we can identify $\Omega^{\infty-n}E$ with the square-zero extension $k \oplus k[n]$ of k. Then the pair $(\mathrm{CAlg}^{\mathrm{aug}}_k, \{E\})$ is a deformation context.

Definition 1.1.5. Let $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ be a deformation context. We will say that a morphism $\phi : A' \rightarrow A$ in Υ is *elementary* if there exists an index $\alpha \in T$, an integer $n > 0$, and a pullback diagram

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "\phi"] & * \arrow[d, "\phi_0"] \\ A \arrow[r] & \Omega^{\infty-n}E_\alpha. \end{tikzcd}$$

Here ϕ_0 corresponds the image $\Omega^{\infty-n}_* E_\alpha$ in the ∞-category of pointed objects Υ_*.

Example 1.1.6. Let k be a field and let $(\Upsilon, \{E\})$ be the deformation context described in Example 1.1.4. Suppose that $\phi : A' \rightarrow A$ is a map between connective objects of $\Upsilon = \mathrm{CAlg}^{\mathrm{aug}}_k$. Using Theorem A.7.4.1.26, we deduce that ϕ is elementary if and only if the following conditions are satisfied:

(a) There exists an integer $n \geq 0$ and an equivalence $\mathrm{fib}(\phi) \simeq k[n]$ in the ∞-category $\mathrm{Mod}_{A'}$ (here we regard k as an object of $\mathrm{Mod}_{A'}$ via the augmentation map $A' \to k$).

(b) If $n = 0$, then the multiplication map $\pi_0 \,\mathrm{fib}(\phi) \otimes \pi_0 \,\mathrm{fib}(\phi) \to \phi_0 \,\mathrm{fib}(\phi)$ vanishes.

If (a) is satisfied for $n = 0$, then we can choose a generator $\overline{x}$ for $\pi_0 \,\mathrm{fib}(\phi)$ having image $x \in \pi_0 A'$. Condition (b') is automatic if $x = 0$. If $a \neq 0$, then the map $\pi_0 \,\mathrm{fib}(\phi) \to \pi_0 A'$ is injective, so condition (b) is equivalent to the requirement that $x^2 = 0$ in $\pi_0 A'$.

Remark 1.1.7. Let $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ be a deformation context, and suppose we are given an object $A \in \Upsilon$. Every elementary map $A' \to A$ in Υ is given by the fiber of a map $A \to \Omega^{\infty - n} E_\alpha$ for some $n > 0$ and some $\alpha \in T$. It follows that the collection of equivalence classes of elementary maps $A' \to A$ is bounded in cardinality.

Definition 1.1.8. Let $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ be a deformation context. We will say that a morphism $\phi : A' \to A$ in Υ is *small* if it can be written as a composition of finitely many elementary morphisms $A' \simeq A_0 \to A_1 \to \cdots \to A_n \simeq A$. We will say that an object $A \in \Upsilon$ is *small* if the map $A \to *$ (which is uniquely determined up to homotopy) is small. We let Υ^{sm} denote the full subcategory of Υ spanned by the small objects.

Example 1.1.9. Let $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ be a deformation context. For every integer $n \geq 0$ and every index $\alpha \in T$, we have a pullback diagram

$$\begin{tikzcd} \Omega^{\infty-n} E_\alpha \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Omega^{\infty-n-1} E_\alpha. \end{tikzcd}$$

It follows that the left vertical map is elementary. In particular, $\Omega^{\infty-n} E_\alpha$ is a small object of Υ.

Remark 1.1.10. Let $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ be a deformation context. It follows from Remark 1.1.7 that the subcategory $\Upsilon^{\mathrm{sm}} \subseteq \Upsilon$ is essentially small.

Proposition 1.1.11. *Let k be a field and let $(\Upsilon, \{E\})$ be the deformation context of Example 1.1.4. Then an object $A \in \Upsilon = \mathrm{CAlg}_k^{\mathrm{aug}}$ is small (in the sense of Definition 1.1.8) if and only if the following conditions are satisfied:*

(1) *The homotopy groups $\pi_n A$ vanish for $n < 0$ and $n \gg 0$.*

(2) *Each homotopy group $\pi_n A$ is finite-dimensional as a vector space over k.*

(3) *The commutative ring $\pi_0 A$ is local with maximal ideal k, and the canonical map $k \to (\pi_0 A)/\mathfrak{m}$ is an isomorphism.*

Proof. Suppose first that A is small, so that there there exists a finite sequence of maps

$$A = A_0 \to A_1 \to \cdots \to A_n \simeq k$$

where each A_i is a square-zero extension of A_{i+1} by $k[m_i]$, for some $n_i \geq 0$. We prove that each A_i satisfies conditions (1), (2), and (3) using descending induction on i. The case $i = n$ is obvious, so let us assume that $i < n$ and that A_{i+1} is known to satisfy conditions (1), (2), and (3). We have a fiber sequence of k-module spectra

$$k[m_i] \to A_i \to A_{i+1}$$

which immediately implies that A_i satisfies (1) and (2). The map $\phi : \pi_0 A_i \to \pi_0 A_{i+1}$ is surjective and $\ker(\phi)^2 = 0$, from which it follows immediately that $\pi_0 A_i$ is local.

Now suppose that A satisfies conditions (1), (2), and (3). We will prove that A is small by induction on the dimension of the k-vector space $\pi_* A$. Let n be the largest integer for which $\pi_n A$ does not vanish. We first treat the case $n = 0$. We will abuse notation by identifying A with the underlying commutative ring $\pi_0 A$. Condition (3) implies that A is a local ring; let $\mathfrak{m}$ denote its maximal ideal. Since A is a finite dimensional algebra over k, we have $\mathfrak{m}^{i+1} \simeq 0$ for $i \gg 0$. Choose i as small as possible. If $i = 0$, then $\mathfrak{m} \simeq 0$ and $A \simeq k$, in which case there is nothing to prove. Otherwise, we can choose a nonzero element $x \in \mathfrak{m}^i \subseteq \mathfrak{m}$. Let A' denote the quotient ring $A/(x)$. It follows from Example 1.1.6 that the quotient map $A \to A'$ is elementary. Since A' is small by the inductive hypothesis, we conclude that A is small.

Now suppose that $n > 0$ and let $M = \pi_n A$, so that M has the structure of a module over the ring $\pi_0 A$. Let $\mathfrak{m} \subseteq \pi_0 A$ be as above, and let i be the least integer such that $\mathfrak{m}^{i+1} M \simeq 0$. Let $x \in \mathfrak{m}^i M$ and let M' be the quotient of M by x, so that we have an exact sequence

$$0 \to k \xrightarrow{x} M \to M' \to 0$$

of modules over $\pi_0 A$. We will abuse notation by viewing this sequence as a fiber sequence of A''-modules, where $A'' = \tau_{\leq n-1} A$. It follows from Theorem A.7.4.1.26 that there is a pullback diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & k \arrow[d] \\ A'' \arrow[r] & k \oplus M[n+1]. \end{tikzcd}$$

Set $A' = A'' \times_{k \oplus M'[n+1]} k$. Then $A \simeq A' \times_{k \oplus k[n+1]} k$ so we have an elementary map $A \to A'$. Using the inductive hypothesis we deduce that A' is small, so that A is also small. $\square$

Remark 1.1.12. Let k be a field and suppose that $A \in \mathrm{CAlg}_k$ satisfies conditions (1), (2), and (3) of Proposition 1.1.11. Then the mapping space $\mathrm{Map}_{\mathrm{CAlg}_k}(A, k)$ is contractible. In particular, A can be promoted (in an essentially unique way) to a small object of $\Upsilon = \mathrm{CAlg}_k^{\mathrm{aug}}$. Moreover, the forgetful functor $\mathrm{CAlg}_k^{\mathrm{aug}} \to \mathrm{CAlg}_k$ is fully faithful when restricted to the full subcategory $\Upsilon^{\mathrm{sm}} \subseteq \Upsilon$. We will denote the essential image of this restriction by $\mathrm{CAlg}_k^{\mathrm{sm}}$. We refer to $\mathrm{CAlg}_k^{\mathrm{sm}}$ as the *∞-category of small $\mathbb{E}_\infty$-algebras over k.*

Remark 1.1.13. Let $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ be a deformation context. Then the collection of small morphisms in Υ is closed under composition. In particular, if $\phi : A' \to A$ is small and A is small, then A' is also small. In particular, if there exists a pullback diagram

$$\begin{tikzcd} B' \arrow[r] \arrow[d] & A' \arrow[d, "\phi"] \\ B \arrow[r] & A \end{tikzcd}$$

where B is small and ϕ is small, then B' is also small.

We are now ready to introduce the main objects of study in this paper.

Definition 1.1.14. Let $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ be a deformation context. A *formal moduli problem* is a functor $X : \Upsilon^{\mathrm{sm}} \to \mathcal{S}$ satisfying the following pair of conditions:

(a) The space $X(*)$ is contractible (here $*$ denotes a final object of Υ).

(b) Let σ :

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & B' \arrow[d, "\phi"] \\ A \arrow[r] & B \end{tikzcd}$$

be a diagram in Υ^{sm}. If σ is a pullback diagram and ϕ is small, then $X(\sigma)$ is a pullback diagram in $\mathcal{S}$.

We let $\mathrm{Moduli}^{\Upsilon}$ denote the full subcategory of $\mathrm{Fun}(\Upsilon^{\mathrm{sm}}, \mathcal{S})$ spanned by the formal moduli problems. We will refer to $\mathrm{Moduli}^{\Upsilon}$ as the *∞-category of formal moduli problems.*

Condition (b) of Definition 1.1.14 has a number of equivalent formulations:

Proposition 1.1.15. *Let $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ be a deformation context, and let $X : \Upsilon^{\mathrm{sm}} \to \mathcal{S}$ be a functor. The following conditions are equivalent:*

(1) *Let σ :*

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & B' \arrow[d, "\phi"] \\ A \arrow[r] & B \end{tikzcd}$$

be a diagram in Υ^{sm}. If σ is a pullback diagram and ϕ is small, then $X(\sigma)$ is a pullback diagram in $\mathcal{S}$.

(2) *Let σ be as in* (1). *If σ is a pullback diagram and ϕ is elementary, then $X(\sigma)$ is a pullback diagram in $\mathcal{S}$.*

(3) *Let σ be as in* (1). *If σ is a pullback diagram and ϕ is the base point morphism $* \to \Omega^{\infty - n} E_\alpha$ for some $\alpha \in T$ and $n > 0$, then $X(\sigma)$ is a pullback diagram in $\mathcal{S}$.*

Proof. The implications $(a) \Rightarrow (b) \Rightarrow (c)$ are clear. The reverse implications follow from Lemma T.4.4.2.1. $\square$

Example 1.1.16. Let $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ be a deformation context, and let $A \in \Upsilon$ be an object. Let $\mathrm{Spec}(A) : \Upsilon^{\mathrm{sm}} \to \mathcal{S}$ be the functor corepresented by A, which is given on small objects of Υ by the formula $\mathrm{Spec}(A)(B) = \mathrm{Map}_{\Upsilon}(A, B)$. Then $\mathrm{Spec}(A)$ is a formal moduli problem. Moreover, the construction $A \mapsto \mathrm{Spec}(A)$ determines a functor $\mathrm{Spec} : \Upsilon^{op} \to \mathrm{Moduli}^{\Upsilon}$.

Remark 1.1.17. Let $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ be a deformation context. The ∞-category $\Upsilon^{\mathrm{sm}} \subseteq \Upsilon$ is essentially small. It follows from Lemmas T.5.5.4.19 and T.5.5.4.18 that the ∞-category $\mathrm{Moduli}^{\Upsilon}$ is an accessible localization of the ∞-category $\mathrm{Fun}(\Upsilon^{\mathrm{sm}}, \mathcal{S})$; in particular, the ∞-category $\mathrm{Moduli}^{\Upsilon}$ is presentable.

Remark 1.1.18. Let $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ be a deformation context, and let $X : \Upsilon^{\mathrm{sm}} \to \mathcal{S}$ be a functor which satisfies the equivalent conditions of Proposition 1.1.15. For every point $\eta \in X(*)$, define a functor $X_\eta : \Upsilon^{\mathrm{sm}} \to \mathcal{S}$ by the formula $X_\eta(A) = X(A) \times_{X(*)} \{\eta\}$. Then X_η is a formal moduli problem. We may therefore identify X as a family of formal moduli problems parametrized by the space $X(*)$. Consequently, condition (a) of Definition 1.1.14 should be regarded as a harmless simplifying assumption.

In the special case where $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ is the deformation context of Example 1.1.4, Definition 1.1.14 agrees with Definition 0.0.8. This is an immediate consequence of the following result:

Proposition 1.1.19. *Let k be a field and let $X : \mathrm{CAlg}_k^{\mathrm{sm}} \to \mathcal{S}$ be a functor. Then conditions* (1), (2), *and* (3) *of Proposition 1.1.15 are equivalent to the following:*

$(*)$ *For every pullback diagram*

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R_0 \arrow[d] \\ R_1 \arrow[r] & R_{01} \end{tikzcd}$$

in $\mathrm{CAlg}_{\mathbf{C}}^{\mathrm{sm}}$ for which the underlying maps $\pi_0 R_0 \to \pi_0 R_{01} \leftarrow \pi_0 R_1$ are surjective, the diagram

$$\begin{tikzcd} X(R) \arrow[r] \arrow[d] & X(R_0) \arrow[d] \\ X(R_1) \arrow[r] & X(R_{01}) \end{tikzcd}$$

is a pullback square.

The proof of Proposition 1.1.19 will require the following elaboration on Proposition 1.1.11:

Lemma 1.1.20. *Let k be a field and let $f : A \to B$ be a morphism in $\mathrm{CAlg}_k^{\mathrm{sm}}$. Then f is small (when regarded as a morphism in $\mathrm{CAlg}_k^{\mathrm{aug}}$) if and only if it induces a surjection of commutative rings $\pi_0 A \to \pi_0 B$.*

Proof. Let K be the fiber of f, regarded as an A-module. If $\pi_0 A \to \pi_0 B$ is surjective, then K is connective. We will prove that f is small by induction on the dimension of the graded vector space $\pi_* K$. If this dimension is zero, then $K \simeq 0$ and f is an equivalence. Assume therefore that $\pi_* K \neq 0$, and let n be the smallest integer such that $\pi_n K \neq 0$. Let $\mathfrak{m}$ denote the maximal ideal of $\pi_0 A$. Then $\mathfrak{m}$ is nilpotent, so $\mathfrak{m}(\pi_n K) \neq \pi_n K$ and we can choose a map of $\pi_0 A$-modules $\phi : \pi_n K \to k$. According to Theorem A.7.4.3.1, we have $(2n+1)$-connective map $K \otimes_A B \to L_{B/A}[-1]$. In particular, we have an isomorphism $\pi_{n+1} L_{B/A} \simeq \mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, \pi_n K)$ so that ϕ determines a map $L_{B/A} \to k[n+1]$. We can interpret this map as a derivation $B \to B \oplus k[n+1]$; let $B' = B \times_{B \oplus k[n+1]} k$. Then f factors as a composition

$$A \xrightarrow{f'} B' \xrightarrow{f''} B.$$

Since the map f'' is elementary, it will suffice to show that f' is small, which follows from the inductive hypothesis. $\square$

Proof of Proposition 1.1.19. The implication $(*) \Rightarrow (3)$ is obvious, and the implication $(1) \Rightarrow (*)$ follows from Lemma 1.1.20. $\square$

Remark 1.1.21. The proof of Proposition 1.1.19 shows that condition $(*)$ is equivalent to the stronger condition that the diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R_0 \arrow[d] \\ R_1 \arrow[r] & R_{01} \end{tikzcd}$$

is a pullback square whenever *one* of the maps $\pi_0 R_0 \to \pi_0 R_{01}$ or $\pi_0 R_1 \to \pi_0 R_{01}$ is surjective.

1.2 The Tangent Complex

Let X be an algebraic variety over the field $\mathbf{C}$ of complex numbers, and let $x : \operatorname{Spec} \mathbf{C} \to X$ be a point of X. A *tangent vector* to X at the point x is a dotted arrow rendering the diagram

$$\begin{tikzcd} \operatorname{Spec} \mathbf{C} \arrow[r, "x"] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} \mathbf{C}[\epsilon]/(\epsilon^2) \arrow[r] \arrow[ur, dashed] & \operatorname{Spec} \mathbf{C} \end{tikzcd}$$

commutative. The collection of tangent vectors to X at x comprise a vector space $T_{X,x}$, which we call the *Zariski tangent space* of X at x. If $\mathcal{O}_{X,x}$ denotes the local ring of X at the point x and $\mathfrak{m} \subseteq \mathcal{O}_{X,x}$ its maximal ideal, then there is a canonical isomorphism of vector spaces $T_{X,\eta} \simeq (\mathfrak{m}/\mathfrak{m}^2)^\vee$.

The tangent space $T_{X,x}$ is among the most basic and useful invariants one can use to study the local structure of an algebraic variety X near a point x. Our goal in this section is to generalize the construction of $T_{X,x}$ to the setting of an arbitrary formal moduli problem, in the sense of Definition 1.1.14. Let us identify X with its functor of points, given by $X(A) = \mathrm{Hom}(\operatorname{Spec} A, X)$ for every $\mathbf{C}$-algebra A (here the Hom is computed in the category of schemes over $\mathbf{C}$). Then $T_{X,x}$ can be described as the fiber of the map $X(\mathbf{C}[\epsilon]/(\epsilon^2)) \to X(\mathbf{C})$ over the point $x \in X(\mathbf{C})$. Note that the commutative ring $\mathbf{C}[\epsilon]/(\epsilon^2)$ is given by $\Omega^\infty E$, where E is the spectrum object of $\mathrm{CAlg}_{\mathbf{C}}^{\mathrm{aug}}$ appearing in Example 1.1.4. This suggests a possible generalization:

Definition 1.2.1. Let $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ be a deformation context, and let $Y : \Upsilon^{\mathrm{sm}} \to \mathcal{S}$ be a formal moduli problem. For each $\alpha \in T$, the *tangent space of Y at α* is the space $Y(\Omega^\infty E_\alpha)$.

There is a somewhat unfortunate aspect to the terminology of Definition 1.2.1. By definition, a formal moduli problem Y is a $\mathcal{S}$-valued functor, so the evaluation of X on any object $A \in \Upsilon^{\mathrm{sm}}$ might be called a "space". The term "tangent space" in algebraic geometry has a different meaning: if X is a complex algebraic variety with a base point x, then we refer to $T_{X,x}$ as the tangent *space* of X not because it is equipped with a topology, but because it has the structure of a vector space over $\mathbf{C}$. In particular, $T_{X,x}$ is equipped with an addition which is commutative and associative. Our next goal is to prove that this phenomenon is quite general: for any formal moduli problem $Y : \Upsilon^{\mathrm{sm}} \to \mathcal{S}$, each tangent space $Y(\Omega^\infty E_\alpha)$ of Y is an infinite loop space, and therefore equipped with a composition law which is commutative and associative up to coherent homotopy.

We begin by recalling some definitions.

Notation 1.2.2. Let $\mathcal{C}$ be an ∞-category which admits finite colimits and $\mathcal{D}$ an ∞-category which admits finite limits. We say that a functor $F : \mathcal{C} \to \mathcal{D}$ is *excisive* if, for every pushout square

$$\begin{tikzcd} C_{01} \arrow[r] \arrow[d] & C_0 \arrow[d] \\ C_1 \arrow[r] & C \end{tikzcd}$$

in $\mathcal{C}$, the diagram

$$\begin{tikzcd} F(C_{01}) \arrow[r] \arrow[d] & F(C_0) \arrow[d] \\ F(C_1) \arrow[r] & F(C) \end{tikzcd}$$

is a pullback square in $\mathcal{D}$. We say that F is *strongly excisive* if it is excisive and carries initial objects of $\mathcal{C}$ to final objects of $\mathcal{D}$.

We let $\mathcal{S}_*^{\mathrm{fin}}$ denote the ∞-category of finite pointed spaces. For any ∞-category $\mathcal{D}$ which admits finite limits, we let $\mathrm{Stab}(\mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{D})$ spanned by the pointed excisive functors. We recall that $\mathrm{Stab}(\mathcal{D})$ is an explicit model for the stabilization of $\mathcal{D}$; in particular, it is a stable ∞-category (see ∞-category, which is model for the stabilization of $\mathcal{D}$ (see Corollary A.1.4.4.14). In particular, we can realize the ∞-category $\mathrm{Sp} = \mathrm{Stab}(\mathcal{S})$ of spectra as the full subcategory of $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S})$ spanned by the strongly excisive functors.

Proposition 1.2.3. *Let $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ be a deformation context. For each $\alpha \in T$, we identify $E_\alpha \in \mathrm{Stab}(\Upsilon)$ with the corresponding functor $\mathcal{S}_*^{fin} \to \Upsilon$. Then:*

(1) *For every map $f : K \to K'$ of pointed finite spaces which induces a surjection $\pi_0 K \to \pi_0 K'$, the induced map $E_\alpha(K) \to E_\alpha(K')$ is a small morphism in Υ.*

(2) *For every pointed finite space K, the object $E_\alpha(K) \in \Upsilon$ is small.*

Proof. We will prove (1); assertion (2) follows by applying (1) to the constant map $K \to *$. Note that f is equivalent to a composition of maps

$$K = K_0 \to K_1 \to \cdots \to K_n = K',$$

where each K_i is obtained from K_{i-1} by attaching a single cell of dimension n_i. Since $\pi_0 K$ surjects onto $\pi_0 K'$, we may assume that each n_i is positive. It follows that we have pushout diagrams of finite pointed spaces

$$\begin{tikzcd} K_{i-1} \arrow[r] \arrow[d] & K_i \arrow[d] \\ * \arrow[r] & S^{n_i}. \end{tikzcd}$$

Since E_α is excisive, we obtain a pullback square

$$\begin{tikzcd}
E_\alpha(K_{i-1}) \arrow[r] \arrow[d] & E_\alpha(K_i) \arrow[d] \\
* \arrow[r] & \Omega^{n_i} E_\alpha,
\end{tikzcd}$$

so that each of the maps $E_\alpha(K_{i-1}) \to E_\alpha(K_i)$ is elementary. $\square$

It follows from Proposition 1.2.3 that if $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ is a deformation context, then each E_α can be regarded as a functor from $\mathcal{S}_*^{\text{fin}}$ to the full subcategory $\Upsilon^{\text{sm}} \subseteq \Upsilon$ spanned by the small object. It therefore makes sense to compose E_α with any formal moduli problem.

Proposition 1.2.4. *Let $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ be a deformation context and let $Y : \Upsilon^{\text{sm}} \to \mathcal{S}$ be a formal moduli problem. For every $\alpha \in T$, the composite functor*

$$\mathcal{S}_*^{fin} \xrightarrow{E_\alpha} \Upsilon^{\text{sm}} \xrightarrow{Y} \mathcal{S}$$

is strongly excisive.

Proof. It is obvious that $Y \circ E_\alpha$ carries initial objects of $\mathcal{S}_*^{\text{fin}}$ to contractible spaces. Suppose we are given a pushout diagram

$$\begin{tikzcd}
K \arrow[r] \arrow[d] & K' \arrow[d] \\
L \arrow[r] & L'
\end{tikzcd}$$

of pointed finite spaces; we wish to show that the diagram σ :

$$\begin{tikzcd}
Y(E_\alpha(K)) \arrow[r] \arrow[d] & Y(E_\alpha(K')) \arrow[d] \\
Y(E_\alpha(L)) \arrow[r] & Y(E_\alpha(L'))
\end{tikzcd}$$

is a pullback square in $\mathcal{S}$. Let K'_+ denote the union of those connected components of K' which meet the image of the map $K \to K'$. There is a retraction of K' onto K'_+, which carries the other connected components of K' to the base point. Define L'_+ and the retraction $L' \to L'_+$ similarly. We have a commutative diagram of pointed finite spaces

$$\begin{tikzcd}
K \arrow[r] \arrow[d] & K' \arrow[r] \arrow[d] & K'_+ \arrow[d] \\
L \arrow[r] & L' \arrow[r] & L'_+
\end{tikzcd}$$

where each square is a pushout, hence a diagram of spaces

$$\begin{tikzcd}
Y(E_\alpha(K)) \arrow[r] \arrow[d] & Y(E_\alpha(K')) \arrow[r] \arrow[d] & Y(E_\alpha(K'_+)) \arrow[d] \\
Y(E_\alpha(L)) \arrow[r] & Y(E_\alpha(L')) \arrow[r] & Y(E_\alpha(L'_+)).
\end{tikzcd}$$

To prove that the left square is a pullback diagrams, it will suffice to show that the right square and the outer rectangle are pullback diagrams. We may therefore reduce to the case where the map $\pi_0 K \to \pi_0 K'$ is surjective. Then the map $\pi_0 L \to \pi_0 L'$ is surjective, so that $E_\alpha(L) \to E_\alpha(L')$ is a small morphism in Υ^{sm} (Proposition 1.2.3). Since E_α is excisive, the diagram

$$\begin{tikzcd} E_\alpha(K) \arrow[r] \arrow[d] & E_\alpha(K') \arrow[d] \\ E_\alpha(L) \arrow[r] & E_\alpha(L') \end{tikzcd}$$

is a pullback square in Υ. Using the assumption that Y is a formal moduli problem, we deduce that σ is a pullback square of spaces. $\square$

Definition 1.2.5. Let $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ be a deformation context, and let $Y : \Upsilon^{\mathrm{sm}} \to \mathcal{S}$ be a formal moduli problem. For each $\alpha \in T$, we let $Y(E_\alpha)$ denote the composite functor

$$\mathcal{S}_*^{\mathrm{fin}} \xrightarrow{E_\alpha} \Upsilon^{\mathrm{sm}} \xrightarrow{Y} \mathcal{S}.$$

We will view $Y(E_\alpha)$ as an object in the ∞-category $\mathrm{Sp} = \mathrm{Stab}(\mathcal{S})$ of spectra, and refer to $Y(E_\alpha)$ as the *tangent complex to* Y at α.

Remark 1.2.6. In the situation of Definition 1.2.5, suppose that T has a single element, so that $\{E_\alpha\}_{\alpha \in T} = \{E\}$ for some $E \in \mathrm{Stab}(\Upsilon)$ (this condition is satisfied in all of the main examples we will study in this paper). In this case, we will omit mention of the index α and simply refer to $Y(E)$ as the *tangent complex* to the formal moduli problem Y.

Remark 1.2.7. Let $Y : \Upsilon^{\mathrm{sm}} \to \mathcal{S}$ be as in Definition 1.2.5. For every index α, we can identify the tangent space $Y(\Omega^\infty E_\alpha)$ at α with the 0th space of the tangent complex $Y(E_\alpha)$. More generally, there are canonical homotopy equivalences

$$Y(\Omega^{\infty - n} E_\alpha) \simeq \Omega^{\infty - n} Y(E_\alpha)$$

for $n \geq 0$.

Example 1.2.8. Let $X = \operatorname{Spec} R$ be an affine algebraic variety over the field $\mathbf{C}$ of complex numbers, and suppose we are given a point x of X (corresponding to an augmentation $\epsilon : R \to \mathbf{C}$ of the $\mathbf{C}$-algebra R). Then X determines a formal moduli problem $X^\wedge : \mathrm{CAlg}^{\mathrm{sm}}_{\mathbf{C}} \to \mathcal{S}$, given by the formula $X^\wedge(A) = \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{aug}}_{\mathbf{C}}}(R, A)$ (here we work in the deformation context $(\mathrm{CAlg}^{\mathrm{aug}}_{\mathbf{C}}, \{E\})$ of Example 1.1.4). Unwinding the definitions, we see that the tangent complex $X^\wedge(E)$ can be identified with the spectrum $\mathrm{Mor}_{\mathrm{Mod}_R}(L_{R/\mathbf{C}}, \mathbf{C})$ classifying maps from the cotangent complex $L_{R/\mathbf{C}}$ into $\mathbf{C}$ (regarded as an R-module via the augmentation ϵ). In particular, the homotopy groups of $X^\wedge(E)$ are given by

$$\pi_i X^\wedge(E) \simeq (\pi_{-i}(k \otimes_R L_{R/\mathbf{C}}))^\vee.$$

It follows that $\pi_i X^\wedge(E)$ vanishes for $i > 0$, and that $\pi_0 X^\wedge(E)$ is isomorphic to the Zariski tangent space $(\mathfrak{m}/\mathfrak{m}^2)^\vee$ of X at the point x. If X is smooth at the point x, then the negative homotopy groups of $\pi_i X^\wedge(E)$ vanish. In general, the homotopy groups $\pi_i X^\wedge(E)$ encode information about the nature of the singularity of X at the point x. One of our goals in this paper is to articulate a sense in which the tangent complex $X^\wedge(E)$ encodes *complete* information about the local structure of X near the point x.

Warning 1.2.9. The terminology of Definition 1.2.5 is potentially misleading. For a general deformation context $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ and formal moduli problem $Y : \Upsilon^{\mathrm{sm}} \to \mathcal{S}$, the tangent complexes $Y(E_\alpha)$ are merely spectra. If k is a field and $(\Upsilon, \{E_\alpha\}_{\alpha \in T}) = (\mathrm{CAlg}^{\mathrm{aug}}_k, \{E\})$ is the deformation context of Example 1.1.4, one can show that the tangent complex $Y(E)$ admits the structure of a k-module spectrum, and can therefore be identified with a chain complex of vector spaces over k. This observation motivates our use of the term "tangent complex." In the general case, it might be more appropriate to refer to $Y(E_\alpha)$ as a "tangent spectrum" to the formal moduli problem Y.

The tangent complex of a formal moduli problem Y is a powerful invariant of Y. We close this section with a simple illustration:

Proposition 1.2.10. *Let $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ be a deformation context and let $u : X \to Y$ be a map of formal moduli problems. Suppose that u induces an equivalence of tangent complexes $X(E_\alpha) \to Y(E_\alpha)$ for each $\alpha \in T$. Then u is an equivalence.*

Proof. Consider an arbitrary object $A \in \Upsilon^{\mathrm{sm}}$, so that there exists a sequence of elementary morphisms

$$A = A_0 \to A_1 \to \cdots \to A_n \simeq *$$

in Υ. We prove that the map $u(A_i) : X(A_i) \to Y(A_i)$ is a homotopy equivalence using descending induction on i, the case $i = n$ being trivial. Assume therefore that $i < n$ and that $u(A_{i+1})$ is a homotopy equivalence. Since $A_i \to A_{i+1}$ is elementary, we have a fiber sequence of maps

$$u(A_i) \to u(A_{i+1}) \to u(\Omega^{\infty - n} E_\alpha)$$

for some $n > 0$ and $\alpha \in T$. To prove that $u(A_i)$ is a homotopy equivalence, it suffices to show that $u(\Omega^{\infty-n} E_\alpha)$ is a homotopy equivalence, which follows immediately from our assumption that u induces an equivalence $X(E_\alpha) \to Y(E_\alpha)$. □

1.3 Deformation Theories

Let $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ be a deformation context. Our main goal in this paper is to obtain an algebraic description of the ∞-category $\mathrm{Moduli}^{\Upsilon} \subseteq \mathrm{Fun}(\Upsilon^{\mathrm{sm}}, \mathcal{S})$ of formal moduli problems. To this end, we would like to have some sort of recognition criterion, which addresses the following question:

(Q) Given an ∞-category Ξ, when does there exist an equivalence $\mathrm{Moduli}^{\Upsilon} \simeq \Xi$?

We take our first cue from Example 1.1.16. To every object $A \in \Upsilon$, we can associate a formal moduli problem $\mathrm{Spec}\, A \in \mathrm{Moduli}^{\Upsilon}$ by the formula $(\mathrm{Spec}\, A)(R) = \mathrm{Map}_{\Upsilon}(A, R)$. Combining this construction with an equivalence $\mathrm{Moduli}^{\Upsilon} \simeq \Xi$, we obtain a functor $\mathfrak{D} : \Upsilon^{op} \to \Xi$. We begin by axiomatizing the properties of this functor:

Definition 1.3.1. Let $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ be a deformation context. A *weak deformation theory* for $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ is a functor $\mathfrak{D} : \Upsilon^{op} \to \Xi$ satisfying the following axioms:

($D1$) The ∞-category Ξ is presentable.

($D2$) The functor $\mathfrak{D}$ admits a left adjoint $\mathfrak{D}' : \Xi \to \Upsilon^{op}$.

($D3$) There exists a full subcategory $\Xi_0 \subseteq \Xi$ satisfying the following conditions:

- (a) For every object $K \in \Xi_0$, the unit map $K \to \mathfrak{D}\mathfrak{D}' K$ is an equivalence.
- (b) The full subcategory Ξ_0 contains the initial object $\emptyset \in \Xi$. It then follows from (a) that $\emptyset \simeq \mathfrak{D}\mathfrak{D}'\emptyset \simeq \mathfrak{D}(*)$, where $*$ denotes the final object of Υ.
- (c) For every index $\alpha \in T$ and every $n \geq 1$, there exists an object $K_{\alpha,n} \in \Xi_0$ and an equivalence $\Omega^{\infty-n} E_\alpha \simeq \mathfrak{D}' K_{\alpha,n}$. It follows that the base point of $\Omega^{\infty-n} E_\alpha$ determines a map

$$v_{\alpha,n} : K_{\alpha,n} \simeq \mathfrak{D}\mathfrak{D}' K_{\alpha,n} \simeq \mathfrak{D}(\Omega^{\infty-n} E_\alpha) \to \mathfrak{D}(*) \simeq \emptyset.$$

- (d) For every pushout diagram

$$\begin{tikzcd} K_{\alpha,n} \arrow[r] \arrow[d, "v_{\alpha,n}"] & K \arrow[d] \\ \emptyset \arrow[r] & K' \end{tikzcd}$$

  where $\alpha \in T$ and $n > 0$, if K belongs to to Ξ_0 then K' also belongs to Ξ_0.

Definition 1.3.1 might seem a bit complicated at a first glance. We can summarize axioms $(D2)$ and $(D3)$ informally by saying that the functor $\mathfrak{D} : \Upsilon^{op} \to \Xi$ is not far from being an equivalence. Axiom $(D2)$ requires that there exists an adjoint $\mathfrak{D}'$ to $\mathfrak{D}$, and axiom $(D3)$ requires that $\mathfrak{D}'$ behave as a homotopy inverse to $\mathfrak{D}$, at least on a subcategory $\Xi_0 \subseteq \Xi$ with good closure properties.

Example 1.3.2. Let k be a field of characteristic zero and let $(\mathrm{CAlg}_k^{\mathrm{aug}}, \{E\})$ be the deformation context described in Example 1.1.4. In §2, we will construct a weak deformation theory $\mathfrak{D} : (\mathrm{CAlg}_k^{\mathrm{aug}})^{op} \to \mathrm{Lie}_k$, where Lie denotes the ∞-category of differential graded Lie algebras over k (Definition 2.1.14). Here the adjoint functor $\mathfrak{D}' : \mathrm{Lie}_k \to (\mathrm{CAlg}_k^{\mathrm{aug}})^{op}$ assigns to each differential graded Lie algebra $\mathfrak{g}_*$ its cohomology Chevalley-Eilenberg complex $C^*(\mathfrak{g}_*)$ (see Construction 2.2.13). In fact, the functor $\mathfrak{D} : (\mathrm{CAlg}_k^{\mathrm{aug}})^{op} \to \mathrm{Lie}\, k$ is even a *deformation theory*: it satisfies condition $(D4)$ appearing in Definition 1.3.9 below.

Remark 1.3.3. In view of Corollary T.5.5.2.9 and Remark T.5.5.2.10, condition $(D2)$ of Definition 1.3.1 is equivalent to the requirement that the functor $\mathfrak{D}$ preserves small limits.

Remark 1.3.4. In the situation of Definition 1.3.1, the objects $K_{\alpha,n} \in \Xi_0$ are determined up to canonical equivalence: it follows from (a) that they are given by $K_{\alpha,n} \simeq \mathfrak{D}\mathfrak{D}' K_{\alpha,n} \simeq \mathfrak{D}(\Omega^{\infty-n} E_\alpha)$. In particular, the objects $\Omega^{\infty-n} E_\alpha$ belong to Ξ_0.

Our next result summarizes some of the basic features of weak deformation theories.

Proposition 1.3.5. *Let $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ be a deformation context and $\mathfrak{D} : \Upsilon^{op} \to \Xi$ a weak deformation theory. Let $\Xi_0 \subseteq \Xi$ be a full subcategory which is stable under equivalence and satisfies condition* (3) *of Definition 1.3.1. Then:*

(1) *The functor $\mathfrak{D}$ carries final objects of Υ to initial objects of Ξ.*

(2) *Let $A \in \Upsilon$ be an object having the form $\mathfrak{D}'(K)$, where $K \in \Xi_0$. Then the unit map $A \to \mathfrak{D}'\mathfrak{D}(A)$ is an equivalence in Υ.*

(3) *If $A \in \Upsilon$ is small, then $\mathfrak{D}(A) \in \Xi_0$ and the unit map $A \to \mathfrak{D}'\mathfrak{D}A$ is an equivalence in Υ.*

(4) *Suppose we are given a pullback diagram σ :*

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & B' \arrow[d, "\phi"] \\ A \arrow[r] & B \end{tikzcd}$$

in Υ, where A and B are small and the morphism ϕ is small. Then $\mathfrak{D}(\sigma)$ is a pushout diagram in Ξ.

Proof. Let $\emptyset$ denote an initial object of Ξ. Then $\emptyset \in \Xi_0$ so that the adjunction map $\emptyset \to \mathfrak{D}\mathfrak{D}'\emptyset$ is an equivalence. Since $\mathfrak{D}' : \Xi \to \Upsilon^{op}$ is left adjoint to $\mathfrak{D}$, it carries $\emptyset$ to a final object $* \in \Upsilon$. This proves (1). To prove (2), suppose that $A = \mathfrak{D}'(K)$ for $K \in \Xi_0$. Then the unit map $u : A \to \mathfrak{D}'\mathfrak{D}A$ has a left homotopy inverse, given by applying $\mathfrak{D}'$ to the the map $v : K \to \mathfrak{D}\mathfrak{D}'K$ in Ξ. Since v is an equivalence (part (a) of Definition 1.3.1), we conclude that u is an equivalence.

We now prove (3). Let $A \in \Upsilon$ be small, so that there exists a sequence of elementary morphisms

$$A = A_0 \to A_1 \to \cdots \to A_n \simeq *.$$

We will prove that $\mathfrak{D}A_i \in \Xi_0$ using descending induction on i. If $i = n$, the desired result follows from (1). Assume therefore that $i < n$, so that the inductive hypothesis guarantees that $\mathfrak{D}(A_{i+1}) \in \Xi_0$. Choose a pullback diagram σ :

$$\begin{tikzcd} A_i \arrow[r] \arrow[d] & * \arrow[d, "\phi"] \\ A_{i+1} \arrow[r, "\psi"] & \Omega^{\infty-n} E_\alpha \end{tikzcd}$$

where $n > 0$, $\alpha \in T$, and ϕ is the base point of $\Omega^{\infty-n} E_\alpha$. Form a pushout diagram τ :

$$\begin{tikzcd}
\mathfrak{D}(\Omega^{\infty-n} E_\alpha) \arrow[r, "\mathfrak{D}\psi"] \arrow[d, "\mathfrak{D}\phi"] & \mathfrak{D} A_{i+1} \arrow[d] \\
\mathfrak{D}(*) \arrow[r] & X
\end{tikzcd}$$

in Ξ. There is an evident transformation $\xi : \sigma \to \mathfrak{D}'(\tau)$ of diagrams in Υ. Since both σ and $\mathfrak{D}'(\tau)$ are pullback diagrams and the objects A_{i+1}, $\Omega^{\infty-n} E_\alpha$, and $*$ belong to the essential image of $\mathfrak{D}'|\,\Xi_0$, it follows from assertion (2) that ξ is an equivalence, so that $A_i \simeq \mathfrak{D}'(X)$. Assumption (d) of Definition 1.3.1 guarantees that $X \in \Xi_0$, so that A_i lies in the essential image of $\mathfrak{D}'|\,\Xi_0$.

We now prove (4). The class of morphisms ϕ for which the conclusion holds (for an arbitrary map $A \to B$ between small objects of Υ) is evidently stable under composition. We may therefore reduce to the case where ϕ is elementary, and further to the case where ϕ is the base point map $* \to \Omega^{\infty-n} E_\alpha$ for some $\alpha \in T$ and some $n > 0$. Arguing as above, we deduce that the pullback diagram σ :

$$\begin{tikzcd}
A' \arrow[r] \arrow[d] & * \arrow[d, "\phi"] \\
A \arrow[r] & \Omega^{\infty-n} E_\alpha
\end{tikzcd}$$

is equivalent to $\mathfrak{D}'(\tau)$, where τ is a diagram in Ξ_0 which is a pushout square in Ξ. Then $\mathfrak{D}(\sigma) \simeq \mathfrak{D}\mathfrak{D}'(\tau) \simeq \tau$ is a pushout diagram, by virtue of condition (a) of Definition 1.3.1. $\square$

Corollary 1.3.6. *Let $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ be a deformation context and $\mathfrak{D} : \Upsilon^{op} \to \Xi$ a weak deformation theory. Let $j : \Xi \to \mathrm{Fun}(\Xi^{op}, \mathcal{S})$ denote the Yoneda embedding. For every object $K \in \Xi$, the composition*

$$\Upsilon^{\mathrm{sm}} \subseteq \Upsilon \xrightarrow{\mathfrak{D}} \Xi^{op} \xrightarrow{j(K)} \mathcal{S}$$

is a formal moduli problem. This construction determines a functor $\Psi : \Xi \to \mathrm{Moduli}^{\Upsilon} \subseteq \mathrm{Fun}(\Upsilon^{\mathrm{sm}}, \mathcal{S})$.

Remark 1.3.7. Corollary 1.3.6 admits a converse. Let $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ be a deformation context. The functor $\mathrm{Spec} : \Upsilon^{op} \to \mathrm{Moduli}$ of Example 1.1.16 satisfies conditions $(D1)$, $(D2)$ and $(D3)$ of Definition 1.3.1, and therefore defines a weak deformation theory (we can define the full subcategory $\mathrm{Moduli}_0 \subseteq \mathrm{Moduli}$ whose existence is required by $(D3)$ to be spanned by objects of the form $\mathrm{Spec}(A)$, where $A \in \Upsilon^{\mathrm{sm}}$).

Combining Corollary 1.3.6 with Proposition 1.2.4, we obtain the following result:

Corollary 1.3.8. *Let $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ be a deformation context and $\mathfrak{D} : \Upsilon^{op} \to \Xi$ a weak deformation theory. For each $\alpha \in T$ and each $K \in \Xi$, the composite map*

$$\mathcal{S}^{fin}_* \xrightarrow{E_\alpha} \Upsilon \xrightarrow{\mathfrak{D}} \Xi^{op} \xrightarrow{j(K)} \mathcal{S}$$

is strongly excisive, and can therefore be identified with a spectrum which we will denote by $e_\alpha(K)$. This construction determines a functor $e_\alpha : \Xi \to \mathrm{Sp} = \mathrm{Stab}(\mathcal{S}) \subseteq \mathrm{Fun}(\mathcal{S}^{fin}_*, \mathcal{S})$.

Definition 1.3.9. Let $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ be a deformation context. A *deformation theory* for $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ is a weak deformation theory $\mathfrak{D} : \Upsilon^{op} \to \Xi$ which satisfies the following additional condition:

$(D4)$ For each $\alpha \in T$, let $e_\alpha : \Xi \to \mathrm{Sp}$ be the functor described in Corollary 1.3.8. Then each e_α preserves small sifted colimits. Moreover, a morphism f in Ξ is an equivalence if and only if each $e_\alpha(f)$ is an equivalence of spectra.

Warning 1.3.10. Let $(\Upsilon, \{X_\alpha\}_{\alpha \in T})$ be a deformation context, and let $\mathrm{Spec} : \Upsilon^{op} \to \mathrm{Moduli}^{\Upsilon}$ be given by the Yoneda embedding (see Remark 1.3.7). The resulting functors $e_\alpha : \mathrm{Moduli}^{\Upsilon} \to \mathrm{Sp}$ are then given by evaluation on the spectrum objects $E_\alpha \in \mathrm{Stab}(\Upsilon)$, and are therefore jointly conservative (Proposition 1.2.10) and preserve filtered colimits. However, it is not clear that Spec is a deformation theory, since the tangent complex constructions $X \mapsto X(E_\alpha)$ do not obviously commute with sifted colimits.

Remark 1.3.11. Let $(\Upsilon, \{E\})$ be a deformation context, let $\mathfrak{D} : \Upsilon^{op} \to \Xi$ be a deformation theory for Υ, and let $e : \Xi \to \mathrm{Sp}$ be as in Corollary 1.3.8. The functor e preserves small limits, and condition $(D4)$ of Definition 1.3.9 implies that e preserves sifted colimits. It follows that e admits a left adjoint $F : \mathrm{Sp} \to \Xi$. The composite functor $e \circ F$ has the structure of a monad A on Sp. Since e is conservative and commutes with sifted colimits, Theorem A.6.2.2.5 gives us an equivalence of ∞-categories $\Xi \simeq \mathrm{LMod}_A(\mathrm{Sp})$ with the ∞-category of algebras over the monad T. In other words, we can think of Ξ as an ∞-category whose objects are spectra which are equipped some additional structure (namely, a left action of the monad A).

More generally, if $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ is a deformation context equipped with a deformation theory $\mathfrak{D} : \Upsilon^{op} \to \Xi$, the same argument supplies an equivalence $\Xi \simeq \mathrm{LMod}_A(\mathrm{Sp}^T)$: that is, we can think of objects of Ξ as determines by a collection of spectra (indexed by T), together with some additional structure.

We can now formulate our main result:

Theorem 1.3.12. *Let $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ be a deformation context and let $\mathfrak{D} : \Upsilon^{op} \to \Xi$ be a deformation theory. Then the functor $\Psi : \Xi \to \mathrm{Moduli}^{\Upsilon}$ of Corollary 1.3.6 is an equivalence of ∞-categories.*

The proof of Theorem 1.3.12 will be given in §1.5.

Remark 1.3.13. Let $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ be a deformation context, let $\mathfrak{D} : \Upsilon^{op} \to \Xi$ a deformation theory, and consider the functor $\Psi : \Xi \to \mathrm{Moduli}$ of Corollary 1.3.6. The composite functor $\Upsilon^{op} \xrightarrow{\mathfrak{D}} \Xi \xrightarrow{\Psi} \mathrm{Moduli}^{\Upsilon}$ carries an object $A \in \Upsilon$ to the formal moduli problem defined by the formula

$$B \mapsto \mathrm{Map}_{\Xi}(\mathfrak{D}B, \mathfrak{D}A) \simeq \mathrm{Map}_{\Upsilon}(A, \mathfrak{D}'\mathfrak{D}(B)).$$

The unit maps $B \to \mathfrak{D}'\mathfrak{D}B$ determines a natural transformation of functors $\beta : \mathrm{Spec} \to \Psi \circ \mathfrak{D}$, where $\mathrm{Spec} : \Upsilon^{op} \to \mathrm{Moduli}^{\Upsilon}$ is as in Example 1.1.16. It follows from Proposition 1.3.5 that the natural transformation β is an equivalence. Combining this with Theorem 1.3.12, we conclude that $\mathfrak{D}$ is equivalent to weak deformation theory $\mathrm{Spec} : \Upsilon^{op} \to \mathrm{Moduli}^{\Upsilon}$ of Remark 1.3.7.

We can summarize the situation as follows: a deformation context $(\Upsilon, \{E_\alpha\}_{\alpha \in T}\}$ admits a deformation theory if and only if, for each $\alpha \in T$, the construction $X \mapsto X(E_\alpha)$ determines a functor $\mathrm{Moduli}^{\Upsilon} \to \mathrm{Sp}$ which commutes with sifted colimits. If this condition is satisfied, then the deformation theory on Υ is unique (up to canonical equivalence), given by the functor $\mathrm{Spec} : \Upsilon^{op} \to \mathrm{Moduli}^{\Upsilon}$.

1.4 Digression: The Small Object Argument

Let $\mathcal{C}$ be a category containing a collection of morphisms $\{f_\alpha : C_\alpha \to D_\alpha\}$, and let $g : X \to Z$ be another morphism in $\mathcal{C}$. Under some mild hypotheses, Quillen's *small object argument* can be used to produce a factorization

$$X \xrightarrow{g'} Y \xrightarrow{g''} Z$$

where g' is "built from" the morphisms f_α, and g'' has the right lifting property with respect to the morphisms f_α (see §T.A.1.2 for a detailed discussion). The small object argument was originally used by Grothendieck to prove that every Grothendieck abelian category has enough injective objects (see [25] or Corollary A.1.3.4.7). It is now a basic tool in the theory of model categories.

Our goal in this section is to carry out an ∞-categorical version of the small object argument (Proposition 1.4.7). We begin by introducing some terminology.

Definition 1.4.1. Let $\mathcal{C}$ be an ∞-category. Let $f : C \to D$ and $g : X \to Y$ be morphisms in $\mathcal{C}$. We will say that g has the *right lifting property* with respect to f if every commutative diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d, "f"] & X \arrow[d, "g"] \\ D \arrow[r] & Y \end{tikzcd}$$

can be extended to a 3-simplex of $\mathcal{C}$, as depicted by the diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d, "f"] & X \arrow[d, "g"] \\ D \arrow[r] \arrow[ur] & Y. \end{tikzcd}$$

In this case, we will also say that f has the *left lifting property* with respect to g.

More generally, if S is any set of morphisms in $\mathcal{C}$, we will say that a morphism g has the *right lifting property* with respect to S if it has the right lifting property with respect to every morphism in S, and that a morphism f has the *left lifting property* with respect to S if f has the left lifting property with respect to every morphism in S.

Definition 1.4.2. Let $\mathcal{C}$ be an ∞-category and let S be a collection of morphisms in $\mathcal{C}$. We will say that a morphism f in $\mathcal{C}$ is a *transfinite pushout of morphisms in S* if there exists an ordinal α and a diagram $F : \mathrm{N}[\alpha] \to \mathcal{C}$ (here $[\alpha]$ denotes the linearly ordered set of ordinals $\{\beta : \beta \leq \alpha\}$) with the following properties:

(1) For every nonzero limit ordinal $\lambda \leq \alpha$, the restriction $F|\,\mathrm{N}[\lambda]$ is a colimit diagram in $\mathcal{C}$.

(2) For every ordinal $\beta < \alpha$, the morphism $F(\beta) \to F(\beta + 1)$ is a pushout of a morphism in S.

(3) The morphism $F(0) \to F(\alpha)$ coincides with f.

Remark 1.4.3. Let $\mathcal{C}$ be an ∞-category, and let S and T be collections of morphisms in $\mathcal{C}$. Suppose that every morphism belonging to T is a transfinite pushout of morphisms in S. If f is a transfinite pushout of morphisms in T, then f is a transfinite pushout of morphisms in S.

Definition 1.4.4. Let $\mathcal{C}$ be an ∞-category and let S be a collection of morphisms in $\mathcal{C}$. We will say that S is *weakly saturated* if it has the following properties:

(1) If f is a morphism in $\mathcal{C}$ which is a transfinite pushout of morphisms in S, then $f \in S$.

(2) The set S is closed under retracts. In other words, if we are given a commutative diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d, "f"] & C' \arrow[r] \arrow[d, "f'"] & C \arrow[d, "f"] \\ D \arrow[r] & D' \arrow[r] & D \end{tikzcd}$$

in which both horizontal compositions are the identity and f' belongs to S, then so does f.

Remark 1.4.5. If $\mathcal{C}$ is the nerve of an ordinary category (which admits small colimits), then Definition 1.4.4 reduces to Definition T.A.1.2.2.

Remark 1.4.6. Let S be a weakly saturated collection of morphisms in an ∞-category $\mathcal{C}$. Any identity map in $\mathcal{C}$ can be written as a transfinite composition of morphisms in S (take $\alpha = 0$ in Definition 1.4.2). Condition (2) of Definition 1.4.4 guarantees that the class of morphisms is stable under equivalence; it follows that every equivalence in $\mathcal{C}$ belongs to S. Condition (1) of Definition 1.4.4 also implies that S is closed under composition (take $\alpha = 2$ in Definition 1.4.2).

We can now formulate our main result, which we will prove at the end of this section.

Proposition 1.4.7 (Small Object Argument)**.** *Let $\mathcal{C}$ be a presentable ∞-category and let S be a small collection of morphisms in $\mathcal{C}$. Then every morphism $f : X \to Z$ admits a factorizaton*

$$X \xrightarrow{f'} Y \xrightarrow{f''} Z$$

where f' is a transfinite pushout of morphisms in S and f'' has the right lifting property with respect to S.

Warning 1.4.8. In contrast with the ordinary categorical setting (see Proposition T.A.1.2.5), the factorization

$$X \xrightarrow{f'} Y \xrightarrow{f''} Z$$

of Proposition 1.4.7 cannot generally be chosen to depend functorially on f.

To apply Proposition 1.4.7, the following observation is often useful:

Proposition 1.4.9. *Let $\mathcal{C}$ be an ∞-category and let T be a collection of morphisms in $\mathcal{C}$. Let S denote the collection of all morphisms in $\mathcal{C}$ which have the left lifting property with respect to T. Then S is weakly saturated.*

Proof. Since the intersection of a collection of weakly saturated collections is weakly saturated, it will suffice to treat the case where T consists of a single morphism $g : X \to Y$. Note that a morphism $f : C \to D$ has the left lifting property with respect to g if and only if, for every lifting of Y to $\mathcal{C}_{f/}$, the induced map $\theta_f : \mathcal{C}_{f//Y} \to \mathcal{C}_{C//Y}$ is surjective on objects which lie over $g \in \mathcal{C}_{/Y}$. Since θ_f is a left fibration, it is a categorical fibration; it therefore suffices to show that object of $\mathcal{C}_{C//Y}$ which lies over g is in the essential image of θ_f. We begin by showing that S is stable under pushouts. Suppose we are given a pushout diagram σ :

$$\begin{tikzcd} C' \arrow[r, "f'"] \arrow[d, "\lambda"] & D' \arrow[d] \\ C \arrow[r, "f"] & D \end{tikzcd}$$

in $\mathcal{C}$, where $f' \in S$. We wish to prove that $f \in S$. Consider a lifting of Y to $\mathcal{C}_{f/}$ which we can lift further to $\mathcal{C}_{\sigma/}$. The map θ_f is equivalent to the left fibration $\mathcal{C}_{\sigma//Y} \to \mathcal{C}_{\lambda/Y}$. Since σ is a pushout diagram, this is equivalent to the map $\theta : \mathcal{C}_{f'//Y} \times_{\mathcal{C}_{C'//Y}} \mathcal{C}_{\lambda//Y} \to \mathcal{C}_{\lambda//Y}$. It will therefore suffice to show that every lifting of g to $\mathcal{C}_{\lambda//Y}$ lies in the essential image of θ, which follows from our assumption that every lifting of g to $\mathcal{C}_{C'//Y}$ lies in the essential image of $\theta_{f'}$.

We now verify condition (1) of Definition 1.4.4. Fix an ordinary α and a diagram $F : [\alpha] \to \mathcal{C}$ satisfying the hypotheses of Definition 1.4.2, and let $f : F(0) \to F(\alpha)$ be the induced map. Choose a lifting of Y to $\mathcal{C}_{f/}$ which we can lift further to $\mathcal{C}_{F/}$. Then θ_f is equivalent to the map $\theta : \mathcal{C}_{F//Y} \to \mathcal{C}_{F(0)//Y}$. It will therefore suffice to show that every lift of g to an object of $\overline{X} \in \mathcal{C}_{F(0)//Y}$ lies in the image of θ. For each $\beta < \alpha$, we let $F_\beta = F|[\beta]$; we will construct a compatible sequence of objects $\overline{X}_\beta \in \mathcal{C}_{F_\beta//Y}$ by induction on β. If $\beta = 0$, we take $\overline{X}_\beta = \overline{X}$. If β is a nonzero limit ordinal, then our assumption that F_β is a colimit diagram guarantees that the map $\mathcal{C}_{F_\beta//Y} \to \varprojlim_{\gamma<\beta} \mathcal{C}_{F_\gamma//Y}$ is a trivial Kan fibration so that $\overline{X}_\beta$ can be defined. It remains to treat the case of a successor ordinal: let $\beta < \alpha$ and assume that $\overline{X}_\beta$ has been defined; we wish to show that the vertex $\overline{X}_\beta$ lies in the image of the map $\theta_\beta : \mathcal{C}_{F_{\beta+1}//Y} \to \mathcal{C}_{F_\beta//Y}$. Let $u : F(\beta) \to F(\beta+1)$ be the morphism determined by F, so that θ_β is equivalent to the map θ_u. Since the image of $\overline{X}_\beta$ in $\mathcal{C}_{F(\beta)//Y}$ lies over g, the existence of the desired lifting follows from our assumption that $u \in S$.

We now verify (2). Consider a diagram $\sigma : \Delta^2 \times \Delta^1 \to \mathcal{C}$, given by

$$\begin{tikzcd} C \arrow[r, "\lambda"] \arrow[d, "f"] & C' \arrow[r] \arrow[d, "f'"] & C \arrow[d, "f"] \\ D \arrow[r] & D' \arrow[r] & D. \end{tikzcd}$$

Assume that $f' \in S$; we wish to prove $f \in S$. Choose a lifting of Y to $\mathcal{C}_{f/}$, and lift Y further to $\mathcal{C}_{\sigma/}$ (here we identify f with $\sigma|(\{2\} \times \Delta^1)$. Let $\overline{X}$ be a lifting of g to $\mathcal{C}_{C//Y}$; we wish to show that $\overline{X}$ lies in the image of θ_f. We can lift $\overline{X}$ further to an object $\widetilde{X} \in \mathcal{C}_{\sigma_0//Y}$, where $\sigma_0 = \sigma|(\Delta^2 \times \{0\})$. Let $\sigma' = \sigma|(\Delta^1 \times \Delta^1)$. The forgetful functor $\theta : \mathcal{C}_{\sigma'//Y} \to \mathcal{C}_{\lambda//Y}$ is equivalent to $\theta_{f'}$, so that the image of $\widetilde{X}$ in $\mathcal{C}_{\lambda//Y}$ lies in the image of θ. It follows immediately that $\overline{X}$ lies in the image of θ_f. □

Corollary 1.4.10. *Let $\mathcal{C}$ be a presentable ∞-category, let S be a small collection of morphisms of $\mathcal{C}$, let T be the collection of all morphisms in $\mathcal{C}$ which have the right lifting property with respect to every morphism in S, and let $S^\vee$ be the collection of all morphisms in $\mathcal{C}$ which have the left lifting property with respect to every morphism in T. Then $S^\vee$ is the smallest weakly saturated collection of morphisms which contains S.*

Proof. Proposition 1.4.9 implies that $S^\vee$ is weakly saturated, and it is obvious that $S^\vee$ contains S. Suppose that $\overline{S}$ is any weakly saturated collection of morphisms which contains S; we will show that $S^\vee \subseteq \overline{S}$. Let $f : X \to Z$ be a morphism in $S^\vee$, and choose a factorization

$$X \stackrel{f'}{\to} Y \stackrel{f''}{\to} Z$$

as in Proposition 1.4.7, so that $f' \in \overline{S}$ and $f'' \in T$. Since $f \in S^\vee$, the diagram

$$\begin{tikzcd}
X \arrow[r, "f'"] \arrow[d, "f"] & Y \arrow[d, "f''"] \\
Z \arrow[r, "\mathrm{id}"] & Z
\end{tikzcd}$$

can be extended to a 3-simplex

$$\begin{tikzcd}
X \arrow[r, "f'"] \arrow[d, "f"] & Y \arrow[d, "f''"] \\
Z \arrow[r, "\mathrm{id}"] \arrow[ur, "g"] & Z.
\end{tikzcd}$$

We therefore obtain a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\mathrm{id}"] \arrow[d, "f"] & X \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\
Z \arrow[r, "g"] & Y \arrow[r, "f''"] & Z
\end{tikzcd}$$

which shows that f is a retract of f' and therefore belongs to $\overline{S}$ as desired. □

Recall that if $\mathcal{C}$ is an ∞-category which admits finite limits and colimits, then every simplicial object $X_\bullet$ of $\mathcal{C}$ determines *latching* and *matching* objects $L_n(X_\bullet), M_n(X_\bullet)$ for $n \geq 0$ (see Remark T.A.2.9.16). The following result will play an important role in our proof of Theorem 1.3.12:

Corollary 1.4.11. *Let $\mathcal{C}$ be a presentable ∞-category and let S be a small collection of morphisms in $\mathcal{C}$. Let Y be any object of $\mathcal{C}$, and let $\phi : \mathcal{C}_{/Y} \to \mathcal{C}$ be the forgetful functor. Then there exists a simplicial object $X_\bullet$ of $\mathcal{C}_{/Y}$ with the following properties:*

(1) *For each $n \geq 0$, let $u_n : L_n(X_\bullet) \to X_n$ be the canonical map. Then $\phi(u_n)$ is a transfinite pushout of morphisms in S.*

(2) *For each $n \geq 0$, let $v_n : X_n \to M_n(X_\bullet)$ be the canonical map in $\mathcal{C}_{/Y}$. Then $\phi(v_n)$ has the right lifting property with respect to every morphism in S.*

Proof. We construct $X_\bullet$ as the union of a compatible family of diagrams $X_\bullet^{(n)} : \mathrm{N}(\mathbf{\Delta}_{\leq n})^{op} \to \mathcal{C}_{/Y}$, which we construct by induction on n. The case $n = -1$ is trivial (since $\mathbf{\Delta}_{\leq -1}$ is empty). Assume that $n \geq 0$ and that $X_\bullet^{(n-1)}$ has been constructed, so that the matching and latching objects $L_n(X)$, $M_n(X)$ are defined and we have a map $t : L_n(X) \to M_n(X)$. Using Proposition T.A.2.9.14, we see that it suffices to construct a commutative diagram

$$\begin{tikzcd} & X_n \arrow[dr, "v_n"] & \\ L_n(X_\bullet) \arrow[ur, "u_n"] \arrow[rr, "t"] & & M_n(X_\bullet) \end{tikzcd}$$

in $\mathcal{C}_{/Y}$. Since the map $\mathcal{C}_{/Y} \to \mathcal{C}$ is a right fibration, this is equivalent to the problem of producing a commutative diagram

$$\begin{tikzcd} & K_n \arrow[dr] & \\ \phi(L_n(X_\bullet)) \arrow[ur, "u"] \arrow[rr, "\phi(t)"] & & \phi(M_n(X_\bullet)) \end{tikzcd}$$

in the ∞-category $\mathcal{C}$. Proposition 1.4.7 guarantees that we are able to make these choices in such a way that (1) and (2) are satisfied. $\square$

Remark 1.4.12. In the situation of Corollary 1.4.11, let $\emptyset$ denote the initial object of $\mathcal{C}$. Then for each $n \geq 0$, the canonical map $w : \emptyset \to \phi(X_n)$ is a transfinite pushout of morphisms in S. To prove this, we let P denote the full subcategory of $\mathbf{\Delta}_{[n]/}$ spanned by the surjective maps $[n] \to [m]$; we will regard P as a partially ordered set. For each upward-closed subset $P_0 \subseteq P$, we let $Z(P_0)$ denote a colimit of the induced diagram

$$\mathrm{N}(P_0)^{op} \longrightarrow \mathrm{N}(\mathbf{\Delta})^{op} \xrightarrow{X_\bullet} \mathcal{C}_{/Y} \xrightarrow{\phi} \mathcal{C}.$$

Then $Z(\emptyset) \simeq \emptyset$ and $Z(P) \simeq \phi(X_n)$. It will therefore suffice to show that if $P_1 \subseteq P$ is obtained from P_0 by adjoining a new element given by $\alpha : [n] \to [m]$, then the induced map $\theta : Z(P_0) \to Z(P_1)$ is a transfinite pushout of morphisms in S. This follows from assertion (1) of Corollary 1.4.11, since θ is a pushout of the map $\phi(u_n) : \phi(L_m(X_\bullet)) \to \phi(X_m)$.

Proof of Proposition 1.4.7. Let $S = \{g_i : C_i \to D_i\}_{i \in I}$. Choose a regular cardinal κ such that each of the objects C_i is κ-compact. We construct a diagram $F : \mathrm{N}[\kappa] \to \mathcal{C}_{/Z}$ as the union of maps $\{F_\alpha : \mathrm{N}[\alpha] \to \mathcal{C}_{/Z}\}_{\alpha \leq \kappa}$; here $[\alpha]$ denotes the linearly ordered set of ordinals $\{\beta : \beta \leq \alpha\}$. The construction proceeds by induction: we let F_0 be the morphism $f : X \to Z$, and for a nonzero limit ordinal $\lambda \leq \kappa$ we let F_λ be a colimit of the diagram obtained by amalgamating themaps $\{F_\alpha\}_{\alpha < \lambda}$. Assume that $\alpha < \kappa$ and that F_α been constructed. Then $F_\alpha(\alpha)$ corresponds to a map $X' \to Z$. Let $T(\alpha)$ be a set of representantives for all equivalence classes of diagrams equivalence classes of diagrams σ_t :

$$\begin{tikzcd} C_t \arrow[r] \arrow[d, "g_t"] & X' \arrow[d] \\ D_t \arrow[r] & Z, \end{tikzcd}$$

where g_t is a morphism belonging to S. Choose a pushout diagram

$$\begin{tikzcd} \coprod_{t \in T(\alpha)} C_t \arrow[r] \arrow[d] & X' \arrow[d] \\ \coprod_{t \in T(\alpha)} D_t \arrow[r] & X'' \end{tikzcd}$$

in $\mathcal{C}_{/Z}$. We regard X'' as an object of $\mathcal{C}_{X'//Z}$. Since the map $(\mathcal{C}_{/Z})_{F_\alpha/} \to \mathcal{C}_{X'//Z}$ is a trivial Kan fibration, we can lift X' to an object of $(\mathcal{C}_{/Z})_{F_\alpha/}$, which determines the desired map $F_{\alpha+1}$.

For each $\alpha \leq \kappa$, let $f_\alpha : Y_\alpha \to Z$ be the image $F(\alpha) \in \mathcal{C}_{/Z}$. Let $Y = Y_\kappa$ and $f'' = f_\kappa$. We claim that f'' has the right lifting property with respect to every morphism in S. In other words, we wish to show that for each $i \in I$ and every map $D_i \to Z$, the induced map

$$\mathrm{Map}_{\mathcal{C}_{/Z}}(D_i, Y) \to \mathrm{Map}_{\mathcal{C}_{/Z}}(C_i, Y)$$

is surjective on connected components. Choose a point $\eta \in \mathrm{Map}_{\mathcal{C}_{/Z}}(C_i, Y)$. Since C_i is κ-compact, the space $\mathrm{Map}_{\mathcal{C}_{/Z}}(C_i, Y)$ can be realized as the filtered colimit of mapping spaces $\varinjlim_\alpha \mathrm{Map}_{\mathcal{C}_{/Z}}(C_i, Y_\alpha)$, so we may assume that η is the image of a point $\eta_\alpha \in \mathrm{Map}_{\mathcal{C}_{/Z}}(C_i, Y_\alpha)$ for some $\alpha < \kappa$. The point η_α determines a commutative diagram

$$\begin{tikzcd} C_i \arrow[r] \arrow[d, "g_i"] & Y_\alpha \arrow[d] \\ D_i \arrow[r] & Z \end{tikzcd}$$

which is equivalent to σ_t for some $t \in T(\alpha)$. It follows that the image of η_α in $\mathrm{Map}_{\mathcal{C}_{/Z}}(C_i, Y_{\alpha+1})$ extends to D_i, so that η lies in the image of the map $\mathrm{Map}_{\mathcal{C}_{/Z}}(D_i, Y_{\alpha+1}) \to \mathrm{Map}_{\mathcal{C}_{/Z}}(C_i, Y)$.

The morphism $F(0) \to F(\kappa)$ in $\mathcal{C}_{/Z}$ induces a morphism $f' : X \to Y$ in $\mathcal{C}$; we will complete the proof by showing that f' is a transfinite pushout of morphisms in S. Using Remark 1.4.3, we are reduced to showing that for each $\alpha < \kappa$, the map $Y_\alpha \to Y_{\alpha+1}$ is a transfinite pushout of morphisms in S. To prove this, choose a well-ordering of $T(\alpha)$ having order type β. For $\gamma < \beta$, let t_γ denote the corresponding element of $T(\alpha)$. We define a functor $G : \mathrm{N}[\beta] \to \mathcal{C}$ so that, for each $\beta' \leq \beta$, we have a pushout diagram

$$\begin{tikzcd} \coprod_{\gamma < \beta'} C_{t_\gamma} \arrow[r] \arrow[d] & Y_\alpha \arrow[d] \\ \coprod_{\gamma < \beta'} D_{t_\gamma} \arrow[r] & G(\beta'). \end{tikzcd}$$

It is easy to see that G satisfies the conditions of Definition 1.4.2 and therefore exhibits $Y_\alpha \to Y_{\alpha+1}$ as a transfinite pushout of morphisms in S. $\square$

1.5 Smooth Hypercoverings

Let $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ be a deformation context which admits a deformation theory $\mathfrak{D} : \Upsilon^{op} \to \Xi$. Our goal in this section is to prove Theorem 1.3.12, which asserts that the construction $K \mapsto \mathrm{Map}_\Xi(\mathfrak{D}(\bullet), K)$ induces an equivalence of ∞-categories $\Psi : \Xi \to \mathrm{Moduli}^\Upsilon$. The key step is to prove that every formal moduli problem X admits a "smooth atlas" (Proposition 1.5.8).

We have seen that if X is an algebraic variety over the field $\mathbf{C}$ of complex numbers and $x : \mathrm{Spec}\,\mathbf{C} \to X$ is a point, then X determines a formal moduli problem $X^\wedge : \mathrm{CAlg}^{\mathrm{sm}}_{\mathbf{C}} \to \mathcal{S}$ (Example 1.2.8). However, Definition 1.1.14 is far more inclusive than this. For example, we can also obtain formal moduli problems by extracting the formal completions of algebraic stacks.

Example 1.5.1. Let $n \geq 0$ be an integer, and let A be a connective $\mathbb{E}_\infty$-ring. We say that an A-module M is *projective of rank n* if $\pi_0 M$ is a projective module over $\pi_0 A$ of rank n, and M is flat over A. Let $X(A)$ denote the subcategory of Mod_A whose objects are modules which are locally free of rank n, and whose morphisms are equivalences of modules. It is not difficult to see that the ∞-category $X(A)$ is an essentially small Kan complex. Consequently, the construction $A \mapsto X(A)$ determines a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$.

Let η denote the point of $X(\mathbf{C})$ corresponding to the complex vector space $\mathbf{C}^n$. We define the *formal completion of X at η* to be the functor $X^\wedge : \mathrm{CAlg}^{\mathrm{sm}}_{\mathbf{C}} \to \mathcal{S}$ given by $X^\wedge(R) = X(R) \times_{X(\mathbf{C})} \{\eta\}$. More informally, $X^\wedge(R)$ is a classifying space for projective R-modules M of rank n equipped with a trivialization $\mathbf{C} \otimes_R M \simeq \mathbf{C}^n$. Then $X^\wedge$ is a formal moduli problem (we will prove a more general statement to this effect in §5.2).

If A is a local commutative ring, then every projective A-module of rank of n is isomorphic to A^n. If X is the functor of Example 1.5.1, we deduce that $X(A)$ can be identified with the classifying space for the group $\mathrm{GL}_n(A)$ of automorphisms of A^n as an A-module. For this reason, the functor X is often denoted by BGL_n. It can be described as the geometric realization (in the ∞-category of functors $F : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ which are sheaves with respect to the Zariski topology) of a simplicial functor $F_\bullet$, given by the formula $F_m(A) = \mathrm{GL}_n(A)^m$, where $\mathrm{GL}_n(A)$ denotes the subspace of $\mathrm{Map}_{\mathrm{Mod}_A}(A^n, A^n)$ spanned by the invertible morphisms. Similarly, the formal completion $X^\wedge$ can be described as the geometric realization of a simplicial functor $F^\wedge_\bullet$, given by $F^\wedge_m(R) = \mathrm{fib}(\mathrm{GL}_n(R) \to \mathrm{GL}_n(\mathbf{C}))$ for $R \in \mathrm{CAlg}^{\mathrm{sm}}_{\mathbf{C}}$ (after passing to formal completions, there is no need to sheafify with respect to the Zariski topology: if $R \in \mathrm{CAlg}^{\mathrm{sm}}_{\mathbf{C}}$, then $\pi_0 R$ is a local Artin ring, so that every projective R-module of rank n is automatically free).

Remark 1.5.2. Example 1.5.1 can be generalized. Suppose that X is an arbitrary Artin stack over $\mathbf{C}$. Then X can be presented by an atlas, which is a (smooth) groupoid object

$$\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} U_1 \rightrightarrows U_0.$$

in the category of $\mathbf{C}$-schemes. Let $\eta_0 : \mathrm{Spec}\, \mathbf{C} \to U_0$ be any point, so that η_0 determines points $\eta_n : \mathrm{Spec}\, \mathbf{C} \to U_n$ for every integer n. We can then define formal moduli problems $U_n^\wedge : \mathrm{CAlg}^{\mathrm{sm}}_{\mathbf{C}} \to \mathcal{S}$ by formally completing each U_n at the point η_n. This gives a simplicial object $U^\wedge_\bullet$ in the ∞-category $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{sm}}_{\mathbf{C}}, \mathcal{S})$. The geometric realization $|U^\wedge_\bullet| \in \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{sm}}_{\mathbf{C}}, \mathcal{S})$ is also a formal moduli problem which we will denote by $X^\wedge$. One can show that it is canonically independent (up to equivalence) of the atlas $U_\bullet$ chosen.

Our first goal in this section is to formulate a converse to Remark 1.5.2. Roughly speaking, we would like to assert that every formal moduli problem $Y : \mathrm{CAlg}^{\mathrm{sm}}_{\mathbf{C}} \to \mathcal{S}$ admits a description that resembles the formal completion of an algebraic stack. However, the precise context of Remark 1.5.2 is too restrictive in several respects:

(a) We can associate formal completions not only to algebraic stacks, but also to *higher* algebraic stacks. Consequently, rather than trying to realize Y as the geometric realization of a groupoid object $Y_\bullet$ of $\mathrm{Moduli} \subseteq \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{sm}}_{\mathbf{C}}, \mathcal{S})$, we will allow more general simplicial objects $Y_\bullet$ of Moduli.

(b) We would like to exhibit Y as the geometric realization of a simplicial object $Y_\bullet$ where each Y_m resembles the formal completion of a $\mathbf{C}$-scheme near some point (which, without loss of generality, we may take to be an affine scheme of the form $\mathrm{Spec}^{\text{ét}} R$). Since the construction of the formal completion makes sense not only for schemes but also for spectral Deligne-Mumford stacks (Example 0.0.10), we should allow the possibility that R is a nondiscrete $\mathbb{E}_\infty$-algebra over $\mathbf{C}$.

(c) If R is an augmented $\mathbb{E}_\infty$-algebra over $\mathbf{C}$, then A determines a formal moduli problem $\mathrm{Spec}\, R : \mathrm{CAlg}^{\mathrm{sm}}_{\mathbf{C}} \to \mathcal{S}$ given by the formula $A \mapsto \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{aug}}_{\mathbf{C}}}(R, A)$. This functor is perhaps better understood (at least in the case where R is Noetherian) as the formal spectrum of $R^\wedge$, where $R^\wedge$ denotes the completion of R along the augmentation ideal in $\pi_0 R$. To incorporate a wider class of examples, we should allow arbitrary (possibly infinite-dimensional) affine formal schemes, not only those which arise as the formal completions of actual schemes.

Let us now explain how to make these ideas more precise. We will work in the setting of an arbitrary deformation context $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$.

Definition 1.5.3. Let $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ be a deformation context. We let $\mathrm{Pro}(\Upsilon^{\mathrm{sm}})$ denote the ∞-category of pro-objects of Υ^{sm}: that is, the smallest full subcategory of $\mathrm{Fun}(\Upsilon^{\mathrm{sm}}, \mathcal{S})^{op}$ which contains all corepresentable functors and is closed under filtered colimits. We will say that a functor $X : \Upsilon^{\mathrm{sm}} \to \mathcal{S}$ is *prorepresentable* if it belongs to the full subcategory $\mathrm{Pro}(\Upsilon^{\mathrm{sm}}) \subseteq \mathrm{Fun}(\Upsilon^{\mathrm{sm}}, \mathcal{S})^{op}$.

Remark 1.5.4. Let $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ be a deformation context. Since filtered colimits in $\mathcal{S}$ are left exact (Example T.7.3.4.4), the full subcategory $\mathrm{Moduli}^{\Upsilon}$ is stable under filtered colimits in $\mathrm{Fun}(\Upsilon^{\mathrm{sm}}, \mathcal{S})$. Since every corepresentable functor is a formal moduli problem (Example 1.1.16), we conclude that $\mathrm{Pro}(\Upsilon^{\mathrm{sm}})^{op}$ is contained in $\mathrm{Moduli}^{\Upsilon}$ (as a full subcategory of $\mathrm{Fun}(\Upsilon^{\mathrm{sm}}, \mathcal{S})$). That is, every prorepresentable functor $\Upsilon^{\mathrm{sm}} \to \mathcal{S}$ is a formal moduli problem.

Our next objective is to introduce a general notion of smooth morphism between two formal moduli problems.

Proposition 1.5.5. *Let $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ be a deformation context, and let $u : X \to Y$ be a map of formal moduli problems $X, Y : \Upsilon^{\mathrm{sm}} \to \mathcal{S}$. The following conditions are equivalent:*

(1) *For every small map $\phi : A \to B$ in Υ^{sm}, u has the right lifting property with respect to $\mathrm{Spec}(\phi) : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$.*

(2) *For every small map $\phi : A \to B$ in Υ^{sm}, the induced map $X(A) \to X(B) \times_{Y(B)} Y(A)$ is surjective on connected components.*

(3) *For every elementary map $\phi : A \to B$ in Υ^{sm}, the induced map $X(A) \to X(B) \times_{Y(B)} Y(A)$ is surjective on connected components.*

(4) *For every $\alpha \in T$ and every $n > 0$, the homotopy fiber of the map $X(\Omega^{\infty - n} E_\alpha) \to Y(\Omega^{\infty-n} E_\alpha)$ (taken over the point determined by the base point of $\Omega^{\infty-n} E_\alpha$) is connected.*

(5) *For every $\alpha \in T$, the map of spectra $X(E_\alpha) \to Y(E_\alpha)$ is connective (that is, it has a connective homotopy fiber).*

Proof. The equivalence (1) $\Leftrightarrow$ (2) is tautological, and the implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are evident. Let S be the collection of all small morphisms $A \to B$ in Υ^{sm} for which the map $X(A) \to X(B) \times_{Y(B)} Y(A)$ is surjective on connected components. The implication (3) $\Rightarrow$ (2) follows from the observation that S is closed under composition, and the implication (4) $\Rightarrow$ (3) from the observation that S is stable under the formation of pullbacks. The equivalence (4) $\Leftrightarrow$ (5) follows from fact that a map of $M \to M'$ of spectra is connective if and only if the induced map $\Omega^{\infty-n} M \to \Omega^{\infty-n} M'$ has connected homotopy fibers for each $n > 0$. $\square$

Definition 1.5.6. Let $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ be a deformation context, and let $u : X \to Y$ be a map of formal moduli problems. We will say that u is *smooth* if it satisfies the equivalent conditions of Proposition 1.5.5. We will say that a formal moduli problem X is smooth if the map $X \to *$ is smooth, where $*$ denotes the final object of $\mathrm{Moduli}^{\Upsilon}$ (that is, the constant functor $\Upsilon^{\mathrm{sm}} \to \mathcal{S}$ taking the value $* \in \mathcal{S}$).

Remark 1.5.7. We can regard condition (5) of Proposition 1.5.5 as providing a differential criterion for smoothness: a map of formal moduli problems $X \to Y$ is smooth if and only if it induces a connective map of tangent complexes $X(E_\alpha) \to Y(E_\alpha)$. This should be regarded as an analogue of the condition that a map of smooth algebraic varieties $f : X \to Y$ induce a surjective map of tangent sheaves $T_X \to f^* T_Y$.

Proposition 1.5.8. *Let $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ be a deformation context and let $X : \Upsilon^{\mathrm{sm}} \to \mathcal{S}$ be a formal moduli problem. Then there exists a simplicial object $X_\bullet$ in $\mathrm{Moduli}^{\Upsilon}_{/X}$ with the following properties:*

(1) *Each X_n is prorepresentable.*

(2) *For each $n \geq 0$, let $M_n(X_\bullet)$ denote the nth matching object of the simplicial object $X_\bullet$ (computed in the ∞-category $\mathrm{Moduli}^{\Upsilon}_{/X}$). Then the canonical map $X_n \to M_n(X_\bullet)$ is smooth.*

In particular, X is equivalent to the geometric realization $|X_\bullet|$ in $\mathrm{Fun}(\Upsilon^{\mathrm{sm}}, \mathcal{S})$.

The proof of Proposition 1.5.8 will use the following simple observation:

Lemma 1.5.9. *Let $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ be a deformation context, and let S be the collection of all morphisms in the ∞-category $\mathrm{Moduli}^{\Upsilon}$ of the form $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$, where the underlying map $A \to B$ is a small morphism in Υ^{sm}. Let $f : X \to Y$ be a morphism in* Moduli*, and suppose that f is a transfinite pushout of morphisms in S. If X is prorepresentable, then Y is prorepresentable.*

Proof. Since the collection of prorepresentable objects of $\mathrm{Moduli}^{\Upsilon}$ is closed under filtered colimits, it will suffice to prove the following:

- $(*)$ If $\phi : A \to B$ is a small morphism in $\mathrm{Moduli}^{\Upsilon}$ and we are given a pushout diagram

$$\begin{tikzcd} \mathrm{Spec}(B) \arrow[r] \arrow[d, "\mathrm{Spec}(\phi)"] & X \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r] & Y \end{tikzcd}$$

  where X is prorepresentable, then Y is also prorepresentable.

To prove $(*)$, we note that X can be regarded as an object of $\mathrm{Pro}(\Upsilon^{\mathrm{sm}})^{op}_{\mathrm{Spec}(B)/} \simeq \mathrm{Ind}((\Upsilon^{\mathrm{sm}}_{/B})^{op})$. In other words, we have $X \simeq \varinjlim_\beta \mathrm{Spec}(B_\beta)$ for some filtered diagram $\{B_\beta\}$ in $\Upsilon^{\mathrm{sm}}_{/B}$. Then

$$Y \simeq \varinjlim(\mathrm{Spec}(B_\beta) \coprod_{\mathrm{Spec}(B)} \mathrm{Spec}(A)).$$

For any formal moduli problem Z, we have

$$\mathrm{Map}_{\mathrm{Moduli}^{\Upsilon}}(\mathrm{Spec}(B_\beta) \coprod_{\mathrm{Spec}(B)} \mathrm{Spec}(A), Z) \simeq Z(B_\beta) \times_{Z(B)} Z(A) \simeq Z(B_\beta \times_B A)$$

(since the map $\phi : A \to B$ is small), so that $\mathrm{Spec}(B_\beta) \coprod_{\mathrm{Spec}(B)} \mathrm{Spec}(A) \simeq \mathrm{Spec}(B_\beta \times_B A)$ is corepresented by an object $B_\beta \times_B A$. It follows that Y is prorepresentable, as desired. $\square$

Proof of Proposition 1.5.8. Let X be an arbitrary formal moduli problem. Applying Corollary 1.4.11, we can choose a simplicial object $X_\bullet$ of $\mathrm{Moduli}^{\Upsilon}_{/X}$ such that each of the maps $X_n \to M_n(X_\bullet)$ is smooth, and each of the maps $L_n(X_\bullet) \to X_n$ is a transfinite pushout of morphisms of the form $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$, where $A \to B$ is an elementary morphism in Υ^{sm}. Using Remark 1.4.12 and Lemma 1.5.9, we conclude that each X_n is prorepresentable. This proves (1) and (2). To prove that $X \simeq |X_\bullet|$ in $\mathrm{Fun}(\Upsilon^{\mathrm{sm}}, \mathcal{S})$, it suffices to observe that condition (2) implies that $X_\bullet(A)$ is a hypercovering of $X(A)$ for every $A \in \Upsilon^{\mathrm{sm}}$. $\square$

We now turn to the proof of Theorem 1.3.12 itself. We will need the following:

Lemma 1.5.10. *Let $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ be a deformation context and let $\mathfrak{D} : \Upsilon^{op} \to \Xi$ be a deformation theory. For every small object $A \in \Upsilon^{\mathrm{sm}}$, $\mathfrak{D}(A)$ is a compact object of the ∞-category Ξ.*

Proof. Since A is small, there exists a sequence of elementary morphisms

$$A = A_0 \to A_1 \to \cdots \to A_n \simeq *$$

in Υ. We will prove that $\mathfrak{D}(A_i)$ is a compact object of Ξ by descending induction on i. When $i = n$, the desired result follows from the observation that $\mathfrak{D}$ carries final objects of Υ to initial objects of Ξ (Proposition 1.3.5). Assume therefore that $i < n$ and that $\mathfrak{D}(A_{i+1}) \in \Xi$ is compact. Since the map $A_i \to A_{i+1}$ is elementary, we have a pullback diagram σ :

$$\begin{tikzcd} A_i \arrow[r] \arrow[d] & * \arrow[d] \\ A_{i+1} \arrow[r] & \Omega^{\infty-n} E_\alpha \end{tikzcd}$$

for some $\alpha \in T$ and some $n > 0$. It follows from Proposition 1.3.5 that $\mathfrak{D}(\sigma)$ is a pushout square in Ξ. Consequently, to show that $\mathfrak{D}(A_i)$ is a compact object of Ξ, it will suffice to show that $\mathfrak{D}(A_{i+1})$, $\mathfrak{D}(*)$, and $\mathfrak{D}(\Omega^{\infty-n}E_\alpha)$ are compact objects of Ξ. In the first two cases, this follows from the inductive hypothesis. For the third, we note that the functor corepresented by $\mathfrak{D}(\Omega^{\infty-n}E_\alpha)$ is given by the composition

$$\Xi \xrightarrow{e_\alpha} \mathrm{Sp} \xrightarrow{\Omega^{\infty-n}} \mathcal{S},$$

where e_α is the functor described in Corollary 1.3.8. Our assumption that $\mathfrak{D}$ is a deformation theory guarantees that e_α commutes with sifted colimits. The functor $\Omega^{\infty-n} : \mathrm{Sp} \to \mathcal{S}$ commutes with filtered colimits, so the composite functor $\Xi \to \mathcal{S}$ commutes with filtered colimits which implies that $\Omega^{\infty-n}E_\alpha$ is a compact object of Ξ. □

We are now ready to prove our main result.

Proof of Theorem 1.3.12. Let $(\Upsilon, \{E_\alpha\}_{\alpha\in T})$ be a deformation context, let $\mathfrak{D} : \Upsilon^{op} \to \Xi$ be a deformation theory, and let $\Psi : \Xi \to \mathrm{Moduli}^{\Upsilon} \subseteq \mathrm{Fun}(\Upsilon^{\mathrm{sm}}, \mathcal{S})$ denote the functor given by the formula $\Psi(K)(A) = \mathrm{Map}_\Xi(\mathfrak{D}(A), K)$. We wish to prove that Ψ is an equivalence of ∞-categories. It is clear that Ψ preserves small limits. It follows from Lemma 1.5.10 that Ψ preserves filtered colimits and is therefore accessible. Using Corollary T.5.5.2.9 we conclude that Ψ admits a left adjoint Φ. To prove that Ψ is an equivalence, it will suffice to show:

(a) The functor Ψ is conservative.

(b) The unit transformation $u : \mathrm{id}_{\mathrm{Moduli}} \to \Psi \circ \Phi$ is an equivalence.

We begin with the proof of (a). Suppose we are given a morphism $f : K \to K'$ in Ξ have that $\Psi(f)$ is an equivalence. In particular, for each $\alpha \in T$ and each $n \geq 0$, we have a homotopy equivalence

$$\mathrm{Map}_\Xi(\mathfrak{D}(\Omega^{\infty-n}E_\alpha), K) \simeq \Psi(K)(\mathfrak{D}\Omega^{\infty-n}E_\alpha) \to \Psi(K')(\mathfrak{D}\Omega^{\infty-n}E_\alpha) \simeq \mathrm{Map}_\Xi(\mathfrak{D}(\Omega^{\infty-n}E_\alpha), K').$$

It follows that $e_\alpha(K) \simeq e_\alpha(K')$, where $e_\alpha : \Xi \to \mathrm{Sp}$ is the functor described in Corollary 1.3.8. Since the functors e_α are jointly conservative (Definition 1.3.9), we conclude that f is an equivalence.

We now prove (b). Let $X \in \mathrm{Moduli}^{\Upsilon}$ be a formal moduli problem; we wish to show that u induces an equivalence $X \to (\Psi \circ \Phi)(X)$. According to Proposition 1.2.10, it suffices to show that for each $\alpha \in T$, the induced map

$$\theta : X(E_\alpha) \to (\Psi \circ \Phi)(X)(E_\alpha) \simeq e_\alpha(\Phi X)$$

is an equivalence of spectra. To prove this, choose a simplicial object $X_\bullet$ of $\mathrm{Moduli}^{\Upsilon}_{/X}$ satisfying the requirements of Proposition 1.5.8. For every object $A \in \Upsilon^{\mathrm{sm}}$, the simplicial space $X_\bullet(A)$ is a hypercovering of $X(A)$ so that the induced map $|X_\bullet(A)| \to X(A)$ is a homotopy equivalence. It follows that X is a colimit of the diagram $X_\bullet$ in the ∞-category $\mathrm{Fun}(\Upsilon^{\mathrm{sm}}, \mathcal{S})$ and therefore also in the ∞-category $\mathrm{Moduli}^{\Upsilon}$. Similarly, $X(E_\alpha)$ is equivalent to the geometric realization $|X_\bullet(E_\alpha)|$ in the ∞-category $\mathrm{Fun}(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{S})$ and therefore also in the ∞-category of spectra. Since Φ preserves small colimits and e_α preserves sifted colimits, we have

$$e_\alpha(\Phi(X)) \simeq e_\alpha(\Phi|X_\bullet|) \simeq |e_\alpha(\Phi X_\bullet)|.$$

It follows that θ is a geometric realization of a simplicial morphism

$$\theta_\bullet : X_\bullet(E_\alpha) \to e_\alpha(\Phi X_\bullet).$$

It will therefore suffice to prove that each θ_n is an equivalence, which is equivalent to the requirement that u induces an equivalence $X_n \to (\Psi \circ \Phi)(X_n)$. In other words, we may replace X by X_n, and thereby reduce to the case where X is prorepresentable. Since the functors Φ and Ψ both commute with filtered colimits, we may further reduce to the case where $X = \mathrm{Spec}(A)$ for some $A \in \Upsilon^{\mathrm{sm}}$. Since $\Phi(\mathrm{Spec}(A)) = \mathfrak{D}(A)$, it suffices to show that for each $B \in \Upsilon^{\mathrm{sm}}$, the map

$$\mathrm{Map}_\Upsilon(A, B) \to \mathrm{Map}_\Xi(\mathfrak{D}(B), \mathfrak{D}(A)) \simeq \mathrm{Map}_\Upsilon(A, \mathfrak{D}'\mathfrak{D}(B))$$

is a homotopy equivalence, which follows immediately from Proposition 1.3.5. □

2 Moduli Problems for Commutative Algebras

Let k be a field of characteristic zero, fixed throughout this section. An $\mathbb{E}_\infty$-algebra $R \in \mathrm{CAlg}_k$ is *small* if it satisfies the conditions of Proposition 1.1.11: that is, if R is connective, $\pi_* R$ is a finite dimensional vector space over k, and $\pi_0 R$ is a local ring with residue field k. We let $\mathrm{CAlg}_k^{\mathrm{sm}}$ denote the full subcategory of CAlg_k spanned by the small $\mathbb{E}_\infty$-algebra over k (we can also identify $\mathrm{CAlg}_k^{\mathrm{sm}}$ with a full subcategory of the ∞-category $\mathrm{CAlg}_k^{\mathrm{aug}}$ of *augmented* $\mathbb{E}_\infty$-algebras over k: see Remark 1.1.12).

Recall that a functor $X : \mathrm{CAlg}_k^{\mathrm{sm}} \to \mathcal{S}$ is called a *formal moduli problem* if it satisfies the following pair of conditions (see Proposition 1.1.19):

(a) The space $X(k)$ is contractible.

(b) For every pullback diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R_0 \arrow[d] \\ R_1 \arrow[r] & R_{01} \end{tikzcd}$$

in $\mathrm{CAlg}_k^{\mathrm{sm}}$, if both the maps $\pi_0 R_0 \to \pi_0 R_{01} \leftarrow \pi_0 R_1$ are surjective, then the diagram of spaces

$$\begin{tikzcd} X(R) \arrow[r] \arrow[d] & X(R_0) \arrow[d] \\ X(R_1) \arrow[r] & X(R_{01}) \end{tikzcd}$$

is also a pullback diagram.

In this section, we will study the full subcategory $\mathrm{Moduli}_k \subseteq \mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{sm}}, \mathcal{S})$ spanned by the formal moduli problems.

We begin by applying the general formalism of §1.2. To every formal moduli problem $X \in \mathrm{Moduli}_k$, we can associate a spectrum $T_X \in \mathrm{Sp}$, which is given informally by the formula $\Omega^{\infty - n} T_X = X(k \oplus k[n])$ for $n \geq 0$. In particular, we can identify the 0th space $\Omega^\infty T_X$ with $X(k[\epsilon]/(\epsilon^2))$, an analogue of the classical Zariski tangent space. We refer to T_X as the *tangent complex* of the formal moduli problem X.

The construction $X \mapsto T_X$ commutes with finite limits. In particular, we have a homotopy equivalence of spectra $T_X[-1] \simeq T_{\Omega X}$, where ΩX denotes the formal moduli problem given by the formula

$$(\Omega X)(R) = \Omega X(R)$$

(note that a choice of point η in the contractible space $X(k)$ determines a base of each $X(R)$, so the loop space $\Omega X(R)$ is well-defined). The formal moduli problem ΩX is equipped with additional structure: it can be regarded as a group object of Moduli_k. It is therefore natural to expect that the tangent complex $T_{\Omega X}$ should behave somewhat like the tangent space to an algebraic group. We can formulate this idea more precisely as follows:

($*$) Let $X \in \mathrm{Moduli}_k$ be a formal moduli problem. Then the shifted tangent complex $T_X[-1] \simeq T_{\Omega X}$ can be identified with the underlying spectrum of a differential graded Lie algebra over k.

Example 2.0.1. Suppose that A is a commutative k-algebra equipped with an augmentation $\epsilon : A \to k$. Then R defines a formal moduli problem X over k, which carries a small $\mathbb{E}_\infty$-algebra R over k to the mapping space $\mathrm{Map}_{\mathrm{CAlg}_k^{\mathrm{aug}}}(A, R)$. When k is of characteristic zero, the tangent complex T_X can be identified with the complex of Andre-Quillen cochains taking values in k. In this case, the existence of a natural differential graded Lie algebra structure on $T_X[-1]$ is proven in [58].

Assertion $(*)$ has a converse: every differential graded Lie algebra $\mathfrak{g}_*$ arises (up to quasi-isomorphism) as the shifted tangent complex $T_X[-1]$ of some $X \in \mathrm{Moduli}_k$. Moreover, the formal moduli problem X is determined by $\mathfrak{g}_*$ up to equivalence. More precisely, we have the following stronger version of Theorem 0.0.13:

Theorem 2.0.2. *Let k be a field of characteristic zero, and let Lie_k denote the ∞-category of differential graded Lie algebras over k (that is, Lie_k is the ∞-category obtained from the ordinary category $\mathrm{Lie}_k^{\mathrm{dg}}$ of differential graded Lie algebras over k by formally inverting all quasi-isomorphisms; see Definition 2.1.14). Then there is an equivalence of ∞-categories $\Psi : \mathrm{Lie}_k \to \mathrm{Moduli}_k$. Moreover, the functor $\mathfrak{g}_* \to T_{\Psi(\mathfrak{g}_*)}[-1]$ is equivalent to the forgetful functor $\mathrm{Lie}_k \to \mathrm{Sp}$ (which carries a differential graded Lie algebra $\mathfrak{g}_*$ to the generalized Eilenberg-MacLane spectrum determined by the underlying chain complex of $\mathfrak{g}_*$).*

Our main goal in this section is to prove Theorem 2.0.2. The first step is to construct the functor $\Psi : \mathrm{Lie}_k \to \mathrm{Moduli}_k$. Let $\mathfrak{g}_*$ be a differential graded Lie algebra over k, and let $R \in \mathrm{CAlg}_k^{\mathrm{sm}}$. Since k has characteristic zero, we can identify R with an (augmented) commutative differential graded algebra over k; let us denote its augmentation ideal by $\mathfrak{m}_R$. The tensor product $\mathfrak{m}_R \otimes_k \mathfrak{g}_*$ then inherits the structure of a differential graded Lie algebra over k. Roughly speaking, $\Psi(\mathfrak{g}_*)(R)$ should be a suitably-defined space of *Maurer-Cartan elements* of the differential graded Lie algebra $\mathfrak{m}_R \otimes_k \mathfrak{g}_*$: that is, the space of solutions to the Maurer-Cartan equation $dx = [x, x]$. There does not seem to be a homotopy-invariant definition for the space $\mathrm{MC}(\mathfrak{g}_*)$ of Maurer-Cartan elements of an arbitrary differential graded Lie algebra over k: the well-definedness of $\mathrm{MC}(\mathfrak{m}_R \otimes_k \mathfrak{g}_*)$ relies on the nilpotence properties of the tensor product $\mathfrak{m}_R \otimes_k \mathfrak{g}_*$ (which follow from our assumption that R is small). Nevertheless, there is a well-defined *bifunctor* $\mathrm{MC} : \mathrm{CAlg}_k^{\mathrm{aug}} \times \mathrm{Lie}_k \to \mathcal{S}$ which is given heuristically by $(R, \mathfrak{g}_*) \mapsto \mathrm{MC}(\mathfrak{m}_R \otimes_k \mathfrak{g}_*)$. This functor can be defined more precisely by the formula

$$\mathrm{MC}(R, \mathfrak{g}_*) = \mathrm{Map}_{\mathrm{Lie}_k}(\mathfrak{D}(R), \mathfrak{g}_*),$$

where $\mathfrak{D} : (\mathrm{CAlg}_k^{\mathrm{aug}})^{op} \to \mathrm{Lie}_k$ is the *Koszul duality* functor that we will describe in §2.3. Roughly speaking, the Koszul dual of an augmented $\mathbb{E}_\infty$-algebra R is a differential graded Lie algebra $\mathfrak{D}(R) \in \mathrm{Lie}_k$ which *corepresents* the functor $\mathfrak{g}_* \mapsto \mathrm{MC}(\mathfrak{m}_R \otimes_k \mathfrak{g}_*)$. However, it will be more convenient for us to describe $\mathfrak{D}(R)$ instead by the functor that *represents*. We will define $\mathfrak{D}$ as the right adjoint to the functor $C^* : \mathrm{Lie}_k \to (\mathrm{CAlg}_k^{\mathrm{aug}})^{op}$, which assigns to each differential graded Lie algebra $\mathfrak{g}_*$ the commutative differential graded algebra $C^*(\mathfrak{g}_*)$ of Lie algebra cochains on $\mathfrak{g}_*$ (see Construction 2.2.13).

Remark 2.0.3. For our purposes, the Maurer-Cartan equation $dx = [x, x]$ (and the associated space $\mathrm{MC}(\mathfrak{g}_*)$ of Maurer-Cartan elements of a differential graded Lie algebra $\mathfrak{g}_*$) are useful heuristics for understanding the functor Ψ appearing in Theorem 2.0.2. They will play no further role in this paper. For a construction of the functor Ψ which makes direct use of the Maurer-Cartan equation, we refer the reader to the work of Hinich (see [26]). We also refer the reader to the work of Goldman and Millson ([22] and [23]).

Let us now outline the contents of this section. We begin in §2.1 with a brief overview of the theory of differential graded Lie algebras and a definition of the ∞-category Lie_k. In §2.2, we will review the homology and cohomology theory of (differential graded) Lie algebras, which are computed by the Chevalley-Eilenberg constructions

$$\mathfrak{g}_* \mapsto C_*(\mathfrak{g}_*) \qquad \mathfrak{g}_* \mapsto C^*(\mathfrak{g}_*).$$

The functor C^* carries quasi-isomorphisms of differential graded Lie algebras to quasi-isomorphisms between (augmented) commutative differential graded algebras, and therefore descends to a (contravariant) functor from the ∞-category Lie_k to the ∞-category $\mathrm{CAlg}_k^{\mathrm{aug}}$. We will show that this functor admits a left adjoint $\mathfrak{D}$ and study its properties. The main point is to show that $\mathfrak{D}$ defines a deformation theory (in the sense of Definition 1.3.9) on the deformation context $(\mathrm{CAlg}_k^{\mathrm{aug}}, \{E\})$ of Example 1.1.4. We will use this fact in §2.3 to deduce Theorem 2.0.2 from Theorem 1.3.12.

If $X : \mathrm{CAlg}_k^{\mathrm{sm}} \to \mathcal{S}$ is a formal moduli problem, then we can introduce an ∞-category $\mathrm{QCoh}(X)$ of *quasi-coherent sheaves* on X. It follows from Theorem 2.0.2 that X is completely determined by a differential graded Lie algebra $\mathfrak{g}_*$ (which is well-defined up to quasi-isomorphism). In §2.4, we will show that $\mathrm{QCoh}(X)$ can be obtained as a full subcategory of the ∞-category of (differential graded) representations of $\mathfrak{g}_*$ (Theorem 2.4.1).

2.1 Differential Graded Lie Algebras

Let k be a field of characteristic zero. Theorem 2.0.2 asserts that the ∞-category Moduli_k of formal moduli problems over k is equivalent to the ∞-category Lie_k of differential graded Lie algebras over k. Our goal in this section is to explain the definition of Lie_k and establish some of its basic properties. We begin by reviewing the definition of the ordinary category $\mathrm{Lie}_k^{\mathrm{dg}}$ of differential graded Lie algebras over k and showing that it admits the structure of a model category (Proposition 2.1.10). Along the way, we will introduce some of the notation and constructions which will play a role in our proof of Theorem 2.0.2.

Notation 2.1.1. Let k be a field. We let $\mathrm{Vect}_k^{\mathrm{dg}}$ denote the category of differential graded vector spaces over k: that is, the category whose objects are chain complexes

$$\cdots \to V_1 \to V_0 \to V_{-1} \to \cdots$$

We will regard $\mathrm{Vect}_k^{\mathrm{dg}}$ as a symmetric monoidal category, whose tensor product is the usual tensor product of chain complexes given by the formula

$$(V \otimes W)_p = \bigoplus_{p=p'+p''} V_{p'} \otimes_k W_{p''},$$

and the symmetry isomorphism $V \otimes W \simeq W \otimes V$ is the sum of the isomorphisms $V_{p'} \otimes_k W_{p''} \simeq W_{p''} \otimes_k V_{p'}$, multiplied by the factor $(-1)^{p'p''}$.

We recall that the category $\mathrm{Vect}_k^{\mathrm{dg}}$ admits a model structure, where:

- (C) A map of chain complexes $f : V_* \to W_*$ is a cofibration it is degreewise monic: that is, each of the induced maps $V_n \to W_n$ is injective.
- (F) A map of chain complexes $f : V_* \to W_*$ is a fibration it is degreewise epic: that is, each of the induced maps $V_n \to W_n$ is surjective..
- (W) A map of chain complexes $f : V_* \to W_*$ is a weak equivalence if it is a *quasi-isomorphism*: that is, if it induces an isomorphism on homology groups $\mathrm{H}_n(V) \to \mathrm{H}_n(W)$ for every integer n.

Moreover, the underlying ∞-category of $\mathrm{Vect}_k^{\mathrm{dg}}$ can be identified with the ∞-category Mod_k of k-module spectra (see Remark A.7.1.1.16).

Notation 2.1.2. Let V be a graded vector space over k. We let $V^\vee$ denote the graded dual of V, given by $(V^\vee)_p = \mathrm{Hom}_k(V_{-p}, k)$. For each integer n, we let $V[n]$ denote the same vector space with grading shifted by n, so that $V[n]_p = V_{p-n}$.

Definition 2.1.3. We let $\mathrm{Alg}_k^{\mathrm{dg}}$ denote the category of associative algebra objects of $\mathrm{Vect}_k^{\mathrm{dg}}$, and $\mathrm{CAlg}_k^{\mathrm{dg}}$ the category of commutative algebra objects of $\mathrm{Vect}_k^{\mathrm{dg}}$. We will refer to objects of $\mathrm{Alg}_k^{\mathrm{dg}}$ as *differential graded algebras* over k and objects of $\mathrm{CAlg}_k^{\mathrm{dg}}$ as *commutative differential graded algebras over k*. According to Propositions A.4.1.4.3 and A.4.4.4.6, $\mathrm{Alg}_k^{\mathrm{dg}}$ and $\mathrm{CAlg}_k^{\mathrm{dg}}$ admit combinatorial model structures, where a map $f : A_* \to B_*$ of (commutative) differential graded algebras is a weak equivalence or fibration if the underlying map of chain complexes is a weak equivalence or fibration.

Remark 2.1.4. In more concrete terms, a differential graded algebra A is a chain complex (A_*, d) together with a unit $1 \in A_0$ and a collection of k-bilinear multiplication maps $A_p \times A_q \to A_{p+q}$ satisfying

$$1x = x1 = x \qquad x(yz) = (xy)z \qquad d(xy) = dxy + (-1)^p xdy$$

for $x \in A_p$, $y \in A_q$, and $z \in A_r$. The differential graded algebra A is commutative if $xy = (-1)^{pq} yx$ for $x \in A_p$, $y \in A_q$.

Definition 2.1.5. A *differential graded Lie algebra* over k is a chain complex $(\mathfrak{g}_*, d)$ of k-vector spaces equipped with a Lie bracket $[,] : \mathfrak{g}_p \otimes_k \mathfrak{g}_q \to \mathfrak{g}_{p+q}$ satisfying the following conditions:

(1) For $x \in \mathfrak{g}_p$ and $y \in \mathfrak{g}_q$, we have $[x,y] + (-1)^{pq}[y,x] = 0$.

(2) For $x \in \mathfrak{g}_p$, $y \in \mathfrak{g}_q$, and $z \in \mathfrak{g}_r$, we have

$$(-1)^{pr}[x,[y,z]] + (-1)^{pq}[y,[z,x]] + (-1)^{qr}[z,[x,y]] = 0.$$

(3) The differential d is a derivation with respect to the Lie bracket. That is, for $x \in \mathfrak{g}_p$ and $y \in \mathfrak{g}_q$, we have

$$d[x,y] = [dx,y] + (-1)^p[x,dy].$$

Given a pair of differential graded Lie algebras $(\mathfrak{g}_*, d)$ and $(\mathfrak{g}'_*, d')$, a *map of differential graded Lie algebras* from $(\mathfrak{g}_*, d)$ to $(\mathfrak{g}'_*, d')$ is a map of chain complexes $F : (\mathfrak{g}_*, d) \to (\mathfrak{g}'_*, d')$ such that $F([x,y]) = [F(x), F(y)]$ for $x \in \mathfrak{g}_p$, $y \in \mathfrak{g}_q$. The collection of all differential graded Lie algebras over k forms a category, which we will denote by $\mathrm{Lie}_k^{\mathrm{dg}}$.

Example 2.1.6. Let A_* be a (possibly nonunital) differential graded algebra over k. Then A_* has the structure of a differential graded Lie algebra, where the Lie bracket

$$[,] : A_p \otimes_k A_q \to A_{p+q}$$

is given by $[x,y] = xy - (-1)^{pq}yx$.

Remark 2.1.7. The construction of Example 2.1.6 determines a forgetful functor $\mathrm{Alg}_k^{\mathrm{dg}} \to \mathrm{Lie}_k^{\mathrm{dg}}$. This functor admits a left adjoint $U : \mathrm{Lie}_k^{\mathrm{dg}} \to \mathrm{Alg}_k^{\mathrm{dg}}$, which assigns to every differential graded Lie algebra $\mathfrak{g}_*$ its *universal enveloping algebra* $U(\mathfrak{g}_*)$. The universal enveloping algebra $U(\mathfrak{g}_*)$ can be described as the quotient of the tensor algebra $\bigoplus_{n \geq 0} \mathfrak{g}_*^{\otimes n}$ by the two-sided ideal generated by all expressions of the form $(x \otimes y) - (-1)^{pq}(y \otimes x) - [x,y]$, where $x \in \mathfrak{g}_p$ and $y \in \mathfrak{g}_q$. The collection of such expressions is stable under the differential on $\bigoplus_{n \geq 0} \mathfrak{g}_*^{\otimes n}$, so that $U(\mathfrak{g}_*)$ inherits the structure of a differential graded algebra.

The universal enveloping algebra $U(\mathfrak{g}_*)$ admits a natural filtration

$$U(\mathfrak{g}_*)^{\leq 0} \subseteq U(\mathfrak{g}_*)^{\leq 1} \subseteq \cdots,$$

where $U(\mathfrak{g}_*)^{\leq n}$ is the image of $\bigoplus_{0 \leq i \leq n} \mathfrak{g}_*^{\otimes i}$ in $U(\mathfrak{g}_*)$. The associated graded algebra of $U(\mathfrak{g}_*)$ is commutative (in the graded sense), so that the canonical map $\mathfrak{g}_* \to U(\mathfrak{g})^{\leq 1}$ induces a map of differential graded algebras $\theta : \mathrm{Sym}^* \mathfrak{g}_* \to \mathrm{gr}\, U(\mathfrak{g}_*)$. According to the Poincare-Birkhoff-Witt theorem, the map θ is an isomorphism (see Theorem 2.3 of [73] for a proof in the setting of differential graded Lie algebras).

Remark 2.1.8. Let $\mathfrak{g}_*$ be a differential graded Lie algebra over k. For each integer n, we let $\psi : \mathfrak{g}_*^{\otimes n} \to U(\mathfrak{g}_*)$ denote the multiplication map. For every permutation σ of the set $\{1, 2, \ldots, n\}$, let ϕ_σ denote the induced automorphism of $\mathfrak{g}_*^{\otimes n}$. The map $\frac{1}{n!} \sum_\sigma \psi \circ \phi_\sigma$ is invariant under precomposition with each of the maps ϕ_σ, and therefore factors as a composition

$$\mathfrak{g}_*^{\otimes n} \to \mathrm{Sym}^n(\mathfrak{g}_*) \xrightarrow{\Psi_n} U(\mathfrak{g}_*)^{\leq n} \subseteq U(\mathfrak{g}_*).$$

We observe that the composite map

$$\mathrm{Sym}^n(\mathfrak{g}_*) \xrightarrow{\Psi_n} U(\mathfrak{g}_*)^{\leq n} \to \mathrm{gr}^n\, U(\mathfrak{g}_*)$$

coincides with the isomorphism of Remark 2.1.7. It follows that the direct sum of the maps Ψ_n determines an isomorphism of chain complexes $\overline{\theta} : \mathrm{Sym}^*(\mathfrak{g}_*) \to U(\mathfrak{g}_*)$.

Definition 2.1.9. Let $f : \mathfrak{g}_* \to \mathfrak{g}'_*$ be a map of differential graded Lie algebras over k. We will say that f is a *quasi-isomorphism* if the underlying map of chain complexes is a quasi-isomorphism: that is, if F induces an isomorphism on homology.

Proposition 2.1.10. *Let k be a field of characteristic zero. Then the category $\mathrm{Lie}_k^{\mathrm{dg}}$ of differential graded Lie algebras over k has the structure of a left proper combinatorial model category, where:*

- (W) *A map of differential graded Lie algebras $f : \mathfrak{g}_* \to \mathfrak{g}'_*$ is a weak equivalence if and only if it is a quasi-isomorphism (Definition 2.1.9).*
- (F) *A map of differential graded Lie algebras $f : \mathfrak{g}_* \to \mathfrak{g}'_*$ is a fibration if and only if it is a fibration of chain complexes: that is, if and only if each of the induced maps $\mathfrak{g}_p \to \mathfrak{g}'_p$ is a surjective map of vector spaces over k.*
- (C) *A map of differential graded Lie algebras $f : \mathfrak{g}_* \to \mathfrak{g}'_*$ is a cofibration if and only if it has the left lifting property with respect to every map of differential graded Lie algebras which is simultaneously a fibration and a weak equivalence.*

Lemma 2.1.11. *Let $f : \mathfrak{g}_* \to \mathfrak{g}'_*$ be a map of differential graded Lie algebras over a field k of characteristic zero. The following conditions are equivalent:*

1. *The map f is a quasi-isomorphism.*
2. *The induced map $U(\mathfrak{g}_*) \to U(\mathfrak{g}'_*)$ is a quasi-isomorphism of differential graded algebras.*

Proof. We note that if $g : V_* \to W_*$ is any map of chain complexes of k-vector spaces, then g is a quasi-isomorphism if and only if g induces a quasi-isomorphism $\mathrm{Sym}^*(V_*) \to \mathrm{Sym}^*(W_*)$. The desired assertion now follows immediately from Remark 2.1.8. □

Proof of Proposition 2.1.10. The forgetful functor $\mathrm{Lie}_k^{\mathrm{dg}} \to \mathrm{Vect}_k^{\mathrm{dg}}$ has a left adjoint (the free Lie algebra functor), which we will denote by $\mathrm{Free} : \mathrm{Vect}_k^{\mathrm{dg}} \to \mathrm{Lie}_k^{\mathrm{dg}}$. For every integer n, let $E(n)_*$ denote the acyclic chain complex

$$\cdots \to 0 \to 0 \to k \simeq k \to 0 \to 0 \to \cdots$$

which is nontrivial only in degrees n and $(n-1)$, and let $\partial\, E(n)_*$ be the subcomplex of $E(n)_*$ which is nontrivial only in degree $(n-1)$. Let C_0 be the collection of morphisms in $\mathrm{Lie}_k^{\mathrm{dg}}$ of the form $\mathrm{Free}(\partial\, E(n)_*) \to \mathrm{Free}(E(n)_*)$, and let W be the collection of all quasi-isomorphisms in $\mathrm{Lie}_k^{\mathrm{dg}}$. We claim that the collection of morphisms C_0 and W satisfy the hypotheses of Proposition T.A.2.6.13:

1. The collection W of quasi-isomorphisms is perfect, in the sense of Definition T.A.2.6.10. This follows immediately from Corollary T.A.2.6.12, applied to the forgetful functor $\mathrm{Lie}_k^{\mathrm{dg}} \to \mathrm{Vect}_k^{\mathrm{dg}}$.
2. The collection of weak equivalences is stable under pushouts of morphisms in C_0. In other words, if $f : \mathfrak{g}_* \to \mathfrak{g}'_*$ is a quasi-isomorphism of differential graded Lie algebras over k and $x \in \mathfrak{g}_{n-1}$ is a cycle classifying a map $\mathrm{Free}(\partial\, E(n)_*) \to \mathfrak{g}_*$, we must show that the induced map
$$\mathfrak{g}_* \coprod_{\mathrm{Free}(\partial\, E(n)_*)} \mathrm{Free}(E(n)_*) \to \mathfrak{g}'_* \coprod_{\mathrm{Free}(\partial\, E(n)_*)} \mathrm{Free}(E(n)_*)$$
is also a quasi-isomorphism of differential graded Lie algebras. Let $A_* = U(\mathfrak{g}_*)$, let $A'_* = U(\mathfrak{g}'_*)$, and let $F : A_* \to A'_*$ be the map induced by f. We will abuse notation and identify x with its image in A_{n-1}. Using Lemma 2.1.11, we see that F is a quasi-isomorphism, and we are reduced to showing that F induces a quasi-isomorphism $B_* \to B'_*$, where B_* is the differential graded algebra obtained from A_* by adjoining a class y in degree n with $dy = x$, and B'_* is defined similarly. To prove this, we note that B_* admits an exhaustive filtration
$$A_* \simeq B_*^{\leq 0} \subseteq B_*^{\leq 1} \subseteq B_*^{\leq 2} \subseteq \cdots$$

where $B^{\leq m}$ is the subspace of B spanned by all expressions of the form $a_0 y a_1 y \cdots y a_k$, where $k \leq m$ and each a_i belongs to the image of A_* in B_*. Similarly, we have a filtration

$$A'_* \simeq B'^{\leq 0}_* \subseteq B'^{\leq 1}_* \subseteq B'^{\leq 2}_* \subseteq \cdots$$

of B'_*. Since the collection of quasi-isomorphisms is stable under filtered colimits, it will suffice to show that for each $m \geq 0$, the map of chain complexes $B^{\leq m}_* \to B'^{\leq m}_*$ is a quasi-isomorphism. The proof proceeds by induction on m, the case $m = 0$ being true by assumption. If $m > 0$, we have a diagram of short exact sequences of chain complexes

$$\begin{tikzcd}
0 \arrow[r] & B^{\leq m-1}_* \arrow[r] \arrow[d] & B^{\leq m}_* \arrow[r] \arrow[d] & B^{\leq n}_*/B^{\leq m-1}_* \arrow[r] \arrow[d, "\phi"] & 0 \\
0 \arrow[r] & B'^{\leq m-1}_* \arrow[r] & B'^{\leq m}_* \arrow[r] & B'^{\leq m}_*/B'^{\leq m-1}_* \arrow[r] & 0.
\end{tikzcd}$$

The inductive hypothesis implies that the left vertical map is a quasi-isomorphism. To complete the inductive step, it will suffice to show that ϕ is a quasi-isomorphism. For this, we observe that the construction $a_0 \otimes \cdots \otimes a_n \mapsto a_0 y a_1 y \cdots y a_m$ determines an isomorphism of chain complexes $A^{\otimes m+1}_* \to B^{\leq m}_*/B^{\leq m-1}_*$, and similarly we have an isomorphism $A'^{\otimes m+1}_* \to B'^{\leq m}_*/B'^{\leq m-1}_*$. Under these isomorphisms, ϕ corresponds to the map $A^{\otimes m+1}_* \to A'^{\otimes m+1}_*$ given by the $(m+1)$st tensor power of F, which is a quasi-isomorphism by assumption.

(3) Let $f : \mathfrak{g}_* \to \mathfrak{g}'_*$ be a map of differential graded Lie algebras which has the right lifting property with respect to every morphism in C_0. We claim that f is a quasi-isomorphism. To prove this, we must show that f induces an isomorphism $\theta_n : \mathrm{H}_n(\mathfrak{g}_*) \to \mathrm{H}_n(\mathfrak{g}'_*)$ for every integer n (here $\mathrm{H}_n(\mathfrak{h}_*)$ denotes the homology of the underlying chain complex of $\mathfrak{h}_*$). We first show that θ_n is surjective. Choose a class $\eta \in \mathrm{H}_n(\mathfrak{g}'_*)$, represented by a cycle $x \in \mathfrak{g}'_n$. Then x determines a map $u : \mathrm{Free}(E(n)_*) \to \mathfrak{g}'_*$ which vanishes on $\mathrm{Free}(\partial\, E(n)_*)$. It follows that $u = f \circ v$, where $v : \mathrm{Free}(E(n)_*) \to \mathfrak{g}_*$ is a map of differential graded Lie algebras which vanishes on $\mathrm{Free}(\partial\, E(n-1)_*)$. The map v is determined by a cycle $\overline{x} \in \mathfrak{g}_n$ which represents a homology class lifting η.

We now prove that θ_n is injective. Let $\eta \in \mathrm{H}_n(\mathfrak{g}_*)$ be a class whose image in $\mathrm{H}_n(\mathfrak{g}'_*)$ vanishes. Then η is represented by a cycle $x \in \mathfrak{g}_n$ such that $f(x) = dy$, for some $y \in \mathfrak{g}'_{n+1}$. Then y determines a map of differential graded Lie algebras $u : \mathrm{Free}(E(n+1)_*) \to \mathfrak{g}'$ such that $u|\,\mathrm{Free}(\partial\, E(n+1)_*)$ lifts to $\mathfrak{g}_*$. It follows that $u = f \circ v$, for some map of differential graded Lie algebras $\mathrm{Free}(E(n+1)_*) \to \mathfrak{g}_*$ such that $v|\,\mathrm{Free}(\partial\, E(n+1)_*)$ classifies x. It follows that x is a boundary, so that $\eta = 0$.

It follows from Proposition T.A.2.6.13 that $\mathrm{Lie}^{\mathrm{dg}}_k$ admits a left proper combinatorial model structure having W as the class of weak equivalences and C_0 as a class of generating cofibrations. To complete the proof, it will suffice to show that a morphism $u : \mathfrak{g}_* \to \mathfrak{g}'_*$ in $\mathrm{Lie}^{\mathrm{dg}}_k$ is a fibration if and only if it is degreewise surjective. Suppose first that u is a fibration. For each integer n, let $i_n : 0 \to \mathrm{Free}(E(n)_*)$ be the evident map of differential graded Lie algebras. Then i_n factors as a composition

$$0 \to 0 \coprod_{\mathrm{Free}(\partial\, E(n-1)_*)} \mathrm{Free}(E(n-1)_*) \simeq \mathrm{Free}(\partial\, E(n)_*) \to \mathrm{Free}(E(n)_*),$$

and is therefore a cofibration. The unit map $k \simeq U(0) \to U(\mathrm{Free}(E(n)_*)) \simeq \bigoplus_{m \geq 0} E(n)^{\otimes m}_*$ is a quasi-isomorphism (since $E(n)$ is acyclic and therefore each $E(n)^{\otimes m}_*$ is acyclic for $m > 0$). It follows that i_n is a trivial cofibration, so that u has the right lifting property with respect to i_n. Unwinding the definitions, we conclude that the map $\mathfrak{g}_n \to \mathfrak{g}'_n$ is surjective.

Now suppose that u is degreewise surjective; we wish to show that u is a fibration. Let S be the collection of all trivial cofibrations in $\mathrm{Lie}^{\mathrm{dg}}_k$ which have the left lifting property with respect to u. Let $f : \mathfrak{h}_* \to \mathfrak{h}''_*$ be a trivial cofibration in $\mathrm{Lie}_k^{\mathrm{dg}}$; we will prove that $f \in S$. Note that f contains each of the trivial cofibrations $i_n : 0 \to \mathrm{Free}(E(n)_*)$ above. Using the small object argument, we can factor f as a composition

$$\mathfrak{h}_* \xrightarrow{f'} \mathfrak{h}'_* \xrightarrow{f''} \mathfrak{h}''_*$$

where $f' \in S$ and f'' has the right lifting property with respect to each of the morphisms i_n: that is, f'' is degreewise surjective. Since f and f' are quasi-isomorphisms, we conclude that f'' is a quasi-isomorphism. It follows that f'' is a trivial fibration in the category of chain complexes and therefore a trivial fibration in the category $\mathrm{Lie}_k^{\mathrm{dg}}$. Since f is a cofibration, the lifting problem

$$\begin{tikzcd}
\mathfrak{h}_* \arrow[r, "f'"] \arrow[d, "f"] & \mathfrak{h}'_* \arrow[d, "f''"] \\
\mathfrak{g}''_* \arrow[r, "\mathrm{id}"] \arrow[ur, dashed] & \mathfrak{g}''_*
\end{tikzcd}$$

admits a solution. We conclude that f is a retract of f', and therefore also belongs to S. $\square$

Remark 2.1.12. The forgetful functor $\mathrm{Alg}_k^{\mathrm{dg}} \to \mathrm{Lie}_k^{\mathrm{dg}}$ of Example 2.1.6 preserves fibrations and weak equivalences, and is therefore a right Quillen functor. It follows that the universal enveloping algebra functor $U : \mathrm{Lie}_k^{\mathrm{dg}} \to \mathrm{Alg}_k^{\mathrm{dg}}$ is a left Quillen functor.

Proposition 2.1.13. *Let $\mathcal{J}$ be a small category such that $\mathrm{N}(\mathcal{J})$ is sifted. The forgetful functor*

$$G : \mathrm{Lie}_k^{\mathrm{dg}} \to \mathrm{Vect}_k^{\mathrm{dg}}$$

preserves $\mathcal{J}$-indexed homotopy colimits.

Proof. Let $G' : \mathrm{Alg}_k^{\mathrm{dg}} \to \mathrm{Vect}_k^{\mathrm{dg}}$ be the forgetful functor. It follows from Remark 2.1.8 that the functor G is a retract of $G' \circ U$. It will therefore suffice to show that $G' \circ U$ preserves $\mathcal{J}$-indexed homotopy colimits. The functor U is a left Quillen functor (Remark 2.1.12) and therefore preserves all homotopy colimits. We are therefore reduced to showing that G' preserves $\mathcal{J}$-indexed homotopy colimits, which is a special case of Lemma A.4.1.4.13. $\square$

Definition 2.1.14. Let k be a field of characteristic zero. We let Lie_k denote the underlying ∞-category of the model category $\mathrm{Lie}_k^{\mathrm{dg}}$. More precisely, Lie_k denotes an ∞-category equipped with a functor $u : \mathrm{N}(\mathrm{Lie}_k^{\mathrm{dg}}) \to \mathrm{Lie}_k$ having the following universal property: for every ∞-category $\mathcal{C}$, composition with u induces an equivalence from $\mathrm{Fun}(\mathrm{Lie}_k, \mathcal{C})$ to the full subcategory of $\mathrm{Fun}(\mathrm{N}(\mathrm{Lie}_k^{\mathrm{dg}}), \mathcal{C})$ spanned by those functors $F : \mathrm{Lie}_k^{\mathrm{dg}} \to \mathcal{C}$ which carry quasi-isomorphisms in $\mathrm{Lie}_k^{\mathrm{dg}}$ to equivalences in $\mathcal{C}$ (see Definition A.1.3.3.2 and Remark A.1.3.3.3). We will refer to Lie_k as the *∞-category of differential graded Lie algebras over k.*

Remark 2.1.15. Using Proposition A.7.1.1.15, we conclude that the underlying ∞-category of the model category $\mathrm{Vect}_k^{\mathrm{dg}}$ can be identified with the ∞-category $\mathrm{Mod}_k = \mathrm{Mod}_k(\mathrm{Sp})$ of k-module spectra. The forgetful functor $\mathrm{Lie}_k^{\mathrm{dg}} \to \mathrm{Vect}_k^{\mathrm{dg}}$ preserves quasi-isomorphisms, and therefore induces a forgetful functor $\mathrm{Lie}_k \to \mathrm{Mod}_k$.

Proposition 2.1.16. *Let k be a field of characteristic zero. Then the ∞-category Lie_k is presentable, and the forgetful functor $\theta : \mathrm{Lie}_k \to \mathrm{Mod}_k$ of Remark 2.1.15 preserves small sifted colimits.*

Proof. The first assertion follows from Proposition A.1.3.3.9. Using Propositions A.1.3.3.11, A.1.3.3.12, and 2.1.13, we conclude that θ preserves colimits indexed by small categories $\mathcal{J}$ such that $\mathrm{N}(\mathcal{J})$ is sifted. Since any filtered ∞-category $\mathcal{I}$ admits a left cofinal map $\mathrm{N}(A) \to \mathcal{I}$ where A is a filtered partially ordered set (Proposition T.5.3.1.16), we conclude that θ preserves small filtered colimits. Since θ also preserves geometric realizations of simplicial objects, it preserves all small sifted colimits (Corollary T.5.5.8.17). $\square$

Remark 2.1.17. The forgetful functor $\theta : \mathrm{Lie}_k \to \mathrm{Mod}_k$ is monadic: that is, θ admits a left adjoint $\mathrm{Free} : \mathrm{Mod}_k \to \mathrm{Lie}_k$, and induces an equivalence of Lie_k with $\mathrm{LMod}_T(\mathrm{Mod}_k)$, where T is the monad on Mod_k given by the composition $\theta \circ \mathrm{Free}$. This follows from Theorem A.6.2.0.6 and Proposition 2.1.16.

2.2 Homology and Cohomology of Lie Algebras

Let $\mathfrak{g}$ be a Lie algebra over a field k and let $U(\mathfrak{g})$ denote its universal enveloping algebra. We can regard k as a (left or right) module over $U(\mathfrak{g})$, with each element of $\mathfrak{g}$ acting trivially on k. The *homology and cohomology* groups of $\mathfrak{g}$ are defined by

$$\mathrm{H}_n(\mathfrak{g}) = \mathrm{Tor}_n^{U(\mathfrak{g})}(k,k) \qquad \mathrm{H}^n(\mathfrak{g}) = \mathrm{Ext}^n_{U(\mathfrak{g})}(k,k).$$

These groups can be described more explicitly as the homology groups of chain complexes $C_*(\mathfrak{g})$ and $C^*(\mathfrak{g})$, called the (homological and cohomological) *Chevalley-Eilenberg complexes* of $\mathfrak{g}$. In this section, we will review the definition of these chain complexes (in the more general setting of differential graded algebras) and establish some of their basic properties. These constructions will play an important role in the construction of the deformation theory $\mathfrak{D} : (\mathrm{CAlg}_k^{\mathrm{aug}})^{op} \to \mathrm{Lie}_k$ required for the proof of Theorem 2.0.2.

Suppose now that $\mathfrak{g}$ is a Lie algebra. In order to compute the groups $\mathrm{H}_*(\mathfrak{g})$ and $\mathrm{H}^*(\mathfrak{g})$, we would like to choose an explicit resolution of the ground field k as a (left) module over the universal enveoping algebra $U(\mathfrak{g})$. We can obtain such a resolution by taking the universal enveloping algebra of an acyclic differential graded Lie algebra which contains $\mathfrak{g}$.

Construction 2.2.1. Let $\mathfrak{g}_*$ be a differential graded Lie algebra over a field k. We define another differential graded Lie algebra $\mathrm{Cn}(\mathfrak{g})_*$ as follows:

(1) For each $n \in \mathbf{Z}$, the vector space $\mathrm{Cn}(\mathfrak{g})_*$ is given by $\mathfrak{g}_n \oplus \mathfrak{g}_{n-1}$. We will denote the elements of $\mathrm{Cn}(\mathfrak{g})_n$ by $x + \epsilon y$, where $x \in \mathfrak{g}_n$ and $y \in \mathfrak{g}_{n-1}$.

(2) The differential on $\mathrm{Cn}(\mathfrak{g})_*$ is given by the formula $d(x + \epsilon y) = dx + y - \epsilon dy$.

(3) The Lie bracket on $\mathrm{Cn}(\mathfrak{g})_*$ is given by $[x + \epsilon y, x' + \epsilon y'] = [x, x'] + \epsilon([y, x'] + (-1)^p [x, y'])$, where $x \in \mathfrak{g}_p$.

We will refer to $\mathrm{Cn}(\mathfrak{g})_*$ as the *cone* on $\mathfrak{g}_*$.

Remark 2.2.2. Let $\mathfrak{g}_*$ be a differential graded Lie algebra over a field k. Then the underlying chain complex $\mathrm{Cn}(\mathfrak{g})_*$ can be identified with the mapping cone for the identity $\mathrm{id} : \mathfrak{g}_* \to \mathfrak{g}_*$. It follows that $\mathrm{Cn}(\mathfrak{g})_*$ is a contractible chain complex. In particular, the map $0 \to \mathrm{Cn}(\mathfrak{g})_*$ is a quasi-isomorphism of differential graded Lie algebras.

Construction 2.2.3. Let $\mathfrak{g}_*$ be a differential graded Lie algebra over a field k. The zero map $\mathfrak{g}_* \to 0$ induces a map of differential graded algebras $U(\mathfrak{g})_*) \to U(0) \simeq k$. There is an evident map of differential graded Lie algebras $\mathfrak{g}_* \to \mathrm{Cn}(\mathfrak{g})_*$. We let $C_*(\mathfrak{g}_*)$ denote the chain complex given by the tensor product $U(\mathrm{Cn}(\mathfrak{g})_*) \otimes_{U(\mathfrak{g}_*)} k$. We will refer to $C_*(\mathfrak{g}_*)$ as the *homological Chevalley-Eilenberg complex of* $\mathfrak{g}_*$.

Remark 2.2.4. Let $\mathfrak{g}_*$ be a differential graded Lie algebra over a field k, and regard the shifted chain complex $\mathfrak{g}_*[1]$ as a graded Lie algebra with a vanishing Lie bracket. There is an evident map of graded Lie algebras (without differential) $\mathfrak{g}_*[1] \to \mathrm{Cn}(\mathfrak{g})_*$. This map induces a map of graded vector spaces $\mathrm{Sym}^*(\mathfrak{g}_*[1]) \simeq U(\mathfrak{g}_*[1]) \to U(\mathrm{Cn}(\mathfrak{g})_*)$. Using the Poincare-Birkhoff-Witt theorem, we obtain an isomorphism of graded right $U(\mathfrak{g}_*)$-modules

$$U(\mathrm{Cn}(\mathfrak{g})_*) \simeq \mathrm{Sym}^*(\mathfrak{g}_*[1]) \otimes_k U(\mathfrak{g}_*),$$

hence an isomorphism of graded vector spaces

$$\phi : \mathrm{Sym}^*(\mathfrak{g}_*[1]) \to C_*(\mathfrak{g}_*).$$

We will often identify $C_*(\mathfrak{g}_*)$ with the symmetric algebra $\mathrm{Sym}^*(\mathfrak{g}_*[1])$ using the isomorphism ϕ. Note that ϕ is not an isomorphism of differential graded vector spaces. Unwinding the definitions, we see that the differential on $C_*(\mathfrak{g}_*)$ is given by the formula

$$\begin{aligned} D(x_1 \dots x_n) &= \sum_{1 \leq i \leq n} (-1)^{p_1 + \cdots + p_{i-1}} x_1 \dots x_{i-1} dx_i x_{i+1} \dots x_n + \\ & \sum_{1 \leq i < j \leq n} (-1)^{p_i(p_{i+1} + \cdots + p_{j-1})} x_1 \dots x_{i-1} x_{i+1} \dots x_{j-1} [x_i, x_j] x_{j+1} \dots x_n. \end{aligned}$$

Remark 2.2.5. Let $\mathfrak{g}_*$ be a differential graded Lie algebra over a field k. The filtration of $\mathrm{Sym}^*(\mathfrak{g}_*)$ by the subsets $\mathrm{Sym}^{\leq n}(\mathfrak{g}_*) \simeq \bigoplus_{i \leq n} \mathrm{Sym}^i \mathfrak{g}_*$ determines a filtration

$$k \simeq C_*^{\leq 0}(\mathfrak{g}_*) \hookrightarrow C_*^{\leq 1}(\mathfrak{g}_*) \hookrightarrow C_*^{\leq 2}(\mathfrak{g}_*) \hookrightarrow \cdots$$

Using the formula for the differential on $C_*(\mathfrak{g}_*)$ given in Remark 2.2.4, we deduce the existence of canonical isomorphisms

$$C_*^{\leq n}(\mathfrak{g}_*)/C_*^{\leq n-1}(\mathfrak{g}_*) \simeq \mathrm{Sym}^n \mathfrak{g}_*$$

in the category of differential graded vector spaces over k.

Proposition 2.2.6. *Let $f : \mathfrak{g}_* \to \mathfrak{g}'_*$ be a quasi-isomorphism between differential graded Lie algebras over a field k of characteristic zero. Then the induced map $C_*(\mathfrak{g}_*) \to C_*(\mathfrak{g}'_*)$ is a quasi-isomorphism of chain complexes.*

Proof. Since the collection of quasi-isomorphisms is closed under filtered colimits, it will suffice to show that the induced map $\theta_n : C_*^{\leq n}(\mathfrak{g}_*) \to C_*^{\leq n}(\mathfrak{g}'_*)$ is a quasi-isomorphism for each $n \geq 0$. We proceed by induction on n. When $n = 0$, the map θ is an isomorphism and there is nothing to prove. Assume therefore that $n > 0$, so that we have a commutative diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & C_*^{\leq n-1}(\mathfrak{g}_*) \arrow[r] \arrow[d, "\theta_{n-1}"] & C_*^{\leq n}(\mathfrak{g}_*) \arrow[r] \arrow[d, "\theta_n"] & \mathrm{Sym}^n(\mathfrak{g}_*[1]) \arrow[r] \arrow[d, "\phi"] & 0 \\
0 \arrow[r] & C_*^{\leq n-1}(\mathfrak{g}'_*) \arrow[r] & C_*^{\leq n}(\mathfrak{g}'_*) \arrow[r] & \mathrm{Sym}^n(\mathfrak{g}'_*[1]) \arrow[r] & 0
\end{tikzcd}$$

Using the inductive hypothesis, we are reduced to showing that the map ϕ is a quasi-isomorphism. Since k is a field of characteristic zero, the map ϕ is a retract of the map $\mathfrak{g}_*^{\otimes n}[n] \to \mathfrak{g}_*'^{\otimes n}[n]$, which is a quasi-isomorphism by virtue of our assumption that f is a quasi-isomorphism. □

If $\mathfrak{g}_*$ is a differential graded Lie algebra, we will refer to the homology groups of the chain complex $C_*(\mathfrak{g}_*)$ as the *Lie algebra homology groups* of $\mathfrak{g}_*$. Our next goal is to show that if $\mathfrak{g}_*$ is free, then the homology of $\mathfrak{g}_*$ is easy to describe.

Proposition 2.2.7. *Let V_* be a differential graded vector spaces over a field k of characteristic zero and let $\mathfrak{g}_*$ be free differential graded Lie algebra generated by V_*. Then the map*

$$\xi : k \oplus V_*[1] \to k \oplus \mathfrak{g}_*[1] \simeq C_*^{\leq 1}(\mathfrak{g}_*) \hookrightarrow C_*(\mathfrak{g}_*)$$

is a quasi-isomorphism of chain complexes over k.

To prove Proposition 2.2.7, we will need a general observations about differential graded algebras and their modules.

Lemma 2.2.8. *Let A_* be a differential graded algebra over a field k, and let $f : M_* \to N_*$ be a map of differential graded right modules over A_*. Assume that:*

(1) *The differential graded module M_* can be written as a union of submodules*

$$0 = M(0)_* \subseteq M(1)_* \subseteq M(2)_* \subseteq \cdots$$

where each successive quotient $M(n)_*/M(n-1)_*$ is isomorphic (as a differential graded A_*-module) to a free differential graded module of the form $\bigoplus_\alpha A_*[e_\alpha]$.

(2) *The chain complex N_* is acyclic.*

Then the map f is nullhomotopic. That is, there exists a map of graded A_*-modules $h : M_* \to N_{*+1}$ satisfying $dh + hd = f$.

Proof. We construct a compatible family of nullhomotopies $h(n) : M(n)_* \to N_{*+1}$ for the maps $f(n) = f|M(n)_*$. When $n = 0$, such a nullhomotopy exists and is unique (since $M(0)_* \simeq 0$). Assume therefore that $n > 0$ and that $h(n-1)$ has been constructed. Condition (1) guarantees that $M(n)_*/M(n-1)_*$ is freely generated (as an A_*-module) by generators $\overline{x}_\alpha \in (M(n)/M(n-1))_{e_\alpha}$. Choose $x_\alpha \in M(n)_{e_\alpha}$ representing $\overline{x}_\alpha$. We compute

$$d(f(x_\alpha) - h(n-1)dx_\alpha) = f(dx_\alpha) - d(h(n-1)dx_\alpha) = h(n-1)d^2 x_\alpha = 0.$$

Since N_* is acyclic, we can choose $y_\alpha \in N_{e_\alpha+1}$ with $dy_\alpha = f(x_\alpha) - h(n-1)dx_\alpha$. We now define $h(n)$ to be the unique map of graded A_*-modules from $M(n)_*$ to N_{*+1} which extends $h(n-1)$ and carries x_α to y_α; it is easy to see that $h(n)$ has the desired properties. □

Lemma 2.2.9. *Let A_* be a differential graded algebra over a field k, and let M_* be a chain complex of differential graded right modules over A_*. Assume that M_* is acyclic and satisfies condition* (1) *of Lemma 2.2.8. Then, for any differential graded left A_*-module N_*, the tensor product $M_* \otimes_{A_*} N_*$ is acyclic.*

Proof. It follows from Lemma 2.2.8 that identity map $\mathrm{id} : M_* \to M_*$ is chain homotopic to zero: that is, there exists a map $h : M_* \to M_{*+1}$ such that $dh + hd = \mathrm{id}$. Then h determines a contracting homotopy for $M_* \otimes_{A_*} N_*$, so that $M_* \otimes_{A_*} N_*$ is also acyclic. □

Proof of Proposition 2.2.7. Note that the universal enveloping algebra $U(\mathfrak{g}_*)$ can be identified with the tensor algebra $T(V_*) \simeq \bigoplus_{n\geq 0} V_*^{\otimes n}$. Let $M_* \subseteq U(\mathrm{Cn}(\mathfrak{g})_*)$ be the right $T(V_*)$-submodule generated by $k \oplus V_*[1]$. Unwinding the definitions, we see that M_* is isomorphic (as a chain complex) to the direct sum $k \oplus M'_*$, where M'_* is isomorphic to mapping cone of the identity map from $\bigoplus_{n\geq 1} V_*^{\otimes n}$ to itself. It follows that the inclusion $k \hookrightarrow M_*$ is a quasi-isomorphism. The composite inclusion $k \hookrightarrow M_* \to U(\mathrm{Cn}(\mathfrak{g})_*)$ is given by applying the universal enveloping algebra functor U to the inclusion of differential graded Lie algebras $0 \to \mathrm{Cn}(\mathfrak{g})_*$, and is therefore a quasi-isomorphism by Remark 2.2.2 and Lemma 2.1.11. It follows that the inclusion $M_* \subseteq U(\mathrm{Cn}(\mathfrak{g})_*)$ is a quasi-isomorphism, so that the quotient $Q_* = U(\mathrm{Cn}(\mathfrak{g})_*)/M_*$ is acyclic. It is not difficult to see that Q_* satisfies hypothesis (1) of Lemma 2.2.8; in particular, Q_* is free as a graded A_*-module. It follows that we have an exact sequence of chain complexes

$$0 \to M_* \otimes_{T(V_*)} k \xrightarrow{\theta} U(\mathrm{Cn}(\mathfrak{g})_*) \otimes_{T(V_*)} k \to Q_* \otimes_{T(V_*)} k \to 0.$$

Lemma 2.2.9 guarantees that $Q_* \otimes_{T(V_*)} k$ is acyclic, so that θ determines a quasi-isomorphism $k \oplus V_*[1] \to C_*(\mathfrak{g}_*)$. □

Notation 2.2.10. It follows from Proposition 2.2.6 that the Chevalley-Eilenberg construction $C_* : \mathrm{Lie}_k^{\mathrm{dg}} \to \mathrm{Vect}_k^{\mathrm{dg}}$ induces a functor of ∞-categories $\mathrm{Lie}_k \to \mathrm{Mod}_k$, which we will also denote by C_*. Note that C_* carries the initial object $0 \in \mathrm{Lie}_k$ to $C_*(0) \simeq k$, and therefore induces a functor $\mathrm{Lie}_k \to (\mathrm{Mod}_k)_{k/}$. We will abuse notation by denoting this functor also by C_*.

Remark 2.2.11. Let $\mathfrak{g}_*$ be a differential graded Lie algebra over a field k of characteristic zero. Then $U(\mathrm{Cn}(\mathfrak{g})_*)$ can be regarded as a cofibrant replacement for k in the model category of differential graded right modules over $U(\mathfrak{g}_*)$. The tensor product functor $M_* \mapsto M_* \otimes_{U(\mathfrak{g}_*)} k$ is a left Quillen functor. It follows that $C_*(\mathfrak{g}_*)$ is an explicit model for the left derived tensor product $k \otimes^L_{U(\mathfrak{g}_*)} k$. Equivalently, the image of $C_*(\mathfrak{g}_*)$ in Mod_k can be identified with the ∞-categorical relative tensor product $k \otimes_A k$, where $A \in \mathrm{Alg}_k$ is the image of $U(\mathfrak{g}_*)$ under the functor $\mathrm{N}(\mathrm{Alg}_k^{\mathrm{dg}}) \to \mathrm{Alg}_k$.

Proposition 2.2.12. *Let k be a field of characteristic zero. Then the functor of ∞-categories $C_* : \mathrm{Lie}_k \to (\mathrm{Mod}_k)_{k/}$ preserves small colimits.*

Proof. In view of Corollary T.4.2.3.11 and Lemma A.1.3.2.9, it will suffice to show that C_* preserves finite coproducts and small sifted colimits. We begin by showing that C_* preseres small sifted colimits. In view of Lemma T.4.4.2.8 and Proposition T.4.3.1.5, it will suffice to show that the composite functor $\mathrm{Lie}_k \to (\mathrm{Mod}_k)_{k/} \to \mathrm{Mod}_k$ preserves small sifted colimits. The proof of Proposition 2.2.6 shows that for each $n \geq 0$, the functor $C_*^{\leq n}$ preserves quasi-isomorphisms and therefore induces a functor of ∞-categories $\mathrm{Lie}_k \to \mathrm{Mod}_k$. Since the collection of quasi-isomorphisms in $\mathrm{Vect}_k^{\mathrm{dg}}$ is closed under filtered colimits, every colimit diagram in $\mathrm{Vect}_k^{\mathrm{dg}}$ indexed by a filtered category determines a homotopy colimit diagram in $\mathrm{Vect}_k^{\mathrm{dg}}$ and therefore a colimit diagram in Mod_k (Proposition A.1.3.3.11). It follows that the functor $C_* : \mathrm{Lie}_k \to \mathrm{Mod}_k$ is a colimit of the functors $C_*^{\leq n} : \mathrm{Lie}_k \to \mathrm{Mod}_k$. Using Proposition T.5.5.2.3, we are reduced to proving that each of the functors $C_*^{\leq n}$ preserves small sifted colimits. We proceed by induction on n, the case $n < 0$ being trivial. Since the field k has characteristic zero, the construction $V_* \mapsto \mathrm{Sym}^n V_*$ preserves quasi-isomorphisms and therefore induces a functor $\mathrm{Sym}^n : \mathrm{Mod}_k \to \mathrm{Mod}_k$. Let $\theta : \mathrm{Lie}_k \to \mathrm{Mod}_k$ be the forgetful functor. Using Remark 2.2.5 and Corollary A.1.3.1.11, we obtain a fiber sequence of functors

$$C_*^{\leq n-1} \to C_*^{\leq n} \to \mathrm{Sym}^n \circ \theta[1]$$

from Lie_k to Mod_k. Since $C_*^{\leq n-1}$ preserves sifted colimits by the inductive hypothesis and $\theta[1]$ preserves sifted colimits by Proposition 2.1.16, it will suffice to show that the functor Sym^n preserves sifted colimits. Since the characteristic of k is zero, the functor Sym^n is a retract of the functor $V_* \mapsto V_*^{\otimes n}$, which evidently preserves sifted colimits.

We now prove that $C_* : \mathrm{Lie}_k \to (\mathrm{Mod}_k)_{k/}$ preserves finite coproducts. Since C_* preserves initial objects by construction, it will suffice to show that C_* preserves pairwise coproducts. That is, we must show that for every pair of differential graded Lie algebras $\mathfrak{g}_*$ and $\mathfrak{g}'_*$ having a coproduct $\mathfrak{g}''_*$ in Lie_k, the diagram σ :

$$\begin{tikzcd}
k \arrow[r] \arrow[d] & C_*(\mathfrak{g}_*) \arrow[d] \\
C_*(\mathfrak{g}'_*) \arrow[r] & C_*(\mathfrak{g}''_*)
\end{tikzcd}$$

is a pushout square in Mod_k.

Let $\mathrm{Free} : \mathrm{Mod}_k \to \mathrm{Lie}_k$ be a left adjoint to the forgetful functor. Using Proposition A.6.2.2.11 and Proposition 2.1.16, we deduce that $\mathfrak{g}_*$ can be obtained as the geometric realization of a simplicial object $(\mathfrak{g}_*)_\bullet$ of Lie_k, where each $(\mathfrak{g}_*)_n$ lies in the essential image of Free. Similarly, we can write $\mathfrak{g}'_*$ as the geometric realization of a simplicial object $(\mathfrak{g}'_*)_\bullet$. Then $(\mathfrak{g}''_*)_\bullet$ is the geometric realization of a simplicial object $(\mathfrak{g}''_*)_\bullet$ of Lie_k, given by $[n] \mapsto (\mathfrak{g}_*)_n \coprod (\mathfrak{g}'_*)_n$. Since the functor C_* commutes with geometric realization of simplicial objects, it will suffice to show that the diagram

$$\begin{tikzcd}
k \arrow[r] \arrow[d] & C_*((\mathfrak{g}_*)_n) \arrow[d] \\
C_*((\mathfrak{g}'_*)_n) \arrow[r] & C_*((\mathfrak{g}''_*)_n)
\end{tikzcd}$$

is a pushout square in Mod_k, for each $n \geq 0$. We may therefore reduce to the case where $\mathfrak{g}_* \simeq \mathrm{Free}(V_*)$, $\mathfrak{g}'_* \simeq \mathrm{Free}(V'_*)$ for some objects $V_*, V'_* \in \mathrm{Mod}_k$. Then $\mathfrak{g}''_* \simeq \mathrm{Free}(V_* \oplus V'_*)$. Using Proposition 2.2.7, we can identify σ with the diagram

$$\begin{tikzcd}
k \arrow[r] \arrow[d] & k \oplus V_*[1] \arrow[d] \\
k \oplus V'_*[1] \arrow[r] & k \oplus V_*[1] \oplus V'_*[1],
\end{tikzcd}$$

which is evidently a pushout square in Mod_k. $\square$

We now turn our attention to the cohomology of (differential graded) Lie algebras.

Construction 2.2.13. Let $\mathfrak{g}_*$ be a differential graded Lie algebra over k. We let $C^*(\mathfrak{g}_*)$ denote the linear dual of the chain complex dual to $C_*(\mathfrak{g}_*)$. We will refer to $C^*(\mathfrak{g}_*)$ as the *cohomological Chevalley-Eilenberg complex* of $\mathfrak{g}_*$. We will identify elements $\lambda \in C^n(\mathfrak{g}_*)$ with the dual space of the degree n part of the graded vector space $\mathrm{Sym}^*(\mathfrak{g}_*[1])$.

There is a natural multiplication on $C^*(\mathfrak{g}_*)$, which carries $\lambda \in C^p(\mathfrak{g}_*)$ and $\mu \in C^q(\mathfrak{g}_*)$ to the element $\lambda\mu \in C^{p+q}(\mathfrak{g}_*)$ characterized by the formula

$$(\lambda\mu)(x_1 \ldots x_n) = \sum_{S,S'} \epsilon(S,S')\lambda(x_{i_1} \ldots x_{i_m})\mu(x_{j_1} \ldots x_{j_{n-m}}).$$

Here $x_i \in \mathfrak{g}_{r_i}$ denotes a sequence of homogeneous elements of $\mathfrak{g}_*$, the sum is taken over all disjoint sets $S = \{i_1 < \ldots < i_m\}$ and $S' = \{j_1 < \ldots < j_{n-m}\}$ range with $S \cup S' = \{1,\ldots,n\}$ and $r_{i_1} + \cdots + r_{i_m} = p$, and $\epsilon(S,S') = \prod_{i\in S', j \in S, i<j}(-1)^{r_i r_j}$. With this multiplication, $C^*(\mathfrak{g}_*)$ has the structure of a commutative differential graded algebra.

Remark 2.2.14. Let $\mathfrak{g}_*$ be a differential graded Lie algebra over a field k of characteristic zero. Unwinding the definitions, we can identify $C^*(\mathfrak{g}_*)$ with the chain complex of right $U(\mathfrak{g}_*)$-linear maps from $U(\mathrm{Cn}(\mathfrak{g})_*)$ into k. Arguing as in Remark 2.2.11, we see that $C^*(\mathfrak{g}_*)$ is a model for the right derived mapping complex of right $U(\mathfrak{g}_*)$-module maps from k to itself.

Remark 2.2.15. Let k be a field of characteristic zero, let V_* be a chain complex of vector spaces over k, and let $\mathfrak{g}_*$ be the free differential graded Lie algebra generated by V_*. The quasi-isomorphism $k \oplus V_*[1] \to C_*(\mathfrak{g}_*)$ of Proposition 2.2.7 induces a quasi-isomorphism of chain complexes

$$C^*(\mathfrak{g}_*) \to k \oplus V_*^\vee[-1],$$

where $V_*^\vee$ denote the dual of the chain complex V_*. In fact, this map is a quasi-isomorphism of commutative differential graded algebras (where we regard $k \oplus V_*^\vee[-1]$ as a trivial square-zero extension of k).

Notation 2.2.16. Let k be a field of characteristic zero. It follows from Proposition 2.2.6 that the construction $\mathfrak{g}_* \mapsto C^*(\mathfrak{g}_*)$ carries quasi-isomorphisms of differential graded Lie algebras to quasi-isomorphisms of commutative differential graded algebras. Consequently, we obtain a functor between ∞-categories $\mathrm{Lie}_k \to \mathrm{CAlg}_k^{op}$, which we will also denote by C^*.

Note that the functor C^* carries the initial object $0 \in \mathrm{Lie}_k$ to the final object $k \in \mathrm{CAlg}_k^{op}$. We therefore obtain a functor

$$\mathrm{Lie}_k \to (\mathrm{CAlg}_k^{op})_{k/} \simeq (\mathrm{CAlg}_k^{\mathrm{aug}})^{op},$$

where $\mathrm{CAlg}_k^{\mathrm{aug}} = (\mathrm{CAlg}_k)_{/k}$ denotes the ∞-category of augmented $\mathbb{E}_\infty$-algebras over k. We will abuse notation by denoting this functor also by C^*.

Proposition 2.2.17. *Let k be a field of characteristic zero. Then the functor $C^* : \mathrm{Lie}_k \to (\mathrm{CAlg}_k^{\mathrm{aug}})^{op}$ preserves small colimits.*

Proof. Using Corollary A.3.2.2.5, we are reduced to proving that the composite functor

$$\mathrm{Lie}_k \xrightarrow{C^*} (\mathrm{CAlg}_k^{\mathrm{aug}})^{op} \longrightarrow (\mathrm{Mod}_k^{op})_{k/}$$

preserves small colimits. We note that this composition can be identified with the functor

$$\mathrm{Lie}_k \xrightarrow{C_*} (\mathrm{Mod}_k)_{k/} \xrightarrow{D} (\mathrm{Mod}_k^{op})_{k/},$$

where D is induced by the k-linear duality functor $V_* \mapsto V_*^\vee$ from $\mathrm{Vect}_k^{\mathrm{dg}}$ to itself. According to Proposition 2.2.12, it will suffice to show that D preserves small colimits. Using Propositions A.1.3.3.10, A.1.3.3.11, and A.1.3.3.12, we are reduced to the problem of showing that the functor $V_* \mapsto V_*^\vee$ carries homotopy colimits in $\mathrm{Vect}_k^{\mathrm{dg}}$ to homotopy limits in $\mathrm{Vect}_k^{\mathrm{dg}}$, which is obvious. □

2.3 Koszul Duality for Differential Graded Lie Algebras

Let k be a field of characteristic zero, and let $C^* : \mathrm{Lie}_k \to (\mathrm{CAlg}_k^{\mathrm{aug}})^{op}$ be the functor constructed in Notation 2.2.16. Proposition 2.2.17 implies that C^* preserves small colimits. Since the ∞-category Lie_k is presentable (Proposition 2.1.16), Corollary T.5.5.2.9 (and Remark T.5.5.2.10) imply that C^* admits a right adjoint $\mathfrak{D} : (\mathrm{CAlg}_k^{\mathrm{aug}})^{op} \to \mathrm{Lie}_k$. We will refer to the functor $\mathfrak{D}$ as *Koszul duality*. The main goal of this section is to prove the following result:

Theorem 2.3.1. *Let k be a field of characteristic zero, and let $(\mathrm{CAlg}_k^{\mathrm{aug}}, \{E\})$ be the deformation context of Example 1.1.4. Then the Koszul duality functor $\mathfrak{D} : (\mathrm{CAlg}_k^{\mathrm{aug}})^{op} \to \mathrm{Lie}_k$ is a deformation theory (see Definition 1.3.9).*

We will then deduce Theorem 2.0.2 by combining Theorems 2.3.1 and 1.3.12.

Remark 2.3.2. Let us temporarily distinguish in notation between the cohomological Chevalley-Eilenberg functor

$$C^* : \mathrm{Lie}_k^{\mathrm{dg}} \to (\mathrm{CAlg}_k^{\mathrm{dg}})^{op}_{/k}$$

of Construction 2.2.13 and the induced functor of ∞-categories $\mathrm{Lie}_k \to (\mathrm{CAlg}_k^{\mathrm{aug}})^{op}$, denoting the latter functor by F. We saw above that F admits a right adjoint, the Koszul duality functor $\mathfrak{D} : (\mathrm{CAlg}_k^{\mathrm{aug}})^{op} \to \mathrm{Lie}_k$. In particular, C^* determines a functor from the homotopy category of $\mathrm{Lie}_k^{\mathrm{dg}}$ to the homotopy category of $(\mathrm{CAlg}_k^{\mathrm{dg}})^{op}_{/k}$ which admits a right adjoint. However, the functor C^* itself does not admit a right adjoint; in particular, it is not a left Quillen functor. Consequently, it is not so easy to describe the functor $\mathfrak{D}$ using the formalism of differential graded Lie algebras. To obtain a more explicit construction of $\mathfrak{D}$, it is convenient to work in the setting of L_∞-algebras. Since we will not need this construction, we do not describe it here.

Remark 2.3.3. We will often abuse notation by identifying the Koszul duality functor $\mathfrak{D} : (\mathrm{CAlg}_k^{\mathrm{aug}})^{op} \to \mathrm{Lie}_k$ with the induced functor between opposite ∞-categories $\mathrm{CAlg}_k^{\mathrm{aug}} \to \mathrm{Lie}_k^{op}$.

The first step in our proof of Theorem 2.3.1 is to show that the Koszul duality functor $\mathfrak{D} : (\mathrm{CAlg}_k^{\mathrm{aug}})^{op} \to \mathrm{Lie}_k$ is a weak deformation theory: that is, it satisfies axioms $(D1)$, $(D2)$, and $(D3)$ of Definition 1.3.1. Axioms $(D1)$ and $(D2)$ are easy: we have already seen that Lie_k is presentable (Proposition 2.1.16), and the functor $\mathfrak{D}$ admits a left adjoint by construction. To verify $(D3)$, we will prove the following:

Proposition 2.3.4. *Let k be a field of characteristic zero and let $\mathfrak{g}_*$ be a differential graded Lie algebra over k. We will say that $\mathfrak{g}_*$ is good if it is cofibrant (with respect to the model structure on $\mathrm{Lie}_k^{\mathrm{dg}}$ described in Proposition 2.1.10) and there exists a graded vector subspace $V_* \subseteq \mathfrak{g}_*$ satisfying the following conditions:*

- (i) *For every integer n, the vector space V_n is finite dimensional.*
- (ii) *For every nonnegative integer n, the vector space V_n is trivial.*
- (iii) *The graded vector space V_* freely generates $\mathfrak{g}_*$ as a graded Lie algebra.*

Let $\mathcal{C}$ be the full subcategory of Lie_k spanned by those objects which can be represented by good objects of $\mathrm{Lie}_k^{\mathrm{dg}}$. Then $\mathcal{C}$ satisfies conditions (a), (b), (c), and (d) of Definition 1.3.1.

The main ingredient in the proof of Proposition 2.3.4 is the following lemma, whose proof we will defer until the end of this section.

Lemma 2.3.5. *Let $\mathfrak{g}_*$ be a differential graded Lie algebra over a field k of characteristic zero. Assume that:*

- (a) *For every integer n, the vector space $\mathfrak{g}_n$ is finite dimensional.*
- (b) *The vector space $\mathfrak{g}_n$ is trivial for $n \geq 0$.*

Then the unit map $u : \mathfrak{g}_* \to \mathfrak{D}C^*(\mathfrak{g})$ is an equivalence in Lie_k.

Proof of Proposition 2.3.4. We verify each condition in turn:

(*a*) Let $\mathfrak{g}_* \in \mathcal{C}$; we wish to prove that the unit map $\mathfrak{g}_* \to \mathfrak{D}C^*(\mathfrak{g})$ is an equivalence in Lie_k. We may assume without loss of generality that $\mathfrak{g}_*$ is good, so there is a graded subspace $V_* \subseteq \mathfrak{g}_*$ satisfying conditions (*i*), (*ii*), and (*iii*). As a graded vector space, $\mathfrak{g}_*$ is isomorphic to a direct summand of the augmentation ideal in $U(\mathfrak{g}) \simeq \bigoplus_{n\geq 0} V_*^{\otimes n}$. It follows that each $\mathfrak{g}_n$ is finite dimensional, and that $\mathfrak{g}_n \simeq 0$ for $n \geq 0$. The desired result now follows from Lemma 2.3.5.

(*b*) The initial object $0 \in \mathrm{Lie}_k$ obviously belongs to $\mathcal{C}$.

(*c*) We must show that for each $n \geq 0$, the square-zero algebra $k \oplus k[n] \in \mathrm{CAlg}_k^{\mathrm{aug}}$ is equivalent to $C^*(\mathfrak{g})$ for some object $\mathfrak{g}_* \in \mathcal{C}$. In fact, we can take $\mathfrak{g}_*$ to be the differential graded Lie algebra freely generated by the complex $k[-n-1]$ (see Remark 2.2.15).

(*d*) Suppose that $n \leq -2$ and that we are given a pushout diagram

$$\begin{tikzcd}
\mathrm{Free}(k[n]) \arrow[r, "\alpha"] \arrow[d, "v"] & \mathfrak{g}_* \arrow[d] \\
0 \arrow[r] & \mathfrak{g}'_*
\end{tikzcd}$$

in the ∞-category Lie_k. Here $\mathrm{Free} : \mathrm{Mod}_k \to \mathrm{Lie}_k$ denotes the left adjoint to the forgetful functor. We wish to show that if $\mathfrak{g}_* \in \mathcal{C}$, then $\mathfrak{g}'_* \in \mathcal{C}$. We may assume without loss of generality that $\mathfrak{g}_*$ is good. Since $\mathrm{Free}(k[n])$ is a cofibrant object of $\mathrm{Lie}_k^{\mathrm{dg}}$ and $\mathfrak{g}_*$ is fibrant, we can assume that α is given by a morphism $\mathrm{Free}(k[n]) \to \mathfrak{g}_*$ in the category $\mathrm{Lie}_k^{\mathrm{dg}}$ (determined by a cycle $x \in \mathfrak{g}_n$). The morphism v in $\mathrm{Lie}\,k$ is represented by the cofibration of differential graded Lie algebras $j : \mathrm{Free}(\partial\, E(n+1)_*) \hookrightarrow \mathrm{Free}(E(n+1)_*)$ (see the proof of Proposition 2.1.10). Form a pushout diagram σ :

$$\begin{tikzcd}
\mathrm{Free}(\partial\, E(n)_*) \arrow[r] \arrow[d, "j"] & \mathfrak{g}_* \arrow[d, "j'"] \\
\mathrm{Free}(E(n+1)_*) \arrow[r] & \mathfrak{h}_*.
\end{tikzcd}$$

Since j is a cofibration and $\mathfrak{g}_*$ is cofibrant, σ is a homotopy pushout diagram in $\mathrm{Lie}_k^{\mathrm{dg}}$, so that $\mathfrak{h}_*$ and $\mathfrak{g}'_*$ are equivalent in Lie_k (Proposition A.1.3.3.11). It will therefore suffice to show that the object $\mathfrak{h}_* \in \mathrm{Lie}_k^{\mathrm{dg}}$ is good.

The differential graded Lie algebra $\mathfrak{h}_*$ is cofibrant by construction. Let $V_* \subseteq \mathfrak{g}$ be a subspace satisfying conditions (*i*), (*ii*), and (*iii*), and let $y \in \mathfrak{h}_{n+1}$ be the image of a generator of $E(n+1)_{n+1}$. Let V'_* be the graded subspace of $\mathfrak{h}_*$ generated by V_* and y. It is trivial to verify that V'_* satisfies conditions (*i*), (*ii*), and (*iii*).

□

Proof of Theorem 2.3.1. Proposition 2.3.4 shows that the functor $\mathfrak{D} : (\mathrm{CAlg}_k^{\mathrm{aug}})^{op} \to \mathrm{Lie}_k$ is a weak deformation theory. We will show that it satisfies axiom (*D*4) of Definition 1.3.9. Let $E \in \mathrm{Stab}(\mathrm{CAlg}_k^{\mathrm{aug}})$ be the spectrum object of Example 1.1.4, so that $\Omega^{\infty - n}E \simeq k \oplus k[n]$. The proof of Proposition 2.3.4 shows that $\mathfrak{D}(E)$ is given by the infinite loop object $\{\mathrm{Free}(k[-n-1])\}_{n\geq 0}$ in Lie_k^{op}; here $\mathrm{Free} : \mathrm{Mod}_k \to \mathrm{Lie}_k$ denotes a left adjoint to the forgetful functor $\theta : \mathrm{Lie}_k \to \mathrm{Mod}_k$. It follows that the functor $e : \mathrm{Lie}_k \to \mathrm{Sp}$ appearing in Definition 1.3.9 is given by $(F \circ \theta)[1]$, where $F : \mathrm{Mod}_k = \mathrm{Mod}_k(\mathrm{Sp}) \to \mathrm{Sp}$ and $\theta : \mathrm{Lie}_k \to \mathrm{Mod}_k$ are the forgetful functors. Since F is conservative and commutes with all colimits, it will suffice to observe that θ is conservative (which is obvious) and preserves sifted colimits (Proposition 2.1.16). □

We are now ready to prove our main result:

Proof of Theorem 2.0.2. Let k be a field of characteristic zero, and let $\Psi : \mathrm{Lie}_k \to \mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{sm}}, \mathcal{S})$ denote the functor given on objects by the formula

$$\Psi(\mathfrak{g}_*)(R) = \mathrm{Map}_{\mathrm{Lie}_k}(\mathfrak{D}(R), \mathfrak{g}_*).$$

Combining Theorems 2.3.1 and 1.3.12, we deduce that Ψ is a fully faithful embedding whose essential image is the full subcategory $\mathrm{Moduli}_k \subseteq \mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{sm}}, \mathcal{S})$ spanned by the formal moduli problems. Let $X \mapsto T_X$ denote tangent complex functor $\mathrm{Moduli}_k \to \mathrm{Sp}$, given by evaluation on the spectrum object $E \in \mathrm{Stab}(\mathrm{CAlg}_k^{\mathrm{sm}})$ appearing in Example 1.1.4. Then the functor $\mathfrak{g}_* \mapsto T_{\Psi(\mathfrak{g}_*)}[-1]$ coincides with the functor $e[-1]$, where $e : \mathrm{Lie}_k \to \mathrm{Sp}$ is the functor appearing in Definition 1.3.9. The proof of Theorem 2.3.1 supplies an equivalence of $e[-1]$ with the forgetful functor

$$\mathrm{Lie}_k \to \mathrm{Mod}_k = \mathrm{Mod}_k(\mathrm{Sp}) \to \mathrm{Sp}\,.$$

□

Corollary 2.3.6. *Let k be a field of characteristic zero and let $X : \mathrm{CAlg}_k^{\mathrm{sm}} \to \mathcal{S}$ be a formal moduli problem. Then following conditions are equivalent:*

(1) *The formal moduli problem X is prorepresentable (see Definition 1.5.3).*

(2) *Let T_X denote the tangent complex of X. Then $\pi_i T_X \simeq 0$ for $i > 0$.*

Proof. Suppose first that X is prorepresentable; we wish to show that the homotopy groups $\pi_i T_X$ vanish for $i > 0$. The construction $X \mapsto \pi_i T_X$ commutes with filtered colimits. It will therefore suffice to show that $\pi_i T_X \simeq 0$ when $X = \mathrm{Spec}\, A$ is the the functor corepresented by an object $A \in \mathrm{CAlg}_k^{\mathrm{sm}}$. This is clear: the homotopy group $\pi_i T_X \simeq \pi_i \mathrm{Map}_{\mathrm{CAlg}_k^{\mathrm{aug}}}(A, k[\epsilon]/(\epsilon^2))$ vanishes because A is connective and $k[\epsilon]/(\epsilon^2)$ is discrete.

We now prove the converse. Let X be a formal moduli problem such that $\pi_i T_X \simeq 0$ for $i > 0$; we wish to prove that X is prorepresentable. Let $\Psi : \mathrm{Lie}_k \to \mathrm{Moduli}_k$ be the equivalence of ∞-categories of Theorem 2.0.2. Then we can assume that $X = \Psi(\mathfrak{g}_*)$ for some differential graded Lie algebra $\mathfrak{g}_*$ satisfying $\mathrm{H}_i(\mathfrak{g}_*) \simeq 0$ for $i \geq 0$ (here we let $\mathrm{H}_i(\mathfrak{g}_*)$ denote the ith homology group of the underlying chain complex of $\mathfrak{g}_*$, rather than the Lie algebra homology of $\mathfrak{g}_*$ computed by the Chevalley-Eilenberg complex $C_*(\mathfrak{g}_*)$ of §2.2).

We now construct a sequence of differential graded Lie algebras

$$0 = \mathfrak{g}(0)_* \to \mathfrak{g}(1)_* \to \mathfrak{g}(2)_* \to \cdots$$

equipped with maps $\phi(i) : \mathfrak{g}(i)_* \to \mathfrak{g}_*$. For every integer n, choose a graded subspace $V_n \subseteq \mathfrak{g}_n$ consisting of cycles which maps isomorphically onto the homology $\mathrm{H}_n(\mathfrak{g}_*)$. Then we can regard V_* as a differential graded vector space with trivial differential. Let $\mathfrak{g}(1)_*$ denote the free differential graded Lie algebra generated by V_*, and $\phi(1) : \mathfrak{g}(1)_* \to \mathfrak{g}_*$ the canonical map. Assume now that $i \geq 1$ and that we have constructed a map $\phi(i) : \mathfrak{g}(i)_* \to \mathfrak{g}_*$ extending $\phi(1)$. Then $\phi(i)$ induces a surjection $\theta : \mathrm{H}_n(\mathfrak{g}(i)_*) \to \mathrm{H}_*(\mathfrak{g}_*)$. Choose a collection of cycles $x_\alpha \in \mathfrak{g}(i)_{n_\alpha}$ whose images form a basis for $\ker(\theta)$. Then we can write $\phi(i)(x_\alpha) = dy_\alpha$ for some $y_\alpha \in \mathfrak{g}_{n_\alpha+1}$. Let $\mathfrak{g}(i+1)_*$ be the differential graded Lie algebra obtained from $\mathfrak{g}(i)_*$ by freely adjoining elements Y_α (in degrees $n_\alpha + 1$) satisfying $dY_\alpha = x_\alpha$. We let $\phi(i+1) : \mathfrak{g}(i+1)_* \to \mathfrak{g}_*$ denote the unique extension of $\phi(i)$ satisfying $\phi(i+1)(Y_\alpha) = y_\alpha$.

We now prove the following assertion for each integer $i \geq 1$:

$(*_i)$ The inclusion $V_{-1} \hookrightarrow \mathfrak{g}(i)_{-1}$ induces an isomorphism $V_{-1} \to \mathrm{H}_{-1}(\mathfrak{g}(i)_*)$, and the groups $\mathfrak{g}(i)_n$ vanish for $n \geq 0$.

Assertion $(*_i)$ is easy when $i = 1$. Let us assume that $(*_i)$ holds, and let θ be defined as above. Then θ is an isomorphism in degrees ≥ -1, so that $\mathfrak{g}(i+1)_*$ is obtained from $\mathfrak{g}(i)_*$ by freely adjoining generators Y_α in degrees ≤ -1. It follows immediately that $\mathfrak{g}(i+1)_n \simeq 0$ for $n \geq 0$. Moreover, we can write

$\mathfrak{g}(i+1)_{-1} \simeq \mathfrak{g}(i)_{-1} \oplus W$, where W is the subspace spanned by elements of the form Y_α where $n_\alpha = -2$. By construction, the differential on $\mathfrak{g}(i+1)_*$ carries W injectively into

$$\mathfrak{g}(i)_{-2}/d\mathfrak{g}(i)_{-1} \subseteq \mathfrak{g}(i+1)_{-2}/d\mathfrak{g}(i)_{-1},$$

so that the Lie algebras $\mathfrak{g}(i+1)_*$ and $\mathfrak{g}(i)_*$ have the same homology in degree -1.

Let $\mathfrak{g}'_*$ denote the colimit of the sequence $\{\mathfrak{g}(i)_*\}_{i \geq 0}$. The evident map $\mathfrak{g}'_* \to \mathfrak{g}_*$ is surjective on homology (since the map $\mathfrak{g}(1)_* \to \mathfrak{g}_*$ is surjective on homology). If $\eta \in \ker(\mathrm{H}_*(\mathfrak{g}'_*) \to \mathrm{H}_*(\mathfrak{g}_*)$, then η is represented by a class $\overline{\eta} \in \ker(\mathrm{H}_*(\mathfrak{g}(i)_*) \to \mathrm{H}_*(\mathfrak{g}_*))$ for $i \gg 0$. By construction, the image of $\overline{\eta}$ vanishes in $\mathrm{H}_*(\mathfrak{g}(i+1)_*)$, so that $\eta = 0$. It follows that the map $\mathfrak{g}'_* \to \mathfrak{g}_*$ is a quasi-isomorphism. Since the collection of quasi-isomorphisms in $\mathrm{Lie}_k^{\mathrm{dg}}$ is closed under filtered colimits, we conclude that $\mathfrak{g}_*$ is a homotopy colimit of the sequence $\{\mathfrak{g}(i)_*\}_{i \geq 0}$ in the model category $\mathrm{Lie}_k^{\mathrm{dg}}$, and therefore a colimit of $\{\mathfrak{g}(i)_*\}_{i \geq 0}$ in the ∞-category Lie_k. Setting $X(i) = \Psi(\mathfrak{g}(i)_*) \in \mathrm{Moduli}_k$, we deduce that $X \simeq \varinjlim X(i)$. To prove that X is prorepresentable, it will suffice to show that each $X(i)$ is prorepresentable.

We now proceed by induction on i, the case $i = 0$ being trivial. To carry out the inductive step, we note that each of the Lie algebras $\mathfrak{g}(i+1)_*$ is obtained from $\mathfrak{g}(i)_*$ by freely adjoining a set of generators $\{Y_\alpha\}_{\alpha \in A}$ of degrees $n_\alpha + 1 \leq -1$, satisfying $dY_\alpha = x_\alpha \in \mathfrak{g}(i)_{n_\alpha}$ (this is obvious when $i = 0$, and follows from $(*_i)$ when $i > 0$). Choose a well-ordering of the set A. For each $\alpha \in A$, we let $\mathfrak{g}_*^{<\alpha}$ denote the Lie subalgebra of $\mathfrak{g}(i+1)_*$ generated by $\mathfrak{g}(i)_*$ and the elements Y_β for $\beta < \alpha$, and let $\mathfrak{g}_*^{\leq \alpha}$ be defined similarly. Set

$$X^{<\alpha} = \Psi(\mathfrak{g}_*^{<\alpha}) \qquad X^{\leq \alpha} = \Psi(\mathfrak{g}_*^{\leq \alpha}).$$

For each $\alpha \in A$, we have a homotopy pushout diagram of differential graded Lie algebras

$$\begin{tikzcd} \mathrm{Free}(\partial\, E(n_\alpha+1)_*) \arrow[r] \arrow[d] & \mathrm{Free}(E(n_\alpha+1)_*) \arrow[d] \\ \mathfrak{g}_*^{<\alpha} \arrow[r] & \mathfrak{g}_*^{\leq\alpha}, \end{tikzcd}$$

hence a pushout diagram of formal moduli problems

$$\begin{tikzcd} \mathrm{Spec}(k \oplus k[n_\alpha+1]) \arrow[r] \arrow[d] & \mathrm{Spec}(k) \arrow[d] \\ X^{<\alpha} \arrow[r] & X^{\leq\alpha}. \end{tikzcd}$$

It follows that the map $X(i) \to X(i+1)$ satisfies the criterion of Lemma 1.5.9. Since $X(i)$ is prorepresentable, we conclude that $X(i+1)$ is prorepresentable. □

The remainder of this section is devoted to the proof of Lemma 2.3.5. We will need a few preliminaries.

Notation 2.3.7. Let $F : (\mathrm{Vect}_k^{\mathrm{dg}})^{op} \to \mathrm{Vect}_k^{\mathrm{dg}}$ be the functor between ordinary categories which carries each chain complex (V_*, d) to the dual chain complex $(V_*^\vee, d^\vee)$, where $V_n^\vee = \mathrm{Hom}_k(V_{-n}, k)$ and the differential $d^\vee$ is characterized by the formula $d^\vee(\lambda)(v) + (-1)^n \lambda(dv) = 0$ for $\lambda \in V_n^\vee$. The construction $V_* \mapsto V_*^\vee$ preserves quasi-isomorphisms and therefore induces a functor $\mathrm{Mod}_k^{op} \to \mathrm{Mod}_k$, which we will denote by $V \mapsto V^\vee$. We will refer to this functor as *k-linear duality.*

Remark 2.3.8. For every pair of k-module spectra $V, W \in \mathrm{Mod}_k$, we have canonical homotopy equivalences

$$\mathrm{Map}_{\mathrm{Mod}_k}(V, W^\vee) \simeq \mathrm{Map}_{\mathrm{Mod}_k}(V \otimes W, k) \simeq \mathrm{Map}_{\mathrm{Mod}_k}(W, V^\vee).$$

It follows that k-linear duality, when regarded as a functor $\mathrm{Mod}_k \to \mathrm{Mod}_k^{op}$, is canonically equivalent to the left adjoint of the k-linear duality functor $\mathrm{Mod}_k^{op} \to \mathrm{Mod}_k$.

Let us now study the composite functor

$$(\mathrm{CAlg}_k^{\mathrm{aug}})^{op} \xrightarrow{\mathfrak{D}} \mathrm{Lie}_k \xrightarrow{\theta} \mathrm{Mod}_k,$$

where θ denotes the forgetful functor. This composition admits a left adjoint

$$\mathrm{Mod}_k \xrightarrow{\mathrm{Free}} \mathrm{Lie}_k \xrightarrow{C^*} (\mathrm{CAlg}_k^{\mathrm{aug}})^{op},$$

which is in turn induced by the map of ordinary categories $\mathrm{Vect}_k^{\mathrm{dg}} \to \mathrm{CAlg}_k^{\mathrm{dg}}$ given by $V_* \mapsto C^*(\mathrm{Free}(V_*))$. Remark 2.2.15 supplies a (functorial) quasi-isomorphism of commutative differential graded algebras

$$C^*(\mathrm{Free}(V_*)) \to k \oplus V_*^\vee[-1].$$

It follows that the underlying functor of ∞-categories $\mathrm{Mod}_k \to (\mathrm{CAlg}_k^{\mathrm{aug}})^{op}$ is given by composing the k-linear duality functor $\mathrm{Mod}_k \to \mathrm{Mod}_k^{op}$ with the functor $\mathrm{Mod}_k^{op} \to (\mathrm{CAlg}_k^{\mathrm{aug}})^{op}$ given by the formation of square-zero extensions $M \mapsto k \oplus M[-1]$. Both of these functors admit left adjoints: in the first case, the left adjoint is given by k-linear duality (Remark 2.3.8), and in the second it is given by the formation of the relative cotangent complex $A \mapsto (L_{A/k} \otimes_A k)[-1] \simeq L_{k/A}$ respectively. We have proven:

Proposition 2.3.9. *Let k be a field of characteristic zero and let $\theta : \mathrm{Lie}_k \to \mathrm{Mod}_k$ be the forgetful functor. Then the composite functor*

$$(\mathrm{CAlg}_k^{\mathrm{aug}})^{op} \xrightarrow{\mathfrak{D}} \mathrm{Lie}_k \xrightarrow{\theta} \mathrm{Mod}_k$$

is given on objects by $A \mapsto L_{k/A}^\vee$.

To prove Lemma 2.3.5, we need to analyze the unit map $\mathfrak{g}_* \to \mathfrak{D}C^*(\mathfrak{g}_*)$ associated to a differential graded Lie algebra $\mathfrak{g}_*$. We begin with a few preliminary remarks regarding explicit models for the cotangent fiber of a commutative differential graded algebra.

Definition 2.3.10. Let A_* be a commutative differential graded algebra over k equipped with an augmentation $u : A_* \to k$. The kernel of u is an ideal $\mathfrak{m}_A \subseteq A_*$. We let $\mathrm{Indec}(A)_*$ denote the quotient $\mathfrak{m}_A/\mathfrak{m}_A^2$, which we regard as a complex of k-vector spaces. We will refer to $\mathrm{Indec}(A)_*$ as the *chain complex of indecomposables in A_**.

Remark 2.3.11. The construction $V_* \mapsto k \oplus V_*$ determines a right Quillen functor from $\mathrm{Vect}_k^{\mathrm{dg}}$ to $(\mathrm{CAlg}_k^{\mathrm{dg}})_{/k}$, whose left adjoint is given by $A_* \mapsto \mathrm{Indec}(A)_*$. It follows that the functor $\mathrm{Indec}(A)_*$ preserves weak equivalences between cofibrant objects of $(\mathrm{CAlg}_k^{\mathrm{dg}})_{/k}$, and induces a functor of ∞-categories $\mathrm{CAlg}_k^{\mathrm{aug}} \to \mathrm{Mod}_k$. This functor is evidently left adjoint to the formation of trivial square-zero extensions, and is therefore given by $A \mapsto L_{A/k} \otimes_A k \simeq L_{k/A}[-1]$. It follows that for every cofibrant augmented commutative differential graded algebra A_*, the canonical map $A_* \to k \oplus \mathrm{Indec}(A)_*$ induces an equivalence $L_{k/A_*}[-1] \simeq L_{A_*/k} \otimes_{A_*} k \to \mathrm{Indec}(A)_*$ in Mod_k (here we abuse notation by identifying A_* with its image in the ∞-category $\mathrm{CAlg}_k^{\mathrm{aug}}$).

Proof of Lemma 2.3.5. Let $\mathfrak{g}_*$ be a differential graded Lie algebra satisfying hypotheses (a) and (b); we wish to show that the unit map $\mathfrak{g}_* \to \mathfrak{D}C^*(\mathfrak{g})$ is an equivalence in the ∞-category Lie_k. Since the forgetful functor $\mathrm{Lie}_k \to \mathrm{Mod}_k$ is conservative, it will suffice to show that u induces an equivalence $\mathfrak{g}_* \to \mathfrak{D}_0 L_{k/C^*(\mathfrak{g}_*)}$ in Mod_k (see Proposition 2.3.9). This map has a predual, given by the map

$$u : L_{C^*(\mathfrak{g}_*)/k} \otimes_{C^*(\mathfrak{g})} k \to \mathfrak{g}_*^\vee[-1].$$

We will prove that u is an equivalence.

Consider the isomorphism of graded vector spaces

$$C^*(\mathfrak{g}_*) \simeq \prod_{n \geq 0} (\mathrm{Sym}^n \, \mathfrak{g}_*[1])^\vee.$$

Choose a basis $\{y_1, \ldots, y_p\}$ for the vector space $\mathfrak{g}_{-1}$, and let $\{x_1, \ldots, x_p\}$ be the dual basis for $\mathfrak{g}_1^\vee$, so that $C^0(\mathfrak{g}_*)$ can be identified with the power series ring $k[[x_1, \ldots, x_p]]$. Let $A_* = \bigoplus_{n \geq 0} (\mathrm{Sym}^n(\mathfrak{g}_*[1]))^\vee$, and regard A_* as a graded subalgebra of $C^*(\mathfrak{g}_*)$. It is easy to see that A_* is a differential graded subalgebra of $C^*(\mathfrak{g}_*)$, and that A_0 contains the polynomial ring $k[x_1, \ldots, x_p]$. Using (a) and (b), we deduce that A_* is a graded polynomial ring generated by $g_*^\vee[-1]$, and that the natural map

$$A_* \otimes_{k[x_1,\ldots,x_p]} k[[x_1, \ldots, x_p]] \to C^*(\mathfrak{g}_*)$$

is an isomorphism of commutative differential graded algebras. Since $k[[x_1, \ldots, x_p]]$ is flat over $k[x_1, \ldots, x_p]$, it follows that for each $n \in \mathbf{Z}$ we have an isomorphism in homology

$$\mathrm{H}_n(A_*) \otimes_{k[x_1,\ldots,x_p]} k[[x_1, \ldots, x_p]] \to \mathrm{H}_n(C^*(\mathfrak{g}_*)),$$

so that the diagram

$$\begin{tikzcd}
k[x_1, \ldots, x_p] \arrow[r] \arrow[d] & k[[x_1, \ldots, x_p]] \arrow[d] \\
A_* \arrow[r] & C^*(\mathfrak{g}_*)
\end{tikzcd}$$

is a pushout square in the ∞-category CAlg_k. We therefore obtain equivalences

$$L_{C^*(\mathfrak{g}_*)/A_*} \otimes_{C^*(\mathfrak{g}_*)} k \simeq L_{k[[x_1,\ldots,x_p]]/k[x_1,\ldots,x_p]} \otimes_{k[[x_1,\ldots,x_p]]} k \simeq L_{R/k} \simeq 0.$$

where R denotes the tensor product $k[[x_1, \ldots, x_p]] \otimes_{k[x_1,\ldots,x_p]} k \simeq k$. It follows that u can be identified with the map $L_{A_*/k} \otimes_{A_*} k \to \mathfrak{g}_*^\vee[-1]$ which classifies the morphism $A_* \to k \oplus \mathfrak{g}_*^\vee[-1] \simeq k \oplus \mathrm{Indec}(A)_*$. Since A_* is a cofibrant differential graded algebra, Remark 2.3.11 implies that u is an equivalence in Mod_k. $\square$

2.4 Quasi-Coherent Sheaves

Let k be a field and let $X : \mathrm{CAlg}_k^{\mathrm{sm}} \to \mathcal{S}$ be a formal moduli problem over k. Following the ideas introduced in §VIII.2.7, we can define a symmetric monoidal ∞-category $\mathrm{QCoh}(X)$ of *quasi-coherent sheaves* on X. Roughly speaking, a quasi-coherent sheaf $\mathcal{F}$ on X is a rule which assigns to each point $\eta \in X(R)$ an R-module $\eta^* \mathcal{F} \in \mathrm{Mod}_R$, which is functorial in the following sense: if $\phi : R \to R'$ is a morphism in $\mathrm{CAlg}_k^{\mathrm{sm}}$ and η' denotes the image of η in $X(R')$, then there is an equivalence $\eta'^* \mathcal{F} \simeq R' \otimes_R \eta^* \mathcal{F}$ in the ∞-category $\mathrm{Mod}_{R'}$.

If the field k has characteristic zero, Theorem 2.0.2 provides an equivalence of ∞-categories $\Psi : \mathrm{Lie}_k \to \mathrm{Moduli}_k$. In particular, every formal moduli problem X is equivalent to $\Psi(\mathfrak{g}_*)$, for some differential graded Lie algebra $\mathfrak{g}_*$ which is well-defined up to quasi-isomorphism. In this section, we will explore the relationship between $\mathfrak{g}_*$ and the ∞-category $\mathrm{QCoh}(X)$. Our main result is the following:

Theorem 2.4.1. *Let k be a field of characteristic zero, let $\mathfrak{g}_*$ be a differential graded Lie algebra over k, and let $X = \Psi(\mathfrak{g}_*)$ be the associated formal moduli problem. Then there is a fully faithful symmetric monoidal embedding* $\mathrm{QCoh}(X) \hookrightarrow \mathrm{Rep}_{\mathfrak{g}_*}$, *where* $\mathrm{Rep}_{\mathfrak{g}_*}$ *denotes the ∞-category of representations of $\mathfrak{g}_*$ (see Notation 2.4.6).*

Remark 2.4.2. It follows from Theorem 2.4.1 that the ∞-category $\mathrm{Rep}_{\mathfrak{g}_*}$ can be regarded as a (symmetric monoidal) *enlargement* of the ∞-category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves on the formal moduli problem determined by $\mathfrak{g}_*$. This enlargement can be described geometrically as the ∞-category of Ind-coherent sheaves on X. We refer the reader to §3.4 for a discussion of Ind-coherent sheaves in the noncommutative setting, and to §3.5 for a noncommutative analogue of Theorem 2.4.1.

We begin with a discussion of representations of differential graded Lie algebras.

Definition 2.4.3. Let k be a field and let $\mathfrak{g}_*$ be a differential graded Lie algebra over k. A *representation* of $\mathfrak{g}_*$ is a differential graded vector space V_* equipped with a map

$$\mathfrak{g}_* \otimes_k V_* \to V_*$$

satisfying the identities

$$[x, y]v = x(yv) + (-1)^{pq} y(xv)$$

for $x \in \mathfrak{g}_p$, $y \in \mathfrak{g}_q$. The representations of $\mathfrak{g}_*$ comprise a category which we will denote by $\mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}_*}$.

Example 2.4.4. For every differential graded vector space V_*, the zero map

$$\mathfrak{g}_* \otimes_k V_* \to V_*$$

exhibits V_* as a representation of $\mathfrak{g}_*$. In particular, taking $V_* = k$ (regarded as a graded vector space concentrated in degree zero), we obtain a representation of $\mathfrak{g}_*$ on k which we call the *trivial representation.*

Note that a representation of a differential graded Lie algebra $\mathfrak{g}_*$ is the same data as a (left) module over the universal enveloping algebra $U(\mathfrak{g}_*)$. Using Proposition A.4.3.3.15, we deduce the following:

Proposition 2.4.5. *Let $\mathfrak{g}_*$ be a differential graded Lie algebra over a field k. Then the category $\mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}_*}$ of representations of $\mathfrak{g}_*$ admits a combinatorial model structure, where:*

(W) A map $f : V_* \to W_*$ of representations of $\mathfrak{g}_*$ is a weak equivalence if and only if it induces an isomorphism on homology.

(F) A map $f : V_* \to W_*$ of representations of $\mathfrak{g}_*$ is a fibration if and only if it is degreewise surjective.

Notation 2.4.6. If $\mathfrak{g}_*$ is a differential graded Lie algebra over a field k, we let $W_{\mathfrak{g}_*}$ denote the collection of all weak equivalences in $\mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}_*}$, and we let

$$\mathrm{Rep}_{\mathfrak{g}_*} = \mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}_*}[W_{\mathfrak{g}_*}^{-1}]$$

denote the ∞-category obtained from $\mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}_*}$ by formally inverting all quasi-isomorphisms: that is, the underlying ∞-category of the model category described in Proposition 2.4.5.

It follows from Theorem A.4.3.3.17 that we can identify $\mathrm{Rep}_{\mathfrak{g}_*}$ with the ∞-category $\mathrm{LMod}_{U(\mathfrak{g}_*)}$ of left modules over the universal enveloping algebra $U(\mathfrak{g}_*)$ (which we regard as an $\mathbb{E}_1$-ring). In particular, $\mathrm{Rep}_{\mathfrak{g}_*}$ is a stable ∞-category.

Construction 2.4.7. Let $\mathfrak{g}_*$ be a differential graded Lie algebra over a field k and let V_* be a representation of $\mathfrak{g}_*$. We let $C^*(\mathfrak{g}_*; V_*)$ denote the differential graded vector space of $U(\mathfrak{g}_*)$-module maps from $U(\mathrm{Cn}(\mathfrak{g})_*)$ into V_*. We will refer to $C^*(\mathfrak{g}_*; V_*)$ as the *cohomological Chevalley-Eilenberg complex of $\mathfrak{g}_*$ with coefficients in V_*.*

Remark 2.4.8. Unwinding the definitions, we see that the graded pieces $C^n(\mathfrak{g}_*; V_*)$ can be identified with the set of graded vector space maps $\mathrm{Sym}^*(\mathfrak{g}_*[1]) \to V_*[-n]$.

We note that $C^*(\mathfrak{g}_*; V_*)$ has the structure of a module over the differential graded algebra $C^*(\mathfrak{g}_*)$. The action is given by k-bilinear maps

$$C^p(\mathfrak{g}_*) \times C^q(\mathfrak{g}_*; V_*) \to C^{p+q}(\mathfrak{g}_*; V_*),$$

which carries a class $\lambda \in C^p(\mathfrak{g}_*)$ and $\mu \in C^q(\mathfrak{g}_*; V_*)$ to the element $\lambda\mu \in C^{p+q}(\mathfrak{g}_*; V_*)$ given by

$$(\lambda\mu)(x_1 \ldots x_n) = \sum_{S,S'} \epsilon(S, S')\lambda(x_{i_1} \ldots x_{i_m})\mu(x_{j_1} \ldots x_{j_{n-m}}),$$

as in Construction 2.2.13.

Remark 2.4.9. It follows from general nonsense that the (differential graded) endomorphism ring of the functor $C^*(\mathfrak{g}_*;\bullet) : \mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}_*} \to \mathrm{Mod}^{\mathrm{dg}}_k$ is isomorphic to the (differential graded) endomorphism ring of $U(\mathrm{Cn}(\mathfrak{g})_*)$ (regarded as a representation of $\mathfrak{g}_*$). In particular, the action of $C^*(\mathfrak{g}_*)$ on $C^*(\mathfrak{g}_*;\bullet)$ arises from an action of $C^*(\mathfrak{g}_*)$ on $U(\mathrm{Cn}(\mathfrak{g})_*)$, which commutes with the left action of $U(\mathfrak{g}_*)$. For an alternative description of this action, we refer the reader to the proof of Proposition 3.3.7.

Proposition 2.4.10. *Let $\mathfrak{g}_*$ be a differential graded Lie algebra over a field k of characteristic zero. Then the functor $V_* \mapsto C^*(\mathfrak{g}_*;V_*)$ preserves quasi-isomorphisms.*

Proof. For each $n \geq 0$ and each $V_* \in \mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}_*}$, we let $F_n(V_*)$ denote the quotient of $C^*(\mathfrak{g}_*;V_*)$ given by maps from $\mathrm{Sym}^{\leq n}(\mathfrak{g}_*[1])$ into V_*. Then $C^*(\mathfrak{g}_*;V_*)$ is given by the inverse limit of a tower of fibrations

$$\cdots \to F_2(V_*) \to F_1(V_*) \to F_0(V_*).$$

It will therefore suffice to show that each of the functors F_n preserves quasi-isomorphisms. We proceed by induction on n. If $n = 0$, then F_n is the identity functor and the result is obvious. Assume therefore that $n > 0$. Let $K : \mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}_*} \to \mathrm{Mod}^{\mathrm{dg}}_k$ be the functor given by the kernel of the surjection $F_n \to F_{n-1}$, so that we have a short exact sequence of functors

$$0 \to K \to F_n \to F_{n-1} \to 0.$$

It will therefore suffice to show that the functor K preserves quasi-isomorphisms. Unwinding the definitions, we see that K carries a representation V_* to the chain complex of Σ_n-equivariant maps from $(\mathfrak{g}_*[1])^{\otimes n}$ into V_*, regarded as objects of $\mathrm{Mod}^{\mathrm{dg}}_k$. Since k has characteristic zero, the functor K is a direct summand of the functor $K' : \mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}_*} \to \mathrm{Mod}^{\mathrm{dg}}_k$, which carries V_* to the chain complex of maps from $(\mathfrak{g}_*[1])^{\otimes n}$ into V_*. This functor evidently preserves quasi-isomorphisms. $\square$

Remark 2.4.11. Let $\mathfrak{g}_*$ be a differential graded Lie algebra over a field k of characteristic zero, and let $\mathrm{Mod}^{\mathrm{dg}}_{C^*(\mathfrak{g}_*)}$ denote the category of differential graded modules over $C^*(\mathfrak{g}_*)$. The functor

$$C^*(\mathfrak{g}_*;\bullet) : \mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}_*} \to \mathrm{Mod}^{\mathrm{dg}}_{C^*(\mathfrak{g}_*)}$$

preserves weak equivalences and fibrations. Moreover, it has a left adjoint F, given by

$$M_* \mapsto U(\mathrm{Cn}(\mathfrak{g})_*) \otimes_{C^*(\mathfrak{g}_*)} M_*$$

(see Remark 2.4.9). It follows that $C^*(\mathfrak{g}_*;\bullet)$ is a right Quillen functor, which induces a map between the underlying ∞-categories $\mathrm{Rep}_{\mathfrak{g}_*} \to \mathrm{Mod}_{C^*(\mathfrak{g}_*)}$. We will generally abuse notation by denoting this functor also by $C^*(\mathfrak{g}_*;\bullet)$. It admits a left adjoint $f : \mathrm{Mod}_{C^*(\mathfrak{g}_*)} \to \mathrm{Rep}_{\mathfrak{g}_*}$ (given by the left derived functor of F).

Proposition 2.4.12. *Let k be a field of characteristic zero and let $\mathfrak{g}_*$ be a differential graded Lie algebra over k. Assume that the underlying graded Lie algebra is freely generated by a finite sequence of homogeneous elements $x_1, \ldots, x_n$ such that each dx_i belongs to the Lie subalgebra of $\mathfrak{g}_*$ generated by $x_1, \ldots, x_{i-1}$. Let $f : \mathrm{Mod}_{C^*(\mathfrak{g}_*)} \to \mathrm{Rep}_{\mathfrak{g}_*}$ denote the left adjoint to the functor $C^*(\mathfrak{g}_*;\bullet)$ (see Remark 2.4.11). Then f is a fully faithful embedding.*

Lemma 2.4.13. *Let k be a field and let A_* be an augmented differential graded algebra over k; we will abuse notation by identifying A_* with its image in Mod_k. Assume that A_* is freely generated (as a graded algebra) by a finite sequence of homogeneous elements $x_1, \ldots, x_n$, such that each dx_i lies in the subalgebra generated by $x_1, \ldots, x_{i-1}$. Then the field k is a compact object of the stable ∞-category LMod_{A_*}.*

Proof. Adding scalars to the elements x_i if necessary, we may assume that the augmentation $A_* \to k$ annihilates each x_i. For $0 \leq i \leq n$, let $M(i)_*$ denote the quotient of A_* by the left ideal generated by the elements $x_1, \ldots, x_i$. We will prove that each $M(i)_*$ is perfect as a left A_*-module; taking $i = n$, this will imply the desired result. The proof proceeds by induction on i. If $i = 0$, then $M(i)_* \simeq A_*$ and the result is obvious. If $i > 0$, then the image of x_i in $M(i-1)_*$ is a cycle. It follows that right multiplication by x_i induces a map of left A_*-modules $A_* \to M(i-1)_*$, fitting into an exact sequence

$$0 \to A_* \xrightarrow{x_i} M(i-1)_* \to M(i)_* \to 0.$$

Since $M(i-1)_*$ is perfect by the inductive hypothesis, we deduce that $M(i)_*$ is perfect. □

Proof of Proposition 2.4.12. We first show that f is fully faithful when restricted to the full subcategory $\mathrm{Mod}^{\mathrm{perf}}_{C^*(\mathfrak{g}_*)} \subseteq \mathrm{Mod}_{C^*(\mathfrak{g}_*)}$ spanned by the perfect $C^*(\mathfrak{g}_*)$-modules. Let M and N be perfect $C^*(\mathfrak{g}_*)$-modules. We wish to show that f induces an isomorphism

$$\theta : \mathrm{Ext}^*_{C^*(\mathfrak{g}_*)}(M, N) \to \mathrm{Ext}^*_{U(\mathfrak{g}_*)}(fM, fN).$$

Regard M as fixed. The collection of those modules N for which θ is an isomorphism is closed under retracts, shifts, and extensions. To prove that θ is an isomorphism for each $N \in \mathrm{Mod}^{\mathrm{perf}}_{C^*(\mathfrak{g}_*)}$, it will suffice to prove that θ is an isomorphism for $N = C^*(\mathfrak{g}_*)$. By the same reasoning, we can reduce to the case where $M = C^*(\mathfrak{g}_*)$. Then $fM \simeq U(\mathrm{Cn}(\mathfrak{g})_*)$ and $fN \simeq U(\mathrm{Cn}(\mathfrak{g}_*) \simeq k$, so that $\mathrm{Ext}^*_{U(\mathfrak{g}_*)}(fM, fN) \simeq \mathrm{Ext}^*_{U(\mathfrak{g}_*)}(U(\mathrm{Cn}(\mathfrak{g})_*), k)$ is canonically isomorphic to the Lie algebra cohomology of $\mathfrak{g}_*$. Under this isomorphism, θ corresponds to the identity map.

We now prove that f is fully faithful in general. Since $\mathrm{Mod}_{C^*(\mathfrak{g}_*)} \simeq \mathrm{Ind}(\mathrm{Mod}^{\mathrm{perf}}_{C^*(\mathfrak{g}_*)})$ and the functor f preserves filtered colimits, it will suffice to show that f carries objects $\mathrm{Mod}^{\mathrm{perf}}_{C^*(\mathfrak{g}_*)}$ to perfect $U(\mathfrak{g}_*)$-modules. The collection of those $M \in \mathrm{Mod}_{C^*(\mathfrak{g}_*)}$ for which fM is perfect is closed under extensions, shifts, and retracts. It will therefore suffice to show that $fC^*(\mathfrak{g}_*) \simeq k$ is perfect as a $U(\mathfrak{g}_*)$-module, which follows from Lemma 2.4.13. □

Lemma 2.4.14. *Let k be a field and let A be a coconnective $\mathbb{E}_1$-algebra over k, equipped with an augmentation $\epsilon : A \to k$. Let $\mathcal{C} \subseteq \mathrm{LMod}_A$ be a full subcategory which contains k (regarded as a left A-module via the augmentation ϵ) and is closed under colimits and extensions. Then $\mathcal{C}$ contains every left A-module whose underlying spectrum is connective.*

Proof. Let M be a left A-module whose underlying spectrum is connective. We will construct a sequence of objects

$$0 = M(0) \to M(1) \to M(2) \to \cdots$$

in $\mathcal{C}$ and a compatible family of maps $\theta(i) : M(i) \to M$ with the following property:

$(*)$ The groups $\pi_j M(i)$ vanish unless $0 \leq j < i$, and the maps $\pi_j M(j) \to \pi_j M$ are isomorphisms for $0 \leq j < i$.

Assume that $i \geq 0$ and that we have already constructed a map $\theta(i)$ satisfying $(*)$. Let $M' = \mathrm{fib}(\theta(i))$, so that $\pi_j M' \simeq 0$ for $j < i-1$. Using Proposition VIII.4.1.9, we can construct a map of left A-modules $N \to M'$ which induces an isomorphism $\pi_{i-1} N \to \pi_{i-1} M'$, with $\pi_j N \simeq 0$ for $j \neq i-1$. Let $M(i+1)$ denote the cofiber of the composite map $N \to M' \to M(i)$. There is an evident map $\theta(i+1) : M(i+1) \to M$ satisfying $(*)$. We will complete the proof by showing that $M(i+1) \in \mathcal{C}$. We have a fiber sequence

$$M(i) \to M(i+1) \to N[1].$$

Lemma VIII.4.3.16 implies that $N[1]$ is equivalent to a direct sum of copies of $k[i]$. Since $\mathcal{C}$ contains k and is closed under colimits, we conclude that $N[1] \in \mathcal{C}$. The module $M(i)$ belong to $\mathcal{C}$ by the inductive hypothesis. Since $\mathcal{C}$ is closed under extensions, we deduce that $M(i+1) \in \mathcal{C}$. □

Notation 2.4.15. Let $\mathfrak{g}_*$ be a differential graded Lie algebra and let V_* be a representation of $\mathfrak{g}_*$. We will say that V_* is *connective* if its image in Mod_k is connective: that is, if the homology groups of the chain complex V_* are concentrated in non-negative degrees. We let $\mathrm{Mod}^{\mathrm{cn}}_{\mathfrak{g}_*}$ denote the full subcategory of $\mathrm{Rep}_{\mathfrak{g}_*}$ spanned by the connective $\mathfrak{g}_*$-modules.

Proposition 2.4.16. *Let $\mathfrak{g}_*$ be as in the statement of Proposition 2.4.12, and assume that each of the generators x_i of $\mathfrak{g}_*$ has negative homological degree. Then the fully faithful embedding $f : \mathrm{Mod}_{C^*(\mathfrak{g}_*)} \to \mathrm{Rep}_{\mathfrak{g}_*}$ induces an equivalence of ∞-categories*

$$\mathrm{Mod}^{\mathrm{cn}}_{C^*(\mathfrak{g}_*)} \to \mathrm{Mod}^{\mathrm{cn}}_{\mathfrak{g}_*}.$$

Proof. Since $C^*(\mathfrak{g}_*)$ is connective, we can characterize as the smallest full subcategory of $\mathrm{Mod}^{\mathrm{cn}}_{C^*(\mathfrak{g}_*)}$ which contains $C^*(\mathfrak{g}_*)$ and is closed under colimits and extensions. It follows that f induces an equivalence from $\mathrm{Mod}^{\mathrm{cn}}_{C^*(\mathfrak{g}_*)}$ to the smallest full subcategory of $\mathrm{Mod}^{\mathrm{cn}}_{\mathfrak{g}_*}$ which contains $fC^*(\mathfrak{g}_*) \simeq k$ and is closed under colimits and extensions. It is clear that this full subcategory is contained in $\mathrm{Mod}^{\mathrm{cn}}_{\mathfrak{g}_*}$, and the reverse inclusion follows from Lemma 2.4.14. □

We next observe that the category $\mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}_*}$ of representations of a differential graded Lie algebra $\mathfrak{g}_*$ is symmetric monoidal. If V_* and W_* are representations of $\mathfrak{g}_*$, then the tensor product $V_* \otimes_k W_*$ can be regarded as a representation of $\mathfrak{g}_*$, with action given by the formula

$$x(v \otimes w) = (xv) \otimes w + (-1)^{pq} v \otimes (xw)$$

for homogeneous elements $x \in \mathfrak{g}_p$, $v \in V_q$, and $w \in W_r$. For fixed $V_* \in \mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}_*}$, the construction $W_* \mapsto V_* \otimes_k W_*$ preserves quasi-isomorphisms. It follows from Proposition A.4.1.3.4 that the underlying ∞-category $\mathrm{Rep}_{\mathfrak{g}_*} = \mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}_*}[W^{-1}_{\mathfrak{g}_*}]$ inherits a symmetric monoidal structure.

Remark 2.4.17. Let $\mathfrak{g}_*$ be a differential graded Lie algebra over a field k. Then the diagram

$$\begin{tikzcd}
\mathrm{Rep}_{\mathfrak{g}_*} \times \mathrm{Rep}_{\mathfrak{g}_*} \arrow[r, "\otimes"] \arrow[d] & \mathrm{Rep}_{\mathfrak{g}_*} \arrow[d] \\
\mathrm{Mod}_k \times \mathrm{Mod}_k \arrow[r, "\otimes"] & \mathrm{Mod}_k
\end{tikzcd}$$

commutes up to equivalence. It follows that the tensor product functor $\otimes : \mathrm{Rep}_{\mathfrak{g}_*} \times \mathrm{Rep}_{\mathfrak{g}_*} \to \mathrm{Rep}_{\mathfrak{g}_*}$ preserves small colimits separately in each variable.

We now wish to study the behavior of the functor $C^*(\mathfrak{g}_*; \bullet)$ with respect to the symmetric monoidal structure defined above. It will be convenient for us to simultaneously study the behavior of this functor with respect to change of differential graded Lie algebra $\mathfrak{g}_*$.

Construction 2.4.18. Let k be a field. We define a category $\mathrm{Rep}^{\otimes}_{\mathrm{dg}}{}^{\otimes}$ as follows:

(1) An object of $\mathrm{Rep}^{\otimes}_{\mathrm{dg}}{}^{\otimes}$ is a tuple $(\mathfrak{g}_*, V^1_*, \ldots, V^n_*)$, where $\mathfrak{g}_*$ is a differential graded Lie algebra over k and each V^i_* is a representation of $\mathfrak{g}_*$.

(2) Given a pair of objects $(\mathfrak{g}_*, V^1_*, \ldots, V^m_*), (\mathfrak{h}_*, W^1_*, \ldots, W^n_*) \in \mathrm{Rep}^{\otimes}_{\mathrm{dg}}{}^{\otimes}$, a morphism

$$(\mathfrak{g}_*, V^1_*, \ldots, V^m_*) \to (\mathfrak{h}_*, W^1_*, \ldots, W^n_*)$$

is given by a map $\alpha : \langle m \rangle \to \langle n \rangle$ of pointed finite sets, a morphism $\phi : \mathfrak{h}_* \to \mathfrak{g}_*$ of differential graded Lie algebras, and, for each $1 \leq j \leq n$, a map $\bigotimes_{\alpha(i)=j} V^i_* \to W^j_*$ of representations of $\mathfrak{h}_*$ (here we regard each V^i_* as a representation of $\mathfrak{h}_*$ via the morphism ϕ).

The category $\mathrm{Rep}^{\otimes}_{\mathrm{dg}}{}^{\otimes}$ is equipped with an evident forgetful functor $\mathrm{Rep}^{\otimes}_{\mathrm{dg}}{}^{\otimes} \to (\mathrm{Lie}^{\mathrm{dg}}_k)^{op} \times \mathcal{F}\mathrm{in}_*$, which induces a coCartesian fibration $\mathrm{N}(\mathrm{Rep}^{\otimes}_{\mathrm{dg}}{}^{\otimes}) \to \mathrm{N}(\mathrm{Lie}^{\mathrm{dg}}_k)^{op} \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$.

For our applications of Construction 2.4.18, we will need the following general result:

Proposition 2.4.19. *Let $p : \mathcal{C} \to \mathcal{D}$ be a coCartesian fibration of ∞-categories. Suppose that we are given, for each $D \in \mathcal{D}$, a collection of morphisms W_D in the fiber $\mathcal{C}_D$. Suppose further that for each morphism $D \to D'$ in $\mathcal{D}$, the induced functor $\mathcal{C}_D \to \mathcal{C}_{D'}$ carries W_D into $W_{D'}$. Let $W = \bigcup_{D \in \mathcal{D}} W_D$. Since p carries each morphism of W to an equivalence in $\mathcal{D}$, it factors as a composition*

$$\mathcal{C} \xrightarrow{\theta} \mathcal{C}[W^{-1}] \xrightarrow{q} \mathcal{D}.$$

Replacing $\mathcal{C}[W^{-1}]$ by an equivalent ∞-category if necessary, we may assume that q is a categorical fibration. Then:

(1) *The map q is a coCartesian fibration.*

(2) *The functor θ carries p-coCartesian morphisms in $\mathcal{C}$ to q-coCartesian morphisms in $\mathcal{C}[W^{-1}]$.*

(3) *For each $D \in \mathcal{D}$, the map θ induces an equivalence $\mathcal{C}[W_D^{-1}] \to (\mathcal{C}[W^{-1}])_D$.*

Proof. Let $\chi : \mathcal{D} \to \mathcal{C}\mathrm{at}_\infty$ classify the Cartesian fibration p. For each $D \in \mathcal{D}$, we have a canonical equivalence $\chi(D) \simeq \mathcal{C}_D$; let W'_D denote the collection of morphisms in $\chi(D)$ whose in $\mathcal{C}_D$ are equivalent to morphisms belonging to W_D. Then the construction $D \mapsto (\chi(D), W)$ determines a functor $\chi_W : \mathcal{D} \to \mathcal{W}\mathcal{C}\mathrm{at}_\infty$, where $\mathcal{W}\mathcal{C}\mathrm{at}_\infty$ is defined as in Construction A.4.1.3.1. Composing with the left adjoint to the inclusion $\mathcal{C}\mathrm{at}_\infty \to \mathcal{W}\mathcal{C}\mathrm{at}_\infty$, we obtain a new functor $\chi' : \mathcal{D} \to \mathcal{C}\mathrm{at}_\infty$, given on objects by $\chi'(D) = \chi(D)[W_D'^{-1}] \simeq \mathcal{C}_D[W_D^{-1}]$. The functor χ' classifies a coCartesian fibration $p' : \mathcal{C}' \to \mathcal{D}$. We have an evident natural transformation $\chi \to \chi'$, which determines a functor $\phi \in \mathrm{Fun}_{\mathcal{D}}(\mathcal{C}, \mathcal{C}')$ which carries p-coCartesian morphisms to p'-coCartesian morphisms. To complete the proof, it will suffice to show that ϕ induces an equivalence $\mathcal{C}[W^{-1}] \to \mathcal{C}'$. Equivalently, we must show that for any ∞-category $\mathcal{E}$, composition with ϕ induces a fully faithful embedding $v : \mathrm{Fun}(\mathcal{C}', \mathcal{E})^{\simeq} \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})^{\simeq}$ whose essential image consists of those functors $F : \mathcal{C} \to \mathcal{E}$ which carry each morphism of W to an equivalence in $\mathcal{E}$.

Evaluation at the vertex $0 \in \Delta^1$ induces a Cartesian fibration $\mathrm{Fun}(\Delta^1, \mathcal{D}) \to \mathcal{D}$. We define a new simplicial set $\mathcal{E}'$ with a map $r : \mathcal{E}' \to \mathcal{D}$ so that the following universal property is satisfied: for every map of simplicial sets $K \to \mathcal{D}$, we have a canonical bijection

$$\mathrm{Hom}_{(\mathcal{S}\mathrm{et}_\Delta)_{/\mathcal{D}}}(K, \mathcal{E}') = \mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(K \times_{\mathrm{Fun}(\{0\}, \mathcal{D})} \mathrm{Fun}(\Delta^1, \mathcal{D}), \mathcal{E}).$$

Using Corollary T.3.2.2.12, we deduce that the map $r : \mathcal{E}' \to \mathcal{D}$ is a coCartesian fibration. The diagonal inclusion $\mathcal{D} \to \mathrm{Fun}(\Delta^1, \mathcal{D})$ induces a map $K \to K \times_{\mathrm{Fun}(\{0\}, \mathcal{D})} \mathrm{Fun}(\Delta^1, \mathcal{D})$ for every map $K \to \mathcal{D}$. Composition with these maps gives a functor $u : \mathcal{E}' \to \mathcal{E}$. We claim:

($*$) Let $\mathrm{Fun}'_{\mathcal{D}}(\mathcal{C}, \mathcal{E}')$ denote the full subcategory of $\mathrm{Fun}_{\mathcal{D}}(\mathcal{C}, \mathcal{E}')$ spanned by those functors which carry p-coCartesian morphisms to r-coCartesian morphisms. Then composition with u induces a trivial Kan fibration $\mathrm{Fun}'_{\mathcal{D}}(\mathcal{C}, \mathcal{E}') \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$

To prove ($*$), we note that $\mathrm{Fun}_{\mathcal{D}}(\mathcal{C}, \mathcal{E}')$ can be identified with the ∞-category

$$\mathrm{Fun}(\mathcal{C} \times_{\mathrm{Fun}(\{0\}, \mathcal{D})} \mathrm{Fun}(\Delta^1, \mathcal{D}), \mathcal{E}).$$

Under this isomorphism, $\mathrm{Fun}'_{\mathcal{D}}(\mathcal{C}, \mathcal{E}')$ can be identified with the full subcategory spanned by those functors F which are right Kan extensions of their restrictions to $\mathcal{C} \hookrightarrow \mathcal{C} \times_{\mathrm{Fun}(\{0\}, \mathcal{D})} \mathrm{Fun}(\Delta^1, \mathcal{D})$. Assertion ($*$) now follows from Proposition T.4.3.2.15. A similar argument gives:

($*$) Let $\mathrm{Fun}'_{\mathcal{D}}(\mathcal{C}', \mathcal{E}')$ denote the full subcategory of $\mathrm{Fun}_{\mathcal{D}}(\mathcal{C}', \mathcal{E}')$ spanned by those functors which carry p'-coCartesian morphisms to r-coCartesian morphisms. Then composition with u induces a trivial Kan fibration $\mathrm{Fun}'_{\mathcal{D}}(\mathcal{C}', \mathcal{E}') \to \mathrm{Fun}(\mathcal{C}', \mathcal{E})$

It follows that we can identify v with the map

$$\mathrm{Fun}'_{\mathcal{D}}(\mathcal{C}', \mathcal{E}')^{\simeq} \to \mathrm{Fun}'_{\mathcal{D}}(\mathcal{C}, \mathcal{E}')^{\simeq}.$$

Let $\nu : \mathcal{D} \to \mathcal{C}\mathrm{at}_\infty$ classify the coCartesian fibration r, so that v is given by the map $\mathrm{Map}_{\mathrm{Fun}(\mathcal{D}, \mathcal{C}\mathrm{at}_\infty)}(\chi', \nu) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{D}, \mathcal{C}\mathrm{at}_\infty)}(\chi, \nu)$ given by composition with the natural transformation α. The desired result now follows from the construction of the natural transformation χ'. $\square$

Corollary 2.4.20. *Let k be a field and let W be the collection of all morphisms in $\mathrm{Rep}_{\mathrm{dg}}^{\otimes \otimes}$ of the form*

$$\alpha : (\mathfrak{g}_*, V_*^1, \ldots, V_*^n) \to (\mathfrak{g}_*, V_*'^1, \ldots, V_*'^n)$$

where the image of α in both $\mathrm{Lie}_k^{\mathrm{dg}}$ and $\mathcal{F}\mathrm{in}_*$ is an identity map, and α induces a quasi-isomorphism $V_*^i \to V_*'^i$ for $1 \leq i \leq n$. Then we have a coCartesian fibration $\mathrm{Rep}_{\mathrm{dg}}^{\otimes \otimes}[W^{-1}] \to \mathrm{N}(\mathrm{Lie}_k^{\mathrm{dg}})^{op} \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$. For every differential graded Lie algebra $\mathfrak{g}_*$ over k, we can identify the fiber $\mathrm{Rep}_{\mathrm{dg}}^{\otimes \otimes}[W^{-1}]_{\mathfrak{g}_*} = \mathrm{Rep}_{\mathrm{dg}}^{\otimes \otimes}[W^{-1}] \times_{\mathrm{N}(\mathrm{Lie}_k^{\mathrm{dg}})^{op}} \{\mathfrak{g}_*\}$ with the symmetric monoidal ∞-category $\mathrm{Rep}_{\mathfrak{g}_*}^{\otimes}$.

The Chevalley-Eilenberg construction $V_* \mapsto C^*(\mathfrak{g}_*; V_*)$ is a lax symmetric monoidal functor. For every pair of representations $V_*, W_* \in \mathrm{Rep}_{\mathfrak{g}_*}^{\mathrm{dg}}$, there is a canonical map

$$C^*(\mathfrak{g}_*; V_*) \otimes_k C^*(\mathfrak{g}_*; W_*) \to C^*(\mathfrak{g}_*; V_* \otimes_k W_*),$$

which classifies bilinear maps

$$C^p(\mathfrak{g}_*; V_*) \times C^q(\mathfrak{g}_*; W_*) \to C^{p+q}(\mathfrak{g}_*; V_* \otimes_k W_*),$$

which carries a class $\lambda \in C^p(\mathfrak{g}_*; V_*)$ and $\mu \in C^q(\mathfrak{g}_*; W_*)$ to the element $\lambda\mu \in C^{p+q}(\mathfrak{g}_*; V_* \otimes_k W_*)$ given by

$$(\lambda\mu)(x_1 \ldots x_n) = \sum_{S, S'} \epsilon(S, S') \lambda(x_{i_1} \ldots x_{i_m}) \otimes \mu(x_{j_1} \ldots x_{j_{n-m}}).$$

Remark 2.4.21. Taking V_* and W_* to be the trivial representation of $\mathfrak{g}_*$, we recover the multiplication on $C^*(\mathfrak{g}_*)$ described in Construction 2.2.13. Taking V_* to be the trivial representation, we recover the action of $C^*(\mathfrak{g}_*)$ on $C^*(\mathfrak{g}_*; W_*)$ described in Remark 2.4.8. It follows from general nonsense that the multiplication maps

$$C^*(\mathfrak{g}_*; V_*) \otimes_k C^*(\mathfrak{g}_*; W_*) \to C^*(\mathfrak{g}_*; V_* \otimes_k W_*)$$

are $C^*(\mathfrak{g}_*)$-bilinear, and therefore descend to give maps

$$C^*(\mathfrak{g}_*; V_*) \otimes_{C^*(\mathfrak{g}_*)} C^*(\mathfrak{g}_*; W_*) \to C^*(\mathfrak{g}_*; V_* \otimes_k W_*).$$

Notation 2.4.22. Let $\mathcal{C}$ be a symmetric monoidal ∞-category. We let $\mathrm{Mod}(\mathcal{C})^{\otimes} = \mathrm{Mod}^{\mathrm{Comm}}(\mathcal{C})^{\otimes}$ be as in Definition A.3.3.3.8: more informally, the objects of $\mathrm{Mod}(\mathcal{C})^{\otimes}$ are given by tuples $(A, M_1, \ldots, M_n)$ where $A \in \mathrm{CAlg}(\mathcal{C})$ and each M_i is a module over A. If $\mathcal{C} = \mathrm{N}(\mathcal{C}_0)$ is isomorphic to the nerve of a symmetric monoidal category $\mathcal{C}_0$, then $\mathrm{Mod}(\mathcal{C})^{\otimes}$ is also isomorphic to the nerve of a category, which we will denote by $\mathrm{Mod}(\mathcal{C}_0)^{\otimes}$.

The lax symmetric monoidal structure on the functor $C^*(\mathfrak{g}_*; \bullet)$, and its dependence on $\mathfrak{g}_*$, are encoded by a map of categories

$$\mathrm{Rep}_{\mathrm{dg}}^{\otimes \otimes} \to \mathrm{Mod}(\mathrm{Mod}_k^{\mathrm{dg}})^{\otimes},$$

given on objects by

$$(\mathfrak{g}_*, V_*^1, \ldots, V_*^n) \mapsto (C^*(\mathfrak{g}_*), C^*(\mathfrak{g}_*; V_*^1), \ldots, C^*(\mathfrak{g}_*; V_*^n)).$$

Composing this with the map of symmetric monoidal functor $\mathrm{N}(\mathrm{Mod}_k^{\mathrm{dg}}) \to \mathrm{Mod}_k$, we obtain a map $\mathrm{N}(\mathrm{Rep}_{\mathrm{dg}}^{\otimes\,\otimes}) \to \mathrm{Mod}(\mathrm{Mod}_k)^{\otimes}$. If the field k has characteristic zero, then Proposition 2.4.10 implies that this functor carries morphisms of W (where W is defined as in Corollary 2.4.20) to equivalences in $\mathrm{Mod}(\mathrm{Mod}_k)^{\otimes}$, and therefore induces a lax symmetric monoidal functor

$$G : \mathrm{N}(\mathrm{Rep}_{\mathrm{dg}}^{\otimes\,\otimes})[W^{-1}] \to \mathrm{N}(\mathrm{Lie}_k^{\mathrm{dg}})^{op} \times_{\mathrm{CAlg}_k} \mathrm{Mod}(\mathrm{Mod}_k)^{\otimes}.$$

Proposition 2.4.23. *Let k be a field of characteristic zero, and consider the commutative diagram*

$$\begin{tikzcd} \mathrm{N}(\mathrm{Rep}_{\mathrm{dg}}^{\otimes\,\otimes})[W^{-1}] \arrow[rr, "G"] \arrow[dr, "p"'] & & \mathrm{N}(\mathrm{Lie}_k^{\mathrm{dg}})^{op} \times_{\mathrm{CAlg}_k} \mathrm{Mod}(\mathrm{Mod}_k)^{\otimes} \arrow[dl, "q"] \\ & \mathrm{N}(\mathrm{Lie}_k^{\mathrm{dg}})^{op} \times \mathrm{N}(\mathcal{F}\mathrm{in}_*). & \end{tikzcd}$$

Then:

(1) *The functor G admits a left adjoint F relative to $\mathrm{N}(\mathrm{Lie}_k^{\mathrm{dg}})^{op} \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ (see Definition A.7.3.2.2).*

(2) *The functor F carries q-coCartesian morphisms to p-coCartesian morphisms.*

Remark 2.4.24. We can summarize Proposition 2.4.23 more informally as follows. For every differential graded Lie algebra $\mathfrak{g}_*$ over k, the construction $V_* \mapsto C^*(\mathfrak{g}_*; V_*)$ determines a lax symmetric monoidal functor from $\mathrm{Rep}_{\mathfrak{g}_*}$ to $\mathrm{Mod}_{C^*(\mathfrak{g}_*)}$. This functor admits a symmetric monoidal left adjoint $f : \mathrm{Mod}_{C^*(\mathfrak{g}_*)} \to \mathrm{Rep}_{\mathfrak{g}_*}$. Moreover, the functor f depends functorially on the differential graded Lie algebra $\mathfrak{g}_*$.

Proof of Proposition 2.4.23. We will prove the existence of F; it will then follow from the fact that F admits a right adjoint relative to $\mathrm{N}(\mathrm{Lie}_k^{\mathrm{dg}})^{op} \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ that F carries q-coCartesian morphisms to p-coCartesian morphisms (see Proposition A.7.3.2.6). To prove the existence of F, we will check that G satisfies the criterion of Proposition A.7.3.2.11. For each differential graded Lie algebra $\mathfrak{g}_*$ and each $\langle n \rangle \in \mathrm{N}(\mathcal{F}\mathrm{in}_*)$, the induced functor

$$G_{\mathfrak{g}_*, \langle n \rangle} : \mathrm{Rep}_{\mathrm{dg}}^{\otimes\,\otimes}[W^{-1}]_{\mathfrak{g}_*, \langle n \rangle} \to (\mathrm{Mod}_{C^*(\mathfrak{g}_*)})^{\otimes}_{\langle n \rangle}$$

is equivalent to a product of n copies of the functor $C^*(\mathfrak{g}_*; \bullet) : \mathrm{Rep}_{\mathfrak{g}_*} \to \mathrm{Mod}_{C^*(\mathfrak{g}_*)}$, and therefore admits a left adjoint $f_{\mathfrak{g}_*}$ by Remark 2.4.11. Unwinding the definitions, we are reduced to proving that for every finite sequence of $C^*(\mathfrak{g}_*)$-modules $M_1, \ldots, M_n$, and every map of differential graded Lie algebras $\mathfrak{h}_* \to \mathfrak{g}_*$, the canonical map

$$f_{\mathfrak{h}_*}(C^*(\mathfrak{h}_*) \otimes_{C^*(\mathfrak{g}_*)} M_1 \otimes_{C^*(\mathfrak{g}_*)} \cdots \otimes_{C^*(\mathfrak{g}_*)} M_n) \to f_{\mathfrak{g}_*}(M_1) \otimes_k \cdots \otimes_k f_{\mathfrak{g}_*}(M_n)$$

is an equivalence. We observe that both sides are compatible with colimits in each M_i (see Remark 2.4.17). Since $\mathrm{Mod}_{C^*(\mathfrak{g}_*)}$ is generated under small colimits by the modules $C^*(\mathfrak{g}_*)[k]$ for $k \in \mathbf{Z}$, we can reduce to the case where $M_i = C^*(\mathfrak{g}_*)$ for $1 \leq i \leq n$. In this case, the result is obvious. $\square$

Construction 2.4.25. Let k be a field. The coCartesian fibration $\mathrm{Mod}(\mathrm{Mod}_k)^{\otimes} \to \mathrm{CAlg}_k \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is classified by a map $\chi : \mathrm{CAlg}_k \to \mathrm{Mon}_{\mathrm{Comm}}(\widehat{\mathcal{C}\mathrm{at}}_\infty) \simeq \mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty)$, which carries an $\mathbb{E}_\infty$-algebra A over k to Mod_A, regarded as a symmetric monoidal ∞-category. Let χ^{sm} denote the restriction of χ to the full subcategory $\mathrm{CAlg}_k^{\mathrm{sm}} \subseteq \mathrm{CAlg}_k$ spanned by the small $\mathbb{E}_\infty$-algebras over k. Applying Theorem T.5.1.5.6, we deduce that χ^{sm} admits an essentially unique factorization as a composition

$$\mathrm{CAlg}_k^{\mathrm{sm}} \xrightarrow{j} \mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{sm}}, \mathcal{S})^{op} \xrightarrow{\mathrm{QCoh}} \mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty),$$

where the functor QCoh preserves small limits. For every functor $X : \mathrm{CAlg}_k^{\mathrm{sm}} \to \mathcal{S}$, we will regard $\mathrm{QCoh}(X) \in \mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty)$ as a symmetric monoidal ∞-category, which we call the *∞-category of quasi-coherent sheaves on X*.

Remark 2.4.26. Let $\mathcal{P}r^{\mathrm{L}} \subseteq \widehat{\mathcal{C}\mathrm{at}}_\infty$ denote the subcategory whose objects are presentable ∞-categories and whose morphisms are colimit preserving functors, and regard $\mathcal{P}r^{\mathrm{L}}$ as a symmetric monoidal ∞-category as explained in §A.6.3.1. Note that the functor χ of Construction 2.4.25 factors through $\mathrm{CAlg}(\mathcal{P}r^{\mathrm{L}}) \subseteq \mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty)$. Since this inclusion preserves small limits, we deduce that the functor

$$\mathrm{QCoh} : \mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{sm}}, \mathcal{S})^{op} \to \mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty)$$

also factors through $\mathrm{CAlg}(\mathcal{P}r^{\mathrm{L}})$. In other words:

(a) For every functor $X : \mathrm{CAlg}_k^{\mathrm{sm}} \to \mathcal{S}$, the ∞-category $\mathrm{QCoh}(X)$ is presentable.

(b) For every functor $X : \mathrm{CAlg}_k^{\mathrm{sm}} \to \mathcal{S}$, the tensor product $\otimes : \mathrm{QCoh}(X) \times \mathrm{QCoh}(X) \to \mathrm{QCoh}(X)$ preserves small colimits separately in each variable.

(c) For every natural transformation $f : X \to Y$ of functors $X, Y : \mathrm{CAlg}_k^{\mathrm{sm}} \to \mathcal{S}$, the induced functor $f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ preserves small colimits.

Remark 2.4.27. Let k be a field and let $X : \mathrm{CAlg}_k^{\mathrm{sm}} \to \mathcal{S}$ be a functor which classifies a right fibration $\mathcal{X} \to \mathrm{CAlg}_k^{\mathrm{sm}}$. Then $\mathrm{QCoh}(X)$ ∞-categories of coCartesian sections of the coCartesian fibration $\mathcal{X} \times_{\mathrm{CAlg}_k} \mathrm{Mod}(\mathrm{Mod}_k) \to \mathcal{X}$. More informally, an object $\mathcal{F} \in \mathrm{QCoh}(X)$ is a rule which assigns to every point $\eta \in X(A)$ an A-module $\mathcal{F}_\eta$, and to every morphism $f : A \to A'$ carrying η to $\eta' \in X(A')$ an equivalence $\mathcal{F}_{\eta'} \simeq A' \otimes_A \mathcal{F}_\eta$.

Construction 2.4.28. Let k be a field of characteristic zero. The coCartesian fibration $\mathrm{Rep}_{\mathrm{dg}}^{\otimes\,\otimes}[W^{-1}] \to \mathrm{N}(\mathrm{Lie}_k^{\mathrm{dg}})^{op} \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ of Corollary 2.4.20 classifies a functor $\overline{\chi}_0 : \mathrm{N}(\mathrm{Lie}_k^{\mathrm{dg}})^{op} \to \mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty)$, given on objects by $\mathfrak{g}_* \mapsto \mathrm{Rep}_{\mathfrak{g}_*}$. If $\phi : \mathfrak{h}_* \to \mathfrak{g}_*$ is a quasi-isomorphism of differential graded Lie algebras, then the induced map $U(\mathfrak{h}_*) \to U(\mathfrak{g}_*)$ is an equivalence in Alg_k, so that the forgetful functor $\mathrm{Rep}_{\mathfrak{g}_*} \to \mathrm{Rep}_{\mathfrak{h}_*}$ is an equivalence of ∞-categories. It follows that $\overline{\chi}_0$ induces a functor $\overline{\chi} : \mathrm{Lie}_k^{op} \to \mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty)$.

We let $\chi_!^{\mathrm{sm}} : \mathrm{CAlg}_k^{\mathrm{sm}} \to \mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty)$ denote the composition of $\overline{\chi}$ with the Koszul duality functor $\mathfrak{D} : (\mathrm{CAlg}_k^{\mathrm{sm}})^{op} \to \mathrm{Lie}_k$ studied in §2.3. Applying Theorem T.5.1.5.6, we deduce that $\chi_!^{\mathrm{sm}}$ admits an essentially unique factorization as a composition

$$\mathrm{CAlg}_k^{\mathrm{sm}} \xrightarrow{j} \mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{sm}}, \mathcal{S})^{op} \xrightarrow{\mathrm{QCoh}^!} \mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty),$$

where j denotes the Yoneda embedding and the functor $\mathrm{QCoh}^!$ preserves small limits.

Remark 2.4.29. If $f : X \to Y$ is a natural transformation between functors $X, Y : \mathrm{CAlg}_k^{\mathrm{sm}} \to \mathcal{S}$, we will denote the induced functor $\mathrm{QCoh}^!(Y) \to \mathrm{QCoh}^!(X)$ by $f^!$.

The functor $\chi_!^{\mathrm{sm}}$ appearing in Construction 2.4.28 factors through subcategory $\mathrm{CAlg}(\mathcal{P}r^{\mathrm{L}}) \subseteq \mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty)$. As in Remark 2.4.26, we deduce that the functor $\mathrm{QCoh}^!$ factors through $\mathrm{CAlg}(\mathcal{P}r^{\mathrm{L}})$. That is, each of the ∞-categories $\mathrm{QCoh}^!(X)$ is presentable, each of the functors $f^! : \mathrm{QCoh}^!(Y) \to \mathrm{QCoh}^!(X)$ preserves small colimits, and the tensor product functors $\mathrm{QCoh}^!(X) \times \mathrm{QCoh}^!(X) \to \mathrm{QCoh}^!(X)$ preserve small colimits separately in each variable.

Remark 2.4.30. For $A \in \mathrm{Alg}_k^{\mathrm{sm}}$, the biduality map $A \to C^*(\mathfrak{D}(A))$ is an equivalence. It follows that the functor χ^{sm} of Construction 2.4.25 is given by the composition

$$\mathrm{CAlg}_k^{\mathrm{sm}} \xrightarrow{\mathfrak{D}} \mathrm{Lie}_k^{op} \xrightarrow{C^*} \mathrm{CAlg}_k^{\mathrm{aug}} \to \mathrm{CAlg}_k \xrightarrow{\chi} \mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty).$$

The functor F of Proposition 2.4.23 induces a natural transformation $\chi^{\mathrm{sm}} \to \chi_!^{\mathrm{sm}}$, and therefore a natural transformation $\mathrm{QCoh} \to \mathrm{QCoh}^!$ of functors $\mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{sm}}, \mathcal{S})^{op} \to \mathrm{CAlg}(\mathcal{P}r^{\mathrm{L}})$.

Let $A \in \mathrm{CAlg}_k^{\mathrm{sm}}$ and let $\mathfrak{g}_* = \mathfrak{D}(A)$ be its Koszul dual. Since A is small, there exists a sequence of maps

$$A = A_n \to A_{n-1} \to \cdots \to A_0 = k$$

where each A_i is a square-zero extension of A_{i-1} by $k[n_i]$ for some $n_i \geq 0$. We therefore have a sequence of differential graded Lie algebras

$$0 = \mathfrak{D}(A_0) \to \mathfrak{D}(A_1) \to \cdots \to \mathfrak{D}(A_n) \simeq \mathfrak{g}_*,$$

where each $\mathfrak{D}(A_i)$ is obtained from $\mathfrak{D}(A_{i-1})$ by adjoining a cell in dimension $-n_i - 1$. It follows that, up to quasi-isomorphism, $\mathfrak{g}_*$ satisfies the hypotheses of Proposition 2.4.12 and Proposition 2.4.16. We conclude that the natural transformation $\chi^{\mathrm{sm}} \to \chi^{\mathrm{sm}}_!$ induces a (symmetric monoidal) fully faithful embedding

$$\chi^{\mathrm{sm}}(A) \simeq \mathrm{Mod}_A \simeq \mathrm{Mod}_{C^*(\mathfrak{g}_*)} \to \mathrm{Rep}_{\mathfrak{g}_*} \simeq \chi^{\mathrm{sm}}_!(A),$$

which restricts to an equivalence of ∞-categories $\mathrm{Mod}^{\mathrm{cn}}_A \to \mathrm{Rep}^{\mathrm{cn}}_{\mathfrak{g}_*}$. It follows that the natural transformation $\mathrm{QCoh} \to \mathrm{QCoh}^!$ determines a (symmetric monoidal) fully faithful embedding $\mathrm{QCoh}(X) \to \mathrm{QCoh}^!(X)$ for each $X \in \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{sm}}_k, \mathcal{S})$.

We now turn to the proof of Theorem 2.4.1. Let k be a field of characteristic zero and let $X : \mathrm{CAlg}^{\mathrm{sm}}_k \to \mathcal{S}$ be a formal moduli problem, given by $\Psi(\mathfrak{g}_*)$ for some differential graded Lie algebra $\mathfrak{g}_*$. Remark 2.4.30 supplies a symmetric monoidal fully faithful embedding $\mathrm{QCoh}(X) \hookrightarrow \mathrm{QCoh}^!(X)$. To prove Theorem 2.4.1, it will suffice to prove the following:

Proposition 2.4.31. *Let k be a field of characteristic zero, let $\mathfrak{g}_*$ be a differential graded Lie algebra over k and let $X = \Psi(\mathfrak{g}_*)$ be the formal moduli problem given by $X(R) = \mathrm{Map}_{\mathrm{Lie}_k}(\mathfrak{D}(R), \mathfrak{g}_*)$. Then there is a canonical equivalence of symmetric monoidal ∞-categories $\mathrm{QCoh}^!(X) \simeq \mathrm{Rep}_{\mathfrak{g}_*}$.*

Lemma 2.4.32. *Let k be a field and let $\nu : \mathrm{Alg}^{op}_k \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ classify the Cartesian fibration $\mathrm{LMod}(\mathrm{Mod}_k) \to \mathrm{Alg}_k$ (so that ν is given by the formula $\nu(A) = \mathrm{LMod}_A$). Then ν preserves K-indexed limits for every weakly contractible simplicial set K.*

Proof. Let $\mathcal{P}\mathrm{r}^{\mathrm{R}}$ denote the subcategory of $\widehat{\mathcal{C}\mathrm{at}}_\infty$ whose objects are presentable ∞-categories and whose morphisms are functors which admit left adjoints, and define $\mathcal{P}\mathrm{r}^{\mathrm{L}} \subseteq \widehat{\mathcal{C}\mathrm{at}}_\infty$ similarly. Note that the functor ν factors through $\mathcal{P}\mathrm{r}^{\mathrm{R}}$, and that the inclusion $\mathcal{P}\mathrm{r}^{\mathrm{R}} \subseteq \widehat{\mathcal{C}\mathrm{at}}_\infty$ preserves small limits (Theorem T.5.5.3.18). It will therefore suffice to show that if K is weakly contractible, then ν carries K-indexed limits in Alg^{op}_k to K-indexed limits in $\mathcal{P}\mathrm{r}^{\mathrm{R}}$. Using the equivalence $\mathcal{P}\mathrm{r}^{\mathrm{L}} \simeq (\mathcal{P}\mathrm{r}^{\mathrm{R}})^{op}$ of Corollary T.5.5.3.4, we can identify ν with a functor $\mu : \mathrm{Alg}_k \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ (the functor μ classifies the coCartesian fibration $\mathrm{LMod}(\mathrm{Mod}_k) \to \mathrm{Alg}_k$). Theorem A.6.3.5.10 implies that the functor $\mathrm{Alg}_k \simeq (\mathrm{Alg}_k)_{k/} \to (\mathcal{P}\mathrm{r}^{\mathrm{L}})_{\mathrm{Mod}_k/}$ admits a right adjoint, and therefore preserves all small colimits. It therefore suffices to verify that the forgetful functor $(\mathcal{P}\mathrm{r}^{\mathrm{L}})_{\mathrm{Mod}_k/} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ preserves K-indexed colimits, which follows from Proposition T.4.4.2.9. $\square$

Lemma 2.4.33. *Let k be a field of characteristic zero, and let $\overline{\chi} : \mathrm{Lie}^{op}_k \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ be as in Construction 2.4.28. Then $\overline{\chi}$ preserves K-indexed limits for every weakly contractible simplicial set K.*

Proof. The functor $\overline{\chi}$ factors as a composition $\mathrm{Lie}^{op}_k \xrightarrow{U} \mathrm{Alg}^{op}_k \xrightarrow{\nu} \widehat{\mathcal{C}\mathrm{at}}_\infty$, where ν preserves K-indexed limits by Lemma 2.4.32, and U preserves all small limits (since it is right adjoint to the forgetful functor $\mathrm{Alg}^{op}_k \to \mathrm{Lie}^{op}_k$). $\square$

Proof of Proposition 2.4.1. Let $\Psi : \mathrm{Lie}_k \to \mathrm{Moduli}_k$ be the equivalence of ∞-categories appearing in Theorem 2.0.2, and let Ψ^{-1} denote a homotopy inverse to Ψ. Let $L : \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{sm}}_k, \mathcal{S}) \to \mathrm{Moduli}_k$ denote a left adjoint to the inclusion functor $\mathrm{Moduli}_k \subseteq \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{sm}}_k, \mathcal{S})$ (see Remark 1.1.17), and let $\widehat{\mathfrak{D}} : \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{sm}}_k, \mathcal{S}) \to \mathrm{Alg}^{\mathrm{aug}}_k$ be the composition $\Psi^{-1} \circ L$. The functor $\widehat{\mathfrak{D}}$ preserves small colimits, and the composition of $\widehat{\mathfrak{D}}$ with the Yoneda embedding $(\mathrm{CAlg}^{\mathrm{sm}}_k)^{op} \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{sm}}_k, \mathcal{S})$ can be identified with the Koszul duality functor $\mathfrak{D} : (\mathrm{Alg}^{\mathrm{sm}}_k)^{op} \to \mathrm{Alg}^{\mathrm{aug}}_k$. Let $\overline{\chi} : \mathrm{Lie}^{op}_k \to \mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty)$ be as in Construction 2.4.28 (given on objects by $\overline{\chi}(\mathfrak{g}_*) = \mathrm{Rep}_{\mathfrak{g}_*}$), and let $F : \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{sm}}_k, \mathcal{S})^{op} \to \mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty)$ denote the composite functor

$$\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{sm}}_k, \mathcal{S})^{op} \xrightarrow{\widehat{\mathfrak{D}}} \mathrm{Lie}^{op}_k \xrightarrow{\overline{\chi}} \mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty).$$

Let $\mathcal{C}$ denote the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{sm}}, \mathcal{S})$ spanned by the corepresentable functors. By construction, the functors F and $\mathrm{QCoh}^!$ agree on the ∞-category $\mathcal{C}$, and by construction $\mathrm{QCoh}^!$ is a right Kan extension of its restriction to $\mathcal{C}$. We therefore have a canonical natural transformation $\alpha : F \to \mathrm{QCoh}^!$. We will prove the following:

(∗) If $X : \mathrm{CAlg}_k^{\mathrm{sm}} \to \mathcal{S}$ is a formal moduli problem, then α induces an equivalence of ∞-categories $F(X) \to \mathrm{QCoh}^!(X))$.

Taking $X = \Psi(\mathfrak{g}_*)$ for $A \in \mathrm{Alg}_k^{\mathrm{aug}}$, we see that (∗) guarantees an equivalence of symmetric monoidal ∞-categories ∞-categories $\mathrm{Rep}_{\mathfrak{g}_*} \simeq F(X) \to \mathrm{QCoh}^!(X)$.

It remains to prove (∗). Let $\mathcal{E} \subseteq \mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{sm}}, \mathcal{S})$ be the full subcategory spanned by those functors X for which α induces an equivalence of ∞-categories $F(X) \to \mathrm{QCoh}^!(X)$. The localization functor $L : \mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{sm}}, \mathcal{S}) \to \mathrm{Moduli}_k$, the equivalence $\Psi^{-1} : \mathrm{Moduli}_k \to \mathrm{Lie}_k$ both preserve small colimits. It follows from Lemma 2.4.33 that the functor $\overline{\chi} : \mathrm{Lie}_k^{op} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ preserves sifted limits, so that F preserves sifted limits. Since the functor $\mathrm{QCoh}^!$ preserves small limits, the ∞-category $\mathcal{E}$ is closed under sifted colimits in $\mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{sm}}, \mathcal{S})$. Since $\mathcal{E}$ contains all corepresentable functors and is closed under filtered colimits, it contains it contains all prorepresentable formal moduli problems (see Definition 1.5.3). Proposition 1.5.8 implies that every formal moduli problem X can be obtained as the geometric realization of a simplicial object $X_\bullet$ of $\mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{sm}}, \mathcal{S})$, where each X_n is prorepresentable. Since $\mathcal{E}$ is closed under geometric realizations in $\mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{sm}}, \mathcal{S})$, we conclude that $X \in \mathcal{E}$ as desired. □

We conclude this section with a simple observation about connectivity conditions. Let $X : \mathrm{CAlg}_k^{\mathrm{sm}} \to \mathcal{S}$ be a formal moduli problem and let $\mathcal{F} \in \mathrm{QCoh}(X)$ be a quasi-coherent sheaf on X, so that $\mathcal{F}$ determines an A-module $\mathcal{F}_\eta$ for every $\eta \in X(A)$ (see Remark 2.4.27). We will say that $\mathcal{F}$ is *connective* if each $\mathcal{F}_\eta \in \mathrm{Mod}_A$ is connective. We let $\mathrm{QCoh}(X)^{\mathrm{cn}}$ denote the full subcategory of $\mathrm{QCoh}(X)$ spanned by the connective objects. It is easy to see that $\mathrm{QCoh}(X)^{\mathrm{cn}}$ is a presentable ∞-category which is closed under colimits and extensions in $\mathrm{QCoh}(X)$, and therefore determines an accessible t-structure on $\mathrm{QCoh}(X)$ (see Proposition A.1.4.5.11).

Proposition 2.4.34. *Let k be a field of characteristic zero, let $\mathfrak{g}_*$ be a differential graded Lie algebra over k and let $X = \Psi(\mathfrak{g}_*)$ be the associated formal moduli problem. Then the fully faithful embedding $\theta : \mathrm{QCoh}(X) \to \mathrm{Rep}_{\mathfrak{g}_*}$ induces an equivalence of ∞-categories $\mathrm{QCoh}(X)^{\mathrm{cn}} \to \mathrm{Rep}_{\mathfrak{g}_*}^{\mathrm{cn}}$.*

Proof. If $\phi : \mathfrak{h}_* \to \mathfrak{g}_*$ is a map of differential graded Lie algebras over k inducing a map of formal moduli problems $Y \to X$, then the diagram

$$\begin{tikzcd} \mathrm{QCoh}(X) \arrow[r] \arrow[d] & \mathrm{Rep}_{\mathfrak{g}_*} \arrow[d] \\ \mathrm{QCoh}(Y) \arrow[r] & \mathrm{Rep}_{\mathfrak{g}_*} \end{tikzcd}$$

commutes up to canonical homotopy. Taking $\mathfrak{h}_* = 0$, we deduce that the composite functor $\mathrm{QCoh}(X) \to \mathrm{Rep}_{\mathfrak{g}_*} \to \mathrm{Mod}_k$ is given by evaluation at the base point $\eta_0 \in X(k)$. In particular, we deduce that θ carries $\mathrm{QCoh}(X)^{\mathrm{cn}}$ into

$$\mathrm{Rep}_{\mathfrak{g}_*}^{\mathrm{cn}} = \mathrm{Rep}_{\mathfrak{g}_*} \times_{\mathrm{Mod}_k} \mathrm{Mod}_k^{\mathrm{cn}}.$$

To complete the proof, it will suffice to show that if $V \in \mathrm{Mod}_{\mathfrak{g}_*}^{\mathrm{cn}}$, then V_* belongs to the essential image of θ. To prove this, it suffices to show that for every point $\eta \in X(A)$ classified by a map of differential graded Lie algebras $\mathfrak{D}(A) \to \mathfrak{g}_*$, the image of V in $\mathrm{Rep}_{\mathfrak{D}(A)}$ belongs to the essential image of the functor $\mathrm{QCoh}(\mathrm{Spec}\, A) \to \mathrm{Rep}_{\mathfrak{D}(A)}$. Since V is connective, this follows from Proposition 2.4.16 (note that $\mathfrak{D}(A)$ satisfies the hypotheses of Propositions 2.4.12 and 2.4.16; see Remark 2.4.30. □

3 Moduli Problems for Associative Algebras

Let A be a connective $\mathbb{E}_\infty$-ring. We say that an A-module spectrum M is *projective of rank* n if the following conditions are satisfied:

(1) The group $\pi_0 M$ is a projective $\pi_0 A$-module of rank n.

(2) For every integer n, the canonical map

$$\mathrm{Tor}_0^{\pi_0 A}(\pi_n A, \pi_0 M) \to \pi_n M$$

is an isomorphism (that is, M is flat over A).

Let $X(A)$ denote the subcategory of Mod_A whose objects are A-modules which are projective of rank n, and whose morphisms are equivalences. Then $X(A)$ is an essentially small Kan complex. The construction $A \mapsto X(A)$ determines a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Let us fix a field k and a point $\eta \in X(k)$, corresponding to a vector space V of dimension n over k. The formal completion of X (at the point η) is the functor $X^\wedge : \mathrm{CAlg}_k^{\mathrm{sm}} \to \mathcal{S}$ given by

$$X^\wedge(R) = X(R) \times_{X(k)} \{\eta\}.$$

More informally, $X^\wedge(R)$ is a classifying space for pairs (M, α), where M is a projective R-module of rank n and $\alpha : k \otimes_R M \simeq V$ is an isomorphism of k-vector spaces. It is not difficult to see that the functor $X^\wedge : \mathrm{CAlg}_k^{\mathrm{sm}} \to \mathcal{S}$ is a formal moduli problem (we will give a proof of a stronger assertion in §5.2).

Assume now that k is a field of characteristic zero. According to Theorem 2.0.2, the functor $X^\wedge : \mathrm{CAlg}_k^{\mathrm{sm}} \to \mathcal{S}$ is determined (up to equivalence) by a differential graded Lie algebra $\mathfrak{g}_*$. Let $T_{X^\wedge}$ denote the tangent complex of $X^\wedge$, so that $T_{X^\wedge}[-1]$ can be identified with the spectrum underlying the chain complex of vector spaces $\mathfrak{g}_*$. Then the space $\Omega^\infty T_{X^\wedge} \simeq X^\wedge(k[\epsilon]/(\epsilon^2))$ can be identified with a classifying space for the groupoid of order deformations of the vector space V: that is, projective $k[\epsilon]/(\epsilon^2)$-modules M equipped with an isomorphism $M/\epsilon M \simeq V$. Since any basis of V can be lifted to a basis for M, this groupoid has only one isomorphism class of objects (which is represented by the module $V[\epsilon]/\epsilon^2 V$). We conclude that $\Omega^\infty T_{X^\wedge}$ is homotopy equivalent to the classifying space BG, where G is the group of automorphisms of $V[\epsilon]/\epsilon^2 V$ which reduce to the identity automorphism modulo ϵ. Every such automorphism can be written uniquely $1 + \epsilon M$, where $M \in \mathrm{End}(V)$ is an endomorphism of V. From this we deduce that the homology of the chain complex $\mathfrak{g}_*$ is isomorphic to $\mathrm{End}(V)$ in degree zero and vanishes in positive degrees. It also vanishes in negative degrees: this follows from the observation that each of the spaces $X^\wedge(k \oplus k[n])$ is connected (any basis for V can be lifted to a basis for any $(k \oplus k[n])$-module deforming V). It follows that $\mathfrak{g}_*$ is quasi-isomorphic to an ordinary Lie algebra $\mathfrak{g}$ over k (concentrated in degree zero), whose underlying abelian group is isomorphic to $\mathrm{End}(V)$. With a bit more effort, we can show that the isomorphism $\mathfrak{g} \simeq \mathrm{End}(V)$ is an isomorphism of Lie algebras: that is, the Lie bracket on $\mathfrak{g}$ can be identified with the usual commutator bracket $[A, B] = AB - BA$ of k-linear endomorphisms of V (see Example 5.2.9).

However, there is more to the story. If $R \in \mathrm{CAlg}_k^{\mathrm{sm}}$, then any connective R-module M equipped with an equivalence $k \otimes_R M \simeq V$ is automatically projective of rank n. We can therefore identify $X^\wedge(R)$ with the fiber product

$$\mathrm{LMod}_R^{\mathrm{cn}} \times_{\mathrm{LMod}_k^{\mathrm{cn}}} \{V\}.$$

This description of $X^\wedge(R)$ makes no reference to the commutativity of R. We can therefore extend $X^\wedge$ to a functor $X_+^\wedge : \mathrm{Alg}_k^{\mathrm{sm}} \to \mathcal{S}$, where $\mathrm{Alg}_k^{\mathrm{sm}}$ denotes the ∞-category of small $\mathbb{E}_1$-algebras over k (see Definition 3.0.1 below). The existence of the extension $X_+^\wedge$ is a special property enjoyed by the formal moduli problem $X^\wedge$. Since $X^\wedge$ is completely determined by the Lie algebra $\mathrm{End}(V)$, we should expect that the existence of $X_+^\wedge$ reflects a special property of $\mathrm{End}(V)$. In fact, there is something special about $\mathrm{End}(V)$: it is the underlying Lie algebra of an associative algebra. We will see that this is a general phenomenon: if $\mathfrak{g}_*$ is a differential graded Lie algebra and $Y : \mathrm{CAlg}_k^{\mathrm{sm}} \to \mathcal{S}$ is the associated formal moduli problem for $\mathbb{E}_\infty$-algebras over k, then Y extends to a formal moduli problem $Y_+ : \mathrm{Alg}_k^{\mathrm{sm}} \to \mathcal{S}$ for $\mathbb{E}_1$-algebras over k if and only if $\mathfrak{g}_*$ is quasi-isomorphic to the underlying Lie algebra of a (nonunital) differential graded algebra A_* (see Example 2.1.6).

Our main goal in this section is to prove an analogue of Theorem 2.0.2 in the setting of noncommutative geometry. Before we can state our result, we need to introduce a bit of terminology.

Definition 3.0.1. Let k be a field. We let Alg_k denote the ∞-category of $\mathbb{E}_1$-algebras over k, and $\mathrm{Alg}_k^{\mathrm{aug}} = (\mathrm{Alg}_k)_{/k}$ the ∞-category of augmented $\mathbb{E}_1$-algebras over k. We will say that an object $A \in \mathrm{Alg}_k$ is *small* if it satisfies the following conditions:

(a) The algebra A is connective: that is, $\pi_i A \simeq 0$ for $i < 0$.

(b) The algebra A is truncated: that is, we have $\pi_i A \simeq 0$ for $i \gg 0$.

(c) Each of the homotopy groups $\pi_i A$ is finite dimensional when regarded as a vector space over field k.

(d) Let $\mathfrak{n}$ denote the radical of the ring $\pi_0 A$ (which is a finite-dimensional associative algebra over k). Then the canonical map $k \to (\pi_0 A)/\mathfrak{n}$ is an isomorphism.

We let $\mathrm{Alg}_k^{\mathrm{sm}}$ denote the full subcategory of Alg_k spanned by the small k-algebras.

Remark 3.0.2. Let k be a field and let $A \in \mathrm{Alg}_k^{\mathrm{sm}}$. It follows from conditions (a) and (d) of Definition 3.0.1 that the mapping space $\mathrm{Map}_{\mathrm{Alg}_k}(A, k)$ is contractible: that is, A admits an essentially unique augmentation. Consequently, the projection map

$$\mathrm{Alg}_k^{\mathrm{sm}} \times_{\mathrm{Alg}_k} \mathrm{Alg}_k^{\mathrm{aug}} \to \mathrm{Alg}_k^{\mathrm{sm}}$$

is an equivalence of ∞-categories. Because of this, we will often abuse notation by identifying $\mathrm{Alg}_k^{\mathrm{sm}}$ with its inverse image in $\mathrm{Alg}_k^{\mathrm{aug}}$.

Definition 3.0.3. Let k be a field and let $X : \mathrm{Alg}_k^{\mathrm{sm}} \to \mathcal{S}$ be a functor. We will say that X is a *formal* $\mathbb{E}_1$ *moduli problem* if it satisfies the following conditions:

(1) The space $X(k)$ is contractible.

(2) For every pullback diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R_0 \arrow[d] \\ R_1 \arrow[r] & R_{01} \end{tikzcd}$$

in $\mathrm{Alg}_k^{\mathrm{sm}}$ for which the underlying maps $\pi_0 R_0 \to \pi_0 R_{01} \leftarrow \pi_0 R_1$ are surjective, the diagram

$$\begin{tikzcd} X(R) \arrow[r] \arrow[d] & X(R_0) \arrow[d] \\ X(R_1) \arrow[r] & X(R_{01}) \end{tikzcd}$$

is a pullback square.

We let $\mathrm{Moduli}_k^{(1)}$ denote the full subcategory of $\mathrm{Fun}(\mathrm{Alg}_k^{\mathrm{sm}}, \mathcal{S})$ spanned by the formal $\mathbb{E}_1$ moduli problems.

We can now state our main result:

Theorem 3.0.4. *Let k be a field. Then there is an equivalence of ∞-categories $\Psi : \mathrm{Alg}_k^{\mathrm{aug}} \to \mathrm{Moduli}_k^{(1)}$.*

Remark 3.0.5. Unlike Theorem 2.0.2, Theorem 3.0.4 does not require any assumptions on the characteristic of the field k.

Like Theorem 2.0.2, Theorem 3.0.4 is a reflection of Koszul duality: this time in the setting of associative algebras. In §3.1, we will introduce the Koszul duality functor

$$\mathfrak{D}^{(1)} : (\mathrm{Alg}_k^{\mathrm{aug}})^{op} \to \mathrm{Alg}_k^{\mathrm{aug}}.$$

Roughly speaking, if $\epsilon : A \to k$ is an augmented $\mathbb{E}_1$-algebra over k, then the Koszul dual $\mathfrak{D}^{(1)}(A)$ is the (derived) endomorphism algebra of k as a (left) A-module. In §3.2, we will show that $\mathfrak{D}^{(1)}$ is a deformation theory (in the sense of Definition 1.3.9). We will then deduce Theorem 3.0.4 from Theorem 1.3.12, using the functor $\Psi : \mathrm{Alg}_k^{\mathrm{aug}} \to \mathrm{Moduli}_k^{(1)}$ given by

$$\Psi(A)(R) = \mathrm{Map}_{\mathrm{Alg}_k^{\mathrm{aug}}}(\mathfrak{D}^{(1)}(R), A).$$

For every field k, there is an evident forgetful functor $\mathrm{CAlg}_k^{\mathrm{sm}} \to \mathrm{Alg}_k^{\mathrm{sm}}$. Composition with this forgetful functor determines a map $\theta : \mathrm{Moduli}_k^{(1)} \to \mathrm{Moduli}_k$. In §3.3, we will show that if the characteristic of k is zero, then θ fits into a commutative diagram of ∞-categories

$$\begin{tikzcd} \mathrm{Alg}_k^{\mathrm{aug}} \arrow[r] \arrow[d, "\theta'"] & \mathrm{Moduli}_k^{(1)} \arrow[d, "\theta"] \\ \mathrm{Lie}_k \arrow[r] & \mathrm{Moduli}_k, \end{tikzcd}$$

where the upper horizontal map is the equivalence of Theorem 3.0.4 and the lower horizontal map is the equivalence of Theorem 2.0.2. Here θ' is a functor which assigns to each augmented $\mathbb{E}_1$-algebra $\epsilon : A \to k$ its augmentation ideal $\mathfrak{m}_A = \mathrm{fib}(\epsilon)$ (Proposition 3.3.2).

If X is a formal $\mathbb{E}_1$-moduli problem over k, then we can associate to X a pair of ∞-categories $\mathrm{QCoh}_L(X)$ and $\mathrm{QCoh}_R(X)$, which we call the *∞-categories of (left and right) quasi-coherent sheaves on X*. Roughly speaking, an object $\mathcal{F} \in \mathrm{QCoh}_L(X)$ is a rule which assigns to each point $\eta \in X(A)$ a left A-module $\mathcal{F}_\eta$, depending functorially on η (and $\mathrm{QCoh}_R(X)$ is defined similarly, using right modules in place of left). In §3.4, we will construct fully faithful embeddings

$$\mathrm{QCoh}_L(X) \hookrightarrow \mathrm{QCoh}^!_L(X) \qquad \mathrm{QCoh}_R(X) \hookrightarrow \mathrm{QCoh}^!_R(X).$$

In §3.5, we will use these constructions to formulate and prove a noncommutative analogue of Theorem 2.4.1: if $X = \Psi(A)$ is the formal moduli problem associated to an augmented $\mathbb{E}_1$-algebra A over k, then there are canonical equivalences of ∞-categories

$$\mathrm{QCoh}^!_L(X) \simeq \mathrm{RMod}_A \qquad \mathrm{QCoh}^!_R(X) \simeq \mathrm{LMod}_A$$

(Theorem 3.5.1). In particular, this gives fully faithful embeddings

$$\mathrm{QCoh}_L(X) \hookrightarrow \mathrm{RMod}_A \qquad \mathrm{QCoh}_R(X) \hookrightarrow \mathrm{LMod}_A,$$

which are equivalences when restricted to connective objects (Proposition 3.5.8).

3.1 Koszul Duality for Associative Algebras

Let k be a field, let A be an associative algebra over k, and let M be a left A-module. The *commutant* B of A in $\mathrm{End}_k(M)$ is defined to be the set of A-linear endomorphisms of M. Then B can be regarded as an associative algebra over k, and M admits the structure of a bimodule over A and B: that is, an action of the tensor product $A \otimes_k B$. In many cases, one can show that the relationship between A and B is symmetric. For example, if A is a finite dimensional central simple algebra over k and M is nonzero and of finite dimension over k, then we can recover A as the commutant of B in $\mathrm{End}_k(M)$.

In this section, we will discuss the operation of Koszul duality in the setting of (augmented) $\mathbb{E}_1$-algebras over a field k. Roughly speaking, Koszul duality can be regarded as a derived version of the formation of commutants. Suppose that A is an $\mathbb{E}_1$-algebra over k equipped with an augmentation $\epsilon : A \to k$. Then ϵ determines an action of A on the k-module $M = k$. The *Koszul dual* of A is an $\mathbb{E}_1$-algebra B over k which classifies A-linear maps from M to itself. We have commuting actions of A and B on M, which can be encoded by a map $\mu : A \otimes_k B \to k$ extending the augmentation ϵ. This suggests the following definition:

($*$) Let A be an augmented $\mathbb{E}_1$-algebra over a field k. Then the *Koszul dual* of A is universal among $\mathbb{E}_1$-algebras B equipped with an augmentation $\mu : A \otimes_k B \to k$ extending the augmentation on A.

Our first goal in this section is to make ($*$) more precise, and show that it determines a (contravariant) functor $\mathfrak{D}^{(1)}$ from the ∞-category $\mathrm{Alg}_k^{\mathrm{aug}}$ of augmented $\mathbb{E}_1$-algebras over k to itself. Every augmentation $\mu : A \otimes_k B \to k$ restricts to augmentations on A and B, and is classified by a map of augmented $\mathbb{E}_1$-algebras $\alpha : B \to \mathfrak{D}^{(1)}(A)$. We will say that μ *exhibits B as a Koszul dual of A* if the map α is an equivalence. The main results of this section establish some basic formal properties of Koszul duality:

(a) Let $\mu : A \otimes_k B \to k$ be an augmentation which exhibits B as the Koszul dual of A. Under some mild hypotheses, there is a close relationship between the ∞-categories LMod_A and LMod_B of (left) modules over A and B, respectively (Theorem 3.1.14).

(b) Let $\mu : A \otimes_k B \to k$ be an augmentation which exhibits B as the Koszul dual of A. Under some mild hypotheses, it follows that μ also exhibits A as a Koszul dual of B (Corollary 3.1.15). In other words, the double commutant map $A \to \mathfrak{D}^{(1)}\mathfrak{D}^{(1)}(A)$ is often an isomorphism.

We begin by introducing some terminology.

Definition 3.1.1. A *pairing of ∞-categories* is a triple $(\mathcal{C}, \mathcal{D}, \lambda : \mathcal{M} \to \mathcal{C} \times \mathcal{D})$, where $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories and μ is a right fibration of ∞-categories.

We will generally abuse notation by denoting a pairing of ∞-categories simply by λ.

Definition 3.1.2. Let $\lambda : \mathcal{M} \to \mathcal{C} \times \mathcal{D}$ be a pairing of ∞-categories, and let $M \in \mathcal{M}$ be an object having image $(C, D) \in \mathcal{C} \times \mathcal{D})$. We will say that M is *left universal* if it is a final object of $\mathcal{M} \times_{\mathcal{C}} \{C\}$, and *right universal* if it is a final object of $\mathcal{M} \times_{\mathcal{D}} \{D\}$. We let $\mathcal{M}^L$ and $\mathcal{M}^R$ denote the full subcategories of $\mathcal{M}$ spanned by the left universal and right universal objects, respectively. We say that λ is *left representable* if, for each object $C \in \mathcal{C}$, there exists a left universal object $M \in \mathcal{M}$ lying over C. We will say that λ is *right representable* if, for each object $D \in \mathcal{D}$, there exists a right universal object $M \in \mathcal{M}$ lying over D.

Construction 3.1.3. Let $\lambda : \mathcal{M} \to \mathcal{C} \times \mathcal{D}$ be a pairing of ∞-categories. λ is classified by a functor $\chi : \mathcal{C}^{op} \times \mathcal{D}^{op} \to \mathcal{S}$. Note that λ is left representable if and only if, for each object $C \in \mathcal{C}$, the induced map $\chi_C : \chi|(\{C\} \times \mathcal{D}^{op}) \to \mathcal{S}$ is representable by an object $D \in \mathcal{D}$. In this case, χ determines a functor $\mathcal{C}^{op} \to \mathrm{Fun}(\mathcal{D}^{op}, \mathcal{S})$ which is given by the composition

$$\mathcal{C}^{op} \xrightarrow{\mathfrak{D}_\lambda} \mathcal{D} \xrightarrow{j} \mathrm{Fun}(\mathcal{D}^{op}, \mathcal{S})$$

for some essentially unique functor $\mathfrak{D}_\lambda : \mathcal{C}^{op} \to \mathcal{D}$ (here $j : \mathcal{D} \to \mathrm{Fun}(\mathcal{D}^{op}, \mathcal{S})$ denotes the Yoneda embedding). We will refer to $\mathfrak{D}_\lambda$ as the *duality functor associated to* λ; it carries each object $C \in \mathcal{C}$ to an object $\mathfrak{D}_\lambda(C)$ which represents the functor χ_C. Similarly, if λ is right representable, then it determines a duality functor $\mathfrak{D}'_\lambda : \mathcal{D}^{op} \to \mathcal{C}$, which we will also refer to as the *duality functor associated to* λ. If λ is both left and right representable, then $\mathfrak{D}_\lambda : \mathcal{C}^{op} \to \mathcal{D}$ is right adjoint to the duality functor $\mathfrak{D}'^{op}_\lambda : \mathcal{D} \to \mathcal{C}^{op}$.

We now specialize to the main example of interest.

Construction 3.1.4. Let k be a field, and let $\mathrm{Alg}_k = \mathrm{Alg}(\mathrm{Mod}_k)$ denote the ∞-category of associative algebra objects of Mod_k, and let $\mathrm{Alg}_k^{\mathrm{aug}} = (\mathrm{Alg}_k)_{/k}$ denote the ∞-category of augmented associative algebra objects of Mod_k. We will regard Alg_k as a symmetric monoidal ∞-category (where the tensor product operation is given objectwise). Let $m : \mathrm{Alg}_k \times \mathrm{Alg}_k \to \mathrm{Alg}_k$ denote the tensor product functor, and let $p_0, p_1 : \mathrm{Alg}_k \times \mathrm{Alg}_k \to \mathrm{Alg}_k$ denote the projection onto the first and second factor, respectively. Since the unit object of Alg_k is an initial object, we have natural transformations $p_0 \stackrel{\alpha_0}{\to} m \stackrel{\alpha_1}{\leftarrow} p_1$, which determine a map $\mathrm{Alg}_k \times \mathrm{Alg}_k \to \mathrm{Fun}(\Lambda^2_2, \mathrm{Alg}_k)$. We let $\mathcal{M}^{(1)}$ denote the fiber product

$$(\mathrm{Alg}_k \times \mathrm{Alg}_k) \times_{\mathrm{Fun}(\Lambda^2_2, \mathrm{Alg}_k)} \mathrm{Fun}(\Lambda^2_2, \mathrm{Alg}_k^{\mathrm{aug}}).$$

More informally, the objects of $\mathcal{M}^{(1)}$ can be identified with triples (A, B, λ), where A and B are $\mathbb{E}_1$-algebras over k, and $\epsilon : A \otimes_k B \to k$ is an augmentation on the tensor product $A \otimes_k B$ (which then induces augmentations on A and B, respectively). Note that evaluation on the vertices $0, 1 \in \Lambda^2_2$ induces a right fibration $\lambda : \mathcal{M}^{(1)} \to \mathrm{Alg}_k^{\mathrm{aug}} \times \mathrm{Alg}_k^{\mathrm{aug}}$.

Proposition 3.1.5. *Let k be a field and let $\lambda : \mathcal{M}^{(1)} \to \mathrm{Alg}_k^{\mathrm{aug}} \times \mathrm{Alg}_k^{\mathrm{aug}}$ be the pairing of ∞-categories described in Construction 3.1.4. Then λ is both left and right representable.*

Proof. We will prove that λ is left representable; the proof of right representability is similar. Fix an object $A \in \mathrm{Alg}_k^{\mathrm{aug}}$, and let $F : (\mathrm{Alg}_k^{\mathrm{aug}})^{op} \to \mathcal{S}$ be the functor given by

$$F(B) = \mathrm{fib}(\mathrm{Map}_{\mathrm{Alg}_k}(A \otimes_k B, k) \to \mathrm{Map}_{\mathrm{Alg}_k}(A, k) \times \mathrm{Map}_{\mathrm{Alg}_k}(B, k)).$$

We wish to show that the functor F is representable by an object of Alg_k. Define $F' : \mathrm{Alg}_k^{op} \to \mathcal{S}$ by the formula

$$F'(B) = \mathrm{fib}(\mathrm{Map}_{\mathrm{Alg}_k}(A \otimes_k B, k) \to \mathrm{Map}_{\mathrm{Alg}_k}(A, k)).$$

Corollary A.6.1.4.13 implies that the functor F' is corepresented by an object $B_0 \in \mathrm{Alg}_k$, given by a centralizer of the augmentation $\epsilon : A \to k$. In particular, we have a point of $\eta \in F'(B_0)$, which determines an augmentation $\mu : A \otimes_k B_0 \to k$. Let us regard B_0 as an augmented algebra object via the composite map $B_0 \to A \otimes_k B_0 \stackrel{\mu}{\to} k$, so that η lifts to a point $\overline{\eta} \in F(B_0)$. To complete the proof, it will suffice to show that for each $B \in \mathrm{Alg}_k^{\mathrm{aug}}$, evaluation on $\overline{\eta}$ induces a homotopy equivalence $\theta : \mathrm{Map}_{\mathrm{Alg}_k^{\mathrm{aug}}}(B, B_0) \to F(B)$. This map fits into a map of fiber sequences

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{Alg}_k^{\mathrm{aug}}}(B, B_0) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{Alg}_k}(B, B_0) \arrow[r] \arrow[d, "\theta'"] & \mathrm{Map}_{\mathrm{Alg}_k}(B, k) \arrow[d, "\theta''"] \\
F(B) \arrow[r] & F'(B) \arrow[r] & \mathrm{Map}_{\mathrm{Alg}_k}(B, k),
\end{tikzcd}$$

where θ' and θ'' are homotopy equivalences (in the first case, this follows from our assumption that η exhibits F' as the functor represented by B_0). $\square$

Definition 3.1.6. Let k be a field. We let $\mathfrak{D}^{(1)} : (\mathrm{Alg}_k^{\mathrm{aug}})^{op} \to \mathrm{Alg}_k^{\mathrm{aug}}$ denote the functor obtained by applying Construction 3.1.3 to the left representable pairing $\lambda : \mathcal{M}^{(1)} \to \mathrm{Alg}_k^{\mathrm{aug}} \times \mathrm{Alg}_k^{\mathrm{aug}}$ of Construction 3.1.4. We will refer to the functor $\mathfrak{D}^{(1)}$ as *Koszul duality*.

Remark 3.1.7. Since the pairing $\lambda : \mathcal{M}^{(1)} \to \mathrm{Alg}_k^{\mathrm{aug}} \times \mathrm{Alg}_k^{\mathrm{aug}}$ of Construction 3.1.4 is both left and right representable, it determines two functors $(\mathrm{Alg}_k^{\mathrm{aug}})^{op} \to \mathrm{Alg}_k^{\mathrm{aug}}$. It follows by symmetry considerations that these functors are (canonically) equivalent to one another; hence there is no risk of confusion if we denote them both by $\mathfrak{D}^{(1)} : (\mathrm{Alg}_k^{\mathrm{aug}})^{op} \to \mathrm{Alg}_k^{\mathrm{aug}}$. It follows from Construction 3.1.3 that $\mathfrak{D}^{(1)}$ is adjoint to itself: more precisely, the functor $\mathfrak{D}^{(1)} : (\mathrm{Alg}_k^{\mathrm{aug}})^{op} \to \mathrm{Alg}_k^{\mathrm{aug}}$ is right adjoint to the induced map between opposite

∞-categories $\mathrm{Alg}_k^{\mathrm{aug}} \to (\mathrm{Alg}_k^{\mathrm{aug}})^{op}$. More concretely, for any pair of objects $A, B \in \mathrm{Alg}_k^{\mathrm{aug}}$ we have a canonical homotopy equivalence

$$\mathrm{Map}_{\mathrm{Alg}_k^{\mathrm{aug}}}(A, \mathfrak{D}^{(1)}(B)) \simeq \mathrm{Map}_{\mathrm{Alg}_k^{\mathrm{aug}}}(B, \mathfrak{D}^{(1)}(A)).$$

In fact, both of these spaces can be identified with the homotopy fiber of the canonical map

$$\mathrm{Map}_{\mathrm{Alg}_k}(A \otimes_k B, k) \to \mathrm{Map}_{\mathrm{Alg}_k}(A, k) \times \mathrm{Map}_{\mathrm{Alg}_k}(B, k).$$

Remark 3.1.8. Let k be a field and let $\mu : A \otimes_k B \to k$ be a morphism in Alg_k, which we can identify with an object of the ∞-category $\mathcal{M}^{(1)}$ of Construction 3.1.4. We say that μ *exhibits* B *as a Koszul dual of* A if it is left universal in the sense of Definition 3.1.2: that is, if it induces an equivalence $B \to \mathfrak{D}^{(1)}(A)$. Similarly, we say that μ *exhibits* A *as a Koszul dual of* B if it is right universal: that is, if it induces an equivalence $A \to \mathfrak{D}^{(1)}(B)$.

Construction 3.1.9. Let k be a field and let $\mu : A \otimes_k B \to k$ be a morphism in Alg_k. Then μ induces a tensor product functor

$$\mathrm{LMod}_A \times \mathrm{LMod}_B \to \mathrm{LMod}_k \simeq \mathrm{Mod}_k .$$

We let LPair_μ denote the fiber product $(\mathrm{LMod}_A \times \mathrm{LMod}_B) \times_{\mathrm{Mod}_k} (\mathrm{Mod}_k)_{/k}$. We can think of the objects of LPair_μ as triples (M, N, ϵ), where M is a left module over A, N is a left module over B, and $\epsilon : M \otimes_k N \to k$ is a map of left modules over $A \otimes_k B$. The projection map

$$\lambda : \mathrm{LPair}_\mu \to \mathrm{LMod}_A \times \mathrm{LMod}_B$$

is a right fibration of ∞-categories, so that we can regard LPair_μ as a pairing of ∞-categories.

Proposition 3.1.10. *Let k be a field, let $\mu : A \otimes_k B \to k$ be a morphism in* Alg_k*, and let $\lambda : \mathrm{LPair}_\mu \to \mathrm{LMod}_A \times \mathrm{LMod}_B$ be the pairing of ∞-categories of Construction 3.1.9. Then λ is both left and right representable.*

Proof. We will prove that λ is left representable; the proof of right representability is similar. Fix an object $M \in \mathrm{LMod}_A$; we wish to show that that the functor $N \mapsto \mathrm{Map}_{\mathrm{LMod}_{A \otimes_k B}}(M \otimes_k N, k)$ is representable by an object of LMod_B. Let us denote this functor by F. According to Proposition T.5.5.2.2, it will suffice to show that the functor F carries colimits in LMod_B to limits in $\mathcal{S}$. This follows from the observation that the construction $N \mapsto M \otimes_k N$ determines a functor $\mathrm{LMod}_B \to \mathrm{LMod}_{A \otimes_k B}$ which commutes with small colimits. □

Notation 3.1.11. Let k be a field and let $\mu : A \otimes_k B \to k$ be a morphism in Alg_k. Combining Proposition 3.1.10 with Construction 3.1.3, we obtain duality functors

$$\mathrm{LMod}_A^{op} \xrightarrow{\mathfrak{D}_\mu} \mathrm{LMod}_B$$

$$\mathrm{LMod}_B^{op} \xrightarrow{\mathfrak{D}'_\mu} \mathrm{LMod}_A .$$

By construction, we have canonical homotopy equivalences

$$\mathrm{Map}_{\mathrm{LMod}_A}(M, \mathfrak{D}'_\mu N) \simeq \mathrm{Map}_{\mathrm{LMod}_{A \otimes_k B}}(M \otimes_k N, k) \simeq \mathrm{Map}_{\mathrm{LMod}_B}(N, \mathfrak{D}_\mu M).$$

Remark 3.1.12. Let k be a field and let $\mu : A \otimes_k B \to k$ be a morphism in Alg_k. The proof of Theorem A.6.1.4.12 shows that μ exhibits B as a Koszul dual of A if and only if it μ exhibits B as a classifying object for morphisms from A to k in $\mathrm{Mod}_A^{\mathcal{A}ss}(\mathrm{Mod}_k) \simeq {}_A\mathrm{BMod}_A(\mathrm{Mod}_k)$ (here we regard ${}_A\mathrm{BMod}_A(\mathrm{Mod}_k)$ as left-tensored over the ∞-category Mod_k). This is equivalent to the condition that ϵ exhibit B as a classifying object for morphisms from $k \simeq A \otimes_A k$ to itself in ${}_A\mathrm{BMod}_k(\mathrm{Mod}_k) \simeq \mathrm{LMod}_A$: that is, that μ induces an equivalence of left B-modules $B \to \mathfrak{D}'_\mu \mathfrak{D}_\mu(B) \simeq \mathfrak{D}_\mu(k)$. Similarly, μ exhibits A as a Koszul dual of B if and only if it induces an equivalence $A \to \mathfrak{D}'_\mu(k)$ of left A-modules.

We would like to use Remark 3.1.12 to verify (in good cases) that the relation of Koszul duality is symmetric. For this, we need to understand the linear duality functors $\mathfrak{D}_\mu$ and $\mathfrak{D}'_\mu$ associated to a pairing $\mu : A \otimes_k B \to k$.

Definition 3.1.13. Let k be a field. An object $A \in \mathrm{Alg}_k$ is *coconnective* if the unit map $k \to A$ exhibits k as a connective cover A. Equivalently, A is coconnective if $\pi_0 A$ is a 1-dimensional vector space over k generated by the unit element, and $\pi_n A \simeq 0$ for $n > 0$.

If $M \in \mathrm{Mod}_k$, we will say that M is *locally finite* if each of the homotopy groups $\pi_n M$ is finite dimensional as a vector space over k. We will say that an object $A \in \mathrm{Alg}_k$ is *locally finite* if it is locally finite when regarded as an object of Mod_k.

Our analysis of the Koszul duality functor rests on the following result:

Theorem 3.1.14. *Let k be a field and let $\mu : A \otimes_k B \to k$ be a morphism in Alg_k. Assume that A is coconnective and that μ exhibits B as a Koszul dual of A. Then:*

(1) *Let M be a left A-module such that $\pi_n M \simeq 0$ for $n > 0$. Then $\pi_n \mathfrak{D}_\mu(M) \simeq 0$ for $n < 0$.*

(2) *The $\mathbb{E}_1$-algebra B is connective.*

(3) *Let N be a connective B-module. Then $\pi_n \mathfrak{D}'_\mu(N) \simeq 0$ for $n > 0$.*

(4) *Let M be as in (1) and assume that M is locally finite. Then the canonical map $M \to \mathfrak{D}'_\mu \mathfrak{D}_\mu M$ is an equivalence in LMod_A.*

Corollary 3.1.15. *Let k be a field, and let $A \in \mathrm{Alg}_k^{\mathrm{aug}}$ be coconnective and locally finite. Then the canonical map $A \to \mathfrak{D}^{(1)} \mathfrak{D}^{(1)}(A)$ is an equivalence. In other words, if $\mu : A \otimes_k B \to k$ exhibits B as a Koszul dual of A, then μ also exhibits A as a Koszul dual of B.*

Proof. Let $\mu : A \otimes_k B \to k$ be a map which exhibits B as a Koszul dual of A. We wish to prove that μ exhibits A as the Koszul dual of B. According to Remark 3.1.12, it will suffice to show that the unit map $A \to \mathfrak{D}'_\mu \mathfrak{D}_\mu(A)$ is an equivalence of left A-modules. Since A is coconnective and locally finite, this follows from Theorem 3.1.14. □

The proof of Theorem 3.1.14 will require some preliminaries. We begin with a variation on Proposition VIII.4.1.9.

Lemma 3.1.16. *Let A be a coconnective $\mathbb{E}_1$-algebra over a field k such that $\pi_{-1} A \simeq 0$, and let M be a left A-module such that $\pi_i M \simeq 0$ for $i > 0$. Assume that A and M are locally finite. Then there exists a sequence of left A-modules*

$$0 = M(0) \to M(1) \to M(2) \to \cdots$$

with the following properties:

(1) *For each $n > 0$, there exists a locally finite object $V(n) \in (\mathrm{Mod}_k)_{\leq -n}$ and a cofiber sequence*

$$A \otimes_k V(n) \to M(n-1) \to M(n)$$

(2) *There exists an equivalence $\theta : \varinjlim M(n) \simeq M$.*

Proof. We construct $M(n)$ using induction on n, beginning with the case $n = 0$ where we set $M(0) = 0$. Assume that $M(n-1) \in (\mathrm{LMod}_A)_{/M}$ has been constructed, and let $V(n)$ denote the underlying k-module of the fiber of the map $M(n-1) \to M$. We then define $M(n)$ to be the cofiber of the induced map $A \otimes_k V(n) \to M(n-1)$. This construction produces a sequence of objects

$$M(0) \to M(1) \to \cdots$$

in $(\mathrm{LMod}_A)_{/M}$, hence a map $\theta : \varinjlim M(n) \to M$. We claim that θ is an equivalence of left A-modules. To prove this, it suffices to show that θ is an equivalence in Mod_k. As an object of Mod_k, we can identify $\varinjlim M(n)$ with the direct limit of the sequence

$$M(0) \to M(0)/V(1) \to M(1) \to M(1)/V(2) \to \cdots$$

It therefore suffices to show that the map $\varinjlim M(i)/V(i+1) \to M$ is an equivalence in Mod_k, which is clear (since each cofiber $M(i)/V(i+1)$ is equivalent to M). This proves (2). We next prove the following by a simultaneous induction on n:

(a_n) The map $M(n) \to M$ induces an isomorphism $\pi_i M(n) \to \pi_i M$ for $i > -n$ and an injection for $i = -n$.

(b_n) Each $M(n)$ is locally finite.

(a'_n) The k-module $V(n+1)$ belongs to $(\mathrm{Mod}_k)_{\leq -n-1}$.

(b'_n) The k-module $V(n+1)$ is locally finite.

Assertions (a_0) and (b_0) are obvious, and the equivalences $(a_n) \Leftrightarrow (a'_n)$ and $(b_n) \Leftrightarrow (b'_n)$ follow from the existence of a long exact sequence

$$\cdots \to \pi_i V(n+1) \to \pi_i M(n) \to \pi_i M \to \pi_{i-1} V(n+1) \to \cdots$$

We will complete the proof by showing that (a'_n) and (b'_n) imply (a'_{n+1}) and (b_{n+1}). Assertion (b_{n+1}) follows from (b'_n) by virtue of the existence of an exact sequence

$$\cdots \to \pi_i(A \otimes_k V(n+1)) \to \pi_i M(n) \to \pi_i M(n+1) \to \pi_{i-1}(A \otimes_k V(n+1)) \to \cdots$$

To prove (a_{n+1}), we note that the identification $M \simeq M(n)/V(n)$ gives a fiber sequence

$$(A/k) \otimes_k V(n) \to M \xrightarrow{\lambda} M(n+1)$$

in Mod_k, where λ is a right inverse to the A-module map $M(n+1) \to M$. We therefore have an equivalence $M(n+1) \simeq M \oplus ((A/k) \otimes_k V(n))[1]$ in Mod_k so that $V(n+1) \simeq (A/k)[1] \otimes_k V(n)$. Since $\pi_i A/k \simeq 0$ for $i \geq -1$, it follows immediately that $(a'_n) \Rightarrow (a'_{n+1})$. $\square$

Lemma 3.1.17. *Let A be a connective $\mathbb{E}_1$-algebra over a field k. Let M and N be left A-modules such that $\pi_m M \simeq 0$ for $m > 0$ and $\pi_m N \simeq 0$ for $m < 0$. Then the canonical map $\theta : \mathrm{Ext}^0_A(M, N) \to \mathrm{Hom}_k(\pi_0 M, \pi_0 N)$ is surjective.*

Proof. We have an evident map of k-module spectra $\pi_0 M \to M$, which determines a map of left A-modules $A \otimes_k (\pi_0 M) \to M$. Let K denote the fiber of this map, so that we have a fiber sequence of spaces

$$\mathrm{Map}_{\mathrm{LMod}_A}(M, N) \xrightarrow{\phi} \mathrm{Map}_{\mathrm{Mod}_k}(\pi_0 M, N) \to \mathrm{Map}_{\mathrm{LMod}_A}(K, N).$$

Since $\pi_m K \simeq 0$ for $m \geq 0$, Proposition VIII.4.1.14 implies that the mapping space $\mathrm{Map}_{\mathrm{LMod}_A}(K, N)$ is connected. It follows that ϕ induces a surjection

$$\mathrm{Ext}^0_A(M, N) \to \mathrm{Ext}^0_A(\pi_0 M, N) \simeq \mathrm{Hom}_k(\pi_0 M, \pi_0 N).$$

$\square$

Lemma 3.1.18. *Let A be a connective $\mathbb{E}_1$-algebra over a field k, and let M be a left A-module such that $\pi_m M \simeq 0$ for $m \neq 0$. Suppose we are given a map of $\mathbb{E}_1$-algebras $A \to k$. Then M lies in the essential image of the forgetful functor $\theta : \mathrm{Mod}_k \simeq \mathrm{LMod}_k \to \mathrm{LMod}_A$.*

Proof. Let $V = \pi_0 M$, and regard V as a discrete k-module spectrum. Lemma 3.1.17 implies that the evident isomorphism $\pi_0\theta(V) \simeq \pi_0 M$ can be lifted to a map of left A-modules $\theta(V) \to M$, which is evidently an equivalence. □

Proof of Theorem 3.1.14. We first prove (1). Let $M \in \mathrm{LMod}_A$ be such that $\pi_m M \simeq 0$ for $m < 0$. Using Proposition VIII.4.1.9 and Remark VIII.4.1.10, we can write M as the colimit of a sequence of A-modules

$$0 = M(0) \to M(1) \to M(2) \to \cdots ,$$

where each $M(n)$ fits into a cofiber sequence $A \otimes_k V(n) \to M(n) \to M(n+1)$ for $V(n) \in (\mathrm{Mod}_k)_{\leq -1}$. Then $\mathfrak{D}_\mu(M)$ is a limit of the tower $\{\mathfrak{D}_\mu(M(n))\}_{n \geq 0}$. It will therefore suffice to prove that $\pi_m \mathfrak{D}_\mu(M(n)) \simeq 0$ for $m < 0$ and that each of the maps $\pi_0\mathfrak{D}_\mu(M(n)) \to \pi_0\mathfrak{D}_\mu(M(n-1))$ is surjective. We have a fiber sequence

$$\mathfrak{D}_\mu(M(n)) \to \mathfrak{D}_\mu(M(n-1)) \to \mathfrak{D}_\mu(A \otimes_k V(n)).$$

It will therefore suffice to show that $\pi_m \mathfrak{D}_\mu(A \otimes_k V(n)) \simeq 0$ for $m \leq 0$. Unwinding the definitions, we must show that if $m \leq 0$, then any map of $A \otimes_k B$-modules from $(A \otimes_k V(n)) \otimes_k B[m]$ to k is nullhomotopic. This is equivalent to the assertion that every map of k-module spectra from $V(n)[m]$ into k is nullhomotopic. Since k is a field, this follows from the observation that $\pi_0 V(n)[m] \simeq \pi_m V(n) \simeq 0$.

Since $\mu : A \otimes_k B \to k$ exhibits B as a Koszul dual of A, the augmentation on B gives a map $k \otimes_k B \to k$ which induces an equvialence $B \to \mathfrak{D}_\mu(k)$. Assertion (2) now follows immediately from assertion (1). We next prove (3). Let $\mathcal{C}$ be the full subcategory of LMod_B spanned by those objects N for which $\pi_n \mathfrak{D}'_\mu(N) \simeq 0$ for $n > 0$. Since $\mathfrak{D}'_\mu : \mathrm{LMod}_B^{op} \to \mathrm{LMod}_A$ preserves small limits, the ∞-category $\mathcal{C}$ is stable under small colimits in LMod_B. To prove that $\mathcal{C}$ contains all connective left B-modules, it will suffice to show that $B \in \mathcal{C}$. This is clear, since $\mathfrak{D}'_\mu(B) \simeq k$.

We now prove (4). Let $M \in \mathrm{LMod}_A$ be locally finite and assume that $\pi_n M \simeq 0$ for $n > 0$. Let K_M denote the fiber of the unit map $u_M : M \to \mathfrak{D}'_\mu \mathfrak{D}_\mu(M)$. Condition (1) implies that $\mathfrak{D}(M)$ is connective, so that $\pi_n \mathfrak{D}'_\mu \mathfrak{D}_\mu(M) \simeq 0$ for $n > 0$ by (3). It follows that $\pi_n K_M \simeq 0$ for $n > 0$. We prove that $\pi_n K_M \simeq 0$ for all n, using descending induction on n. Using Proposition VIII.4.1.9, we can choose a map of left A-modules $u : M' \to M$ which induces an isomorphism $\pi_m M' \to \pi_m M$ for $m < 0$ and satisfies $\pi_m M' \simeq 0$ for $m \geq 0$. Let M'' denote the cofiber of u, so that $\pi_m M'' \simeq 0$ for $m \neq 0$ and therefore Lemma 3.1.18 guarantees that M'' is a direct sum of (finitely many) copies of k. The condition that ϵ exhibits B as a Koszul dual of A guarantees that $B \simeq \mathfrak{D}_\mu(k)$ and therefore the unit map $u_k : k \simeq \mathfrak{D}'_\mu(B) \to \mathfrak{D}'_\mu \mathfrak{D}_\mu(k)$ is an equivalence. It follows that $u_{M''}$ is an equivalence. The cofiber sequence

$$M \to M'' \to M'[1]$$

induces an equivalence $K_M \simeq K_{M'[1]}[-1]$. The inductive hypothesis implies that $\pi_{n+1} K_{M'[1]} \simeq 0$, so that $\pi_n K_M \simeq 0$ as desired. □

In §4, we will need the following stronger version of Corollary 3.1.15:

Proposition 3.1.19. *Let k be a field and suppose given a finite collection of maps$\{\mu_i : A_i \otimes_k B_i \to k\}_{1 \leq i \leq m}$ in Alg_k. Assume that each A_i is coconnective and locally finite and that each μ_i exhibits B_i as a Koszul dual of A_i. Let $A = \bigotimes_i A_i$, $B = \bigotimes_i B_i$, and let $\mu : A \otimes_k B \to k$ be the tensor product of the maps μ_i. Then μ exhibits A as the Koszul dual of B.*

Warning 3.1.20. In the situation of Proposition 3.1.19, it is not necessarily true that μ exhibits B as the Koszul dual of A. For example, suppose that $m = 2$ and that $A_1 = A_2 = k \oplus k[-1]$, endowed with the square-zero algebra structure. In this case, the Koszul dual of A_1 can be identified with the power series ring $k[[x_1]]$, regarded as a discrete $\mathbb{E}_1$-algebra. Similarly, the Koszul dual of A_2 can be identified with $k[[x_2]]$, and the Koszul dual of the tensor product $A_1 \otimes_k A_2$ is given by $k[[x_1, x_2]]$. The canonical map $\theta : k[[x_1]] \otimes_k k[[x_2]] \to k[[x_1, x_2]]$ is not an isomorphism: however, Proposition 3.1.19 guarantees that θ induces an equivalence after applying the Koszul duality functor.

Proof. For $1 \leq i \leq m$, let $\mathfrak{D}_i : \mathrm{LMod}_{A_i}^{op} \to \mathrm{LMod}_{B_i}$ be the duality functor determined by μ_i, and let $\mathfrak{D}'_\mu : \mathrm{LMod}_B^{op} \to \mathrm{LMod}_A$ be the duality functor associated to μ. For every sequence of objects $\vec{M} = \{M_i \in \mathrm{LMod}_{A_i}\}$, we have a canonical map $u_{\vec{M}} : M_1 \otimes_k \cdots \otimes_k M_m \to \mathfrak{D}'_\mu(\mathfrak{D}_1 M_1 \otimes_k \cdots \otimes_k \mathfrak{D}_m M_m)$. We will prove the following:

$(*)$ If $\vec{M} = \{M_i\}_{1 \leq i \leq m} \in \prod_i \mathrm{LMod}_{A_i}$ is such that the homotopy groups $\pi_n M_i \simeq 0$ vanish for $i > 0$ and each M_i is locally finite, then $u_{\vec{M}}$ is an equivalence.

Fix $0 \leq m' \leq m$. We will show that assertion $(*)$ holds under the additional assumption that $M_i \simeq k$ for $i > m'$. The proof proceeds by induction on m'. If $m' = 0$, then each $M_i \simeq k$ and the desired result follows immediately from our assumption that each μ_i exhibits B_i as a Koszul dual of A_i. Let us therefore assume that $m' > 0$ and that condition $(*)$ holds whenever $M_i \simeq k$ for $i < m'$.

Note that if $\vec{M}$ satisfies the hypotheses of $(*)$, then Theorem 3.1.14 guarantees that each $\mathfrak{D}_i(M_i)$ is connective and therefore that $\pi_n \mathfrak{D}'_\mu(\mathfrak{D}_1 M_1 \otimes_k \cdots \otimes_k \mathfrak{D}_m M_m) \simeq 0$ for $n > 0$. Let $K_{\vec{M}}$ denote the fiber of $u_{\vec{M}}$, so that $\pi_n K_{\vec{M}} \simeq 0$ for $n > 0$. We prove that $\pi_n K_{\vec{M}} \simeq 0$ for all n, using descending induction on n. Using Proposition VIII.4.1.9, we can choose a map of left A-modules $v : M' \to M_{m'}$ which induces an isomorphism $\pi_p M' \to \pi_p M_{m'}$ for $p < 0$ and satisfies $\pi_p M' \simeq 0$ for $p \geq 0$. Let M'' denote the cofiber of v, so that $\pi_p M'' \simeq 0$ for $p \neq 0$ and therefore Lemma 3.1.18 guarantees that M'' is a direct sum of (finitely many) copies of k. Let $\vec{M}''$ be the sequence of modules obtained from $\vec{M}$ by replacing $M_{m'}$ with M'', and let $\vec{N}$ be the sequence of modules obtained from $\vec{M}$ by replacing $M_{m'}$ with $M'[1]$. The inductive hypothesis implies that $K_{\vec{M}''} \simeq 0$. Using the cofiber sequence

$$M_{m'} \to M'' \to M'[1],$$

we obtain an equivalence $K_{\vec{M}} \simeq K_{\vec{N}}[-1]$, so that $\pi_n K_{\vec{M}} \simeq \pi_{n+1} K_{\vec{N}}$ is trivial by the other inductive hypothesis. □

3.2 Formal Moduli Problems for Associative Algebras

Let k be a field. In this section, we will use the Koszul duality functor $\mathfrak{D}^{(1)} : (\mathrm{Alg}_k^{\mathrm{aug}})^{op} \to \mathrm{Alg}_k^{\mathrm{aug}}$ to construct an equivalence of ∞-categories $\mathrm{Alg}_k^{\mathrm{aug}} \simeq \mathrm{Moduli}_k^{(1)}$, and thereby obtain a proof of Theorem 3.0.4. The main point is to show that $\mathfrak{D}^{(1)}$ is a deformation theory (in the sense of Definition 1.3.9). We begin by introducing a variation on Example 1.1.4:

Construction 3.2.1. Let k be a field. Theorem A.7.3.5.14 gives an equivalence of the stabilization $\mathrm{Stab}(\mathrm{Alg}_k^{\mathrm{aug}})$ with the ∞-category ${}_k\mathrm{BMod}_k(\mathrm{Mod}_k) \simeq \mathrm{Mod}_k$ of k-module spectra. Let $E \in \mathrm{Stab}(\mathrm{Alg}_k^{\mathrm{aug}})$ correspond to the unit object $k \in \mathrm{Mod}_k$ under this identification (so we have $\Omega^{\infty-n} E \simeq k \oplus k[n]$ for every integer n). We regard $(\mathrm{Alg}_k^{\mathrm{aug}}, \{E\})$ as a deformation context (see Definition 1.1.3).

Our first goal in this section is to show that the deformation context $(\mathrm{Alg}_k^{\mathrm{aug}}, \{E\})$ of Construction 3.2.1 allows us to recover the notion of small k-algebra and formal $\mathbb{E}_1$ moduli problem via the general formalism laid out in §1.1.

Proposition 3.2.2. *Let k be a field and let $(\mathrm{Alg}_k^{\mathrm{aug}}, \{E\})$ be the deformation context of Construction 3.2.1. Then an object $A \in \mathrm{Alg}_k^{\mathrm{aug}}$ is small (in the sense of Definition 1.1.8) if and only if its image in Alg_k is small (in the sense of Definition 3.0.1). That is, A is small if and only if it satisfies the following conditions:*

(a) The algebra A is connective: that is, $\pi_i A \simeq 0$ for $i < 0$.

(b) The algebra A is truncated: that is, we have $\pi_i A \simeq 0$ for $i \gg 0$.

(c) Each of the homotopy groups $\pi_i A$ is finite dimensional when regarded as a vector space over field k.

(d) Let $\mathfrak{n}$ denote the radical of the ring $\pi_0 A$ (which is a finite-dimensional associative algebra over k). Then the canonical map $k \to (\pi_0 A)/\mathfrak{n}$ is an isomorphism.

Proof. Suppose first that there there exists a finite sequence of maps

$$A = A_0 \to A_1 \to \cdots \to A_n \simeq k$$

where each A_i is a square-zero extension of A_{i+1} by $k[n_i]$, for some $n_i \geq 0$. We prove that each A_i satisfies conditions (a) through (d) using descending induction on i. The case $i = n$ is obvious, so let us assume that $i < n$ and that A_{i+1} is known to satisfy conditions (a) through (d). We have a fiber sequence of k-module spectra

$$k[n_i] \to A_i \to A_{i+1}$$

which immediately implies that A_i satisfies (a), (b), and (c). To prove (d), we note that the map $\phi : \pi_0 A_i \to \pi_0 A_{i+1}$ is surjective and $\ker(\phi)^2 = 0$, so that the quotient of $\pi_0 A_i$ by its radical agrees with the quotient of $\pi_0 A_{i+1}$ by its radical.

Now suppose that A satisfies conditions (a) through (d). We will prove that A is small by induction on the dimension of the k-vector space $\pi_* A$. Let n be the largest integer for which $\pi_n A$ does not vanish. We first treat the case $n = 0$. We will abuse notation by identifying A with the underlying associative ring $\pi_0 A$. Let $\mathfrak{n}$ denote the radical of A. If $\mathfrak{n} = 0$, then condition (d) implies that $A \simeq k$ so there is nothing to prove. Otherwise, we can view $\mathfrak{n}$ as a nonzero module over the associative algebra $A \otimes_k A^{op}$. It follows that there exists a nonzero element $x \in \mathfrak{n}$ which is annihilated by $\mathfrak{n} \otimes_k \mathfrak{n}$. Using (d) again, we deduce that the subspace $kx \subseteq A$ is a two-sided ideal of A. Let A' denote the quotient ring A/kx. Theorem A.7.4.1.23 implies that A is a square-zero extension of A' by k. The inductive hypothesis implies that A' is small, so that A is also small.

Now suppose that $n > 0$ and let $M = \pi_n A$. Then M is a nonzero bimodule over the finite dimensional k-algebra $\pi_0 A$. It follows that there is a nonzero element $x \in M$ which is annihilated (on both sides) by the action of the radical $\mathfrak{n} \subseteq \pi_0 A$. Let M' denote the quotient of M by the bimodule generated by x (which, by virtue of (d), coincides with kx), and let $A'' = \tau_{\leq n-1} A$. It follows from Theorem A.7.4.1.23 that there is a pullback diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & k \arrow[d] \\ A'' \arrow[r] & k \oplus M[n+1]. \end{tikzcd}$$

Set $A' = A'' \times_{k \oplus M'[n+1]} k$. Then $A \simeq A' \times_{k \oplus k[n+1]} k$ so we have an elementary map $A \to A'$. Using the inductive hypothesis we deduce that A' is small, so that A is also small. $\square$

We will also need a noncommutative analogue of Lemma 1.1.20:

Proposition 3.2.3. *Let k be a field and let $f : A \to B$ be a morphism in $\mathrm{Alg}_k^{\mathrm{sm}}$. Then f is small (when regarded as a morphism in $\mathrm{Alg}_k^{\mathrm{aug}}$) if and only if it induces a surjection of associative rings $\pi_0 A \to \pi_0 B$.*

Proof. The "only if" direction is obvious. For the converse, suppose that $\pi_0 A \to \pi_0 B$ is surjective, so that the fiber $I = \mathrm{fib}(f)$ is connective. We will prove that f is small by induction on the dimension of the graded vector space $\pi_* I$. If this dimension is zero, then $I \simeq 0$ and f is an equivalence. Assume therefore that $\pi_* I \neq 0$, and let n be the smallest integer such that $\pi_n I \neq 0$. Let $L_{B/A}$ denote the relative cotangent complex of B over A in the setting of $\mathbb{E}_1$-algebras, regarded as an object of ${}_B\mathrm{BMod}_B(\mathrm{Mod}_k)$. Remark A.7.4.1.12 supplies a fiber sequence

$$L_{B/A} \to B \otimes_A B \to B.$$

In the ∞-category LMod_B, this sequence splits; we therefore obtain an equivalence of left B-modules

$$L_{B/A} \simeq \mathrm{cofib}(B \to B \otimes_A B) \simeq B \otimes_A \mathrm{cofib}(A \to B) \simeq B \otimes_A I[1].$$

The kernel of the map $\pi_0 A \to \pi_0 B$ is contained in the radical of $\pi_0 A$ and is therefore a nilpotent ideal. It follows that $\pi_{n+1} L_{B/A} \simeq \mathrm{Tor}_0^{\pi_0 B}(\pi_0 A, \pi_n I)$ is a nonzero quotient of of $\pi_n I$. Let us regard $\pi_{n+1} L_{B/A}$

as a bimodule over $\pi_0 B$, and let $\mathfrak{n}$ be the radical of $\pi_0 B$. Since $\mathfrak{n}$ is nilpotent, the two-sided submodule $\mathfrak{n}(\pi_{n+1} L_{B/A}) + (\pi_{n+1} L_{B/A})\mathfrak{n}$ does not coincide with $\pi_{n+1} L_{B/A}$. It follows that there exists a map of $\pi_0 B$-bimodules $\pi_{n+1} L_{B/A} \to k$, which determines a map $L_{B/A} \to k[n+1]$ in the ∞-category ${}_B\mathrm{BMod}_B(\mathrm{Mod}_k)$. We can interpret this map as a derivation $B \to B \oplus k[n+1]$. Let $B' = B \times_{B \oplus k[n+1]} k$ be the associated square-zero extension of B by $k[n]$. Then f factors as a composition

$$A \xrightarrow{f'} B' \xrightarrow{f''} B.$$

Since the map f'' is elementary, it will suffice to show that f' is small, which follows from the inductive hypothesis. $\square$

Corollary 3.2.4. *Let k be a field and let $X : \mathrm{Alg}_k^{\mathrm{sm}} \to \mathcal{S}$ be a functor. Then X belongs to the full subcategory $\mathrm{Moduli}_k^{(1)}$ of Definition 3.0.3 if and only if it is a formal moduli problem in the sense of Definition 1.1.14.*

Proof. The "if" direction follows immediately from Proposition 3.2.3. For the converse, suppose that X satisfies the conditions of Definition 3.0.3; we wish to show that X is a formal moduli problem. According to Proposition 1.1.15, it will suffice to show that for every pullback diagram in $\mathrm{Alg}_k^{\mathrm{sm}}$

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ k \arrow[r] & k \oplus k[n] \end{tikzcd}$$

satisfying $n > 0$, the associated diagram of spaces

$$\begin{tikzcd} X(A) \arrow[r] \arrow[d] & X(B) \arrow[d] \\ X(k) \arrow[r] & X(k \oplus k[n]) \end{tikzcd}$$

is also a pullback square. This follows immediately from condition (2) of Definition 3.0.3. $\square$

We can now state the main result of this section:

Theorem 3.2.5. *Let k be a field. Then the Koszul duality functor*

$$\mathfrak{D}^{(1)} : (\mathrm{Alg}_k^{\mathrm{aug}})^{op} \to \mathrm{Alg}_k^{\mathrm{aug}}$$

is a deformation theory (on the deformation context $(\mathrm{Alg}_k^{\mathrm{aug}}, \{E\})$ of Construction 3.2.1).

Proof of Theorem 3.0.4. Let k be a field of characteristic zero, and let $\Psi : \mathrm{Alg}_k^{\mathrm{aug}} \to \mathrm{Fun}(\mathrm{Alg}_k^{\mathrm{sm}}, \mathcal{S})$ be the functor given on objects by the formula $\Psi(A)(R) = \mathrm{Map}_{\mathrm{Alg}_k^{\mathrm{aug}}}(\mathfrak{D}^{(1)}(R), A)$. Combining Theorems 3.2.5 and 1.3.12, we deduce that Ψ is a fully faithful embedding whose essential image is the full subcategory $\mathrm{Moduli}_k^{(1)} \subseteq \mathrm{Fun}(\mathrm{Alg}_k^{\mathrm{sm}}, \mathcal{S})$ spanned by the formal $\mathbb{E}_1$ moduli problems. $\square$

Proof of Theorem 3.2.5. The ∞-category $\mathrm{Alg}_k^{\mathrm{aug}}$ is presentable by Corollary A.3.2.3.5, and $\mathfrak{D}^{(1)}$ admits a left adjoint by Remark 3.1.7. Let $\Xi_0 \subseteq \mathrm{Alg}_k^{\mathrm{aug}}$ be the full subcategory spanned by those algebras which are coconnective and locally finite (see Definition 3.1.13). We will complete the proof that $\mathfrak{D}^{(1)}$ is a weak deformation theory by showing that the subcategory Ξ_0 satisfies conditions (a) thorugh (d) of Definition 1.3.1:

(a) For every object $A \in \Xi_0$, the unit map $A \to \mathfrak{D}^{(1)}\mathfrak{D}^{(1)}(A)$ is an equivalence. This follows from Corollary 3.1.15.

(*b*) The full subcategory Ξ_0 contains the initial object $k \in \mathrm{Alg}_k^{\mathrm{aug}}$. This is clear from the definitions.

(*c*) For each $n \geq 1$, there exists an object $K_n \in \Xi_0$ and an equivalence $k \oplus k[n] \simeq \mathfrak{D}^{(1)}(K_n)$. In fact, we can take K_n to be the free algebra $\bigoplus_{m \geq 0} V^{\otimes m}$ generated by $V = k[-n-1]$ (this is a consequence of Proposition 4.5.6, but is also not difficult to verify by direct calculation).

(*d*) Let $n \geq 1$ and suppose we are given a pushout diagram σ :

$$\begin{tikzcd} K_n \arrow[r] \arrow[d, "\phi"] & k \arrow[d] \\ A \arrow[r] & A' \end{tikzcd}$$

in $\mathrm{Alg}_k^{\mathrm{aug}}$, where K_n is as in (*c*). We must show that if $A \in \Xi_0$, then $A' \in \Xi_0$. Note that σ is also a pushout diagram in Alg_k. We will make use of the fact that Alg_k is the underlying ∞-category of the model category $\mathrm{Alg}_k^{\mathrm{dg}}$ of differential graded algebras over k (for a different argument which does not use the theory of model categories, we refer the reader to the proof of Theorem 4.5.5). Choose a cofibrant differential graded algebra A_* representing A, and let B_* denote the free differential graded algebra generated by a class x in degree $(-n-1)$. Since B_* is cofibrant and A_* is fibrant, the map $\phi : K_n \to A$ can be represented by a map $\phi_0 : B_* \to A_*$ of differential graded algebras, which is determined by the element $x' = \phi_0(x) \in A_{-n-1}$. Let B'_* denote the free differential graded algebra generated by the chain complex $E(-n)_*$ (see the proof of Proposition 2.1.10): in other words, B'_* is obtained from B_* by freely adjoining an element $y \in B'_{-n}$ satisfying $dy = x$. Then B'_* is quasi-isomorphic to the ground field k. Let $\psi_0 : B_* \to B'_*$ be the evident inclusion, and form a pushout diagram σ_0 :

$$\begin{tikzcd} B_* \arrow[r, "\psi_0"] \arrow[d, "\phi_0"] & B'_* \arrow[d] \\ A_* \arrow[r] & A'_* \end{tikzcd}$$

in the category $\mathrm{Alg}_k^{\mathrm{dg}}$. Since A_* is cofibrant and ψ_0 is a cofibration, the diagram σ_0 is also a homotopy pushout square, so that the image of σ_0 in Alg_k is equivalent to the diagram σ. It follows that the differential graded algebra A'_* represents A'. We can describe A'_* explicitly as the differential graded algebra obtained from A_* by adjoining an element y' in degree $-n$ satisfying $dy' = x'$. As a chain complex, A'_* can be written as a union of an increasing family of subcomplexes

$$A_* = A'^0_* \subseteq A'^1_* \subseteq A'^2_* \subseteq \cdots,$$

where A'^m_* denote the graded subspace of A'_* generated by products of the form $a_0 y a_1 y \cdots a_{m_1} y a_m$. The successive quotients for this filtration are given by $A'^m_*/A'^{m-1}_* \simeq A_*^{\otimes m+1}[-nm]$. It follows that the homology groups of A'_*/A_* can be computed by means of a (convergent) spectral sequence $\{E_r^{p,q}, d_r\}_{r \geq 2}$ with

$$E_2^{p,q} \simeq \begin{cases} 0 & \text{if } p \leq 0 \\ (\mathrm{H}_*(A_*)^{\otimes p+1})_{q+p+np} & \text{if } p \geq 1. \end{cases}$$

Since A is coconnective and $n > 0$, the groups $E_2^{p,q}$ vanish unless $p + q \leq -np < 0$. It follows that each homology group $\mathrm{H}_m(A'_*/A_*)$ admits a finite filtration by subquotients of the vector spaces $E_2^{p,q}$ with $p + q = m$, each of which is finite dimensional (since A is locally finite), and that the groups $\mathrm{H}_m(A'_*/A_*)$ vanish for $m \geq 0$. Using the long exact seuqence

$$\cdots \to \mathrm{H}_m(A_*) \to \mathrm{H}_m(A'_*) \to \mathrm{H}_m(A'_*/A_*) \to \mathrm{H}_{m-1}(A_*) \to \cdots,$$

we deduce that $\mathrm{H}_m(A'_*)$ is finite dimensional for all m and isomorphic to $\mathrm{H}_m(A_*)$ for $m \geq 0$, from which it follows immediately that $A' \in \Xi_0$.

We now complete the proof by showing that the weak deformation theory $\mathfrak{D}^{(1)}$ satisfies axiom $(D4)$ of Definition 1.3.9. For $n \geq 1$, and $A \in \mathrm{Alg}_k^{\mathrm{aug}}$, we have a canonical homotopy equivalence

$$\Psi(A)(\Omega^{\infty-n}(E)) = \mathrm{Map}_{\mathrm{Alg}_k^{\mathrm{aug}}}(\mathfrak{D}^{(1)}(k \oplus k[n]), A) \simeq \mathrm{Map}_{\mathrm{Alg}_k^{\mathrm{aug}}}(K_n, A) \simeq \Omega^{\infty-n-1} \mathrm{fib}(A \to k).$$

These maps determine an equivalence from the functor $e : \mathrm{Alg}_k^{\mathrm{aug}} \to \mathrm{Sp}$ with $I[1]$, where $I : \mathrm{Alg}_k^{\mathrm{aug}} \to \mathrm{Sp}$ denotes the functor which assigns to each augmented algebra $\epsilon : A \to k$ its augmentation ideal $\mathrm{fib}(\epsilon)$. This functor is evidently conservative, and preserves sifted colimits by Proposition A.3.2.3.1. $\square$

Remark 3.2.6. Let k be a field, let $\epsilon : A \to k$ be an object of $\mathrm{Alg}_k^{\mathrm{aug}}$, and let $X = \Psi(A)$ denote the formal $\mathbb{E}_1$ moduli problem associated to A via the equivalence of Theorem 3.0.4. The proof of Theorem 3.2.5 shows that the shifted tangent complex $X(E)[-1] \in \mathrm{Sp}$ can be identified with the augmentation ideal $\mathrm{fib}(\epsilon)$ of A.

We close this section by proving a noncommutative analogue of Corollary 2.3.6.

Proposition 3.2.7. *Let k be a field and let $X : \mathrm{Alg}_k^{\mathrm{sm}} \to \mathcal{S}$ be a formal $\mathbb{E}_1$ moduli problem over k. The following conditions are equivalent:*

(1) *The functor X is prorepresentable (see Definition 1.5.3).*

(2) *Let $X(E)$ denote the tangent complex of X. Then $\pi_i X(E) \simeq 0$ for $i > 0$.*

(3) *The functor X has the form $\Psi(A)$, where $A \in \mathrm{Alg}_k^{\mathrm{aug}}$ is coconnective and $\Psi : \mathrm{Alg}_k^{\mathrm{aug}} \to \mathrm{Moduli}_k^{(1)}$ is the equivalence of Theorem 3.0.4.*

Proof. The equivalence of (2) and (3) follows from Remark 3.2.6. We next prove that (1) $\Rightarrow$ (2). Since the construction $X \mapsto X(E)$ commutes with filtered colimits, it will suffice to show that $\pi_i X(E) \simeq 0$ for $i > 0$ in the case when $X = \mathrm{Spec}\, R$ is representable by an object $R \in \mathrm{Alg}_k^{\mathrm{sm}}$. In this case, we can write $X = \Psi(A)$ where $A = \mathbb{D}^{(1)}(R)$ belongs to the full subcategory $\Xi_0 \subseteq \mathrm{Alg}_k^{\mathrm{aug}}$ appearing in the proof of Theorem 3.2.5. In particular, A is coconnective, so that X satisfies condition (3) (and therefore also condition (2)).

We now complete the proof by showing that (3) $\Rightarrow$ (1). Let $A \in \mathrm{Alg}_k^{\mathrm{aug}}$ be coconnective. Choose a representative of A by a cofibrant differential graded algebra $A_* \in \mathrm{Alg}_k^{\mathrm{dg}}$. Since A_* is cofibrant, we may assume that the augmentation of A is determined by an augmentation of A_*. We now construct a sequence of differential graded algebras

$$k = A(-1)_* \to A(0)_* \to A(1)_* \to A(2)_* \to \cdots$$

equipped with maps $\phi(i) : A(i)_* \to A_*$. For each $n < 0$, choose a graded subspace $V_n \subseteq A_n$ consisting of cycles which maps isomorphically onto the homology $\mathrm{H}_n(A_*)$. We regard V_* as a differential graded vector space with trivial differential (which vanishes in nonnegative degrees). Let $A(0)_*$ denote the free differential graded Lie algebra generated by V_*, and $\phi(0) : A(0)_* \to A_*$ the evident map. Assume now that we have constructed a map $\phi(i) : A(i)_* \to A_*$ extending $\phi(1)$. Since A is coconnective, the map $\theta : \mathrm{H}_n(A(i)_*) \to \mathrm{H}_*(A_*)$ is surjective. Choose a collection of cycles $x_\alpha \in A(i)_{n_\alpha}$ whose images form a basis for $\ker(\theta)$. Then we can write $\phi(i)(x_\alpha) = dy_\alpha$ for some $y_\alpha \in A_{n_\alpha+1}$. Let $A(i+1)_*$ be the differential graded algebra obtained from $A(i)_*$ by freely adjoining elements Y_α (in degrees $n_\alpha + 1$) satisfying $dY_\alpha = x_\alpha$. We let $\phi(i+1) : A(i+1)_* \to A_*$ denote the unique extension of $\phi(i)$ satisfying $\phi(i+1)(Y_\alpha) = y_\alpha$.

We now prove the following assertion for each integer $i \geq 0$:

$(*_i)$ The inclusion $V_{-1} \hookrightarrow A(i)_{-1}$ induces an isomorphism $V_{-1} \to \mathrm{H}_{-1}(A(i)_*)$, the unit map $k \to A(i)_0$ is an isomorphism, and $A(i)_j \simeq 0$ for $j > 0$.

Assertion $(*_i)$ is obvious when $i = 0$. Let us assume that $(*_i)$ holds, and let θ be defined as above. Then θ is an isomorphism in degrees ≥ -1, so that $A(i+1)_*$ is obtained from $A(i)_*$ by freely adjoining generators Y_α in degrees ≤ -1. It follows immediately that $A(i+1)_j \simeq 0$ for $j > 0$ and that the unit map $k \to A(i+1)_0$ is an isomorphism. Moreover, we can write $A(i+1)_{-1} \simeq A(i)_{-1} \oplus W$, where W is the subspace spanned by elements of the form Y_α where $n_\alpha = -2$. By construction, the differential on $A(i+1)_*$ carries W injectively into

$$A(i)_{-2}/dA(i)_{-1} \subseteq A(i+1)_{-2}/dA(i)_{-1},$$

so that the differential graded algebras $A(i+1)_*$ and $A(i)_*$ have the same homology in degree -1.

Let A'_* denote the colimit of the sequence $\{A(i)_*\}_{i \geq 0}$. The evident map $\mathfrak{g}'_* \to \mathfrak{g}_*$ is surjective on homology (since the map $A(0)_* \to \mathfrak{g}_*$ is surjective on homology). If $\eta \in \ker(\mathrm{H}_*(A'_*) \to \mathrm{H}_*(A_*))$, then η is represented by a class $\overline{\eta} \in \ker(\mathrm{H}_*(A(i)_*) \to \mathrm{H}_*(A_*))$ for $i \gg 0$. By construction, the image of $\overline{\eta}$ vanishes in $\mathrm{H}_*(A(i+1)_*)$, so that $\eta = 0$. It follows that the map $A'_* \to A_*$ is a quasi-isomorphism. Since the collection of quasi-isomorphisms in $\mathrm{Alg}_k^{\mathrm{dg}}$ is closed under filtered colimits, we conclude that A_* is a homotopy colimit of the sequence $\{A(i)_*\}_{i \geq 0}$ in the model category $\mathrm{Alg}_k^{\mathrm{dg}}$. Let $A(i) \in \mathrm{Alg}_k^{\mathrm{aug}}$ be the image of the differential graded algebra $A(i)_*$ (equipped with the augmentation determined by the map $\phi(i) : A(i)_* \to A_*$), so that $A \simeq \varinjlim A(i)$ in $\mathrm{Alg}_k^{\mathrm{aug}}$. Setting $X(i) = \Psi(A(i)_*) \in \mathrm{Moduli}_k^{(1)}$, we deduce that $X \simeq \varinjlim X(i)$. To prove that X is prorepresentable, it will suffice to show that each $X(i)$ is prorepresentable.

We now proceed by induction on i, the case $i = -1$ being trivial. To carry out the inductive step, we note that each of the Lie algebras $A(i+1)_*$ is obtained from $A(i)_*$ by freely adjoining a set of generators $\{Y_\alpha\}_{\alpha \in S}$ of degrees $n_\alpha + 1 \leq -1$, satisfying $dY_\alpha = x_\alpha \in A(i)_{n_\alpha}$. Choose a well-ordering of the set S. For each $\alpha \in S$, we let $A_*^{<\alpha}$ denote the Lie subalgebra of $A(i+1)_*$ generated by $A(i)_*$ and the elements Y_β for $\beta < \alpha$, and let $A_*^{\leq \alpha}$ be defined similarly. Set

$$X^{<\alpha} = \Psi(A_*^{<\alpha}) \qquad X^{\leq \alpha} = \Psi(A_*^{\leq \alpha}).$$

For each integer n, let $B(n)_*$ be the free differential graded algebra generated by a class x in degree n and $B'(n)_*$ the free differential graded algebra generated by a class x in degree n and a class y in degree $n+1$ satisfying $dy = x$. For each $\alpha \in S$, we have a homotopy pushout diagram of differential graded algebras

$$\begin{tikzcd} B(n)_* \arrow[r] \arrow[d] & B'(n)_* \arrow[d] \\ A_*^{<\alpha} \arrow[r] & A_*^{\leq \alpha}, \end{tikzcd}$$

hence a pushout diagram diagram of formal $\mathbb{E}_1$ moduli problems

$$\begin{tikzcd} \mathrm{Spec}(k \oplus k[n_\alpha + 1]) \arrow[r] \arrow[d] & \mathrm{Spec}(k) \arrow[d] \\ X^{<\alpha} \arrow[r] & X^{\leq \alpha}. \end{tikzcd}$$

It follows that the map $X(i) \to X(i+1)$ satisfies the criterion of Lemma 1.5.9. Since $X(i)$ is prorepresentable, we conclude that $X(i+1)$ is prorepresentable. □

3.3 Comparison of Commutative and Associative Deformation Theories

Let k be a field of characteristic zero and let $X : \mathrm{Alg}_k^{\mathrm{sm}} \to \mathcal{S}$ be a formal $\mathbb{E}_1$-moduli problem. The forgetful functor $\mathrm{CAlg}_k \to \mathrm{Alg}_k$ carries small $\mathbb{E}_\infty$-algebras over k to small $\mathbb{E}_1$-algebras over k, and therefore induces a forgetful functor $\theta : \mathrm{CAlg}_k^{\mathrm{sm}} \to \mathrm{Alg}_k^{\mathrm{sm}}$. The composite functor $(X \circ \theta) : \mathrm{CAlg}_k^{\mathrm{sm}} \to \mathcal{S}$ is a formal moduli problem over k. Consequently, composition with θ determines a functor $\phi : \mathrm{Moduli}_k^{(1)} \to \mathrm{Moduli}_k$. Theorems 3.0.4 and 2.0.2 supply equivalences of ∞-categories

$$\mathrm{Alg}_k^{\mathrm{aug}} \simeq \mathrm{Moduli}_k^{(1)} \qquad \mathrm{Lie}_k \simeq \mathrm{Moduli}_k,$$

so that we can identify ϕ with a functor $\phi' : \mathrm{Alg}_k^{\mathrm{aug}} \to \mathrm{Lie}_k$. Our goal in this section is to give an explicit description of the functor ϕ'.

Recall that the ∞-category Alg_k of $\mathbb{E}_1$-algebras over k can be identified with the underlying ∞-category of the model category $\mathrm{Alg}_k^{\mathrm{dg}}$ of differential graded algebras over k (Proposition A.7.1.4.6). Let $(\mathrm{Alg}_k^{\mathrm{dg}})_{/k}$ denote the the category of *augmented* differential graded algebras over k. Then $(\mathrm{Alg}_k^{\mathrm{dg}})_{/k}$ inherits a model structure, and (because $k \in \mathrm{Alg}_k^{\mathrm{dg}}$ is fibrant) the underlying ∞-category of $(\mathrm{Alg}_k^{\mathrm{dg}})_{/k}$ can be identified with $\mathrm{Alg}_k^{\mathrm{aug}}$. For every object $\epsilon : A_* \to k$ of $(\mathrm{Alg}_k^{\mathrm{dg}})_{/k}$, we let $\mathfrak{m}_{A_*} = \ker(\epsilon)$ denote the augmentation ideal of A_*. Then $\mathfrak{m}_{A_*}$ inherits the structure of a nonunital differential graded algebra over k. In particular, we can view $\mathfrak{m}_{A_*}$ as a differential graded Lie algebra over k (see Example 2.1.6). The construction $A_* \mapsto \mathfrak{m}_{A_*}$ determines a functor $(\mathrm{Alg}_k^{\mathrm{dg}})_{/k} \to \mathrm{Lie}_k^{\mathrm{dg}}$, which carries quasi-isomorphisms to quasi-isomorphisms. We therefore obtain an induced functor of ∞-categories $\psi : \mathrm{Alg}_k^{\mathrm{aug}} \to \mathrm{Lie}\, k$. We will prove that the functors $\psi, \phi' : \mathrm{Alg}_k^{\mathrm{aug}} \to \mathrm{Lie}_k$ are equivalent to one another. We can state this result more precisely as follows:

Theorem 3.3.1. *Let k be a field of characteristic zero. The diagram of ∞-categories*

$$\begin{tikzcd} \mathrm{Alg}_k^{\mathrm{aug}} \arrow[r, "\psi"] \arrow[d] & \mathrm{Lie}_k \arrow[d] \\ \mathrm{Moduli}_k^{(1)} \arrow[r, "\phi"] & \mathrm{Moduli}_k \end{tikzcd}$$

commutes (up to canonical homotopy). Here ϕ and ψ are the functors described above, and the vertical maps are the equivalences provided by Theorems 2.0.2 and 3.0.4.

To prove Theorem 3.3.1, we need to construct a homotopy between two functors $\mathrm{Alg}_k^{\mathrm{aug}} \to \mathrm{Moduli}_k \subseteq \mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{sm}}, \mathcal{S})$. Equivalently, we must construct a homotopy between the functors

$$F, F' : \mathrm{Alg}_k^{\mathrm{aug}} \times \mathrm{CAlg}_k^{\mathrm{sm}} \to \mathcal{S}$$

given by

$$F(A, R) = \mathrm{Map}_{\mathrm{Lie}_k}(\mathfrak{D}(R), \psi(A)) \qquad F'(A, R) = \mathrm{Map}_{\mathrm{Alg}_k^{\mathrm{aug}}}(\mathfrak{D}^{(1)}(R), A).$$

Composing the Koszul duality functor $\mathfrak{D} : (\mathrm{CAlg}_k^{\mathrm{aug}})^{op} \to \mathrm{Lie}_k$ with the equivalence of ∞-categories $\mathrm{Lie}_k \simeq \mathrm{Moduli}_k$, we obtain the functor $\mathrm{Spec} : (\mathrm{CAlg}_k^{\mathrm{aug}})^{op} \to \mathrm{Moduli}_k$ of Example 1.1.16. It follows from Yoneda's lemma that this functor is fully faithful when restricted to $(\mathrm{CAlg}_k^{\mathrm{sm}})^{op}$, so that $\mathfrak{D}$ induces an equivalence from $(\mathrm{CAlg}_k^{\mathrm{aug}})^{op}$ onto its essential image $\mathcal{C} \subseteq \mathrm{Lie}_k$. The inverse of this equivalence is given by $\mathfrak{g}_* \mapsto C^*(\mathfrak{g}_*)$. It follows that we can identify F and F' with functors $G, G' : \mathrm{Alg}_k^{\mathrm{aug}} \times \mathcal{C}^{op} \to \mathcal{S}$, given by the formulas

$$G(A, \mathfrak{g}_*) = \mathrm{Map}_{\mathrm{Lie}_k}(\mathfrak{g}_*, \psi(A)) \qquad G'(A, \mathfrak{g}_*) = \mathrm{Map}_{\mathrm{Alg}_k^{\mathrm{aug}}}(\mathfrak{D}^{(1)} C^*(\mathfrak{g}_*), A).$$

Note that the forgetful functor $(\mathrm{Alg}_k^{\mathrm{dg}})_{/k} \to \mathrm{Lie}_k^{\mathrm{dg}}$ is a right Quillen functor, with left adjoint given by the universal enveloping algebra construction $\mathfrak{g}_* \mapsto U(\mathfrak{g}_*)$ of Remark 2.1.7. It follows that the functor ψ admits a left adjoint $\mathrm{Lie}_k \to \mathrm{Alg}_k^{\mathrm{aug}}$, which we will also denote by U. Then the functor $G : \mathrm{Alg}_k^{\mathrm{aug}} \times \mathcal{C}^{op} \to \mathcal{S}$ can be described by the formula $G(A, \mathfrak{g}_*) = \mathrm{Map}_{\mathrm{Alg}_k^{\mathrm{aug}}}(U(\mathfrak{g}_*), A)$. Theorem 3.3.1 is therefore a consequence of the following assertion:

Proposition 3.3.2. *Let k be a field of characteristic zero. Then the diagram of ∞-categories*

$$\begin{tikzcd} (\mathrm{Lie}_k)^{op} \arrow[r, "C^*"] \arrow[d, "U"] & \mathrm{CAlg}_k^{\mathrm{aug}} \arrow[d] \\ (\mathrm{Alg}_k^{\mathrm{aug}})^{op} \arrow[r, "\mathfrak{D}^{(1)}"] & \mathrm{Alg}_k^{\mathrm{aug}} \end{tikzcd}$$

commutes up to canonical homotopy.

The proof of Proposition 3.3.2 will require a brief digression.

Definition 3.3.3. Let $\lambda : \mathcal{M} \to \mathcal{C} \times \mathcal{D}$ and $\lambda' : \mathcal{M}' \to \mathcal{C}' \times \mathcal{D}'$ be pairings of ∞-categories. A *morphism of pairings* from λ to λ' is a triple of maps

$$\alpha : \mathcal{M} \to \mathcal{M}' \qquad \beta : \mathcal{C} \to \mathcal{C}' \qquad \gamma : \mathcal{D} \to \mathcal{D}'$$

such that the diagram

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\alpha"] \arrow[d, "\lambda"] & \mathcal{M}' \arrow[d, "\lambda'"] \\ \mathcal{C} \times \mathcal{D} \arrow[r, "(\beta,\gamma)"] & \mathcal{C}' \times \mathcal{D}' \end{tikzcd}$$

commutes up to homotopy. Assume that λ and λ' are left representable. We will say that a morphism of pairings (α, β, γ) is *left representable* if it carries left universal objects of $\mathcal{M}$ to left universal objects of $\mathcal{M}'$.

Proposition 3.3.4. *Let $\lambda : \mathcal{M} \to \mathcal{C} \times \mathcal{D}$ and $\lambda' : \mathcal{M}' \to \mathcal{C}' \times \mathcal{D}'$ be left representable pairings of ∞-categories, which induce functors $\mathfrak{D}_\lambda : \mathcal{C}^{op} \to \mathcal{D}$ and $\mathfrak{D}_{\lambda'} : \mathcal{C}'^{op} \to \mathcal{D}'$. Let (α, β, γ) from λ to λ'. Then the diagram*

$$\begin{tikzcd} \mathcal{C}^{op} \arrow[r, "\mathfrak{D}_\lambda"] \arrow[d, "\beta"] & \mathcal{D} \arrow[d, "\gamma"] \\ \mathcal{C}'^{op} \arrow[r, "\mathfrak{D}_{\lambda'}"] & \mathcal{D}' \end{tikzcd}$$

commutes up to canonical homotopy.

Proof. The right fibrations λ and λ' are classified by functors

$$\mathcal{C}^{op} \times \mathcal{D}^{op} \to \mathcal{S} \qquad \mathcal{C}'^{op} \times \mathcal{D}'^{op} \to \mathcal{S},$$

which we can identify with maps $\chi : \mathcal{C}^{op} \to \mathrm{Fun}(\mathcal{D}^{op}, \mathcal{S})$ and $\chi' : \mathcal{C}'^{op} \to \mathrm{Fun}(\mathcal{D}'^{op}, \mathcal{S})$. Let $G : \mathrm{Fun}(\mathcal{D}'^{op}, \mathcal{S}) \to \mathrm{Fun}(\mathcal{C}'^{op}, \mathcal{S})$ be the functor given by composition with β. Then α induces a natural transformation $\chi \to G \circ \chi' \circ \beta$. Let F denote a left adjoint to G, so that we obtain a natural transformation $u : F \circ \chi \to \chi' \circ \beta$ of functors from $\mathcal{C}^{op}$ to $\mathrm{Fun}(\mathcal{D}'^{op}, \mathcal{S})$. Let $j_{\mathcal{D}} : \mathcal{D} \to \mathrm{Fun}(\mathcal{D}^{op}, \mathcal{S})$ and $j_{\mathcal{D}'} : \mathcal{D}' \to \mathrm{Fun}(\mathcal{D}'^{op}, \mathcal{S})$ denote the Yoneda embeddings. Then $\chi \simeq j_{\mathcal{D}} \circ \mathfrak{D}_\lambda$ and $\chi' \simeq j_{\mathcal{D}'} \circ \mathfrak{D}_{\lambda'}$, and Proposition T.5.2.6.3 gives an equivalence $F \circ j_{\mathcal{D}} \simeq j_{\mathcal{D}'} \circ \gamma$. Then u determines a natural transformation

$$j_{\mathcal{D}'} \circ \gamma \circ \mathfrak{D}_\lambda \simeq F \circ j_{\mathcal{D}} \circ \mathfrak{D}_\lambda \simeq F \circ \chi \xrightarrow{u} \chi' \circ \beta \simeq j_{\mathcal{D}'} \circ \mathfrak{D}_{\lambda'} \circ \beta.$$

Since $j_{\mathcal{D}'}$ is fully faithful, this is the image of the a natural transformation of functors $\gamma \circ \mathfrak{D}_\lambda \to \mathfrak{D}_{\lambda'} \circ \beta$. Our assumption that α carries left universal objects of $\mathcal{M}$ to left universal objects of $\mathcal{M}'$ implies that this natural transformation is an equivalence. $\square$

Construction 3.3.5. Let $\mathcal{C}$ be a category. We define a new category $\mathrm{TwArr}(\mathcal{C})$ as follows:

- (a) An object of $\mathrm{TwArr}(\mathcal{C})$ is given by a triple (C, D, ϕ), where $C \in \mathcal{C}$, $D \in \mathcal{D}$, and $\phi : C \to D$ is a morphism in $\mathcal{C}$.
- (b) Given a pair of objects $(C, D, \phi), (C', D', \phi') \in \mathrm{TwArr}(\mathcal{C})$, a morphism from (C, D, ϕ) to (C', D', ϕ') consists of a pair of morphisms $\alpha : C \to C'$, $\beta : D' \to D$ for which the diagram

$$\begin{tikzcd} C \arrow[r, "\phi"] \arrow[d, "\alpha"] & D \\ C' \arrow[r, "\phi'"] & D' \arrow[u, "\beta"] \end{tikzcd}$$

  commutes.

(c) Given a pair of morphisms

$$(C, D, \phi) \xrightarrow{(\alpha,\beta)} (C', D', \phi') \xrightarrow{(\alpha',\beta')} (C'', D'', \phi'')$$

in $\mathrm{TwArr}(\mathcal{C})$, the composition of (α', β') with (α, β) is given by $(\alpha' \circ \alpha, \beta \circ \beta')$.

We will refer to $\mathrm{TwArr}(\mathcal{C})$ as the *twisted arrow category* of $\mathcal{C}$. The construction $(C, D, \phi) \mapsto (C, D)$ determines a forgetful functor $\lambda : \mathrm{TwArr}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{op}$ which exhibits $\mathrm{TwArr}(\mathcal{C})$ as fibered in sets over $\mathcal{C} \times \mathcal{C}^{op}$ (the fiber of λ over an object $(C, D) \in \mathcal{C} \times \mathcal{C}^{op}$ can be identified with the set $\mathrm{Hom}_{\mathcal{C}}(C, D)$). It follows that the induced map

$$\lambda : \mathrm{N}(\mathrm{TwArr}(\mathcal{C})) \to \mathrm{N}(\mathcal{C}) \times \mathrm{N}(\mathcal{C})^{op}$$

is a pairing of ∞-categories. This pairing is both left and right representable, and the associated duality functors

$$\mathfrak{D}_\lambda : \mathrm{N}(\mathcal{C})^{op} \to \mathrm{N}(\mathcal{C})^{op} \qquad \mathfrak{D}'_\lambda : \mathrm{N}(\mathcal{C}) \to \mathrm{N}(\mathcal{C})$$

are equivalent to the identity.

Remark 3.3.6. We will discuss an ∞-categorical version of Construction 3.3.5 in §4.2.

We will deduce Proposition 3.3.2 from the following:

Proposition 3.3.7. *Let k be a field of characteristic zero and let $\mathcal{M}^{(1)} \to \mathrm{Alg}_k^{\mathrm{aug}} \times \mathrm{Alg}_k^{\mathrm{aug}}$ be the pairing of ∞-categories of Construction 3.1.4. There exists a left representable map of pairings*

$$\begin{tikzcd}
\mathrm{N}(\mathrm{TwArr}(\mathrm{Lie}_k^{\mathrm{dg}})) \arrow[d, "\lambda"] & & \mathcal{M}^{(1)} \arrow[d] \\
\mathrm{N}(\mathrm{Lie}_k^{\mathrm{dg}}) \times \mathrm{N}(\mathrm{Lie}_k^{\mathrm{dg}})^{op} \arrow[rr, "U \times C^*"] & & \mathrm{Alg}_k^{\mathrm{aug}} \times \mathrm{Alg}_k^{\mathrm{aug}} .
\end{tikzcd}$$

Here U and C^* denote the (covariant and contravariant) functors from $\mathrm{N}(\mathrm{Lie}_k^{\mathrm{dg}})$ to $\mathrm{Alg}_k^{\mathrm{aug}}$ induced by the universal enveloping algebra and cohomological Chevalley-Eilenberg constructions, respectively.

Proof of Proposition 3.3.2. As noted in Construction 3.3.5, the pairing of ∞-categories $\mathrm{N}(\mathrm{TwArr}(\mathrm{Lie}_k^{\mathrm{dg}})) \to \mathrm{N}(\mathrm{Lie}_k^{\mathrm{dg}}) \times \mathrm{N}(\mathrm{Lie}_k^{\mathrm{dg}})^{op}$ induces the identity functor $\mathrm{id} : \mathrm{N}(\mathrm{Lie}_k^{\mathrm{dg}})^{op} \to \mathrm{N}(\mathrm{Lie}_k^{\mathrm{dg}})^{op}$. Applying Proposition 3.3.4 to the morphism of pairings T of Proposition 3.3.7, we obtain an equivalence between the functors $C^* \circ \mathrm{id}, \mathfrak{D}^{(1)} \circ U : \mathrm{N}(\mathrm{Lie}_k^{\mathrm{dg}})^{op} \to \mathrm{Alg}_k^{\mathrm{aug}}$. Since the canonical map

$$\mathrm{Fun}(\mathrm{Lie}_k^{op}, \mathrm{Alg}_k^{op}) \to \mathrm{Fun}(\mathrm{N}(\mathrm{Lie}_k^{\mathrm{dg}})^{op}, \mathrm{Alg}_k^{\mathrm{aug}})$$

is fully faithful, we obtain an equivalence between the functors $C^*, \mathfrak{D}^{(1)} \circ U : \mathrm{Lie}_k^{op} \to \mathrm{Alg}_k^{\mathrm{aug}}$. $\square$

Proof of Proposition 3.3.7. Let $\mathfrak{g}_*$ be a differential graded Lie algebra and let $\mathrm{Cn}(\mathfrak{g})_*$ be as in Construction 2.2.1. The universal enveloping algebra $U(\mathrm{Cn}(\mathfrak{g})_*)$ has the structure of a (differential graded) Hopf algebra, where the comultiplication is determined by the requirement that the image of $\mathrm{Cn}(\mathfrak{g})_*$ consists of primitive elements. In particular, we have a counit map $\epsilon : \mathrm{Cn}(\mathfrak{g})_* \to k$. Let $\mathrm{End}(U\,\mathrm{Cn}(\mathfrak{g})_*)$ denote the chain complex of $U(\mathrm{Cn}(\mathfrak{g})_*)$-comodule maps from $U(\mathrm{Cn}(\mathfrak{g})_*)$ to itself. Since $U(\mathrm{Cn}(\mathfrak{g})_*)$ is cofree as a comodule over itself, composition with the counit map $\epsilon : U(\mathrm{Cn}(\mathfrak{g})_*) \to k$ induces an isomorphism θ from $\mathrm{End}(U\,\mathrm{Cn}(\mathfrak{g})_*)$ to the k-linear dual of $U\,\mathrm{Cn}(\mathfrak{g})_*$. We regard $\mathrm{End}(U\,\mathrm{Cn}(\mathfrak{g})_*)$ as endowed with the *opposite* of the evident differential graded Lie algebra structure, so that $U\,\mathrm{Cn}(\mathfrak{g})_*$ has the structure of a right module over $\mathrm{End}(U\,\mathrm{Cn}(\mathfrak{g})_*)$. Let $\mathrm{End}_{\mathfrak{g}}(U\,\mathrm{Cn}(\mathfrak{g})_*)$ denote the subcomplex of $\mathrm{End}(U\,\mathrm{Cn}(\mathfrak{g})_*)$ consisting of right $U(\mathfrak{g}_*)$-module maps, so that θ restricts to an isomorphism from $\mathrm{End}_{\mathfrak{g}}(U\,\mathrm{Cn}(\mathfrak{g})_*)$ to the k-linear dual $C^*(\mathfrak{g}_*)$ of $C_*(\mathfrak{g}_*) \simeq U\,\mathrm{Cn}(\mathfrak{g})_* \otimes_{U(\mathfrak{g}_*)} k$. It is not difficult to verify that this isomorphism is compatible with the multiplication on $C^*(\mathfrak{g})$ described in Construction 2.2.13. It follows that $U\,\mathrm{Cn}(\mathfrak{g})_*$ is equipped with a right action of $C^*(\mathfrak{g}_*)$, which is compatible with the right action of $U(\mathfrak{g}_*)$ on $U\,\mathrm{Cn}(\mathfrak{g})_*$. Let $M_*(\mathfrak{g}_*)$ denote the k-linear dual of $U\,\mathrm{Cn}(\mathfrak{g})_*$. Then M_* is a contravariant functor, which carries a differential graded Lie algebra $\mathfrak{g}_*$ to a chain complex equipped with commuting right actions of $U(\mathfrak{g}_*)$ and $C^*(\mathfrak{g}_*)$. Moreover, the unit map $k \to UE(\mathfrak{g}_*)$ determines a quasi-isomorphism $\epsilon_{\mathfrak{g}_*} : M_*(\mathfrak{g}_*) \to k$.

Note that the initial object $k \in \mathrm{Alg}_k^{(1)}$ can be identified with a classifying object for endomorphisms of the unit object $k \in \mathrm{Mod}_k$. Using Theorem A.6.1.2.34 and Proposition A.6.1.2.39, we can identify $\mathrm{Alg}_k^{\mathrm{aug}}$ with the fiber product $\mathrm{LMod}(\mathrm{Mod}_k) \times_{\mathrm{Mod}_k} \{k\}$. Let $\mathcal{X} \subseteq (\mathrm{Mod}_k)_{/k}$ denote the full subcategory spanned by the final objects, so that we have an equivalence of ∞-categories

$$\alpha : \mathcal{M}^{(1)} \simeq (\mathrm{Alg}_k \times \mathrm{Alg}_k) \times_{\mathrm{Alg}_k} \mathrm{LMod}(\mathrm{Mod}_k) \times_{\mathrm{Mod}_k} \mathcal{X}.$$

We define a more rigid analogue of $\mathcal{M}^{(1)}$ as follows: let $\mathcal{Y} \subseteq (\mathrm{Vect}_k^{\mathrm{dg}})_{k/}$ be the full subcategory spanned by the quasi-isomorphisms of chain complexes $V_* \to k$ and let $\mathcal{C}$ denote the category

$$\mathrm{Alg}_k^{\mathrm{dg}} \times \mathrm{Alg}_k^{\mathrm{dg}} \times_{\mathrm{Alg}_k^{\mathrm{dg}}} \mathrm{LMod}(\mathrm{Vect}_k^{\mathrm{dg}}) \times_{\mathrm{Vect}_k^{\mathrm{dg}}} \mathcal{Y},$$

so that α determines a functor $T'' : \mathrm{N}(\mathcal{C}) \to \mathcal{M}^{(1)}$. We will define T as a composition

$$\mathrm{TwArr}(\mathrm{N}(\mathrm{Lie}_k^{\mathrm{dg}})) \xrightarrow{T'} \mathrm{N}(\mathcal{C}) \xrightarrow{T''} \mathcal{M}^{(1)}.$$

Here the functor T' assigns to each map $\gamma : \mathfrak{h}_* \to \mathfrak{g}_*$ of differential graded Lie algebras the object of $\mathcal{C}$ given by $(U(\mathfrak{h}_*), C^*(\mathfrak{g}_*), M_*(\mathfrak{g}_*), \epsilon_{\mathfrak{g}_*})$, where $M_*(\mathfrak{g}_*)$ is regarded as a left module over $U(\mathfrak{h}_*) \otimes_k C^*(\mathfrak{g}_*)$ by combining the commuting left actions of $U(\mathfrak{g}_*)$ and $C^*(\mathfrak{g}_*)$ on $M_*(\mathfrak{g}_*)$ (and composing with the map γ).

We now claim that the diagram σ :

$$\begin{tikzcd} \mathrm{N}(\mathrm{TwArr}(\mathrm{Lie}_k^{\mathrm{dg}})) \arrow[r, "T"] \arrow[d, "\lambda"] & \mathcal{M}^{(1)} \arrow[d] \\ \mathrm{N}(\mathrm{Lie}_k^{\mathrm{dg}}) \times \mathrm{N}(\mathrm{Lie}_k^{\mathrm{dg}})^{op} \arrow[r, "U \times C^*"] & \mathrm{Alg}_k^{\mathrm{aug}} \times \mathrm{Alg}_k^{\mathrm{aug}} \end{tikzcd}$$

commutes up to canonical homotopy. Consider first the composition of T with the map $\mathcal{M}^{(1)} \to \mathrm{Alg}_k^{\mathrm{aug}}$ given by projection onto the first factor. Unwinding the definitions, we see that this map is given by the composing the equivalence $\xi : \mathrm{LMod}(\mathrm{Mod}_k) \times_{\mathrm{Mod}_k} \mathcal{X} \simeq \mathrm{Alg}_k^{\mathrm{aug}}$ with the functor $T_0' : \mathrm{N}(\mathrm{TwArr}(\mathrm{Lie}_k^{\mathrm{dg}})) \to \mathrm{LMod}(\mathrm{Mod}_k) \times_{\mathrm{Mod}_k} \mathcal{X}$ given by

$$T_0'(\gamma : \mathfrak{h}_* \to \mathfrak{g}_*) = (U(\mathfrak{h}_*), M_*(\mathfrak{g}_*), \epsilon_{\mathfrak{g}_*}).$$

The counit map $U(\mathfrak{g}_*) \to k$ determines a quasi-isomorphism of $U(\mathfrak{h}_*)$-modules $k \to M_*(\mathfrak{g}_*)$, so that T_0' is equivalent to the functor $\overline{T}_0'$ given by $\overline{T}_0'(\gamma : \mathfrak{h}_* \to \mathfrak{g}_*) = (U(\mathfrak{h}_*), k, \mathrm{id}_k)$, which (after composing with ξ) can be identified with the map $\mathrm{N}(\mathrm{TwArr}(\mathrm{Lie}_k^{\mathrm{dg}})) \to \mathrm{N}(\mathrm{Lie}_k^{\mathrm{dg}}) \xrightarrow{U} \mathrm{Alg}_k^{\mathrm{aug}}$. Now consider the composition of T with the map $\mathcal{M}^{(1)} \to \mathrm{Alg}_k^{\mathrm{aug}}$ given by projection onto the second factor. This functor is given by composing the equivalence ξ with the functor $T_1' : \mathrm{N}(\mathrm{TwArr}(\mathrm{Lie}_k^{\mathrm{dg}})) \to \mathrm{LMod}(\mathrm{Mod}_k) \times_{\mathrm{Mod}_k} \mathcal{X}$ given by $T_1'(\gamma : \mathfrak{h}_* \to \mathfrak{g}_*) = (C^*(\mathfrak{g}), M_*(\mathfrak{g}_*), \epsilon_{\mathfrak{g}_*})$. Note that $\epsilon_{\mathfrak{g}}$ is a map of $C^*(\mathfrak{g})$-modules and therefore determines an equivalence of T_1' with the functor $\overline{T}_1'$ given by $\overline{T}_1'(\gamma : \mathfrak{h}_* \to \mathfrak{g}_*) = (C^*(\mathfrak{g}), k, \mathrm{id}_k)$. It follows that the composition of T_1' with ξ can be identified with the composition

$$\mathrm{N}(\mathrm{TwArr}(\mathrm{Lie}_k^{\mathrm{dg}})) \to \mathrm{N}(\mathrm{Lie}_k^{\mathrm{dg}})^{op} \xrightarrow{C^*} \mathrm{Alg}_k^{\mathrm{aug}}.$$

This proves the homotopy commutativity of the diagram σ. After replacing T by an equivalent functor, we can assume that the diagram σ is commutative.

It remains to show that σ determines a left representable map between pairings of ∞-categories. Let $\mathfrak{g}_*$ be a differential graded Lie algebra, and let $\mathrm{End}(M_*(\mathfrak{g}_*))$ denote the differential graded algebra of endomorphisms of the chain complex $M_*(\mathfrak{g}_*)$. Since $M_*(\mathfrak{g}_*)$ is quasi-isomorphic to k, the unit map $k \to \mathrm{End}(M_*(\mathfrak{g}_*))$ is a quasi-isomorphism of differential graded algebras. Unwinding the definitions, we must show that the map $\theta : U(\mathfrak{g}_*) \otimes_k C^*(\mathfrak{g}_*) \to \mathrm{End}(M_*(\mathfrak{g}))$ exhibits $C^*(\mathfrak{g}_*)$ as Koszul dual (as an $\mathbb{E}_1$-algebra) to $U(\mathfrak{g}_*)$. Let A_* denote the differential graded algebra of endomorphisms of $U\,\mathrm{Cn}(\mathfrak{g})_*$ (as a chain complex). Then θ factors as a composition

$$U(\mathfrak{g}_*) \otimes_k C^*(\mathfrak{g}_*) \xrightarrow{\theta'} A_* \xrightarrow{\theta''} \mathrm{End}(M_*(\mathfrak{g}_*))$$

where θ'' is a quasi-isomorphism. It will therefore suffice to show that θ' exhibits $C^*(\mathfrak{g}_*)$ as Koszul dual to $U(\mathfrak{g}_*)$. Since $UE_*(\mathfrak{g}_*)$ is a free $U(\mathfrak{g}_*)$-module, this is equivalent to the requirement that θ' induces a quasi-isomorphism $\phi : C^*(\mathfrak{g}_*) \to W_*$, where W_* is the differential graded algebra of right $U(\mathfrak{g}_*)$-module maps from $U\,\mathrm{Cn}(\mathfrak{g})_*$ to itself. This is clear, since ϕ admits a left inverse given by composition with the quasi-isomorphism $U\,\mathrm{Cn}(\mathfrak{g})_* \to k$. $\square$

3.4 Quasi-Coherent and Ind-Coherent Sheaves

Let k be a field and let $X : \mathrm{CAlg}_k^{\mathrm{sm}} \to \mathcal{S}$ be a formal moduli problem. In §2.4, we introduced a symmetric monoidal ∞-category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves on X. Our goal in this section is to study analogous definitions in the noncommutative setting. In this case, there is no symmetric monoidal structure and it is important to distinguish between left and right modules. Consequently, if $X : \mathrm{Alg}_k^{\mathrm{sm}} \to \mathcal{S}$ is a formal $\mathbb{E}_1$-moduli problem, then there are two natural analogues of the ∞-category $\mathrm{QCoh}(X)$. We will denote these ∞-categories by $\mathrm{QCoh}_L(X)$ and $\mathrm{QCoh}_R(X)$, and refer to them as the ∞-categories of (left and right) *quasi-coherent sheaves* on X. We will also study noncommutative counterparts of the fully faithful embedding $\mathrm{QCoh}(X) \to \mathrm{QCoh}^!(X)$ of Remark 2.4.30.

We will devote most of our attention to the case where $X = \operatorname{Spec} A$ is corepresented by a small $\mathbb{E}_1$-algebra A over k. At the end of this section, we will explain how to extrapolate our discussion to the general case (Construction 3.4.20).

Definition 3.4.1. Let k be a field. We will say that an object $M \in \mathrm{Mod}_k$ is *small* if it is perfect as a k-module: that is, if $\pi_* M$ has finite dimension over k. If R is an $\mathbb{E}_1$-algebra over k and M is a right or left module over R, we will say that M is *small* if it is small when regarded as an object of Mod_k. We let $\mathrm{LMod}_R^{\mathrm{sm}}$ denote the full subcategory of LMod_R spanned by the small left R-modules, and $\mathrm{RMod}_R^{\mathrm{sm}}$ the full subcategory of RMod_R spanned by the small right R-modules.

Remark 3.4.2. Let k be a field, let R be an augmented $\mathbb{E}_1$-algebra over k. Assume that R is connective and that the kernel I of the augmentation map $\pi_0 R \to k$ is a nilpotent ideal in $\pi_0 R$. Then an object $M \in \mathrm{LMod}_R$ is small if and only if it belongs to the the smallest stable full subcategory $\mathcal{C} \subseteq \mathrm{LMod}_R$ which contains $k \simeq (\pi_0 R)/I$ and is closed under equivalence. The "if" direction is obvious (and requires no assumptions on R), since the full subcategory $\mathrm{LMod}_R^{\mathrm{sm}} \subseteq \mathrm{LMod}_R$ is stable, closed under retracts, and contains k. For the converse, suppose that M is small; we prove that $M \in \mathcal{C}$ using induction on the dimension of the k-vector space $\pi_* M$. If $M \simeq 0$ there is nothing to prove. Otherwise, there exists some largest integer n such that $\pi_n M$ is nonzero. Since I is nilpotent, there exists a nonzero element $x \in \pi_n M$ which is annihilated by I. Then multiplication by x determines a map of discrete R-modules $k \to \pi_n M$, which in turn determines a fiber sequence of R-modules

$$k[n] \to M \to M'.$$

The inductive hypothesis guarantees that $M' \in \mathcal{C}$ and it is clear that $k[n] \in \mathcal{C}$, so that $M \in \mathcal{C}$ as desired.

We first show that if R is an $\mathbb{E}_1$-algebra over a field k, then k-linear duality determines a contravariant equivalence between the ∞-categories $\mathrm{LMod}_R^{\mathrm{sm}}$ and $\mathrm{RMod}_R^{\mathrm{sm}}$.

Lemma 3.4.3. *Let k be a field and let R be an $\mathbb{E}_1$-algebra over k. Define a functor $\lambda : \mathrm{RMod}_R^{op} \times \mathrm{LMod}_R^{op} \to \mathcal{S}$ by the formula $\lambda(M, N) = \mathrm{Map}_{\mathrm{Mod}_k}(M \otimes_R N, k)$. Then:*

(1) *For every right R-module M, let $\lambda_M : \mathrm{LMod}_R^{op} \to \mathcal{S}$ be the restriction of λ to $\{M\} \times \mathrm{LMod}_R^{op}$. Then λ_M is a representable functor.*

(1′) *Let $\mu : \mathrm{RMod}_R^{op} \to \mathrm{Fun}(\mathrm{LMod}_R^{op}, \mathcal{S})$ be given by $\mu(M)(N) = \lambda(M, N)$. Then μ is homotopic to a composition*

$$\mathrm{RMod}_R^{op} \xrightarrow{\mu_0} \mathrm{LMod}_R \xrightarrow{j} \mathrm{Fun}(\mathrm{LMod}_R^{op}, \mathcal{S}),$$

where j denotes the Yoneda embedding.

(2) *For every left R-module M, let $\lambda_N : \mathrm{RMod}_R^{op} \to \mathcal{S}$ be the restriction of λ to $\mathrm{RMod}_R^{op} \times \{N\}$. Then λ_N is a representable functor.*

(2′) *Let $\mu' : \mathrm{LMod}_R^{op} \to \mathrm{Fun}(\mathrm{RMod}_R^{op}, \mathcal{S})$ be given by $\mu'(N)(M) = \lambda(M, N)$. Then μ' is homotopic to a composition*

$$\mathrm{LMod}_R^{op} \xrightarrow{\mu'_0} \mathrm{RMod}_R \xrightarrow{j} \mathrm{Fun}(\mathrm{RMod}_R^{op}, \mathcal{S}),$$

where j denotes the Yoneda embedding.

(3) *The functors μ_0 and μ'_0 determine mutually inverse equivalences between the ∞-categories $\mathrm{RMod}_R^{\mathrm{sm}}$ and $(\mathrm{LMod}_R^{\mathrm{sm}})^{op}$.*

Proof. We first note that (1′) and (2′) are reformulations of (1) and (2). We will prove (1); the proof of (2) is similar. Let $M \in \mathrm{RMod}_R$; we wish to show that λ_M is a representable functor. Since LMod_R is a presentable ∞-category, it will suffice to show that the functor λ_M preserves small limits (Proposition T.5.5.2.2). This is clear, since the functor $N \mapsto M \otimes_R N$ preserves small colimits.

Let $\mu_0 : \mathrm{RMod}_R^{op} \to \mathrm{LMod}_R$ and $\mu'_0 : \mathrm{LMod}_R^{op} \to \mathrm{RMod}_R$ be as in (1′) and (2′). We note that μ'_0 can be identified with the right adjoint to μ_0^{op}. Let $M \in \mathrm{RMod}_R$. For every integer n, we have canonical isomorphisms

$$\pi_n \mu_0(M) \simeq \pi_0 \mathrm{Map}_{\mathrm{RMod}_R}(R[n], \mu_0(M)) \simeq \pi_0 \mathrm{Map}_{\mathrm{Mod}_k}(M \otimes_R R[n], k) \simeq (\pi_{-n} M)^\vee,$$

where $(\pi_n M)^\vee$ denotes the k-linear dual of the vector space $\pi_{-n} M$. It follows that μ_0 carries $(\mathrm{RMod}_R^{\mathrm{sm}})^{op}$ into $\mathrm{LMod}_R^{\mathrm{sm}}$. Similarly, μ'_0 carries $(\mathrm{LMod}_R^{\mathrm{sm}})^{op}$ into $\mathrm{RMod}_R^{\mathrm{sm}}$. To prove (3), it will suffice to show that for every pair of objects $M \in \mathrm{RMod}_R^{\mathrm{sm}}$, $N \in \mathrm{LMod}_R^{\mathrm{sm}}$, the unit maps

$$M \to \mu'_0 \mu_0(M) \qquad N \to \mu_0 \mu'_0(N)$$

are equivalences in RMod_R and LMod_R, respectively. Passing to homotopy groups, we are reduced to proving that the double duality maps

$$\pi_n M \to ((\pi_n M)^\vee)^\vee \qquad \pi_n N \to ((\pi_n N)^\vee)^\vee$$

are isomorphisms for every integer n. This follows from the finite-dimensionality of the vector spaces $\pi_n M$ and $\pi_n N$. □

Definition 3.4.4. Let k be a field and let $R \in \mathrm{Alg}_k^{\mathrm{sm}}$ be a small $\mathbb{E}_1$-algebra over k. We let $\mathrm{LMod}_R^!$ denote the full subcategory of $\mathrm{Fun}(\mathrm{RMod}_R^{\mathrm{sm}}, \mathcal{S})$ spanned by the left exact functors, and $\mathrm{RMod}_R^!$ the full subcategory of $\mathrm{Fun}(\mathrm{LMod}_R^{\mathrm{sm}}, \mathcal{S})$ spanned by the left exact functors. We will refer to $\mathrm{LMod}_R^!$ as the *∞-category of* Ind-*coherent left R-modules*, and $\mathrm{RMod}_R^!$ as the *∞-category of* Ind-*coherent right R-modules.*

Remark 3.4.5. Using the equivalence $\mathrm{RMod}_R^{\mathrm{sm}} \simeq (\mathrm{LMod}_R^{\mathrm{sm}})^{op}$ of Lemma 3.4.3, we obtain equivalences of ∞-categories

$$\mathrm{LMod}_R^! \simeq \mathrm{Ind}(\mathrm{LMod}_R^{\mathrm{sm}}) \qquad \mathrm{RMod}_R^! \simeq \mathrm{Ind}(\mathrm{RMod}_R^{\mathrm{sm}}).$$

Our next goal is to explain the dependence of the ∞-categories $\mathrm{LMod}_R^!$ and $\mathrm{RMod}_R^!$ on the choice of algebra $R \in \mathrm{Alg}_k^{\mathrm{sm}}$. This will require a bit of a digression.

Construction 3.4.6. Let $p : X \to S$ be a map of simplicial sets. We define a new simplicial set $\mathrm{Dl}(p)$ equipped with a map $\mathrm{Dl}(p) \to S$ so that the following universal property is satisfied: for every map of simplicial sets $K \to S$, we have a bijection

$$\mathrm{Hom}_{(\mathcal{S}\mathrm{et}_\Delta)_{/S^{op}}}(K, \mathrm{Dl}(p)) \simeq \mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(K \times_S X, \mathcal{S}).$$

Note that for each vertex $s \in S$, the fiber $\mathrm{Dl}(p)_s = \mathrm{Dl}(p) \times_S \{s\}$ is canonically isomorphic to the presheaf ∞-category $\mathrm{Fun}(X_s, \mathcal{S})$.

Assume that p is an inner fibration. We let $\mathrm{Dl}^0(p)$ the full simplicial subset of $\mathrm{Dl}(p)$ spanned by those vertices which correspond to corepresentable functors $X_s \to \mathcal{S}$, for some $s \in S$. If each of the ∞-categories X_s admits finite limits, we let $\mathrm{Dl}^{\mathrm{lex}}(p)$ denote the full simplicial subset of $\mathrm{Dl}(p)$ spanned by those vertices which correspond to left exact functors $X_s \to \mathcal{S}$, for some vertex $s \in S$,

Remark 3.4.7. Let $p : X \to S$ be an inner fibration and assume that each of the fibers X_s is an ∞-category which admits finite limits. Then for each vertex $s \in S$, we have a canonical isomorphism $\mathrm{Dl}^{\mathrm{lex}}(p)_s \simeq \mathrm{Ind}(X_s^{op})$.

Proposition 3.4.8. *Let $p : X \to S$ be a map of simplicial sets. Then:*

(1) *If p is a Cartesian fibration, then the map $\mathrm{Dl}(p) \to S$ is a coCartesian fibration. Moreover, for every edge $e : s \to s'$ in S, the induced functor*

$$\mathrm{Fun}(X_s, \mathcal{S}) \simeq \mathrm{Dl}(p)_s \to \mathrm{Dl}(p)_{s'} \simeq \mathrm{Fun}(X_{s'}, \mathcal{S})$$

is given by composition with the pullback functor $e^* : X_{s'} \to X_s$ determined by p.

(2) *If p is a coCartesian fibration, then the map $\mathrm{Dl}(p) \to S$ is a Cartesian fibration. Moreover, for every edge $e : s \to s'$ in S, the induced functor*

$$\mathrm{Fun}(X_{s'}, \mathcal{S}) \simeq \mathrm{Dl}(p)_{s'} \to \mathrm{Dl}(p)_s \simeq \mathrm{Fun}(X_s, \mathcal{S})$$

is given by composition with the functor $e_! : X_s \to X_{s'}$ determined by p.

(3) *Suppose p is a Cartesian fibration, that each fiber X_s of p admits finite limits and that for every edge $e : s \to s'$ in S, the pullback functor $e^* : X_{s'} \to X_s$ is left exact. Then the map $\mathrm{Dl}^{\mathrm{lex}}(p) \to S$ is a coCartesian fibration. Moreover, for every edge $e : s \to s'$ in S, the induced functor*

$$\mathrm{Ind}(X_s^{op}) \simeq \mathrm{Dl}^{\mathrm{lex}}(p)_s \to \mathrm{Dl}^{\mathrm{lex}}(p)_{s'} \simeq \mathrm{Ind}(X_{s'}^{op})$$

is given by composition with the pullback functor e^*.

(4) *If p is a coCartesian fibration, then the canonical map $q : \mathrm{Dl}(p) \to S$ is a coCartesian fibration, which restricts to a coCartesian fibration $\mathrm{Dl}^0(p) \to S$. If each fiber X_s admits finite limits, then q also restricts to a coCartesian fibration $\mathrm{Dl}^{\mathrm{lex}}(p) \to S$.*

Proof. Assertion (1) and (3) follow from Corollary T.3.2.2.12. The implication (1) $\Rightarrow$ (2) is immediate. We now prove (4). Assume that p is a coCartesian fibration. Then (2) implies that $\mathrm{Dl}(p) \to S$ is a Cartesian fibration, and that each edge $e : s \to s'$ induces a pullback functor $\mathrm{Dl}(p)_{s'} \to \mathrm{Dl}(p)_s$ which preserves small limits and filteredcolimits Using Corollary T.5.5.2.9, we deduce that this pullback functor admits a left adjoint $\mathrm{Fun}(X_s, \mathcal{S}) \to \mathrm{Fun}(X_{s'}, \mathcal{S})$, which is given by left Kan extension along the functor $e_! : X_s \to X_{s'}$. Corollary T.5.2.2.5 implies that the forgetful functor $q : \mathrm{Dl}(p) \to S$ is also a coCartesian fibration. Since the operation of left Kan extension carries corepresentable functors to corepresentable functors, we conclude that q restricts to a coCartesian fibration $q_0 : \mathrm{Dl}^0(p) \to S$ (and that a morphism in $\mathrm{Dl}^0(p)$ is q_0-coCartesian if and only if it is q-coCartesian). Now suppose that each fiber X_s of p admits finite limits. For each $s \in S$, the ∞-category $\mathrm{Dl}^{\mathrm{lex}}(p)_s = \mathrm{Ind}(X_s^{op})$ can be identified with the full subcategory of $\mathrm{Dl}(p)_s = \mathcal{P}(X_s^{op})$ generated by $\mathrm{Dl}^0(p)_s$ under filtered colimits. If $e : s \to s'$ is an edge of S, then the functor $e_! : \mathrm{Dl}(p)_s \to \mathrm{Dl}(p)_{s'}$ preserves small filtered colimits and carries $\mathrm{Dl}^0(p)_s$ into $\mathrm{Dl}^0(p)_{s'}$, and therefore carries $\mathrm{Dl}^{\mathrm{lex}}(p)_s$ into $\mathrm{Dl}^{\mathrm{lex}}(p)_{s'}$. It follows that q restricts to a coCartesian fibration $\mathrm{Dl}^{\mathrm{lex}}(p) \to S$. $\square$

Remark 3.4.9. Let $p : X \to S$ be a coCartesian fibration of simplicial sets. For each $s \in S$, the fiber $\mathrm{Dl}^0(p)_s$ is isomorphic to the essential image of the Yoneda embedding $X_s^{op} \to \mathrm{Fun}(X_s, \mathcal{S})$, and therefore equivalent to X_s^{op}. In fact, we can be more precise: if p is classified by a map $\chi : S \to \mathcal{C}\mathrm{at}_\infty$, then the coCartesian fibration $\mathrm{Dl}^0(p) \to S$ is classified by the functor $e \circ \chi$, where e is the equivalence of $\mathcal{C}\mathrm{at}_\infty$ with itself which carries each ∞-category to its opposite.

Proposition 3.4.10. *Let $p : X \to S$ be a coCartesian fibration of simplicial sets. Assume that:*

(a) *For each $s \in S$, the fiber X_s is compactly generated.*

(b) *For every morphism edge $e : s \to s'$ in S, the induced functor $X_s \to X_{s'}$ preserves compact objects and small colimits.*

Let X^c denote the full simplicial subset of X spanned by those vertices which are compact objects of X_s for some $s \in S$. Let $p^{op} : X^{op} \to S^{op}$ be the induced map between opposite simplicial sets, and let $p_c^{op} : (X^c)^{op} \to S^{op}$ be the restriction of p^{op}. Then the restriction map $\phi : \mathrm{Dl}^0(p^{op}) \subseteq \mathrm{Dl}^{\mathrm{lex}}(p^{op}) \to \mathrm{Dl}^{\mathrm{lex}}(p_c^{op})$ is an equivalence of coCartesian fibrations over S^{op}.

Proof. It follows from (a) and (b) that each of the functors $e_! : X_s \to X_{s'}$ admits a right adjoint e^*, so that p is a Cartesian fibration and therefore p^{op} is a coCartesian fibration. Applying Proposition 3.4.8, we conclude that the projection map $q : \mathrm{Dl}^0(p^{op}) \to S^{op}$ is a coCartesian fibration. It follows from (b) that the projection $X^c \to S$ is a coCartesian fibration whose fibers admit finite colimits, and that for every edge $e : s \to s'$ in S the induced functor $X_s^c \to X_{s'}^c$ preserves finite colimits. Applying Proposition 3.4.8 again, we deduce that the map $q' : \mathrm{Dl}^{\mathrm{lex}}(p_c^{op}) \to S^{op}$ is a coCartesian fibration. We next claim that ϕ carries q-coCartesian morphisms to q'-coCartesian morphisms. Unwinding the definitions, this amounts to the following claim: if $\overline{e} : x \to x'$ is a p-Cartesian edge lifting $e : s \to s'$, then $\overline{e}$ induces an equivalence $h_{x'} \circ e_! \to h_x$ of functors $X_s^{op} \to \mathcal{S}$, where $h_x : X_s^{op} \to \mathcal{S}$ is the functor represented by x and $h_{x'} : X_{s'}^{op} \to \mathcal{S}$ is the functor represented by x'. This is an immediate consequence of the definitions.

To complete the proof, it will suffice to show that for every vertex $s \in S$, the functor ϕ induces an equivalence of ∞-categories $\mathrm{Dl}^0(p^{op})_s \to \mathrm{Dl}^{\mathrm{lex}}(p_c^{op})$. That is, we must show that the composite functor

$$\psi : X_s \to \mathrm{Fun}(X_s^{op}, \mathcal{S}) \to \mathrm{Fun}((X_s^c)^{op}, \mathcal{S}) = \mathrm{Ind}(X_s^c)$$

is an equivalence of ∞-categories. This is clear, since ψ is right adjoint to the canonical map $\mathrm{Ind}(X_s^c) \to X_s$ (which is an equivalence by virtue of our assumption that X_s is compactly generated). $\square$

Construction 3.4.11. Let k be a field and let $\mathrm{LMod}(\mathrm{Mod}_k)$ and $\mathrm{RMod}(\mathrm{Mod}_k)$ denote the ∞-categories of left and right module objects of the symmetric monoidal ∞-category Mod_k. That is, LMod_k is an ∞-category whose objects are pairs (R, M), where $R \in \mathrm{Alg}_k$ and M is a left R-module, and RMod_k is an ∞-category whose objects are pairs (R, M) where $R \in \mathrm{Alg}_k$ and M is a left R-module. We let $\mathrm{LMod}^{\mathrm{sm}}(\mathrm{Mod}_k)$ denote the full subcategory of $\mathrm{LMod}(\mathrm{Mod}_k)$ spanned by those pairs (R, M) where $R \in \mathrm{Alg}_k^{\mathrm{sm}}$ and $M \in \mathrm{LMod}_R^{\mathrm{sm}}$, and define $\mathrm{RMod}^{\mathrm{sm}}(\mathrm{Mod}_k) \subseteq \mathrm{RMod}(\mathrm{Mod}_k)$ similarly. We have evident forgetful functors

$$\mathrm{LMod}^{\mathrm{sm}}(\mathrm{Mod}_k) \xrightarrow{q} \mathrm{Alg}_k^{\mathrm{sm}} \xleftarrow{q'} \mathrm{RMod}^{\mathrm{sm}}(\mathrm{Mod}_k).$$

We set

$$\mathrm{RMod}^!(\mathrm{Mod}_k) = \mathrm{Dl}^{\mathrm{lex}}(q) \qquad \mathrm{LMod}^!(\mathrm{Mod}_k) = \mathrm{Dl}^{\mathrm{lex}}(q'),$$

so that we have evident forgetful functors

$$\mathrm{RMod}^!(\mathrm{Mod}_k) \to \mathrm{Alg}_k^{\mathrm{sm}} \leftarrow \mathrm{LMod}^!(\mathrm{Mod}_k).$$

Remark 3.4.12. It follows from Proposition 3.4.8 that the forgetful functors

$$\mathrm{LMod}^!(\mathrm{Mod}_k) \to \mathrm{Alg}_k^{\mathrm{sm}} \leftarrow \mathrm{RMod}^!(\mathrm{Mod}_k)$$

are coCartesian fibrations. For every object $R \in \mathrm{Alg}_k^{\mathrm{sm}}$, we can identify the fiber $\mathrm{RMod}^!(\mathrm{Mod}_k) \times_{\mathrm{Alg}_k^{\mathrm{sm}}} \{R\}$ with the ∞-category $\mathrm{RMod}_R^!$ of Definition 3.4.4, and $\mathrm{LMod}^!(\mathrm{Mod}_k) \times_{\mathrm{Alg}_k^{\mathrm{sm}}} \{R\}$ with the ∞-category $\mathrm{LMod}_R^!$ of Definition 3.4.4. If $f : R \to R'$ is a morphism in $\mathrm{Alg}_k^{\mathrm{sm}}$, then f induces functors

$$\mathrm{RMod}_R^! \to \mathrm{RMod}_{R'}^! \qquad \mathrm{LMod}_R^! \to \mathrm{LMod}_{R'}^!,$$

both of which we will denote by $f^!$.

We next explain how to regard the ∞-categories $\mathrm{RMod}_R^!$ and $\mathrm{LMod}_R^!$ as enlargements of the ∞-categories LMod_R and RMod_R, respectively. Let $\mathrm{LMod}_R^{\mathrm{perf}}$ and $\mathrm{RMod}_R^{\mathrm{perf}}$ denote the full subcategories of LMod_R and RMod_R spanned by the perfect R-modules. If R is small, we have evident inclusions

$$\mathrm{RMod}_R^{\mathrm{perf}} \subseteq \mathrm{RMod}_R^{\mathrm{sm}} \qquad \mathrm{LMod}_R^{\mathrm{perf}} \subseteq \mathrm{LMod}_R^{\mathrm{sm}}.$$

The first inclusion induces a fully faithful embedding

$$\Phi_R : \mathrm{RMod}_R \simeq \mathrm{Ind}(\mathrm{RMod}_R^{\mathrm{perf}}) \hookrightarrow \mathrm{Ind}(\mathrm{RMod}_R^{\mathrm{sm}}) \simeq \mathrm{RMod}_R^!.$$

However, the functors Φ_R are badly behaved in some respects. For example, if $f : R \to R'$ is a morphism in $\mathrm{Alg}_k^{\mathrm{sm}}$, then the diagram of ∞-categories

$$\begin{tikzcd} \mathrm{RMod}_R \arrow[r, "f^*"] \arrow[d, "\Phi_R"] & \mathrm{RMod}_{R'} \arrow[d, "\Phi_{R'}"] \\ \mathrm{RMod}_R^! \arrow[r, "f^!"] & \mathrm{RMod}_{R'}^! \end{tikzcd}$$

generally does not commute up to homotopy (here f^* denotes the base change functor $M \mapsto M \otimes_R R'$). In what follows, we will instead consider the fully faithful embeddings $\Psi_R : \mathrm{RMod}_R \to \mathrm{RMod}_R^!$ given by

$$\mathrm{RMod}_R \simeq \mathrm{Ind}(\mathrm{RMod}_R^{\mathrm{perf}}) \simeq \mathrm{Ind}((\mathrm{LMod}_R^{\mathrm{perf}})^{op}) \hookrightarrow \mathrm{Ind}((\mathrm{LMod}_R^{\mathrm{sm}})^{op}) \simeq \mathrm{Ind}(\mathrm{RMod}_R^{\mathrm{sm}}) \simeq \mathrm{RMod}_R^!.$$

Our next goal is to give a description of this functor which is manifestly compatible with base change in the algebra $R \in \mathrm{Alg}_k^{\mathrm{sm}}$.

Construction 3.4.13. Let k be a field, and let $\lambda : \mathrm{RMod}(\mathrm{Mod}_k) \times_{\mathrm{Alg}_k} \mathrm{LMod}^{\mathrm{sm}}(\mathrm{Mod}_k) \to \mathcal{S}$ be the functor given by

$$\lambda(M, R, N) = \mathrm{Map}_{\mathrm{Mod}_k}(k, M \otimes_R N).$$

If we fix M and R, then the functor $N \mapsto \mathrm{Map}_{\mathrm{Mod}_k}(k, M \otimes_R N)$ is left exact. It follows that λ is determines a functor $\Psi : \mathrm{RMod}(\mathrm{Mod}_k) \times_{\mathrm{Alg}_k} \mathrm{Alg}_k^{\mathrm{sm}} \to \mathrm{RMod}^!(\mathrm{Mod}_k)$.

Proposition 3.4.14. *Let k be a field, and consider the diagram*

$$\begin{tikzcd} \mathrm{RMod}(\mathrm{Mod}_k) \times_{\mathrm{Alg}_k} \mathrm{Alg}_k^{\mathrm{sm}} \arrow[rr, "\Psi"] \arrow[dr, "p"'] & & \mathrm{RMod}^!(\mathrm{Mod}_k) \arrow[dl, "q"] \\ & \mathrm{Alg}_k^{\mathrm{sm}} & \end{tikzcd}$$

where p and q are the forgetful functors and Ψ is defined as in Construction 3.4.13. Then:

(1) *The functor Ψ carries p-coCartesian morphisms to q-coCartesian morphisms.*

(2) *For every object $R \in \mathrm{Alg}_k^{\mathrm{sm}}$, the induced functor $\Psi_R : \mathrm{RMod}_R \to \mathrm{RMod}_R^!$ preserves small colimits.*

(3) *The functor Ψ is fully faithful.*

Proof. We first prove (1). Let $\alpha : (M, R) \to (M', R')$ be a p-coCartesian morphism in $\mathrm{RMod}(\mathrm{Mod}_k)$. We wish to prove that $\Psi(\alpha)$ is q-coCartesian. Unwinding the definitions, we must show that for every small R'-module N, the canonical map

$$\mathrm{Map}_{\mathrm{Mod}_k}(k, M \otimes_R N) \to \mathrm{Map}_{\mathrm{Mod}_k}(k, M' \otimes_{R'} N)$$

is an equivalence. This is clear, since the map $M \otimes_R N \to M' \otimes_{R'} N$ is an equivalence.

We now prove (2). Fix an object $R \in \mathrm{Alg}_k^{\mathrm{sm}}$. For every $N \in \mathrm{LMod}_R^{\mathrm{sm}}$, the functor $M \mapsto \mathrm{Map}_{\mathrm{Mod}_k}(k, M \otimes_R N)$ commutes with filtered colimits and finite limits. It follows that Ψ_R commutes with filtered colimits and finite limits. Since Ψ_R is a left exact functor between stable ∞-categories, it is also right exact. We conclude that Ψ_R commutes with filtered colimits and finite colimits, and therefore with all small colimits.

We now prove (3). By virtue of (1), it will suffice to prove that for $R \in \mathrm{Alg}_k^{\mathrm{sm}}$ the functor $\Psi_R : \mathrm{RMod}_R \to \mathrm{RMod}_R^!$ is fully faithful. Using (2) and Proposition T.5.3.5.11, we are reduced to proving that the restriction $\Psi_R | \mathrm{RMod}_R^{\mathrm{perf}}$ is fully faithful. Note that if M be a perfect right R-module and $M^\vee$ its R-linear dual (regarded as a perfect left R-module), then $\Psi_R(M)$ is the functor corepresented by $M^\vee \in \mathrm{LMod}_R^{\mathrm{perf}} \subseteq \mathrm{LMod}_R^{\mathrm{sm}}$. We can therefore identify $\Psi_R | \mathrm{RMod}_R^{\mathrm{perf}}$ with the composition of fully faithful embeddings

$$\mathrm{RMod}_R^{\mathrm{perf}} \simeq (\mathrm{LMod}_R^{\mathrm{perf}})^{op} \subseteq (\mathrm{LMod}_R^{\mathrm{sm}})^{op} \xhookrightarrow{j} \mathrm{RMod}_R^!,$$

(here j denotes the Yoneda embedding). □

Remark 3.4.15. Construction 3.4.13 and Proposition 3.4.14 have evident dual versions, which give a fully faithful embedding $\mathrm{LMod}(\mathrm{Mod}_k) \times_{\mathrm{Alg}_k} \mathrm{Alg}_k^{\mathrm{sm}} \to \mathrm{LMod}^!(\mathrm{Mod}_k)$.

Our next goal is to say something about the essential image of the functor $\Psi_R : \mathrm{RMod}_R \to \mathrm{RMod}_R^!$ of Proposition 3.4.14.

Definition 3.4.16. Let R be a small $\mathbb{E}_1$-algebra over a field k, and let $\epsilon : R \to k$ be the augmentation. We will say that an object $M \in \mathrm{RMod}_R^!$ is *connective* if $\epsilon^! M$ is a connective object of $\mathrm{Mod}_k \simeq \mathrm{Mod}_k^!$. We let $\mathrm{RMod}_R^{!,\mathrm{cn}}$ denote the full subcategory of $\mathrm{RMod}_R^!$ spanned by the connective objects. Similarly, we define a full subcategory $\mathrm{LMod}_R^{!,\mathrm{cn}} \subseteq \mathrm{LMod}_R^!$.

Remark 3.4.17. Let R be a small $\mathbb{E}_1$-algebra over a field k. It follows from Proposition T.5.4.6.6 that $\mathrm{RMod}_R^{!,\mathrm{cn}}$ is an accessible subcategory of $\mathrm{RMod}_R^!$, which is evidently closed under small colimits and extensions. Applying Proposition A.1.4.5.11, we conclude that there exists a t-structure on the stable ∞-category $\mathrm{RMod}_R^!$ with $(\mathrm{RMod}_R^!)_{\geq 0} = \mathrm{RMod}_R^{!,\mathrm{cn}}$.

Proposition 3.4.18. *Let k be a field, let $R \in \mathrm{Alg}_k^{\mathrm{sm}}$. Then the fully faithful embedding $\Psi_R : \mathrm{RMod}_R \to \mathrm{RMod}_R^!$ of Proposition 3.4.14 restricts to an equivalence of ∞-categories $\mathrm{RMod}_R^{\mathrm{cn}} \hookrightarrow \mathrm{RMod}_R^{!,\mathrm{cn}}$.*

Proof. Let $\epsilon : R \to k$ be the augmentation map and let $M \in \mathrm{RMod}_R^!$. We wish to show that $\epsilon^! M$ is connective if and only if $M \simeq \Psi_R(M')$ for some $M' \in \mathrm{RMod}_R^{\mathrm{cn}}$. The "if" direction is clear: if $M \simeq \Psi_R(M')$, we have equivalences

$$\epsilon^! M \simeq \epsilon^! \Psi_R(M') \simeq \Psi_k(\epsilon^* M') \simeq k \otimes_R M'.$$

For the converse, assume that $\epsilon^! M$ is connective. Let $\mathcal{C} \subseteq \mathrm{RMod}_R^!$ denote the essential image of $\Psi_R | \mathrm{RMod}_R^{\mathrm{cn}}$; we wish to prove that $M \in \mathcal{C}$. It follows from Proposition 3.4.14 that $\mathcal{C}$ is closed under colimits and extensions in $\mathrm{RMod}_R^!$.

We begin by constructing a sequence of objects

$$0 = M(0) \to M(1) \to M(2) \to \cdots$$

in $\mathcal{C}$ and a compatible family of maps $\theta(i) : M(i) \to M$ with the following property:

($*$) The groups $\pi_j \epsilon^! M(i)$ vanish unless $0 \leq j < i$, and the maps $\pi_j \epsilon^! M(j) \to \pi_j \epsilon^! M$ are isomorphisms for $0 \leq j < i$.

Assume that $i \geq 0$ and that we have already constructed a map $\theta(i)$ satisfying ($*$). Let $M' = \mathrm{cofib}(\theta(i))$, so that $\pi_j \epsilon^! M' \simeq 0$ for $j < i$. Let us regard M' as a functor $\mathrm{LMod}^{\mathrm{sm}}_R \to \mathcal{S}$, so that $M'(k[m])$ is $(m+i)$-connective for every integer i. It follows by induction that for every m-connective object $N \in \mathrm{LMod}^{\mathrm{sm}}_R$, the space $M'(N)$ is $(m+i)$-connective. In particular, $M'(N)$ is connected when N denotes the cofiber of the map $R[-i] \to k[-i]$. Using the fiber sequence

$$M'(R[-i]) \to M'(k[-i]) \to M'(\mathrm{cofib}(R[-i] \to k[-i])),$$

we deduce that the map $\pi_0 M'(R[-i]) \to \pi_0 M'(k[-i])$ is surjective. Let $K = \Psi_R(R) \in \mathrm{RMod}^!_R$. Then $K[i] = \Psi_R(R[i])$ can be identifed with the functor corepresented by $R[-i]$. We have proven the following:

($*'$) For every point $\eta \in \pi_i \epsilon^! M' \simeq \pi_0 M'(k'[-i])$, there exists a map $K[i] \to M$ such that η belongs to the image of the induced map $k \simeq \pi_i \epsilon^! K[i] \to \pi_i \epsilon^! M'$.

Choose a basis $\{v_\alpha\}_{\alpha \in S}$ for the k-vector space $\pi_i \epsilon^! M' \simeq \pi_i \epsilon^! M$. Applying ($*'$) repeatedly, we obtain a map $v : \bigoplus_{\alpha \in S} K[i] \to M'$. Let $M'' = \mathrm{cofib}(v)$ and let $M(i+1)$ denote the fiber of the composite map $M \to M' \to M''$. We have a fiber sequence

$$M(i) \to M(i+1) \to \bigoplus_{\alpha \in S} K[i].$$

Since $\mathcal{C}$ is closed under colimits and extensions (and contains $K[i] \simeq \Psi_R R[i]$), we conclude that $M(i+1) \in \mathcal{C}$. Using the long exact sequence of homotopy groups

$$\pi_j \epsilon^! M(i) \to \pi_j \epsilon^! M(i+1) \to \pi_j \epsilon^! \bigoplus_{\alpha \in S} K[i] \to \pi_{j-1} \epsilon^! M(i),$$

we deduce that the canonical map $M(i+1) \to M$ satisfies condition ($*$).

Let $M(\infty) = \varinjlim M(i)$. Since $\mathcal{C}$ is closed under colimits, we deduce that $M(\infty) \in \mathcal{C}$. Using ($*$) (and the vanishing of the groups $\pi_j \epsilon^! M$ for $j < 0$), we deduce that w induces an equivalence $e^* M(\infty) \to e^* M$. Identifying M and $M(\infty)$ with left exact functors $\mathrm{LMod}^{\mathrm{sm}}_R \to \mathcal{S}$, we conclude that w induces a homotopy equivalence $M(\infty)(k[j]) \to M(k[j])$ for every integer j. Since M and $M(\infty)$ are left exact, the collection of those objects $N \in \mathrm{LMod}^{\mathrm{sm}}_R$ for which $M(\infty)(N) \to M(N)$ is a homotopy equivalence is closed under finite limits. Using Lemma 3.4.2, we deduce that every object $N \in \mathrm{LMod}^{\mathrm{sm}}_R$ has this property, so that w is an equivalence and $M \simeq M(\infty) \in \mathcal{C}$ as desired. $\square$

Remark 3.4.19. Let R be a small $\mathbb{E}_1$-algebra over a field k. The natural t-structure on the ∞-category RMod_R is right complete. It follows from Proposition 3.4.18 that the fully faithful embedding $\Psi_R : \mathrm{RMod}_R \to \mathrm{RMod}^!_R$ induces an equivalence from RMod_R to the right completion of $\mathrm{RMod}^!_R$.

Construction 3.4.20. Let k be a field. The coCartesian fibrations

$$\mathrm{RMod}(\mathrm{Mod}_k) \times_{\mathrm{Alg}_k} \mathrm{Alg}^{\mathrm{sm}}_k \xrightarrow{p_R} \mathrm{Alg}^{\mathrm{sm}}_k \xleftarrow{q_R} \mathrm{RMod}^!(\mathrm{Mod}_k)$$

are classified by functors $\chi, \chi_! : \mathrm{Alg}^{\mathrm{sm}}_k \to \widehat{\mathcal{C}\mathrm{at}}_\infty$. Since $\widehat{\mathcal{C}\mathrm{at}}_\infty$ admits small limits, Theorem T.5.1.5.6 implies that χ and $\chi_!$ admit (essentially unique) factorizations as compositions

$$\mathrm{Alg}^{\mathrm{sm}}_k \xrightarrow{j} \mathrm{Fun}(\mathrm{Alg}^{\mathrm{sm}}_k, \mathcal{S})^{op} \xrightarrow{\mathrm{QCoh}_R} \widehat{\mathcal{C}\mathrm{at}}_\infty$$

$$\mathrm{Alg}^{\mathrm{sm}}_k \xrightarrow{j} \mathrm{Fun}(\mathrm{Alg}^{\mathrm{sm}}_k, \mathcal{S})^{op} \xrightarrow{\mathrm{QCoh}^!_R} \widehat{\mathcal{C}\mathrm{at}}_\infty$$

where j denotes the Yoneda embedding and the functors QCoh_R and $\mathrm{QCoh}^!_R$ preserve small limits. Similarly, the coCartesian fibrations

$$\mathrm{LMod}(\mathrm{Mod}_k) \times_{\mathrm{Alg}_k} \mathrm{Alg}^{\mathrm{sm}}_k \to \mathrm{Alg}^{\mathrm{sm}}_k \to \mathrm{LMod}^!(\mathrm{Mod}_k)$$

are classified by maps $\chi', \chi'_! : \mathrm{Alg}^{\mathrm{sm}}_k \to \widehat{\mathcal{C}\mathrm{at}}_\infty$, which admit factorizations

$$\mathrm{Alg}^{\mathrm{sm}}_k \xrightarrow{j} \mathrm{Fun}(\mathrm{Alg}^{\mathrm{sm}}_k, \mathcal{S})^{op} \xrightarrow{\mathrm{QCoh}_L} \widehat{\mathcal{C}\mathrm{at}}_\infty$$

$$\mathrm{Alg}^{\mathrm{sm}}_k \xrightarrow{j} \mathrm{Fun}(\mathrm{Alg}^{\mathrm{sm}}_k, \mathcal{S})^{op} \xrightarrow{\mathrm{QCoh}^!_L} \widehat{\mathcal{C}\mathrm{at}}_\infty.$$

For each functor $X : \mathrm{Alg}^{\mathrm{sm}}_k \to \mathcal{S}$, we will refer to $\mathrm{QCoh}_L(X)$ and $\mathrm{QCoh}_R(X)$ as the ∞-categories of (left and right) *quasi-coherent sheaves on* X. Similarly, we will refer to $\mathrm{QCoh}^!_L(X)$ and $\mathrm{QCoh}^!_R(X)$ as the ∞-categories of (left and right) Ind-*coherent sheaves on* X.

Remark 3.4.21. For every functor $X : \mathrm{Alg}^{\mathrm{sm}}_k \to \mathcal{S}$, the ∞-categories $\mathrm{QCoh}_L(X)$, $\mathrm{QCoh}_R(X)$, $\mathrm{QCoh}^!_L(X)$, and $\mathrm{QCoh}^!_R(X)$ are presentable and stable.

Remark 3.4.22. Let k be a field, let $X : \mathrm{Alg}^{\mathrm{sm}}_k \to \mathcal{S}$ be a functor which classifies a right fibration $\mathcal{X} \to \mathrm{Alg}^{\mathrm{sm}}_k$. Then $\mathrm{QCoh}_R(X)$ and $\mathrm{QCoh}^!_R(X)$ can be identified with the ∞-categories of coCartesian sections of the coCartesian fibrations

$$\mathcal{X} \times_{\mathrm{Alg}_k} \mathrm{RMod}(\mathrm{Mod}_k) \to \mathcal{X} \leftarrow \mathcal{X} \times_{\mathrm{Alg}^{\mathrm{sm}}_k} \mathrm{RMod}^!(\mathrm{Mod}_k).$$

More informally, an object $\mathcal{F} \in \mathrm{QCoh}_R(X)$ is a rule which assigns to every point $\eta \in X(A)$ a right A-module $\mathcal{F}_\eta$, and to every morphism $f : A \to A'$ carrying η to $\eta' \in X(A')$ an equivalence $\mathcal{F}_{\eta'} \simeq \mathcal{F}_\eta \otimes_A A'$. Similarly, an object of $\mathcal{G} \in \mathrm{QCoh}^!_R(X)$ is a rule which assigns to every point $\eta \in X(A)$ an Ind-coherent right R-module $\mathcal{G}_\eta \in \mathrm{RMod}^!_A$, and to every morphism $f : A \to A'$ carrying η to $\eta' \in X(R')$ an equivalence $\mathcal{G}_{\eta'} \simeq f^! \, \mathcal{G}_\eta$. The ∞-categories $\mathrm{QCoh}_L(X)$ and $\mathrm{QCoh}^!_L(X)$ admit similar descriptions, using left modules in place of right modules.

Notation 3.4.23. By construction, the ∞-categories $\mathrm{QCoh}_R(X)$, $\mathrm{QCoh}_L(X)$, $\mathrm{QCoh}^!_R(X)$, and $\mathrm{QCoh}^!_L(X)$ depend contravariantly on the object $X \in \mathrm{Fun}(\mathrm{Alg}^{\mathrm{sm}}_k, \mathcal{S})$. If $\alpha : X \to Y$ is a natural transformation, we will denote the resulting functors by

$$\alpha^* : \mathrm{QCoh}_R(Y) \to \mathrm{QCoh}_R(X) \qquad \alpha^! : \mathrm{QCoh}^!_R(Y) \to \mathrm{QCoh}^!_R(X)$$

$$\alpha^* : \mathrm{QCoh}_L(Y) \to \mathrm{QCoh}_L(X) \qquad \alpha^! : \mathrm{QCoh}^!_L(Y) \to \mathrm{QCoh}^!_L(X).$$

Remark 3.4.24. Let k be a field. The fully faithful embedding Ψ of Proposition 3.4.14 induces a natural transformation $\mathrm{QCoh}_R \to \mathrm{QCoh}^!_R$ of functors $\mathrm{Fun}(\mathrm{Alg}^{\mathrm{sm}}_k, \mathcal{S}) \to \widehat{\mathcal{C}\mathrm{at}}_\infty$. For every functor $X : \mathrm{Alg}^{\mathrm{sm}}_k \to \mathcal{S}$, we obtain a fully faithful embedding $\mathrm{QCoh}_R(X) \to \mathrm{QCoh}^!_R(X)$ which preserves small colimits. Moreover, if $\alpha : X \to Y$ is a natural transformation of functors, we obtain a diagram of ∞-categories

$$\begin{tikzcd}
\mathrm{QCoh}_R(Y) \arrow[r] \arrow[d, "\alpha^*"] & \mathrm{QCoh}^!_R(Y) \arrow[d, "\alpha^!"] \\
\mathrm{QCoh}_R(X) \arrow[r] & \mathrm{QCoh}^!_R(X)
\end{tikzcd}$$

which commutes up to canonical homotopy. Similarly, we have fully faithful embedding $\mathrm{QCoh}_L(X) \to \mathrm{QCoh}^!_L(X)$, which depend functorially on X in the same sense.

Notation 3.4.25. Let k be a field and let $X : \mathrm{Alg}_k^{\mathrm{sm}} \to \mathcal{S}$ be a functor. We will say that an object $\mathcal{F} \in \mathrm{QCoh}_R(X)$ is *connective* if $\mathcal{F}_\eta \in \mathrm{RMod}_A$ is connective for every point $\eta \in X(A)$ (see Notation 3.4.22). We let $\mathrm{QCoh}_R(X)^{\mathrm{cn}}$ denote the full subcategory of $\mathrm{QCoh}_R(X)$ spanned by the connective right quasi-coherent sheaves, and define a full subcategory $\mathrm{QCoh}_L(X)^{\mathrm{cn}} \subseteq \mathrm{QCoh}_L(X)$ similarly.

We will say that an object $\mathcal{G} \in \mathrm{QCoh}_R^!$ is *connective* if, for every point $\eta \in X(k)$, the object $\mathcal{G}_\eta \in \mathrm{RMod}_k^! \simeq \mathrm{Mod}_k$ is connective. We let $\mathrm{QCoh}_R^!(X)^{\mathrm{cn}}$ denote the full subcategory of $\mathrm{QCoh}_R^!(X)$ spanned by the connective objects, and define $\mathrm{QCoh}_L^!(X)^{\mathrm{cn}} \subseteq \mathrm{QCoh}_L^!(X)$ similarly.

Remark 3.4.26. Let $X : \mathrm{Alg}_k^{\mathrm{sm}} \to \mathcal{S}$ be a functor. The full subcategories

$$\mathrm{QCoh}_L(X)^{\mathrm{cn}} \subseteq \mathrm{QCoh}_L(X) \qquad \mathrm{QCoh}_R(X)^{\mathrm{cn}} \subseteq \mathrm{QCoh}_R(X)$$

$$\mathrm{QCoh}_L^!(X)^{\mathrm{cn}} \subseteq \mathrm{QCoh}_L^!(X) \qquad \mathrm{QCoh}_R^!(X)^{\mathrm{cn}} \subseteq \mathrm{QCoh}_R^!(X)$$

of Notation 3.4.25 are presentable and closed under extensions and small colimits. It follows from Proposition A.1.4.5.11 that they determine t-structures on $\mathrm{QCoh}_L(X)$, $\mathrm{QCoh}_R(X)$, $\mathrm{QCoh}_L^!(X)$, and $\mathrm{QCoh}_R^!(X)$.

Using Proposition 3.4.18, we immediately deduce the following:

Proposition 3.4.27. *Let k be a field and let $X : \mathrm{Alg}_k^{\mathrm{sm}} \to \mathcal{S}$ be a functor. Then the fully faithful embeddings*

$$\mathrm{QCoh}_L(X) \hookrightarrow \mathrm{QCoh}_L^!(X) \qquad \mathrm{QCoh}_R(X) \hookrightarrow \mathrm{QCoh}_R^!(X)$$

of Remark 3.4.24 induce equivalences of ∞-categories

$$\mathrm{QCoh}_L(X)^{\mathrm{cn}} \simeq \mathrm{QCoh}_L^!(X)^{\mathrm{cn}} \qquad \mathrm{QCoh}_R(X)^{\mathrm{cn}} \simeq \mathrm{QCoh}_R^!(X)^{\mathrm{cn}}.$$

3.5 Koszul Duality for Modules

Our goal in this section is to prove the following non-commutative analogue of Theorem 2.4.1:

Theorem 3.5.1. *Let k be a field, let $A \in \mathrm{Alg}_k^{\mathrm{aug}}$, and let $X : \mathrm{Alg}_k^{\mathrm{sm}} \to \mathcal{S}$ be the formal $\mathbb{E}_1$ moduli problem associated to A (see Theorem 3.0.4). Then there are canonical equivalences of ∞-categories*

$$\mathrm{QCoh}_L^!(X) \simeq \mathrm{RMod}_A \qquad \mathrm{QCoh}_R^!(X) \simeq \mathrm{LMod}_A\,.$$

In particular, we have fully faithful embeddings

$$\mathrm{QCoh}_L(X) \hookrightarrow \mathrm{RMod}_A \qquad \mathrm{QCoh}_R(X) \hookrightarrow \mathrm{LMod}_A\,.$$

The main ingredient in the proof of Theorem 3.5.1 is the following result.

Proposition 3.5.2. *Let k be a field, and let $\chi_! : \mathrm{Alg}_k^{\mathrm{sm}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ be as in Construction 3.4.20 (given on objects by $\chi_!(R) = \mathrm{RMod}_R^!$), and let $\chi' : \mathrm{Alg}_k^{op}$ be the functor which classifies the Cartesian fibration $\mathrm{LMod}(\mathrm{Mod}_k) \to \mathrm{Alg}_k$ (given by the formula $\chi'(A) = \mathrm{LMod}_A$). Then $\chi_!$ is homotopic to the composition*

$$\mathrm{Alg}_k^{\mathrm{sm}} \to \mathrm{Alg}_k^{\mathrm{aug}} \xrightarrow{\mathfrak{D}^{(1)}} (\mathrm{Alg}_k^{\mathrm{aug}})^{op} \to \mathrm{Alg}_k^{op} \xrightarrow{\chi'} \widehat{\mathcal{C}\mathrm{at}}_\infty.$$

In particular, for every $R \in \mathrm{Alg}_k^{\mathrm{sm}}$, there is a canonical equivalence of ∞-categories $\mathrm{RMod}_R^! \simeq \mathrm{LMod}_{\mathfrak{D}^{(1)}(R)}$.

Before giving the proof of Proposition 3.5.2, let us explain how it leads to a proof of Theorem 3.5.1.

Proof of Theorem 3.5.1. We will construct the equivalence $\mathrm{QCoh}_R^!(X) \simeq \mathrm{LMod}_A$; the construction of the equivalence $\mathrm{QCoh}_L^!(X) \simeq \mathrm{RMod}_A$ is similar. Let k be a field and let $\mathrm{QCoh}_R^! : \mathrm{Fun}(\mathrm{Alg}_k^{\mathrm{sm}}, \mathcal{S})^{op} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ be as in Construction 3.4.20. Let $\Psi : \mathrm{Alg}_k^{\mathrm{aug}} \to \mathrm{Moduli}_k^{(1)}$ be the equivalence of ∞-categories provided by Theorem 3.0.4, and let Ψ^{-1} denote a homotopy inverse to Ψ. Let $L : \mathrm{Fun}(\mathrm{Alg}_k^{\mathrm{sm}}, \mathcal{S}) \to \mathrm{Moduli}_k^{(1)}$ denote a left adjoint to the inclusion functor $\mathrm{Moduli}_k^{(1)} \subseteq \mathrm{Fun}(\mathrm{Alg}_k^{\mathrm{sm}}, \mathcal{S})$ (see Remark 1.1.17), and let $\widehat{\mathfrak{D}}^{(1)} : \mathrm{Fun}(\mathrm{Alg}_k^{\mathrm{sm}}, \mathcal{S}) \to \mathrm{Alg}_k^{\mathrm{aug}}$ be the composition of $\Psi^{-1} \circ L$. The functor $\widehat{\mathfrak{D}}^{(1)}$ preserves small colimits, and the composition of $\widehat{\mathfrak{D}}^{(1)}$ with the Yoneda embedding $(\mathrm{Alg}_k^{\mathrm{sm}})^{op} \to \mathrm{Fun}(\mathrm{Alg}_k^{\mathrm{sm}}, \mathcal{S})$ can be identified with the Koszul duality functor $\mathfrak{D}^{(1)} : (\mathrm{Alg}_k^{\mathrm{sm}})^{op} \to \mathrm{Alg}_k^{\mathrm{aug}}$. Let $\chi' : \mathrm{Alg}_k^{op} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ be as in Proposition 3.5.2 (given on objects by $\chi'(A) = \mathrm{LMod}_A$), and let $F : \mathrm{Fun}(\mathrm{Alg}_k^{\mathrm{sm}}, \mathcal{S})^{op} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ denote the composite functor

$$\mathrm{Fun}(\mathrm{Alg}_k^{\mathrm{sm}}, \mathcal{S})^{op} \xrightarrow{\widehat{\mathfrak{D}}^{(1)}} (\mathrm{Alg}_k^{\mathrm{aug}})^{op} \to \mathrm{Alg}_k^{op} \xrightarrow{\chi'} \widehat{\mathcal{C}\mathrm{at}}_\infty.$$

Let $\mathcal{C}$ denote the full subcategory of $\mathrm{Fun}(\mathrm{Alg}_k^{\mathrm{sm}}, \mathcal{S})$ spanned by the corepresentable functors. Proposition 3.5.2 implies that there is an equivalence of functors $\alpha_0 : F|\,\mathcal{C}^{op} \to \mathrm{QCoh}^!_R\,|\,\mathcal{C}^{op}$. Since $\mathrm{QCoh}^!_R$ is a right Kan extension of its restriction to $\mathcal{C}^{op}$, the equivalence α_0 extends to a natural transformation $F \to \mathrm{QCoh}^!_R$. We will prove:

(∗) If $X : \mathrm{Alg}_k^{\mathrm{sm}} \to \mathcal{S}$ is a formal $\mathbb{E}_1$ moduli problem, then α induces an equivalence of ∞-categories $F(X) \to \mathrm{QCoh}^!_R(X)$.

Taking $X = \Psi(A)$ for $A \in \mathrm{Alg}_k^{\mathrm{aug}}$, we see that (∗) guarantees an equivalence of ∞-categories

$$\beta_A : \mathrm{LMod}_A \simeq \chi'(\Psi^{-1}\Psi(A)) \simeq F(\Psi(A)) \to \mathrm{QCoh}^!_R(X).$$

It remains to prove (∗). Let $\mathcal{E} \subseteq \mathrm{Fun}(\mathrm{Alg}_k^{\mathrm{sm}}, \mathcal{S})$ be the full subcategory spanned by those functors X for which α induces an equivalence of ∞-categories $F(X) \to \mathrm{QCoh}^!_R(X)$. The localization functor $L : \mathrm{Fun}(\mathrm{Alg}_k^{\mathrm{sm}}, \mathcal{S}) \to \mathrm{Moduli}_k^{(1)}$, the equivalence $\Psi^{-1} : \mathrm{Moduli}_k^{(1)} \to \mathrm{Alg}_k^{\mathrm{aug}}$, and the forgetful functor $\mathrm{Alg}_k^{\mathrm{aug}} \to \mathrm{Alg}_k$ preserve small colimits. Lemma 2.4.32 implies that the functor $\chi' : \mathrm{Alg}_k^{op} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ preserves sifted limits. It follows that the functor F preserves sifted limits. Since $\mathrm{QCoh}^!_R$ preserves small limits, the ∞-category $\mathcal{E}$ is closed under sifted colimits in $\mathrm{Fun}(\mathrm{Alg}_k^{\mathrm{sm}}, \mathcal{S})$. Since $\mathcal{E}$ contains all corepresentable functors and is closed under filtered colimits, it contains it contains all prorepresentable formal moduli problems (see Definition 1.5.3). Proposition 1.5.8 implies that every formal $\mathbb{E}_1$ moduli problem X can be obtained as the geometric realization of a simplicial object $X_\bullet$ of $\mathrm{Fun}(\mathrm{Alg}_k^{\mathrm{sm}}, \mathcal{S})$, where each X_n is prorepresentable. Since $\mathcal{E}$ is closed under geometric realizations in $\mathrm{Fun}(\mathrm{Alg}_k^{\mathrm{sm}}, \mathcal{S})$, we conclude that $X \in \mathcal{E}$ as desired. □

We now turn to the proof of Proposition 3.5.2. Consider first the functor $\chi' : \mathrm{Alg}_k^{op} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ classifying the Cartesian fibration $p : \mathrm{LMod}(\mathrm{Mod}_k) \to \mathrm{Alg}_k$. Using Remark 3.4.9, we see that χ' also classifies the coCartesian fibration $\mathrm{Dl}^0(p^{op}) \to \mathrm{Alg}_k^{op}$. Let $\mathrm{LMod}^{\mathrm{perf}}(\mathrm{Mod}_k)$ denote the full subcategory of $\mathrm{LMod}(\mathrm{Mod}_k)$ spanned by those pairs (A, M), where $A \in \mathrm{Alg}_k$ and M is a perfect left module over A. Let p_{perf} denote the restriction of p to $\mathrm{LMod}(\mathrm{Mod}_k)$. Proposition 3.4.10 supplies an equivalence of $\mathrm{Dl}^0(p^{op}) \simeq \mathrm{Dl}^{\mathrm{lex}}(p^{op}_{\mathrm{perf}})$ of coCartesian fibrations over Alg_k^{op}. By construction, $\chi_! : \mathrm{Alg}_k^{\mathrm{sm}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ classifies the coCartesian fibration $\mathrm{RMod}^!(\mathrm{Mod}_k) = \mathrm{Dl}^{\mathrm{lex}}(q) \to \mathrm{Alg}_k^{\mathrm{sm}}$, where q denotes the Cartesian fibration $\mathrm{LMod}^{\mathrm{sm}}(\mathrm{Mod}_k) \to \mathrm{Alg}_k^{\mathrm{sm}}$. Consequently, Proposition 3.5.2 is a consequence of the following:

Proposition 3.5.3. *Let k be a field and let $\mathfrak{D} : \mathrm{Alg}_k^{\mathrm{sm}} \to \mathrm{Alg}_k^{op}$ denote the composition*

$$\mathrm{Alg}_k^{\mathrm{sm}} \hookrightarrow \mathrm{Alg}_k^{\mathrm{aug}} \xrightarrow{\mathfrak{D}^{(1)}} (\mathrm{Alg}_k^{\mathrm{aug}})^{op} \to \mathrm{Alg}_k^{op},$$

where $\mathfrak{D}^{(1)}$ denotes the Koszul duality functor of Definition 3.1.6. Then there is a pullback diagram of ∞-categories

$$\begin{tikzcd} \mathrm{LMod}^{\mathrm{sm}}(\mathrm{Mod}_k) \arrow[r, "\mathfrak{D}'"] \arrow[d] & \mathrm{LMod}^{\mathrm{perf}}(\mathrm{Mod}_k)^{op} \arrow[d] \\ \mathrm{Alg}_k^{\mathrm{sm}} \arrow[r, "\mathfrak{D}"] & \mathrm{Alg}_k^{op}. \end{tikzcd}$$

We now proceed to construct the diagram appearing in the statement of Proposition 3.5.3.

Construction 3.5.4. Fix a field k. We let $\mathcal{M}^{(1)} \to \mathrm{Alg}_k^{\mathrm{aug}} \times \mathrm{Alg}_k^{\mathrm{aug}}$ be the pairing of ∞-categories defined in Construction 3.1.4. The objects of $\mathcal{M}^{(1)}$ are given by triple (A, B, ϵ), where $A, B \in \mathrm{Alg}_k$ and $\epsilon : A \otimes_k B \to k$ is an augmentation on $A \otimes_k B$ (which then determines augmentations on A and B). Set

$$\overline{\mathcal{M}} = \mathrm{LMod}(\mathrm{Mod}_k) \times_{\mathrm{Alg}_k} \mathcal{M}^{(1)} \times_{\mathrm{Alg}_k} \mathrm{LMod}(\mathrm{Mod}_k),$$

so that $\overline{\mathcal{M}}$ is an ∞-category whose objects can be identified with quintuples (A, B, ϵ, M, N), where $A, B \in \mathrm{Alg}_k$, $\epsilon : A \otimes_k B \to k$ is an augmentation, $M \in \mathrm{LMod}_A$, and $N \in \mathrm{LMod}_B$. There is an evident functor $\chi : \overline{\mathcal{M}}^{op} \to \mathcal{S}$, given on objects by the formula

$$\chi(A, B, \epsilon, M, N) = \mathrm{Map}_{\mathrm{LMod}_{A \otimes_k B}}(M \otimes_k N, k)$$

Then χ classifies a right fibration $\mathcal{M}^{\mathcal{LM}} \to \overline{\mathcal{M}}$. Let $\mathrm{LMod}^{\mathrm{aug}}(\mathrm{Mod}_k)$ denote the ∞-category

$$\mathrm{LMod}(\mathrm{Mod}_k) \times_{\mathrm{Alg}_k} \mathrm{Alg}_k^{\mathrm{aug}},$$

so that the forgetful functor $\mathcal{M}^{\mathcal{LM}} \to \mathrm{LMod}^{\mathrm{aug}}(\mathrm{Mod}_k) \times \mathrm{LMod}^{\mathrm{aug}}(\mathrm{Mod}_k)$ is a right fibration and therefore determines a pairing of $\mathrm{LMod}^{\mathrm{aug}}(\mathrm{Mod}_k)$ with itself.

Proposition 3.5.5. *Let k be a field, let $\lambda : \mathcal{M}^{(1)} \to \mathrm{Alg}_k^{\mathrm{aug}} \times \mathrm{Alg}_k^{\mathrm{aug}}$ be the pairing of Construction 3.1.4 and $\lambda' : \mathcal{M}^{\mathcal{LM}} \to \mathrm{LMod}^{\mathrm{aug}}(\mathrm{Mod}_k) \times \mathrm{LMod}^{\mathrm{aug}}(\mathrm{Mod}_k)$ the pairing of Construction 3.5.4. Then λ' is both left and right representable. Moreover, the forgetful functor $\mathcal{M}^{\mathcal{LM}} \to \mathcal{M}^{(1)}$ is both left and right representable.*

Proof. We will show that λ' is left representable and $\mathcal{M}^{\mathcal{LM}} \to \mathcal{M}^{(1)}$ is left representable; the corresponding assertions for right representability will follow by symmetry. Fix an object $A \in \mathrm{Alg}_k^{\mathrm{aug}}$ and a left A-module M. Let $B = \mathfrak{D}^{(1)}(A)$ be the Koszul dual of A and $\epsilon : A \otimes_k B \to k$ the canonical map. Proposition 3.1.10 implies that ϵ determines a duality functor $\mathfrak{D}_\epsilon : \mathrm{LMod}_A^{op} \to \mathrm{LMod}_B$. We let $N = \mathfrak{D}_\epsilon(M)$, so that there is a canonical map of left $A \otimes_k B$-modules $\mu : M \otimes_k N \to k$. The quintuple (A, B, ϵ, M, N) is an object of the ∞-category $\overline{\mathcal{M}}$ of Construction 3.5.4, and μ determines a lifting to an object $X \in \mathcal{M}^{\mathcal{LM}}$. We complete the proof by observing that X is left universal and has left universal image in $\mathcal{M}^{(1)}$. $\square$

It follows from Proposition 3.5.5 that the pairing $\mathcal{M}^{\mathcal{LM}} \to \mathrm{LMod}^{\mathrm{aug}}(\mathrm{Mod}_k) \times \mathrm{LMod}^{\mathrm{aug}}(\mathrm{Mod}_k)$ determines a duality functor $\mathfrak{D}^{\mathcal{LM}} : \mathrm{LMod}^{\mathrm{aug}}(\mathrm{Mod}_k) \to \mathrm{LMod}^{\mathrm{aug}}(\mathrm{Mod}_k)^{op}$. Let $\mathfrak{D}'$ denote the composite map

$$\mathrm{LMod}^{\mathrm{sm}}(\mathrm{Mod}_k) \to \mathrm{LMod}^{\mathrm{aug}}(\mathrm{Mod}_k) \xrightarrow{\mathfrak{D}^{\mathcal{LM}}} \mathrm{LMod}^{\mathrm{aug}}(\mathrm{Mod}_k)^{op} \to \mathrm{LMod}(\mathrm{Mod}_k)^{op}.$$

By construction, we have a commutative diagram σ :

$$\begin{tikzcd}
\mathrm{LMod}^{\mathrm{sm}}(\mathrm{Mod}_k) \arrow[r, "\mathfrak{D}'"] \arrow[d, "p"] & \mathrm{LMod}(\mathrm{Mod}_k)^{op} \arrow[d, "q"] \\
\mathrm{Alg}_k^{\mathrm{sm}} \arrow[r, "\mathfrak{D}"] & \mathrm{Alg}_k^{op} .
\end{tikzcd}$$

We next claim that the functor $\mathfrak{D}'$ carries p-Cartesian morphisms to q-Cartesian morphisms. Unwinding the definitions, we must show that if $f : R \to R'$ is a morphism in $\mathrm{Alg}_k^{\mathrm{sm}}$ and M is a small left R'-module, then the canonical map

$$\theta_M : \mathfrak{D}^{(1)}(R) \otimes_{\mathfrak{D}^{(1)}(R')} \mathfrak{D}_{\mu'}(M) \to \mathfrak{D}_\mu(M)$$

is an equivalence, where $\mathfrak{D}_\mu : \mathrm{LMod}_R^{op} \to \mathrm{LMod}_{\mathfrak{D}^{(1)}(R)}$ and $\mathfrak{D}_{\mu'} : \mathrm{LMod}_R^{op} \to \mathrm{LMod}_{\mathfrak{D}^{(1)}(R)}$ are the duality functors determined by the pairings $\mu : R \otimes_k \mathfrak{D}^{(1)}(R) \to k$ and $\mu' : R' \otimes_k \mathfrak{D}^{(1)}(R') \to k$. The modules

$M \in \mathrm{LMod}_{R'}$ for which θ_M is an equivalence span a stable subcategory of $\mathrm{LMod}_{R'}$ which includes k, and therefore contains all small R'-modules (Lemma 3.4.2).

To complete the proof of Proposition 3.5.3, it suffices to show that the functor $\mathfrak{D}'$ carries $\mathrm{LMod}^{\mathrm{sm}}(\mathrm{Mod}_k)$ into $\mathrm{LMod}^{\mathrm{perf}}(\mathrm{Mod}_k)^{op}$ and induces an equivalence of ∞-categories

$$\mathrm{LMod}^{\mathrm{sm}}(\mathrm{Mod}_k) \to \mathrm{LMod}^{\mathrm{perf}}(\mathrm{Mod}_k)^{op} \times_{\mathrm{Alg}_k^{op}} \mathrm{Alg}_k^{\mathrm{sm}}.$$

Using Corollary T.2.4.4.4, we are reduced to proving that $\mathfrak{D}'$ induces an equivalence of ∞-categories

$$\mathrm{LMod}_R^{\mathrm{sm}} \to (\mathrm{LMod}^{\mathrm{perf}}_{\mathfrak{D}^{(1)}(R)})^{op}$$

for every $R \in \mathrm{Alg}_k^{\mathrm{sm}}$. This is a consequence of Remark 3.4.2 together with the following more general assertion:

Proposition 3.5.6. *Let k be a field and let $\mu : A \otimes_k B \to k$ be a map of $\mathbb{E}_1$-algebras over k which exhibits B as a Koszul dual of A. Then the duality functor $\mathfrak{D}_\mu : \mathrm{LMod}_A^{op} \to \mathrm{LMod}_B$ restricts to an equivalence $\mathcal{C} \to \mathrm{LMod}_B^{\mathrm{perf}}$, where $\mathcal{C}$ denotes the smallest stable subcategory of LMod_A which contains k (regarded as a left A-module via the augmentation $A \to A \otimes_k B \xrightarrow{\mu} k$) and is closed under retracts.*

Proof. Let $\mathfrak{D}'_\mu : \mathrm{LMod}_B^{op} \to \mathrm{LMod}_A$ be as in Notation 3.1.11, and let $\mathcal{D}$ denote the full subcategory of LMod_A spanned by those objects M for which the unit map $M \to \mathfrak{D}'_\mu \mathfrak{D}_\mu(M)$ is an equivalence in LMod_A. It is clear that $\mathcal{D}$ is a stable subcategory of LMod_A which is closed under retracts. Since μ exhibits B as a Koszul dual of A, the subcategory $\mathcal{D}$ contains k so that $\mathcal{C} \subseteq \mathcal{D}$. It follows that the functor $\mathfrak{D}_\mu | \mathcal{C}$ is fully faithful. Moreover, the essential image of $\mathfrak{D}_\mu | \mathcal{C}$ is the smallest stable full subcategory of LMod_B which contains $\mathfrak{D}_\mu(k) \simeq B$ and is closed under retracts: this is the full subcategory $\mathrm{LMod}_B^{\mathrm{perf}} \subseteq \mathrm{LMod}_B$. □

Remark 3.5.7. Let k be a field of characteristic zero and let $\theta : \mathrm{CAlg}_k^{\mathrm{sm}} \to \mathrm{Alg}_k^{\mathrm{sm}}$ denote the forgetful functor. Let $X : \mathrm{Alg}_k^{\mathrm{sm}} \to \mathcal{S}$ be a formal $\mathbb{E}_1$ moduli problem over k, so that $X \circ \theta$ is a formal moduli problem over k. For each $R \in \mathrm{CAlg}_k^{\mathrm{sm}}$, we have a canonical equivalence of ∞-categories $\mathrm{Mod}_R \simeq \mathrm{RMod}_{\theta(R)}$. Passing to the inverse limit over points $\eta \in X(\theta(R))$, we obtain a functor $\mathrm{QCoh}_R(X) \to \mathrm{QCoh}(X \circ \theta)$. According to Theorem 3.0.4, there exists an augmented $\mathbb{E}_1$-algebra A over k such that X is given by the formula $X(R) = \mathrm{Map}_{\mathrm{Alg}_k^{\mathrm{aug}}}(\mathfrak{D}^{(1)}(R), A)$. Let $\mathfrak{m}_A$ denote the augmentation ideal of A. Regard $\mathfrak{m}_A$ as an object of Lie_k, so that $X \circ \theta$ is given by the formula

$$(X \circ \theta)(R) = \mathrm{Map}_{\mathrm{Lie}_k}(\mathfrak{D}(R), \mathfrak{m}_A)$$

(see Theorem 3.3.1). Theorems 2.4.1 and 3.5.1 determine fully faithful embeddings

$$\mathrm{QCoh}_R(X) \hookrightarrow \mathrm{LMod}_A \qquad \mathrm{QCoh}(X \circ \theta) \hookrightarrow \mathrm{Rep}_{\mathfrak{m}_A}.$$

We have an evident map of $\mathbb{E}_1$-algebras $U(\mathfrak{m}_A) \to A$, which determines a forgetful functor $\mathrm{LMod}_A \to \mathrm{LMod}_{U(\mathfrak{m}_A)} \simeq \mathrm{Rep}_{\mathfrak{m}_A}$. With some additional effort, one can show that the diagram

$$\begin{tikzcd} \mathrm{QCoh}_R(X) \arrow[r] \arrow[d] & \mathrm{LMod}_A \arrow[d] \\ \mathrm{QCoh}(X \circ \theta) \arrow[r] & \mathrm{Rep}_{\mathfrak{m}_A} \end{tikzcd}$$

commutes up to canonical homotopy. That is, the algebraic models for quasi-coherent sheaves provided by Theorems 2.4.1 and 3.5.1 in the commutative and noncommutative settings are compatible with one another.

We conclude this section with a discussion of the exactness properties of equivalences

$$\mathrm{QCoh}^!_L(X) \simeq \mathrm{RMod}_A \qquad \mathrm{QCoh}^!_R(X) \simeq \mathrm{LMod}_A$$

appearing in Theorem 3.5.1. Let $\mathrm{RMod}_A^{\mathrm{cn}}$ and $\mathrm{LMod}_A^{\mathrm{cn}}$ denote the full subcategories spanned by those right and left A-modules whose underlying spectra are connective. Note that the equivalences of Theorem 3.5.1 depend functorially on A, and when $A = k$ they are equivalent to the identity functor from the ∞-category Mod_k to itself. Let $*$ denote the final object of $\mathrm{Moduli}_k^{(1)}$, so that we have a canonical map of formal moduli problems $* \to X$ (induced by the map of augmented $\mathbb{E}_1$-algebras $k \to A$). It follows that Theorem 3.5.1 gives an equivalence

$$\begin{aligned} \mathrm{QCoh}^!_L(X)^{\mathrm{cn}} &\simeq \mathrm{QCoh}^!_L(X) \times_{\mathrm{QCoh}^!_L(*)} \mathrm{QCoh}^!_L(*)^{\mathrm{cn}} \\ &\simeq \mathrm{RMod}_A \times_{\mathrm{RMod}_k} \mathrm{RMod}_k^{\mathrm{cn}} \\ &\simeq \mathrm{RMod}_A^{\mathrm{cn}} \end{aligned}$$

and, by symmetry, an equivalence $\mathrm{QCoh}^!_R(X)^{\mathrm{cn}} \simeq \mathrm{LMod}_A^{\mathrm{cn}}$. Combining this observation with Proposition 3.4.27, we obtain the following result:

Proposition 3.5.8. *Let k be a field, let $A \in \mathrm{Alg}_k^{\mathrm{aug}}$, and let $X : \mathrm{Alg}_k^{\mathrm{sm}} \to \mathcal{S}$ be the formal $\mathbb{E}_1$ moduli problem associated to A (see Theorem 3.0.4). Then the fully faithful embeddings*

$$\mathrm{QCoh}_L(X) \hookrightarrow \mathrm{RMod}_A \qquad \mathrm{QCoh}_R(X) \hookrightarrow \mathrm{LMod}_A .$$

of Theorem 3.5.1 restrict to equivalences of ∞-categories

$$\mathrm{QCoh}_L(X)^{\mathrm{cn}} \simeq \mathrm{RMod}_A^{\mathrm{cn}} \qquad \mathrm{QCoh}_R(X)^{\mathrm{cn}} \simeq \mathrm{LMod}_A^{\mathrm{cn}} .$$

Warning 3.5.9. If A is an arbitrary $\mathbb{E}_1$-ring, then the full subcategories

$$\mathrm{LMod}_A^{\mathrm{cn}} = \mathrm{LMod}_A \times_{\mathrm{Sp}} \mathrm{Sp}^{\mathrm{cn}} \subseteq \mathrm{LMod}_A \qquad \mathrm{RMod}_A^{\mathrm{cn}} = \mathrm{RMod}_A \times_{\mathrm{Sp}} \mathrm{Sp}^{\mathrm{cn}} \subseteq \mathrm{RMod}_A$$

are presentable, closed under small colimits, and closed under extensions. It follows from Proposition A.1.4.5.11 that LMod_A and RMod_A admit t-structures with

$$(\mathrm{LMod}_A)_{\geq 0} = \mathrm{LMod}_A^{\mathrm{cn}} \qquad (\mathrm{RMod}_A)_{\geq 0} = \mathrm{RMod}_A^{\mathrm{cn}} .$$

However, it is often difficult to describe the subcategories $(\mathrm{LMod}_A)_{\leq 0} \subseteq \mathrm{LMod}_A$ and $(\mathrm{RMod}_A)_{\leq 0} \subseteq \mathrm{RMod}_A$. In particular, they generally do not coincide with the subcategories

$$\mathrm{LMod}_A \times_{\mathrm{Sp}} \mathrm{Sp}_{\leq 0} \subseteq \mathrm{LMod}_A \qquad \mathrm{RMod}_A \times_{\mathrm{Sp}} \mathrm{Sp}_{\leq 0} \subseteq \mathrm{RMod}_A$$

unless the $\mathbb{E}_1$-ring A is connective.

4 Moduli Problems for $\mathbb{E}_n$-Algebras

Let k be a field. In §2 and §3 we studied the ∞-categories Moduli_k and $\mathrm{Moduli}_k^{(1)}$ consisting of formal moduli problems defined for commutative and associative algebras over k, respectively. In the ∞-categorical context, there is a whole hierarchy of algebraic notions in between these two extremes. Recall that the commutative ∞-operad can be identified with the colimit of a sequence

$$\mathcal{A}\mathrm{ss}^{\otimes} \simeq \mathbb{E}_1^{\otimes} \to \mathbb{E}_2^{\otimes} \to \mathbb{E}_3^{\otimes} \to \cdots ,$$

where $\mathbb{E}_n^{\otimes}$ denotes the Boardman-Vogt ∞-operad of little n-cubes (see Corollary A.5.1.1.5). Consequently, the ∞-category CAlg_k of $\mathbb{E}_\infty$-algebras over k can be identified with the limit of a tower of ∞-categories

$$\cdots \to \mathrm{Alg}_k^{(3)} \to \mathrm{Alg}_k^{(2)} \to \mathrm{Alg}_k^{(1)} \simeq \mathrm{Alg}_k,$$

where $\mathrm{Alg}_k^{(n)}$ denotes the ∞-category of $\mathbb{E}_n$-algebras over k. Our goal in this section is to prove a generalization of Theorem 3.0.4 in the setting of $\mathbb{E}_n$-algebras, for an arbitrary integer $n \geq 0$. To formulate our result, we need a bit of terminology.

Definition 4.0.1. Let k be a field, let $n \geq 1$, and let A be an $\mathbb{E}_n$-algebra over k. We will say that A is *small* if its image in Alg_k is small, in the sense of Definition 3.0.1. We let $\mathrm{Alg}_k^{(n),\mathrm{sm}}$ denote the full subcategory of $\mathrm{Alg}_k^{\mathrm{sm}}$ spanned by the small $\mathbb{E}_n$-algebras over k.

Remark 4.0.2. Let $n \geq 1$ and let A be an $\mathbb{E}_n$-algebra over k. Then A is small if and only if it is connective, $\pi_* A$ is finite-dimensional over k, and the unit map $k \to (\pi_0 A)/\mathfrak{m}$ is an isomorphism, where $\mathfrak{m}$ denotes the radical of $\pi_0 A$.

Remark 4.0.3. Let k be a field and let A be an $\mathbb{E}_n$-algebra over k, for $n \geq 0$. An *augmentation* on A is a map of $\mathbb{E}_n$-algebras $\epsilon : A \to k$. We let $\mathrm{Alg}_k^{(n),\mathrm{aug}} = (\mathrm{Alg}_k^{(n)})_{/k}$ denote the ∞-category of augmented $\mathbb{E}_n$-algebras over k. Note that if $n \geq 1$ and $A \in \mathrm{Alg}_k^{(n)}$ is small, then the space $\mathrm{Map}_{\mathrm{Alg}_k^{(n)}}(A, k)$ of augmentations on A is contractible. It follows that the projection map

$$\mathrm{Alg}_k^{(n),\mathrm{aug}} \times_{\mathrm{Alg}_k^{(n)}} \mathrm{Alg}_k^{(n),\mathrm{sm}} \to \mathrm{Alg}_k^{(n),\mathrm{sm}}$$

is an equivalence of ∞-categories. We will henceforth abuse notation by identifying $\mathrm{Alg}_k^{(n),\mathrm{sm}}$ with its inverse image in $\mathrm{Alg}_k^{(n),\mathrm{aug}}$.

Remark 4.0.4. It will be convenient to have a version of Definition 3.0.1 also in the case $n = 0$. We therefore adopt the following convention: we will say that an augmented $\mathbb{E}_0$-algebra A over k is *small* if A is connective and $\pi_* A$ is a finite dimensional vector space over k. We let $\mathrm{Alg}_k^{(0),\mathrm{sm}}$ denote the full subcategory of $\mathrm{Alg}_k^{(0),\mathrm{aug}}$ spanned by the small augmented $\mathbb{E}_0$-algebras over k.

Notation 4.0.5. Let k be a field, let $n \geq 0$, and let $\epsilon : A \to k$ be an augmented $\mathbb{E}_n$-algebra over k. We let $\mathfrak{m}_A$ denote the fiber of the map ϵ in the stable ∞-category Mod_k. We will refer to $\mathfrak{m}_A$ as the *augmentation ideal* of A. The construction $(\epsilon : A \to k) \mapsto \mathfrak{m}_A$ determines a functor

$$\mathfrak{m} : \mathrm{Alg}_k^{(n),\mathrm{aug}} \to \mathrm{Mod}_k .$$

In the case $n = 0$, this functor is an equivalence of ∞-categories.

Definition 4.0.6. Let k be a field, let $n \geq 0$ be an integer and let $X : \mathrm{Alg}_k^{(n),\mathrm{sm}} \to \mathcal{S}$ be a functor. We will say that X is a *formal* $\mathbb{E}_n$ *moduli problem* if it satisfies the following conditions:

(1) The space $X(k)$ is contractible.

(2) For every pullback diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R_0 \arrow[d] \\ R_1 \arrow[r] & R_{01} \end{tikzcd}$$

in $\mathrm{Alg}_k^{\mathrm{sm}}$ for which the underlying maps $\pi_0 R_0 \to \pi_0 R_{01} \leftarrow \pi_0 R_1$ are surjective, the diagram

$$\begin{tikzcd} X(R) \arrow[r] \arrow[d] & X(R_0) \arrow[d] \\ X(R_1) \arrow[r] & X(R_{01}) \end{tikzcd}$$

is a pullback square.

We let $\mathrm{Moduli}_k^{(n)}$ denote the full subcategory of $\mathrm{Fun}(\mathrm{Alg}_k^{(n),\mathrm{sm}}, \mathcal{S})$ spanned by the formal $\mathbb{E}_n$ moduli problems.

Example 4.0.7. It is not difficult to show that a functor $X : \mathrm{Alg}_k^{(0),\mathrm{sm}} \to \mathcal{S}$ is a formal $\mathbb{E}_0$ moduli problem if and only if it is strongly excisive (see Definition A.1.4.4.4): that is, if and only if X carries the initial object of $\mathrm{Alg}_k^{(0),\mathrm{sm}}$ to a final object of $\mathcal{S}$, and carries pushout squares to pullback squares.

We are now ready to formulate our main result.

Theorem 4.0.8. *Let k be a field and let $n \geq 0$ be an integer. Then there is an equivalence of ∞-categories $\Psi : \mathrm{Alg}_k^{(n),\mathrm{aug}} \to \mathrm{Moduli}_k^{(n)}$. Moreover, the diagram*

$$\begin{tikzcd} \mathrm{Alg}_k^{(n),\mathrm{aug}} \arrow[r, "\Psi"] \arrow[d, "\mathfrak{m}"] & \mathrm{Moduli}_k^{(n)} \arrow[d, "T[-n]"] \\ \mathrm{Mod}_k \arrow[r] & \mathrm{Sp} \end{tikzcd}$$

commutes up to homotopy, where $T : \mathrm{Moduli}_k^{(n)} \to \mathrm{Sp}$ denotes the tangent complex functor (so that $\Omega^{\infty - m} T_X \simeq X(k \oplus k[m])$ for $m \geq 0$) and $\mathfrak{m} : \mathrm{Alg}_k^{(n),\mathrm{aug}} \to \mathrm{Mod}_k$ is the augmentation ideal functor of Notation 4.0.5.

Example 4.0.9. When $n = 1$, Theorem 4.0.8 follows from Theorem 3.0.4 and Remark 3.2.6.

Remark 4.0.10. Suppose that k is a field of characteristic zero. For each $n \geq 0$, there is an evident forgetful functor $\mathrm{CAlg}_k^{\mathrm{sm}} \to \mathrm{Alg}_k^{(n),\mathrm{sm}}$, which induces a forgetful functor $\theta : \mathrm{Moduli}_k^{(n)} \to \mathrm{Moduli}_k$. Using the equivalences

$$\mathrm{Lie}_k \simeq \mathrm{Moduli}_k \qquad \mathrm{Moduli}_k^{(n)} \simeq \mathrm{Alg}_k^{(n),\mathrm{aug}}$$

of Theorems 2.0.2 and 4.0.8, we can identify θ with a map $\mathrm{Alg}_k^{(n),\mathrm{aug}} \to \mathrm{Lie}_k$. We can summarize the situation informally as follows: if A is an augmented $\mathbb{E}_n$-algebra over k, then the shifted augmentation ideal $\mathfrak{m}_A[n-1]$ inherits the structure of a differential graded Lie algebra over k. In particular, at the level of homotopy groups we obtain a Lie bracket operation

$$[,] : \pi_p \mathfrak{m}_A \times \pi_q \mathfrak{m}_A \to \pi_{p+q+n-1} \mathfrak{m}_A.$$

One can show that this Lie bracket is given by the *Browder operation* on $\mathfrak{m}_A$. If $\mathrm{Free} : \mathrm{Mod}_k \to \mathrm{Alg}_k^{(n)}$ denotes the free algebra functor (left adjoint to the forgetful functor $\mathrm{Alg}_k^{(n)} \to \mathrm{Mod}_k$), then the Browder operation is universally represented by the the map ϕ appearing in the cofiber sequence of augmented $\mathbb{E}_n$-algebras

$$\mathrm{Free}(k[p+q+n-1]) \xrightarrow{\phi} \mathrm{Free}(k[p] \oplus k[q]) \to \mathrm{Free}(k[p]) \otimes_k \mathrm{Free}(k[q])$$

supplied by Theorem A.5.1.5.1.

The appearance of the theory of $\mathbb{E}_n$-algebras on both sides of the equivalence

$$\mathrm{Alg}_k^{(n),\mathrm{aug}} \simeq \mathrm{Moduli}_k^{(n)} \subseteq \mathrm{Fun}(\mathrm{Alg}_k^{(n),\mathrm{sm}}, \mathcal{S})$$

is somewhat striking: it is a reflection of the Koszul self-duality of the little cubes operads $\mathbb{E}_n^{\otimes}$ (see [17]). In particular, there is a Koszul duality functor

$$\mathfrak{D}^{(n)} : (\mathrm{Alg}_k^{(n),\mathrm{aug}})^{op} \to \mathrm{Alg}_k^{(n),\mathrm{aug}}.$$

This functor is not difficult to define directly : if A is an augmented $\mathbb{E}_n$-algebra over k, then $\mathfrak{D}^{(n)}(A)$ is universal among $\mathbb{E}_n$-algebras over k such that the tensor product $A \otimes_k \mathfrak{D}^{(n)}(A)$ is equipped with an augmentation extending the augmentation on A. The equivalence Ψ appearing in the statement of Theorem 4.0.8 carries an augmented $\mathbb{E}_n$-algebra A to the functor X given by the formula

$$X(R) = \mathrm{Map}_{\mathrm{Alg}_k^{(n),\mathrm{aug}}}(\mathfrak{D}^{(n)}(R), A).$$

In §4.5, we will prove that the Koszul duality functor $\mathfrak{D}^{(n)}$ is a deformation theory (in the sense of Definition 1.3.9), so that Theorem 4.0.8 is a consequence of Theorem 1.3.12. The main point is to produce a full subcategory $\Xi_0 \subseteq \mathrm{Alg}_k^{(n),\mathrm{aug}}$ which satisfies axiom $(D3)$ of Definition 1.3.1. We will define Ξ_0 to be the full subcategory of $\mathrm{Alg}_k^{(n),\mathrm{aug}}$ spanned by those augmented $\mathbb{E}_n$ algebras A which satisfy suitable finiteness and coconnectivity conditions. We will then need to prove two things:

(a) The full subcategory $\Xi_0 \subseteq \mathrm{Alg}_k^{(n),\mathrm{aug}}$ has good closure properties.

(b) For every object $A \in \Xi_0$, the biduality map $A \to \mathfrak{D}^{(n)}\mathfrak{D}^{(n)}(A)$ is an equivalence.

The verification of (a) comes down to connectivity properties of free algebras over the $\mathbb{E}_n$-operad. We will establish these properties in §4.1, using topological properties of configuration spaces of points in Euclidean space.

The proof of (b) is more involved, and requires us to have a good understanding of the Koszul duality functor $\mathfrak{D}^{(n)}$. Let us begin with the case $n = 1$, which we have already studied in §3.1. Let A be an $\mathbb{E}_1$-algebra over a field k. Then the Koszul dual $\mathfrak{D}^{(1)}(A)$ can be described as a classifying object for A-linear maps from k to itself, or equivalently as the k-linear dual of the object $k \otimes_A k$. In §4.3, we will show that the algebra structure on $\mathfrak{D}^{(1)}(A)$ can be obtained by dualizing a *coalgebra* structure on $\mathrm{Bar}(A) = k \otimes_A k$: in particular, we have a comultiplication given by

$$\mathrm{Bar}(A) = k \otimes_A k \simeq k \otimes_A A \otimes_A k \to k \otimes_A k \otimes_A k \simeq \mathrm{Bar}(A) \otimes_k \mathrm{Bar}(A).$$

The proof will require an ∞-categorical generalization of the twisted arrow category introduced in Construction 3.3.5, which we will study in §4.2.

The bar construction $A \mapsto \mathrm{Bar}(A) = k \otimes_A k$ is in some ways better behaved than the Koszul duality functor $\mathfrak{D}^{(1)}$: for example, it is a symmetric monoidal functor, while $\mathfrak{D}^{(1)}$ is not (see Warning 3.1.20). In §4.4, we will use this observation to analyze the Koszul duality functor $\mathfrak{D}^{(n)}$ for a general integer n, using induction on n. Using Theorem A.5.1.2.2, we can identify the ∞-category $\mathrm{Alg}_k^{(n+1)}$ with $\mathrm{Alg}(\mathrm{Alg}_k^{(n)})$, the ∞-category of associative algebra objects of $\mathrm{Alg}_k^{(n)}$. If A is an augmented $\mathbb{E}_{n+1}$-algebra, then we can apply the bar construction to obtain a coalgebra object of $\mathrm{Alg}_k^{(n),\mathrm{aug}}$. It is not difficult to show that the Koszul duality functor

$$\mathfrak{D}^{(n-1)} : (\mathrm{Alg}_k^{(n),\mathrm{aug}})^{op} \to \mathrm{Alg}_k^{(n),\mathrm{aug}}$$

is lax monoidal. We will show that the composite map

$$\begin{aligned}
(\mathrm{Alg}_k^{(n+1),\mathrm{aug}})^{op} &\simeq \mathrm{Alg}(\mathrm{Alg}_k^{(n),\mathrm{aug}})^{op} \\
&\stackrel{\mathrm{Bar}}{\to} \mathrm{Alg}((\mathrm{Alg}_k^{(n),\mathrm{aug}})^{op}) \\
&\stackrel{\mathfrak{D}^{(n)}}{\to} \mathrm{Alg}(\mathrm{Alg}_k^{(n),\mathrm{aug}}) \\
&\simeq \mathrm{Alg}_k^{(n+1),\mathrm{aug}}
\end{aligned}$$

can be identified with the Koszul duality functor $\mathfrak{D}^{(n+1)}$. This will allow us to deduce results about the Koszul duality functors $\mathfrak{D}^{(n)}$ from analogous facts about the Koszul duality functor $\mathfrak{D}^{(1)}$, and in particular to deduce (a) from Corollary 3.1.15 (see Theorem 4.4.5).

Remark 4.0.11. Let $X : \mathrm{Alg}_k^{(n),\mathrm{sm}} \to \mathcal{S}$ be a formal $\mathbb{E}_n$-moduli problem for $n \geq 1$. Using the ideas introduced in §3.4, we can define ∞-categories $\mathrm{QCoh}_L(X)$ and $\mathrm{QCoh}_L^!(X)$ of quasi-coherent and Ind-coherent sheaves on X, respectively. According to Theorem 4.0.8, the functor X is given by the formula

$$X(R) = \mathrm{Map}_{\mathrm{Alg}_k^{(n),\mathrm{aug}}}(\mathfrak{D}^{(n)}(R), A)$$

for some (essentially unique) augmented $\mathbb{E}_n$-algebra A over k. Since the bar construction $B \mapsto \mathrm{Bar}(B)$ is symmetric monoidal, it carries augmented $\mathbb{E}_m$-algebras over k to augmented $\mathbb{E}_{m-1}$-algebras over k. It follows that the iterated bar construction $\mathrm{Bar}^{n-1}(A)$ admits the structure of an $\mathbb{E}_1$-algebra. In this context, we have the following version of Theorem 2.4.1: there is an equivalence of ∞-categories $\mathrm{QCoh}^!_L(X) \simeq \mathrm{RMod}_{\mathrm{Bar}^{n-1}(A)}$ (and therefore also a fully faithful embedding $\mathrm{QCoh}_L(X) \hookrightarrow \mathrm{RMod}_{\mathrm{Bar}^{n-1}(A)}$). Moreover, one can show that this is an equivalence of $\mathbb{E}_{n-1}$-monoidal ∞-categories (here the $\mathbb{E}_{n-1}$-monoidal structure on $\mathrm{RMod}_{\mathrm{Bar}^{n-1}(A)}$ arises from the fact that $\mathrm{Bar}^{n-1}(A)$ can be regarded as an $\mathbb{E}_{n-1}$-algebra object of Alg_k^{op}).

4.1 Coconnective $\mathbb{E}_n$-Algebras

Let k be a field and let $n \geq 0$ be an integer. Our goal in this section is to study some finiteness and coconnectivity conditions on $\mathbb{E}_n$-algebras over k which will play a role in our proof of Theorem 4.0.8.

Definition 4.1.1. Let k be a field and let A be an $\mathbb{E}_0$-algebra over k: that is, a k-module equipped with a unit map $e : k \to A$. Let m be an integer. We will say that A is *m-coconnective* if the homotopy groups $\pi_i \operatorname{cofib}(e)$ vanish for $i > -m$.

More generally, if A is an $\mathbb{E}_n$-ring equipped with a map of $\mathbb{E}_n$-rings $k \to A$, we will say that A is *m-coconnective* if it is m-coconnective when regarded as an $\mathbb{E}_0$-algebra over k (here we do not require that A is an $\mathbb{E}_n$-algebra over k, though this will always be satisfied in cases of interest to us).

Remark 4.1.2. If A is an $\mathbb{E}_1$-algebra over a field k, then A is coconnective (in the sense of Definition 3.1.13) if and only if it is 1-coconnective (in the sense of Definition 4.1.1).

Remark 4.1.3. If $m > 0$, then an $\mathbb{E}_n$-algebra A over k is m-coconnective if and only if the unit map $k \to A$ induces an isomorphism $k \to \pi_0 A$, and the homotopy groups $\pi_i A$ vanish for $i > 0$ and $-m < i < 0$.

Notation 4.1.4. Let k be a field and let $n \geq 0$ be an integer. We let $\mathrm{Free}^{(n)} : \mathrm{Mod}_k \to \mathrm{Alg}_k^{(n)}$ denote a left adjoint to the forgetful functor $\mathrm{Alg}_k^{(n)} \to \mathrm{Mod}_k$. For any object $V \in \mathrm{Mod}_k$, the free algebra $\mathrm{Free}^{(n)}(V)$ is equipped with a canonical augmentation $\epsilon : \mathrm{Free}^{(n)}(V) \to k$, corresponding to the zero morphism. $V \to k$ in Mod_k.

Our main result can be stated as follows:

Theorem 4.1.5. *Let k be a field, let A be an $\mathbb{E}_n$-algebra over k, and let $m \geq n$ be an integer. Suppose we are given a map $\phi : V \to A$ in Mod_k, where $\pi_i V \simeq 0$ for $i \geq -m$, and form a pushout diagram*

$$\begin{tikzcd} \mathrm{Free}^{(n)}(V) \arrow[r, "\phi'"] \arrow[d, "\epsilon"] & A \arrow[d] \\ k \arrow[r] & A' \end{tikzcd}$$

where ϕ' is the map of $\mathbb{E}_n$-algebras determined by ϕ and ϵ is the augmentation of Notation 4.1.4. If A is m-coconnective, then A' is also m-coconnective.

Our proof of Theorem 4.1.5 is somewhat indirect. We will first show that the conclusion of Theorem 4.1.5 is valid under an additional hypothesis on A (Proposition 4.1.13). We will then use this variant of Theorem 4.1.5 to show that the additional hypothesis is automatically satisfied (Proposition 4.1.14). First, we need to introduce a bit of terminology.

Notation 4.1.6. Let k be a field and let A be an $\mathbb{E}_n$-algebra over k. We let $\mathrm{Mod}_A^{\mathbb{E}_n} = \mathrm{Mod}_A^{\mathbb{E}_n}(\mathrm{Mod}_k)$ denote the ∞-category of $\mathbb{E}_n$-modules over A (see §A.3.3.3). Note that $\mathrm{Mod}_A^{\mathbb{E}_n}$ is a presentable ∞-category (Theorem A.3.4.4.2) and the forgetful functor $\theta : \mathrm{Mod}_A^{\mathbb{E}_n} \to \mathrm{Mod}_k$ is conservative and preserves small limits and colimits (Corollaries A.3.4.3.3, A.3.4.3.6, and A.3.4.4.6). It follows that $\mathrm{Mod}_A^{\mathbb{E}_n}$ is a stable ∞-category. The composite functor

$$\mathrm{Mod}_A^{\mathbb{E}_n} \xrightarrow{\theta} \mathrm{Mod}_k \to \mathrm{Sp} \xrightarrow{\Omega^\infty} \mathcal{S}$$

preserves small limits and filtered colimits, and is therefore corepresentable by an object $M \in \mathrm{Mod}_A^{\mathbb{E}_n}$ (Proposition T.5.5.2.7). Since θ is conservative, the object M generates $\mathrm{Mod}_A^{\mathbb{E}_n}$ in the following sense: an object $N \in \mathrm{Mod}_A^{\mathbb{E}_n}$ vanishes if and only if the abelian groups $\mathrm{Ext}^n_{\mathrm{Mod}_A^{\mathbb{E}_n}}(M, N)$ vanish for every integer n. Applying Theorem A.7.1.2.1 (and its proof), we see that there exists an $\mathbb{E}_1$-ring $\int A$ and an equivalence of ∞-categories $\mathrm{LMod}_{\int A} \simeq \mathrm{Mod}_A^{\mathbb{E}_n}$ carrying $\int A$ to the module M (this latter condition is equivalent to the requirement that the composition

$$\mathrm{LMod}_{\int A} \simeq \mathrm{Mod}_A^{\mathbb{E}_n} \to \mathrm{Mod}_k \to \mathrm{Sp}$$

is equivalent to the forgetful functor $\mathrm{LMod}_{\int A} \to \mathrm{Sp}$). The $\mathbb{E}_1$-ring A can be characterized (up to equivalence) as the $\mathbb{E}_1$-ring of endomorphisms of M in the stable ∞-category $\mathrm{Mod}_A^{\mathbb{E}_n}$.

Remark 4.1.7. In the situation of Notation 4.1.6, let $\mathrm{Mod}^{\mathbb{E}_n}$ denote the ∞-category of pairs (A, M), where A is an $\mathbb{E}_n$-algebra over k and M is an $\mathbb{E}_n$-module over A. We have a presentable fibration $\mathrm{Mod}^{\mathbb{E}_n} \to \mathrm{Alg}_k^{(n)}$, classified by a functor $\chi : \mathrm{Alg}_k^{(n)} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$; here $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ denotes the ∞-category whose objects are presentable ∞-categories and whose morphisms are functors which preserve small colimits. Since each $\mathrm{Mod}_A^{\mathbb{E}_n}$ is stable, the functor χ factors as a composition

$$\mathrm{Alg}_k^{(n)} \xrightarrow{\chi'} \mathrm{Mod}_{\mathrm{Sp}}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$$

(see Proposition A.6.3.2.13). Since χ carries the inital object $k \in \mathrm{Alg}_k^{(n)}$ to Mod_k (see Proposition A.3.4.2.1), the canonical map $\mathrm{Sp} \to \mathrm{Mod}_k$ allows us to factor χ through a functor $\chi'' : \mathrm{Alg}_k^{(n)} \to \mathrm{Mod}_{\mathrm{Sp}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})_{\mathrm{Sp}\,/}$. According to Theorem A.6.3.5.5, the construction $B \mapsto \mathrm{LMod}_B$ determines a fully faithful embedding $\mathrm{Alg}(\mathrm{Sp}) \to \mathrm{Mod}_{\mathrm{Sp}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})_{\mathrm{Sp}\,/}$, and Notation 4.1.6 implies that the functor χ'' factors through the essential image of this embedding. It follows that we can regard the construction $A \mapsto \int A$ as a functor $\int : \mathrm{Alg}_k^{(n)} \to \mathrm{Alg}(\mathrm{Sp})$.

Remark 4.1.8. Let k be a field, and regard k as an $\mathbb{E}_n$-algebra over itself. Then the forgetful functor $\mathrm{Mod}_k^{\mathbb{E}_n} \to \mathrm{Mod}_k$ is an equivalence of ∞-categories (Proposition A.3.4.2.1), so that we have a canonical equivalence of $\mathbb{E}_1$-rings $k \simeq \int k$. For any $\mathbb{E}_n$-algebra A over k, the unit map $k \to A$ is a map of $\mathbb{E}_n$-algebras, and therefore induces a map of $\mathbb{E}_1$-rings $k \simeq \int k \to \int A$. In particular, the homotopy groups $\pi_* A$ can be regarded as vector spaces over the field k.

With more effort, one can show that the map $k \to \int A$ is central: that is, $\int A$ can be regarded as an $\mathbb{E}_1$-algebra over k. We will not need this fact.

Example 4.1.9. If $n = 0$ and A is an $\mathbb{E}_n$-algebra over k, then the forgetful functor $\mathrm{Mod}_A^{\mathbb{E}_n} \to \mathrm{Mod}_k$ is an equivalence Proposition A.3.3.3.19). It follows that the map $k \to \int A$ of Remark 4.1.8 is an equivalence. That is, $\int : \mathrm{Alg}_k^{(0)} \to \mathrm{Alg}^{(1)}$ can be identified with the constant functor taking the value k.

Example 4.1.10. If $n = 1$ and $A \in \mathrm{Alg}_k^{(n)}$, then there is a canonical equivalence of ∞-categories $\mathrm{Mod}_A^{\mathbb{E}_1} \simeq {}_A\mathrm{BMod}_A(\mathrm{Mod}_k)$ (Theorem A.4.3.4.28). Using Corollary A.6.3.6.12, we obtain a canonical equivalence of $\mathbb{E}_1$-rings $\int A \simeq A \otimes_k A^{\mathrm{rev}}$, where A^{rev} denotes the $\mathbb{E}_1$-algebra A equipped with the opposite multiplication.

More generally, for any integer $n \geq 1$, the inclusion of ∞-operads $\mathbb{E}_1^{\otimes} \to \mathbb{E}_n^{\otimes}$ determines a forgetful functor $\mathrm{Alg}_A^{\mathbb{E}_n} \to \mathrm{Alg}_A^{\mathbb{E}_1}$ which induces a map of $\mathbb{E}_1$-rings $A \otimes_k A^{\mathrm{rev}} \to \int A$. We may therefore regard $\int A$ as an A-A bimodule object of Mod_k.

Remark 4.1.11. Let k be a field and let A be an $\mathbb{E}_n$-algebra over k. One can show that the $\mathbb{E}_1$-ring $\int A$ is given by the topological chiral homology $\int_{\mathbf{R}^n - \{0\}} A$ defined in §A.5.3.2). We will make no use of this description in what follows.

Remark 4.1.12. Let A be an $\mathbb{E}_n$-algebra over a field k. Theorem A.7.3.6.1 supplies a fiber sequence of $\mathbb{E}_n$-modules over A

$$L_{A/k}[n-1] \to \int A \to A,$$

where $L_{A/k}$ denotes the relative cotangent complex of A over k in the setting of $\mathbb{E}_n$-algebras. If A is connective, then $L_{A/k}$ is also connective. For $n \geq 1$, it follows that $\int A$ is also connective (this is also true for $n = 0$; see Example 4.1.9), so that the ∞-category $\mathrm{Mod}_A^{\mathbb{E}_n}(\mathrm{Mod}_k) \simeq \mathrm{Mod}_{\int A}$ inherits a t-structure from the t-structure on Mod_k. For $n \geq 2$, we deduce also that the map $\pi_0 \int A \to \pi_0 A$ is an isomorphism, so that the forgetful functor $\mathrm{Mod}_A^{\mathbb{E}_n}(\mathrm{Mod}_k) \to \mathrm{LMod}_A$ induces an equivalence of abelian categories $\mathrm{Mod}_A^{\mathbb{E}_n}(\mathrm{Mod}_k)^{\heartsuit} \simeq \mathrm{LMod}_A^{\heartsuit}$.

Theorem 4.1.5 is an immediate consequence of the following pair of results:

Proposition 4.1.13. *Let k be a field, let A be an $\mathbb{E}_n$-algebra over k, and let $m \geq n$ be an integer. Suppose we are given a map $\phi : V \to A$ in Mod_k, where $\pi_i V \simeq 0$ for $i \geq -m$, and form a pushout diagram*

$$\begin{tikzcd} \mathrm{Free}^{(n)}(V) \arrow[r, "\phi'"] \arrow[d, "\epsilon"] & A \arrow[d] \\ k \arrow[r] & A' \end{tikzcd}$$

where ϕ' is the map of $\mathbb{E}_n$-algebras determined by ϕ and ϵ is the augmentation of Notation 4.1.4. Assume that A is m-coconnective and that $\int A$ is $(m+1-n)$-coconnective. Then A' is m-coconnective, and $\int A'$ is $(m+1-n)$-coconnective. Moreover, if A and $\int A$ are locally finite, then A' and $\int A'$ are locally finite.

Proposition 4.1.14. *Let A be an $\mathbb{E}_n$-algebra over a field k, and assume that A is m-coconnective for $m \geq n$. Then the $\mathbb{E}_1$-ring $\int A$ is $(m+1-n)$-coconnective.*

The proof of Propositions 4.1.13 and 4.1.14 rely on having a good understanding of the free algebra functor $\mathrm{Free}^{(n)} : \mathrm{Mod}_k \to \mathrm{Alg}_k^{(n)}$. Recall that for $V \in \mathrm{Mod}_k$, the underlying k-module spectrum of $\mathrm{Free}^{(n)}(V)$ is given by

$$\bigoplus_{m \geq 0} \mathrm{Sym}^m_{\mathbb{E}_n}(V),$$

where $\mathrm{Sym}^m_{\mathbb{E}_n}(V)$ is the colimit of a diagram indexed by the full subcategory $K_{m,n} \subseteq (\mathbb{E}_n^{\otimes})_{/\langle 1 \rangle}$ spanned by the active morphisms $\langle m \rangle \to \langle 1 \rangle$ in the ∞-operad $\mathbb{E}_n^{\otimes}$ (see Proposition A.3.1.3.11). We will need the following geometric fact, which will be proven at the end of this section:

Lemma 4.1.15. *Let m and n be positive integers. Then the Kan complex $K_{m,n}$ defined above is homotopy equivalent to $\mathrm{Sing}(X)$, where X is a finite CW complex of dimension $\leq (m-1)(n-1)$.*

Lemma 4.1.16. *Let A be a coconnective $\mathbb{E}_1$-algebra over a field k such that $\pi_{-1} A \simeq 0$. Let M be a left A-module, let N be a right A-module. Suppose that A, M, and N are locally finite, and that $\pi_i M \simeq \pi_i N \simeq 0$ for $i > 0$. Then $N \otimes_A M$ is locally finite.*

Proof. Let $\{M(n)\}_{n \geq 0}$ be as in Lemma 3.1.16. Then $\pi_i(N \otimes_A M) \simeq \varinjlim \pi_i(N \otimes_A M(n))$. We have cofiber sequences

$$A \otimes_k V(n) \to M(n-1) \to M(n)$$

where $V(n) \in (\mathrm{Mod}_k)_{\leq -n}$, whence a cofiber sequences

$$N \otimes_k V(n) \to N \otimes_A M(n-1) \to N \otimes_A M(n)$$

in Mod_k. Since each $\pi_i V(n)$ is finite-dimensional, the homotopy groups of $N \otimes_k V(n)$ are finite-dimensional. It follows by induction on n that $N \otimes_A M(n)$ is locally finite. Since $\pi_i(N \otimes_k V(n)) \simeq 0$ for $i > -n$, the maps $\pi_i(N \otimes_A M(n-1)) \to \pi_i(N \otimes_A M(n))$ are bijective for $i > -n+1$. It follows that $\pi_i(N \otimes_A M) \simeq \varinjlim \pi_i(N \otimes_A M(n))$ is also a finite dimensional vector space over k. $\square$

Lemma 4.1.17. *Let k be a field and let $n \geq 0$. The functor $\int : \mathrm{Alg}_k^{(n)} \to \mathrm{Alg}^{(1)}$ preserves small sifted colimits.*

Proof. Let $\Theta : \mathrm{Alg}^{(1)} \to \mathrm{Mod}_{\mathrm{Sp}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})_{\mathrm{Sp}/}$ be the fully faithful embedding $B \mapsto \mathrm{LMod}_B$ of Theorem A.6.3.5.5. To prove that the functor $\int$ preseves small sifted colimits, it will suffice to show that the composite functor $\Theta \circ \int$ preserves small sifted colimits. Since every sifted simplicial set is contractible, it suffices to show that the induced map $\mathrm{Alg}_k^{(n)} \to \mathrm{Mod}_{\mathrm{Sp}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ preserves small sifted colimits (Proposition T.4.4.2.9). Using Theorem A.7.3.5.14, we see that this functor classifies the stable envelope of the Cartesian fibration θ : $\mathrm{Fun}(\Delta^1, \mathrm{Alg}_k^{(n)}) \to \mathrm{Fun}(\{1\}, \mathrm{Alg}_k^{(n)})$, classified by the functor $\xi : \mathrm{Alg}_k^{(n)} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ given informally by $A \mapsto (\mathrm{Alg}_k^{(n)})_{/A}$. It will therefore suffice to show that ξ preserves small sifted colimits. Using Theorem T.5.5.3.18, we are reduced to showing that the composite functor $\mathrm{Alg}_k^{(n)} \xrightarrow{\xi} \mathcal{P}\mathrm{r}^{\mathrm{L}} \simeq \mathcal{P}\mathrm{r}^{\mathrm{R}op} \to \widehat{\mathcal{C}\mathrm{at}}_{\infty}^{op}$ preserves small sifted colimits. This functor also classifies the forgetful functor θ (this time as a Cartesian fibration). Fix a sifted ∞-category K and a colimit diagram $\overline{f} : K^{\triangleright} \to \mathrm{Alg}_k^{(n)}$; we wish to show that the Cartesian fibration $\theta' : \mathrm{Fun}(\Delta^1, \mathrm{Alg}_k^{(n)}) \times_{\mathrm{Fun}(\{1\}, \mathrm{Alg}_k^{(n)})} K^{\triangleright}$ is classified by limit diagram $(K^{\triangleright})^{op} \to \widehat{\mathcal{C}\mathrm{at}}_{\infty}$. Let $A \in \mathrm{Alg}_k^{(n)}$ denote the image under $\overline{f}$ of the cone point of $K^{\triangleright}$, and for each vertex $v \in K$ let $A_v = \overline{f}(v)$. According to Lemma VII.5.17, it will suffice to verify the following:

(*a*) The pullback functors $q_v : (\mathrm{Alg}_k^{(n)})_{/A} \to (\mathrm{Alg}_k^{(n)})_{/A_v}$, given by $B \mapsto A_v \times_A B$, are jointly conservative. Since K is nonempty, it will suffice to show that for each $v \in K$, the pullback q_v is conservative. To this end, suppose we are given a map $\alpha : B \to B'$ in $(\mathrm{Alg}_k^{(n)})_{/A}$ such that $q_v(\alpha)$ is an equivalence. Since the fiber of α (as a map of spectra) is equivalent to the fiber of $q_v(\alpha)$ (by virtue of Corollary A.3.2.2.5), we conclude that α is an equivalence as well.

(*b*) Let $h \in \mathrm{Fun}_{\mathrm{Alg}_k^{(n)}}(K, \mathrm{Fun}(\Delta^1, \mathrm{Alg}_k^{(n)}))$ be a map which carries each edge of K to a θ-Cartesian morphism in the ∞-category $\mathrm{Fun}(\Delta^1, \mathrm{Alg}_k^{(n)})$, corresponding to a natural transformation $\{B_v \to A_v\}_{v \in K}$, and let $\overline{h} \in \mathrm{Fun}_{\mathrm{Alg}_k^{(n)}}(K^{\triangleright}, \mathrm{Alg}_k^{(n)})$ be an θ-colimit diagram extending h; we wish to show that $\overline{h}$ carries each edge of $K^{\triangleright}$ to a θ-Cartesian morphism in $\mathrm{Fun}(\Delta^1, \mathrm{Alg}_k^{(n)})$. Unwinding the definitions, we must show that if $B = \varinjlim B_v$, then for each $v \in K$ the diagram σ :

$$\begin{tikzcd} B_v \arrow[r] \arrow[d] & B \arrow[d] \\ A_v \arrow[r] & A \end{tikzcd}$$

is a pullback square in $\mathrm{Alg}_k^{(n)}$. For each $v \in K$, let I_v denote the fiber of the map $B_v \to A_v$ in Mod_k, and let I be the fiber of the map $B \to A$; we wish to show that each of the canonical maps $I_v \to I$ is an equivalence in Mod_k. Our assumption on h guarantees that the diagram $v \mapsto I_v$ carries each edge of K to an equivalence in Mod_k. It will therefore suffice to show that the canonical map $\varinjlim I_k \to I$ is an equivalence in Mod_k. Since Mod_k is stable, the formation of fibers commutes with colimits; it will therefore suffice to show that A and B are colimits of the diagrams $\{A_v\}_{v \in K}$ and $\{B_v\}_{v \in K}$ in Mod_k, respectively. Since K is sifted, this follows from the fact that the forgetful functor $\mathrm{Alg}_k^{(n)} \to \mathrm{Mod}_k$ preserves sifted colimits (Proposition A.3.2.3.1).

□

Remark 4.1.18. Let $\mathcal{C}$ be a presentable ∞-category equipped with an $\mathbb{E}_n$-monoidal structure, and if $n > 0$ assume that the tensor product on $\mathcal{C}$ preserves coproducts separately in each variable. Let $K_{m,n}$ be as in the statement of Lemma 4.1.15, and let $K_{m,n}(S)$ denote the fiber product $K_{m,n} \times_{\mathrm{N}(\mathcal{J})} \mathrm{N}(\mathcal{J}(S))$. According to Proposition A.3.1.3.11, the free algebra functor $\mathrm{Free}^{(n)} : \mathcal{C} \to \mathrm{Alg}_{/\mathbb{E}_n}(\mathcal{C})$ carries an object $X \in \mathcal{C}$ to the coproduct $\coprod_{m \geq 0} \mathrm{Sym}^{(n)}_{\mathbb{E}_n}(X)$, where $\mathrm{Sym}^{(n)}_{\mathbb{E}_n}(X)$ denotes the colimit of a diagram $\phi_X : K_{m,n} \to \mathcal{C}$.

Suppose now that $X \in \mathcal{C}$ is a coproduct of objects $\{X_s\}_{s \in S}$. Let $\mathcal{J}$ be the subcategory of $\mathcal{F}\mathrm{in}_*$ consisting of the object $\langle m \rangle$ and its automorphisms. Let $\mathcal{J}(S)$ be the category whose objects are maps of sets $\{1, \ldots, m\} \to S$, and whose morphisms are given by commutative diagrams

$$\begin{tikzcd} \{1,\ldots,m\} \arrow[rr, "\nu"] \arrow[dr] & & \{1,\ldots,m\} \arrow[dl] \\ & S & \end{tikzcd}$$

where ν is bijective. There is an evident forgetful functor $q : \mathcal{J}(S) \to \mathcal{J}$. Then ϕ_X can be identified with the left Kan extension (along q) of a functor $\phi_{\{X_s\}_{s \in S}} : \mathcal{J}(S) \to \mathcal{J}$, where $\overline{\beta}$ carries an operation $\gamma : \langle m \rangle \to \langle 1 \rangle$ in $\mathbb{E}_n^{\otimes}$ and a map $\alpha : \langle m \rangle^{\circ} \to S$ to the object $\gamma_!(X_{\alpha(1)}, \ldots, X_{\alpha(m)}) \in \mathcal{C}$ (here $\gamma_! : \mathcal{C}^m \to \mathcal{C}$ denotes the functor determined by the $\mathbb{E}_n$-monoidal structure on $\mathcal{C}$).

For every map $\mu : S \to \mathbf{Z}_{\geq 0}$ satisfying $\sum_{s \in S} \mu(s) = m < \infty$, let $\mathcal{J}(S, \mu)$ denote the full subcategory of $\mathcal{J}(S)$ spanned by those maps $\alpha : \langle m \rangle^{\circ} \to S$ such that $\alpha^{-1}\{s\}$ has cardinality $\mu(s)$, and let $K_{m,n}(S, \mu)$ denote the fiber product $K_{m,n}(S) \times_{\mathrm{N}(\mathcal{J}(S))} \mathrm{N}(\mathcal{J}(S, \mu))$. Then $K_{m,n}(S)$ is a disjoint union of the Kan complexes $K_{m,n}(S, \mu)$. It follows that $\mathrm{Sym}^m_{\mathbb{E}_n}(X)$ is a coproduct $\coprod_{\mu} \varinjlim \overline{\beta} | K_{m,n}(\mu)$. Note that if $T \subseteq S$ and $\mu(s) = 0$ for $s \notin T$, then there is a canonical equivalence

$$\phi_{\{X_s\}_{s \in S}} | K_{m,n}(S, \mu) \simeq \phi_{\{X_s\}_{s \in T}} | K_{m,n}(T, \mu | T).$$

It follows that if the cardinality of S is larger than m, then $\mathrm{Sym}^m_{\mathbb{E}_n}(X)$ can be written as a coproduct of objects, each of which is a summand of $\mathrm{Sym}^m_{\mathbb{E}_n}(\coprod_{t \in S - \{s\}} X_t)$ for some $s \in S$.

Proof of Proposition 4.1.13. We will assume $n > 0$ (otherwise the result is trivial). Let $\phi'_0 : \mathrm{Free}^{(n)}(V) \to A$ be the map of $\mathbb{E}_n$-algebras induced by the zero map $V \to A$, so that A' can be identified with the colimit of the coequalizer diagram

$$\mathrm{Free}^{(n)}(V) \underset{\phi'_0}{\overset{\phi'}{\rightrightarrows}} A.$$

given by a map $u_0 : \mathrm{N}(\mathbf{\Delta}_{s, \leq 1})^{op} \to \mathrm{Alg}^{(n)}_k$. Let $u : \mathrm{N}(\mathbf{\Delta})^{op} \to \mathrm{Alg}^{(n)}_k$ be a left Kan extension of u_0 along the inclusion $\mathrm{N}(\mathbf{\Delta}_{s, \leq 1})^{op} \hookrightarrow \mathrm{N}(\mathbf{\Delta})^{op}$, so that u determines a simplicial object $A_{\bullet}$ in $\mathrm{Alg}^{(n)}_k$ with $A' \simeq |A_{\bullet}|$ and $A_p \simeq A \coprod \mathrm{Free}(V^p)$ for all $p \geq 0$. Let $R = \int A$ so that $\mathrm{LMod}_R \simeq \mathrm{Mod}^{\mathbb{E}_n}_A(\mathrm{Mod}_k)$ is equipped with an $\mathbb{E}_n$-monoidal structure, where the tensor product is given by the relative tensor product over A. We will need the following estimate:

$(*)$ For each integer $a > 0$, the iterated tensor product $R^{\otimes a}$ belongs to $(\mathrm{Mod}_k)_{\leq 0}$. Moreover, if A and $\int A$ are locally finite, then $R^{\otimes a}$ is locally finite.

Suppose first that $n = 1$, so that $R \simeq A \otimes_k A^{\mathrm{rev}}$ (Example 4.1.10). Then $R^{\otimes a}$ can be identified with an iterated tensor product $A \otimes_k A \otimes_k \cdots \otimes_k A$ and assertion $(*)$ is obvious. We may therefore assume that $n \geq 2$. In this case, A is $m \geq n \geq 2$-coconnective, so the desired result follows from Corollary VIII.4.1.11 and Lemma 4.1.16.

Let $V' = R \otimes_k V$ denote the image of V in $\mathrm{Mod}^{\mathbb{E}_n}_A$. For each $p \geq 0$, Corollary A.3.4.1.5 allows us to identify A_p with the free $\mathbb{E}_n$-algebra generated by V' in the $\mathbb{E}_n$-monoidal ∞-category $\mathrm{Mod}^{\mathbb{E}_n}_A$. Using Proposition A.3.1.3.11, we obtain an equivalence

$$A_p \simeq \bigoplus_{a \geq 0} \mathrm{Sym}^a_{\mathbb{E}_n} V'^p$$

(where the symmetric powers are computed in $\mathrm{Mod}_A^{\mathbb{E}_n}$). Let Q denote the cofiber of the map $A \rightarrow A'$ in the stable ∞-category $\mathrm{Mod}_A^{\mathbb{E}_n}(\mathrm{Mod}_k)$, so that Q is given by the geometric realization of a simplicial object $Q_\bullet$ with $Q_p \simeq \bigoplus_{a>0} \mathrm{Sym}^a_{\mathbb{E}_n} V'^p$. To show that A' is m-coconnective, it will suffice to show that $\pi_i Q \simeq 0$ for $i > -m$.

Using Remark A.1.2.4.5, we obtain a spectral sequence $\{E_r^{p,q}\}_{r\geq 1}$ converging to $\pi_{p+q}Q$, where $E_1^{*,q}$ is the normalized chain complex associated to the simplicial k-vector space

$$[p] \mapsto \bigoplus_{a>0} \pi_q \mathrm{Sym}^a_{\mathbb{E}_n}(V'^p).$$

It follows from Remark 4.1.18 that the summand $\pi_q \mathrm{Sym}^a_{\mathbb{E}_n}(V'^p)$ lies in the image of the degeneracy maps of this simplicial vector space whenever $p > a$.

Note that $\mathrm{Sym}^a_{\mathbb{E}_n}(V'^p)$ is the colimit of a diagram $\phi_{V'} : K_{a,n} \rightarrow \mathrm{Mod}_A^{\mathbb{E}_n}(\mathrm{Mod}_k)$ whose value on each vertex is given by $(V'^p)^{\otimes a}$, where $K_{a,n}$ is the Kan complex appearing in the statement of Lemma 4.1.15. Since $\pi_i V \simeq 0$ for $i > -m-1$, condition $(*)$ guarantees that $\pi_i (V'^p)^{\otimes a} \simeq 0$ for $i > (-m-1)a$, and that the homotopy groups of $(V'^p)^{\otimes a}$ are finitely generated vector spaces over k provided that A and R are locally finite. Combining this with Lemma 4.1.15, we deduce that if $a > 0$, then $\pi_q \mathrm{Sym}^a_{\mathbb{E}_n}(V'^p)$ vanishes for

$$q > (a-1)(n-1) + (-m-1)a = 1 - 2a - m + (n-m)(a-1)$$

and thus for $q > 1 - 2a - m$.

If the vector space $E_1^{p,q}$ is nonzero, there must be an integer $a > 0$ such that $p \leq a$ and $q \leq 1 - 2a - m$, so that $p + q \leq 1 - a - m \leq -m$. This proves that $\pi_i Q \simeq 0$ for $i > p + q$, so that A' is m-coconnective. For any integer i, the inequality $i = p + q \leq 1 - a - m$ implies that a is bounded above by $1 - m - i$, so that $\pi_i Q$ admits a finite filtration whose associated graded vector space consists of subquotients of $\pi_{i-p} \mathrm{Sym}^a_{\mathbb{E}_n} V'^p$ where $a \leq 1 - m - i$ and $p \leq a$. It follows that if A and R are locally finite, then Q is also locally finite, so that A' is locally finite.

To complete the proof, we must show that $\int A'$ is $(1+m-n)$-coconnective, and that $\int A'$ is locally finite if A and $\int A$ are locally finite. According to Lemma 4.1.17, $\int A'$ can be identified with the geometric realization of the simplicial $\mathbb{E}_1$-ring $\int A_\bullet$. Let B be an arbitrary $\mathbb{E}_n$-ring, let $W \in \mathrm{Mod}_k$ and let $W' = (\int B) \otimes_k W$ denote the image of W in $\mathrm{Mod}_B^{\mathbb{E}_n} \simeq \mathrm{LMod}_{\int B}$. Then the coproduct $B \coprod \mathrm{Free}^{(n)}(W)$ can be identified with the free $\mathbb{E}_n$-algebra in $\mathrm{Mod}_B^{\mathbb{E}_n}$ generated by W', which is given by $\bigoplus_{a\geq 0} \mathrm{Sym}^a_{\mathbb{E}_n}(W')$. If we let $Z(W)$ denote the cofiber in Mod_k of the map $B \rightarrow B \coprod \mathrm{Free}^{(n)}(W)$, then we obtain an equivalence $\int B \simeq \varinjlim_{b\mapsto\infty} \Omega^b Z(\Sigma^b k)$. Taking $B = A_p$, we obtain an equivalence

$$\int A_p \simeq \varinjlim_{b\mapsto\infty} \Omega^b (\bigoplus_{a\geq 0} \mathrm{Sym}^a_{\mathbb{E}_n}(V'^p \oplus \Sigma^b k))/(\bigoplus_{a\geq 0} \mathrm{Sym}^a_{\mathbb{E}_n}(V'^p)).$$

Remark 4.1.18 gives a canonical decomposition

$$\mathrm{Sym}^a_{\mathbb{E}_n}(V'^p \oplus \Sigma^b k) \simeq \bigoplus_{a=a'+a''} F_{a',a''}(V'^p, \Sigma^b k).$$

Note that $F_{a-1,1}$ is an exact functor of the second variable, and that if $a'' \geq 2$, then the colimit

$$\varinjlim \Omega^b F_{a',a''}(V'^p, \Sigma^b k)$$

vanishes. We therefore obtain an equivalence $\int A_p \simeq \bigoplus_{a>0} F_{a-1,1}(V'^p, k)$. Unwinding the definitions, we see that $F_{a-1,1}(X, Y)$ is given by the colimit of a diagram $\widetilde{K}_{a,n} \rightarrow \mathrm{Mod}_A^{\mathbb{E}_n}(\mathrm{Mod}_k)$, which carries each vertex to the iterated tensor product $(V'^p)^{\otimes a-1} \otimes_A R$, here $\widetilde{K}_{a,n}$ is a finite-sheeted covering space of $K_{m,n}$ and therefore equivalent to the singular complex of a finite CW complex of dimension $\leq (a-1)(n-1)$ (Lemma 4.1.15). Since condition $(*)$ implies that $\pi_i (V'^p)^{\otimes a-1} \otimes_A R$ vanishes for $i > (-m-1)(a-1)$, we conclude that $\pi_q F_{a-1,1}(V'^p, k) \simeq 0$ for $q > (-m-1)(a-1) + (n-1)(a-1) = (n-m-2)(a-1)$ and therefore for $q > 2-2a$. Moreover, if A and $\int A$ are locally finite, then each $\pi_q F_{a-1,1}(V'^p, k)$ is a finite dimensional vector space over k.

Let Q' denote the spectrum given by the cofiber of the map $\int A \to \int A'$, so that Q' is the geometric realization of a simplicial spectrum $Q'_\bullet$ given by

$$Q'_p \simeq \bigoplus_{a \geq 2} F_{a-1,1}(V'^p, k).$$

Using Remark A.1.2.4.5, we obtain a spectral sequence $\{E_r'^{p,q}\}_{r \geq 1}$ converging to $\pi_{p+q} Q'$, where $E_1'^{*,q}$ is the normalized chain complex associated to the simplicial k-vector space

$$[p] \mapsto \bigoplus_{a \geq 2} \pi_q F_{a-1,1}(V'^p, k).$$

Arguing as above, we deduce that the summand $\pi_q F_{a-1,1}(V'^p, k)$ lies in the image of the degeneracy maps of this vector space whenever $p \geq a$. It follows that if $E_1'^{p,q}$ is nonzero, then there exists an integer $a \geq 2$ such that $p < a$ and $q \leq 2 - 2a$, so that $p + q < 2 - a \leq 0$. It follows that $\pi_i Q' \simeq 0$ for $i \geq 0$, from which we immediately conclude that $\int A'$ is 1-coconnective. For any integer i, the inequality $i = p + q \leq 2 - a$ implies that a is bounded above by $2 - i$, so that $\pi_i Q'$ admits a finite filtration whose associated graded vector space consists of subquotients of $\pi_{i-p} F_{a-1,1}(V'^p, k)$ where $2 \leq a \leq 2 - i$ and $p < a$. If A and $\int A$ are locally finite, then these subquotients are necessarily finite dimensional, so that each $\pi_i Q'$ is a finite dimensional vector space. It then follows that $\int A'$ is locally finite as desired. □

Proof of Proposition 4.1.14. Let A be an $\mathbb{E}_n$-algebra over a field k, and assume that A is m-coconnective for $m \geq n$. We wish to show that $\int A$ is $(m-n+1)$-coconnective. The result is trivial if $n = 0$ (Example 4.1.9); we will therefore assume that $n \geq 1$. We construct a sequence of maps $A(0) \to A(1) \to \cdots$ in $(\mathrm{Alg}_k^{(n)})_{/A}$ by induction. Let $A(0) = k$. Assuming that $A(i)$ has already been defined, we let $V(i)$ denote the fiber of the map $A(i) \to A$ (in Mod_k) and define $A(i+1)$ so that there is a pushout square

$$\begin{tikzcd}
\mathrm{Free}^{(n)}(V(i)) \arrow[r, "\phi'"] \arrow[d, "\epsilon"] & A(i) \arrow[d] \\
k \arrow[r] & A(i+1)
\end{tikzcd}$$

as in the statement of Proposition 4.1.13. We prove the following statements by induction on i:

- (a_i) The $\mathbb{E}_n$-algebra $A(i)$ is m-coconnective.
- (b_i) The map $\pi_{-m} A(i) \to \pi_{-m} A$ is injective.
- (c_i) The $\mathbb{E}_1$-algebra $\int A$ is $(m+1-n)$-coconnective.
- (d_i) We have $\pi_j V(i) \simeq 0$ for $j \geq -m$.

It is clear that conditions (a_0), (b_0), and (c_0) are satisfied. Note that (a_i) and (b_i) imply (d_i) and that (a_i), (c_i) and (d_i) imply (a_{i+1}) and (c_{i+1}) by Proposition 4.1.13. It will therefore suffice to show that (a_i), (b_i), (c_i), and (d_i) imply condition (b_{i+1}). As in the proof of Proposition 4.1.13, we can identify $A(i+1)$ with the geometric realization of a simplicial object $A_\bullet$ of $\mathrm{Alg}_k^{(n)}$, with $A_p \simeq A(i) \coprod \mathrm{Free}^{(n)}(V(i)^p)$. Let Q denote the cofiber of the map $A(i) \to A(i+1)$ (as an object of $\mathrm{Mod}_{A_i}^{\mathbb{E}_n}(k)$) we have a canonical map $\phi : Q \to \mathrm{cofib}(A(i) \to A) \simeq V(i)[1]$. We wish to prove that ϕ induces an injection $\pi_{-n} Q \to \pi_{-n-1} V(i)$, which follows immediately by inspecting the spectral sequence $\{E_r^{p,q}\}_{r \geq 1}$ appearing in the proof of Proposition 4.1.13.

We now claim that the canonical map $\theta : \varinjlim A(i) \to A$ is an equivalence. Combining this with assertions (c_i) and Lemma 4.1.17, we conclude that $\int A \simeq \varinjlim \int A(i)$ is $(m+1-n)$-connective as desired. To prove that θ is an equivalence, we note that the image of $A(i)$ in Mod_k can be identified with the colimit of the sequence

$$k \simeq A(0) \to A(0)/V(0) \to A(1) \to A(1)/V(1) \to \cdots,$$

where each cofiber $A(i)/V(i)$ is equivalent to A. $\square$

We complete this section by giving the proof of Lemma 4.1.15.

Lemma 4.1.19. *Fix an integer $b \geq 0$, and let Q_b denote the set of sequences $(I_1, \ldots, I_b)$, where each $I_j \subseteq (-1,1)$ is a closed interval, and we have $x < y$ whenever $x \in I_i$, $y \in I_j$, and $i < j$. We regard Q as a partially ordered set, where $(I_1, \ldots, I_b) \leq (I'_1, \ldots, I'_b)$ if $I_j \subseteq I'_j$ for $1 \leq i \leq j$. Then the nerve $\mathrm{N}(Q_b)$ is weakly contractible.*

Proof. The proof proceeds by induction on b. If $b = 0$, then Q_b has a single element and there is nothing to prove. Otherwise, we observe that "forgetting" the last coordinate induces a Cartesian fibration $q : \mathrm{N}(Q_b) \to \mathrm{N}(Q_{b-1})$. We will prove that the fibers of q are weakly contractible, so q is left cofinal (Lemma T.4.1.3.2) and therefore a weak homotopy equivalence. Fix an element $x = ([t_1, t'_1], [t_2, t'_2], \ldots, [t_{b-1}.t'_{b-1}]) \in Q_{b-1}$. Then $q^{-1}\{x\}$ can be identified with the nerve of the partially ordered set $Q' = \{(t_b, t'_b) : t'_{b-1} < t_b < t'_b < 1\}$, where $(t_b, t'_b) \leq (s_b, s'_b)$ if $t_b \geq s_b$ and $t'_b \leq s'_b$.

The map $(t_b, t'_b) \mapsto t'_b$ is a monotone map from Q' to the open interval $(t'_{b-1}, 1)$. This map determines a coCartesian fibration $q' : \mathrm{N}(Q') \to \mathrm{N}(t'_{b-1}, 1)$. The fiber of q' over a point s can be identified with the *opposite* of the nerve of the interval (t'_{b-1}, s), and is therefore weakly contractible. It follows that q' is a weak homotopy equivalence, so that $\mathrm{N}(Q')$ is weakly contractible as desired. $\square$

Proof of Lemma 4.1.15. For every topological space X, let $\mathrm{Sym}^m(X)$ denote the quotient of X^m by the action of the symmetric group Σ_m, and let $\mathrm{Conf}_m(X)$ denote the subspace of $\mathrm{Sym}^m(X)$ corresponding to m-tuples of *distinct* points in X. Let $\square^n = (-1,1)^n$ denote an open cube of dimension n. Using a variant of Lemma A.5.1.1.3, we obtain a homotopy equivalence $K_{m,n} \simeq \mathrm{Sing}(\mathrm{Conf}_m(\square^n))$. It will therefore suffice to show that the configuration space $\mathrm{Conf}_m(\square^n)$ is homotopy equivalent to a finite CW complex of dimension $\leq (m-1)(n-1)$. If $n = 1$, then $\mathrm{Conf}_m(\square^n)$ is contractible and there is nothing to prove. We prove the result in general by induction on n. Let us therefore assume that $K_{m',n-1}$ is equivalent to the singular complex of a CW complex of dimension $\leq (m'-1)(n-2)$ for every integer $m' \geq 1$.

Let us identify $\square^n$ with a product $\square^{n-1} \times (-1,1)$, and let $p_0 : \square^n \to \square^{n-1}$ and $p_1 : \square^n \to (-1,1)$ be the projection maps. If $I \subseteq (-1,1)$ is a disjoint union of finitely many closed intervals (with nonempty interiors), we let $[t] \in \pi_0 I$ denote the connected component containing t for each $t \in I$. Then $\pi_0 I$ inherits a linear ordering, with $[t] < [t']$ if and only if $t < t'$ and $[t] \neq [t']$. Let P denote the partially ordered set of triples $(I, \sim, \mu)$, where $I \subseteq (-1,1)$ is a nonempty disjoint union of finitely many closed intervals, $\sim$ is an equivalence relation on $\pi_0 I$ such that $x < y < z$ and $x \sim z$ implies $x \sim y \sim z$, and $\mu : \pi_0 I \to \mathbf{Z}_{>0}$ is a positive integer-valued function such that $\sum_{x \in \pi_0 I} \mu(x) = m$. We regard $(I, \sim, \mu)$ as a partially ordered set, with $(I, \sim, \mu) \leq (I', \sim', \mu')$ if $I \subseteq I'$, $\mu'(x) = \sum_y \mu(y)$ where the sum is taken over all $y \in \pi_0 I$ contained in x, and $[s] \sim' [t]$ implies $[s] \sim [t]$ for all $s, t \in I$. For every pair $(I, \sim, \mu) \in P$, we let $Z(I, \sim, \mu)$ be the open subset of $\mathrm{Conf}_m(\square^n)$ consisting of subsets $S \subseteq \square^n$ which are contained in $\square^{n-1} \times I^{\circ}$ (here I° denotes the interior of I), have the property that if $s, s' \in S$ and $[p_1(s)] \sim [p_1(s')]$, then either $s = s'$ or $p_0(s) \neq p_0(s')$, and satisfy $\mu(x) = |\{s \in S : p_1(s) \in x\}|$ for $x \in \pi_0 I$.

We next claim:

$(*)$ The Kan complex $\mathrm{Sing}(\mathrm{Conf}_m(\square^n))$ is a homotopy colimit of the diagram of simplicial sets

$$\{\mathrm{Sing}(Z(I, \sim, \mu))\}_{(I,\sim,\mu) \in P}.$$

To prove this, it will suffice (by Theorem A.A.3.1) to show that for each point $S \in \mathrm{Conf}_m(\square^n)$, the partially ordered set $P_S = \{(I, \sim, \mu) \in P : S \in Z(I, \sim, \mu)\}$ has weakly contractible nerve. Let Q denote the collection of all equivalence relations $\sim$ on S with the following properties:

(i) If $p_1(s) \leq p_1(s') \leq p_1(s'')$ and $s \sim s''$, then $s \sim s' \sim s''$.

(ii) If $s \sim s'$, then either $s = s'$ or $p_0(s) \neq p_0(s')$.

We regard Q as a partially ordered set with respect to refinement. Pullback of equivalence relations determines a forgetful functor $\phi : \mathrm{N}(P_S) \to \mathrm{N}(Q)^{op}$. It is easy to see that μ is a Cartesian fibration. The simplicial set $\mathrm{N}(Q)$ is weakly contractible, since Q has a smallest element (given by the equivalence relation where $s \sim s'$ if and only if $s = s'$). We will complete the proof of $(*)$ by showing that the fibers of ϕ are weakly contractible, so that ϕ is left cofinal (Lemma T.4.1.3.2) and therefore a weak homotopy equivalence.

Fix an equivalence relation $\sim \in Q$. Unwinding the definitions, we see that $\phi^{-1}\{\sim\}$ can be identified with the nerve of the partially ordered set R consisting of those subsets $I \subseteq (-1, 1)$ satisfying the following conditions:

(a) The set I is a disjoint union of closed intervals in $(-1, 1)$.

(b) The set I contains $p_1(S)$, and p_1 induces a surjection $S \to \pi_0 I$.

(c) If $p_1(s)$ and $p_1(s')$ belong to the same connected component of I, then $s \sim s'$.

To see that $\mathrm{N}(R)$ is contractible, it suffices to observe that the partially ordered set R^{op} is filtered: it has a cofinal subset given by sets of the form $\bigcup_{s \in S}[p_1(s) - \epsilon, p_1(s) + \epsilon]$ for $\epsilon > 0$. This completes the proof of $(*)$.

We define a category $\mathcal{J}$ as follows:

- An object of $\mathcal{J}$ is a triple $([a], \sim, \mu)$ where $[a] \in \mathbf{\Delta}$, $\sim$ is an equivalence relation on $[a]$ such that $i < j < k$ and $i \sim k$ implies that $i \sim j \sim k$, and $\mu : [a] \to \mathbf{Z}_{>0}$ is a function satisfying $m = \sum_{0 \leq i \leq a} \mu(i)$.
- A morphism from $([a], \sim, \mu)$ to $([a'], \sim', \mu')$ in $\mathcal{J}$ is a nondecreasing map $\alpha : [a] \to [a']$ such that $\alpha(j) \sim \alpha(j')$ implies $j \sim j'$ and $\mu'(j) = \sum_{\alpha(i) = j} \mu(i)$.

There is an evident forgetful functor $q : P \to \mathcal{J}$, which carries a pair $(I, \sim, \mu)$ to $(\pi_0 I, \sim, \mu)$ where we abuse notation by identifying $\pi_0 I$ with the linearly ordered set $[a]$ for some $a \geq 0$. Let $Z' : \mathcal{J} \to \mathcal{S}\mathrm{et}_{\Delta}$ be a homotopy left Kan extension of Z along q. For each object $([a], \sim, \mu) \in \mathcal{J}$, we can identify $Z'([a], \sim, \mu)$ with the homotopy colimit of the diagram $Z|P_{[a], \sim, \mu}$, where $P_{[a], \sim, \mu}$ denotes the partially ordered set of quadruples $(I, \lambda, \sim', \mu')$ where $I \subseteq (-1, 1)$ is a disjoint union of closed intervals, $\lambda : \pi_0 I \to [a]$ is nondecreasing surjection, $\sim'$ is an equivalence relation on $\pi_0 I$ such that $\lambda([t]) \sim \lambda([t'])$ implies $[t] \sim' [t']$, and $\mu' : \pi_0 I \to \mathbf{Z}_{>0}$ is a map satisfying $\mu(i) = \sum_{\lambda(x) = i} \mu'(x)$ for $0 \leq i \leq a$. Let $P'_{[a], \sim, \mu}$ be the subset of $P_{[a], \sim, \mu}$ consisting of those quadruples $(I, \lambda, \sim', \mu')$ where λ is a bijection and $\sim'$ is the pullback of $\sim$ along λ. The inclusion $\mathrm{N}(P'_{[a], \sim, \mu}) \to \mathrm{N}(P_{[a], \sim, \mu})$ admits a left adjoint and is therefore left cofinal. It follows that $Z'([a], \sim, \mu)$ can be identified with a homotopy colimit of the diagram $Z|P'_{[a], \sim, \mu}$. Note that Z carries each morphism in $P'_{[a], \sim, \mu}$ to a homotopy equivalence of Kan complexes. Since $P'_{[a], \sim}$ is weakly contractible (Lemma 4.1.19), we conclude that the map $Z(I, \sim', \mu') \to Z'([a], \sim, \mu)$ is a weak homotopy equivalence for any $(I, \lambda, \sim', \mu') \in P'_{[a], \sim, \mu}$.

It follows from condition $(*)$ that $\mathrm{Sing}(K_{m,n})$ can be identified with a homotopy colimit of the diagram Z'. We may assume without loss of generality that Z' takes values in Kan complexes, so that Z' determines a map of ∞-categories $\mathrm{N}(\mathcal{J}) \to \mathcal{S}$. We will abuse notation by denoting this map also by Z'. Note that if $([a], \sim, \mu) \in \mathcal{J}$, then $m = \sum_{0 \leq i \leq a} \mu(i) \geq a + 1$. For each object $([a], \sim, \mu) \in \mathcal{J}$, we define the *complexity* $d([a], \sim, \mu)$ to be the sum $|[a]/\sim| + \sum_{0 \leq i \leq a}(\mu(i) - 1)$. Since $[a]/\sim$ has at least one element, $d([a], \sim, \mu)$ is bounded below by 1 and bounded above by $|[a]| + \sum_{0 \leq i \leq a} \mu(i) - 1 = \sum_{0 \leq i \leq a} \mu(i) = m$. Note that for every nonidentity morphism $J \to J'$ in $\mathcal{J}$, we have $d(J) < d(J')$. It follows that every nondegenerate simplex in $\mathrm{N}(\mathcal{J})$, corresponding to a sequence of nonidentity morphisms $J_0 \to \cdots \to J_b$ in $\mathcal{J}$, is bounded in length by $b \leq m - 1$. It follows immediately that the simplicial set $\mathrm{N}(\mathcal{J})$ has only finitely many nondegenerate simplices. We will prove that for every finite simplicial subset $A \subseteq \mathrm{N}(\mathcal{J})$, the colimit of the diagram $Z'|A$ is homotopy equivalent to the singular complex of finite CW complex of dimension $\leq (n-1)(m-1)$. This is obvious when $A = \emptyset$, and when $A = \mathrm{N}(\mathcal{J})$ it implies the desired result. To carry out the inductive step, assume that A is nonempty so that there is a pushout diagram

$$\begin{tikzcd} \partial \Delta^b \arrow[r] \arrow[d] & \Delta^b \arrow[d, "\sigma"] \\ A_0 \arrow[r] & A \end{tikzcd}$$

for some smaller simplicial subset $A_0 \subseteq \mathrm{N}(\mathcal{J})$. The simplex σ carries the initial vertex $0 \in \Delta^b$ to an object $([a], \sim, \mu) \in \mathcal{J}$, and we have a pushout diagram

$$\begin{tikzcd} Z'([a], \sim, \mu) \times \partial \Delta^b \arrow[r] \arrow[d] & Z'([a], \sim, \mu) \times \Delta^b \arrow[d] \\ \varinjlim(Z'|A_0) \arrow[r] & \varinjlim(Z'|A) \end{tikzcd}$$

where $\varinjlim(Z'|A_0)$ is homotopy equivalent to the singular complex of a finite CW complex of dimension $\leq (n-1)(m-1)$. Let $I \subseteq (-1, 1)$ be a finite union of $a+1$ closed intervals and identify $\pi_0 I$ with $[a]$, so that $Z'([a], \sim, \mu) \simeq Z(I, \sim, \mu)$ is homotopy equivalent to the product $\prod_{x \in [a]/\sim} \mathrm{Conf}_{\mu_x}(\square^{n-1})$, where $\mu_x = \sum_{i \in x} \mu(i)$. Using the inductive hypothesis, we deduce that $Z'([a], \sim, \mu)$ is homotopy equivalent to the nerve of a CW complex of dimension $(m-d)(n-2)$, where d is the cardinality of the quotient $[a]/\sim$. It follows that $\varinjlim(Z'|A)$ is homotopy equivalent to the singular complex of a CW complex having dimension at most the maximum of $(m-1)(n-1)$ and $(m-d)(n-2)+b \leq (m-1)(n-2)+(m-1) = (m-1)(n-1)$, as desired. $\square$

4.2 Twisted Arrow ∞-Categories

Let $\mathcal{C}$ be an ∞-category. Recall that a functor $X : \mathcal{C}^{op} \to \mathcal{S}$ is *representable* if there exists an object $C \in \mathcal{C}$ and a point $\eta \in F(C)$ such that evaluation on η induces a homotopy equivalence $\mathrm{Map}_{\mathcal{C}}(C', C) \to F(C')$ for each $C' \in \mathcal{C}'$. An ∞-categorical version of Yoneda's lemma asserts that there is a fully faithful embedding $j : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$ (Proposition T.5.1.3.1), whose essential image is the full subcategory of $\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$ spanned by the representable functors. The functor j classifies a map $\mu : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{S}$, given informally by the formula $(C, D) \mapsto \mathrm{Map}_{\mathcal{C}}(C, D)$. In [40], we gave an explicit construction of μ by choosing an equivalence of $\mathcal{C}$ with the nerve of a fibrant simplicial cateogry (see §T.5.1.3).

Our goal in this section is to give another construction of μ, which does not rely on the theory of simplicial categories. For this, we will need an ∞-categorical version of Construction 3.3.5: to any ∞-category $\mathcal{C}$, we will associate a new ∞-category $\mathrm{TwArr}(\mathcal{C})$, called the *twisted arrow ∞-category* of $\mathcal{C}$. Roughly speaking, the objects of $\mathrm{TwArr}(\mathcal{C})$ are morphisms $f : C \to D$ in $\mathcal{C}$, and morphisms in $\mathrm{TwArr}(\mathcal{C})$ are given by commutative diagrams

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d] & D \\ C' \arrow[r, "f'"] & D. \arrow[u] \end{tikzcd}$$

We will give a precise definition of $\mathrm{TwArr}(\mathcal{C})$ below (Construction 4.2.1) and prove that the construction $(f : C \to D) \mapsto (C, D)$ determines a right fibration $\lambda : \mathrm{TwArr}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{op}$ (Proposition 4.2.3). The right fibration λ is classified by a functor $\mu : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{S}$, which we can view in turn as a functor $\mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$. We will show that this functor is equivalent to the Yoneda embedding (Proposition 4.2.5); in particular, it is fully faithful and its essential image is the collection of representable functors $F : \mathcal{C}^{op} \to \mathcal{S}$.

The twisted arrow ∞-category $\mathrm{TwArr}(\mathcal{C})$ will play an important role when we discuss the bar construction in §4.3. For our applications, it is important to know that the construction $\mathcal{C} \mapsto \mathrm{TwArr}(\mathcal{C})$ is functorial and commutes with small limits. To prove this, it will be convenient to describe $\mathrm{TwArr}(\mathcal{C})$ by means of a universal property. We will discuss two such descriptions at the end of this section (see Corollary 4.2.11).

Construction 4.2.1. If I is a linearly ordered set, we let I^{op} denote the same set with the opposite ordering. If I and J are linearly ordered sets, we let $I \star J$ denote the coproduct $I \coprod J$, equipped with the unique linear ordering which restricts to the given linear orderings of I and J, and satisfies $i \leq j$ for $i \in I$ and $j \in J$. Let $\mathbf{\Delta}$ denote the category of combinatorial simplices: that is, the category whose objects sets of the form $[n] = \{0, 1, \ldots, n\}$ for $n \geq 0$, and whose morphisms are nondecreasing maps between such sets. Then $\mathbf{\Delta}$ is equivalent to the larger category consisting of all nonempty finite linearly ordered sets. The construction $I \mapsto I \star I^{op}$ determines a functor Q from the category $\mathbf{\Delta}$ to itself, given on objects by $[n] \mapsto [2n+1]$. If $\mathcal{C}$ is a simplicial set (regarded as a functor $\mathbf{\Delta}^{op} \to \mathcal{S}$), we let $\mathrm{TwArr}(\mathcal{C})$ denote the simplicial set given by

$$[n] \mapsto \mathcal{C}(Q[n]) = \mathcal{C}([2n+1]).$$

Let $\mathcal{C}$ be an ∞-category. By construction, the vertices of $\mathrm{TwArr}(\mathcal{C})$ are edges $f : C \to D$ in $\mathcal{C}$. More generally, the n-simplices of $\mathrm{TwArr}(\mathcal{C})$ are given by $(2n+1)$-simplices of $\mathcal{C}$, which it may be helpful to depict as diagrams

$$\begin{tikzcd} C_0 \arrow[r] \arrow[d] & C_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & C_n \arrow[d] \\ D_0 & D_1 \arrow[l] & \cdots \arrow[l] & D_n. \arrow[l] \end{tikzcd}$$

Example 4.2.2. Let $\mathcal{C}$ be an ordinary category, and let $\mathrm{TwArr}(\mathcal{C})$ be the twisted arrow category of $\mathcal{C}$ introduced in Construction 3.3.5. Then there is a canonical isomorphism of simplicial sets

$$\mathrm{N}(\mathrm{TwArr}(\mathcal{C})) \simeq \mathrm{TwArr}(\mathrm{N}(\mathcal{C})).$$

Consequently, we can think of Construction 4.2.1 as a generalization of Construction 3.3.5 to the ∞-categorical context.

Let $\mathcal{C}$ be an arbitrary simplicial set. For any linearly ordered set I, we have canonical inclusions

$$I \hookrightarrow I \star I^{op} \hookleftarrow I^{op}.$$

Composition with these inclusions determines maps of simplicial sets

$$\mathcal{C} \leftarrow \mathrm{TwArr}(\mathcal{C}) \rightarrow \mathcal{C}^{op}.$$

Proposition 4.2.3. *Let $\mathcal{C}$ be an ∞-category. Then the canonical map $\lambda : \mathrm{TwArr}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{op}$ is a right fibration of simplicial sets. In particular, $\mathrm{TwArr}(\mathcal{C})$ is also an ∞-category.*

Proof. We must show that the map λ has the right lifting property with respect to the inclusion of simplicial sets $\Lambda^n_i \hookrightarrow \Delta^n$ for $0 < i \leq n$. Unwinding the definitions, we must show that every lifting problem of the form

$$\begin{tikzcd} K \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \Delta^{2n+1} \arrow[r] \arrow[ur, dashed] & \Delta^0 \end{tikzcd}$$

admits a solution, where K denotes the simplicial subset of Δ^{2n+1} consisting of those faces σ which satisfy one of the following three conditions:

- The vertices of σ are contained in the set $\{0, \ldots, n\}$.
- The vertices of σ are contained in the set $\{n+1, \ldots, 2n+1\}$.

- There exists an integer $j \neq i$ such that $0 \leq j \leq n$ and neither j nor $2n+1-j$ is a vertex of σ.

Since $\mathcal{C}$ is an ∞-category, it will suffice to show that the inclusion $K \hookrightarrow \Delta^{2n+1}$ is an inner anodyne map of simplicial sets.

Let us say that a face σ of Δ^{2n+1} is *primary* if it does not belong to K and does not contain any vertex in the set $\{0, 1, \ldots, i-1\}$, and *secondary* if it does not belong to K and does contain a vertex in the set $\{0, 1, \ldots, i-1\}$. Let S be the collection of all simplices of Δ^{2n+1} which are either primary and do not contain the vertex i, or secondary and do not contain the vertex $2n+1-i$. If $\sigma \in S$, we let σ' denote the face obtained from σ by adding the vertex $2n+1-i$ if σ is primary, and by adding the vertex i if σ is secondary. Note that every face of Δ^{2n+1} either belongs to K, belongs to S, or has the form σ' for a unique $\sigma \in S$.

Choose an ordering $\{\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_m\}$ of S with the following properties:

- If $p \leq q$, then the dimension of σ_p is less than or equal to the dimension of σ_q.
- If $p \leq q$, the simplices σ_p and σ_q have the same dimension, and σ_q is primary, then σ_p is also primary.

For $0 \leq q \leq m$, let K_q denote the simplicial subset of Δ^{2n+1} obtained from K by adjoining the simplices σ_p and σ'_p for $1 \leq p \leq q$. We have a sequence of inclusions

$$K = K_0 \hookrightarrow K_1 \hookrightarrow \cdots \hookrightarrow K_m = \Delta^{2n+1}.$$

It will therefore suffice to show that each of the maps $K_{q-1} \hookrightarrow K_q$ is inner anodyne. Let d denote the dimension of the simplex σ'_q. It now suffices to observe that there is a pushout diagram of simplicial sets

$$\begin{tikzcd} \Lambda^d_j \arrow[r] \arrow[d] & K_{q-1} \arrow[d] \\ \Delta^d \arrow[r, "\sigma_q"] & K_q, \end{tikzcd}$$

where $0 < j < d$. $\square$

It follows from Proposition 4.2.3 that we can view the map $\lambda : \mathrm{TwArr}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{op}$ as a pairing of ∞-categories, in the sense of Definition 3.1.1.

Proposition 4.2.4. *Let $\mathcal{C}$ be an ∞-category. Then the pairing $\lambda : \mathrm{TwArr}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{op}$ is both left and right representable. Moreover, the following conditions on an object $M \in \mathrm{TwArr}(\mathcal{C})$ are equivalent:*

(a) *The object M is left universal (in the sense of Definition 3.1.2).*

(b) *The object M is right universal (in the sense of Definition 3.1.2).*

(c) *When viewed as a morphism in the ∞-category $\mathcal{C}$, the object M is an equivalence.*

Proof. We will prove that (c) $\Rightarrow$ (b). Then for every object $C \in \mathcal{C}$, the identity morphism id_C is a right universal object of $\mathrm{TwArr}(\mathcal{C})$ lying over $C \in \mathcal{C}^{op}$, which proves that λ is right representable. Since a right universal object of $\mathrm{TwArr}(\mathcal{C})$ lying over $C \in \mathcal{C}^{op}$ is determined uniquely up to equivalence, we may also conclude that (b) $\Rightarrow$ (c). By symmetry, we can also conclude that (a) $\Leftrightarrow$ (c) and that the pairing λ is left representable.

Fix an object $D \in \mathcal{C}^{op}$, and let $\mathrm{TwArr}(\mathcal{C})_D$ denote the fiber product $\mathrm{TwArr}(\mathcal{C}) \times_{\mathcal{C}^{op}} \{D\}$. Then λ induces a right fibration of simplicial sets $\lambda_D : \mathrm{TwArr}(\mathcal{C})_D \to \mathcal{C}$. We wish to prove that if M is an object of $\mathrm{TwArr}(\mathcal{C})_D$ given by an equivalence $f : C \to D$ in $\mathcal{C}$, then M represents the right fibration λ_D.

For every linearly ordered set I, there is an evident map of linearly ordered sets $I \star I^{op} \to I \star *$, depending functorially on I. Composing with these maps, we obtain a functor $\psi : \mathcal{C}_{/D} \to \mathrm{TwArr}(\mathcal{C})_D$. This map is bijective on vertices (vertices of both $\mathcal{C}_{/D}$ and $\mathrm{TwArr}(\mathcal{C})_D$ can be identified with edges $f : C \to D$ of the simplicial set $\mathcal{C}$). Since the right fibration $\mathcal{C}_{/D} \to \mathcal{C}$ is representable by any equivalence $f : C \to D$ (see the proof of Proposition T.4.4.4.5), it will suffice to show that the ψ is an equivalence of ∞-categories.

We now define an auxiliary simplicial set $\mathcal{M}$ as follows. For every $[n] \in \mathbf{\Delta}$, we let $\mathcal{M}([n])$ denote the subset of $\mathcal{C}([n] \star [0] \star [n]^{op})$ consisting of those $(2n+2)$-simplices of $\mathcal{C}$ whose restriction to $[0] \star [n]^{op}$ is the constant $(n+1)$-simplex at the vertex D. The inclusions of linearly ordered sets

$$[n] \star [0] \hookrightarrow [n] \star [0] \star [n]^{op} \hookleftarrow [n] \star [n]^{op}$$

induce maps of simplicial sets

$$\mathcal{C}_{/D} \xleftarrow{\phi} \mathcal{M} \xrightarrow{\phi'} \mathrm{TwArr}(\mathcal{C})_D.$$

The map $\psi : \mathcal{C}_{/D} \to \mathrm{TwArr}(\mathcal{C})$ can be obtained by composing ϕ' with a section of ϕ. To prove that ψ is a categorical equivalence, it will suffice to show that ϕ and ϕ' are categorical equivalences. We will complete the proof by showing that ϕ and ϕ' are trivial Kan fibrations.

We first show that ϕ is a trivial Kan fibration: that is, that ϕ has the right lifting property with respect to every inclusion $\partial\,\Delta^n \hookrightarrow \Delta^n$. Unwinding the definitions, we are reduced to solving a lifting problem of the form

$$\begin{tikzcd}
K \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\
\Delta^n \star \Delta^0 \star \Delta^n \arrow[r] \arrow[ur, dashed] & \Delta^0
\end{tikzcd}$$

where K denotes the simplicial subset of $\Delta^n \star \Delta^0 \star \Delta^n \simeq \Delta^{2n+2}$ spanned by $\Delta^n \star \Delta^0$, $\Delta^0 \star \Delta^n$, and $\Delta^I \star \Delta^0 \star \Delta^{I^{op}}$ for every proper subset $I \subsetneq [n]$. Since $\mathcal{C}$ is an ∞-category, it suffices to show that the inclusion $K \hookrightarrow \Delta^n \star \Delta^0 \star \Delta^n$ is a categorical equivalence.

Lemma T.5.4.5.10 implies that the composite map

$$(\Delta^n \star \Delta^0) \coprod_{\Delta^0} (\Delta^0 \star \Delta^n) \xhookrightarrow{i} K \to \Delta^n \star \Delta^0 \star \Delta^n$$

is a categorical equivalence. It will therefore suffice to show that the map i is a categorical equivalence. Let K_0 denote the simplicial subset of K spanned by those faces of the form $\Delta^I \star \Delta^0 \star \Delta^{I^{op}}$, where I is a proper subset of $[n]$. We have a pushout diagram of simplicial sets

$$\begin{tikzcd}
(\partial\,\Delta^n \star \Delta^0) \coprod_{\Delta^0} (\Delta^0 \star \partial\,\Delta^n) \arrow[d, "i_0", "i"'] & K_0 \arrow[d] \\
(\Delta^n \star \Delta^0) \coprod_{\Delta^0} (\Delta^0 \star \Delta^n) & K.
\end{tikzcd}$$

Since the Joyal model structure is left proper, we are reduced to proving that the map i_0 is a categorical equivalence. We can write i_0 as a homotopy colimit of morphisms of the form

$$(\Delta^I \star \Delta^0) \coprod_{\Delta^0} (\Delta^0 \star \Delta^{I^{op}}) \to \Delta^I \star \Delta^0 \star \Delta^{I^{op}},$$

where I ranges over all proper subsets of $[n]$. Since each of these maps is a categorical equivalence (Lemma T.5.4.5.10), we conclude that i_0 is a categorical equivalence as desired.

We now prove that ϕ' is a trivial Kan fibration. We must show that ϕ' has the right lifting property with respect to every inclusion of simplicial sets $\partial\,\Delta^n \hookrightarrow \Delta^n$. To prove this, we must show that every lifting problem of the form

$$\begin{tikzcd}
L \arrow[r] \arrow[d, "f_0"] & \mathcal{C} \arrow[d] \\
\Delta^n \star \Delta^0 \star \Delta^n \arrow[r] \arrow[ur, dashed] & \Delta^0
\end{tikzcd}$$

has a solution, where L denotes the simplicial subset of $\Delta^n \star \Delta^0 \star \Delta^n \simeq \Delta^{2n+2}$ given by the union of $\Delta^0 \star \Delta^n$, $\Delta^n \star \Delta^n$, and K_0, and f_0 is a map whose restriction to $\Delta^0 \star \Delta^n$ is constant.

Let σ be a face of Δ^{2n+2} which does not belong to L. Let $i(\sigma)$ denote the first vertex of Δ^{2n+2} which belongs to σ. Since σ does not belong to $\Delta^0 \star \Delta^n \subseteq L$, we must have $i(\sigma) \leq n$. For $j < i(\sigma)$, we have $j \notin \sigma$. Since σ is not contained in K_0, we conclude that $2n+2-j \in \sigma$. Let us say that σ is *large* if it contains the vertex $2n+2-i(\sigma)$, and *small* if it does not contain the vertex $2n+2-i(\sigma)$. Let S be the collection of small faces of Δ^{2n+2} (which are not contained in L). For each $\sigma \in S$, we let σ' denote the face obtained from σ by adding the vertex $2n+2-i(\sigma)$. Choose an ordering of $S = \{\sigma_1, \sigma_2, \ldots, \sigma_m\}$ with the following properties:

(*a*) If $p \leq q$, then the dimension of σ_p is less than or equal to the dimension of σ_q.

(*b*) If $p \leq q$ and the simplices σ_p and σ_q have the same dimension, then $i(\sigma_q) \leq i(\sigma_p)$.

For $0 \leq q \leq m$, let L_q denote the simplicial subset of Δ^{2n+2} obtained from L by adding the faces σ_p and σ'_p for $p \leq q$. We have a sequence of inclusions

$$L = L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots \subseteq L_m = \Delta^{2n+2}.$$

To complete the proof, it will suffice to show that the map $f_0 : L_0 \to \mathcal{C}$ can be extended to a compatible sequence of maps $\{f_q : L_q \to \mathcal{C}\}_{0 \leq q \leq m}$. We proceed by induction. Assume that $q > 0$ and that $f_{q-1} : L_{q-1} \to \mathcal{C}$ has already been constructed. Let d be the dimension of σ'_q, and observe that there is a pushout diagram of simplicial sets

$$\begin{tikzcd} \Lambda^d_{d-i(\sigma_q)} \arrow[r] \arrow[d] & L_{q-1} \arrow[d] \\ \Delta^d \arrow[r] & L_q. \end{tikzcd}$$

Consequently, to prove the existence of f_q, it will suffice to show that the map $f_{q-1}|\Lambda^d_{d-i(\sigma_q)}$ can be extended to a d-simplex of $\mathcal{C}$. Since $d > i(\sigma_q)$, the existence of such an extension follows from the assumption that $\mathcal{C}$ is an ∞-category provided that $i(\sigma_q) > 0$. In the special case $i(\sigma) = 0$, it suffices to show that the map $f_{q-1}|\Lambda^d_d$ carries the final edge of Λ^d_d to an equivalence in $\mathcal{C}$. This follows from our assumption that $f_0|(\Delta^0 \star \Delta^n)$ is a constant map (note that σ'_q automatically contains the vertices $n+1$ and $2n+2$, so that the final edge of σ'_q is contained in $\Delta^0 \star \Delta^n \subseteq \Delta^n \star \Delta^0 \star \Delta^n \simeq \Delta^{2n+2}$). $\square$

Proposition 4.2.5. *Let $\mathcal{C}$ be an ∞-category, and let $\chi : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{S}$ classify the right fibration $\lambda : \mathrm{TwArr}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{op}$. The map $\mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$ determined by χ is homotopic to the Yoneda embedding (see §T.5.1.3).*

Remark 4.2.6. Let $\mathcal{C}$ be an ∞-category and let $\lambda : \mathrm{TwArr}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{op}$ be the pairing of Proposition 4.2.3. Proposition 4.2.4 implies that λ is right and left representable, so that Construction 3.1.3 yields a pair of adjoint functors

$$\mathfrak{D}^{op}_\lambda : \mathcal{C} \to \mathcal{C} \qquad \mathfrak{D}'_\lambda : \mathcal{C} \to \mathcal{C}\,.$$

Proposition 4.2.5 asserts that these functors are homotopic to the identity.

Proof. We begin by recalling the construction of the Yoneda embedding $j : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$. Choose a fibrant simplicial category $\mathcal{D}$ and an equivalence of ∞-categories $\psi : \mathcal{C} \to \mathrm{N}(\mathcal{D})$. The construction $(D, D') \mapsto \mathrm{Map}_{\mathcal{D}}(D, D')$ determines a simplicial functor $\mathcal{F} : \mathcal{D}^{op} \times \mathcal{D} \to \mathcal{K}\mathrm{an}$, where $\mathcal{K}\mathrm{an}$ denote the (simplicial) category of Kan complexes. Passing to nerves, we obtain a functor

$$\mu : \mathcal{C} \times \mathcal{C}^{op} \to \mathrm{N}(\mathcal{D}) \times \mathrm{N}(\mathcal{D})^{op} \simeq \mathrm{N}(\mathcal{D} \times \mathcal{D}^{op}) \to \mathrm{N}(\mathcal{K}\mathrm{an}) = \mathcal{S}$$

which we can identify with Yoneda embedding $\mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$. We are therefore reduced to proving that the functor μ classifies the right fibration $\mathrm{TwArr}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{op}$.

Let $\phi : \mathfrak{C}[\mathcal{C} \times \mathcal{C}^{op}] \to \mathcal{D} \times \mathcal{D}^{op}$ be the equivalence of simplicial categories determined by ψ, and let

$$\mathrm{Un}_{\phi} : \mathcal{S}\mathrm{et}_{\Delta}^{\mathcal{D}^{op} \times \mathcal{D}} \to (\mathcal{S}\mathrm{et}_{\Delta})_{/\,\mathcal{C} \times \mathcal{C}^{op}}$$

denote the unstraightening functor defined in §T.2.2.1. To complete the proof, it will suffice to construct an equivalence $\beta : \mathrm{TwArr}(\mathcal{C}) \to \mathrm{Un}_{\phi}(\mathcal{F})$ of right fibrations over $\mathcal{C} \times \mathcal{C}^{op}$.

We begin by constructing the map β. Let $\mathcal{E}$ be the simplicial category obtained from $\mathcal{D} \times \mathcal{D}^{op}$ by adjoining a new element v, with mapping spaces given by

$$\mathrm{Map}_{\mathcal{E}}(v, (D, D')) = \emptyset \qquad \mathrm{Map}_{\mathcal{E}}((D, D'), v) = \mathrm{Map}_{\mathcal{D}}(D, D').$$

Unwinding the definitions, we see that giving the map β is equivalent to constructing a map $\gamma : \mathrm{TwArr}(\mathcal{C})^{\triangleright} \to \mathrm{N}(\mathcal{E})$ carrying the cone point of $\mathrm{TwArr}(\mathcal{C})^{\triangleright}$ to v and such that $\gamma | \mathrm{TwArr}(\mathcal{C})$ is given by the composition

$$\mathrm{TwArr}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{op} \to \mathrm{N}(\mathcal{D} \times \mathcal{D}^{op}) \hookrightarrow \mathrm{N}(\mathcal{E}).$$

To describe the map γ, it suffices to define the composite map

$$\gamma_{\sigma} : \Delta^{n+1} \xrightarrow{\sigma^{\triangleright}} \mathrm{TwArr}(\mathcal{C})^{\triangleright} \xrightarrow{\gamma} \mathrm{N}(\mathcal{E})$$

for every n-simplex $\sigma : \Delta^n \to \mathrm{TwArr}(\mathcal{C})$. Let $\mathfrak{C} : \mathcal{S}\mathrm{et}_{\Delta} \to \mathcal{C}\mathrm{at}_{\Delta}$ denote the left adjoint to the simplicial nerve functor. We will define γ_{σ} as the adjoint of a map of simplicial categories $\mathfrak{C}[\Delta^{n+1}] \to \mathcal{E}$, carrying the final vertex of Δ^{n+1} to v and given on $\mathfrak{C}[\Delta^n]$ by the composite map

$$\mathfrak{C}[\Delta^n] \xrightarrow{\sigma \times \sigma^{op}} \mathfrak{C}[\mathcal{C}] \times \mathfrak{C}[\mathcal{C}]^{op} \to \mathcal{D} \times \mathcal{D}^{op} \subseteq \mathcal{E}\,.$$

We can identify σ with a map $\Delta^{2n+1} \to \mathcal{C}$, which induces a functor of simplicial categories

$$\nu_{\sigma} : \mathfrak{C}[\Delta^{2n+1}] \to \mathfrak{C}(\mathcal{C}) \to \mathcal{D}\,.$$

To complete the definition of γ_{σ}, it suffices to describe the induced maps

$$\mathrm{Map}_{\mathfrak{C}[\Delta^{n+1}]}(i, n+1) \to \mathrm{Map}_{\mathcal{E}}(\gamma_{\sigma}(i), v) = \mathrm{Map}_{\mathcal{D}}(\nu_{\sigma}(i), \nu_{\sigma}(2n+1-i))$$

for $0 \leq i \leq n$. These maps will be given by a composition

$$\mathrm{Map}_{\mathfrak{C}[\Delta^{n+1}]}(i, n+1) \xrightarrow{\alpha} \mathrm{Map}_{\mathfrak{C}[\Delta^{2n+1}]}(i, 2n+1-i) \xrightarrow{\nu_{\sigma}} \mathrm{Map}_{\mathcal{D}}(\nu_{\sigma}(i), \nu_{\sigma}(2n+1-i)).$$

Recall that for $0 \leq j \leq k \leq m$, the mapping space $\mathrm{Map}_{\mathfrak{C}[\Delta^m]}(j, k)$ can be identified with the nerve of the partially ordered collection of subsets of $[m]$ having infimum j and supremum k (see Definition T.1.1.5.1). Under this identification, α corresponds to the map of partially ordered sets given by

$$S \cup \{n+1\} \mapsto S \cup \{2n+1-j : j \in S\}.$$

It is not difficult to see that these maps determine a simplicial functor $\mathfrak{C}[\Delta^{n+1}] \to \mathcal{E}$, giving a map of simplicial sets $\gamma_{\sigma} : \Delta^{n+1} \to \mathrm{N}(\mathcal{E})$. The construction is functorial in σ, and therefore arises from the desired map $\gamma : \mathrm{TwArr}(\mathcal{C})^{\triangleright} \to \mathrm{N}(\mathcal{E})$.

It remains to prove that β is a homotopy equivalence. Since the maps

$$\mathrm{TwArr}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{op} \leftarrow \mathrm{Un}_{\phi}(\mathcal{F})$$

are right fibrations, it will suffice to prove that β induces a homotopy equivalence

$$\beta_{C,C'} : \mathrm{TwArr}(\mathcal{C}) \times_{\mathcal{C} \times \mathcal{C}^{op}} \{(C, C')\} \to (\mathrm{Un}_{\phi}\, \mathcal{F}) \times_{\mathcal{C} \times \mathcal{C}'} \{(C, C')\}$$

for every pair of objects $C, C' \in \mathcal{C}$. Consider the map

$$u : \mathcal{C}_{/C'} \to \mathrm{TwArr}(\mathcal{C}) \times_{\mathcal{C}^{op}} \{C'\}$$

appearing in the proof of Proposition 4.2.4. Since u induces a homotopy equivalence

$$\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(C, C') = \{C\} \times_{\mathcal{C}} \mathcal{C}_{/C'} \to \mathrm{TwArr}(\mathcal{C}) \times_{\mathcal{C} \times \mathcal{C}^{op}} \{(C, C')\},$$

we are reduced to proving that the composite map

$$\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(C, C') \to \mathrm{TwArr}(\mathcal{C}) \times_{\mathcal{C} \times \mathcal{C}^{op}} \{(C, C')\} \to (\mathrm{Un}_{\phi}\, \mathcal{F}) \times_{\mathcal{C} \times \mathcal{C}'} \{(C, C')\}$$

is a homotopy equivalence. This follows from Proposition T.2.2.4.1. $\square$

Our next goal is to characterize the twisted arrow ∞-category $\mathrm{TwArr}(\mathcal{C})$ by a universal property. In fact, we will give two such universal properties.

Construction 4.2.7. Let $\lambda : \mathcal{M} \to \mathcal{C} \times \mathcal{D}$ and $\lambda' : \mathcal{M}' \to \mathcal{C}' \times \mathcal{D}'$ be pairings of ∞-categories. We define a simplicial set $\mathrm{Map}_{\mathrm{CPair}_{\Delta}}(\lambda, \lambda')$ so that the following universal property is satisfied: for every simplicial set K, there is a canonical bijection between $\mathrm{Hom}_{\mathcal{S}\mathrm{et}_{\Delta}}(K, \mathrm{Map}_{\mathrm{CPair}_{\Delta}}(\lambda, \lambda'))$ and the set of triples (α, β, γ) where $\alpha : K \times \mathcal{C} \to \mathcal{C}'$ is a map of simplicial sets carrying each edge of K to an equivalence in $\mathrm{Fun}(\mathcal{C}, \mathcal{C}')$, $\beta : K \times \mathcal{D} \to \mathcal{D}'$ is a map of simplicial sets carrying each edge of K to an equivalence in $\mathrm{Fun}(\mathcal{D}, \mathcal{D}')$, and $\gamma : K \times \mathcal{M} \to \mathcal{M}'$ is a map fitting into a commutative diagram

$$\begin{tikzcd}
K \times \mathcal{M} \arrow[r, "\gamma"] \arrow[d, "\lambda"] & \mathcal{M}' \arrow[d, "\lambda'"] \\
K \times K \times \mathcal{C} \times \mathcal{D} \arrow[r, "\alpha \times \beta"] & \mathcal{C}' \times \mathcal{D}'
\end{tikzcd}$$

(it then follows automatically that γ carries each edge of K to an equivalence in $\mathrm{Fun}(\mathcal{M}, \mathcal{M}')$).

We let CPair_{Δ} be the simplicial category whose objects are pairings of ∞-categories $\lambda : \mathcal{M} \to \mathcal{C} \times \mathcal{D}$, with morphism spaces given as above. Then CPair_{Δ} is a fibrant simplicial category (see Lemma 4.2.15 below); we let $\mathrm{CPair} = \mathrm{N}(\mathrm{CPair}_{\Delta})$ denote the associated ∞-category. We will refer to CPair as *the ∞-category of pairings of ∞-categories.*

Let CPair^R denote the subcategory of CPair whose objects are right representable pairings of ∞-categories $\lambda : \mathcal{M} \to \mathcal{C} \times \mathcal{D}$ and whose morphisms are right representable morphisms between pairings (see Definition 3.3.3). We will refer to CPair^R as the *∞-category of right representable pairings of ∞-categories.*

Remark 4.2.8. If $\lambda : \mathcal{M} \to \mathcal{C} \times \mathcal{D}$ and $\lambda' : \mathcal{M}' \to \mathcal{C}' \times \mathcal{D}'$ are pairings of ∞-categories, then giving an edge $\alpha : \lambda \to \lambda'$ in the ∞-category CPair is equivalent to giving a morphism of pairings from λ to λ', in the sense of Definition 3.3.3.

Remark 4.2.9. It follows from Proposition T.4.2.4.4 that CPair is equivalent to the full subcategory of $\mathrm{Fun}(\Lambda^2_0, \mathcal{C}\mathrm{at}_{\infty})$ spanned by those diagrams $\mathcal{C} \leftarrow \mathcal{M} \to \mathcal{D}$ for which the induced map $\mathcal{M} \to \mathcal{C} \times \mathcal{D}$ is equivalent to a right fibration. This subcategory is a localization of $\mathrm{Fun}(\Lambda^2_0, \mathcal{C}\mathrm{at}_{\infty})$; in particular, we can identify CPair with a full subcategory of $\mathrm{Fun}(\Lambda^2_0, \mathcal{C}\mathrm{at}_{\infty})$ which is closed under small limits.

Proposition 4.2.10. *Let $\mathcal{C}$ be an ∞-category and let $\lambda : \mathrm{TwArr}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{op}$ be the pairing of Proposition 4.2.3. Let $\mu : \mathcal{M} \to \mathcal{D} \times \mathcal{E}$ be an arbitrary right representable pairing of ∞-categories. Then the evident maps*

$$\mathrm{Map}_{\mathrm{CPair}^R}(\lambda, \mu) \to \mathrm{Map}_{\mathcal{C}\mathrm{at}_{\infty}}(\mathcal{C}^{op}, \mathcal{E}) \qquad \mathrm{Map}_{\mathrm{CPair}^R}(\mu, \lambda) \to \mathrm{Map}_{\mathcal{C}\mathrm{at}_{\infty}}(\mathcal{D}, \mathcal{C})$$

are homotopy equivalences.

Before giving the proof of Proposition 4.2.10, let us describe some of its consequences.

Corollary 4.2.11. *Let $\phi, \psi : \mathrm{CPair}^R \to \mathcal{C}\mathrm{at}_\infty$ be the forgetful functors given on objects by the formulas*

$$\phi(\lambda : \mathcal{M} \to \mathcal{C} \times \mathcal{D}) = \mathcal{C} \qquad \psi(\lambda : \mathcal{M} \to \mathcal{C} \times \mathcal{D}) = \mathcal{D}\,.$$

Then:

(1) *The functor ϕ admits a right adjoint, given at the level of objects by $\mathcal{C} \mapsto \mathrm{TwArr}(\mathcal{C})$.*

(2) *The functor ψ admits a left adjoint, given at the level of objects by $\mathcal{D} \mapsto \mathrm{TwArr}(\mathcal{D}^{op})$.*

Proof. Combine Propositions 4.2.10 and T.5.2.4.2. □

Remark 4.2.12. Let us say that a pairing of ∞-categories is *perfect* if it is equivalent (in the ∞-category CPair) to a pairing of the form $\mathrm{TwArr}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{op}$, for some ∞-category $\mathcal{C}$. We let $\mathrm{CPair}^{\mathrm{perf}}$ denote the subcategory of CPair whose objects are perfect pairings of ∞-categories and whose morphisms are right representable morphisms of pairings (note that if λ and λ' are perfect pairings of ∞-categories, then Proposition 4.2.4 implies that a morphism of pairings from λ to λ' is left representable if and only if it is right representable). It follows from Corollary 4.2.11 that the full subcategory $\mathrm{CPair}^{\mathrm{perf}} \subseteq \mathrm{CPair}^R$ is both a localization and a colocalization of CPair^R. Moreover, the forgetful functors $\phi, \psi : \mathrm{CPair}^R \to \mathcal{C}\mathrm{at}_\infty$ restrict to equivalences $\mathrm{CPair}^{\mathrm{perf}} \to \mathcal{C}\mathrm{at}_\infty$. Composing these equivalences, we obtain an equivalence of ∞-categories from $\mathcal{C}\mathrm{at}_\infty$ to itself, given at the level of objects by $\mathcal{C} \mapsto \mathcal{C}^{op}$.

Remark 4.2.13. Let $\mu : \mathcal{M} \to \mathcal{C} \times \mathcal{D}$ be a right representable pairing of ∞-categories, and let $\lambda : \mathrm{TwArr}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{op}$ be the pairing of Proposition 4.2.3. Using Proposition 4.2.10, we can lift the identity functor $\mathrm{id}_{\mathcal{C}}$ to a right representable morphism of pairings $(\alpha, \beta, \gamma) : \mu \to \lambda$. For every object $D \in \mathcal{D}$, the induced map

$$\gamma_D : \mathcal{M} \times_{\mathcal{D}} \{D\} \to \mathrm{TwArr}(\mathcal{C}) \times_{\mathcal{C}^{op}} \{\beta(D)\}$$

is a map between representable right fibrations over $\mathcal{C}$ which preserves final objects, and therefore an equivalence of ∞-categories. It follows that the diagram

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\gamma"] \arrow[d] & \mathrm{TwArr}(\mathcal{C}) \arrow[d] \\ \mathcal{D} \arrow[r, "\beta"] & \mathcal{C}^{op} \end{tikzcd}$$

is homotopy Cartesian (note that the vertical maps are Cartesian fibrations, so that this condition can be tested fiberwise).

Corollary 4.2.14. *Let $\mu : \mathcal{M} \to \mathcal{C} \times \mathcal{D}$ be a pairing of ∞-categories. The following conditions are equivalent:*

(1) *The pairing μ is perfect.*

(2) *The pairing μ is both left and right representable, and an object of $\mathcal{M}$ is left universal if and only if it is right universal.*

(3) *The pairing μ is both left and right representable, and the adjoint functors*

$$\mathfrak{D}_\mu^{op} : \mathcal{C} \to \mathcal{D}^{op} \qquad \mathfrak{D}'_\mu : \mathcal{D}^{op} \to \mathcal{C}$$

of Construction 3.1.3 are mutually inverse equivalences.

Proof. We first prove that conditions (2) and (3) are equivalent. Assume that μ is both left and right representable. Let $C \in \mathcal{C}$, and choose a left universal object $M \in \mathcal{M}$ lying over C. Let $D = \mathfrak{D}_\mu(C)$ be the image of M in $\mathcal{D}$, and choose a right universal object $N \in \mathcal{M}$ lying over D. Then N is a final object of $\mathcal{M} \times_{\mathcal{D}} \{D\}$, so there is a canonical map $u_0 : M \to N$ in $\mathcal{M}$. Unwinding the definitions, we see that the image of u_0 in $\mathcal{C}$ can be identified with the unit map $u : C \to \mathfrak{D}'_\mu \mathfrak{D}_\mu^{op}(C)$. Since μ is a right fibration and the image of u_0 in $\mathcal{D}$ is an equivalence, we conclude that u is an equivalence if and only if u_0 is an equivalence. That is, u is an equivalence if and only if M is also a right universal object of $\mathcal{M}$. This proves the following:

(∗) The unit map $\mathrm{id}_{\mathcal{C}} \to \mathfrak{D}'_\mu \circ \mathfrak{D}^{op}_\mu$ is an equivalence if and only if every left universal object of $\mathcal{M}$ is also right universal.

The same argument proves:

(∗′) The counit map $\mathfrak{D}^{op}_\mu \circ \mathfrak{D}'_\mu \to \mathrm{id}_{\mathcal{D}^{op}}$ is an equivalence if and only if every right universal object of $\mathcal{M}$ is also left universal.

Combining (∗) and (∗′), we deduce that conditions (2) and (3) are equivalent.

The implication (1) ⇒ (2) follows from Proposition 4.2.4. We will complete the proof by showing that (3) ⇒ (1). Let $\lambda : \mathrm{TwArr}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{op}$ be the pairing of Proposition 4.2.3. Since λ is right representable, the identity functor $\mathrm{id}_{\mathcal{C}}$ can be lifted to a right representable morphism of pairings $(\mathrm{id}_{\mathcal{C}}, \beta, \gamma) : \mu \to \lambda$. We wish to prove that β and γ are equivalences. Since the diagram

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\gamma"] \arrow[d] & \mathrm{TwArr}(\mathcal{C}) \arrow[d] \\ \mathcal{D} \arrow[r, "\beta"] & \mathcal{C}^{op} \end{tikzcd}$$

is homotopy Cartesian (Remark 4.2.13), it will suffice to show that β is an equivalence of ∞-categories. Using Remark 4.2.6 and Proposition 3.3.4, we see that the diagram of ∞-categories

$$\begin{tikzcd} \mathcal{D}^{op} \arrow[r, "\mathfrak{D}'_\lambda"] \arrow[d, "\beta"] & \mathcal{C} \arrow[d, "\mathrm{id}"] \\ \mathcal{C} \arrow[r, "\mathrm{id}"] & \mathcal{C} \end{tikzcd}$$

commutes up to homotopy. It follows that β is homotopic to $\mathfrak{D}'_\lambda$, which is an equivalence by virtue of assumption (3). □

We now turn to the proof of Proposition 4.2.10. We begin with a general discussion of the mapping spaces in the ∞-category CPair. Suppose we are given pairings of ∞-categories

$$\lambda : \mathcal{M} \to \mathcal{C} \times \mathcal{D} \qquad \lambda' : \mathcal{M}' \to \mathcal{C}' \times \mathcal{D}'.$$

We have an evident map of simplicial sets

$$\theta : \mathrm{Map}_{\mathrm{CPair}_\Delta}(\lambda, \lambda') \to \mathrm{Fun}(\mathcal{C}, \mathcal{C}')^{\simeq} \times \mathrm{Fun}(\mathcal{D}, \mathcal{D}')^{\simeq}.$$

Lemma 4.2.15. *In the situation described above, the map θ is a Kan fibration. In particular, the mapping space $\mathrm{Map}_{\mathrm{CPair}_\Delta}(\lambda, \lambda')$ is a Kan complex.*

Proof. Since $\mathrm{Fun}(\mathcal{C}, \mathcal{C}')^{\simeq} \times \mathrm{Fun}(\mathcal{D}, \mathcal{D}')^{\simeq}$ is a Kan complex, it will suffice to show that the map θ is a right fibration (Lemma T.2.1.3.3). We will prove that θ has the right lifting property with respect to every right anodyne map of simplicial sets $i : A \to B$. Fix a map $B \to \mathrm{Fun}(\mathcal{C}, \mathcal{C}')^{\simeq} \times \mathrm{Fun}(\mathcal{D}, \mathcal{D}')^{\simeq}$, and let $\mathcal{N}$ denote the fiber product $(\mathcal{C} \times \mathcal{D} \times B) \times_{\mathcal{C}' \times \mathcal{D}'} \mathcal{M}'$. Unwinding the definitions, we are reduced to solving a lifting problem of the form

$$\begin{tikzcd} A \times \mathcal{M} \arrow[r, "i'"] \arrow[d] & \mathcal{N} \arrow[d, "q"] \\ B \times \mathcal{M} \arrow[r] \arrow[ur, dashed] & B \times \mathcal{C} \times \mathcal{D}. \end{tikzcd}$$

Since q is a pullback of λ', it is a right fibration. It will therefore suffice to show i' is right anodyne, which follows from Corollary T.2.1.2.7. □

Our next step is to analyze the fibers of Kan fibration $\theta : \mathrm{Map}_{\mathrm{CPair}_{\Delta}}(\lambda, \lambda') \to \mathrm{Fun}(\mathcal{C}, \mathcal{C}')^{\simeq} \times \mathrm{Fun}(\mathcal{D}, \mathcal{D}')^{\simeq}$. Fix a pair of functors $\alpha : \mathcal{C} \to \mathcal{C}'$ and $\beta : \mathcal{D} \to \mathcal{D}'$. Unwinding the definitions, we see that the fiber $\theta^{-1}\{(\alpha, \beta)\}$ is the ∞-category $\mathrm{Fun}_{\mathcal{C}' \times \mathcal{D}'}(\mathcal{M}, \mathcal{M}')$. Let $\chi : \mathcal{C}^{op} \times \mathcal{D}^{op} \to \mathcal{S}$ classify the right fibration λ, and let $\chi' : \mathcal{C}'^{op} \times \mathcal{D}'^{op} \to \mathcal{S}$ classify the right fibration λ'. Then $\mathrm{Fun}_{\mathcal{C}' \times \mathcal{D}'}(\mathcal{M}, \mathcal{M}')$ is homotopy equivalent to the mapping space $\mathrm{Map}_{\mathrm{Fun}(\mathcal{C}^{op} \times \mathcal{D}^{op}, \mathcal{S})}(\chi, \chi' \circ (\alpha \times \beta))$. Let $\mathcal{P}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$ and define $\mathcal{P}(\mathcal{C}')$ similarly, so that χ and χ' can be identified with maps $\nu : \mathcal{D}^{op} \to \mathcal{P}(\mathcal{C})$ and $\nu' : \mathcal{D}'^{op} \to \mathcal{P}(\mathcal{C}')$. We then have

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C}^{op} \times \mathcal{D}^{op}, \mathcal{S})}(\chi, \chi' \circ (\alpha \times \beta)) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{D}^{op}, \mathcal{P}(\mathcal{C}))}(\nu, G \circ \nu' \circ \beta),$$

where $G : \mathcal{P}(\mathcal{C}') \to \mathcal{P}(\mathcal{C})$ is the map given by composition with α. Note that G admits a left adjoint $\overline{\alpha} : \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{C}')$, which fits into a commutative diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d, "\alpha"] & \mathcal{P}(\mathcal{C}) \arrow[d, "\overline{\alpha}"] \\
\mathcal{D} \arrow[r] & \mathcal{P}(\mathcal{D}),
\end{tikzcd}$$

where the horizontal maps are given by the Yoneda embeddings (see Proposition T.5.2.6.3). Combining this observation with the analysis above, we obtain a homotopy equivalence

$$\theta^{-1}\{(\alpha, \beta)\} = \mathrm{Map}_{\mathrm{Fun}(\mathcal{D}^{op}, \mathcal{P}(\mathcal{C}'))}(\overline{\alpha} \circ \nu, \nu' \circ \beta).$$

Let us now specialize to the case where the pairings $\lambda : \mathcal{M} \to \mathcal{C} \times \mathcal{D}$ and $\lambda' : \mathcal{M}' \to \mathcal{C}' \times \mathcal{D}'$ are right representable. In this case, the functors ν and ν' admit factorizations

$$\mathcal{D}^{op} \stackrel{\mathfrak{D}'_{\lambda}}{\to} \mathcal{C} \to \mathcal{P}(\mathcal{C})$$

$$\mathcal{D}'^{op} \stackrel{\mathfrak{D}'_{\lambda'}}{\to} \mathcal{C}' \to \mathcal{P}(\mathcal{C}')$$

(see Construction 3.1.3). We may therefore identify $\theta^{-1}\{(\alpha, \beta)\}$ with the mapping space

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{D}^{op}, \mathcal{C}')}(\alpha \circ \mathfrak{D}'_{\lambda}, \mathfrak{D}'_{\lambda'} \circ \beta).$$

Under this identification, the subspace

$$\mathrm{Map}_{\mathrm{CPair}^R}(\lambda, \lambda') \times_{\mathrm{Fun}(\mathcal{C}, \mathcal{C}')^{\simeq} \times \mathrm{Fun}(\mathcal{D}, \mathcal{D}')^{\simeq}} \{(\alpha, \beta)\}$$

corresponds the summand of $\mathrm{Map}_{\mathrm{Fun}(\mathcal{D}^{op}, \mathcal{C}')}(\alpha \circ \mathfrak{D}'_{\lambda}, \mathfrak{D}'_{\lambda'} \circ \beta)$ spanned by the equivalences $\alpha \circ \mathfrak{D}'_{\lambda} \simeq \mathfrak{D}'_{\lambda'} \circ \beta$ (see Proposition 3.3.4 and its proof).

Proof of Proposition 4.2.10. Let $\mathcal{C}$ be an ∞-category, let $\lambda : \mathrm{TwArr}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{op}$ be the pairing of Proposition 4.2.3, and let $\mu : \mathcal{M} \to \mathcal{D} \times \mathcal{E}$ be an arbitrary right representable pairing of ∞-categories. We first show that the forgetful functor

$$\mathrm{Map}_{\mathrm{CPair}^R}(\lambda, \mu) \to \mathrm{Map}_{\mathcal{C}\mathrm{at}_{\infty}}(\mathcal{C}^{op}, \mathcal{E})$$

is a homotopy equivalence. Let $\mathrm{Map}_{\mathrm{CPair}^R_{\Delta}}(\lambda, \mu)$ denote the full simplicial subset of $\mathrm{Map}_{\mathrm{CPair}_{\Delta}}(\lambda, \mu)$ spanned by the right representable morphisms of pairings. It follows from Lemma 4.2.15 that the map of simplicial sets

$$\phi : \mathrm{Map}_{\mathrm{CPair}^R_{\Delta}}(\lambda, \mu) \to \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{E})^{\simeq}$$

is a Kan fibration. It will therefore suffice to show that the fibers of ϕ are contractible. Fix a functor $\beta : \mathcal{C}^{op} \to \mathcal{E}$, so that we have a Kan fibration of simplicial sets $u : \phi^{-1}\{\beta\} \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\simeq}$. Combining Remark 4.2.6 with the analysis given above, we see that the fiber of u over a functor $\alpha : \mathcal{C} \to \mathcal{D}$ can be identified with the summand of $\mathrm{Map}_{\mathrm{Fun}(\mathcal{C}, \mathcal{D})}(\alpha, \mathfrak{D}'_{\mu} \circ \beta)$ spanned by the equivalences. It follows that u is a right fibration represented by the object $\mathfrak{D}'_\mu \circ \beta \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})$, so that the fiber $\phi^{-1}\{\beta\}$ is equivalent to $\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\simeq}_{/(\mathfrak{D}'_\mu \circ \beta)}$ and therefore contractible.

We now show that the forgetful functor $\mathrm{Map}_{\mathrm{CPair}^R}(\mu, \lambda) \to \mathrm{Map}_{\mathcal{C}\mathrm{at}_\infty}(\mathcal{D}, \mathcal{C})$ is a homotopy equivalence. For this, it suffices to show that the Kan fibration of simplicial sets $\psi : \mathrm{Map}_{\mathrm{CPair}^R_\Delta}(\mu, \lambda) \to \mathrm{Fun}(\mathcal{D}, \mathcal{C})^{\simeq}$ has contractible fibers. Fix a functor $\alpha : \mathcal{D} \to \mathcal{C}$, so that we have a Kan fibration $v : \psi^{-1}\{\alpha\} \to \mathrm{Fun}(\mathcal{E}, \mathcal{C}^{op})^{\simeq}$. Using Remark 4.2.6 and the above analysis, we see that the fiber of v over a map $\beta : \mathcal{E} \to \mathcal{C}^{op}$ can be identified with the summand of the mapping space $\mathrm{Map}_{\mathrm{Fun}(\mathcal{E}^{op}, \mathcal{C})}(\alpha \circ \mathfrak{D}'_\mu, \beta)$. It follows that v is a left fibration represented by the object $\alpha \circ \mathfrak{D}'_\mu \in \mathrm{Fun}(\mathcal{E}^{op}, \mathcal{C})$, so that the fiber $\psi^{-1}\{\alpha\}$ is equivalent to $\mathrm{Fun}(\mathcal{E}^{op}, \mathcal{C})^{\simeq}_{\alpha \circ \mathfrak{D}'_\mu /}$ and therefore contractible. $\square$

4.3 The Bar Construction

Let $\mathcal{C}$ be a monoidal ∞-category, let $A \in \mathrm{Alg}(\mathcal{C})$ be an algebra object of $\mathcal{C}$, and let $\epsilon : A \to \mathbf{1}$ be an augmentation on the algebra A. If $\mathcal{C}$ admits geometric realizations of simplicial objects, then the *bar construction* on A is defined to be the relative tensor product $\mathbf{1} \otimes_A \mathbf{1}$ (here we regard $\mathbf{1}$ as both a right and left module over A, by means of the augmentation ϵ): that is, the geometric realization of the simplicial object

$$\cdots \rightrightarrows A \otimes A \rightrightarrows A \longrightarrow \mathbf{1}.$$

We will denote this geometric realization by $\mathrm{Bar}\, A$. Our goal in this section is to study some of the properties of the construction $A \mapsto \mathrm{Bar}\, A$.

(a) Consider the composition

$$\begin{aligned} \mathrm{Bar}\, A &= \mathbf{1} \otimes_A \mathbf{1} \\ &\simeq \mathbf{1} \otimes_A A \otimes_A \mathbf{1} \\ &\to \mathbf{1} \otimes_A \mathbf{1} \otimes_A \mathbf{1} \\ &\simeq \mathbf{1} \otimes_A \mathbf{1} \otimes_{\mathbf{1}} \mathbf{1} \otimes_A \mathbf{1} \\ &\xrightarrow{\alpha} (\mathbf{1} \otimes_A \mathbf{1}) \otimes (\mathbf{1} \otimes_A \mathbf{1}) \\ &= \mathrm{Bar}\, A \otimes \mathrm{Bar}\, A. \end{aligned}$$

(here the map α is an equivalence if we assume that the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realizations of simplicial objects). We can view this composite map as giving a comultiplication $\Delta : \mathrm{Bar}\, A \to \mathrm{Bar}\, A \otimes \mathrm{Bar}\, A$. We will show that this comultiplication is coherently associative: that is, it exhibits $\mathrm{Bar}\, A$ as an associative algebra object in the monoidal ∞-category $\mathcal{C}^{op}$. Moreover, this algebra object is equipped with a canonical augmentation, given by the morphism

$$\mathbf{1} \simeq \mathbf{1} \otimes \mathbf{1} \to \mathbf{1} \otimes_A \mathbf{1} = \mathrm{Bar}\, A$$

in $\mathcal{C}$. We may therefore identify the construction $A \mapsto \mathrm{Bar}\, A$ with a functor

$$\mathrm{Bar} : \mathrm{Alg}^{\mathrm{aug}}(\mathcal{C})^{op} \to \mathrm{Alg}^{\mathrm{aug}}(\mathcal{C}^{op}).$$

(b) Assume that $\mathcal{C}$ admits totalizations of cosimplicial objects. Then $\mathcal{C}^{op}$ admits geometric realizations of simplicial objects, so that we can apply the bar construction to augmented associative algebra objects of $\mathcal{C}^{op}$. We therefore obtain a functor

$$\mathrm{CoBar} : \mathrm{Alg}^{\mathrm{aug}}(\mathcal{C}^{op}) \to \mathrm{Alg}^{\mathrm{aug}}(\mathcal{C})^{op},$$

which we will refer to as the *cobar construction*. If C is an augmented algebra object of $\mathcal{C}^{op}$ (which we can think of as a augmented coalgebra object of $\mathcal{C}$), then $\mathrm{CoBar}\, C$ is given by the totalization of a cosimplicial diagram

$$\mathbf{1} \longrightarrow C \rightrightarrows C \otimes C \rightrightarrows \cdots$$

We will show that the functor $\mathrm{CoBar} : \mathrm{Alg}^{\mathrm{aug}}(\mathcal{C}^{op}) \to \mathrm{Alg}^{\mathrm{aug}}(\mathcal{C})^{op}$ is adjoint to the bar construction $\mathrm{Bar} : \mathrm{Alg}^{\mathrm{aug}}(\mathcal{C})^{op} \to \mathrm{Alg}^{\mathrm{aug}}(\mathcal{C}^{op})$.

To simplify the discussion, we note that there is no harm in assuming that the unit object $\mathbf{1} \in \mathcal{C}$ is both initial and final. This can always be achieved by replacing $\mathcal{C}$ by $\mathcal{C}_{\mathbf{1}//\mathbf{1}}$ (which inherits a monoidal structure: see §A.2.2.2). We have canonical equivalences

$$\mathrm{Alg}(\mathcal{C}_{\mathbf{1}//\mathbf{1}}) \simeq \mathrm{Alg}^{\mathrm{aug}}(\mathcal{C}) \qquad \mathrm{Alg}(\mathcal{C}^{op}_{\mathbf{1}//\mathbf{1}}) \simeq \mathrm{Alg}^{\mathrm{aug}}(\mathcal{C}^{op}),$$

which allows us to ignore augmentations in the discussion below.

Let us now describe the adjunction appearing in assertion (b) more explicitly. Suppose we are given an algebra object $A \in \mathrm{Alg}^{\mathrm{aug}}(\mathcal{C})$ and a coalgebra object $C \in \mathrm{Alg}^{\mathrm{aug}}(\mathcal{C}^{op})$. According to (b), we should have a canonical homotopy equivalence

$$\mathrm{Map}_{\mathrm{Alg}(\mathcal{C})}(A, \mathrm{CoBar}\, C) \simeq \mathrm{Map}_{\mathrm{Alg}(\mathcal{C}^{op})}(C, \mathrm{Bar}\, A).$$

We will prove this by identifying both sides with a classifying space for liftings of the pair $(A, C) \in \mathrm{Alg}(\mathcal{C} \times \mathcal{C}^{op})$ to an algebra object of the twisted arrow ∞-category $\mathrm{TwArr}(\mathcal{C})$.

Theorem 4.3.1. *Let $\mathcal{C}$ be a monoidal ∞-category, so that $\mathcal{C}^{op}$ and $\mathrm{TwArr}(\mathcal{C})$ inherit the structure of monoidal ∞-categories (see Example 4.3.6). Assume that the unit object $\mathbf{1} \in \mathcal{C}$ is both initial and final. Then:*

(1) *The induced map $\lambda : \mathrm{Alg}(\mathrm{TwArr}(\mathcal{C})) \to \mathrm{Alg}(\mathcal{C}) \times \mathrm{Alg}(\mathcal{C}^{op})$ is a pairing of ∞-categories.*

(2) *Assume that the unit object $\mathbf{1} \in \mathcal{C}$ is final (so that every algebra object of $\mathcal{C}$ is equipped with a canonical augmentation) and that $\mathcal{C}$ admits geometric realizations of simplicial objects. Then the pairing λ is left representable, and therefore determines a functor $\mathfrak{D}_\lambda : \mathrm{Alg}(\mathcal{C})^{op} \to \mathrm{Alg}(\mathcal{C}^{op})$. The composite functor*

$$\mathrm{Alg}(\mathcal{C})^{op} \xrightarrow{\mathfrak{D}_\lambda} \mathrm{Alg}(\mathcal{C}^{op}) \to \mathcal{C}^{op}$$

is given by $A \mapsto \mathrm{Bar}\, A$.

(3) *Assume that the unit object $\mathbf{1}$ is initial (so that every coalgebra object of $\mathcal{C}$ is equipped with a canonical augmentation) and that $\mathcal{C}$ admits totalizations of cosimplicial objects. Then the pairing λ is right representable, and therefore determined a functor $\mathfrak{D}'_\lambda : \mathrm{Alg}(\mathcal{C}^{op})^{op} \to \mathrm{Alg}(\mathcal{C})$. The composite functor*

$$\mathrm{Alg}(\mathcal{C}^{op})^{op} \xrightarrow{\mathfrak{D}'_\lambda} \mathrm{Alg}(\mathcal{C}) \to \mathcal{C}$$

is given by $A \mapsto \mathrm{CoBar}\, A$.

Remark 4.3.2. Assertion (3) of Theorem 4.3.1 follows from assertion (2), applied to the opposite ∞-category $\mathcal{C}^{op}$.

Remark 4.3.3. In the situation of Theorem 4.3.1, suppose that the unit object $\mathbf{1}$ is both initial and final, and that $\mathcal{C}$ admits both geometric realizations of simplicial objects and totalizations of cosimplicial objects. Then the pairing $\lambda : \mathrm{Alg}(\mathrm{TwArr}(\mathcal{C})) \to \mathrm{Alg}(\mathcal{C}) \times \mathrm{Alg}(\mathcal{C}^{op})$ is both left and right representable. We therefore obtain adjoint functors

$$\mathrm{Alg}(\mathcal{C}) \underset{\mathfrak{D}'_\lambda}{\overset{\mathfrak{D}^{op}_\lambda}{\rightleftarrows}} \mathrm{cAlg}(\mathcal{C}),$$

given by the bar and cobar constructions, where $\mathrm{cAlg}(\mathcal{C}) = \mathrm{Alg}(\mathcal{C}^{op})^{op}$ is the ∞-category of coalgebra objects of $\mathcal{C}$.

More generally, if $\mathcal{C}$ is an arbitrary monoidal ∞-category which admits geometric realizations of simplicial objects and totalizations of cosimplicial objects, then by applying Theorem 4.3.1 to the ∞-category $\mathcal{C}_{\mathbf{1}//\mathbf{1}}$ we obtain an adjunction

$$\mathrm{Alg}^{\mathrm{aug}}(\mathcal{C}) \underset{\mathrm{CoBar}}{\overset{\mathrm{Bar}}{\rightleftarrows}} \mathrm{cAlg}^{\mathrm{aug}}(\mathcal{C}).$$

The proof of Theorem 4.3.1 will require some general remarks about pairings between monoidal ∞-categories.

Definition 4.3.4. Let $\mathcal{O}^{\otimes}$ be an ∞-operad. A *pairing of $\mathcal{O}$-monoidal ∞-categories* is a triple

$$(p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}, q : \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}, \lambda^{\otimes} : \mathcal{M}^{\otimes} \to \mathcal{C}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{D}^{\otimes})$$

where p and q exhibit $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ as $\mathcal{O}$-monoidal ∞-categories and $\lambda^{\otimes} : \mathcal{M}^{\otimes} \to \mathcal{C}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{D}^{\otimes}$ is a $\mathcal{O}$-monoidal functor which is a categorical fibration and which induces a right fibration $\lambda_X : \mathcal{M}_X \to \mathcal{C}_X \times \mathcal{D}_X$ after taking the fiber over any object $X \in \mathcal{O}$.

Remark 4.3.5. In the situation of Definition 4.3.4, we will generally abuse terminology by simply referring to the $\mathcal{O}$-monoidal functor $\lambda^{\otimes} : \mathcal{M}^{\otimes} \to \mathcal{C}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{D}^{\otimes}$ as a pairing of monoidal ∞-categories. In the special case where $\mathcal{O}^{\otimes} = \mathcal{A}\mathrm{ss}^{\otimes}$ is the associative ∞-operad, we will refer to $\lambda^{\otimes}$ simply as a *pairing of monoidal ∞-categories*. If $\mathcal{O}^{\otimes} = \mathrm{Comm}^{\otimes}$ is the commutative ∞-operad, we will refer to $\lambda^{\otimes}$ as a *pairing of symmetric monoidal ∞-categories*.

Example 4.3.6. Recall that the forgetful functor

$$(\lambda : \mathcal{M} \to \mathcal{C} \times \mathcal{D}) \mapsto \mathcal{C}$$

induces an equivalence $\mathrm{CPair}^{\mathrm{perf}} \to \mathcal{C}\mathrm{at}_{\infty}$, whose homotopy inverse is given on objects by $\mathcal{C} \mapsto (\lambda : \mathrm{TwArr}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{op})$ (see Remark 4.2.12). Let $\mathcal{O}^{\otimes}$ be an ∞-operad and let $\mathcal{C}$ be a $\mathcal{O}$-monoidal ∞-category, which we can identify with a $\mathcal{O}$-monoid object in the ∞-category $\mathcal{C}\mathrm{at}_{\infty}$. It follows that $\mathrm{TwArr}(\mathcal{C})$ admits the structure of a $\mathcal{O}$-monoid object of $\mathrm{CPair}^{\mathrm{perf}}$, which we can identify with a pairing of $\mathcal{O}$-monoidal ∞-categories

$$\mathrm{TwArr}(\mathcal{C})^{\otimes} \to \mathcal{C}^{\otimes} \times_{\mathcal{O}^{\otimes}} (\mathcal{C}^{op})^{\otimes}.$$

Remark 4.3.7. Let $\lambda^{\otimes} : \mathcal{M}^{\otimes} \to \mathcal{C}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{D}^{\otimes}$ be a pairing of $\mathcal{O}$-monoidal ∞-categories. Then the induced map $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{M}) \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) \times \mathrm{Alg}_{/\mathcal{O}}(\mathcal{D})$ is a pairing of ∞-categories. This follows immediately from Corollary A.3.2.2.3.

In particular, if $\lambda^{\otimes}$ is a pairing of monoidal ∞-categories, then it induces a pairing

$$\mathrm{Alg}(\lambda) : \mathrm{Alg}(\mathcal{M}) \to \mathrm{Alg}(\mathcal{C}) \times \mathrm{Alg}(\mathcal{D}).$$

The key step in the proof of Theorem 4.3.1 is to establish a criterion for verifying that $\mathrm{Alg}(\lambda)$ is left (or right) representable.

Proposition 4.3.8. *Let $\lambda^{\otimes} : \mathcal{M}^{\otimes} \to \mathcal{C}^{\otimes} \times_{\mathcal{A}\mathrm{ss}^{\otimes}} \mathcal{D}^{\otimes}$ be a pairing of monoidal ∞-categories. Assume that:*

(1) *If $\mathbf{1}$ denotes the unit object of $\mathcal{D}$, then the right fibration $\mathcal{M} \times_{\mathcal{D}} \{\mathbf{1}\} \to \mathcal{C}$ is a categorical equivalence.*

(2) *The underlying pairing $\lambda : \mathcal{M} \to \mathcal{C} \times \mathcal{D}$ is left representable.*

(3) *The ∞-category $\mathcal{D}$ admits totalizations of cosimplicial objects.*

Then the induced pairing $\mathrm{Alg}(\lambda) : \mathrm{Alg}(\mathcal{M}) \to \mathrm{Alg}(\mathcal{C}) \times \mathrm{Alg}(\mathcal{D})$ is left representable.

The proof of Proposition 4.3.8 will occupy our attention for most of this section. We begin by treating an easy special case.

Proposition 4.3.9. *Let $\lambda^{\otimes} : \mathcal{M}^{\otimes} \to \mathcal{C}^{\otimes} \times_{\mathcal{A}\mathrm{ss}^{\otimes}} \mathcal{D}^{\otimes}$ be a pairing between monodial ∞-categories, and assume that the underlying pairing of ∞-categories $\lambda : \mathcal{M} \to \mathcal{C} \times \mathcal{D}$ is left representable. Let $A \in \mathrm{Alg}(\mathcal{C})$ be a trivial algebra object of $\mathcal{C}$ (see §A.3.2.1). Then:*

(1) *There exists a left universal object of $\mathrm{Alg}(\mathcal{M})$ lying over $A \in \mathrm{Alg}(\mathcal{C})$.*

(2) *An object $M \in \mathrm{Alg}(\mathcal{M})$ lying over $A \in \mathrm{Alg}(\mathcal{C})$ is left universal if and only if the image of M in $\mathcal{M}$ is left universal (with respect to the pairing $\lambda : \mathcal{M} \to \mathcal{C} \times \mathcal{D}$).*

Proof. We can identify $\mathrm{Alg}(\mathcal{M}) \times_{\mathrm{Alg}(\mathcal{C})} \{A\}$ with $\mathrm{Alg}(\mathcal{N})$, where $\mathcal{N}^{\otimes}$ denotes the monoidal ∞-category $\mathcal{M}^{\otimes} \times_{\mathcal{C}^{\otimes}} \mathcal{A}\mathrm{ss}^{\otimes}$. An object of $\mathrm{Alg}(\mathcal{M}) \times_{\mathrm{Alg}(\mathcal{C})} \{A\}$ is left universal if and only if it is a final object of $\mathrm{Alg}(\mathcal{N})$. Since λ is left representable, $\mathcal{N}$ has a final object. Assertions (1) and (2) are therefore immediate consequences of Corollary A.3.2.2.5. □

To prove Proposition 4.3.8 in general, we must show that an arbitrary algebra object $A \in \mathrm{Alg}(\mathcal{C})$ can be lifted to a left universal object of $\mathrm{Alg}(\mathcal{M})$. This object is not as easy to find: for example, its image in $\mathcal{M}$ is generally not left universal for the underlying pairing $\lambda : \mathcal{M} \to \mathcal{C} \times \mathcal{D}$. In order to construct it, we would like to reduce to the situation where A is a trivial algebra object of $\mathcal{C}$. We will accomplish this by replacing $\mathcal{C}$ by another monoidal ∞-category having A as the unit object: namely, the ∞-category $\mathrm{Mod}_A^{\mathcal{A}\mathrm{ss}}(\mathcal{C}) \simeq {}_A\mathrm{BMod}_A(\mathcal{C})$ (see Theorem A.4.3.4.28).

Lemma 4.3.10. *Let $\lambda^{\otimes} : \mathcal{M}^{\otimes} \to \mathcal{C}^{\otimes} \times_{\mathcal{A}\mathrm{ss}^{\otimes}} \mathcal{D}^{\otimes}$ be a pairing of monoidal ∞-categories, and let $M \in \mathrm{Alg}(\mathcal{M})$ have image $(A, B) \in \mathrm{Alg}(\mathcal{C}) \times \mathrm{Alg}(\mathcal{D})$. Then the induced map*

$$ {}_M\mathrm{BMod}_M(\mathcal{M})^{\otimes} \to {}_A\mathrm{BMod}_A(\mathcal{C})^{\otimes} \times_{\mathcal{A}\mathrm{ss}^{\otimes}} {}_B\mathrm{BMod}_B(\mathcal{D})^{\otimes} $$

is also a pairing of monoidal ∞-categories.

Proof. It will suffice to show that the map ${}_M\mathrm{BMod}_M(\mathcal{M}) \to {}_A\mathrm{BMod}_A(\mathcal{C}) \times {}_B\mathrm{BMod}_B(\mathcal{D})$ is a right fibration. This map is a pullback of the categorical fibration

$$ \theta : \mathrm{BMod}(\mathcal{M}) \to (\mathrm{BMod}(\mathcal{C}) \times \mathrm{BMod}(\mathcal{D})) \times_{\mathrm{Alg}(\mathcal{C}) \times \mathrm{Alg}(\mathcal{D})} \mathrm{Alg}(\mathcal{M}). $$

It will therefore suffice to show that θ is a right fibration. Let

$$ \theta' : (\mathrm{BMod}(\mathcal{C}) \times \mathrm{BMod}(\mathcal{D})) \times_{\mathrm{Alg}(\mathcal{C})^2 \times \mathrm{Alg}(\mathcal{D})^2} \mathrm{Alg}(\mathcal{M})^2 \to \mathrm{BMod}(\mathcal{C}) \times \mathrm{BMod}(\mathcal{D}) $$

be the projection map. Since θ is a categorical fibration, it will suffice to show that θ' and $\theta' \circ \theta$ are right fibrations. The map θ' is a pullback of the forgetful functor $\mathrm{Alg}(\lambda) : \mathrm{Alg}(\mathcal{M}) \to \mathrm{Alg}(\mathcal{C}) \times \mathrm{Alg}(\mathcal{D})$. It will therefore suffice to show that $\mathrm{Alg}(\lambda)$ and $\theta' \circ \theta$ are right fibrations, which follows immediately from Corollary A.3.2.2.3. □

Proposition 4.3.11. *Let $\lambda^{\otimes} : \mathcal{M}^{\otimes} \to \mathcal{C}^{\otimes} \times_{\mathcal{A}\mathrm{ss}^{\otimes}} \mathcal{D}^{\otimes}$ be a pairing of monoidal ∞-categories. Let $M \in \mathrm{Alg}(\mathcal{M})$ have image $(A, B) \in \mathrm{Alg}(\mathcal{C}) \times \mathrm{Alg}(\mathcal{D})$, and identify A and B with their images in $\mathcal{C}$ and $\mathcal{D}$, respectively. Assume that B is a trivial algebra object of $\mathcal{D}$ and that, for every object $C \in \mathcal{C}$, the Kan complex $\lambda^{-1}\{(C, B)\} \subseteq \mathcal{M}$ is contractible. Then the forgetful functor $\mathrm{Alg}({}_M\mathrm{BMod}_M(\mathcal{M})) \to \mathrm{Alg}(\mathcal{M})$ carries left universal objects of $\mathrm{Alg}({}_M\mathrm{BMod}_M(\mathcal{M}))$ to left universal objects of $\mathrm{Alg}(\mathcal{M})$.*

Proof. It will suffice to show that for every $A' \in \mathrm{Alg}({}_A\mathrm{BMod}_A(\mathcal{C})) \simeq \mathrm{Alg}(\mathcal{C})_{A/}$ having image $A'_0 \in \mathrm{Alg}(\mathcal{C})$, the left fibration $\mathrm{Alg}({}_M\mathrm{BMod}_M(\mathcal{M})) \times_{\mathrm{Alg}({}_A\mathrm{BMod}_A(\mathcal{C}))} \{A'\} \to \mathrm{Alg}(\mathcal{M}) \times_{\mathrm{Alg}(\mathcal{C})} \{A'_0\}$ is an equivalence of ∞-categories (and therefore carries final objects to final objects). Since B is a trivial algebra object of $\mathcal{D}$, the forgetful functor $\mathrm{Alg}({}_B\mathrm{BMod}_B(\mathcal{D})) \to \mathrm{Alg}(\mathcal{D})$ is an equivalence of ∞-categories. It will therefore suffice to show that for each $B' \in \mathrm{Alg}({}_B\mathrm{BMod}_B(\mathcal{D}))$ having image B'_0 in $\mathrm{Alg}({}_B\mathrm{BMod}_B(\mathcal{D}))$, the induced map

$$ \mathrm{Alg}({}_M\mathrm{BMod}_M(\mathcal{M})) \times_{\mathrm{Alg}({}_A\mathrm{BMod}_A(\mathcal{C})) \times \mathrm{Alg}({}_B\mathrm{BMod}_B(\mathcal{D}))} \{(A', B')\} \to \mathrm{Alg}(\mathcal{M}) \times_{\mathrm{Alg}(\mathcal{C}) \times \mathrm{Alg}(\mathcal{D})} \{(A'_0, B'_0)\} $$

is a homotopy equivalence of Kan complexes. For this, it suffices to show that $M \in \mathrm{Alg}(\mathcal{M})$ is a p-initial object, where $p : \mathrm{Alg}(\mathcal{M}) \to \mathrm{Alg}(\mathcal{C}) \times \mathrm{Alg}(\mathcal{D})$ denotes the projection. Since p is a right fibration, it suffices to verify that M is an initial object of $\mathrm{Alg}(\mathcal{M}) \times_{\mathrm{Alg}(\mathcal{C}) \times \mathrm{Alg}(\mathcal{D})} \{(A, B)\}$. This ∞-category is is given by a homotopy fiber of the map $\phi : \mathrm{Alg}(\mathcal{N}) \to \mathrm{Alg}(\mathcal{C})$, where $\mathcal{N}^{\otimes} = \mathcal{M}^{\otimes} \times^{\otimes}_{\mathcal{D}} \mathcal{A}\mathrm{ss}^{\otimes}$. We now observe that ϕ is a categorical equivalence, since the monoidal functor $\mathcal{N} \to \mathcal{C}$ is a categorical equivalence (by virtue of the fact that it is a right fibration whose fibers are contractible Kan complexes). □

Notation 4.3.12. Let $\lambda^{\otimes} : \mathcal{M}^{\otimes} \to \mathcal{C}^{\otimes} \times_{\mathcal{A}\mathrm{ss}^{\otimes}} \mathcal{D}^{\otimes}$ be a pairing of monoidal ∞-categories. If $M \in \mathrm{Alg}(\mathcal{M})$ has image $(A, B) \in \mathrm{Alg}(\mathcal{C}) \times \mathrm{Alg}(\mathcal{D})$, we let λ_M denote the induced pairing ${}_M\mathrm{BMod}_M(\mathcal{M}) \to {}_A\mathrm{BMod}_A(\mathcal{C}) \times {}_B\mathrm{BMod}_B(\mathcal{D})$.

Lemma 4.3.13. *Let $\lambda^{\otimes} : \mathcal{M}^{\otimes} \to \mathcal{C}^{\otimes} \times_{\mathcal{A}\mathrm{ss}^{\otimes}} \mathcal{D}^{\otimes}$ be a pairing of monoidal ∞-categories, and let $M \in \mathrm{Alg}(\mathcal{M})$ be an object having image $(A, B) \in \mathrm{Alg}(\mathcal{C}) \times \mathrm{Alg}(\mathcal{D})$, where B is a trivial algebra object of $\mathcal{D}$. Let $F : \mathcal{M} \to {}_M\mathrm{BMod}_M(\mathcal{M})$ be a left adjoint to the forgetful functor, given by $V \mapsto M \otimes V \otimes M$ (Corollary A.4.3.3.14). Then F carries left universal objects of $\mathcal{M}$ (with respect to the pairing $\lambda : \mathcal{M} \to \mathcal{C} \times \mathcal{D}$) to left universal objects of ${}_M\mathrm{BMod}_M(\mathcal{M})$ (with respect to the pairing $\lambda_M : {}_M\mathrm{BMod}_M(\mathcal{M}) \to {}_A\mathrm{BMod}_A(\mathcal{C}) \times {}_B\mathrm{BMod}_B(\mathcal{D})$).*

Proof. Let $F' : \mathcal{C} \to {}_A\mathrm{BMod}_A(\mathcal{C})$ and $F'' : \mathcal{D} \to {}_B\mathrm{BMod}_B(\mathcal{D})$ be left adjoints to the forgetful functors $G' : {}_A\mathrm{BMod}_A(\mathcal{C}) \to \mathcal{C}$ and $G'' : {}_B\mathrm{BMod}_B(\mathcal{D}) \to \mathcal{D}$. We may assume without loss of generality that the diagram

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "F"] \arrow[d] & {}_M\mathrm{BMod}_M(\mathcal{M}) \arrow[d] \\
\mathcal{C} \times \mathcal{D} \arrow[r, "F' \times F''"] & {}_A\mathrm{BMod}_A(\mathcal{C}) \times {}_B\mathrm{BMod}_B(\mathcal{D})
\end{tikzcd}$$

is commutative. For each $C \in \mathcal{C}$, F induces a functor

$$f : \mathcal{M} \times_{\mathcal{C}} \{C\} \to {}_M\mathrm{BMod}_M(\mathcal{M}) \times_{{}_A\mathrm{BMod}_A(\mathcal{C})} \{F'(C)\}.$$

We note that f has a right adjoint g, given by composing the forgetful functor ${}_M\mathrm{BMod}_M(\mathcal{M}) \times_{{}_A\mathrm{BMod}_A(\mathcal{C})} \{F'(C)\} \to \mathcal{M} \times_{\mathcal{C}} \{(G' \circ F')(C)\}$ with the pullback functor $\mathcal{M} \times_{\mathcal{C}} \{(G' \circ F'(C)\} \to \mathcal{M} \times_{\mathcal{C}} \{C\}$ associated to the unit map $C \to (G' \circ F')(C)$. To show that f preserves final objects, it will suffice to show that g is a homotopy inverse to f. Let $u : \mathrm{id} \to g \circ f$ be the unit map. For every object $V \in \mathcal{M} \times_{\mathcal{C}} \{C\}$ having image $D \in \mathcal{D}$, the unit map $u_V : V \to (g \circ f)(V)$ has image in $\mathcal{D}$ equivalent to the unit map $D \to (G'' \circ F'')(D) \simeq B \otimes D \otimes B$ in $\mathcal{D}$. Since B is a trivial algebra, we conclude that the image of u_V in $\mathcal{D}$ is an equivalence. Because the map $\mathcal{M} \times_{\mathcal{C}} \{C\} \to \mathcal{D}$ is a right fibration, we conclude that u_V is an equivalence. A similar argument shows that the counit map $v : f \circ g \to \mathrm{id}$ is an equivalence of functors, so that g is homotopy inverse to f as desired. ☐

Lemma 4.3.14. *Let $f : \overline{\mathcal{C}} \to \mathcal{C}$ be a right fibration of ∞-categories, classified by a map $\chi : \mathcal{C}^{op} \to \mathcal{S}$ and suppose we are given a diagram $\overline{p} : K^{\triangleright} \to \mathcal{C}$. The following conditions are equivalent:*

(1) *For every commutative diagram σ :*

$$\begin{tikzcd}
K \arrow[r, "q"] \arrow[d] & \overline{\mathcal{C}} \arrow[d, "f"] \\
K^{\triangleright} \arrow[r, "\overline{p}"] \arrow[ur, dashed, "\overline{q}"] & \mathcal{C}
\end{tikzcd}$$

there exists an extension $\overline{q}$ as indicated, which is an f-colimit diagram.

(2) *The restriction $\chi|(K^{\triangleright})^{op}$ is a limit diagram in $\mathcal{S}$.*

If $\overline{p}$ is a colimit diagram in $\mathcal{C}$, then these conditions are equivalent to the following:

(3) *For every diagram σ as in (1), the diagram $q : K \to \overline{\mathcal{C}}$ can be extended to a colimit diagram in $\overline{\mathcal{C}}$, whose image in $\mathcal{C}$ is also a colimit diagram.*

Proof. The equivalence of (1) and (2) follows from Lemma VII.5.17, and the equivalence of (1) and (3) from Proposition T.4.3.1.5. ☐

We will need the following refinement of Corollary A.4.2.3.5:

Lemma 4.3.15. *Let $\mathcal{M}$ be an ∞-category which is left-tensored over a monoidal ∞-category $\mathcal{C}$, let A be an algebra object of $\mathcal{C}$, and let $\theta : \mathrm{LMod}_A(\mathcal{M}) \to \mathcal{M}$ denote the forgetful functor. Let $p : K \to \mathrm{LMod}_A(\mathcal{M})$ be a diagram and let $p_0 = \theta \circ p$. Suppose that p_0 can be extended to an operadic colimit diagram $\overline{p}_0 : K^{\triangleleft} \to \mathcal{M}$ (in other words, $\overline{p}_0$ has the property that for every object $C \in \mathcal{C}$, the composite map*

$$K^{\triangleright} \stackrel{\overline{p}_0}{\to} \mathcal{M} \stackrel{C\otimes}{\to} \mathcal{M}$$

is also a colimit diagram in $\mathcal{M}$). Then:

(1) *The diagram p extends to a colimit diagram $\overline{p} : K^{\triangleleft} \to \mathrm{LMod}_A(\mathcal{M})$.*

(2) *Let $\overline{p} : K^{\triangleleft} \to \mathrm{LMod}_A(\mathcal{M})$ be an arbitrary extension of p. Then $\overline{p}$ is a colimit diagram if and only if $\theta \circ \overline{p}$ is a colimit diagram.*

Proof. Let $q : \mathcal{M}^{\circledast} \to \mathcal{C}^{\circledast}$ be the locally coCartesian fibration defined in Notation A.4.2.2.16. The algebra object A determines a map $\mathrm{N}(\mathbf{\Delta})^{op} \to \mathcal{C}^{\circledast}$. Let $\mathcal{X}$ denote the fiber product $\mathrm{N}(\mathbf{\Delta})^{op} \times_{\mathcal{C}^{\circledast}} \mathcal{M}^{\circledast}$ and let $\mathrm{LMod}_A(\mathcal{M}) \to {}^{\Delta}\mathrm{LMod}_A(\mathcal{M}) \subseteq \mathrm{Fun}_{\mathrm{N}(\mathbf{\Delta})^{op}}(\mathrm{N}(\mathbf{\Delta})^{op}, \mathcal{X})$ be the equivalence of Corollary A.4.2.2.15. It follows from Lemma A.3.2.2.9 that the composite map $K \to \mathrm{LMod}_A(\mathcal{M}) \to {}^{\Delta}\mathrm{LMod}_A(\mathcal{M})$ can be extended to a colimit diagram in $\mathrm{Fun}_{\mathrm{N}(\mathbf{\Delta})^{op}}(\mathrm{N}(\mathbf{\Delta})^{op}, \mathcal{X})$, that this diagram factors through ${}^{\Delta}\mathrm{LMod}_A(\mathcal{M})$, and that its image in $\mathcal{M}$ is a colimit diagram. This proves (1) and the "only if" direction of (2). To prove the "if" direction of (2), let us suppose we are given an arbitrary extension $\overline{p} : K^{\triangleleft} \to \mathrm{LMod}_A(\mathcal{M})$, carrying the cone point to a left A-module M. Then $\overline{p}$ determines a map $\alpha : \varinjlim(p) \to M$. If the image of $\overline{p}$ in $\mathcal{M}$ is a colimit diagram, then the image of α in $\mathcal{M}$ is a colimit diagram. Since the forgetful functor $\mathrm{LMod}_A(\mathcal{M}) \to \mathcal{M}$ is conservative (Corollary A.4.2.3.2), we deduce that α is an equivalence. It follows that $\overline{p}$ is a colimit diagram as desired. □

Example 4.3.16. Let $\mathcal{M}$ be an ∞-category which is left tensored over a monoidal ∞-category $\mathcal{C}^{\otimes}$, let $A \in \mathrm{Alg}(\mathcal{C})$, and $\theta : \mathrm{LMod}_A(\mathcal{M}) \to \mathcal{M}$ denote the forgetful functor. If $M_\bullet$ is a θ-split simplicial object of $\mathrm{LMod}_A(\mathcal{M})$, then the underlying map $\mathrm{N}(\mathbf{\Delta})^{op} \to \mathrm{LMod}_A(\mathcal{M})$ satisfies the hypothesis of Lemma 4.3.15. It follows that $M_\bullet$ admits a geometric realization in $\mathrm{LMod}_A(\mathcal{M})$ which is preserved by the forgetful functor θ.

Example 4.3.17. Let $\mathcal{M}$ be an ∞-category which is bitensored over the a pair of monoidal ∞-categories $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$, and suppose we are given algebra objects $A \in \mathrm{Alg}(\mathcal{C})$ and $B \in \mathrm{Alg}(\mathcal{D})$. Let $\theta : \mathrm{RMod}_B(\mathcal{M}) \to \mathcal{M}$ denote the forgetful functor, and regard $\mathrm{RMod}_B(\mathcal{M})$ as an ∞-category left-tensored over $\mathcal{C}$ (see §A.4.3.2). Let $\mu : {}_A\mathrm{BMod}_B(\mathcal{M}) \simeq \mathrm{LMod}_A(\mathrm{RMod}_B(\mathcal{M})) \to \mathrm{RMod}_B(\mathcal{M})$ be the forgetful functor, and let $M_\bullet$ be a simplicial object of ${}_A\mathrm{BMod}_B(\mathcal{M})$. Assume that $M_\bullet$ is $\theta \circ \mu$-split. Then $\mu(M_\bullet)$ is a θ-split simplicial object of $\mathrm{RMod}_B(\mathcal{M})$. It follows from Example 4.3.16 that $\mu(M_\bullet)$ admits a geometric realization in $\mathrm{RMod}_B(\mathcal{M})$. For every object $C \in \mathcal{C}$, the diagram

$$\begin{tikzcd}
\mathrm{RMod}_B(\mathcal{M}) \arrow[r, "C\otimes"] \arrow[d, "\theta"] & \mathrm{RMod}_B(\mathcal{M}) \arrow[d, "\theta"] \\
\mathcal{M} \arrow[r, "C\otimes"] & \mathcal{M}
\end{tikzcd}$$

commutes up to homotopy. It follows that the formation of the geometric realization of $\mu(M_\bullet)$ is preserved by operation of tensor product with C. Applying Lemma 4.3.15, we deduce that $M_\bullet$ admits a geometric realization in ${}_A\mathrm{BMod}_B(\mathcal{M})$, which is preserved by the forgetful functor μ. This proves the following:

($*$) Let $M_\bullet$ be ν-split simplicial object of ${}_A\mathrm{BMod}_B(\mathcal{M})$, where $\nu = \theta \circ \mu : {}_A\mathrm{BMod}_B(\mathcal{M}) \to \mathcal{M}$ is the forgetful functor. Then $M_\bullet$ admits a geometric realization in ${}_A\mathrm{BMod}_B(\mathcal{M})$, which is preserved by the functor ν.

Remark 4.3.18. In the situation of Example 4.3.17, the forgetful functor $\nu : {}_A\mathrm{BMod}_B(\mathcal{M}) \to \mathcal{M}$ is conservative (since the functors θ and μ are conservative, by Corollary A.4.2.3.2). Applying Theorem A.6.2.2.5, we deduce that the functor ν is *monadic*: that is, it exhibits ${}_A\mathrm{BMod}_B(\mathcal{M})$ as the ∞-category of representations of a monad T on the ∞-category $\mathcal{M}$. The underlying functor of T is given on objects by $M \mapsto A \otimes M \otimes B$.

Remark 4.3.19. Let $G : \mathcal{C} \to \mathcal{D}$ be any monadic functor between ∞-categories, so that $\mathcal{C} \simeq \mathrm{Mod}_T(\mathcal{D})$ for some monad T on $\mathcal{D}$. Let F be a left adjoint to G. Every object $C \in \mathcal{C}$ can be (canonically) realized as the geometric realization of a G-split simplicial object $C_\bullet$ of $\mathcal{C}$, where each $C_n \simeq F \circ T^n \circ G(C)$ lies in the essential image of F.

Lemma 4.3.20. *Let $f : \overline{\mathcal{C}} \to \mathcal{C}$ be a right fibration of ∞-categories and let $\overline{C}_\bullet$ be a simplicial object of $\overline{\mathcal{C}}$. If $f(\overline{C}_\bullet)$ is a split simplicial object of $\mathcal{C}$, then $\overline{C}_\bullet$ is a split simplicial object of $\overline{\mathcal{C}}_\bullet$.*

Proof. This is an immediate consequence of the characterization of split simplicial objects given in Corollary A.6.2.1.7. □

Lemma 4.3.21. *Let $\lambda^\otimes : \mathcal{M}^\otimes \to \mathcal{C}^\otimes \times_{\mathcal{A}\mathrm{ss}^\otimes} \mathcal{D}^\otimes$ be a pairing of monoidal ∞-categories, and let $M \in \mathrm{Alg}(\mathcal{M})$ be an object having image $(A, B) \in \mathrm{Alg}(\mathcal{C}) \times \mathrm{Alg}(\mathcal{D})$. Assume that:*

(1) *The object $B \in \mathrm{Alg}(\mathcal{D})$ is a trivial algebra in $\mathcal{D}$.*

(2) *The pairing $\lambda : \mathcal{M} \to \mathcal{C} \times \mathcal{D}$ is left representable.*

(3) *The ∞-category $\mathcal{D}$ admits totalizations of cosimplicial objects.*

Then the induced pairing $\lambda_M : {}_M\mathrm{BMod}_M(\mathcal{M}) \to {}_A\mathrm{BMod}_A(\mathcal{C}) \times {}_B\mathrm{BMod}_B(\mathcal{D})$ is left representable.

Proof. Fix an object $C \in {}_A\mathrm{BMod}_A(\mathcal{C})$; we wish to show that C can be lifted to a left universal object of ${}_M\mathrm{BMod}_M(\mathcal{M})$. Let $p' : {}_A\mathrm{BMod}_A(\mathcal{C}) \to \mathcal{C}$ denote the forgetful functor. Using Example 4.3.17, we deduce that there is a p'-split simplicial object $C_\bullet$ of ${}_A\mathrm{BMod}_A(\mathcal{C})$ having colimit C, where each C_n lies in the essential image of the left adjoint to p'.

Fix an object $D \in \mathcal{D}$, let $\mathcal{M}_D$ denote the fiber product $\mathcal{M} \times_{\mathcal{D}} \{D\}$, and consider the induced right fibration $\theta : \mathcal{M}_D \to \mathcal{C}$. Condition (1) implies that D lifts uniquely to an object $\overline{D} \in {}_B\mathrm{BMod}_B(\mathcal{D})$. Set $\mathcal{N} = {}_M\mathrm{BMod}_M(\mathcal{M}) \times_{{}_B\mathrm{BMod}_B(\mathcal{D})} \{D'\}$, so that the projection map $\mathcal{N} \to {}_A\mathrm{BMod}_A(\mathcal{C})$ is a right fibration classified by a map $\chi_D : {}_A\mathrm{BMod}_A(\mathcal{C})^{op} \to \mathcal{S}$. We claim that the canonical map $\chi_D(C) \to \varinjlim \chi_D(C_\bullet)$ is a homotopy equivalence. To prove this, it will suffice (Lemma 4.3.14) to show that for every simplicial object $N_\bullet$ of $\mathcal{N}$ lifting $C_\bullet$, there exists a geometric realization $|N_\bullet|$ which is preserved by the forgetful functor $q : \mathcal{N} \to {}_A\mathrm{BMod}_A(\mathcal{C})$. Let $p : \mathcal{N} \to \mathcal{M}_D$ denote the forgetful functor. Since $q' : \mathcal{M}_D \to \mathcal{C}$ is a right fibration, it follows from Lemma 4.3.20 that $p(N_\bullet)$ is a split simplicial object of $\mathcal{M}_D$. Since $\mathcal{N}$ can be identified with an ∞-category of bimodule objects of $\mathcal{M}_D$, Example 4.3.17 implies that $N_\bullet$ has a colimit N in $\mathcal{N}$ such that $p(N) \simeq |p(N_\bullet)|$. Since $p(N_\bullet)$ is split, we conclude that the colimit of $N_\bullet$ is preserved by $q' \circ p \simeq p' \circ q$. Using Lemma 4.3.20 again, we conclude that the colimit of $N_\bullet$ is preserved by q.

The pairing λ_M is classified by a functor $\chi' : {}_A\mathrm{BMod}_A(\mathcal{C})^{op} \to \mathrm{Fun}({}_B\mathrm{BMod}_B(\mathcal{D})^{op}, \mathcal{S})$. It follows from the above arguments that $\chi'(C) \simeq \varprojlim \chi'(C_\bullet)$. We wish to prove that $\chi'(C)$ is representable. Using condition (3), we are reduced to proving that each $\chi'(C_n)$ is a representable functor. This follows immediately from (2) and Lemma 4.3.13. □

Proof of Proposition 4.3.8. Let $A \in \mathrm{Alg}(\mathcal{C})$; we wish to show that there is a left universal object of $\mathrm{Alg}(\mathcal{M})$ lying over A. Let B be a trivial algebra object of $\mathcal{D}$, so that condition (1) implies that the right fibration $\mathcal{M} \times_{\mathcal{D}} \{B\} \to \mathcal{C}$ is an equivalence of (monoidal) ∞-categories. It follows that the pair (A, B) can be lifted to an object $M \in \mathrm{Alg}(\mathcal{M})$ in an essentially unique way. Using Proposition 4.3.11, we are reduced to proving that there exists a left universal object of $\mathrm{Alg}({}_M\mathrm{BMod}_M(\mathcal{M}))$ lying over $A \in \mathrm{Alg}({}_A\mathrm{BMod}_A(\mathcal{C}))$. Since A is the unit object of ${}_A\mathrm{BMod}_A(\mathcal{C})$, it suffices to lift A to a left universal object of ${}_M\mathrm{BMod}_M(\mathcal{M})$ (Proposition 4.3.9). The existence of such a lift now follows from Lemma 4.3.21. □

Remark 4.3.22. Let $\lambda^\otimes : \mathcal{M}^\otimes \to \mathcal{C}^\otimes \times_{\mathcal{A}\mathrm{ss}^\otimes} \mathcal{D}^\otimes$ be a pairing of monoidal ∞-categories satisfying the hypotheses of Proposition 4.3.8. Then $\mathrm{Alg}(\lambda) : \mathrm{Alg}(\mathcal{M}) \to \mathrm{Alg}(\mathcal{C}) \times \mathrm{Alg}(\mathcal{D})$ is a left representable pairing, and therefore induces a duality functor $\mathfrak{D}_{\mathrm{Alg}(\lambda)} : \mathrm{Alg}(\mathcal{C})^{op} \to \mathrm{Alg}(\mathcal{D})$. By unravelling the proof, we can obtain a more explicit description of this functor. Namely, let B be a trivial algebra object of $\mathcal{D}$ and lift the pair (A, B) to an algebra object $M \in \mathrm{Alg}(\mathcal{M})$. Then, as an object of $\mathcal{D}$, we can identify $\mathfrak{D}_{\mathrm{Alg}(\lambda)}(A)$ with

$\mathfrak{D}_{\lambda_M}(A)$. To compute the latter, we need to resolve $A \in {}_A\mathrm{BMod}_A(\mathcal{C})$ by free bimodules. The equivalence $A \simeq A \otimes_A A$ gives a simplicial resolution $A_\bullet$ of A by free bimodules $A_m \simeq A \otimes (A^{\otimes m}) \otimes A$. Then

$$\mathfrak{D}_{\mathrm{Alg}(\lambda)}(A) \simeq \mathfrak{D}_{\lambda_M}(A) \simeq \varprojlim \mathfrak{D}_{\lambda_M}(A_\bullet) \simeq \varprojlim_{[m] \in \mathbf{\Delta}} \mathfrak{D}_\lambda(A^{\otimes m}),$$

where the last equivalence follows from Proposition 4.3.11.

Proof of Theorem 4.3.1. Let $\mathcal{C}$ be a monoidal ∞-category and let $\mathrm{Alg}(\lambda) : \mathrm{Alg}(\mathrm{TwArr}(\mathcal{C})) \to \mathrm{Alg}(\mathcal{C}) \times \mathrm{Alg}(\mathcal{C}^{op})$ be the canonical map. Assertion (1) of the Theorem follows from Remark 4.3.7, which asserts that λ is a pairing of ∞-categories. To complete the proof, it will suffice to verify assertion (2) (Remark 4.3.2). Assume that the unit object $\mathbf{1} \in \mathcal{C}$ is final and that $\mathcal{C}$ admits geometric realizations of simplicial objects. Then the pairing $\lambda : \mathrm{TwArr}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{op}$ satisfies the hypotheses of Proposition 4.3.8, so that λ is left representable. In particular, we have a duality functor $\mathfrak{D}_{\mathrm{Alg}(\lambda)} : \mathrm{Alg}(\mathcal{C})^{op} \to \mathrm{Alg}(\mathcal{C}^{op})$.

Note that the underlying pairing of ∞-categories $\lambda : \mathrm{TwArr}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{op}$ induces the identity map $\mathfrak{D}_\lambda : \mathcal{C}^{op} \to \mathcal{C}^{op}$ (Remark 4.2.6). Let $A \in \mathrm{Alg}(\mathcal{C})$ be arbitrary and let $M \in \mathrm{Alg}(\mathrm{TwArr}(\mathcal{C}))$ be as in the proof of Proposition 4.3.8. Then the duality functor

$$\mathfrak{D}_{\lambda_M} : {}_A\mathrm{BMod}_A(\mathcal{C})^{op} \to \mathcal{C}^{op}$$

is right adjoint to the forgetful functor ${}_A\mathrm{BMod}_A(\mathcal{C})^{op} \to \mathcal{C}^{op}$, and therefore given by the two-sided bar construction $C \mapsto \mathbf{1} \otimes_A C \otimes_A \mathbf{1}$. In particular, we see that $\mathfrak{D}_{\mathrm{Alg}(\lambda)}$ carries A to the object

$$\mathbf{1} \otimes_A A \otimes_A \mathbf{1} \simeq \mathbf{1} \otimes_A \mathbf{1} \in \mathcal{C}.$$

□

4.4 Koszul Duality for $\mathbb{E}_n$-Algebras

Our goal in this section is to study the operation of Koszul duality in the setting of augmented $\mathbb{E}_n$-algebras over a field k. More precisely, we will construct a self-adjoint functor

$$\mathfrak{D}^{(n)} : (\mathrm{Alg}_k^{(n),\mathrm{aug}})^{op} \to \mathrm{Alg}_k^{(n),\mathrm{aug}}.$$

Our main result asserts that for large class of augmented $\mathbb{E}_n$-algebras A, the unit map $A \to \mathfrak{D}^{(n)}\mathfrak{D}^{(n)}A$ is an equivalence (Theorem 4.4.5).

We begin with the definition of the Koszul duality functor $\mathfrak{D}^{(n)}$. Let A be an $\mathbb{E}_n$-algebra over a field k. An *augmentation* on A is a map of $\mathbb{E}_n$-algebras $A \to k$. We let $\mathrm{Aug}(A) = \mathrm{Map}_{\mathrm{Alg}_k^{(n)}}(A, k)$ denote the space of augmentations on A. If we are given a pair of augmented $\mathbb{E}_n$-algebras $\epsilon : A \to k$ and $\epsilon' : B \to k$, we let $\mathrm{Pair}(A, B)$ denote the homotopy fiber of the map $\mathrm{Aug}(A \otimes_k B) \to \mathrm{Aug}(A) \times \mathrm{Aug}(B)$, taken over the point (ϵ, ϵ'). More informally, we can describe $\mathrm{Pair}(A, B)$ as the space of augmentations on $A \otimes_k B$ which extend the given augmentations on A and B. We will refer to the points of $\mathrm{Pair}(A, B)$ as *pairings* of A with B. The starting point for the theory of Koszul duality is the following fact:

Proposition 4.4.1. *Let k be a field, $n \geq 0$ an integer, and A be an augmented $\mathbb{E}_n$-algebra over k. Then the construction $B \mapsto \mathrm{Pair}(A, B)$ determines a representable functor from $(\mathrm{Alg}_k^{(n),\mathrm{aug}})^{op}$ into $\mathcal{S}$. That is, there exists an augmented $\mathbb{E}_n$-algebra $\mathfrak{D}^{(n)}(A)$ and a pairing $\nu : A \otimes_k \mathfrak{D}^{(n)}(A) \to k$ with the following universal property: for every augmented $\mathbb{E}_n$-algebra B, composition with ν induces a homotopy equivalence*

$$\mathrm{Map}_{\mathrm{Alg}_k^{(n),\mathrm{aug}}}(B, \mathfrak{D}^{(n)}(A)) \to \mathrm{Pair}(A, B).$$

In the situation of Proposition, we will refer to $\mathfrak{D}^{(n)}(A)$ as the *Koszul dual* of A. The construction $A \mapsto \mathfrak{D}^{(n)}(A)$ determines a functor $\mathfrak{D}^{(n)} : (\mathrm{Alg}_k^{(n),\mathrm{aug}})^{op} \to \mathrm{Alg}_k^{(n),\mathrm{aug}}$, which we will refer to as *Koszul duality*.

Example 4.4.2. Suppose that $n = 0$. Then the construction $V \mapsto k \oplus V$ defines an equivalence from the ∞-category Mod_k of k-module spectra to the ∞-category $\mathrm{Alg}^{(n)}_{\mathrm{aug}}$. If V and W are objects of Mod_k, then a pairing of V with W is a k-linear map

$$\phi : (k \oplus V) \otimes_k (k \oplus W) \simeq k \oplus V \oplus W \oplus (V \otimes_k W) \to k$$

equipped with homotopies $\phi|k \simeq \mathrm{id}$, $\phi|V \simeq 0 \simeq \phi|W$. It follows that we can identify $\mathrm{Pair}(k \oplus V, k \oplus W)$ with the space $\mathrm{Map}_{\mathrm{Mod}_k}(V \otimes_k W, k)$. It follows that the Koszul duality functor $\mathfrak{D}^{(0)}$ is given by $k \oplus V \mapsto k \oplus V^\vee$, where $V^\vee$ is the k-linear dual of V (with homotopy groups given by $\pi_i V^\vee \simeq \mathrm{Hom}_k(\pi_{-i}V, k)$).

Example 4.4.3. When $n = 1$, the Koszul duality functor $\mathfrak{D}^{(1)} : (\mathrm{Alg}^{\mathrm{aug}}_k)^{op} \to \mathrm{Alg}^{\mathrm{aug}}_k$ agrees with the functor studied in §3.1.

Remark 4.4.4. The construction $A, B \mapsto \mathrm{Pair}(A, B)$ is symmetric in A and B. Consequently, for any pair of augmented $\mathbb{E}_n$-algebras A and B, we have homotopy equivalences

$$\mathrm{Hom}_{\mathrm{Alg}^{(n),\mathrm{aug}}_k}(B, \mathfrak{D}^{(n)}(A)) \simeq \mathrm{Pair}(A, B) \simeq \mathrm{Pair}(B, A) \simeq \mathrm{Hom}_{\mathrm{Alg}^{(n),\mathrm{aug}}_k}(A, mathfrakD^{(n)}(B)).$$

In particular, the tautological pairing $A \otimes_k \mathfrak{D}^{(n)}(A) \to k$ can be identified with a point of $\mathrm{Pair}(\mathfrak{D}^{(n)}(A), A)$, which is classified by a *biduality map* $u_A : A \to \mathfrak{D}^{(n)}\mathfrak{D}^{(n)}(A)$. Our main goal in this section is to study conditions which guarantee that u_A is an equivalence.

We can now state the main result of this section:

Theorem 4.4.5. *Let $n \geq 0$ and let A be an augmented $\mathbb{E}_n$-algebra over a field k. If A is n-coconnective and locally finite, then the biduality map $u_A : A \to \mathfrak{D}^{(n)}\mathfrak{D}^{(n)}(A)$ is an equivalence of augmented $\mathbb{E}_n$-algebras over k.*

The remainder of this section is devoted to the proofs of Proposition 4.4.1 and Theorem 4.4.5. We begin with Proposition 4.4.1. If A is an augmented $\mathbb{E}_n$-algebra over a field k, it is easy to deduce the existence of the Koszul dual $\mathfrak{D}^{(n)}(A)$ from the general formalism developed in §A.6.1.4: we can describe $\mathfrak{D}^{(n)}(A)$ as a centralizer of the augmentation map $\epsilon : A \to k$ (see Example A.6.1.4.14). We will give a somewhat different proof here, one which suggests methods of calculating with the Koszul duality functor $\mathfrak{D}^{(n)}$ (which will be needed in the proof of Theorem 4.4.5). Our first step is to translate the definition of Koszul duality into the language of pairings of monoidal ∞-categories, developed in §4.3.

Construction 4.4.6. Fix a field k. We let $\mathrm{Alg}^{(0)}_k \simeq (\mathrm{Mod}_k)_{k/}$ denote the ∞-category whose objects are k-module spectra A equipped with a unit map $k \to A$. We will identify k with the initial object of $\mathrm{Alg}^{(0)}_k$, and let $\mathrm{Alg}^{(0),\mathrm{aug}}_k$ denote the ∞-category $(\mathrm{Alg}^{(0)}_k)_{/k}$. There is a canonical equivalence of ∞-categories $\mathrm{Mod}_k \to \mathrm{Alg}^{(0),\mathrm{aug}}_k$, given by $V \mapsto k \oplus V$.

The symmetric monoidal structure on Mod_k endows $\mathrm{Alg}^{(0)}_k$ with a symmetric monoidal structure (see §A.2.2.2). Let $m : \mathrm{Alg}^{(0)}_k \times \mathrm{Alg}^{(0)}_k \to \mathrm{Alg}^{(0)}_k$ be the tensor product functor, and let $p_0, p_1 : \mathrm{Alg}^{(0)}_k \times \mathrm{Alg}^{(0)}_k \to \mathrm{Alg}^{(0)}_k$ be the projection maps onto the first and second factor, respectively. Since the unit object of $\mathrm{Alg}^{(0)}_k$ is initial, there are natural transformations $p_0 \stackrel{\alpha_0}{\to} m \stackrel{\alpha_1}{\leftarrow} p_1$, given by a map $\mathrm{Alg}^{(0)}_k \times \mathrm{Alg}^{(0)}_k \to \mathrm{Fun}(\Lambda^2_2, \mathrm{Alg}^{(0)}_k)$. We let $\mathcal{M}$ denote the fiber product

$$(\mathrm{Alg}^{(0)}_k \times \mathrm{Alg}^{(0)}_k) \times_{\mathrm{Fun}(\Lambda^2_2, \mathrm{Alg}^{(0)}_k)} \mathrm{Fun}(\Lambda^2_2, \mathrm{Alg}^{(0),\mathrm{aug}}_k).$$

There is an evident pair of forgetful functors

$$\mathrm{Alg}^{(0),\mathrm{aug}}_k \leftarrow \mathcal{M} \to \mathrm{Alg}^{(0),\mathrm{aug}}_k$$

which determine a right fibration $\lambda : \mathcal{M} \to \mathrm{Alg}^{(0),\mathrm{aug}}_k \times \mathrm{Alg}^{(0),\mathrm{aug}}_k$.

The symmetric monoidal structure on Mod_k induces a symmetric monoidal structure on the ∞-categories $\mathrm{Alg}_k^{(0)}$, $\mathrm{Alg}_k^{(0),\mathrm{aug}}$, and $\mathcal{M}$. The forgetful functor λ promotes to a symmetric monoidal functor

$$\lambda^{\otimes} : \mathcal{M}^{\otimes} \to (\mathrm{Alg}_k^{(0),\mathrm{aug}})^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} (\mathrm{Alg}_k^{(0),\mathrm{aug}})^{\otimes}.$$

For every integer $n \geq 0$, we let $\mathcal{M}^{(n)} = \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{M})$. Theorem A.5.1.2.2 gives a canonical equivalence $\mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Alg}_k^{(0)}) \simeq \mathrm{Alg}_k^{(n)}$, which in turn induces an equivalence of ∞-categories $\mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Alg}_k^{(0),\mathrm{aug}}) \simeq \mathrm{Alg}_k^{(n),\mathrm{aug}}$. It follows that λ induces a right fibration $\lambda^{(n)} : \mathcal{M}^{(n)} \to \mathrm{Alg}_k^{(n),\mathrm{aug}} \times \mathrm{Alg}_k^{(n),\mathrm{aug}}$ (see Example 4.3.6).

Remark 4.4.7. In the special case $n = 1$, the right fibration

$$\lambda^{(n)} : \mathcal{M}^{(n)} \to \mathrm{Alg}_k^{(n),\mathrm{aug}} \times \mathrm{Alg}_k^{(n),\mathrm{aug}}$$

agrees with the pairing of Construction 3.1.4.

It is not difficult to see that right fibration $\lambda^{(n)} : \mathcal{M}^{(n)} \to \mathrm{Alg}_k^{(n),\mathrm{aug}} \times \mathrm{Alg}_k^{(n),\mathrm{aug}}$ is classified by the functor $\mathbf{Pr} : (\mathrm{Alg}_k^{(n),\mathrm{aug}})^{op} \times (\mathrm{Alg}_k^{(n),\mathrm{aug}})^{op} \to \mathcal{S}$ introduced at the beginning of this section. We may therefore reformulate Proposition 4.4.1 as follows:

Proposition 4.4.8. *Let k be a field and let $n \geq 0$ be an integer. Then the pairing $\lambda^{(n)} : \mathcal{M}^{(n)} \to \mathrm{Alg}_k^{(n),\mathrm{aug}} \times \mathrm{Alg}_k^{(n),\mathrm{aug}}$ of Construction 4.4.6 is both right and left representable.*

Proof. We will show that $\lambda^{(n)}$ is left representable; the proof of right representability is identical. We proceed by induction on n, the case $n = 0$ being trivial (see Example 4.4.2). To carry out the inductive step, let us assume that $\lambda^{(n)}$ is left representable; we wish to prove that $\lambda^{(n+1)}$ is left representable. Let $\mathcal{M} \to \mathrm{Alg}_k^{(0),\mathrm{aug}} \times \mathrm{Alg}_k^{(0),\mathrm{aug}}$ be as in Construction 4.4.6. The symmetric monoidal structure on Mod_k induces a symmetric monoidal structure on $\mathcal{M}$ and $\mathrm{Alg}_k^{(0),\mathrm{aug}}$, hence (symmetric) monoidal structures on $\mathcal{M}^{(n)}$ and $\mathrm{Alg}_k^{(n),\mathrm{aug}}$. It follows that $\lambda^{(n)}$ can be regarded as a pairing of monoidal ∞-categories

$$\lambda^{(n),\otimes} : \mathcal{M}^{(n),\otimes} \to (\mathrm{Alg}_k^{(n),\mathrm{aug}})^{\otimes} \times (\mathrm{Alg}_k^{(n),\mathrm{aug}})^{\otimes}.$$

Using Theorem A.5.1.2.2, we can identify $\lambda^{(n+1)}$ with the induced pairing

$$\mathrm{Alg}(\lambda^{(n)}) : \mathrm{Alg}(\mathcal{M}^{(n)}) \to \mathrm{Alg}(\mathrm{Alg}_k^{(n),\mathrm{aug}}) \times \mathrm{Alg}(\mathrm{Alg}_k^{(n),\mathrm{aug}}).$$

We will prove that this pairing is left representable by verifying the hypotheses of Proposition 4.3.8:

(1) The right fibration $\mathcal{M}^{(n)} \times_{\mathrm{Alg}_k^{(n),\mathrm{aug}}} \{k\} \to \mathrm{Alg}_k^{(n),\mathrm{aug}}$ is a categorical equivalence. Unwinding the definitions, we must show that the Koszul dual $\mathfrak{D}^{(n)}(k)$ is equivalent to the final object $k \in \mathrm{Alg}_k^{(n),\mathrm{aug}}$, which follows immediately from the definitions.

(2) The pairing $\lambda^{(n)}$ is left representable: this follows from the inductive hypothesis.

(3) The ∞-category $\mathrm{Alg}_k^{(n),\mathrm{aug}}$ admits totalizations of cosimplicial objects. In fact, $\mathrm{Alg}_k^{(n),\mathrm{aug}}$ is a presentable ∞-category, and therefore admits all limits and colimits.

□

Let us now outline our strategy for proving Theorem 4.4.5. Our proof will proceed by induction on n. The main idea is to understand the Koszul duality functor $\mathfrak{D}^{(n+1)}$ as a mixture of Koszul duality for $\mathfrak{D}^{(n)}$ and the bar construction studied in §4.3. For this, we would like to compare the pairing of monoidal ∞-categories

$$\lambda^{(n),\otimes} : \mathcal{M}^{(n),\otimes} \to (\mathrm{Alg}_k^{(n),\mathrm{aug}})^{\otimes} \times (\mathrm{Alg}_k^{(n),\mathrm{aug}})^{\otimes}$$

appearing in the proof of Proposition 4.4.8 with the pairing of monoidal ∞-categories

$$\mathrm{TwArr}(\mathrm{Alg}_k^{(n),\mathrm{aug}})^{\otimes} \to (\mathrm{Alg}_k^{(n),\mathrm{aug}})^{\otimes} \times (\mathrm{Alg}_k^{(n),\mathrm{aug}})^{op,\otimes}$$

of Example 4.3.6. We will do so by constructing a morphism of monoidal pairings

$$\begin{tikzcd} \mathrm{TwArr}(\mathrm{Alg}_k^{(n),\mathrm{aug}})^{\otimes} \arrow[rr] \arrow[d] & & \mathcal{M}^{(n),\otimes} \arrow[d] \\ (\mathrm{Alg}_k^{(n),\mathrm{aug}})^{\otimes} \times (\mathrm{Alg}_k^{(n),\mathrm{aug}})^{\otimes} \arrow[rr, "\mathrm{id} \times \mathfrak{D}^{(n)}"] & & (\mathrm{Alg}_k^{(n),\mathrm{aug}})^{\otimes} \times (\mathrm{Alg}_k^{(n),\mathrm{aug}})^{op,\otimes}. \end{tikzcd}$$

If we ignore the monoidal structures, this morphism of pairings can be obtained by invoking the universal property of $\mathrm{TwArr}(\mathrm{Alg}_k^{(n),\mathrm{aug}})$ (Proposition 4.2.10). However, we will need to work harder, because the horizontal maps appearing in the above diagram are not monoidal functors (recall that the Koszul duality functor $\mathfrak{D}^{(n)}$ does not commute with tensor products in general; see Warning 3.1.20). To carry out the construction, we will need a relative version of the twisted arrow construction $\mathcal{C} \mapsto \mathrm{TwArr}(\mathcal{C})$ introduced in §4.2.

Construction 4.4.9. Let $\lambda : \mathcal{M} \to \mathcal{C} \times \mathcal{D}$, be a pairing of ∞-categories, classified by a functor $\chi : \mathcal{D}^{op} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}) = \mathcal{P}(\mathcal{C})$. Let $j : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ be the Yoneda embedding, and set

$$\mathcal{C}_\lambda = \mathcal{C} \times_{\mathrm{Fun}(\{0\} \times \mathcal{C}^{op}, \mathcal{S})} \mathrm{Fun}(\Delta^1 \times \mathcal{C}^{op}, \mathcal{S}) \times_{\mathrm{Fun}(\{1\} \times \mathcal{C}^{op}, \mathcal{S})} \mathcal{D}^{op}.$$

Let $e_0 : \mathcal{C}_\lambda \to \mathcal{C}$ and $e_1 : \mathcal{C}_\lambda \to \mathcal{D}^{op}$ be the two projection maps, so that we have a natural transformation $\alpha : (j \circ e_0) \to (\chi \circ e_1)$ of functors $\mathcal{C}_\lambda \to \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$. The functor $j \circ e_0$ classifies a right fibration $\mu : \mathrm{TwArr}_\lambda(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}_\lambda^{op}$, which we regard as a pairing of ∞-categories. We will refer to $\mathrm{TwArr}_\lambda(\mathcal{C})$ as the *∞-category of twisted arrows of $\mathcal{C}$ relative to λ*.

Note that α classifies a map $\gamma : \mathrm{TwArr}_\lambda(\mathcal{C}) \to \mathcal{M} \times_{\mathcal{D}} \mathcal{C}_\lambda^{op}$ of right fibrations over $\mathcal{C} \times \mathcal{C}_\lambda^{op}$. We therefore obtain a morphism of pairings

$$\begin{tikzcd} \mathrm{TwArr}_\lambda(\mathcal{C}) \arrow[r, "\gamma"] \arrow[d, "\mu"] & \mathcal{M} \arrow[d, "\lambda"] \\ \mathcal{C} \times \mathcal{C}_\lambda^{op} \arrow[r, "\mathrm{id} \times e_1"] & \mathcal{C} \times \mathcal{D}. \end{tikzcd}$$

Example 4.4.10. In the setting of Construction 4.4.9, suppose that $\mathcal{D} = \Delta^0$ and that λ is the identity map from $\mathcal{C}$ to itself. In this case, the evaluation map $e_0 : \mathcal{C}_\lambda \to \mathcal{C}$ is an equivalence, and the right fibration $\mathrm{TwArr}_\lambda(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}_\lambda^{op}$ classifies the Yoneda pairing

$$\mathcal{C}^{op} \times \mathcal{C}_\lambda \simeq \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{S}.$$

Applying Proposition 4.2.5, we deduce that the pairing $\mathrm{TwArr}_\lambda(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}_\lambda^{op}$ is equivalent to the pairing $\mathrm{TwArr}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{op}$ of Construction 4.2.3 (this can also be deduced by comparing the universal properties of $\mathrm{TwArr}(\mathcal{C})$ and $\mathrm{TwArr}_\lambda(\mathcal{C})$ given by Proposition 4.2.10 and 4.4.11, respectively).

Proposition 4.4.11. *Let $\lambda : \mathcal{M} \to \mathcal{C} \times \mathcal{D}$ be a pairing of ∞-categories, let $\mu : \mathrm{TwArr}_\lambda(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}_\lambda^{op}$ be as in Construction 4.4.9. Then:*

(1) *The pairing μ is right representable.*

(2) *Let $\lambda' : \mathcal{M}' \to \mathcal{C}' \times \mathcal{D}'$ be an arbitrary right representable pairing of ∞-categories. Then composition with the canonical morphism $\mu \to \lambda$ induces a homotopy equivalence*

$$\theta : \mathrm{Map}_{\mathrm{CPair}^R}(\lambda', \mu) \to \mathrm{Map}_{\mathrm{CPair}}(\lambda, \mu).$$

Corollary 4.4.12. *The inclusion functor* $\mathrm{CPair}^R \hookrightarrow \mathrm{CPair}$ *admits a right adjoint, given on objects by the construction*

$$(\lambda : \mathcal{M} \to \mathcal{C} \times \mathcal{D}) \mapsto (\mathrm{TwArr}_\lambda(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}_\lambda^{op}).$$

Proof of Proposition 4.4.11. We have a commutative diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{CPair}^R}(\lambda', \mu) \arrow[rr, "\theta"] \arrow[dr, "q"'] & & \mathrm{Map}_{\mathrm{CPair}}(\lambda', \lambda) \arrow[dl, "p"] \\
& \mathrm{Fun}(\mathcal{C}', \mathcal{C})^{\simeq} \times \mathrm{Fun}(\mathcal{D}', \mathcal{D})^{\simeq}. &
\end{tikzcd}$$

To prove that θ is a homotopy equivalence, it will suffice to show that θ induces a homotopy equivalence of homotopy fibers over any pair of functors $(F : \mathcal{C}' \to \mathcal{C}, G : \mathcal{D}' \to \mathcal{D})$. It now suffice to observe that both homotopy fibers can be identified with the mapping space $\mathrm{Map}_{\mathrm{Fun}(\mathcal{C}'^{op} \times \mathcal{D}'^{op}, \mathcal{S})}(\chi', \chi \circ (F \times G))$, where χ and χ' classify the right fibrations λ and λ', respectively. □

Remark 4.4.13. Let $\lambda : \mathcal{M} \to \mathcal{C} \times \mathcal{D}$ be a pairing of ∞-categories and let $\mu : \mathrm{TwArr}_\lambda(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}_\lambda^{op}$ be the pairing of Construction 4.4.9. Assume that λ is left representable, so that the duality functor $\mathfrak{D}_\lambda : \mathcal{C}^{op} \to \mathcal{D}$ is defined. Unwinding the definitions, we see that $\mathcal{C}_\lambda^{op}$ is equivalent to the ∞-category $\mathcal{C}^{op} \times_{\mathcal{D}} \mathrm{Fun}(\Delta^1, \mathcal{D})$ whose objects are triples (C, D, ϕ) where $C \in \mathcal{C}^{op}$, $D \in \mathcal{D}$, and $\phi : D \to \mathfrak{D}_\lambda(C)$ is a morphism in $\mathcal{D}$. In particular, the forgetful functor $\mathcal{C}_\lambda^{op} \to \mathcal{C}^{op}$ admits a fully faithful left adjoint L, whose essential image is spanned by those triples (C, D, ϕ) where $\phi : D \to \mathfrak{D}_\lambda(C)$ is an equivalence in $\mathcal{D}$. We will denote this essential image by $(\mathcal{C}_\lambda^0)^{op}$, and we let $\mathrm{TwArr}_\lambda^0(\mathcal{C})$ denote the inverse image of $\mathcal{C} \times (\mathcal{C}_\lambda^0)^{op}$ in $\mathrm{TwArr}_\lambda(\mathcal{C})$.

Note that $(\mathcal{C}_\lambda^0)^{op}$ is a localization of $\mathcal{C}_\lambda^{op}$. Moreover, if f is a morphsim in $\mathcal{C}_\lambda^{op}$, then Lf is an equivalence if and only if the image of f in $\mathcal{C}^{op}$ is an equivalence.

Remark 4.4.14. Suppose we are given a morphism of pairings

$$\begin{tikzcd}
\mathcal{M} \arrow[r] \arrow[d, "\lambda"] & \mathcal{M}' \arrow[d, "\lambda'"] \\
\mathcal{C} \times \mathcal{D} \arrow[r] & \mathcal{C} \times \mathcal{D}'.
\end{tikzcd}$$

We then obtain an induced right representable morphism of pairings

$$\begin{tikzcd}
\mathrm{TwArr}_\lambda(\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{TwArr}_{\lambda'}(\mathcal{C}) \arrow[d] \\
\mathcal{C} \times \mathcal{C}_\lambda^{op} \arrow[r] & \mathcal{C} \times \mathcal{C}_{\lambda'}^{op}.
\end{tikzcd}$$

Taking $\mathcal{D}' = \Delta^0$ and $\mathcal{M}' = \mathcal{C}$, we obtain a morphism of pairings

$$\begin{tikzcd}
\mathrm{TwArr}_\lambda(\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{TwArr}(\mathcal{C}) \arrow[d] \\
\mathcal{C} \times \mathcal{C}_\lambda^{op} \arrow[r] & \mathcal{C} \times \mathcal{C}^{op}
\end{tikzcd}$$

(see Example 4.4.10). If λ is left representable, this morphism restricts to an equivalence

$$\begin{tikzcd}
\mathrm{TwArr}_\lambda^0(\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{TwArr}(\mathcal{C}) \arrow[d] \\
\mathcal{C} \times (\mathcal{C}_\lambda^0)^{op} \arrow[r] & \mathcal{C} \times \mathcal{C}^{op},
\end{tikzcd}$$

where the pairing on the left is defined as in Remark 4.4.13.

Construction 4.4.15. Suppose we are given a pairing of symmetric monoidal ∞-categories

$$\lambda^{\otimes} : \mathcal{M}^{\otimes} \to \mathcal{C}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathcal{D}^{\otimes},$$

which we can identify with a commtuative monoid object of the ∞-category CPair^R. Applying the right adjoint to the inclusion $\mathrm{CPair}^R \hookrightarrow \mathrm{CPair}$, we see that the pairing $\mathrm{TwArr}_\lambda(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{op}_\lambda$ of Construction 4.4.9 can be promoted to a commutative monoid object of CPair^R, corresponding to another pairing of symmetric monoidal ∞-categories

$$\mu^{\otimes} : \mathrm{TwArr}_\lambda(\mathcal{C})^{\otimes} \to \mathcal{C}^{\otimes} \times (\mathcal{C}^{op}_\lambda)^{\otimes}.$$

We obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{TwArr}(\mathcal{C})^{\otimes} \arrow[d] & \mathrm{TwArr}_\lambda(\mathcal{C})^{\otimes} \arrow[l] \arrow[r, "\gamma"] \arrow[d] & \mathcal{M}^{\otimes} \arrow[d] \\
\mathcal{C}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} (\mathcal{C}^{op})^{\otimes} & \mathcal{C}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} (\mathcal{C}^{op}_\lambda)^{\otimes} \arrow[l] \arrow[r] & \mathcal{C}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathcal{D}^{\otimes}
\end{tikzcd}$$

where the horizontal maps are symmetric monoidal functors.

Now suppose that λ is left representable. The localization functor L appearing in Remark 4.4.13 is compatible with the symmetric monoidal structure on $\mathcal{C}^{op}_\lambda$ (in the sense of Definition A.2.2.1.6), so that the full subcategory $\mathcal{C}^0_\lambda)^{op} \subseteq (\mathcal{C}^0_\lambda)^{op}$ inherits a symmetric monoidal structure. Moreover, since the projection map $\mathcal{C}^{op}_\lambda \to \mathcal{C}^{op}$ carries L-equivalences to equivalences, it induces a symmetric monoidal functor $\beta : (\mathcal{C}^0_\lambda)^{op,\otimes} \to (\mathcal{C}^{op})^{\otimes}$. Since the underlying functor $(\mathcal{C}^0_\lambda)^{op} \to \mathcal{C}^{op}$ is an equivalence, we conclude that β is an equivalence. Let $\mathrm{TwArr}^0_\lambda(\mathcal{C})^{\otimes}$ denote the fiber product

$$\mathrm{TwArr}_\lambda(\mathcal{C}) \times_{(\mathcal{C}^{op}_\lambda)^{\otimes}} (\mathcal{C}^0_\lambda)^{op,\otimes},$$

so that we have an equivalence of symmetric monoidal pairings

$$\begin{tikzcd}
\mathrm{TwArr}(\mathcal{C})^{\otimes} \arrow[d] & \mathrm{TwArr}^0_\lambda(\mathcal{C})^{\otimes} \arrow[l] \arrow[d] \\
\mathcal{C}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} (\mathcal{C}^{op})^{\otimes} & \mathcal{C}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} (\mathcal{C}^0_\lambda)^{op,\otimes} \arrow[l]
\end{tikzcd}$$

Composing a homotopy inverse of this equivalence with γ, we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{TwArr}(\mathcal{C})^{\otimes} \arrow[r] \arrow[d] & \mathcal{M}^{\otimes} \arrow[d] \\
\mathcal{C}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} (\mathcal{C}^{op})^{\otimes} \arrow[r] & \mathcal{C}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathcal{D}^{\otimes} .
\end{tikzcd}$$

Note that the horizontal maps in this diagram are merely lax symmetric monoidal functors in general.

Remark 4.4.16. We can informally summarize the conclusion of Construction 4.4.15 as follows: if we given a pairing of symmetric monoidal ∞-categories

$$\lambda^{\otimes} : \mathcal{M}^{\otimes} \to \mathcal{C}^{\otimes} \times_{\mathcal{A}\mathrm{ss}^{\otimes}} \mathcal{D}^{\otimes}$$

for which the underlying pairing $\lambda : \mathcal{M} \to \mathcal{C} \times \mathcal{D}$ is left dualizable, then the duality map $\mathfrak{D}_\lambda : \mathcal{C}^{op} \to \mathcal{D}$ of Construction 3.1.3 has the structure of a lax symmetric monoidal functor.

Remark 4.4.17. Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category. Assume that the unit object $\mathbf{1} \in \mathcal{C}$ is final and that $\mathcal{C}$ admits geometric realizations of simplicial objects. Let $\lambda^{\otimes} : \mathrm{TwArr}(\mathcal{C})^{\otimes} \to \mathcal{C}^{\otimes} \times_{\mathrm{Comm}^{\otimes}} (\mathcal{C}^{op})^{\otimes}$ be the induced pairing of symmetric monoidal ∞-categories (see Example 4.3.6), so that $\lambda^{\otimes}$ determines a duality functor $\mathfrak{D}_{\mathrm{Alg}(\lambda)} : \mathrm{Alg}(\mathcal{C})^{op} \to \mathrm{Alg}(\mathcal{C}^{op})$ (Proposition 4.3.8). Note that $\lambda^{\otimes}$ induces a pairing of symmetric monoidal ∞-categories. $\mathrm{Alg}(\lambda)^{\otimes} : \mathrm{Alg}(\mathrm{TwArr}(\mathcal{C}))^{\otimes} \to \mathrm{Alg}(\mathcal{C})^{\otimes} \times \mathrm{Alg}(\mathcal{C}^{op})^{\otimes}$. It follows from Remark 4.4.16 that we can identify $\mathfrak{D}_{\mathrm{Alg}(\lambda)}$ with a lax symmetric monoidal functor from $\mathrm{Alg}(\mathcal{C})^{op}$ to $\mathrm{Alg}(\mathcal{C})^{op}$. Concretely, this structure arises from the observation that for every pair of algebra objects $A, B \in \mathrm{Alg}(\mathcal{C}) \simeq \mathrm{Alg}^{\mathrm{aug}}(\mathcal{C})$, there is a canonical equivalence

$$\alpha : \mathrm{Bar}_A(\mathbf{1}, \mathbf{1})_\bullet \otimes \mathrm{Bar}_B(\mathbf{1}, \mathbf{1})_\bullet \to \mathrm{Bar}_{A \otimes B}(\mathbf{1}, \mathbf{1})_\bullet$$

of simplicial objects of $\mathcal{C}$. If we assume that the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realizations of simplicial objects, then α induces an equivalence $\mathrm{Bar}(A) \otimes \mathrm{Bar}(B) \to \mathrm{Bar}(A \otimes B)$, so that the lax symmetric monoidal functor $\mathfrak{D}_{\mathrm{Alg}(\lambda)} : \mathrm{Alg}(\mathcal{C})^{op} \to \mathrm{Alg}(\mathcal{C}^{op})$ is actually symmetric monoidal.

In the situation of Construction 4.4.15, we obtain an induced morphism of pairings between algebra objects τ :

$$\begin{tikzcd}
\mathrm{Alg}(\mathrm{TwArr}(\mathcal{C})) \arrow[r, "\beta"] \arrow[d] & \mathrm{Alg}(\mathcal{M}) \arrow[d] \\
\mathrm{Alg}(\mathcal{C}) \times \mathrm{Alg}(\mathcal{C}^{op}) \arrow[r] & \mathrm{Alg}(\mathcal{C}) \times \mathrm{Alg}(\mathcal{D}).
\end{tikzcd}$$

Proposition 4.3.8 supplies conditions which guarantee that the vertical maps in this diagram are left representable pairings. For applications to Theorem 4.4.5, we also need a criterion which will guarantee that the map β preserves left universal objects. For this, we have the following somewhat technical result:

Proposition 4.4.18. *Suppose we are given pairings of monoidal ∞-categories*

$$\lambda^{\otimes} : \mathcal{M}^{\otimes} \to \mathcal{C}^{\otimes} \times_{\mathcal{A}\mathrm{ss}^{\otimes}} \mathcal{D}^{\otimes} \qquad \lambda'^{\otimes} : \mathcal{M}'^{\otimes} \to \mathcal{C}^{\otimes} \times_{\mathcal{A}\mathrm{ss}^{\otimes}} \mathcal{D}'^{\otimes}.$$

Let $\alpha : \mathcal{C}^{\otimes} \to \mathcal{C}'^{\otimes}$, $\beta : \mathcal{D}^{\otimes} \to \mathcal{D}'^{\otimes}$ and $\gamma : \mathcal{M}^{\otimes} \to \mathcal{M}'^{\otimes}$ be maps of planar ∞-operads which render the diagram

$$\begin{tikzcd}
\mathcal{M}^{\otimes} \arrow[r, "\gamma"] \arrow[d, "\lambda^{\otimes}"] & \mathcal{M}'^{\otimes} \arrow[d, "\lambda'^{\otimes}"] \\
\mathcal{C}^{\otimes} \times_{\mathcal{A}\mathrm{ss}^{\otimes}} \mathcal{D}^{\otimes} \arrow[r, "\alpha \times \beta"] & \mathcal{C}'^{\otimes} \times_{\mathcal{A}\mathrm{ss}^{\otimes}} \mathcal{D}'^{\otimes}
\end{tikzcd}$$

commutes. Assume that:

(1) *If $\mathbf{1}_{\mathcal{D}}$ and $\mathbf{1}_{\mathcal{D}'}$ are the unit objects of $\mathcal{D}$ and $\mathcal{D}'$, respectively, then the right fibrations $\mathcal{M} \times_{\mathcal{D}} \{\mathbf{1}_{\mathcal{D}}\} \to \mathcal{C}$ and $\mathcal{M}' \times_{\mathcal{D}'} \{\mathbf{1}_{\mathcal{D}'}\} \to \mathcal{C}'$ are categorical equivalences.*

(2) *The pairings $\lambda : \mathcal{M} \to \mathcal{C} \times \mathcal{D}$ and $\lambda' : \mathcal{M}' \to \mathcal{C}' \times \mathcal{D}'$ are left representable.*

(3) *The ∞-categories $\mathcal{D}$ and $\mathcal{D}'$ admit totalizations of cosimplicial objects.*

(4) *The map of planar ∞-operads α is monoidal, and the map β preserves unit objects.*

(5) *The underlying functor $\mathcal{D} \to \mathcal{D}'$ preserves totalizations of cosimplicial objects.*

(6) *The underlying morphism of pairings*

$$\begin{tikzcd}
\mathcal{M} \arrow[r] \arrow[d, "\lambda"] & \mathcal{M}' \arrow[d, "\lambda'"] \\
\mathcal{C} \times \mathcal{D} \arrow[r] & \mathcal{C}' \times \mathcal{D}'
\end{tikzcd}$$

is left representable: that is, γ carries left universal objects of $\mathcal{M}$ to left universal objects of $\mathcal{M}'$.

Then the induced morphism of pairings

$$\begin{tikzcd} \mathrm{Alg}(\mathcal{M}) \arrow[r] \arrow[d, "\mathrm{Alg}(\lambda)"] & \mathrm{Alg}(\mathcal{M}') \arrow[d, "\mathrm{Alg}(\lambda')"] \\ \mathrm{Alg}(\mathcal{C}) \times \mathrm{Alg}(\mathcal{D}) \arrow[r] & \mathrm{Alg}(\mathcal{C}') \times \mathrm{Alg}(\mathcal{D}') \end{tikzcd}$$

is left representable. In particular, the diagram

$$\begin{tikzcd} \mathrm{Alg}(\mathcal{C})^{op} \arrow[r] \arrow[d, "\mathfrak{D}_{\mathrm{Alg}(\lambda)}"] & \mathrm{Alg}(\mathcal{C}')^{op} \arrow[d, "\mathfrak{D}_{\mathrm{Alg}(\lambda')}"] \\ \mathrm{Alg}(\mathcal{D}) \arrow[r] & \mathrm{Alg}(\mathcal{D}') \end{tikzcd}$$

commutes up to canonical homotopy (see Proposition 3.3.4).

Example 4.4.19. Let $F : \mathcal{C}^{\otimes} \to \mathcal{C}'^{\otimes}$ be a monoidal functor between monoidal ∞-categories. Assume that the underlying ∞-categories $\mathcal{C}$ and $\mathcal{C}'$ admit geometric realizations, that the underlying functor $\mathcal{C} \to \mathcal{C}'$ preserves geometric realizations, and that the unit objects of $\mathcal{C}$ and $\mathcal{C}'$ are final. Then F induces a morphism between pairings of monoidal ∞-categories

$$\begin{tikzcd} \mathrm{TwArr}(\mathcal{C})^{\otimes} \arrow[r] \arrow[d, "\lambda^{\otimes}"] & \mathrm{TwArr}(\mathcal{C}')^{\otimes} \arrow[d, "\lambda'^{\otimes}"] \\ \mathcal{C}^{\otimes} \times (\mathcal{C}^{op})^{\otimes} \arrow[r] & \mathcal{C}'^{\otimes} \times (\mathcal{C}'^{op})^{\otimes} \end{tikzcd}$$

and therefore a morphism of pairings

$$\begin{tikzcd} \mathrm{Alg}(\mathrm{TwArr}(\mathcal{C})) \arrow[r] \arrow[d, "\mathrm{Alg}(\lambda)"] & \mathrm{Alg}(\mathrm{TwArr}(\mathcal{C}')) \arrow[d] \\ \mathrm{Alg}(\mathcal{C}) \times \mathrm{Alg}(\mathcal{C}^{op}) \arrow[r] & \mathrm{Alg}(\mathcal{C}') \times \mathrm{Alg}(\mathcal{C}'^{op}). \end{tikzcd}$$

Theorem 4.3.1 shows that the pairings $\mathrm{Alg}(\lambda)$ and $\mathrm{Alg}(\lambda')$ are left representable, and Proposition 4.4.18 shows that the functor $\mathrm{Alg}(\mathrm{TwArr}(\mathcal{C})) \to \mathrm{Alg}(\mathrm{TwArr}(\mathcal{C}'))$ preserves left universal objects. Using Proposition 3.3.4 we see that the diagram

$$\begin{tikzcd} \mathrm{Alg}(\mathcal{C})^{op} \arrow[r, "\mathfrak{D}_{\mathrm{Alg}(\lambda)}"] \arrow[d] & \mathrm{Alg}(\mathcal{C}^{op}) \arrow[d] \\ \mathrm{Alg}(\mathcal{C}')^{op} \arrow[r, "\mathfrak{D}_{\mathrm{Alg}(\lambda')}"] & \mathrm{Alg}(\mathcal{C}'^{op}) \end{tikzcd}$$

commutes up to canonical homotopy.

If we assume that $\mathcal{C}$ and $\mathcal{C}'$ admit totalizations of cosimplicial objects, that the underlying functor $\mathcal{C} \to \mathcal{C}'$ preserves totalizations of cosimplicial objects, and that the unit objects of $\mathcal{C}$ and $\mathcal{C}'$ are initial, then the same arguments show that the diagram

$$\begin{tikzcd} \mathrm{Alg}(\mathcal{C}^{op}) \arrow[r, "\mathfrak{D}'_{\mathrm{Alg}(\lambda)}"] \arrow[d] & \mathrm{Alg}(\mathcal{C})^{op} \arrow[d] \\ \mathrm{Alg}(\mathcal{C}'^{op}) \arrow[r, "\mathfrak{D}'_{\mathrm{Alg}(\lambda')}"] & \mathrm{Alg}(\mathcal{C}')^{op} \end{tikzcd}$$

commutes up to canonical homotopy.

Example 4.4.20. Let $\lambda^{\otimes} : \mathcal{M}^{\otimes} \to \mathcal{C}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathcal{D}^{\otimes}$ be a pairing of symmetric monoidal ∞-categories. Assume that the underlying pairing $\lambda : \mathcal{M} \to \mathcal{C} \times \mathcal{D}$ is left representable, and consider the diagram σ :

$$\begin{tikzcd}
\mathrm{TwArr}(\mathcal{C})^{\otimes} \arrow[r] \arrow[d, "\mu^{\otimes}"] & \mathcal{M}^{\otimes} \arrow[d, "\lambda^{\otimes}"] \\
\mathcal{C}^{\otimes} \times_{\mathcal{A}\mathrm{ss}^{\otimes}} (\mathcal{C}^{op})^{\otimes} \arrow[r] & \mathcal{C}^{\otimes} \times_{\mathcal{A}\mathrm{ss}^{\otimes}} \mathcal{D}^{\otimes}
\end{tikzcd}$$

of Construction 4.4.15. The underlying map of pairings

$$\begin{tikzcd}
\mathrm{TwArr}(\mathcal{C}) \arrow[r] \arrow[d, "\mu"] & \mathcal{M} \arrow[d, "\lambda"] \\
\mathcal{C} \times \mathcal{C}^{op} \arrow[r] & \mathcal{C} \times \mathcal{D}
\end{tikzcd}$$

is left represntable by construction. Assume that the following further conditions are satisfied:

- (i) The ∞-category $\mathcal{C}$ admits geometric realizations of simplicial objects.
- (ii) The ∞-category $\mathcal{D}$ admits totalizations of cosimplicial objects.
- (iii) The duality functor $\mathfrak{D}_{\lambda} : \mathcal{C}^{op} \to \mathcal{D}$ preserves totalizations of cosimplicial objects.
- (iv) Let $\mathbf{1}_{\mathcal{C}}$ and $\mathbf{1}_{\mathcal{D}}$ be the unit objects of $\mathcal{C}$ and $\mathcal{D}$, respectively. Then $\mathbf{1}_{\mathcal{C}}$ and $\mathbf{1}_{\mathcal{D}}$ are final objects of $\mathcal{C}$ and $\mathcal{D}$, and the right fibrations $\mathcal{M} \times_{\mathcal{C}} \{\mathbf{1}_{\mathcal{C}}\} \to \mathcal{D}$ and $\mathcal{M} \times_{\mathcal{D}} \{\mathbf{1}_{\mathcal{D}}\} \to \mathcal{C}$ are categorical equivalences.

It follows from (iv) that the duality functor $\mathfrak{D}_{\lambda}$ carries the unit object of $\mathcal{C}$ to the unit object of $\mathcal{D}$, so that the hypotheses of Proposition 4.4.18 are satisfied. It follows that the morphism of pairings

$$\begin{tikzcd}
\mathrm{Alg}(\mathrm{TwArr}(\mathcal{C})) \arrow[r] \arrow[d, "\mathrm{Alg}(\mu)"] & \mathrm{Alg}(\mathcal{M}) \arrow[d, "\mathrm{Alg}(\lambda)"] \\
\mathrm{Alg}(\mathcal{C}) \times \mathrm{Alg}(\mathcal{C}^{op}) \arrow[r] & \mathrm{Alg}(\mathcal{C}) \times \mathrm{Alg}(\mathcal{D})
\end{tikzcd}$$

is left representable. In particular, the duality functor $\mathfrak{D}_{\mathrm{Alg}(\lambda)} : \mathrm{Alg}(\mathcal{C})^{op} \to \mathrm{Alg}(\mathcal{D})$ is given by the composition

$$\mathrm{Alg}(\mathcal{C})^{op} \xrightarrow{\mathfrak{D}_{\mathrm{Alg}(\mu)}} \mathrm{Alg}(\mathcal{C}^{op}) \xrightarrow{\phi} \mathrm{Alg}(\mathcal{D})$$

where $\mathfrak{D}_{\mathrm{Alg}(\mu)}$ is given by the bar construction of §4.3, and ϕ is given by composition with the lax symmetric monoidal functor $(\mathcal{C}^{op})^{\otimes} \to \mathcal{D}^{\otimes}$ of Remark 4.4.16 (given on objects $C \mapsto \mathfrak{D}_{\lambda}(C)$).

Proof of Proposition 4.4.18. We wish to show that the functor $\mathrm{Alg}(\mathcal{M}) \to \mathrm{Alg}(\mathcal{M}')$ determined by γ carries left universal objects to left universal objects. Let $A \in \mathrm{Alg}(\mathcal{C})$, let $B \in \mathrm{Alg}(\mathcal{D})$ be a trivial algebra so that (by (1)) the pair (A, B) can be lifted to an object $M \in \mathrm{Alg}(\mathcal{M})$ in an essentially unique way. Let $A' \in \mathrm{Alg}(\mathcal{C}')$, $B' \in \mathrm{Alg}(\mathcal{D}')$, and $M' \in \mathrm{Alg}(\mathcal{M}')$ be the images of A, B, and M; condition (4) guarantees that B' is a trivial algebra object of $\mathcal{D}'$. Using Propositions 4.3.9 and 4.3.11, we see that it suffices to show that the induced functor ${}_M\mathrm{BMod}_M(\mathcal{M}) \to {}_{M'}\mathrm{BMod}_{M'}(\mathcal{M}')$ preserves left universal objects. In other words, we must show that for $C \in {}_A\mathrm{BMod}_A(\mathcal{C})$ having image $C' \in {}_{A'}\mathrm{BMod}_{A'}(\mathcal{C}')$, the canonical map $u_C : \beta(\mathfrak{D}_{\lambda_M}(C)) \to \mathfrak{D}_{\lambda'_{M'}}(C')$ is an equivalence in $\mathcal{D}'$. Let $\theta : {}_A\mathrm{BMod}_A(\mathcal{C}) \to \mathcal{C}$ be the forgetful functor and choose a θ-split simplicial object $C_\bullet$ with $C \simeq |C_\bullet|$ such that each C_n belongs to the essential image of the left adjoint of θ. Let $C'_\bullet$ be the image of $C_\bullet$ in ${}_{A'}\mathrm{BMod}_{A'}(\mathcal{C}')$ and let $\theta' : {}_{A'}\mathrm{BMod}_{A'}(\mathcal{C}') \to \mathcal{C}'$ be the forgetful functor. The simplicial object $\theta'(C'_\bullet) = \alpha(\theta(C_\bullet))$ is split with colimit $\theta'(C') \simeq \alpha(\theta(C))$. It follows from Example 4.3.17 that the canonical map $|C'_\bullet| \to C'$ is an equivalence. Moreover, assumption (4) implies that each C'_n lies in the essential image of the left adjoint to θ'. Arguing as in proof of Lemma 4.3.21, we conclude that the maps

$$\mathfrak{D}_{\lambda_M} C \to \varprojlim \mathfrak{D}_{\lambda_M} C_\bullet \qquad \mathfrak{D}_{\lambda'_{M'}} C' \to \varprojlim \mathfrak{D}_{\lambda'_{M'}} C'_\bullet$$

are equivalences. Combining this with (5), we conclude that u_C is the totalization of the diagram $[n] \mapsto u_{C_n}$. It will therefore suffice to prove that u_{C_n} is an equivalence for each $n \geq 0$. We may therefore replace C by C_n and thereby reduce to the case where $C = \phi(\overline{C})$, where $\phi : \mathcal{C} \to {}_A\mathrm{BMod}_A(\mathcal{C})$ is a left adjoint to θ. Let $\phi' : \mathcal{C}' \to {}_{A'}\mathrm{BMod}_{A'}(\mathcal{C}')$ be a left adjoint to θ', so that condition (4) implies that $C' \simeq \phi'(\overline{C}')$ where $\overline{C}' = \alpha(\overline{C})$. Using Lemma 4.3.13, we are reduced to showing that the induced map $\beta(\mathfrak{D}_\lambda(\overline{C})) \to \mathfrak{D}_{\lambda'}(\overline{C}')$ is an equivalence, which follows immediately from (6). $\square$

We now return to the proof of Theorem 4.4.5. Let A be an augmented $\mathbb{E}_n$-algebra over a field k; we wish to prove that (under suitable assumptions) the biduality map $A \to \mathfrak{D}^{(n)}\mathfrak{D}^{(n)}(A)$ is an equivalence. This is equivalent to the requirement that, for every augmented $\mathbb{E}_n$-algebra B over k, the canonical map

$$\mathrm{Map}_{\mathrm{Alg}_k^{(n),\mathrm{aug}}}(B, A) \to \mathrm{Map}_{\mathrm{Alg}_k^{(n),\mathrm{aug}}}(B, \mathfrak{D}^{(n)}\mathfrak{D}^{(n)}(A)) \simeq \mathrm{Map}_{\mathrm{Alg}_k^{(n),\mathrm{aug}}}(\mathfrak{D}^{(n)}(A), \mathfrak{D}^{(n)}(B))$$

is a homotopy equivalence. Our strategy is to prove this using induction on n. To make the induction work, we will need to prove the following slightly stronger assertion (which immediately implies Theorem 4.4.5):

Proposition 4.4.21. *Let k be a field, let $n \geq 0$ be an integer, and suppose we are given a finite collection $\{A_1, \ldots, A_m\}$ of augmented $\mathbb{E}_n$-algebras over k. Let B be an arbitrary augmented $\mathbb{E}_n$-algebra over k. If each A_i is n-coconnective and locally finite, then the canonical map*

$$\mathrm{Map}_{\mathrm{Alg}_k^{(n),\mathrm{aug}}}(B, A_1 \otimes_k \cdots \otimes_k A_m) \to \mathrm{Map}_{\mathrm{Alg}_k^{(n),\mathrm{aug}}}(\mathfrak{D}^{(n)}A_1 \otimes_k \cdots \otimes_k \mathfrak{D}^{(n)}A_m, \mathfrak{D}^{(n)}B)$$

is a homotopy equivalence.

Remark 4.4.22. The statement Proposition 4.4.21 can be reformulated as saying that the canonical map

$$A_1 \otimes_k \cdots \otimes_k A_m \to \mathfrak{D}^{(n)}(\mathfrak{D}^{(n)}A_1 \otimes_k \cdots \otimes_k \mathfrak{D}^{(n)}A_m)$$

is an equivalence of augmented $\mathbb{E}_n$-algebras over k.

Warning 4.4.23. In the situation of Proposition 4.4.21, the tensor product $A = A_1 \otimes_k \cdots \otimes_k A_m$ is also a locally finite n-coconnective augmented $\mathbb{E}_n$-algebra over k, so that (by Theorem 4.4.5) the biduality map $A \to \mathfrak{D}^{(n)}\mathfrak{D}^{(n)}A$ is an equivalence. It follows that the map

$$A_1 \otimes_k \cdots \otimes_k A_m \to \mathfrak{D}^{(n)}(\mathfrak{D}^{(n)}A_1 \otimes_k \cdots \otimes_k \mathfrak{D}^{(n)}A_m)$$

can be identified with the Koszul dual of a map

$$\theta : \mathfrak{D}^{(n)}A_1 \otimes_k \cdots \otimes_k \mathfrak{D}^{(n)}A_m \to \mathfrak{D}^{(n)}(A).$$

With some further assumptions, one can show that θ is an equivalence (and thereby deduce Proposition 4.4.21 from Theorem 4.4.5). For example, θ is an equivalence if each A_i is $(n+1)$-coconnective. However, θ is not an equivalence in general.

The proof of Proposition 4.4.21 relies on the following general observation.

Lemma 4.4.24. *Let $F : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ be a lax symmetric monoidal functor between symmetric monoidal ∞-categories $p : \mathcal{C}^\otimes \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ and $q : \mathcal{D}^\otimes \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$. Let $\mathcal{C}_0$ be a full subcategory of $\mathcal{C}$ satisfying the following condition:*

($*$) *For every sequence of objects $\{C_i\}_{1 \leq i \leq m}$ of $\mathcal{C}_0$ and every object $C' \in \mathcal{C}$, the canonical map*

$$\mathrm{Map}_{\mathcal{C}}(\bigotimes_i C_i, C') \to \mathrm{Map}_{\mathcal{D}}(\bigotimes_i F(C_i), F(C'))$$

is a homotopy equivalence.

Let $\mathcal{O}^{\otimes}$ be an ∞-operad, and suppose we are given a sequence of algebra objects $\{A_i \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})\}_{1 \leq i \leq n}$ such that, for each $X \in \mathcal{O}$ and $1 \leq i \leq n$, we have $A_i(X) \in \mathcal{C}_0$. Then for every object $B \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$, the canonical map Then F induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})}(\bigotimes_i A_i, B) \to \mathrm{Map}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{D})}(\bigotimes_i F(A_i), F(B)).$$

Proof. Let $\Delta^1 \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ classify the unique active morphism $\langle n \rangle \to \langle 1 \rangle$, and let $\overline{\mathcal{C}} = \mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\Delta^1, \mathcal{C}^{\otimes})$. In what follows, we will abuse notation by identifying $\mathcal{C}^{\otimes}_{\langle n \rangle}$ with $\mathcal{C}^n$. The ∞-category $\overline{\mathcal{C}}$ inherits a symmetric monoidal structure from $\mathcal{C}$, and we have symmetric monoidal forgetful functors $(\mathcal{C}^n)^{\otimes} \leftarrow \overline{\mathcal{C}}^{\otimes} \to \mathcal{C}^{\otimes}$. The sequence $(A_1, \ldots, A_n)$ can be identified with a $\mathcal{O}$-algebra object of $\mathcal{C}^n$, and B determines a map $\mathcal{O}^{\otimes} \to \mathcal{C}^{\otimes}$. We let $\mathcal{C}'^{\otimes} = \overline{\mathcal{C}}^{\otimes} \times_{(\mathcal{C}^n)^{\otimes} \times \mathcal{C}^{\otimes}} \mathcal{O}^{\otimes}$, so that we have a fibration of ∞-operads $\mathcal{C}'^{\otimes} \to \mathcal{O}^{\otimes}$ and $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}')$ can be identified with the mapping space $\mathrm{Map}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})}(\bigotimes_i A_i, B)$. We define a fibration of ∞-operads $\mathcal{D}'^{\otimes} \to \mathcal{O}^{\otimes}$ similarly, so that $\mathrm{Map}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{D})}(\bigotimes_i F(A_i), F(B)) \simeq \mathrm{Alg}_{/\mathcal{O}}(\mathcal{D}')$. We wish to show that F induces a homotopy equivalence of Kan complexes $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}') \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{D}')$. For this, it suffices to show that for every map of simplicial sets $K \to \mathcal{O}^{\otimes}$, the induced map $\theta : \mathrm{Fun}_{\mathcal{O}^{\otimes}}(K, \mathcal{C}'^{\otimes}) \to \mathrm{Fun}_{\mathcal{O}^{\otimes}}(K, \mathcal{D}'^{\otimes})$ is a homotopy equivalence of Kan complexes. Working simplex-by-simplex, we can assume that $K = \Delta^p$. Then the inclusion $K' = \Delta^{\{0,1\}} \coprod_{\{1\}} \cdots \coprod_{\{p-1\}} \Delta^{\{p-1,p\}} \hookrightarrow K$ is a categorical equivalence; we may therefore replace K by K'. Working simplex-by-simplex again, we can assume that $K = \Delta^p$ for $p = 0$ or $p = 1$. When $p = 0$, the desired result follows immediately from ($*$). In the case $p = 1$, the map $\Delta^p \to \mathcal{O}^{\otimes}$ determines a morphism $\overline{\alpha} : X \to Y$ in $\mathcal{O}^{\otimes}$. Let $\alpha : \langle m \rangle \to \langle m' \rangle$ be the image of $\overline{\alpha}$ in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$, so that $X \simeq \bigoplus_{j \in \langle m \rangle^{\circ}} X_j$ and $Y = \bigoplus_{j' \in \langle m' \rangle^{\circ}} Y_{j'}$ for some objects $X_j, Y_{j'} \in \mathcal{O}$. Unwinding the definitions, we see that $\mathrm{Fun}_{\mathcal{O}^{\otimes}}(\Delta^p, \mathcal{C}'^{\otimes})$ is given by the homotopy limit of the diagram

$$\begin{tikzcd} & \prod_{j' \in \langle m' \rangle^{\circ}} \mathrm{Map}_{\mathcal{C}}(\bigotimes_{1 \leq i \leq n} A_i(Y_{j'}), B(Y_{j'})) \arrow[d] \\ \prod_{j \in \langle m \rangle^{\circ}} \mathrm{Map}_{\mathcal{C}}(\bigotimes_{1 \leq i \leq n} A_i(X_j), B(X_j)) \arrow[r] & \prod_{j' \in \langle m' \rangle^{\circ}} \mathrm{Map}_{\mathcal{C}}(\bigotimes_{1 \leq i \leq n, \alpha(j) = j'} A_i(X_j), B(Y_{j'})). \end{tikzcd}$$

Similarly, $\mathrm{Fun}_{\mathcal{O}^{\otimes}}(\Delta^p, \mathcal{D}'^{\otimes})$ can be identified with the homotopy limit of the diagram

$$\begin{tikzcd} & \prod_{j' \in \langle m' \rangle^{\circ}} \mathrm{Map}_{\mathcal{D}}(\bigotimes_{1 \leq i \leq n} FA_i(Y_{j'}), FB(Y_{j'})) \arrow[d] \\ \prod_{j \in \langle m \rangle^{\circ}} \mathrm{Map}_{\mathcal{D}}(\bigotimes_{1 \leq i \leq n} FA_i(X_j), FB(X_j)) \arrow[r] & \prod_{j' \in \langle m' \rangle^{\circ}} \mathrm{Map}_{\mathcal{D}}(\bigotimes_{1 \leq i \leq n, \alpha(j) = j'} FA_i(X_j), FB(Y_{j'})). \end{tikzcd}$$

It now follows from ($*$) that θ is a homotopy equivalence as desired. $\square$

Proof of Proposition 4.4.21. We proceed by induction on n. Assume first that $n = 0$. For every vector space V over k, let $V^{\vee} = \mathrm{Hom}_k(V, k)$ denote the dual vector space. For any object $A \in \mathrm{Alg}_k^{(0),\mathrm{aug}}$, we have canonical isomorphisms $\pi_p \mathfrak{D}^{(0)}(A) \simeq (\pi_{-p} A)^{\vee}$ (see Example 4.4.2). Using Remark 4.4.22, we are reduced to proving that if $\{A_i\}_{1 \leq i \leq m}$ is a finite collection of locally finite 0-connective objects of $\mathrm{Alg}_k^{(0),\mathrm{aug}}$ the canonical map

$$\bigoplus_{p=p_1+\cdots+p_m} \bigotimes_i (\pi_{p_i} A_i) \to (\bigoplus_{p=p_1+\cdots+p_m} (\bigotimes_i (\pi_{p_i} A_i)^\vee))^\vee$$

is an isomorphism for every integer p. Since $\pi_{p_i} A_i \simeq 0$ for $p_i > 0$, each of the direct sums is essentially finite, and the desired result follows immediately from the fact that each $\pi_{p_i} A_i$ is a finite dimensional vector space over k.

The case $n = 1$ follows from Proposition 3.1.19. Let us now suppose that $n \geq 1$ and that Proposition 4.4.21 is valid for $\mathbb{E}_n$-algebras; we prove that it is also valid for $\mathbb{E}_{n+1}$-algebras. Arguing as in the proof of Proposition 4.4.8, we see that the right fibration $\lambda^{(n)} : \mathcal{M}^{(n)} \to \mathrm{Alg}_k^{(n),\mathrm{aug}} \times \mathrm{Alg}_k^{(n),\mathrm{aug}}$ can be promoted to a pairing of symmetric monoidal ∞-categories

$$(\lambda^{(n)})^{\otimes} : (\mathcal{M}^{(n)})^{\otimes} \to (\mathrm{Alg}_k^{(n),\mathrm{aug}})^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} (\mathrm{Alg}_k^{(n),\mathrm{aug}})^{\otimes},$$

and Theorem A.5.1.2.2 allows us to identify $\lambda^{(n+1)}$ with the induced pairing

$$\mathrm{Alg}(\mathcal{M}^{(n)}) \to \mathrm{Alg}(\mathrm{Alg}_k^{(n),\mathrm{aug}}) \times \mathrm{Alg}(\mathrm{Alg}_k^{(n),\mathrm{aug}}).$$

Remark 4.4.16 allows us to regard $\mathfrak{D}^{(n)}$ as a lax symmetric monoidal functor. Moreover, Example 4.4.20 shows that $\mathfrak{D}^{(n+1)}$ is equivalent to the composition

$$\mathrm{Alg}(\mathrm{Alg}_k^{(n),\mathrm{aug}})^{op} \xrightarrow{G} \mathrm{Alg}((\mathrm{Alg}_k^{(n),\mathrm{aug}})^{op}) \xrightarrow{G'} \mathrm{Alg}(\mathrm{Alg}_k^{(n),\mathrm{aug}})$$

where G is given by the bar construction of §4.3 and G' is induced by $\mathfrak{D}^{(n)}$. Note that G is given on objects by the formula $G(A) = k \otimes_A k$.

Assume that A is $(n+1)$-connective and locally finite. We have a cofiber sequence of A-modules $A \to k \to Q$ where $\pi_i Q \simeq 0$ for $i > -n$. Using Corollary VIII.4.1.11, we deduce that $\pi_i(k \otimes_A Q) \simeq 0$ for $i > -n$ so that $G(A)$ is n-coconnective. Moreover, Lemma 4.1.16 shows that $G(A)$ is locally finite (here we use our assumption that $n \geq 1$).

Using the inductive hypothesis together with Lemma 4.4.24, we deduce that for any sequence $\{A_i\}_{1 \leq i \leq m}$ of $(n+1)$-connective, locally finite objects of $\mathrm{Alg}_k^{(n+1),\mathrm{aug}}$ and any object $C \in \mathrm{Alg}((\mathrm{Alg}_k^{(n),\mathrm{aug}})^{op})$, the canonical map

$$\mathrm{Map}_{\mathrm{Alg}((\mathrm{Alg}_k^{(n),\mathrm{aug}})^{op})}(G(A_1) \otimes \cdots G(A_m), C) \to \mathrm{Map}_{\mathrm{Alg}(\mathrm{Alg}_k^{(n),\mathrm{aug}})}((G'G)(A_1) \otimes \cdots \otimes (G'G)(A_m), G'(C))$$

is a homotopy equivalence. Consequently, it will suffice to show that for $B \in \mathrm{Alg}(\mathrm{Alg}_k^{(n),\mathrm{aug}})$, the functor G induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{Alg}(\mathrm{Alg}_k^{(n),\mathrm{aug}})}(B, A_1 \otimes \cdots \otimes A_m) \to \mathrm{Map}_{\mathrm{Alg}((\mathrm{Alg}_k^{(n),\mathrm{aug}})^{op})}(G(A_1) \otimes \cdots \otimes G(A_m), G(B)).$$

The formula $G(A) \simeq k \otimes_A k$ shows that G is a monoidal functor, so that $G(A_1) \otimes \cdots \otimes G(A_m) \simeq G(A)$ with $A \simeq A_1 \otimes \cdots \otimes A_m$. Note that A is locally finite and $(n+1)$-coconnective. Let F denote a left adjoint to G (given by the cobar construction). Using Remark 4.4.22, we are reduced to proving that the counit map $(F \circ G)(A) \to A$ is an equivalence in $\mathrm{Alg}(\mathrm{Alg}_k^{(n),\mathrm{aug}})^{op}$.

The monoidal ∞-category $\mathrm{Alg}_k^{(0),\mathrm{aug}}$ admits geometric realizations and totalizations and the unit object is a zero object, so the cobar and bar constructions yield a pair of adjoint functors

$$\mathrm{Alg}((\mathrm{Alg}_k^{(0),\mathrm{aug}})^{op}) \underset{G_0}{\overset{F_0}{\rightleftarrows}} \mathrm{Alg}(\mathrm{Alg}_k^{(0),\mathrm{aug}})^{op}.$$

Let $\phi : \mathrm{Alg}_k^{(n),\mathrm{aug}} \to \mathrm{Alg}_k^{(0),\mathrm{aug}}$ denote the forgetful functor. Then ϕ is a (symmetric) monoidal functor which preserves geometric realizations of simplicial objects and totalizations of cosimplicial objects, so that ϕ is compatible with the bar and cobar constructions (Example 4.4.19). It will therefore suffice to show that the counit map $(F_0 \circ G_0)(\phi A) \to \phi(A)$ is an equivalence in $\mathrm{Alg}(\mathrm{Alg}_k^{(0),\mathrm{aug}})^{op}$. Equivalently, it suffices to show that for each $R \in \mathrm{Alg}(\mathrm{Alg}_k^{(0),\mathrm{aug}})$, the canonical map

$$\mathrm{Map}_{\mathrm{Alg}(\mathrm{Alg}_k^{(0),\mathrm{aug}})}(R, \phi A) \to \mathrm{Map}_{\mathrm{Alg}((\mathrm{Alg}_k^{(0),\mathrm{aug}})^{op})}(G_0(\phi A), G_0(R))$$

is a homotopy equivalence. Let $G_0' : \mathrm{Alg}((\mathrm{Alg}_k^{(0),\mathrm{aug}})^{op}) \to \mathrm{Alg}(\mathrm{Alg}_k^{(0),\mathrm{aug}})$ be the functor given by composition with the lax monoidal functor $\mathfrak{D}^{(0)}$. Using the inductive hypothesis and Lemma 4.4.24, we deduce that G_0' induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{Alg}((\mathrm{Alg}_k^{(0),\mathrm{aug}})^{op})}(G_0(\phi A), G_0(R)) \to \mathrm{Map}_{\mathrm{Alg}(\mathrm{Alg}_k^{(0),\mathrm{aug}})}((G_0' G_0)(\phi A), (G_0' G_0)(R)).$$

It will therefore suffice to show that the composite map

$$\begin{aligned} \mathrm{Map}_{\mathrm{Alg}(\mathrm{Alg}_k^{(0),\mathrm{aug}})}(R, \phi A) &\to \mathrm{Map}_{\mathrm{Alg}(\mathrm{Alg}_k^{(0),\mathrm{aug}})}((G_0' G_0)(\phi_A), (G_0' G_0)(R)) \\ &\simeq \mathrm{Map}_{\mathrm{Alg}_k^{(1),\mathrm{aug}}}(\mathfrak{D}^{(1)}(\phi A), \mathfrak{D}^{(1)}(R)) \end{aligned}$$

is a homotopy equivalence. Since $n \geq 1$, this follows from our inductive hypothesis. □

4.5 Deformation Theory of $\mathbb{E}_n$-Algebras

Let k be a field. Our goal in this section is to prove Theorem 4.0.8, which asserts that the ∞-category $\mathrm{Moduli}_k^{(n)}$ of formal $\mathbb{E}_n$ moduli problems over k is equivalent to the ∞-category $\mathrm{Alg}_k^{(n),\mathrm{aug}}$ of augmented $\mathbb{E}_n$-algebras over k. We first introduce a suitable deformation context, and show that our discussion fits into the general paradigm described in §1.1. We will then prove that the Koszul duality functor $\mathfrak{D}^{(n)} : (\mathrm{Alg}_k^{(n),\mathrm{aug}})^{op} \to \mathrm{Alg}_k^{(n),\mathrm{aug}}$ of §4.4 is a deformation theory, in the sense of Definition 1.3.9 (Theorem 4.5.5). We will then use this result to deduce Theorem 4.0.8 from Theorem 1.3.12.

We begin by introducing the relevant deformation context. Let k be a field and let $n \geq 0$ be an integer. Using Theorem A.7.3.5.14 and Proposition A.3.4.2.1, we obtain equivalences of ∞-categories

$$\mathrm{Stab}(\mathrm{Alg}_k^{(n),\mathrm{aug}}) \simeq \mathrm{Mod}_k^{\mathbb{E}_n}(\mathrm{Mod}_k) \simeq \mathrm{Mod}_k .$$

In particular, we can identify the unit object $k \in \mathrm{Mod}_k$ with a spectrum object $E \in \mathrm{Stab}(\mathrm{Alg}_k^{(n),\mathrm{aug}})$, given informally by $\Omega^{\infty - m} E = k \oplus k[m]$. We regard the pair $(\mathrm{Alg}_k^{(n),\mathrm{aug}}, \{E\})$ as a deformation context.

We will need the following generalization of Proposition 3.2.2:

Proposition 4.5.1. *Let k be a field, let $n \geq 1$, and let $(\mathrm{Alg}_k^{(n)\,\mathrm{aug}}, \{E\})$ be the deformation context defined above. Then an object $A \in \mathrm{Alg}_k^{(n),\mathrm{aug}}$ is small (in the sense of Definition 1.1.8) if and only if its image in $\mathrm{Alg}_k^{(n)}$ is small (in the sense of Definition 3.0.1). That is, A is small if and only if it satisfies the following conditions:*

- (a) *The algebra A is connective: that is, $\pi_i A \simeq 0$ for $i < 0$.*
- (b) *The algebra A is truncated: that is, we have $\pi_i A \simeq 0$ for $i \gg 0$.*
- (c) *Each of the homotopy groups $\pi_i A$ is finite dimensional when regarded as a vector space over field k.*
- (d) *Let $\mathfrak{n}$ denote the radical of the ring $\pi_0 A$ (which is a finite-dimensional associative algebra over k). Then the canonical map $k \to (\pi_0 A)/\mathfrak{n}$ is an isomorphism.*

Remark 4.5.2. Proposition 4.5.1 is also valid in the case $n = 0$, provided that we adopt the convention of Remark 4.0.4. That is, an object $A \in \mathrm{Alg}_k^{(0),\mathrm{aug}}$ is small (in the sense of Definition 1.1.8) if and only if it connective and $\pi_* A$ is a finite-dimensional vector space over k.

Proof. The "only if" direction follows from Proposition 3.2.2 (note that if A is small as an augmented $\mathbb{E}_n$-algebra, then its image in $\mathrm{Alg}_k^{(n),\mathrm{aug}}$ is also small). To prove the converse, suppose that $A \in \mathrm{Alg}_k^{(n),\mathrm{aug}}$ satisfies conditions (a) through (d). We wish to prove that there exists a finite sequence of maps

$$A = A_0 \to A_1 \to \cdots \to A_d \simeq k$$

where each A_i is a square-zero extension of A_{i+1} by $k[m_i]$, for some $m_i \geq 0$. If $n = 1$, this follows from Proposition 3.2.2. Let us therefore assume that $n \geq 2$. We proceed by induction on the dimension of the k-vector space $\pi_* A$.

Let m be the largest integer for which $\pi_m A$ does not vanish. We first treat the case $m = 0$. We will abuse notation by identifying A with the underlying commutative ring $\pi_0 A$. Let $\mathfrak{n}$ denote the radical of A. If $\mathfrak{n} = 0$, then condition (d) implies that $A \simeq k$ so there is nothing to prove. Otherwise, we can view $\mathfrak{n}$ as a nonzero module over the commutative ring A. It follows that there exists a nonzero element $x \in \mathfrak{n}$ which is annihilated by $\mathfrak{n}$. Using (d) again, we deduce that the subspace $kx \subseteq A$ is an ideal of A. Let A' denote the quotient ring A/kx. Theorem A.7.4.1.26 implies that A is a square-zero extension of A' by k. The inductive hypothesis implies that A' is small, so that A is also small.

Now suppose that $m > 0$ and let $M = \pi_m A$. Then M is a nonzero module over the finite dimensional k-algebra $\pi_0 A$. It follows that there is a nonzero element $x \in M$ which is annihilated by the action of the radical $\mathfrak{n} \subseteq \pi_0 A$. Let M' denote the quotient of M by the submodule generated by x (which, by virtue of (d), coincides with kx), and let $A'' = \tau_{\leq n-1} A$. It follows from Theorem A.7.4.1.26 that there is a pullback diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & k \arrow[d] \\ A'' \arrow[r] & k \oplus M[m+1]. \end{tikzcd}$$

Set $A' = A'' \times_{k \oplus M'[m+1]} k$. Then $A \simeq A' \times_{k \oplus k[m+1]} k$, so we have an elementary map $A \to A'$. Using the inductive hypothesis we deduce that A' is small, so that A is also small. $\square$

Proposition 4.5.3. *Let k be a field and let $f : A \to B$ be a morphism in $\mathrm{Alg}_k^{(n),\mathrm{sm}}$. Then f is small (when regarded as a morphism in $\mathrm{Alg}_k^{(n),\mathrm{aug}}$) if and only if it induces a surjection $\pi_0 A \to \pi_0 B$.*

Proof. If $n = 1$, the desired result follows from Proposition 3.2.3. We will assume that $n \geq 2$, and leave the case $n = 0$ to the reader. The "only if" direction follows from Proposition 3.2.3 (note that if f is small, then the induced map between the underlying $\mathbb{E}_1$-algebras is also small). We first treat the case where $B \simeq A \oplus M[j]$, for some $M \in \mathrm{Mod}_A^{\mathbb{E}_n}(\mathrm{Mod}_k)^{\heartsuit}$ and some $j \geq 1$. According to Remark 4.1.12, the abelian category $\mathrm{Mod}_A^{\mathbb{E}_n}(\mathrm{Mod}_k)^{\heartsuit}$ is equivalent to the category of modules over the commutative ring $\pi_0 B$. Since M is finite dimensional as a vector space over k, it admits a finite filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_m = M,$$

where each of the successive quotients M_i/M_{i-1} is isomorphic to k. This filtration determines a factorization of f as a composition

$$A \simeq A \oplus M_0[j] \to A \oplus M_1[j] \to \cdots \to A \oplus M_m[j] = B.$$

Each of the maps $A \oplus M_i[j] \to A \oplus M_{i+1}[j]$ is elementary, so that f is small.

We now treat the general case. Note that the map $\pi_0 A \times_{\pi_0 B} B \to B$ is a pullback of the map $\pi_0 A \to \pi_0 B$, and therefore a small extension (the map $\pi_0 A \to \pi_0 B$ is even a small extension of $\mathbb{E}_\infty$-algebras over k, by Lemma 1.1.20). It will therefore suffice to show that the map $A \to \pi_0 A \times_{\pi_0 B} B$ is a small extension. We will prove that each of the maps

$$\tau_{\leq j} A \times_{\tau_{\leq j} B} B \to \pi_0 A \times_{\pi_0 B} B$$

is a small extension; taking $j \gg 0$ we will obtain the desired result. The proof proceeds by induction on j, the case $j = 0$ being trivial. Assume that $j > 0$; by the inductive hypothesis, we are reduced to proving that the map

$$\theta : \tau_{\leq j} A \times_{\tau_{\leq j} B} B \to \tau_{\leq j-1} A \times_{\tau_{\leq j-1} B} B$$

is small.

We can factor θ as a composition

$$\tau_{\leq j} A \times_{\tau_{\leq j} B} B \xrightarrow{\theta'} \tau_{\leq j} A \times_{\tau_{\leq j-1} B} B \xrightarrow{\theta''} \tau_{\leq j-1} A \times_{\tau_{\leq j-1} B} B.$$

The map θ'' is a pullback of the truncation map $u : \tau_{\leq j} A \to \tau_{\leq j-1} A$. It follows from Corollary A.7.4.1.28 that u exhibits $\tau_{\leq j} A$ as a square-zero extension of $\tau_{\leq j-1} A$, so that we have a pullback square

$$\begin{tikzcd}
\tau_{\leq j} A \arrow[r, "u"] \arrow[d] & \tau_{\leq j-1} A \arrow[d] \\
\tau_{\leq j-1} A \arrow[r, "u_0"] & (\tau_{\leq j-1} A) \oplus (\pi_j A)[j+1].
\end{tikzcd}$$

Here the map u_0 is small by the argument given above, so that u is small and therefore θ'' is small. We will complete the proof by showing that θ' is small. Note that θ' is a pullback of the diagonal map

$$\delta : \tau_{\leq j} B \to \tau_{\leq j} B \times_{\tau_{\leq j-1} B} \tau_{\leq j} B.$$

Since $\tau_{\leq j} B$ is a square-zero extension of $\tau_{\leq j-1} B$ by $(\pi_j B)[j]$ (Corollary A.7.4.1.28), the truncation map $\tau_{\leq j} B \to \tau_{\leq j-1} B$ is a pullback of the canonical map $\tau_{\leq j-1} B \to \tau_{\leq j-1} B \oplus (\pi_j B)[j+1]$. It follows that δ' is a pullback of the map

$$\delta' : \tau_{\leq j-1} B \to \tau_{\leq j-1} B \times_{\tau_{\leq j-1} B \oplus (\pi_j B)[j+1]} \tau_{\leq j-1} B \simeq \tau_{\leq j-1} B \oplus (\pi_j B)[j].$$

Since $j \geq 1$, the first part of the proof shows that δ' is small. $\square$

Corollary 4.5.4. *Let k be a field, let $n \geq 0$ be an integer, and let and let $X : \mathrm{Alg}_k^{(n),\mathrm{sm}} \to \mathcal{S}$ be a functor. Then X belongs to the full subcategory $\mathrm{Moduli}_k^{(n)}$ of Definition 4.0.6 if and only if it is a formal moduli problem in the sense of Definition 1.1.14.*

Proof. The "if" direction follows immediately from Proposition 4.5.3. For the converse, suppose that X satisfies the conditions of Definition 4.0.6; we wish to show that X is a formal moduli problem. According to Proposition 1.1.15, it will suffice to show that for every pullback diagram in $\mathrm{Alg}_k^{(n),\mathrm{sm}}$

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
k \arrow[r] & k \oplus k[m]
\end{tikzcd}$$

satisfying $m > 0$, the associated diagram of spaces

$$\begin{tikzcd}
X(A) \arrow[r] \arrow[d] & X(B) \arrow[d] \\
X(k) \arrow[r] & X(k \oplus k[m])
\end{tikzcd}$$

is also a pullback square. This follows immediately from condition (2) of Definition 4.0.6. $\square$

The main result of this section is the following:

Theorem 4.5.5. *Let k be a field and let $n \geq 0$ be an integer. Then the Koszul duality functor*

$$\mathfrak{D}^{(n)} : (\mathrm{Alg}_k^{(n),\mathrm{aug}})^{op} \to \mathrm{Alg}_k^{(n),\mathrm{aug}}$$

is a deformation theory (in the sense of Definition 1.3.9).

We will give the proof of Theorem 4.5.5 at the end of this section. It relies on the following property of the Koszul duality functor:

Proposition 4.5.6. *Let k be a field, and let $\mathrm{Free}^{(n)} : \mathrm{Mod}_k \to \mathrm{Alg}_k^{(n)}$ be a left adjoint to the forgetful functor (so that $\mathrm{Free}^{(n)}$ assigns to every k-module spectrum V the free $\mathbb{E}_n$-algebra $\bigoplus_{n\geq 0} \mathrm{Sym}^n_{\mathbb{E}_k}(V)$). Note that $\mathrm{Free}^{(n)}(0) \simeq k$, so that $\mathrm{Free}^{(n)}$ determines a functor $\mathrm{Free}^{\mathrm{aug}} : \mathrm{Mod}_k \simeq (\mathrm{Mod}_k)_{/0} \to (\mathrm{Alg}_k^{(n)})_{/k} \simeq \mathrm{Alg}_k^{(n),\mathrm{aug}}$. Let $\mathfrak{D}^{(n)} : (\mathrm{Alg}_k^{(n),\mathrm{aug}})^{op} \to \mathrm{Alg}_k^{(n),\mathrm{aug}}$ be the Koszul duality functor. Then the composition $\mathfrak{D}^{(n)} \circ \mathrm{Free}^{\mathrm{aug}}$ is equivalent to the functor $\mathrm{Mod}_k^{op} \to \mathrm{Alg}_k^{(n),\mathrm{aug}}$ given by $V \mapsto k \oplus V^\vee[-n]$, where $V^\vee$ denotes the k-linear dual of V.*

Proof. The functor $\mathfrak{D}^{(n)} \circ \mathrm{Free}^{\mathrm{aug}}$ admits a left adjoint and is therefore left exact. Since Mod_k is stable, Proposition A.1.4.4.10 implies that $\mathfrak{D}^{(n)} \circ \mathrm{Free}^{(n)}$ factors as a composition

$$\mathrm{Mod}_k^{op} \xrightarrow{T} \mathrm{Stab}(\mathrm{Alg}_k^{(n),\mathrm{aug}}) \xrightarrow{\Omega^\infty} \mathrm{Alg}_k^{(n),\mathrm{aug}}.$$

Note that the stabilization $\mathrm{Stab}(\mathrm{Alg}_k^{(n),\mathrm{aug}})$ is equivalent to Mod_k, and that under this equivalence the functor $\Omega^\infty : \mathrm{Stab}(\mathrm{Alg}_k^{(n),\mathrm{aug}}) \to \mathrm{Alg}_k^{(n),\mathrm{aug}}$ is given by the formation of square-zero extensions $V \mapsto k \oplus V$ (Theorem A.7.3.5.7). It follows that we can identify T with the functor $\mathrm{Mod}_k^{op} \to \mathrm{Mod}_k$ given by the composition

$$\mathrm{Mod}_k^{op} \xrightarrow{\mathrm{Free}^{(n)}} (\mathrm{Alg}_k^{(n),\mathrm{aug}})^{op} \xrightarrow{\mathfrak{D}^{(n)}} \mathrm{Alg}_k^{(n),\mathrm{aug}} \xrightarrow{I} \mathrm{Mod}_k,$$

where I denotes the functor which carries each augmented $\mathbb{E}_n$-algebra A to its augmentation ideal. The composition $I \circ \mathfrak{D}^{(n)}$ assigns to each augmented $\mathbb{E}_n$-algebra B its shifted tangent fiber $\mathrm{Mor}_{\mathrm{Mod}_B^{\mathbb{E}_n}}(L_{B/k}[n], k)$ (see Example A.7.3.6.7), so that the composition $I \circ \mathfrak{D}^{(n)} \circ \mathrm{Free}^{(n)}$ is given by $V \mapsto V[n]^\vee \simeq V^\vee[-n]$. □

Proof of Theorem 4.0.8. Let k be a field and let $n \geq 0$ be an integer. Define a functor $\Psi : \mathrm{Alg}_k^{(n),\mathrm{aug}} \to \mathrm{Fun}(\mathrm{Alg}_k^{(n),\mathrm{sm}}, \mathcal{S})$ by the formula

$$\Psi(A)(R) = \mathrm{Map}_{\mathrm{Alg}_k^{(n),\mathrm{aug}}}(\mathfrak{D}^{(n)}(R), A).$$

Combining Theorem 4.5.5, Theorem 1.3.12, and Corollary 4.5.4, we deduce that Ψ is a fully faithful embedding whose essential image is the full subcategory $\mathrm{Moduli}_k^{(n)} \subseteq \mathrm{Fun}(\mathrm{Alg}_k^{(n),\mathrm{sm}}, \mathcal{S})$ spanned by the formal moduli problems. If $m \geq 0$, then Proposition 4.1.13 implies that $\mathrm{Free}^{\mathrm{aug}}(k[-m-n])$ is n-coconnective and locally finite, so the the biduality map

$$\mathrm{Free}^{\mathrm{aug}}(k[-m-n]) \to \mathfrak{D}^{(n)}\mathfrak{D}^{(n)} \mathrm{Free}^{\mathrm{aug}}(k[-m-n])$$

is an equivalence (Theorem 4.4.5). Using Proposition 4.5.6, we obtain canonical homotopy equivalences

$$\begin{aligned}
\Psi(A)(k \oplus k[m]) &\simeq \Psi(A)(\mathfrak{D}^{(n)} \mathrm{Free}^{\mathrm{aug}}(k[-m-n])) \\
&\to \mathrm{Map}_{\mathrm{Alg}_k^{(n),\mathrm{aug}}}(\mathfrak{D}^{(n)}\mathfrak{D}^{(n)} \mathrm{Free}^{\mathrm{aug}}(k[-m-n]), A) \\
&\to \mathrm{Map}_{\mathrm{Alg}_k^{(n),\mathrm{aug}}}(\mathrm{Free}^{\mathrm{aug}}(k[-m-n]), A) \\
&\simeq \Omega^{\infty-m-n}\mathfrak{m}_A,
\end{aligned}$$

where $\mathfrak{m}_A$ denotes the augmentation ideal of A. These equivalences are natural in m, and therefore give rise to an equivalence of spectra $T_{\Psi(A)} \simeq \mathfrak{m}_A[n]$ (depending functorially on A). □

Example 4.5.7. Suppose that $n = 0$ in the situation of Theorem 4.0.8. Then the Koszul duality functor $\mathfrak{D}^{(n)} : (\mathrm{Alg}_k^{(0),\mathrm{aug}})^{op} \mapsto \mathrm{Alg}_k^{(0),\mathrm{aug}}$ is given by $k \oplus V \mapsto k \oplus V^\vee$ (see Example 4.4.2). It follows that the functor $\Psi : \mathrm{Alg}_k^{(0),\mathrm{aug}} \to \mathrm{Moduli}_k^{(0)}$ is given by

$$\Psi_{k\oplus W}(k \oplus V) = \mathrm{Map}_{\mathrm{Alg}_k^{(0),\mathrm{aug}}}(k \oplus V^\vee, k \oplus W) \simeq \mathrm{Map}_{\mathrm{Mod}_k}(V^\vee, W) \simeq \Omega^\infty(V \otimes_k W).$$

Here the last equivalence depends on the fact that V is a dualizable object of Mod_k (since V is a perfect k-module).

We may summarize the situation as follows: every object $W \in \mathrm{Mod}_k$ determines a formal $\mathbb{E}_0$ moduli problem, given by the formula $k \oplus V \mapsto \Omega^\infty(V \otimes_k W)$. Moreoever, every formal $\mathbb{E}_0$ moduli problem arises in this way, up to equivalence.

Proof of Theorem 4.5.5. Let k be a field and let $n \geq 0$. We wish to prove that the Koszul duality functor $\mathfrak{D}^{(n)} : \mathfrak{D}^{(n)} : (\mathrm{Alg}_k^{(n),\mathrm{aug}})^{op} \to \mathrm{Alg}_k^{(n),\mathrm{aug}}$ satisfies axioms $(D1)$ through $(D4)$ of Definitions 1.3.1 and 1.3.9:

$(D1)$ The ∞-category $\mathrm{Alg}_k^{(n),\mathrm{aug}}$ is presentable: this follows from Corollary A.3.2.3.5.

$(D2)$ The functor $\mathfrak{D}^{(n)}$ admits a left adjoint. In fact, this left adjoint is given by the opposite of $\mathfrak{D}^{(n)}$ (see Remark 4.4.4).

$(D3)$ Let $\Xi_0 \subseteq \mathrm{Alg}_k^{(n),\mathrm{aug}}$ be the full subcategory spanned by those augmented $\mathbb{E}_n$-algebras A over k, where A is coconnective and both A and $\int A$ are locally finite. We will verify that this subcategory satisfies the requirements of Definition 1.3.1:

- (a) For every object $A \in \Xi_0$, the biduality map $A \to \mathfrak{D}^{(n)}\mathfrak{D}^{(n)}(A)$ is an equivalence. This follows from Theorem 4.4.5.
- (b) The subcategory Ξ_0 contains the initial object $k \in \mathrm{Alg}_k^{(n),\mathrm{aug}}$.
- (c) For each $m \geq 1$, there exists an object $K_m \in \Xi_0$ and an equivalence $\alpha : k \oplus k[m] \simeq \mathfrak{D}^{(n)} K_m$. In fact, we can take K_m to be the free $\mathbb{E}_n$-algebra generated by $k[-m-n]$. This belongs to Ξ_0 by virtue of Proposition 4.1.13, and Proposition 4.5.6 supplies the equivalence α.
- (d) For every pushout diagram

$$\begin{tikzcd} K_m \arrow[r] \arrow[d, "\epsilon"] & A \arrow[d] \\ k \arrow[r] & A', \end{tikzcd}$$

  where $A \in \Xi_0$ and ϵ is the canonical augmentation on K_m, the object A' also belongs to Ξ_0. This follows immediately from Propositions 4.1.14 and 4.1.13.

$(D4)$ Arguing as in the proof of Theorem 4.0.8, we see that the functor $e : \mathrm{Alg}_k^{(n),\mathrm{aug}} \to \mathrm{Sp}$ appearing in Definition 1.3.9 is given by $A \mapsto \mathfrak{m}_A[n]$, where $\mathfrak{m}_A$ denotes the augmentation ideal of A. This functor is obviously conservative, and preserves sifted colimits by Proposition A.3.2.3.1.

□

We close this section by proving a generalization of Proposition 3.2.7:

Proposition 4.5.8. *Let k be a field and let $X : \mathrm{Alg}_k^{(n),\mathrm{sm}} \to \mathcal{S}$ be a formal $\mathbb{E}_n$ moduli problem over k. The following conditions are equivalent:*

(1) *The functor X is prorepresentable (see Definition 1.5.3).*

(2) *Let $X(E)$ denote the tangent complex of X. Then $\pi_i X(E) \simeq 0$ for $i > 0$.*

(3) *The functor X has the form $\Psi(A)$, where $A \in \mathrm{Alg}_k^{(n),\mathrm{aug}}$ is n-coconnective coconnective and $\Psi : \mathrm{Alg}_k^{(n),\mathrm{aug}} \to \mathrm{Moduli}_k^{(n)}$ is the equivalence of Theorem 4.0.8.*

Lemma 4.5.9. *Let A be an augmented $\mathbb{E}_n$-algebra over a field k. If A is connective, then the Koszul dual $\mathfrak{D}^{(n)}(A)$ is n-coconnective.*

Proof. Let $\mathrm{Mod}_A^{\mathbb{E}_n}$ denote the ∞-category of $\mathbb{E}_n$-modules over A in the ∞-category Mod_k, and regard $\mathrm{Mod}_A^{\mathbb{E}_n}$ as tensored over Mod_k. As an object of Mod_k, we can identify $\mathfrak{D}^{(n)}(A)$ as a classifying object (in Mod_k) for morphisms from A to k in $\mathrm{Mod}_A^{\mathbb{E}_n}$ (see Example A.6.1.4.14 and Theorem A.6.1.4.27). Theorem A.7.3.6.1 supplies fiber sequence

$$\int A \to A \to L_{A/k}[n]$$

n $\mathrm{Mod}_A^{\mathbb{E}_n}$, where $L_{A/k}$ denote the relative cotangent complex of A over k as an $\mathbb{E}_n$-algebra. We therefore obtain a fiber sequence

$$\mathrm{Mor}(L_{A/k}[n], k) \to \mathfrak{D}^{(n)}(A) \overset{\epsilon_A}{\to} k$$

in Mod_k. The map ϵ_A depends functorially on A and is an equivalence in the case $A = k$, and can therefore be identified with the augmentation on $\mathfrak{D}^{(n)}(A)$. We may therefore identify the augmentation ideal $\mathfrak{m}_{\mathfrak{D}^{(n)}(A)}$ with a classifying object for morphisms from $L_{A/k}[n]$ to k in $\mathrm{Mod}_A^{\mathbb{E}_n}$. To prove that $\mathfrak{D}^{(n)}$ is n-coconnective, it suffices to show that the mapping space

$$\mathrm{Map}_{\mathrm{Mod}_k^{\mathbb{E}_n}}(L_{A/k}, k) \simeq \mathrm{Map}_{\mathrm{Alg}_k^{(n),\mathrm{aug}}}(A, k[\epsilon]/(\epsilon^2))$$

is discrete. This is clear, since A is connective and $k[\epsilon]/(\epsilon^2)$ is discrete. □

Proof of Proposition 4.5.8. The equivalence of (2) and (3) follows from the observation that for $X = \Psi(A)$, we have $\pi_i X(E) \simeq \pi_{i-n}\mathfrak{m}_A$, where $\mathfrak{m}_A$ is the augmentation ideal of A. We next prove that (1) $\Rightarrow$ (2). Since the construction $X \mapsto X(E)$ commutes with filtered colimits, we may reduce to the case where $X = \mathrm{Spec}\, R$ is representable by an object $R \in \mathrm{Alg}_k^{(n),\mathrm{sm}}$. Then R is connective and the desired result follows from Lemma 4.5.9.

We now complete the proof by showing that (3) $\Rightarrow$ (1). Let $A \in \mathrm{Alg}_k^{(n),\mathrm{aug}}$ be n-coconnective, and choose a sequence of maps

$$k = A(0) \to A(1) \to A(2) \to \cdots$$

as in the proof of Proposition 4.1.14. Then $A = \varinjlim A(i)$, so that $X \simeq \varinjlim X(i)$ with $X(i) = \Psi(A(i))$. To prove that X is prorepresentable, it will suffice to show that each $X(i)$ is prorepresentable. We proceed by induction on i, the case $i = 0$ being trivial.

Assume that $X(i)$ is prorepresentable. By construction, we have a pushout diagram

$$\begin{tikzcd} \mathrm{Free}^{(n)}(V) \arrow[r] \arrow[d] & A(i) \arrow[d] \\ k \arrow[r] & A(i+1) \end{tikzcd}$$

where $\pi_j V \simeq 0$ for $j \geq -n$. For $m \geq n$, form a pushout diagram

$$\begin{tikzcd} \mathrm{Free}^{(n)}(\tau_{\geq -m} V) \arrow[r] \arrow[d] & A(i) \arrow[d] \\ k \arrow[r] & A(i,m), \end{tikzcd}$$

so that $A(i+1) \simeq \varinjlim_m A(i,m)$. Then $X(i+1) \simeq \varinjlim_m \Psi(A(i,m))$, so we are reduced to proving that each $\Psi(A(i,m))$ is prorepresentable. We proceed by induction on m. If $m = n$, then $A(i,m) \simeq A(i)$ and the desired result follows from our inductive hypothesis. Assume that $m > n$ and that $\Psi(A(i,m-1))$ is prorepresentable. Let $W = \pi_{-m} V$, so that we have a pushout diagram

$$\begin{tikzcd} \mathrm{Free}^{(n)}(W[-m]) \arrow[r] \arrow[d] & A(i,m-1) \arrow[d] \\ k \arrow[r] & A(i,m). \end{tikzcd}$$

Write W as a union of its finite-dimensional subspaces $\{W_\alpha\}$. For every finite dimensional subspace $W_\alpha \subseteq W$, form a pushout diagram

$$\begin{tikzcd} \mathrm{Free}^{(n)}(W_\alpha[-m]) \arrow[r] \arrow[d] & A(i,m-1) \arrow[d] \\ k \arrow[r] & A(i,W_\alpha). \end{tikzcd}$$

Then $\Psi(A(i,m))$ is a filtered colimit of the objects $\Psi(A(i,W_\alpha))$. It will therefore suffice to show that each $\Psi(A(i,W_\alpha))$ is prorepresentable. We proceed by induction on the dimension of W_α; if that dimension is zero, then $A(i,W_\alpha) \simeq A(i,m-1)$ and the result is clear. If W_α has positive dimension, then we can choose a subspace W'_α of codimension 1. Then we have a pushout diagram

$$\begin{tikzcd} \mathrm{Free}^{(n)}(k[-m]) \arrow[r] \arrow[d] & A(i,W'_\alpha) \arrow[d] \\ k \arrow[r] & A(i,W_\alpha), \end{tikzcd}$$

hence a pushout diagram of formal moduli problems

$$\begin{tikzcd} \mathrm{Spec}(k \oplus k[m-n]) \arrow[r] \arrow[d] & A(i,W'_\alpha) \arrow[d] \\ \mathrm{Spec}(k) \arrow[r] & \Psi(A(i,W_\alpha)). \end{tikzcd}$$

We conclude by invoking Lemma 1.5.9. $\square$

5 Examples of Moduli Problems

Let k be a field, and let $\mathcal{C}$ be a k-linear ∞-category (that is, a presentable ∞-category equipped with an action of the monoidal ∞-category Mod_k of k-module spectra: see Definition VII.6.2). To $\mathcal{C}$ we can associate a variety of deformation problems associated to $\mathcal{C}$:

(a) Fix an object $C \in \mathcal{C}$, and let $R \in \mathrm{CAlg}_k^{\mathrm{sm}}$ be a small $\mathbb{E}_\infty$-algebra over k. A *deformation of C over R* is an object $C_R \in \mathrm{Mod}_R(\mathcal{C})$, together with an equivalence $C \simeq k \otimes_R C_R$. Let $X(R)$ denote the ∞-category $\mathrm{Mod}_R(\mathcal{C}) \times_{\mathcal{C}} \{C\}$ of deformations of C over R.

(b) For $R \in \mathrm{CAlg}_k^{\mathrm{sm}}$, a *deformation of* $\mathcal{C}$ *over* R is an R-linear ∞-category $\mathcal{C}_R$ equipped with an equivalence $\mathcal{C} \simeq \mathrm{Mod}_k \otimes_{\mathrm{Mod}_R} \mathcal{C}_R$. Let $Y(R) = \mathrm{LinCat}_R \times_{\mathrm{LinCat}_k} \{\mathcal{C}\}$ denote a classifying space for R-linear ∞-categories.

Our goal in this section is to analyze the deformation functors X and Y using the theory of formal moduli problems developed earlier in this paper. However, we immediately encounter an obstacle: the functors X and Y need not satisfy the axioms described in Proposition 1.1.19. Suppose, for example, that we are given a pullback diagram σ :

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R_0 \arrow[d] \\ R_1 \arrow[r] & R_{01} \end{tikzcd}$$

in $\mathrm{CAlg}_k^{\mathrm{sm}}$, where the maps $\pi_0 R_0 \to \pi_0 R_{01} \leftarrow \pi_0 R_1$ are surjective. If $C \in \mathcal{C}$ and C_R is a deformation of C over R, then C_R is uniquely determined by the objects $C_{R_0} = R_0 \otimes_R C_R$, $C_{R_1} = R_1 \otimes_R C_{R_1}$, together with the evident equivalence

$$R_{01} \otimes_{R_0} C_{R_0} \simeq R_{01} \otimes_R C_R \simeq R_{01} \otimes_{R_1} C_{R_1}$$

(see Proposition 5.2.2). More precisely, the functor X described in (a) determines a fully faithful embedding (of Kan complexes)

$$X(R) \to X(R_0) \times_{X(R_{01})} X(R_1).$$

The functor Y described in (b) is even more problematic: the map

$$Y(R) \to Y(R_0) \times_{Y(R_{01})} Y(R_1)$$

need not be fully faithful in general, but always has discrete homotopy fibers (Proposition 5.3.3): that is, we can regard $Y(R)$ as a covering space of $Y(R_0) \times_{Y(R_{01})} Y(R_1)$. To accommodate these examples, it is useful to introduce a weaker version of the axiomatics developed in §1. For every integer $n \geq 0$, we will define the notion of a *n-proximate formal moduli problem* (Definition 5.1.5). When $n = 0$, we recover the notion of formal moduli problem introduced in Definition 1.1.14. The requirement that a functor Z be an n-proximate formal moduli problem becomes increasingly weak as n grows. Nonetheless, we will show that an n-proximate formal moduli problem Z is not far from being a formal moduli problem: namely, there exists an (essentially unique) formal moduli problem $Z^\wedge$ and a natural transformation $Z \to Z^\wedge$ such that, for every test algebra R, the map of spaces $Z(R) \to Z^\wedge(R)$ has $(n-1)$-truncated homotopy fibers (Theorem 5.1.9).

In §5.2, we will turn our attention to the functor X described above, which classifies the deformations of a fixed object $C \in \mathcal{C}$. We begin by observing that the definition of $X(R)$ does not require the assumption that R is commutative. Rather, the functor X is naturally defined on the ∞-category $\mathrm{Alg}_k^{\mathrm{sm}}$ of small $\mathbb{E}_1$-algebras over k. We may therefore regard the construction $R \mapsto X(R)$ as a functor $X : \mathrm{Alg}_k^{\mathrm{sm}} \to \mathcal{S}$, which we will prove is a 1-proximate formal moduli problem (Corollary 5.2.5). Using Theorem 5.1.9, we can choose an embedding of X into a formal moduli problem $X^\wedge : \mathrm{Alg}_k^{\mathrm{sm}} \to \mathcal{S}$. According to Theorem 3.0.4 (and its proof), the functor $X^\wedge$ is given by $X^\wedge(R) = \mathrm{Map}_{\mathrm{Alg}_k^{\mathrm{aug}}}(\mathfrak{D}^{(1)}(R), A)$, for some augmented $\mathbb{E}_1$-algebra A over k. Our main result (Theorem 5.2.8) characterizes this algebra: the augmentation ideal $\mathfrak{m}_A$ can be identified (as a nonunital $\mathbb{E}_1$-algebra) with the endomorphism algebra of the object $C \in \mathcal{C}$.

Remark 5.0.1. Efimov, Lunts, and Orlov have made an extensive study of a variant of the deformation functor X described above. We refer the reader to [10], [11], and [12] for details. The global structure of moduli spaces of objects of (well-behaved) differential graded categories is treated in [67].

In §5.3, we will study the functor Y which classifies deformations of∞-category $\mathcal{C}$ itself. Once again, the definition of the space $Y(R)$ does not require the assumption that R is commutative. To define $Y(R)$, we only need to be able to define the notion of an R-linear ∞-category. This requires a monoidal structure on the ∞-category LMod_R of left R-modules, and such a monoidal structure exists for every $\mathbb{E}_2$-algebra R over k. We may therefore regard the construction $R \mapsto Y(R)$ as a functor $Y : \mathrm{Alg}_k^{(2),\mathrm{sm}} \to \mathcal{S}$, which we will prove to be a 2-proximate formal moduli problem (Corollary 5.3.8). Using Theorem 5.1.9, we deduce the existence of a formal moduli problem $Y^\wedge : \mathrm{Alg}_k^{(2),\mathrm{sm}} \to \mathcal{S}$ and a natural transformation $Y \to Y^\wedge$ which induces a covering map $Y(R) \to Y^\wedge(R)$ for each $R \in \mathrm{Alg}_k^{(2),\mathrm{sm}}$. According to Theorem 4.0.8 (and its proof), the functor $Y^\wedge$ is given by $Y^\wedge(R) = \mathrm{Map}_{\mathrm{Alg}_k^{(2),\mathrm{aug}}}(\mathfrak{D}^{(2)}(R), A)$ for some augmented $\mathbb{E}_2$-algebra A over k. Once again, our main result gives an explicit description of the algebra A: its augmentation ideal $\mathfrak{m}_A$ can be identified (as a nonunital $\mathbb{E}_2$-algebra) with the Hochschild cochain complex $\mathrm{HC}^*(\mathcal{C})$ of the ∞-category $\mathcal{C}$ (Theorem 5.3.16).

Remark 5.0.2. For a more extensive discussion of the deformation theory of differential graded categories, we refer the reader to [30]. See also [37] and [38].

Remark 5.0.3. It is possible to treat the functors X and Y introduced above simultaneously. Let Υ denote the ∞-category whose objects are pairs (A_1, A_2), where A_2 is an augmented $\mathbb{E}_2$-algebra over k and A_1 is an $\mathbb{E}_1$-algebra over A_2 equipped with a map $A_1 \to k$ of $\mathbb{E}_1$-algebras over A_2. We have spectrum objects $E_1, E_2 \in \mathrm{Stab}(\Upsilon)$, given by

$$\Omega^{\infty-n} E_1 = (k \oplus k[n], k) \qquad \Omega^{\infty-n} E_2 = (k, k \oplus k[n]).$$

Let us regard $(\Upsilon, \{E_1, E_2\})$ as a deformation context (in the sense of Definition 1.1.3).

Let $\mathcal{C}$ be a k-linear ∞-category and let $C \in \mathcal{C}$ be an object. Given a pair $(R_1, R_2) \in \Upsilon$, we let $Z(R_1, R_2)$ denote a classifying space for pairs $(C_1, \mathcal{C}_2)$, where $\mathcal{C}_2$ is an R_2-linear ∞-category deforming $\mathcal{C}$, and $C_1 \in \mathrm{LMod}_{R_1}(\mathcal{C}_2)$ is an object deforming C. The construction $(R_1, R_2) \mapsto Z(R_1, R_2)$ determines a 2-proximate formal moduli problem. Using Theorem 5.1.9 we can complete Z to a formal moduli problem $Z^\wedge : \Upsilon^{\mathrm{sm}} \to \mathcal{S}$.

Using a generalization of the techniques studied in §3 and §4, one can combine the Koszul duality functors

$$\mathfrak{D}^{(1)} : (\mathrm{Alg}_k^{\mathrm{aug}})^{op} \to \mathrm{Alg}_k^{\mathrm{aug}} \qquad \mathfrak{D}^{(2)} : (\mathrm{Alg}_k^{(2),\mathrm{aug}})^{op} \to \mathrm{Alg}_k^{(2),\mathrm{aug}}$$

to obtain a deformation theory $\mathfrak{D} : (\Upsilon)^{op} \to \Upsilon$. Using Theorem 1.3.12, we see that the formal moduli problem $Z^\wedge$ is determined by an object $(A_1, A_2) \in \Upsilon$. One can show that the augmentation ideals $\mathfrak{m}_{A_1}$ and $\mathfrak{m}_{A_2}$ are given by the endomorphism algebra $\mathrm{End}(C)$ and the Hochschild cochain complex $\mathrm{HC}^*(\mathcal{C})$ of $\mathcal{C}$, respectively (note that $\mathrm{HC}^*(\mathcal{C})$ acts centrally on $\mathrm{End}(C)$).

At the cost of a bit of information, we can be much more concrete. The construction $R \mapsto Z^\wedge(R, R)$ determines a formal $\mathbb{E}_2$ moduli problem $F : \mathrm{Alg}_k^{(2),\mathrm{aug}} \to \mathcal{S}$; for each $R \in \mathrm{Alg}_k^{(2),\mathrm{sm}}$ we have a fiber sequence

$$X^\wedge(R) \to F(R) \to Y^\wedge(R),$$

where $X^\wedge$ and $Y^\wedge$ are the formal $\mathbb{E}_1$ and $\mathbb{E}_2$ moduli problems described above. Applying Theorem 4.0.8, we deduce that F is given by the formula $F(R) = \mathrm{Map}_{\mathrm{Alg}_k^{(2),\mathrm{aug}}}(\mathfrak{D}^{(2)}(R), A)$ for some augmented $\mathbb{E}_2$-algebra A over k. Then the augmentation ideal $\mathfrak{m}_A$ can be identified with the fiber of the natural map $\mathrm{HC}^*(\mathcal{C}) \to \mathrm{End}(C)$.

5.1 Approximations to Formal Moduli Problems

The notion of formal moduli problem introduced in Definition 1.1.14 is a very general one, which includes (as a special case) the formal completion of any reasonable algebro-geometric object at a point (see Example 0.0.10). However, there are also many functors $X : \mathrm{CAlg}_k^{\mathrm{cn}} \to \mathcal{S}$ which are of interest in deformation theory, which do not quite satisfy the requirements of being a formal moduli problem. The deformation functors that we will study in §5.2 and §5.3 are of this nature. In this section, we will introduce a generalization of the notion of formal moduli problem (see Definition 5.1.5) which incorporates these examples as well.

We begin by reviewing some general terminology.

Definition 5.1.1. Let $n \geq -2$ be an integer. We will say that a diagram of spaces

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

is n-*Cartesian* if the induced map $\phi : X' \to X \times_Y Y'$ is n-truncated (that is, the homotopy fibers of ϕ are n-truncated).

Example 5.1.2. If $n = -2$, then a commutative diagram of spaces is n-Cartesian if and only if it is a pullback square.

The following lemma summarizes some of the basic transitivity properties of Definition 5.1.1:

Lemma 5.1.3. *Let $n \geq -2$ be an integer, and suppose we are given a commutative diagram*

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[d] \\ X \arrow[r] & Y \arrow[r] & Z \end{tikzcd}$$

in $\mathcal{S}$. If the right square is n-Cartesian, then the outer square is n-Cartesian if and only if the left square is n-Cartesian.

Using Lemma 5.1.3, we immediately deduce the following generalization of Proposition 1.1.15.

Proposition 5.1.4. *Let $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ be a deformation context, and let $X : \Upsilon^{\mathrm{sm}} \to \mathcal{S}$ be a functor. Let $n \geq 0$ be an integer. The following conditions are equivalent:*

(1) *Let σ :*

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & B' \arrow[d, "\phi"] \\ A \arrow[r] & B \end{tikzcd}$$

be a diagram in Υ^{sm}. If σ is a pullback diagram and ϕ is small, then $X(\sigma)$ is an $(n-2)$-Cartesian diagram in $\mathcal{S}$. pullback diagram in $\mathcal{S}$.

(2) *Let σ be as in (1). If σ is a pullback diagram and ϕ is elementary, then $X(\sigma)$ is an $(n-2)$-Cartesian diagram in $\mathcal{S}$.*

(3) *Let σ be as in (1). If σ is a pullback diagram and ϕ is the base point morphism $* \to \Omega^{\infty - m} E_\alpha$ for some $\alpha \in T$ and $m > 0$, then $X(\sigma)$ is an $(n-2)$-Cartesian diagram in $\mathcal{S}$.*

Definition 5.1.5. Let $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ be a deformation context and let $n \geq 0$ be an integer. We will say that a functor $X : \Upsilon^{\mathrm{sm}} \to \mathcal{S}$ is a n-*proximate formal moduli problem* if $X(*)$ is contractible and X satisfies the equivalent conditions of Proposition 5.1.4.

Example 5.1.6. A functor $X : \Upsilon^{\mathrm{sm}} \to \mathcal{S}$ is a 0-proximate formal moduli problem if and only if it is a formal moduli problem, in the sense of Definition 1.1.14.

Remark 5.1.7. Let $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ be a deformation context, and suppose we are given a pullback diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

in $\mathrm{Fun}(\Upsilon^{\mathrm{sm}}, \mathcal{S})$. If X and Y are n-proximate formal moduli problems and Y is an $(n+1)$-proximate formal moduli problem, then X' is an n-proximate formal moduli problem.

Definition 5.1.8. Let $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ be a deformation context and let $f : X \to Y$ be a natural transformation between functors $X, Y : \Upsilon^{\mathrm{sm}} \to \mathcal{S}$. We will say that f is *n-truncated* if the induced map $X(A) \to Y(A)$ is n-truncated, for each $A \in \mathrm{Art}$.

We can now state the main result of this section:

Theorem 5.1.9. *Let $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ be a deformation context which admits a deformation theory, and let $X : \Upsilon^{\mathrm{sm}} \to \mathcal{S}$ be a functor such that $X(*)$ is contractible (here $*$ denotes the final object of Υ). The following conditions are equivalent:*

(1) *The functor X is an n-proximate formal moduli problem.*

(2) *There exists an $(n-2)$-truncated map $f : X \to Y$, where Y is an n-proximate formal moduli problem.*

(3) *Let L denote a left adjoint to the inclusion* $\mathrm{Moduli}^{\Upsilon} \subseteq \mathrm{Fun}(\Upsilon^{\mathrm{sm}}, \mathcal{S})$ *(see Remark 1.1.17). Then the unit map $X \to LX$ is $(n-2)$-truncated.*

The proof of Theorem 5.1.9 will require some preliminaries. Let us identify $\mathrm{Sp} = \mathrm{Stab}(\mathcal{S})$ with the ∞-category of strongly excisive functors $\mathcal{S}_*^{\mathrm{fin}} \to \mathcal{S}$. Let $L_0 : \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S}) \to \mathrm{Sp}$ denote a left adjoint to the inclusion. If $X : \Upsilon^{\mathrm{sm}} \to \mathcal{S}$ is a functor, then the composition $X \circ E_\alpha$ determines a functor $\mathcal{S}_*^{\mathrm{fin}} \to \mathcal{S}$, and therefore a spectrum $L_0(X \circ E_\alpha)$.

Remark 5.1.10. Suppose that $F : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{S}$ is a functor which preserves final objects. Using the results of §A.1.5.2, we see that $L_0 F \in \mathrm{Sp}$ is described by the formula

$$(L_0 F)(K) = \varinjlim_n \Omega^n L_0(\Sigma^n K).$$

In particular, the functor L_0 is left exact when restricted to the full subcategory $\mathrm{Fun}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S}) \subseteq \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S})$ spanned by those functors which preserve final objects.

Remark 5.1.11. Let $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ be a deformation context which admits a deformation theory, and let $X : \Upsilon^{\mathrm{sm}} \to \mathcal{S}$ be an n-proximate formal moduli problem. It follows from Theorem 5.1.9 that there exists an $(n-2)$-truncated natural transformation $\alpha : X \to Y$, where Y is a formal moduli problem. In fact, the formal moduli problem Y (and the natural transformation α) are uniquely deterined up to equivalence. To prove this, we note that α factors as a composition

$$X \xrightarrow{\beta} LX \xrightarrow{\gamma} Y,$$

where β is $(n-2)$-truncated and γ is a map between formal moduli problems. For each $\alpha \in T$ and each $m \geq 0$, we have homotopy equivalences

$$\Omega^n X(\Omega^{\infty - m - n} E_\alpha) \simeq \Omega^n LX(\Omega^{\infty - m - n} E_\alpha) \simeq LX(\Omega^{\infty - m} E_\alpha)$$

$$\Omega^n X(\Omega^{\infty - m - n} E_\alpha) \simeq \Omega^n Y(\Omega^{\infty - m - n} E_\alpha) \simeq Y(\Omega^{\infty - m} E_\alpha).$$

From this it follows that γ induces an equivalence $LX(\Omega^{\infty - m} E_\alpha) \to Y(\Omega^{\infty - m} E_\alpha)$. Since LX and Y are formal moduli problems, we conclude that γ is an equivalence.

The main ingredient in the proof of Theorem 5.1.9 is the following lemma:

Lemma 5.1.12. *Let $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ be a deformation context which admits a deformation theory and let L denote a left adjoint to the inclusion functor* $\mathrm{Moduli}^{\Upsilon} \subseteq \mathrm{Fun}(\Upsilon^{\mathrm{sm}}, \mathcal{S})$. *Suppose that $X : \Upsilon^{\mathrm{sm}} \to \mathcal{S}$ is an n-proximate formal moduli problem for some $n \geq 0$.*

For each $\alpha \in T$, the canonical map $X(E_\alpha) \to (LX)(E_\alpha)$ induces an equivalence of spectra $L_0(X \circ E_\alpha) \to (LX) \circ E_\alpha$.

Proof. The proof proceeds by induction on n. In this case $n = 0$, $X \simeq LX$ and $X \circ E_\alpha$ is a spectrum, so there is nothing to prove. Assume therefore that $n > 0$. Let $\mathcal{C}$ be the full subcategory of $\mathrm{Fun}(\Upsilon^{\mathrm{sm}}, \mathcal{S})$ spanned by the $(n-1)$-proximate formal moduli problems, and let $\mathcal{C}_{/X}$ denote the fiber product $\mathcal{C} \times_{\mathrm{Fun}(\Upsilon^{\mathrm{sm}}, \mathcal{S})} \mathrm{Fun}(\Upsilon^{\mathrm{sm}}, \mathcal{S})_{/X}$. By the inductive hypothesis, the map $L_0(Y \circ E_\alpha) \to (LY) \circ E_\alpha$ is an equivalence of spectra for each $Y \in \mathcal{C}_{/X}$. It will therefore suffice to prove the following assertions (for each $\alpha \in T$):

(a) The spectrum $L_0(X \circ E_\alpha)$ is a colimit of the diagram $\{L_0(Y \circ E_\alpha)\}_{Y \in \mathcal{C}_{/X}}$ in the ∞-category Sp.

(b) The spectrum $(LX) \circ E_\alpha$ is a colimit of the diagram $\{(LY) \circ E_\alpha\}_{Y \in \mathcal{C}_{/X}}$ in the ∞-category Sp.

To prove (a), we note that L_0 preserves colimits (being a left adjoint) and that the construction $Y \mapsto Y(E_\alpha)$ carries colimit diagrams in $\mathrm{Fun}_*(\Upsilon^{\mathrm{sm}}, \mathcal{S})$ to colimit diagrams in $\mathrm{Fun}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S})$, where $\mathrm{Fun}_*(\Upsilon^{\mathrm{sm}}, \mathcal{S})$ denotes the full subcategory of $\mathrm{Fun}(\Upsilon^{\mathrm{sm}}, \mathcal{S})$ spanned by those functors which preserve final objects and $\mathrm{Fun}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S})$ is defined similarly. It will therefore suffice to show that X is a colimit of the diagram $\mathcal{C}_{/X} \to \mathrm{Fun}(\Upsilon^{\mathrm{sm}}, \mathcal{S})$. (and therefore also of the underlying functor $\mathcal{C}_{/X} \to \mathrm{Fun}_*(\Upsilon^{\mathrm{sm}}, \mathcal{S})$). We prove a more general assertion: namely, that the identity functor from $\mathrm{Fun}(\Upsilon^{\mathrm{sm}}, \mathcal{S})$ to itself is a left Kan extension of the inclusion $\mathcal{C} \to \mathrm{Fun}(\Upsilon^{\mathrm{sm}}, \mathcal{S})$. This follows from Proposition T.4.3.2.8 and Lemma T.5.1.5.3, since $\mathcal{C}$ contains the corepresentable functor $\mathrm{Spec}\, R$ for each $R \in \Upsilon^{\mathrm{sm}}$.

We now prove (b). Fix an index $\alpha \in T$, and let $F : \mathcal{C}_{/X} \to \mathrm{Sp}$ be the functor given by $Y \mapsto (LY) \circ E_\alpha$. Let $*$ denote the final object of Υ and let $X_0 = \mathrm{Spec}(*)$ denote the functor corepresented by $*$, so that X_0 is an initial object of $\mathrm{Fun}_*(\Upsilon^{\mathrm{sm}}, \mathcal{S})$. Let $X_\bullet$ denote the Čech nerve of the map $X_0 \to X$, and let $\mathcal{C}^0_{/X}$ denote the full subcategory of $\mathcal{C}_{/X}$ spanned by those maps $Y \to X$ which factor through X_0. We first prove:

($*$) The functors $L|\,\mathcal{C}_{/X} : \mathcal{C}_{/X} \to \mathrm{Moduli}^{\Upsilon}$ and $F : \mathcal{C}_{/X} \to \mathrm{Sp}$ are left Kan extensions of $L|\,\mathcal{C}^0_{/X}$ and $F|\,\mathcal{C}^0_{/X}$.

To prove ($*$), choose an object $Y \in \mathcal{C}_{/X}$, and let $\mathcal{C}^0_{/Y}$ be the full subcategory of $\mathcal{C}_{/Y}$ spanned by those morphisms $Z \to Y$ which factor through $Y_0 = X_0 \times_X Y$. We wish to prove that $LY \in \mathrm{Moduli}^{\Upsilon}$ and $FY \in \mathrm{Sp}$ are colimits of the diagrams $L|\,\mathcal{C}^0_{/Y}$ and $F|\,\mathcal{C}^0_{/Y}$, respectively. Let $Y_\bullet$ denote the simplicial object of $\mathcal{C}^0_{/Y}$ given by the Cech nerve of the map $Y_0 \to Y$, so that $Y_n \simeq X_n \times_X Y$. The construction $[n] \mapsto Y_n$ determines a left cofinal map $\mathrm{N}(\boldsymbol{\Delta})^{op} \to \mathcal{C}^0_{/Y}$; it will therefore suffice to show that the canonical maps

$$u : |LY_\bullet| \to LY \qquad v : |FY_\bullet| \to FY$$

are equivalences. Using Theorem 1.3.12 and condition (4) of Definition 1.3.9, we deduce that the construction $Z \mapsto Z \circ E_\alpha$ determines a functor $\mathrm{Moduli}^{\Upsilon} \to \mathrm{Sp}$ which commutes with sifted colimits. Consequently, to prove that u is an equivalence, it will suffice to show that v is an equivalence for every choice of index $\alpha \in T$. It follows from Remark 5.1.7 that each Y_m is an $(n-1)$-proximate formal moduli problem. Using the inductive hypothesis, we are reduced to showing that the canonical map $\theta : |L_0(Y_\bullet \circ E_\alpha)| \to L_0(Y(E_\alpha))$ is an equivalence of spectra. Note that $Y_\bullet \circ E_\alpha$ is the Čech nerve of the natural transformation

$$Y_0 \circ E_\alpha \to Y \circ E_\alpha$$

in the ∞-category $\mathrm{Fun}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S})$. Since the functor L_0 is left exact when restricted to $\mathrm{Fun}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S})$ (Remark 5.1.10), we conclude that $L_0(Y_\bullet(E_\alpha))$ is a Čech nerve of the map $L_0(Y_0 \circ E_\alpha) \to L_0(Y \circ E_\alpha))$, so that θ is an equivalence as desired.

To prove (b), we must show that $(LX) \circ E_\alpha$ is a colimit of the diagram F. Since F is a left Kan extension of $F|\,\mathcal{C}^0_{/X}$, it will suffice to show that $(LX) \circ E_\alpha)$ is a colimit of the diagram $F|\,\mathcal{C}^0_{/X}$ (Lemma T.4.3.2.7). The simplicial object $X_\bullet$ determines a left cofinal map $\mathrm{N}(\boldsymbol{\Delta})^{op} \to \mathcal{C}^0_{/X}$. We are therefore reduced to proving that the map $|(LX_\bullet) \circ E_\alpha| \to (LX) \circ E_\alpha$ is an equivalence of spectra. Since the construction $Z \mapsto Z \circ E_\alpha$ determines a functor $\mathrm{Moduli}^{\Upsilon} \to \mathrm{Sp}$ which preserves sifted colimits, it will suffice to show that $|LX_\bullet| \simeq LX$ in $\mathrm{Moduli}^{\Upsilon}$. This is equivalent to the assertion that LX is a colimit of the diagram $L|\,\mathcal{C}^0_{/X}$. Using ($*$) and Lemma T.4.3.2.7, we are reduced to proving that LX is a colimit of the diagram $L|\,\mathcal{C}_{/X}$. Since L preserves small colimits, this follows from the fact that X is a colimit of the inclusion functor $\mathcal{C}_{/X} \hookrightarrow \mathrm{Fun}(\Upsilon^{\mathrm{sm}}, \mathcal{S})$. $\square$

Lemma 5.1.13. *Let $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ be a deformation context and let $X : \Upsilon^{\rm sm} \to \mathcal{S}$ be an n-proximate formal moduli problem. For each $\alpha \in T$ and each $m \geq 0$, the canonical map $\theta : X(\Omega^{\infty - m} E_\alpha) \to \Omega^{\infty - m} L_0(X \circ E_\alpha))$ is $(n-2)$-truncated.*

Proof. We observe that θ is a filtered colimit of a sequence of morphisms

$$\theta_{m'} : X(\Omega^{\infty - m} E_\alpha) \to \Omega^{m'} X(\Omega^{\infty - m - m'} E_\alpha).$$

It will therefore suffice to show that each $\theta_{m'}$ is $(n-2)$-truncated. Each $\theta_{m'}$ is given by a composition of a finite sequence of morphisms $X(\Omega^{\infty - p} E_\alpha) \to \Omega X(\Omega^{\infty - p - 1} E_\alpha)$, which is $(n-2)$-truncated by virtue of our assumption that X is an n-proximate formal moduli problem. □

Lemma 5.1.14. *Let $(\Upsilon, \{E_\alpha\}_{\alpha \in T})$ be a deformation context and let $f : X \to Y$ be a natural transformation between n-proximate formal moduli problems $X, Y : \Upsilon^{\rm sm} \to \mathcal{S}$. Assume that, for every index $\alpha \in T$ and each $m \geq 0$, the map of spaces $X(\Omega^{\infty - m} E_\alpha) \to Y(\Omega^{\infty - m} E_\alpha)$ is $(n-2)$-truncated. Then, for each $A \in \Upsilon^{\rm sm}$, the map $X(A) \to Y(A)$ is $(n-2)$-truncated.*

Proof. Since A is small we can choose a sequence of elementary morphisms

$$A = A_0 \to A_1 \to \cdots \to A_p \simeq *.$$

We will prove that the map $\theta_i : X(A_i) \to Y(A_i)$ is $(n-2)$-truncated by descending induction on i. The case $i = p$ is clear (since θ is a morphism between contractible spaces and therefore a homotopy equivalence). Assume therefore that $i < p$ and that θ_{i+1} is $(n-2)$-truncated. Since the map $A_i \to A_{i+1}$ is elementary, we have a fiber sequence

$$A_i \to A_{i+1} \to \Omega^{\infty - m} E_\alpha$$

in $\Upsilon^{\rm sm}$. Let F be the homotopy fiber of the map $X(A_{i+1}) \to X(\Omega^{\infty - m} E_\alpha)$, and let F' be the homotopy fiber of the map $Y(A_{i+1}) \to Y(\Omega^{\infty - m} E_\alpha)$. We have a map of fiber sequences

$$\begin{tikzcd}
F \arrow[r] \arrow[d, "\psi"] & X(A_{i+1}) \arrow[r] \arrow[d, "\theta_{i+1}"] & X(\Omega^{\infty - m} E_\alpha \arrow[d, "\phi"] \\
F' \arrow[r] & Y(A_{i+1}) \arrow[r] & Y(\Omega^{\infty - m} E_\alpha).
\end{tikzcd}$$

Since ϕ is $(n-2)$-truncated by assumption and θ_{i+1} is $(n-2)$-truncated by the inductive hypothesis, we conclude that ψ is $(n-2)$-truncated. The map θ_i factors as a composition

$$X(A_i) \xrightarrow{\theta_i'} Y(A_i) \times_{F'} F \xrightarrow{\theta_i''} Y(A_i),$$

where θ_i'' is a pullback of ψ and therefore $(n-2)$-truncated. It will therefore suffice to show that θ_i' is $(n-2)$-truncated. Since Y is an n-proximate formal moduli problem, the map $Y(A_i) \to F'$ is $(n-2)$-truncated, so the projection $Y(A_i) \times_{F'} F \to F$ is $(n-2)$-truncated. It will therefore suffice to show that the composite map

$$X(A_i) \xrightarrow{\theta_i'} Y(A_i) \times_{F'} F \to F$$

is $(n-2)$-truncated, which follows from our assumption that X is an n-proximate formal moduli problem. □

Proof of Theorem 5.1.9. The implication (3) ⇒ (2) is obvious. We next show that (2) ⇒ (1). Assume that $X(*)$ is contractible and that there exists an $(n-2)$-truncated map $f : X \to Y$, where Y is an n-proximate formal moduli problem. We wish to show that X is an n-proximate formal moduli problem. Choose a pullback diagram

$$\begin{tikzcd}
A' \arrow[r] \arrow[d] & A \arrow[d, "\phi"] \\
B' \arrow[r] & B
\end{tikzcd}$$

in Υ^{sm} where ϕ is small; we wish to show that left square in the diagram of spaces

$$\begin{tikzcd} X(A') \arrow[r] \arrow[d] & X(A) \arrow[r] \arrow[d] & Y(A) \arrow[d] \\ X(B') \arrow[r] & X(B) \arrow[r] & Y(B) \end{tikzcd}$$

is $(n-2)$-Cartesian. Our assumption that f is $(n-2)$-truncated guarantees that the right square is $(n-2)$-Cartesian; it will therefore suffice to show that the outer square is $(n-2)$-Cartesian (Lemma 5.1.3). Using Lemma 5.1.3 again, we are reduced to showing that both the left and right squares in the diagram

$$\begin{tikzcd} X(A') \arrow[r] \arrow[d] & Y(A') \arrow[r] \arrow[d] & Y(A) \arrow[d] \\ X(B') \arrow[r] & Y(B') \arrow[r] & Y(B) \end{tikzcd}$$

are n-Cartesian. For the left square, this follows from our assumption that f is $(n-2)$-truncated; for the right square, it follows from our assumption that Y is an n-proximate formal moduli problem.

We now complete the proof by showing that (1) $\Rightarrow$ (3). Assume that X is an n-proximate formal moduli problem; we wish to show that the map $X \to LX$ is $(n-2)$-truncated. According to Lemma 5.1.14, it will suffice to show that the map $\phi : X(\Omega^{\infty-m} E_\alpha) \to LX(\Omega^{\infty-m} E_\alpha)$ is $(n-2)$-truncated for each $\alpha \in T$ and each $m \geq 0$. Using Lemma 5.1.12, we can identify ϕ with the canonical map $X(\Omega^{\infty-m} E_\alpha) \to \Omega^{\infty-m} L_0(X \circ E_\alpha))$, which is $(n-2)$-truncated by Lemma 5.1.13. □

5.2 Deformations of Objects

Let X be an algebraic variety defined over a field k, and let $\mathcal{E}$ be an algebraic vector bundle on X. A *first order deformation* of $\mathcal{E}$ is an algebraic vector bundle $\overline{\mathcal{E}}$ over the scheme $\overline{X} = X \times_{\operatorname{Spec} k} \operatorname{Spec} k[\epsilon]/(\epsilon^2)$, together with an isomorphism $i^* \overline{\mathcal{E}} \to \mathcal{E}$ (where i denotes the closed immersion $X \hookrightarrow \overline{X}$). Standard arguments in deformation theory show that the collection of isomorphism classes of first-order deformations can be identified with the cohomology group $\mathrm{H}^1(X; \mathrm{End}(\mathcal{E}))$, while the automorphism of each first order deformation of $\mathcal{E}$ is given by $\mathrm{H}^0(X; \mathrm{End}(\mathcal{E}))$.

Our goal in this section is to place the above observations into a more general context.

- (a) In the definition above, we can replace the ring of dual numbers $k[\epsilon]/(\epsilon^2)$ by an arbitrary $R \in \mathrm{CAlg}_k^{\mathrm{sm}}$ to obtain a notion of a *deformation of* $\mathcal{E}$ *over* R. Let $\mathrm{ObjDef}_{\mathcal{E}}(R)$ denote a classifying space for deformations of $\mathcal{E}$ over R. Then $\mathrm{ObjDef}_{\mathcal{E}}$ can be regarded as a functor $\mathrm{CAlg}_k^{\mathrm{sm}} \to \mathcal{S}$. We will see below that this functor is a formal moduli problem.

- (b) The isomorphisms

$$\mathrm{H}^0(X; \mathrm{End}(\mathcal{E})) \simeq \pi_1 \operatorname{ObjDef}_{\mathcal{E}}(k[\epsilon]/(\epsilon^2)) \qquad \mathrm{H}^1(X; \mathrm{End}(\mathcal{E})) \simeq \pi_0 \operatorname{ObjDef}_{\mathcal{E}}(k[\epsilon]/(\epsilon^2))$$

  follow from an identification of the cochain complex $C^*(X; \mathrm{End}(\mathcal{E}))$ with the (shifted) tangent complex $T_{\mathrm{ObjDef}_{\mathcal{E}}}[-1]$.

- (c) The definition of $\mathrm{ObjDef}_{\mathcal{E}}(R)$ does not require the commutativity of R. Consequently, we can extend the domain of definition of $\mathrm{ObjDef}_{\mathcal{E}}$ to $\mathrm{Alg}_k^{\mathrm{sm}}$, and thereby regard $\mathrm{ObjDef}_{\mathcal{E}}$ as a formal $\mathbb{E}_1$-moduli problem (see Definition 3.0.3). We will see that the identification $C^*(X; \mathrm{End}(\mathcal{E})) \simeq T_{\mathrm{ObjDef}_{\mathcal{E}}}[-1]$ is multiplicative: that is, it can be regarded as an equivalence of nonunital $\mathbb{E}_1$-algebras (where $T_{\mathrm{ObjDef}_{\mathcal{E}}}[-1]$ is equipped with the nonunital $\mathbb{E}_1$-algebra structure given by Remark 3.2.6).

(d) The definition of the formal moduli problem $\mathrm{ObjDef}_{\mathcal{E}}$ depends only on the algebraic vector bundle $\mathcal{E}$ as an object of the stable ∞-category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves on X. We will therefore consider the more general problem of deforming an object M of an arbitrary k-linear ∞-category $\mathcal{C}$.

We begin by introducing some notation.

Construction 5.2.1. Let k be a field, let $\mathcal{C}$ be a k-linear ∞-category, and let $M \in \mathcal{C}$ be an object. We let $\mathrm{RMod}(\mathcal{C})$ denote the ∞-category of pairs (A, M_A) where $A \in \mathrm{Alg}_k$ and M_A is a right A-module object of $\mathcal{C}$. The forgetful functor $q : \mathrm{RMod}(\mathcal{C}) \to \mathrm{Alg}_k$ is a coCartesian fibration. We let $\mathrm{RMod}^{\mathrm{coCart}}(\mathcal{C})$ denote the subcategory of $\mathrm{RMod}(\mathcal{C})$ spanned by the q-coCartesian morphisms, so that q restricts to a left fibration $\mathrm{RMod}^{\mathrm{coCart}}(\mathcal{C}) \to \mathrm{Alg}_k$. We will abuse notation by identifying M with an object of $\mathrm{RMod}^{\mathrm{coCart}}(\mathcal{C})$ (via the equivalence $\mathrm{RMod}_k(\mathcal{C}) \to \mathcal{C}$). We let $\mathrm{Defor}[M] = \mathrm{RMod}^{\mathrm{coCart}}(\mathcal{C})_{/M}$. We will refer to $\mathrm{Defor}[M]$ as the *∞-category of deformations of M*.

There is an evident forgetful functor $\theta : \mathrm{Defor}[M] \to \mathrm{Alg}_k^{\mathrm{aug}}$. The fiber of θ over an augmented k-algebra A can be identified with the ∞-category of pairs (M_A, μ), where $M_A \in \mathrm{RMod}_A(\mathcal{C})$ and $\mu : M_A \otimes_A k \to M$ is an equivalence in $\mathcal{C}$. The map θ is a left fibration, classified by a functor $\chi : \mathrm{Alg}_k^{\mathrm{aug}} \to \widehat{\mathcal{S}}$; here $\widehat{\mathcal{S}}$ denotes the ∞-category of spaces which are not necessarily small.

Let $\mathcal{C}$ be as in Construction 5.2.1. Proposition IX.7.4 implies that for every pullback diagram

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & A \arrow[d] \\ B' \arrow[r] & B \end{tikzcd}$$

in Alg_k, the induced functor $\mathrm{RMod}_{A'}(\mathcal{C}) \to \mathrm{RMod}_A(\mathcal{C}) \times_{\mathrm{RMod}_B(\mathcal{C})} \mathrm{RMod}_{B'}(\mathcal{C})$ is fully faithful. This immediately implies the following:

Proposition 5.2.2. *Let k be a field, $\mathcal{C}$ a k-linear ∞-category, $M \in \mathcal{C}$ an object, and let $\chi : \mathrm{Alg}_k^{\mathrm{aug}} \to \widehat{\mathcal{S}}$ be as in Construction 5.2.1. Then, for every pullback diagram*

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & A \arrow[d] \\ B' \arrow[r] & B \end{tikzcd}$$

in $\mathrm{Alg}_k^{\mathrm{aug}}$, the induced map $\chi(A') \to \chi(A) \times_{\chi(B)} \chi(B')$ induces a homotopy equivalent from $\chi(A')$ to a summand of $\chi(A) \times_{\chi(B)} \chi(B')$.

Corollary 5.2.3. *Let k be a field, $\mathcal{C}$ a k-linear ∞-category, $M \in \mathcal{C}$ an object, and let $\chi : \mathrm{Alg}_k^{\mathrm{aug}} \to \widehat{\mathcal{S}}$ be as in Construction 5.2.1. Then:*

(1) *The space $\chi(k)$ is contractible.*

(2) *Let $V \in \mathrm{Mod}_k$. Then the space $\chi(k \oplus V)$ is essentially small.*

(3) *Let $A \in \mathrm{Alg}_k^{\mathrm{aug}}$ be small. Then the space $\chi(A)$ is essentially small.*

Proof. Assertion (1) is immediate from the definitions. To prove (2), we note that for each $A \in \mathrm{Alg}_k^{\mathrm{aug}}$, the space $\chi(A)$ is locally small (when regarded as an ∞-category). We have a pullback diagram

$$\begin{tikzcd} k \oplus V \arrow[r] \arrow[d] & k \arrow[d] \\ k \arrow[r] & k \oplus V[1] \end{tikzcd}$$

so that Proposition 5.2.2 guarantees that $\chi(k \oplus V)$ is a summand of $\Omega\chi(k \oplus V[1])$, and therefore essentially small. We now prove (3). Assume that A is small, so that there exists a finite sequence of maps

$$A \simeq A_0 \to A_1 \to \cdots \to A_n \simeq k$$

and pullback diagrams

$$\begin{tikzcd} A_i \arrow[r] \arrow[d] & k \arrow[d] \\ A_{i+1} \arrow[r] & k \oplus k[m_i]. \end{tikzcd}$$

Using (1), (2), and Proposition 5.2.2, we deduce that each $\chi(A_i)$ is essentially small by descending induction on i. □

Notation 5.2.4. Let k be a field, $\mathcal{C}$ a k-linear ∞-category, and $M \in \mathcal{C}$ an object. Let $\chi : \mathrm{Alg}_k^{\mathrm{aug}} \to \widehat{\mathcal{S}}$ be the functor classifying the left fibration $\mathrm{Defor}[M] \to \mathrm{Alg}_k^{\mathrm{aug}}$ of Construction 5.2.1. We let ObjDef_M denote the restriction of χ to the full subcategory $\mathrm{Alg}_k^{\mathrm{sm}} \subseteq \mathrm{Alg}_k^{\mathrm{aug}}$ spanned by the small $\mathbb{E}_1$-algebras over k. Using Corollary 5.2.3, we can identify ObjDef_M with a functor from $\mathrm{Alg}_k^{\mathrm{sm}}$ to $\mathcal{S}$.

More informally: the functor ObjDef_M assigns to each $R \in \mathrm{Alg}_k^{\mathrm{sm}}$ a classifying space for pairs (M_R, μ), where $M_R \in \mathrm{RMod}_R(\mathcal{C})$ and $\mu : M_R \otimes_R k \to M$ is an equivalence in $\mathcal{C}$.

Combining Corollary 5.2.3 and Proposition 5.2.2, we obtain the following:

Corollary 5.2.5. *Let k be a field, $\mathcal{C}$ a k-linear ∞-category, and $M \in \mathcal{C}$ an object. Then the functor $\mathrm{ObjDef}_M : \mathrm{Alg}_k^{\mathrm{sm}} \to \mathcal{S}$ is a 1-proximate formal moduli problem (see Definition 5.1.5).*

Notation 5.2.6. Let k be a field, and let $L : \mathrm{Fun}(\mathrm{Alg}_k^{\mathrm{sm}}, \mathcal{S}) \to \mathrm{Moduli}_k^{(1)}$ denote a left adjoint to the inclusion. If $\mathcal{C}$ is a k-linear ∞-category and $M \in \mathcal{C}$ is an object, we let $\mathrm{ObjDef}_M^{\wedge}$ denote the formal $\mathbb{E}_1$-moduli problem $L(\mathrm{ObjDef}_M)$. By construction, we have a natural transformation $\mathrm{ObjDef}_M \to \mathrm{ObjDef}_M^{\wedge}$. It follows from Theorem 5.1.9 that this natural transformation exhibits $\mathrm{ObjDef}_M(R)$ as a summand of $\mathrm{ObjDef}_M^{\wedge}(R)$, for each $R \in \mathrm{Alg}_k^{\mathrm{sm}}$. Moreover, $\mathrm{ObjDef}_M^{\wedge}$ is characterized up to equivalence by this property: see Remark 5.1.11.

Notation 5.2.7. Let k be a field, let $\mathcal{C}$ be a k-linear ∞-category, and let $M \in \mathcal{C}$ be an object. We let $\mathrm{End}(M)$ denote a classifying object for endomorphisms of M. That is, $\mathrm{End}(M)$ is an object of Mod_k equipped with a map $a : M \otimes \mathrm{End}(M) \to M$ in $\mathcal{C}$ having the following universal property: for every object $V \in \mathrm{Mod}_k$, composition with a induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{Mod}_k}(V, \mathrm{End}(M)) \simeq \mathrm{Map}_{\mathcal{C}}(M \otimes V, M).$$

The existence of the object $\mathrm{End}(M)$ follows from Proposition A.4.2.1.33. Moreover, it follows from the results of §A.6.1.2 show that we can regard $\mathrm{End}(M)$ as an $\mathbb{E}_1$-algebra over k, and M as a right module over $\mathrm{End}(M)$. In what follows, it will be more convenient to view $\mathrm{End}(M)$ as a *nonunital* $\mathbb{E}_1$-algebra over k, which can be identified with the augmentation ideal in the augmented $\mathbb{E}_1$-algebra $k \oplus \mathrm{End}(M)$.

We can now formulate the main result of this section.

Theorem 5.2.8. *Let k be a field, let $\mathcal{C}$ be a k-linear ∞-category, let $M \in \mathcal{C}$ be an object, and let $\Psi : \mathrm{Alg}_k^{\mathrm{aug}} \to \mathrm{Moduli}_k^{(1)}$ be the equivalence of ∞-categories of Theorem 3.0.4. Then there is a canonical equivalence $\Psi(k \oplus \mathrm{End}(M)) \simeq \mathrm{ObjDef}_M^{\wedge}$.*

Example 5.2.9. Let k be a field and regard Mod_k as a k-linear ∞-category. Let V be a finite-dimensional vector space over k, and define ObjDef_V as above. We will see below that ObjDef_V is a formal $\mathbb{E}_1$ moduli problem (Proposition 5.2.14), so that $\mathrm{ObjDef}_V \,|\, \mathrm{CAlg}_k^{\mathrm{sm}}$ is a formal moduli problem over k. Assume now that k has characteristic zero, and let $\Phi : \mathrm{Lie}_k \to \mathrm{Moduli}_k$ be the equivalence of Theorem 2.0.2. Combining Theorems 5.2.8 and 3.3.1, we deduce that $\mathrm{ObjDef}_V \,|\, \mathrm{CAlg}_k^{\mathrm{sm}}$ corresponds, under the equivalence Φ, to the matrix algebra $\mathrm{End}(V)$ (equipped with its usual Lie bracket).

Remark 5.2.10. Let k be a field, let $\mathcal{C}$ be a k-linear ∞-category, and let $M \in \mathcal{C}$ be an object. For each $R \in \mathrm{Alg}_k^{\mathrm{sm}}$, Theorem A.6.3.4.1 yields a canonical homotopy equivalence

$$\mathrm{ObjDef}_M(R) = \mathrm{RMod}_R(\mathcal{C}) \times_{\mathcal{C}} \{M\} \simeq \mathrm{Map}_{\mathrm{LinCat}_k}(\mathrm{LMod}_R, \mathcal{C}) \times_{\mathcal{C}^{\simeq}} \{M\}.$$

It follows that $\mathrm{ObjDef}_M(R)$ depends only on the k-linear ∞-category LMod_R, together with the distinguished object $k \in \mathrm{LMod}_R$ (given by the augmentation $R \to k$), so that the construction $R \mapsto \mathrm{ObjDef}_M(R)$ enjoys some extra functoriality. This special feature of $\mathrm{ObjDef}_M(R)$ is reflected in the structure of the associated formal $\mathbb{E}_1$ moduli problem $\mathrm{ObjDef}^{\wedge}_M(R)$: according to Theorem 5.2.8, $\mathrm{ObjDef}^{\wedge}_M(R)$ has the form $\Psi(A)$, where $A \in \mathrm{Alg}_k^{\mathrm{aug}}$ is an augmented $\mathbb{E}_1$-algebra over k whose augmentation ideal $\mathfrak{m}_A \simeq \mathrm{End}(M)$ is itself unital.

Theorem 5.2.8 is a consequence of a more precise assertion (Proposition 5.3.16) which describes the equivalence $\Psi(k \oplus \mathrm{End}(M)) \simeq \mathrm{ObjDef}^{\wedge}_M$ explicitly. Before we can formulate it, we need to introduce a bit more notation.

Construction 5.2.11. Let k be a field, let $\mathcal{C}$ be a k-linear ∞-category, and let $M \in \mathcal{C}$ be an object. We let $\lambda : \mathcal{M}^{(1)} \to \mathrm{Alg}_k^{\mathrm{aug}} \times \mathrm{Alg}_k^{\mathrm{aug}}$ be the pairing of Construction 4.4.6, so that we can identify objects of $\mathcal{M}^{(1)}$ with triples (A, B, ϵ) where $A, B \in \mathrm{Alg}_k$ and $\epsilon : A \otimes_k B \to k$ is an augmentation. Given an object $(A, B, \epsilon) \in \mathcal{M}^{(1)}$ and an object $(A, M_A, \mu) \in \mathrm{Defor}[M]$, we regard $M_A \otimes_k B$ as an object of ${}_B\mathrm{BMod}_{A \otimes_k B}(\mathcal{C})$, so that

$$(M \otimes_k B) \otimes_{A \otimes_k B} k$$

can be identified with an object of $\mathrm{LMod}_B(\mathcal{C})$ whose image in $\mathcal{C}$ is given by $M_A \otimes_A k \simeq M$. This construction determines a functor

$$\mathrm{Defor}[M] \times_{\mathrm{Alg}_k^{\mathrm{aug}}} \mathcal{M}^{(1)} \to \mathrm{LMod}(\mathcal{C}) \times_{\mathcal{C}} \{M\}$$

(see Corollary A.6.1.2.40). Let $\mathrm{LMod}^{\mathrm{aug}}(\mathcal{C})$ denote the fiber product $\mathrm{Alg}_k^{\mathrm{aug}} \times_{\mathrm{Alg}_k} \mathrm{LMod}(\mathcal{C})$. The induced map $\mathrm{Defor}[M] \times_{\mathrm{Alg}_k^{\mathrm{aug}}} \mathcal{M}^{(1)} \to \mathrm{Defor}[M] \times (\mathrm{LMod}^{\mathrm{aug}}(\mathcal{C}) \times_{\mathcal{C}} \{M\})$ factors as a composition

$$\mathrm{Defor}[M] \times_{\mathrm{Alg}_k^{\mathrm{aug}}} \mathcal{M}^{(1)} \xrightarrow{i} \widetilde{\mathcal{M}}^{(1)} \xrightarrow{\lambda'} \mathrm{Defor}[M] \times (\mathrm{LMod}^{\mathrm{aug}}(\mathcal{C}) \times_{\mathcal{C}} \{M\})$$

where i is an equivalence of ∞-categories and λ' is a categorical fibration. It is not difficult to see that λ' is a left representable pairing of ∞-categories, which induces a duality functor

$$\mathfrak{D}^{(1)}_M : \mathrm{Defor}[M]^{op} \to \mathrm{LMod}^{\mathrm{aug}}(\mathcal{C}) \times_{\mathcal{C}} \{M\}.$$

Concretely, the functor $\mathfrak{D}^{(1)}_M$ assigns to each object $(A, M_A, \mu) \in \mathrm{Defor}[M]^{op}$ the object $(\mathfrak{D}^(1)(A), M)$, where we regard M as a left $\mathfrak{D}^{(1)}$-module object of $\mathcal{C}$ via the equivalence

$$M \simeq M_A \otimes_A k \simeq (M \otimes_k \mathfrak{D}^{(1)}(A)) \otimes_{A \otimes_k \mathfrak{D}^{(1)}(A)} k.$$

Using Corollary A.6.1.2.40, we have a canonical equivalence of ∞-categories $\eta : \mathrm{LMod}(\mathcal{C}) \times_{\mathcal{C}} \{M\} \simeq (\mathrm{Alg}_k)_{/\mathrm{End}(M)}$. Then η induces an equivalence

$$\mathrm{LMod}^{\mathrm{aug}}(\mathcal{C}) \times_{\mathcal{C}} \{M\} \simeq (\mathrm{Alg}_k^{\mathrm{aug}})_{/(k \oplus \mathrm{End}(M))}.$$

Combining this equivalence with Construction 5.2.11, we obtain a diagram of ∞-categories

$$\begin{tikzcd}
\mathrm{Defor}[M]^{op} \arrow[r] \arrow[d] & (\mathrm{Alg}_k^{\mathrm{aug}})_{/k \oplus \mathrm{End}(M)} \arrow[d] \\
(\mathrm{Alg}_k^{\mathrm{aug}})^{op} \arrow[r] & \mathrm{Alg}_k^{\mathrm{aug}}
\end{tikzcd}$$

which commutes up to canonical homotopy, where the vertical maps are right fibrations. This diagram determines a natural transformation $\beta : \mathrm{ObjDef}_M \to X$, where $X : \mathrm{Art}^{(1)} \to \mathcal{S}$ is the functor given by the formula $X(A) = \mathrm{Map}_{\mathrm{Alg}_k^{\mathrm{aug}}}(\mathfrak{D}^{(1)}(A), k \oplus \mathrm{End}(M)) \simeq \mathrm{Map}_{\mathrm{Alg}_k}(\mathfrak{D}^{(1)}(A), \mathrm{End}(M))$.

Theorem 5.2.8 is a consequence of Remark 5.1.11 together with the following result:

Proposition 5.2.12. *Let k be a field, $\mathcal{C}$ a k-linear ∞-category, and $M \in \mathcal{C}$ an object. Then the natural transformation $\beta : \mathrm{ObjDef}_M \to X$ of Construction 5.2.11 exhibits $\mathrm{ObjDef}_M(R)$ as a summand of $X(R)$ for each $R \in \mathrm{Alg}_k^{\mathrm{sm}}$.*

Proposition 5.2.12 asserts that β induces an equivalence of formal moduli problems $\overline{\beta} : \mathrm{ObjDef}_M^{\wedge} \to X$. According to Proposition 1.2.10, it suffices to show that $\overline{\beta}$ induces an equivalence of tangent complexes. Using the description of the tangent complex of $\mathrm{ObjDef}_M^{\wedge}$ supplied by Lemma 5.1.12, we are reduced to proving the following special case of Proposition 5.2.12:

Proposition 5.2.13. *Let k be a field, $\mathcal{C}$ a k-linear ∞-category, and let $M \in \mathcal{C}$ be an object. For each $m \geq 0$, the natural transformation $\beta : \mathrm{ObjDef}_M \to X$ of Construction 5.2.11 induces a (-1)-truncated map*

$$\mathrm{ObjDef}_M(k \oplus k[m]) \to \mathrm{Map}_{\mathrm{Alg}_k^{\mathrm{aug}}}(\mathfrak{D}^{(1)}(k \oplus k[m]), k \oplus \mathrm{End}(M)) \simeq \mathrm{Map}_{\mathrm{Alg}_k}(\mathfrak{D}^{(1)}(k \oplus k[m]), \mathrm{End}(M)).$$

Proof. We have a commutative diagram

$$\begin{tikzcd}
\mathrm{ObjDef}_M(k \oplus k[m]) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{Alg}_k}(\mathfrak{D}^{(1)}(k \oplus k[m]), \mathrm{End}(M)) \arrow[d] \\
\Omega \, \mathrm{ObjDef}_M(k \oplus k[m+1]) \arrow[r, "\theta"] & \Omega \, \mathrm{Map}_{\mathrm{Alg}_k}(\mathfrak{D}^{(1)}(k \oplus k[m+1]), \mathrm{End}(M)),
\end{tikzcd}$$

where the left vertical map is (-1)-truncated by Corollary 5.2.5 and the right vertical map is a homotopy equivalence. It will therefore suffice to show that θ is a homotopy equivalence. Let $A = k \oplus k[m+1]$ and let $M_A = M \otimes_k A \in \mathrm{RMod}_A(\mathcal{C})$. We can identify the domain of θ with the homotopy fiber of the map $\xi : \mathrm{Map}_{\mathrm{RMod}_A(\mathcal{C})}(M_A, M_A) \to \mathrm{Map}_{\mathcal{C})}(M, M)$. We have a fiber sequence

$$M[m+1] \to M_A \to M$$

in $\mathrm{RMod}_A(\mathcal{C})$, where A acts on M via the augmentation map $A \to k$. It follows that the homotopy fiber of ξ is given by

$$\begin{aligned}
\mathrm{Map}_{\mathrm{RMod}_A(\mathcal{C})}(M_A, M[m+1]) &\simeq \mathrm{Map}_{\mathcal{C}}(M_A \otimes_A k, M[m+1]) \\
&\simeq \mathrm{Map}_{\mathcal{C}}(M, M[m+1]) \\
&\simeq \mathrm{Map}_{\mathrm{Mod}_k}(k[-m-1], \mathrm{End}(M))
\end{aligned}$$

The map θ is induced by a morphism $\nu : k[-m-1] \to \mathfrak{D}^{(1)}(k \oplus k[m])$ in Mod_k. Let $\mathrm{Free}^{(1)} : \mathrm{Mod}_k \to \mathrm{Alg}_k$ be a left adjoint to the forgetful functor, so that ν determines an augmentation $(k \oplus k[m]) \otimes_k \mathrm{Free}^{(1)}(k[-m-1]) \to k$. This pairing exhibits $k \oplus k[m]$ as a Koszul dual of $\mathrm{Free}^{(1)}(k[-m-1])$, and therefore also exhibits $\mathrm{Free}^{(1)}(k[-m-1])$ as a Koszul dual of $k \oplus k[m]$ (Theorem 3.1.14). It follows immediately that θ is a homotopy equivalence. □

We conclude this section with a few observations concerning the discrepancy between the deformation functor ObjDef_M and the associated formal moduli problem $\mathrm{ObjDef}_M^{\wedge}$ introduced in Notation 5.2.6. Under favorable circumstances, one can show that these two functors are equivalent:

Proposition 5.2.14. *Let k be a field, let $\mathcal{C}$ be a k-linear ∞-category, and let M be an object. Assume that $\mathcal{C}$ admits a left complete t-structure and that M is connective. Then the functor $\mathrm{ObjDef}_M : \mathrm{Alg}_k^{\mathrm{sm}} \to \mathcal{S}$ of Notation 5.2.4 is a formal moduli problem.*

Proof. Let $A \in \mathrm{Alg}_k^{\mathrm{sm}}$. We first show that if $(M, \mu) \in \mathrm{ObjDef}_M(A)$, then $M \in \mathrm{RMod}_A(\mathcal{C})_{\geq 0}$. Since A is small, we can choose a finite sequence of maps

$$A \simeq A_0 \to A_1 \to \cdots \to A_n \simeq k$$

and pullback diagrams

$$\begin{tikzcd} A_i \arrow[r] \arrow[d] & k \arrow[d] \\ A_{i+1} \arrow[r] & k \oplus k[m_i] \end{tikzcd}$$

for some integers $m_i > 0$. We prove by descending induction on i that $M \otimes_A A_i$ belongs to $\mathrm{RMod}_{A_i}(\mathcal{C})_{\geq 0}$. In this case $i = n$, this follows from our assumption that M_n is connective. If $i < n$, it follows from the inductive hypothesis since we have a fiber sequence

$$M[m_i - 1] \to M \otimes_A A_i \to M \otimes_A A_{i+1}$$

in $\mathcal{C}$.

Proposition IX.7.6 implies that if

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & A \arrow[d, "f"] \\ B' \arrow[r, "g"] & B \end{tikzcd}$$

is a pullback diagram in $\mathrm{Alg}_k^{\mathrm{sm}}$ where the maps f and g induce surjections $\pi_0 A \to \pi_0 B \leftarrow \pi_0 B'$, then the functor

$$\mathrm{RMod}_{A'}(\mathcal{C})_{\geq 0} \to \mathrm{RMod}_A(\mathcal{C})_{\geq 0} \times_{\mathrm{RMod}_B(\mathcal{C})_{\geq 0}} \mathrm{RMod}_{B'}(\mathcal{C})_{\geq 0}$$

is an equivalence of ∞-categories. It follows immediately that

$$\begin{tikzcd} \mathrm{ObjDef}_M(A') \arrow[r] \arrow[d] & \mathrm{ObjDef}_M(A) \arrow[d] \\ \mathrm{ObjDef}_M(B') \arrow[r] & \mathrm{ObjDef}_M(B) \end{tikzcd}$$

is a pullback diagram in $\mathcal{S}$. $\square$

Corollary 5.2.15. *Let k be a field, let $\mathcal{C}$ be a k-linear ∞-category which admits a left complete t-structure, and let $M \in \mathcal{C}_{\geq 0}$. Then the natural transformation $\beta : \mathrm{ObjDef}_M \to X$ of Construction 5.2.11 is an equivalence. In other words, $\mathrm{ObjDef}_M : \mathrm{Alg}_k^{\mathrm{sm}} \to \mathcal{S}$ is the formal moduli problem which corresponds, under the equivalence of Theorem 3.0.4, to the augmented algebra $k \oplus \mathrm{End}(M)$.*

Proof. Combine Theorem 5.2.8, Corollary 5.2.5, and Theorem 5.1.9. $\square$

Remark 5.2.16. Let k be a field. For $R \in \mathrm{Alg}_k^{\mathrm{sm}}$, let $\mathrm{RMod}_R^!$ denote the ∞-category of Ind-coherent right R-modules over R (see §3.4). One can show that $\mathrm{RMod}_R^!$ has the structure of a k-linear ∞-category (depending functorially on R). For any k-linear ∞-category $\mathcal{C}$, let $\mathrm{RMod}_R^!(\mathcal{C})$ denote the relative tensor product $\mathcal{C} \otimes_{\mathrm{Mod}_k} \mathrm{RMod}_R^!$. The equivalence $\mathrm{RMod}_R^! \simeq \mathrm{LMod}_{\mathfrak{D}^{(1)}(R)}$ of Proposition 3.5.2 is k-linear, and (combined with Theorem A.6.3.4.6) determines an equivalence $\mathrm{RMod}_R^!(\mathcal{C}) \simeq \mathrm{LMod}_{\mathfrak{D}^{(1)}(R)}(\mathcal{C})$. Let $M \in \mathcal{C}$ and let $X : \mathrm{Alg}_k^{\mathrm{sm}} \to \mathcal{S}$ be the formal moduli problem described in Construction 5.2.11. Then X is given by the formula

$$\begin{aligned} X(R) &= \mathrm{Map}_{\mathrm{Alg}_k^{\mathrm{aug}}}(\mathfrak{D}^{(1)}(R), k \oplus \mathrm{End}(M)) \\ &\simeq \mathrm{Map}_{\mathrm{Alg}_k}(\mathfrak{D}^{(1)}(R), \mathrm{End}(M)) \\ &\simeq \mathrm{LMod}_{\mathfrak{D}^{(1)}(R)}(\mathcal{C}) \times_{\mathcal{C}} \{M\} \\ &\simeq \mathrm{RMod}_R^!(\mathcal{C}) \times_{\mathcal{C}} \{M\}. \end{aligned}$$

In other words, the formal moduli problem X assigns to each $R \in \mathrm{Alg}_k^{\mathrm{sm}}$ a classifying space for pairs (M, α), where $M \in \mathrm{RMod}^!_R(\mathcal{C})$ and α is an equivalence of M with the image of $M \otimes_R k$ of M in the ∞-category $\mathcal{C}$. The (-1)-truncated map $\mathrm{ObjDef}_M \hookrightarrow \mathrm{ObjDef}_M^\wedge \simeq X$ is induced by fully faithful embedding $\mathrm{RMod}_R(\mathcal{C}) \hookrightarrow \mathrm{RMod}^!_R(\mathcal{C})$, which are in turn determined by the fully faithful embeddings $\mathrm{RMod}_R \hookrightarrow \mathrm{RMod}^!_R$ of Proposition 3.4.14. From this point of view, we can view Proposition 5.2.14 as a generalization of Proposition 3.4.18: it asserts that the fully faithful embedding $\mathrm{RMod}_R(\mathcal{C}) \hookrightarrow \mathrm{RMod}^!_R(\mathcal{C})$ induces an equivalence on connective objects (where we declare an object of $\mathrm{RMod}^!_R(\mathcal{C})$ to be connective if its image in $\mathcal{C}$ is connective).

5.3 Deformations of Categories

Let k be a field and let $\mathcal{C}$ be a k-linear ∞-category. In §5.2, we studied the problem of deforming a fixed object $M \in \mathcal{C}$. In this section, we will study the deformation theory of the ∞-category $\mathcal{C}$ itself. For every small $\mathbb{E}_2$-algebra R over k, we will define a classifying space $\mathrm{CatDef}_{\mathcal{C}}(R)$ for R-linear ∞-categoriers $\mathcal{C}_R$ equipped with an equivalence $\mathcal{C} \simeq \mathrm{Mod}_k \otimes_{\mathrm{Mod}_R} \mathcal{C}_R$. We will show that, modulo size issues, the construction $R \mapsto \mathrm{CatDef}_{\mathcal{C}}(R)$ is a 2-proximate formal moduli problem (Corollary 5.3.8; in good cases, we can say even more: see Proposition 5.3.21 and Theorem 5.3.33). Using Theorem 5.1.9, we deduce that there is a 0-truncated natural transformation $\mathrm{CatDef}_{\mathcal{C}} \to \mathrm{CatDef}_{\mathcal{C}}^\wedge$, where $\mathrm{CatDef}_{\mathcal{C}}^\wedge$ is a formal $\mathbb{E}_2$ moduli problem (which is uniquely determined up to equivalence: see Remark 5.1.11). According to Theorem 4.0.8, the formal moduli problem $\mathrm{CatDef}_{\mathcal{C}}^\wedge$ is given by $R \mapsto \mathrm{Map}_{\mathrm{Alg}_k^{(2),\mathrm{aug}}}(\mathfrak{D}^{(2)}(R), A)$ for an essentially unique augmented $\mathbb{E}_2$-algebra A over k. The main result of this section identifies the augmentation ideal $\mathfrak{m}_A$ (as a nonunital $\mathbb{E}_2$-algebra) with the *k-linear center* of the ∞-category $\mathcal{C}$ (Theorem 5.3.16): in other words, with the chain complex of Hochschild cochains on $\mathcal{C}$.

We begin with a more precise description of the deformation functor $\mathrm{CatDef}_{\mathcal{C}}$.

Notation 5.3.1. Let k be an $\mathbb{E}_\infty$-ring. We let $\mathrm{LinCat}_k \simeq \mathrm{Mod}_{\mathrm{Mod}_k}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ denote the ∞-category of k-linear ∞-categories, which we regard as as a symmetric monoidal ∞-category. According to Theorem A.6.3.5.14, the construction $A \mapsto \mathrm{LMod}_A(\mathrm{Mod}_k)$ determines a symmetric monoidal functor $\mathrm{Alg}_k \to \mathrm{LinCat}_k$. Passing to algebra objects, we obtain a functor $\mathrm{Alg}_k^{(2)} \simeq \mathrm{Alg}(\mathrm{Alg}_k) \to \mathrm{Alg}(\mathrm{LinCat}_k)$. We set

$$\mathrm{LCat}(k) = \mathrm{Alg}_k^{(2)} \times_{\mathrm{Alg}(\mathrm{LinCat}_k)} \mathrm{LMod}(\mathrm{LinCat}_k)$$

$$\mathrm{RCat}(k) = \mathrm{Alg}_k^{(2)} \times_{\mathrm{Alg}(\mathrm{LinCat}_k)} \mathrm{RMod}(\mathrm{LinCat}_k).$$

The objects of $\mathrm{LCat}(k)$ are pairs $(A, \mathcal{C})$ where A is an $\mathbb{E}_2$-algebra over k and $\mathcal{C}$ is an A-linear ∞-category (that is, an ∞-category left-tensored over LMod_A). Similarly, the objects of $\mathrm{RCat}(k)$ are pairs $(B, \mathcal{C})$ where B is an $\mathbb{E}_2$-algebra over k and $\mathcal{C}$ is an ∞-category right-tensored over LMod_B.

Construction 5.3.2. Let k be a field, and let $q : \mathrm{LCat}(k) \to \mathrm{Alg}_k^{(2)}$ be the evident coCartesian fibration. We let $\mathrm{LCat}(k)^{\mathrm{coCart}}$ denote the subcategory of $\mathrm{LCat}(k)$ spanned by the q-coCartesian morphisms, so that q restricts to a left fibration $\mathrm{LCat}(k)^{\mathrm{coCart}} \to \mathrm{Alg}_k^{(2)}$.

Let $\mathcal{C}$ be a k-linear ∞-category and regard $(k, \mathcal{C})$ as an object of $\mathrm{LCat}(k)$. We let $\mathrm{Defor}[\mathcal{C}]$ denote the ∞-category $\mathrm{LCat}(k)^{\mathrm{coCart}}_{/(k,\mathcal{C})}$. We will refer to $\mathrm{Defor}[\mathcal{C}]$ as the *∞-category of deformations of* $\mathcal{C}$. There is an evident forgetful functor $\theta : \mathrm{Defor}[\mathcal{C}] \to \mathrm{Alg}_k^{(2),\mathrm{aug}}$. The fiber of θ over an augmented k-algebra A can be identified with the ∞-category of pairs $(\mathcal{C}_A, \mu)$, where $\mathcal{C}_A$ is an A-linear ∞-category and μ is a k-linear equivalence

$$\mathrm{LMod}_k(\mathcal{C}_A) \simeq \mathrm{Mod}_k \otimes_{\mathrm{LMod}_A} \mathcal{C}_A \to \mathcal{C}\,.$$

The map θ is a left fibration, classified by a functor $\chi : \mathrm{Alg}_k^{(2),\mathrm{aug}} \to \widehat{\mathcal{S}}$; here $\widehat{\mathcal{S}}$ denotes the ∞-category of spaces which are not necessarily small. We let $\mathrm{CatDef}_{\mathcal{C}}$ denote the composition of the functor χ with the fully faithful embedding $\mathrm{Alg}_k^{(2),\mathrm{sm}} \to \mathrm{Alg}_k^{(2),\mathrm{aug}}$.

Let us now fix a k-linear ∞-category $\mathcal{C}$ and study the properties of the functor $\chi : \mathrm{Alg}_k^{(2),\mathrm{aug}} \to \widehat{\mathcal{S}}$ introduced in Construction 5.3.2. We begin with a simple observation. Let A be an $\mathbb{E}_2$-ring and let $\mathcal{C}_A$ be an A-linear ∞-category. For every map of $\mathbb{E}_2$-rings $A \to B$, let $\mathcal{C}_B = \mathrm{LMod}_B \otimes_{\mathrm{LMod}_A} \mathcal{C} \simeq \mathrm{LMod}_B(\mathcal{C})$. Proposition IX.7.4 implies that if we are given a pullback diagram σ : $\mathbb{E}_2$-rings

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A' \arrow[d] \\ B \arrow[r] & B', \end{tikzcd}$$

of $\mathbb{E}_2$-rings then the induced map

$$\mathcal{C}_A \to \mathcal{C}_{A'} \times_{\mathcal{C}_{B'}} \mathcal{C}_B$$

is fully faithful. Let $\mathcal{D}_A$ be another A-linear ∞-category. For every map of $\mathbb{E}_2$-rings $A \to B$, we let $\mathcal{D}_B \simeq \mathrm{LMod}_B(\mathcal{D})$ be defined as above, and $\mathrm{Fun}_B(\mathcal{D}_B, \mathcal{C}_B)$ denote the ∞-category of LMod_B-linear functors from $\mathcal{C}_B$ to $\mathcal{D}_B$ which preserve small colimits, so there is a canonical equivalence $\mathrm{Fun}_B(\mathcal{D}_B, \mathcal{C}_B) \simeq \mathrm{Fun}_A(\mathcal{D}_A, \mathcal{C}_B)$. It follows that if σ is a pullback diagram as above, then it induces a fully faithful functor

$$\mathrm{Fun}_A(\mathcal{D}_A, \mathcal{C}_A) \to \mathrm{Fun}_{A'}(\mathcal{D}_{A'}, \mathcal{C}_{A'}) \times_{\mathrm{Fun}_{B'}(\mathcal{D}_{B'}, \mathcal{C}_{B'})} \mathrm{Fun}_B(\mathcal{D}_B, \mathcal{C}_B).$$

This immediately implies the following result:

Proposition 5.3.3. *Let k be a field, let $\mathcal{C}$ be a k-linear ∞-category, and let $\chi : \mathrm{Alg}_k^{(2),\mathrm{aug}} \to \widehat{\mathcal{S}}$ be as in Construction 5.3.2. Then for every pullback diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A' \arrow[d] \\ B \arrow[r] & B' \end{tikzcd}$$

in $\mathrm{Alg}_k^{(2),\mathrm{aug}}$, the induced map $\theta : \chi(A) \to \chi(A') \times_{\chi(B')} \chi(B)$ is 0-truncated (in other words, the homotopy fibers of θ are discrete, up to homotopy).

Variant 5.3.4. Let k be a field, $\mathcal{C}$ a k-linear ∞-category, and κ a regular cardinal such that $\mathcal{C}$ is κ-compactly generated. For each $A \in \mathrm{Alg}_k^{(2),\mathrm{aug}}$, we let $\chi_\kappa(A)$ denote the summand of $\chi(A)$ spanned by those pairs $(\mathcal{C}_A, \mu)$ where $\mathcal{C}_A$ is κ-compactly generated. We can regard χ_κ as a functor $\mathrm{Alg}_k^{(2),\mathrm{aug}} \to \widehat{\mathcal{S}}$. It follows immediately from Proposition 5.3.3 that for each pullback diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A' \arrow[d] \\ B \arrow[r] & B' \end{tikzcd}$$

in $\mathrm{Alg}_k^{(2),\mathrm{aug}}$, the induced map

$$\chi_\kappa(A) \to \chi_\kappa(A') \times_{\chi_\kappa(B')} \chi_\kappa(B)$$

has discrete homotopy fibers. We claim that each of these homotopy fibers is essentially small. Unwinding the definitions, we must show that for every compatible triple of κ-compactly generated ∞-categories $\mathcal{C}_{A'} \in \mathrm{LinCat}_{A'}$, $\mathcal{C}_{B'} \in \mathrm{LinCat}_{B'}$, and $\mathcal{C}_B \in \mathrm{LinCat}_B$, there is only a bounded number of equivalence classes of full A-linear subcategories $\mathcal{C}_A \subseteq \mathcal{C}_{A'} \times_{\mathcal{C}_{B'}} \mathcal{C}_B$ which are κ-compactly generated and induce equivalences

$$\mathcal{C}_{A'} \simeq \mathrm{LMod}_{A'}(\mathcal{C}_A) \qquad \mathcal{C}_B \simeq \mathrm{LMod}_B(\mathcal{C}_A).$$

For in this case, $\mathcal{C}_A$ must be generated generated (under κ-filtered colimits) by some subcategory of the essentially small ∞-category $\mathcal{C}_{A'}^\kappa \times_{\mathcal{C}_{B'}^\kappa} \mathcal{C}_B^\kappa$, where $\mathcal{C}_{A'}^\kappa$ denotes the full subcategory of $\mathcal{C}_{A'}$ spanned by the κ-compact objects, and $\mathcal{C}_{B'}^\kappa$ and $\mathcal{C}_B^\kappa$ are defined similarly.

Corollary 5.3.5. *Let k be a field, let $\mathcal{C}$ be a k-linear ∞-category, and let $\chi : \mathrm{Alg}_k^{(2),\mathrm{aug}} \to \widehat{\mathcal{S}}$ be as in Construction 5.3.2. Then:*

(1) *The space $\chi(k)$ is contractible.*

(2) *Let $V \in \mathrm{Mod}_k$. Then $\chi(k \oplus V)$ is locally small, when regarded an ∞-category. In other words, each path component of $\chi(k \oplus V)$ is essentially small.*

(3) *Let $A \in \mathrm{Alg}_k^{(2),\mathrm{aug}}$ be small. Then the space $\chi(A)$ is locally small.*

Proof. Assertion (1) is immediate from the definitions. To prove (2), we note that for each $A \in \mathrm{Alg}_k^{(2),\mathrm{aug}}$ and every point $\eta \in \chi(A)$ corresponding to a pair $(\mathcal{C}_A, \mu)$, the space $\Omega^2(\chi(A), \eta)$ can be identified with the homotopy fiber of the restriction map

$$\mathrm{Map}_{\mathrm{Fun}_A(\mathcal{C}_A, \mathcal{C}_A)}(\mathrm{id}, \mathrm{id}) \to \mathrm{Map}_{\mathrm{Fun}_k(\mathcal{C}, \mathcal{C})}(\mathrm{id}, \mathrm{id})$$

and is therefore essentially small. We have pullback diagrams

$$\begin{tikzcd} k \oplus V \arrow[r] \arrow[d] & k \arrow[d] \\ k \arrow[r] & k \oplus V[1] \end{tikzcd} \qquad \begin{tikzcd} k \oplus V[1] \arrow[r] \arrow[d] & k \arrow[d] \\ k \arrow[r] & k \oplus V[2] \end{tikzcd}$$

so that Proposition 5.3.3 guarantees that the map $\chi(k \oplus V) \to \Omega^2 \chi(k \oplus V[2])$ has discrete homotopy fibers. It follows that each path component of $\chi(k \oplus V)$ is a connected covering space of the essentially small space $\Omega^2 \chi(k \oplus V[2])$, and is therefore essentially small.

We now prove (3). Assume that A is small, so that there exists a finite sequence of maps

$$A \simeq A_0 \to A_1 \to \cdots \to A_n \simeq k$$

and pullback diagrams

$$\begin{tikzcd} A_i \arrow[r] \arrow[d] & k \arrow[d] \\ A_{i+1} \arrow[r] & k \oplus k[m_i]. \end{tikzcd}$$

We prove that $\chi(A_i)$ is locally small using descending induction on i. Using (1), (2), and the inductive hypothesis, we deduce that $X = \chi(k) \times_{\chi(k \oplus k[m_i])} \chi(A_{i+1})$ is locally small. Proposition 5.3.3 implies that the map $\chi(A_{i+1}) \to X$ has discrete homotopy fibers. It follows that every path component of $\chi(A_{i+1})$ is a connected covering space of a path component of X, and therefore essentially small. $\square$

Variant 5.3.6. Let k be a field, $\mathcal{C}$ a k-linear ∞-category, and κ a regular cardinal such that $\mathcal{C}$ is κ-compactly generated. Let $\chi_\kappa : \mathrm{Alg}_k^{(2),\mathrm{aug}} \to \widehat{\mathcal{S}}$ be as in Variation 5.3.4. The proof of Corollary 5.3.5 yields the following results:

(1′) The space $\chi_\kappa(k)$ is contractible.

(2′) Let $V \in \mathrm{Mod}_k$. Then $\chi_\kappa(k \oplus V)$ is essentially small.

(3′) Let $A \in \mathrm{Alg}_k^{(2),\mathrm{aug}}$ be small. Then the space $\chi(A)$ is essentially small.

Notation 5.3.7. Let k be a field and $\mathcal{C}$ a k-linear ∞-category. We let $\chi : \mathrm{Alg}_k^{(2),\mathrm{aug}} \to \widehat{\mathcal{S}}$ be as in Construction 5.3.2, and let $\mathrm{CatDef}_{\mathcal{C}} : \mathrm{Alg}_k^{(2),\mathrm{sm}} \to \widehat{\mathcal{S}}$ denote the composition of χ with the fully faithful embedding $\nu : \mathrm{Alg}_k^{(2),\mathrm{sm}} \hookrightarrow \mathrm{Alg}_k^{(2),\mathrm{aug}}$. If κ is a regular cardinal such that $\mathcal{C}$ is κ-compactly generated, we let $\mathrm{CatDef}_{\mathcal{C},\kappa}$ denote the composition $\chi_\kappa \circ \nu$, where χ_κ is as in Variation 5.3.4. It follows from Variation 5.3.6 that we can identify $\mathrm{CatDef}_{\mathcal{C},\kappa}$ with a functor $\mathrm{Alg}_k^{(2),\mathrm{sm}} \to \mathcal{S}$ (and the functor $\mathrm{CatDef}_{\mathcal{C}}$ is given by the filtered colimit of the transfinite sequence of functors $\{\mathrm{CatDef}_{\mathcal{C},\kappa}\}$, where κ ranges over all small regular cardinals).

Corollary 5.3.8. *Let k be a field and $\mathcal{C}$ a k-linear ∞-category. Then there exists a formal moduli problem $\mathrm{CatDef}^{\wedge}_{\mathcal{C}} : \mathrm{Alg}^{(2),\mathrm{sm}}_k \to \mathcal{S}$ and a natural transformation $\alpha : \mathrm{CatDef}_{\mathcal{C}} \to \mathrm{CatDef}^{\wedge}_{\mathcal{C}}$ which is 0-truncated. In particular, we can regard $\mathrm{CatDef}_{\mathcal{C}} : \mathrm{Alg}^{(2),\mathrm{sm}}_k \to \widehat{\mathcal{S}}$ as a 2-proximate formal moduli problem after a change of universe (see Theorem 5.1.9).*

Proof. Combining Corollary 5.3.5, Proposition 5.3.3, and Theorem 5.1.9, we deduce the existence of a formal moduli problem $\mathrm{CatDef}^{\wedge}_{\mathcal{C}} : \mathrm{Alg}^{(2),\mathrm{sm}}_k \to \widehat{\mathcal{S}}$ and a 0-truncated natural transformation $\alpha : \mathrm{CatDef}_{\mathcal{C}} \to \mathrm{CatDef}^{\wedge}_{\mathcal{C}}$. For each $m \geq 0$, we see that the space

$$\mathrm{CatDef}^{\wedge}_{\mathcal{C}}(k \oplus k[m]) \simeq \Omega^2 \mathrm{CatDef}^{\wedge}_{\mathcal{C}}(k \oplus k[m+2]) \simeq \Omega^2 \mathrm{CatDef}_{\mathcal{C}}(k \oplus k[m+2])$$

is essentially small (see the proof of Corollary 5.3.5). For an arbitrary object $A \in \mathrm{Alg}^{(2),\mathrm{sm}}_k$, we can choose a finite sequence of maps $A = A_0 \to A_1 \to \cdots \to A_n \simeq k$ and pullback diagrams

$$\begin{tikzcd} A_i \arrow[r] \arrow[d] & k \arrow[d] \\ A_{i+1} \arrow[r] & k \oplus k[m_i]. \end{tikzcd}$$

Using the fact that $\mathrm{CatDef}^{\wedge}_{\mathcal{C}}$ is a formal moduli problem, we deduce that each $\mathrm{CatDef}^{\wedge}_{\mathcal{C}}(A_i)$ is essentially small by descending induction on i, so that $\mathrm{CatDef}^{\wedge}_{\mathcal{C}}(A)$ is essentially small. □

Remark 5.3.9. In the situation of Corollary 5.3.8, let κ be a regular cardinal such that $\mathcal{C}$ is κ-compactly generated. Then the composite map

$$\mathrm{CatDef}_{\mathcal{C},\kappa} \to \mathrm{CatDef}_{\mathcal{C}} \to \mathrm{CatDef}^{\wedge}_{\mathcal{C}}$$

is 0-truncated, so that $\mathrm{CatDef}_{\mathcal{C},\kappa}$ is a 2-proximate formal moduli problem by Theorem 5.1.9.

Our next goal is to describe the formal $\mathbb{E}_2$-moduli problem $\mathrm{CatDef}^{\wedge}_{\mathcal{C}}$ more explicitly. Using Theorem 4.0.8 (and its proof), we see that the functor $\mathrm{CatDef}^{\wedge}_{\mathcal{C}}$ is given by

$$\mathrm{CatDef}^{\wedge}_{\mathcal{C}}(R) = \mathrm{Map}_{\mathrm{Alg}^{(2),\mathrm{aug}}_k}(\mathfrak{D}^{(2)}(R), k \oplus \mathfrak{m}),$$

for some nonunital $\mathbb{E}_2$-algebra $\mathfrak{m}$ over k. We would like to make the dependence of $\mathfrak{m}$ on $\mathcal{C}$ more explicit.

Definition 5.3.10. Let k be an $\mathbb{E}_\infty$-ring and let $\mathcal{C}$ be a k-linear ∞-category. We let $\mathrm{RCat}(k)_{\mathcal{C}}$ denote the fiber product $\mathrm{RCat}(k) \times_{\mathrm{LinCat}_k} \{\mathcal{C}\}$. We will say that an object $(B, \mathcal{C}) \in \mathrm{RCat}(k)_{\mathcal{C}}$ of $\mathrm{RCat}(k)_{\mathcal{C}}$ *exhibits B as the k-linear center of* $\mathcal{C}$ if $(B, \mathcal{C})$ is a final object of $\mathrm{RCat}(k)_{\mathcal{C}}$.

Remark 5.3.11. In the situation of Definition 5.3.10 Corollary A.6.1.2.42 implies that the forgetful functor $\mathrm{RMod}(\mathrm{LinCat}_k) \times_{\mathrm{LinCat}_k} \{\mathcal{C}\} \to \mathrm{Alg}(\mathrm{LinCat}_k)$ is a right fibration. It follows that the map $q : \mathrm{RCat}(k)_{\mathcal{C}} \to \mathrm{Alg}^{(2)}_k$ is a right fibration, so that an object $(B, \mathcal{C}) \in \mathrm{RCat}(k)_{\mathcal{C}}$ is final if and only if the right fibration q is represented by the object $B \in \mathrm{Alg}^{(2)}_k$. In other words, the k-linear center B of $\mathcal{C}$ can be characterized by the following universal property: for every $A \in \mathrm{Alg}^{(2)}_k$, the space $\mathrm{Map}_{\mathrm{Alg}^{(2)}_k}(A, B)$ can be identified with the space $\mathrm{RMod}_{\mathrm{LMod}_A}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \times_{\mathrm{LinCat}_k} \{\mathcal{C}\}$ of k-linear right actions of LMod_A on $\mathcal{C}$.

Proposition 5.3.12. *Let k be an $\mathbb{E}_\infty$-ring and let $\mathcal{C}$ be a k-linear ∞-category. Then there exists an object $(B, \mathcal{C}) \in \mathrm{RCat}(k)_{\mathcal{C}}$ which exhibits B as a k-linear center of $\mathcal{C}$.*

Proof. Let $\mathcal{E}$ be an endomorphism object of $\mathcal{C}$ in LinCat_k: that is, $\mathcal{E}$ is the ∞-category of k-linear functors from $\mathcal{C}$ to itself. We regard $\mathcal{E}$ as a monoidal ∞-category via order-reversed composition, so that $\mathcal{C}$ is a right $\mathcal{E}$-module object of LinCat_k. According to Theorem A.6.3.5.10, the symmetric monoidal functor $\mathrm{Alg}_k \to (\mathrm{LinCat}_k)_{\mathrm{Mod}_k /}$ admits a right adjoint G. It follows that G induces a right adjoint G' to the functor

$$\mathrm{Alg}_k^{(2)} \simeq \mathrm{Alg}(\mathrm{Alg}_k) \to \mathrm{Alg}((\mathrm{LinCat}_k)_{\mathrm{Mod}_k /}) \simeq \mathrm{Alg}(\mathrm{LinCat}_k),$$

and we can define $B = G'(\mathcal{E})$. □

Remark 5.3.13. Let k be an $\mathbb{E}_\infty$-ring and $\mathcal{C}$ a k-linear ∞-category The proofs of Proposition 5.3.12 and Theorem A.6.3.5.10 furnish a somewhat explicit description of the k-linear center B of $\mathcal{C}$, at least as an $\mathbb{E}_1$-algebra over k: it can be described as the endomorphism ring of the identity functor $\mathrm{id}_{\mathcal{E}} \in \mathcal{E}$, where $\mathcal{E}$ is the ∞-category of k-linear functors from $\mathcal{C}$ to itself.

Example 5.3.14. Let k be an $\mathbb{E}_\infty$-ring, let $R \in \mathrm{Alg}_k$ be an $\mathbb{E}_1$-algebra over k, and let $\mathfrak{Z}(R) = \mathfrak{Z}_{\mathbb{E}_1}(R) \in \mathrm{Alg}_k^{(2)}$ be a center of R (see Definition A.6.1.4.10). Then $\mathfrak{Z}(R)$ is a k-linear center of the ∞-category $\mathrm{RMod}_R(\mathrm{Mod}_k)$.

Remark 5.3.15. Let k be an $\mathbb{E}_\infty$-ring, $\mathcal{C}$ a k-linear ∞-category, and $A \in \mathrm{Alg}_k^{(2)}$ a k-linear center of $\mathcal{C}$. The homotopy groups $\pi_n A$ are often called the *Hochschild cohomology groups* of $\mathcal{C}$. In the special case where $\mathcal{C} = \mathrm{LMod}_R(\mathrm{Mod}_k)$ for some $R \in \mathrm{Alg}_k$, Example 5.3.14 allows us to identify $\pi_n A$ with the group $\mathrm{Ext}^{-n}_{{}_R\mathrm{BMod}_R(\mathrm{Mod}_k)}(R, R)$.

We are now ready to formulate the main result of this section.

Theorem 5.3.16. *Let k be a field and let $\mathcal{C}$ be a k-linear ∞-category. Then the functor $\mathrm{ObjDef}^\wedge_{\mathcal{C}} : \mathrm{Alg}_k^{(2),\mathrm{sm}} \to \mathcal{S}$ is given by*

$$\mathrm{ObjDef}^\wedge_{\mathcal{C}}(R) = \mathrm{Map}_{\mathrm{Alg}_k^{(2),\mathrm{aug}}}(\mathfrak{D}^{(2)}(R), k \oplus \mathfrak{Z}(\mathcal{C})) \simeq \mathrm{Map}_{\mathrm{Alg}_k^{(2)}}(\mathfrak{D}^{(2)}(R), \mathfrak{Z}(\mathcal{C})).$$

where $\mathfrak{Z}(\mathcal{C})$ denotes a k-linear center of $\mathcal{C}$.

Using Remark 5.1.11, we see that Theorem 5.3.16 is equivalent to the following:

Proposition 5.3.17. *Let k be a field, let $\mathcal{C}$ be a k-linear ∞-category, and let $\mathfrak{Z}(\mathcal{C}) \in \mathrm{Alg}_k^{(2)}$ denote a k-linear center of $\mathcal{C}$. Let $X : \mathrm{Alg}_k^{(2),\mathrm{sm}} \to \mathcal{S}$ be the functor given by the formula $X(R) = \mathrm{Map}_{\mathrm{Alg}_k^{(2)}}(\mathfrak{D}^{(2)}(R), B)$. Then there exists a 0-truncated natural transformation $\beta : \mathrm{CatDef}_{\mathcal{C}} \to X$.*

The first step in our proof of Proposition 5.3.17 is to construct the natural transformation $\mathrm{CatDef}_{\mathcal{C}} \to X$.

Construction 5.3.18. Let k be a field and let $\mathcal{C}$ be a k-linear ∞-category. We let $\lambda^{(2)} : \mathcal{M}^{(2)} \to \mathrm{Alg}_k^{(2),\mathrm{aug}} \times \mathrm{Alg}_k^{(2),\mathrm{aug}}$ be the pairing of Construction 4.4.6, so that we can identify objects of $\mathcal{M}^{(2)}$ with triples (A, B, ϵ) where $A, B \in \mathrm{Alg}_k^{(2)}$ and $\epsilon : A \otimes_k B \to k$ is an augmentation. Given an object $(A, B, \epsilon) \in \mathcal{M}^{(2)}$ and an object $(A, \mathcal{C}_A, \mu) \in \mathrm{Defor}[\mathcal{C}]$, we regard $\mathcal{C}_A \otimes \mathrm{LMod}_B$ as an object of

$${}_{\mathrm{LMod}_A \otimes \mathrm{LMod}_B}\mathrm{BMod}_{\mathrm{LMod}_B}(\mathrm{LinCat}_k),$$

so that

$$\mathrm{Mod}_k \otimes_{\mathrm{LMod}_A \otimes \mathrm{LMod}_B} (\mathcal{C}_A \otimes \mathrm{LMod}_B)$$

can be identified with an object of $\mathrm{RMod}_{\mathrm{LMod}_B}(\mathrm{Mod}_k)$ whose image in LinCat_k is given by

$$\mathrm{Mod}_k \otimes_{\mathrm{LMod}_A} \mathcal{C}_A \simeq \mathcal{C}\,.$$

This construction determines a functor

$$\mathrm{Defor}[\mathcal{C}] \times_{\mathrm{Alg}_k^{(2),\mathrm{aug}}} \mathcal{M}^{(2)} \to \mathrm{RMod}(\mathrm{LinCat}_k) \times_{\mathrm{LinCat}_k} \{\mathcal{C}\}.$$

The induced map

$$\mathrm{Defor}[\mathcal{C}] \times_{\mathrm{Alg}_k^{(2),\mathrm{aug}}} \mathcal{M}^{(2)} \to \mathrm{Defor}[\mathcal{C}] \times (\mathrm{Alg}_k^{(2),\mathrm{aug}} \times_{\mathrm{Alg}(\mathrm{LinCat}_k)} \mathrm{RMod}(\mathrm{LinCat}_k) \times_{\mathrm{LinCat}_k} \{\mathcal{C}\})$$

factors as a composition

$$\mathrm{Defor}[\mathcal{C}] \times_{\mathrm{Alg}_k^{\mathrm{aug}}} \mathcal{M}^{(2)} \xrightarrow{i} \widetilde{\mathcal{M}}^{(2)} \xrightarrow{\lambda'} \mathrm{Defor}[\mathcal{C}] \times (\mathrm{Alg}_k^{(2),\mathrm{aug}} \times_{\mathrm{Alg}(\mathrm{LinCat}_k)} \mathrm{RMod}(\mathrm{LinCat}_k) \times_{\mathrm{LinCat}_k} \{\mathcal{C}\})$$

where i is an equivalence of ∞-categories and λ' is a categorical fibration. It is not difficult to see that λ' is a left representable pairing of ∞-categories, which induces a duality functor

$$\mathfrak{D}_{\mathcal{C}}^{(2)} : \mathrm{Defor}[\mathcal{C}]^{op} \to \mathrm{Alg}_k^{(2),\mathrm{aug}} \times_{\mathrm{Alg}(\mathrm{LinCat}_k)} \mathrm{RMod}(\mathrm{LinCat}_k) \times_{\mathrm{LinCat}_k} \{\mathcal{C}\}.$$

Concretely, the functor $\mathfrak{D}_{\mathcal{C}}^{(2)}$ assigns to each object $(A, \mathcal{C}_A, \alpha) \in \mathrm{Defor}[\mathcal{C}]^{op}$ the object $(\mathfrak{D}^{(2)}(A), \mathcal{C})$, where we regard $\mathcal{C}$ as right-tensored over $\mathrm{LMod}_{\mathfrak{D}^{(2)}(A)}$ via the k-linear equivalence

$$\mathcal{C} \simeq \mathrm{Mod}_k \otimes_{\mathrm{LMod}_A \otimes \mathrm{LMod}_B} (\mathcal{C}_A \otimes \mathrm{LMod}_B).$$

Let k be a field and let $\mathcal{C}$ be a k-linear ∞-category. We let $\mathfrak{Z}(\mathcal{C})$ denote a k-linear center of $\mathcal{C}$ (Definition 5.3.10), so that we have a canonical equivalence of ∞-categories

$$\eta : \mathrm{Alg}_k^{(2)} \times_{\mathrm{Alg}(\mathrm{LinCat}_k)} \mathrm{RMod}(\mathrm{LinCat}_k) \times_{\mathrm{LinCat}_k} \mathcal{C} \simeq (\mathrm{Alg}_k^{(2)})_{/\mathfrak{Z}(\mathcal{C})}.$$

Composing η with the functor $\mathfrak{D}_{\mathcal{C}}^{(2)}$, we obtain a diagram of ∞-categories

$$\begin{tikzcd}
\mathrm{Defor}[\mathcal{C}]^{op} \arrow[r] \arrow[d] & (\mathrm{Alg}_k^{(2),\mathrm{aug}})_{/(k \oplus \mathfrak{Z}(\mathcal{C}))} \arrow[d] \\
(\mathrm{Alg}_k^{(2),\mathrm{aug}})^{op} \arrow[r, "\mathfrak{D}^{(2)}"] & \mathrm{Alg}_k^{(2),\mathrm{aug}}
\end{tikzcd}$$

which commutes up to canonical homotopy, where the vertical maps are right fibrations. This diagram determines a natural transformation $\beta : \mathrm{ObjDef}_{\mathcal{C}} \to X$, where $X : \mathrm{Alg}_k^{(2),\mathrm{sm}} \to \mathcal{S}$ is the functor given by the formula $X(R) = \mathrm{Map}_{\mathrm{Alg}_k^{(2)}}(\mathfrak{D}^{(2)}(A), \mathfrak{Z}(\mathcal{C}))$.

We will prove Proposition 5.3.17 (and therefore also Theorem 5.3.16) by showing that the natural transformation β of Construction 5.3.18 is 0-truncated. Since the functor X is a formal $\mathbb{E}_2$-moduli problem, β induces a natural transformation $\overline{\beta} : \mathrm{CatDef}_{\mathcal{C}}^{\wedge} \to X$. We wish to prove that $\overline{\beta}$ is an equivalence (which implies Proposition 5.3.17, by virtue of Theorem 5.1.9). According to Proposition 1.2.10, it suffices to show that $\overline{\beta}$ induces an equivalence of tangent complexes. Using the description of the tangent complex of $\mathrm{CatDef}_{\mathcal{C}}^{\wedge}$ supplied by Lemma 5.1.12, we are reduced to proving the following special case of Proposition 5.3.17:

Proposition 5.3.19. *Let k be a field and $\mathcal{C}$ a k-linear ∞-category. For each $m \geq 0$, the natural transformation $\beta : \mathrm{ObjDef}_{\mathcal{C}} \to X$ of Construction 5.3.18 induces a 0-truncated map*

$$\mathrm{ObjDef}_{\mathcal{C}}(k \oplus k[m]) \to \mathrm{Map}_{\mathrm{Alg}_k^{(2)}}(\mathfrak{D}^{(2)}(k \oplus k[m]), \mathfrak{Z}(\mathcal{C})).$$

Proof. We have a commutative diagram

$$\begin{tikzcd}
\mathrm{ObjDef}_{\mathcal{C}}(k \oplus k[m]) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{Alg}_k^{(2)}}(\mathfrak{D}^{(2)}(k \oplus k[m]), \mathfrak{Z}(\mathcal{C})) \arrow[d] \\
\Omega^2 \mathrm{ObjDef}_{\mathcal{C}}(k \oplus k[m+2]) \arrow[r, "\theta"] & \Omega^2 \mathrm{Map}_{\mathrm{Alg}_k^{(2)}}(\mathfrak{D}^{(2)}(k \oplus k[m+2]), \mathfrak{Z}(\mathcal{C})),
\end{tikzcd}$$

where the left vertical map is 0-truncated by Corollary 5.3.8 and the right vertical map is a homotopy equivalence. It will therefore suffice to show that θ is a homotopy equivalence. Let $A = k \oplus k[m+2]$, let $\mathcal{C}_A = \mathrm{LMod}_A \otimes \mathcal{C} \simeq \mathrm{RMod}_A(\mathcal{C})$, let $\mathcal{E}$ be the ∞-category of k-linear functors from $\mathcal{C}$ to itself, and let $\mathcal{E}_A$ be the ∞-category of LMod_A-linear functors from $\mathcal{C}_A$ to itself, so that there is a canonical equivalence $\gamma : \mathcal{E}_A \simeq \mathrm{LMod}_A(\mathcal{E})$. Let $\mathrm{id} \in \mathcal{E}$ denote the identity functor from $\mathcal{C}$ to itself. Under the equivalence γ, the identity functor from $\mathcal{C}_A$ to itself can be identified with the free module $A \otimes \mathrm{id} \in \mathrm{LMod}_A(\mathcal{E})$. Unwinding the definitions, we see that the domain of θ can be identified with the homotopy fiber of the map

$$\xi : \mathrm{Map}_{\mathcal{E}_A}(A \otimes \mathrm{id}, A \otimes \mathrm{id}) \simeq \mathrm{Map}_{\mathcal{E}}(\mathrm{id}, A \otimes \mathrm{id}) \to \mathrm{Map}_{\mathcal{E})}(\mathrm{id}, \mathrm{id}).$$

We have a canonical fiber sequence

$$\mathrm{id}[m+2] \to A \otimes \mathrm{id} \to \mathrm{id}$$

in $\mathcal{E}$, so that the homotopy fiber of ξ is given by

$$\mathrm{Map}_{\mathcal{E}}(\mathrm{id}, \mathrm{id}[m+2]) \simeq \mathrm{Map}_{\mathrm{Mod}_k}(k[-m-2], \mathfrak{Z}(\mathcal{C})).$$

The map θ is induced by a morphism $\nu : k[-m-2] \to \mathfrak{D}^{(2)}(k \oplus k[m])$ in Mod_k. Let $\mathrm{Free}^{(2)} : \mathrm{Mod}_k \to \mathrm{Alg}_k^{(2)}$ be a left adjoint to the forgetful functor, so that ν determines an augmentation $(k \oplus k[m]) \otimes_k \mathrm{Free}^{(2)}(k[-m-2]) \to k$. The proof of Proposition 4.5.6 shows that this pairing exhibits $\mathrm{Free}^{(2)}(k[-m-2])$ as the Koszul dual of $k \oplus k[m]$, from which it immediately follows that θ is a homotopy equivalence. □

Our goal, for the remainder of this section, is to describe the formal moduli problem $\mathrm{CatDef}_{\mathcal{C}}^{\wedge}$ more explicitly in terms of the ∞-category $\mathcal{C}$. Assume that $\mathcal{C}$ is compactly generated. Let ω denote the first infinite cardinal and let $\mathrm{CatDef}_{\mathcal{C},\omega}$ be the deformation functor of Notation 5.3.7 (so that $\mathrm{CatDef}_{\mathcal{C},\omega}$ classifies *compactly generated* deformations of $\mathcal{C}$). Our main result (Theorem 5.3.33) asserts that, under some rather restrictive assumptions, the composite map

$$\mathrm{CatDef}_{\mathcal{C},\omega} \to \mathrm{CatDef}_{\mathcal{C}} \to \mathrm{CatDef}_{\mathcal{C}}^{\wedge}$$

is an equivalence of functors. Since the natural transformation $\mathrm{CatDef}_{\mathcal{C},\omega} \to \mathrm{CatDef}_{\mathcal{C}}^{\wedge}$ is 0-truncated, this is equivalent to the assertion that $\mathrm{CatDef}_{\mathcal{C},\omega}$ is itself a formal moduli problem (see Remark 5.1.11). The functor $\mathrm{CatDef}_{\mathcal{C},\omega}$ is automatically a 2-proximate formal moduli problem (Remark 5.3.9). Our first step is to obtain a criterion which guarantees that $\mathrm{CatDef}_{\mathcal{C},\omega}$ is a 1-proximate formal moduli problem. First, we need to introduce a bit of terminology.

Definition 5.3.20. Let $\mathcal{C}$ be a presentable stable ∞-category. We let $\mathcal{C}^c$ denote the full subcategory of $\mathcal{C}$ spanned by the compact objects of $\mathcal{C}$. We will say that $\mathcal{C}$ is *tamely compactly generated* if it satisfies the following conditions:

- (a) The ∞-category $\mathcal{C}$ is compactly generated (that is, $\mathcal{C} \simeq \mathrm{Ind}(\mathcal{C}^c)$).
- (b) For every pair of compact objects $C, D \in \mathcal{C}$, the groups $\mathrm{Ext}^n_{\mathcal{C}}(C, D)$ vanish for $n \gg 0$.

Proposition 5.3.21. *Let k be a field, let $\mathcal{C}$ be a k-linear ∞-category which is tamely compactly generated, and let $\mathrm{CatDef}_{\mathcal{C},\omega} : \mathrm{Alg}_k^{(2),\mathrm{sm}} \to \mathcal{S}$ be as in Notation 5.3.7. Then $\mathrm{CatDef}_{\mathcal{C},\omega}$ is a 1-proximate formal moduli problem.*

The proof of Proposition 5.3.21 will require some preliminaries.

Notation 5.3.22. Let R be an $\mathbb{E}_2$-ring and let $\mathcal{C}$ be an R-linear ∞-category. For every pair of objects $C, D \in \mathcal{C}$, we let $\mathrm{Mor}_{\mathcal{C}}(C, D) \in \mathrm{LMod}_R$ be a classifying object for morphisms from C to D. This object is characterized (up to canonical equivalence) by the requirement that there exists a map $e : \mathrm{Mor}_{\mathcal{C}}(C, D) \otimes C \to D$ such that, for every $M \in \mathrm{LMod}_R$, the composite map

$$\mathrm{Map}_{\mathrm{LMod}_R}(M, \mathrm{Mor}_{\mathcal{C}}(C, D)) \to \mathrm{Map}_{\mathcal{C}}(M \otimes C, \mathrm{Mor}_{\mathcal{C}}(C, D) \otimes C) \xrightarrow{e\circ} \mathrm{Map}_{\mathcal{C}}(M \otimes C, D)$$

is a homotopy equivalence.

Lemma 5.3.23. *Let R be an $\mathbb{E}_2$-ring and let $\mathcal{C}$ be an R-linear ∞-category. If $C \in \mathcal{C}$ is compact, then the construction $D \mapsto \mathrm{Mor}_{\mathcal{C}}(C, D)$ determines a colimit-preserving functor $\mathcal{C} \to \mathrm{LMod}_R$.*

Proof. It is clear that the construction $D \mapsto \mathrm{Mor}_{\mathcal{C}}(C, D)$ commutes with limits and is therefore an exact functor. To prove that it preserves colimits, it suffices to show that it preserves filtered colimits. For this, it suffices to show that the construction $D \mapsto \Omega^\infty \mathrm{Mor}_{\mathcal{C}}(C, D)$ preserves filtered colimits (as a functor from $\mathcal{C}$ to $\mathcal{S}$), which is equivalent to the requirement that C is compact. □

Let R and $\mathcal{C}$ be as in Notation 5.3.22. Given an object $N \in \mathrm{LMod}_R$, the induced map

$$N \otimes \mathrm{Mor}_{\mathcal{C}}(C, D) \otimes C \xrightarrow{\mathrm{id}_N \otimes e} N \otimes D$$

is classified by a map $\lambda : N \otimes \mathrm{Mor}_{\mathcal{C}}(C, D) \to \mathrm{Mor}_{\mathcal{C}}(C, N \otimes D)$.

Lemma 5.3.24. *Let R be an $\mathbb{E}_2$-ring and let $\mathcal{C}$ be an R-linear ∞-category. Let $C, D \in \mathcal{C}$ and let $N \in \mathrm{LMod}_R$. If C is a compact object of $\mathcal{C}$, then the map $\lambda : N \otimes \mathrm{Mor}_{\mathcal{C}}(C, D) \to \mathrm{Mor}_{\mathcal{C}}(C, N \otimes D)$ is an equivalence.*

Proof. Using Lemma 5.3.23, we deduce that the functor $N \mapsto \mathrm{Mor}_{\mathcal{C}}(C, N \otimes D)$ preserves small colimits. It follows that the collection of objects $N \in \mathrm{LMod}_R$ such that λ is an equivalence is closed under colimits in LMod_R. We may therefore suppose that $N \simeq R[n]$ for some integer n, in which case the result is obvious. □

Lemma 5.3.25. *Suppose we are given a map of $\mathbb{E}_2$-rings $R \to R'$, let $\mathcal{C}$ be an R-linear ∞-category, and let $\mathcal{C}' = \mathrm{LMod}_{R'} \otimes_{\mathrm{LMod}_R} \mathcal{C} \simeq \mathrm{LMod}_{R'}(\mathcal{C})$. Let $F : \mathcal{C} \to \mathcal{C}'$ be a left adjoint to the forgetful functor $G : \mathcal{C}' \to \mathcal{C}$, so that F is given by given by $C \mapsto R' \otimes C$. For every pair of objects $C, D \in \mathcal{C}$, the map $\mathrm{Mor}_{\mathcal{C}}(C, D) \otimes C \to D$ induces a map*

$$(R' \otimes_R \mathrm{Mor}_{\mathcal{C}}(C, D)) \otimes F(C) \simeq F(\mathrm{Mor}_{\mathcal{C}}(C, D) \otimes C) \to F(D),$$

which is classified by a map $\alpha : R' \otimes_R \mathrm{Mor}_{\mathcal{C}}(C, D) \to \mathrm{Mor}_{\mathcal{C}'}(F(C), F(D))$. If $C \in \mathcal{C}$ is compact, then α is an equivalence.

Proof. The image of α under the forgetful functor $\mathrm{LMod}_{R'} \to \mathrm{LMod}_R$ coincides with the equivalence $R' \otimes_R \mathrm{Mor}_{\mathcal{C}}(C, D) \to \mathrm{Mor}_{\mathcal{C}}(C, R' \otimes D)$ of Lemma 5.3.24. □

Lemma 5.3.26. *Suppose given a pullback diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ A' \arrow[r] & B' \end{tikzcd}$$

of $\mathbb{E}_2$-rings. Let $\mathcal{C}_A$ be an A-linear ∞-category, let $\mathcal{C}_B = \mathrm{LMod}_B \otimes_{\mathrm{LMod}_A} \mathcal{C}_A \simeq \mathrm{LMod}_B(\mathcal{C}_A)$, and define $\mathcal{C}_{A'}$ and $\mathcal{C}_{B'}$ similarly. An object $C \in \mathcal{C}_A$ is compact if and only if its images in $\mathcal{C}_B$ and $\mathcal{C}_{A'}$ are compact.

Proof. The "only if" direction is obvious, since the forgetful functors $\mathcal{C}_B \to \mathcal{C}_A \leftarrow \mathcal{C}_{A'}$ preserve filtered colimits. For the converse, suppose that $C \in \mathcal{C}_A$ has compact images $C_B \in \mathcal{C}_B$ and $C_{A'} \in \mathcal{C}_{A'}$. Then the image of C in $\mathcal{C}_B \times_{\mathcal{C}_{B'}} \mathcal{C}_{A'}$ is compact. Since the natural map $\mathcal{C}_A \to \mathcal{C}_B \times_{\mathcal{C}_{B'}} \mathcal{C}_{A'}$ is fully faithful (Proposition IX.7.4) and preserves filtered colimits, we conclude that C is compact. □

Lemma 5.3.27. *Let $f : A \to B$ be a map of connective $\mathbb{E}_2$-rings, and let $\mathcal{C}_A$ be an A-linear ∞-category which is tamely compactly generated. Then $\mathcal{C}_B = \mathrm{LMod}_B(\mathcal{C})$ is tamely compactly generated.*

Proof. We note that $\mathcal{C}_B$ is compactly generated: in fact, $\mathcal{C}_B$ is generated under small colimits by the essential image of the composite functor map $\mathcal{C}_A^c \hookrightarrow \mathcal{C}_A \xrightarrow{F} \mathcal{C}_B$, which consists of compact objects (since F is left adjoint to a forgetful functor). It follows that the ∞-category $\mathcal{C}_B^c$ is the smallest stable full subcategory of $\mathcal{C}_B$ which contains $F(\mathcal{C}_A^c)$ and is closed under retracts. Let $\mathcal{X} \subseteq \mathcal{C}_B$ be the full subcategory spanned by those objects C such that for every $D \in \mathcal{C}_B^c$, we have $\mathrm{Ext}^n_{\mathcal{C}_B}(C, D) \simeq 0$ for $n \gg 0$. It is easy to see that $\mathcal{X}$ is stable and closed under retracts. Consequently, to show that $\mathcal{C}_B^c \subseteq \mathcal{X}$, it will suffice to show that $F(C_0) \in \mathcal{X}$ for each $C_0 \in \mathcal{C}_A^c$. Let us regard C_0 as fixed, and let $\mathcal{Y}$ be the full subcategory of $\mathcal{C}_B$ spanned by those objects D for which the groups $\operatorname{Ext}^n_{\mathcal{C}_B}(F(C_0), D)$ vanish for $n \gg 0$. Since $\mathcal{Y}$ is stable and closed under retracts, it will suffice to show that $F(D_0) \in \mathcal{Y}$ for each $D_0 \in \mathcal{C}_A^c$. In other words, we are reduced to proving that the homotopy groups $\pi_{-n} \operatorname{Mor}_{\mathcal{C}_B}(F(C_0), F(D_0))$ vanish for $n \gg 0$. Using Lemma 5.3.25, we must show that $\pi_{-n}(B \otimes_A \operatorname{Mor}_{\mathcal{C}_A}(C_0, D_0))$ vanishes for $n \gg 0$. Since A and B are connective, this follows from the fact that $\pi_{-n} \operatorname{Mor}_{\mathcal{C}_A}(C_0, D_0) \simeq 0$ for $n \gg 0$ (since $\mathcal{C}_A$ is tamely compactly generated). □

Lemma 5.3.28. *Let A be a connective $\mathbb{E}_2$-ring and let $\mathcal{C}_A$ be an A-linear ∞-category which is tamely compactly generated. For every map of $\mathbb{E}_2$-rings $A \to R$, we let $\mathcal{C}_R$ denote the ∞-category $\operatorname{LMod}_R \otimes_{\operatorname{LMod}_A} \mathcal{C}_A \simeq \operatorname{LMod}_R(\mathcal{C}_A)$. Suppose we are given a pullback diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ A' \arrow[r] & B' \end{tikzcd}$$

of connective $\mathbb{E}_2$-rings which induces surjective maps $\pi_0 B \to \pi_0 B'$ and $\pi_0 A' \to \pi_0 B'$. Then the induced map $\theta^c : \mathcal{C}_A^c \to \mathcal{C}_B^c \times_{\mathcal{C}_{B'}^c} \mathcal{C}_{A'}^c$ is an equivalence of ∞-categories.

Proof. The functor θ^c is given by the restriction of a functor $\theta : \mathcal{C}_A \to \mathcal{C}_B \times_{\mathcal{C}_{B'}} \mathcal{C}_{A'}$, which is fully faithful by Proposition IX.7.4; this proves that θ^c is fully faithful. We will show that θ is essentially surjective. We can identify objects of $\mathcal{C}_B \times_{\mathcal{C}_{B'}} \mathcal{C}_{A'}$ with triples $(C_B, C_{A'}, \eta)$ where $C_B \in \mathcal{C}_B^c$, $C_{A'} \in \mathcal{C}_{A'}^c$, and η is an equivalence $B' \otimes_B C_B \simeq B' \otimes_{A'} C_{A'}$. In this case, we will denote $B' \otimes_B C_B \simeq B' \otimes_{A'} C_{A'}$ by $C_{B'}$. Note that θ admits a right adjoint G, given by $(C_B, C_{A'}, \eta) \mapsto C_B \times_{C_{B'}} C_{A'}$. In view of Lemma 5.3.26, it will suffice to show that the counit transformation $v : \theta \circ G \to \mathrm{id}$ is an equivalence when restricted to objects of $\mathcal{C}_B^c \times_{\mathcal{C}_{B'}^c} \mathcal{C}_{A'}^c$. Choose such an object $(C_B, C_{A'}, \eta)$ (so that C_B and $C_{A'}$ are compact) and let $C_A = C_B \times_{C_{B'}} C_{A'}$; we wish to show that the canonical maps

$$\phi : B \otimes_A C_A \to C_B \qquad \phi' : A' \otimes_A C_A C_{A'}$$

are equivalences. We will show that ϕ is an equivalence; the argument that ϕ' is an equivalence is similar. Let $\mathcal{X} \subseteq \mathcal{C}_B$ be the full subcategory spanned by those objects $D_B \in \mathcal{C}_B$ such that ϕ induces an equivalence $\phi_0 : \operatorname{Mor}_{\mathcal{C}_B}(D_B, B \otimes_A C_A) \to \operatorname{Mor}_{\mathcal{C}_B}(D_B, C_B)$. We wish to show that $\mathcal{X} = \mathcal{C}_B$. Since $\mathcal{X}$ is closed under small colimits, it will suffice to show that $\mathcal{X}$ contains $B \otimes_A D_A$ for every compact object $D_A \in \mathcal{C}_A$. Let $D_{A'}$ and $D_{B'}$ be the images of D_A in $\mathcal{C}_{A'}$ and $\mathcal{C}_{B'}$, respectively. Using Lemma 5.3.24, we can identify ϕ_0 with the canonical map $B \otimes_A \operatorname{Mor}_{\mathcal{C}_A}(D_A, C_A) \to \operatorname{Mor}_{\mathcal{C}_B}(D_B, C_B)$. Note that we have a pullback diagram

$$\begin{tikzcd} \operatorname{Mor}_{\mathcal{C}_A}(D_A, C_A) \arrow[r] \arrow[d] & \operatorname{Mor}_{\mathcal{C}_B}(D_B, C_B) \arrow[d] \\ \operatorname{Mor}_{\mathcal{C}_{A'}}(D_{A'}, C_{A'}) \arrow[r] & \operatorname{Mor}_{\mathcal{C}_{B'}}(D_{B'}, C_{B'}) \end{tikzcd}$$

and that Lemma 5.3.24 guarantees that the underlying maps

$$B' \otimes_{A'} \operatorname{Mor}_{\mathcal{C}_{A'}}(D_{A'}, C_{A'}) \to \operatorname{Mor}_{\mathcal{C}_{B'}}(D_{B'}, C_{B'}) \leftarrow B' \otimes_B \operatorname{Mor}_{\mathcal{C}_B}(D_B, C_B)$$

are equivalences. It will therefore suffice to show that there exists an integer n such that $\operatorname{Mor}_{\mathcal{C}_B}(D_B, C_B)$ and $\operatorname{Mor}_{\mathcal{C}_{A'}}(D_{A'}, C_{A'})$ belong to $(\operatorname{LMod}_B)_{\geq n}$ and $(\operatorname{LMod}_{A'})_{\geq n}$, respectively (Proposition IX.7.6). This follows immediately from Lemma 5.3.27. □

Notation 5.3.29. Let $\operatorname{LinCat}^{\mathrm{tcg}}$ be the subcategory of LinCat whose objects are pairs $(A, \mathcal{C})$, where A is a connective $\mathbb{E}_2$-ring and $\mathcal{C}$ is a tamely compactly generated A-linear ∞-category, and whose morphisms are maps $(A, \mathcal{C}) \to (A', \mathcal{C}')$ such that the underlying functor $\mathcal{C} \to \mathcal{C}'$ carries compact objects of $\mathcal{C}$ to compact objects of $\mathcal{C}'$. It follows from Lemma 5.3.27 that the forgetful functor $\mathrm{LinCat}^{\mathrm{tcg}} \to \mathrm{Alg}^{(2),\mathrm{cn}}$ is a coCartesian fibration. This coCartesian fibration is classified by a functor $\chi^{\mathrm{tcg}} : \mathrm{Alg}^{(2),\mathrm{cn}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$.

Lemma 5.3.30. *Let $\chi^{\mathrm{tcg}} : \mathrm{Alg}^{(2),\mathrm{cn}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ be as in Notation 5.3.29. Suppose we are given a pullback diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ A' \arrow[r] & B' \end{tikzcd}$$

of connective $\mathbb{E}_2$-rings which induces surjective maps $\pi_0 B \to \pi_0 B'$ and $\pi_0 A' \to \pi_0 B'$. Then the induced map $F : \chi^{\mathrm{tcg}}(A) \to \chi^{\mathrm{tcg}}(B) \times_{\chi^{\mathrm{tcg}}(B')} \chi^{\mathrm{tcg}}(A')$ is fully faithful.

Proof. We can identify objects of the fiber product $\chi^{\mathrm{tcg}}(B) \times_{\chi^{\mathrm{tcg}}(B')} \chi^{\mathrm{tcg}}(A')$ with triples $(\mathcal{C}_B, \mathcal{C}_{A'}, \eta)$ where $\mathcal{C}_B \in \mathrm{LinCat}^{\mathrm{tcg}}_B$, $\mathcal{C}_{A'} \in \mathrm{LinCat}^{\mathrm{tcg}}_A$, and η is an equivalence $\mathrm{LMod}_{B'}(\mathcal{C}_B) \simeq \mathrm{LMod}_{B'}(\mathcal{C}_A)$. Given such an object, we let $\mathcal{C}_{B'}$ denote the ∞-category $\mathrm{LMod}_{B'}(\mathcal{C}_B) \simeq \mathrm{LMod}_{B'}(\mathcal{C}_A)$. The functor F admits a right adjoint G, which carries a triple $(\mathcal{C}_B, \mathcal{C}_{A'}, \eta)$ to the full subcategory of $\mathcal{C}_B \times_{\mathcal{C}_{B'}} \mathcal{C}_{A'}$ generated under small colimits by $\mathcal{C}^c_B \times_{\mathcal{C}^c_{B'}} \mathcal{C}^c_{A'}$. We wish to show that the unit map $u : \mathrm{id} \to G \circ F$ is an equivalence. In other words, we wish to show that if $(\mathcal{C}_B, \mathcal{C}_{A'}, \eta) = F(\mathcal{C}_A)$ for some tamely compactly generated A-linear ∞-category $\mathcal{C}_A$, then the canonical map $\mathcal{C}_A \to \mathcal{C}_B \times_{\mathcal{C}_{B'}} \mathcal{C}_{A'}$ is fully faithful and its essential image is the subcategory generated by $\mathcal{C}^c_B \times_{\mathcal{C}^c_{B'}} \mathcal{C}^c_A$ under small colimits. This follows immediately from Lemma 5.3.28. □

Lemma 5.3.31. *Let k be a field and let $A \in \mathrm{Alg}^{(2),\mathrm{sm}}_k$ be a small $\mathbb{E}_2$-algebra over k. Let $\mathcal{C}_A$ be a compactly generated A-linear ∞-category. Then $\mathcal{C}_A$ is tamely compactly generated if and only if $\mathrm{LMod}_k(\mathcal{C}_A) \simeq \mathrm{LMod}_k \otimes_{\mathrm{LMod}_A} \mathcal{C}_A$ is tamely compactly generated.*

Proof. The "only if" direction follows from Lemma 5.3.27. For the converse, suppose that $\mathrm{LMod}_k(\mathcal{C}_A)$ is tamely compactly generated. Let $\mathcal{X} \subseteq \mathcal{C}_A$ be the full subcategory spanned by those objects D such that, for each $C \in \mathcal{C}^c_A$, the groups $\mathrm{Ext}^n_{\mathcal{C}_A}(C, D)$ vanish for $n \gg 0$. Note that if $C, D \in \mathcal{C}^c_A$, then

$$\mathrm{Ext}^n_{\mathcal{C}_A}(C, k \otimes D) \simeq \mathrm{Ext}^n_{\mathrm{LMod}_k(\mathcal{C}_A)}(k \otimes C, k \otimes D)$$

vanishes for $n \gg 0$. It follows that $\mathcal{X}$ contains $k \otimes D$ for each $D \in \mathcal{C}^c_A$. Since A is small, we can choose a finite sequence

$$A = A_0 \to \cdots \to A_n$$

and pullback diagrams

$$\begin{tikzcd} A_i \arrow[r] \arrow[d] & k \arrow[d] \\ A_{i+1} \arrow[r] & k \oplus k[m_i]. \end{tikzcd}$$

In particular, we have fiber sequences of A-modules

$$A_i \to A_{i+1} \to k[m_i].$$

It follows by descending induction on i that $A_i \otimes D \in \mathcal{X}$ for each $D \in \mathcal{C}^c_A$. Taking $i = 0$, we deduce that $\mathcal{C}^c_A \subseteq \mathcal{X}$ as desired. □

Proof of Proposition 5.3.21. Combine Variation 5.3.6 with Lemmas 5.3.30 and 5.3.31. □

We can improve further on Proposition 5.3.21 if we are willing to to impose some stronger conditions on the k-linear ∞-category $\mathcal{C}$.

Definition 5.3.32. Let $\mathcal{C}$ be a presentable stable ∞-category. We will say that an object $C \in \mathcal{C}$ is *unobstructible* if C is compact and the groups $\mathrm{Ext}^n_{\mathcal{C}}(C, C)$ vanish for $n \geq 2$.

Theorem 5.3.33. *Let k be a field and let $\mathcal{C}$ be a k-linear ∞-category. Assume that $\mathcal{C}$ is tamely compactly generated and that there exists a collection of unobstructible objects $\{C_\alpha\}$ which generates $\mathcal{C}$ under small colimits. Then the functor $\mathrm{CatDef}_{\mathcal{C},\omega} : \mathrm{Alg}_k^{(2),\mathrm{sm}} \to \mathcal{S}$ of Proposition 5.3.21 is a formal $\mathbb{E}_2$ moduli problem.*

Corollary 5.3.34. *Let k be a field and let $\mathcal{C}$ be a k-linear ∞-category. Assume that $\mathcal{C}$ is tamely compactly generated and that there exists a collection of unobstructible objects $\{C_\alpha\}$ which generates $\mathcal{C}$ under small colimits. Then the composite map*

$$\mathrm{CatDef}_{\mathcal{C},\omega} \to \mathrm{CatDef}_{\mathcal{C}} \to \mathrm{CatDef}^{\wedge}_{\mathcal{C}}$$

is an equivalence. Consequently, the functor $\mathrm{CatDef}_{\mathcal{C},\omega}$ is given by

$$\mathrm{CatDef}_{\mathcal{C},\omega}(R) = \mathrm{Map}_{\mathrm{Alg}_k^{(2)}}(\mathfrak{D}^{(2)}(R), \mathfrak{Z}(\mathcal{C})),$$

where $\mathfrak{Z}(\mathcal{C})$ denotes the k-linear center of $\mathcal{C}$.

Proof. Combine Theorems 5.3.33 and 5.3.16 with Remarks 5.3.9 and 5.1.11. □

The proof of Theorem 5.3.33 will require some preliminaries. Our first lemma gives an explanation for the terminology of Definition 5.3.32.

Lemma 5.3.35. *Let k be a field, let $f : A \to A'$ be a small morphism between augmented $\mathbb{E}_2$-algebras over k. Let $\mathcal{C}_A$ be a tamely compactly generated A-linear ∞-category, let $\mathcal{C}_{A'} = \mathrm{LMod}_{A'}(\mathcal{C}_A)$, and let $\mathcal{C} = \mathrm{LMod}_k(\mathcal{C}_A)$. Suppose that $C \in \mathcal{C}_{A'}$ is a compact object whose image in $\mathcal{C}$ is unobstructible. Then there exists a compact object $C_A \in \mathcal{C}_A$ and an equivalence $C_{A'} \simeq A \otimes \mathcal{C}_A$ in $\mathcal{C}_{A'}$.*

Proof. Let $C \in \mathcal{C}$ denote the image of $C_{A'}$. Since f is small, we can choose a finite sequence of morphisms

$$A = A_0 \to \cdots \to A_n \simeq A'$$

and pullback diagrams

$$\begin{tikzcd} A_i \arrow[r] \arrow[d] & k \arrow[d] \\ A_{i+1} \arrow[r] & k \oplus k[m_i] \end{tikzcd}$$

in $\mathrm{Alg}_k^{(2),\mathrm{aug}}$, where each $m_i \geq 1$. We prove by descending induction on i that $C_{A'}$ can be lifted to a compact object $C_i \in \mathrm{LMod}_{A_i}(\mathcal{C}_A)$, the case $i = n$ being trivial. Assume that C_{i+1} has been constructed. Let $\mathcal{C}' = \mathrm{LMod}_{k \oplus k[m_i]}(\mathcal{C})$. According to Lemma 5.3.28, we have an equivalence of ∞-categories

$$\mathcal{C}_i^c \to \mathcal{C}_{i+1}^c \times_{\mathcal{C}'^c} \mathcal{C}^c.$$

Consequently, to show that C_{i+1} can be lifted to an object $C_i \in \mathcal{C}_i^c$, it will suffice to show that C_{i+1} and C have the same image in $\mathcal{C}'^c$. This is a special case of the following assertion:

($*$) Let $X, Y \in \mathcal{C}'$ be objects having image $C \in \mathcal{C}$. Then there is an equivalence $X \simeq Y$ in $\mathcal{C}'$.

To prove ($*$), we let $\mathrm{ObjDef}_C : \mathrm{Art}^{(1)} \to \mathcal{S}$ be defined as in Notation 5.2.4; we wish to prove that any two points of the space $\mathrm{ObjDef}_C(k \oplus k[m_i])$ belong to the same path component. According to Proposition 5.2.13, $\mathrm{ObjDef}_C(k \oplus k[m_i])$ can be identified with a summand of the mapping space $\mathrm{Map}_{\mathrm{Alg}_k^{(1)}}(\mathfrak{D}^{(1)}(k \oplus k[m_i]), \mathrm{End}(C))$. Since the Koszul dual $\mathfrak{D}^{(1)}(k \oplus k[m_i])$ is the free associative algebra generated by $k[-m_i - 1]$, we have a canonical isomorphism

$$\pi_0 \mathrm{Map}_{\mathrm{Alg}_k^{(1)}}(\mathfrak{D}^{(1)}(k \oplus k[m_i]), \mathrm{End}(C)) \simeq \pi_{-m_i-1} \mathrm{End}(C) \simeq \mathrm{Ext}^{m_i+1}_{\mathcal{C}}(C, C).$$

These groups vanish by virtue of our assumption that C is unobstructible. □

Remark 5.3.36. In the situation of Lemma 5.3.35, if we assume that $\mathrm{Ext}^1_{\mathcal{C}}(C, C)$ vanishes, then lifting of $C_{A'}$ to $\mathcal{C}_A$ is unique up to equivalence: that is, C is *undeformable*.

Lemma 5.3.37. *Let k be a field, let $f : A \in \mathrm{Alg}_k^{(2),\mathrm{aug}}$ be a small augmented $\mathbb{E}_2$-algebra over k, let $\mathcal{C}_A$ be a tamely compactly generated A-linear ∞-category, and let $\mathcal{C} = \mathrm{LMod}_k(\mathcal{C}_A)$. Let $\{C_\alpha\}$ be a collection of objects of $\mathcal{C}$ which generates $\mathcal{C}$ under small colimits, and let $\{\overline{C}_\alpha\}$ be a collection of objects of $\mathcal{C}_A$ with $C_\alpha \simeq k \otimes \overline{C}_\alpha$. Then the collection $\{\overline{C}_\alpha\}$ generates $\mathcal{C}_A$ under small colimits.*

Proof. Let $\mathcal{X}$ be the full subcategory of $\mathcal{C}_A$ generated by $\{\overline{C}_\alpha\}$ under small colimits. Then $\mathcal{X}$ contains $M \otimes \overline{C}_\alpha$ for each $M \in \mathrm{LMod}_A$. Taking $M = k$, we deduce that $\mathcal{X}$ contains the images of the objects $\{C_\alpha\}$ under the forgetful functor $\theta : \mathcal{C} \to \mathcal{C}_A$. Since θ preserves small colimits, it follows that $\mathcal{X}$ contains the essential image of θ. In particular, $k \oplus C \in \mathcal{X}$ for each $C \in \mathcal{C}_A$. Since A is small, we can choose a finite sequence

$$A = A_0 \to \cdots \to A_n$$

and pullback diagrams

$$\begin{tikzcd} A_i \arrow[r] \arrow[d] & k \arrow[d] \\ A_{i+1} \arrow[r] & k \oplus k[m_i]. \end{tikzcd}$$

It follows by descending induction on i that $\mathcal{X}$ contains $A_i \otimes C$ for each $C \in \mathcal{C}_A$. Taking $i = 0$, we deduce that $\mathcal{X} = \mathcal{C}_A$. $\square$

Proof of Theorem 5.3.33. Proposition 5.3.21 implies that $\mathrm{CatDef}_{\mathcal{C},\omega}$ is a 1-proximate formal moduli problem. Suppose we are given a pullback diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ A' \arrow[r] & B' \end{tikzcd}$$

in $\mathrm{Alg}_k^{(2),\mathrm{sm}}$ which induces surjective maps $\pi_0 B \to \pi_0 B'$ and $\pi_0 A' \to \pi_0 B'$. Then the map

$$\theta : \mathrm{CatDef}_{\mathcal{C},\omega}(A) \to \mathrm{CatDef}_{\mathcal{C},\omega}(B) \times_{\mathrm{CatDef}_{\mathcal{C},\omega}(B')} \mathrm{CatDef}_{\mathcal{C},\omega}(A')$$

is (-1)-truncated, and we wish to show that it is a homotopy equivalence. Fix a point of the fiber product $\mathrm{CatDef}_{\mathcal{C},\omega}(B) \times_{\mathrm{CatDef}_{\mathcal{C},\omega}(B')} \mathrm{CatDef}_{\mathcal{C},\omega}(A')$, which determines a pair $(\mathcal{C}_{A'}, \mathcal{C}_B, \eta)$ where $\mathcal{C}_{A'}$ is a compactly generated A'-linear ∞-category, $\mathcal{C}_B$ is a compactly generated B-linear ∞-category, η is an equivalence $\mathrm{LMod}_{B'}(\mathcal{C}_{A'}) \simeq \mathrm{LMod}_{B'}(\mathcal{C}_B)$, and let $\mathcal{C}_{B'}$ denote the ∞-category $\mathrm{LMod}_{B'}(\mathcal{C}_{A'}) \simeq \mathrm{LMod}_{B'}(\mathcal{C}_B)$. Let $\mathcal{C}_A$ denote the full subcategory of $\mathcal{C}_B \times_{\mathcal{C}_{B'}} \mathcal{C}_{A'}$ generated under small colimits by $\mathcal{C}^c_B \times_{\mathcal{C}^c_{B'}} \mathcal{C}^c_{A'}$. We will show that $\theta(\mathcal{C}_A) \simeq (\mathcal{C}_B, \mathcal{C}_{A'}, \eta)$. Unwinding the definitions, it suffices to show that the canonical maps $q : \mathrm{LMod}_B(\mathcal{C}_A) \to \mathcal{C}_B$ and $q' : \mathrm{LMod}_{A'}(\mathcal{C}_A) \to \mathcal{C}_{A'}$ are categorical equivalences. We will show that q is an equivalence; the proof for q' is similar.

We first claim that q is fully faithful. It will suffice to show that q is fully faithful when restricted to compact objects. Since the collection of compact objects of $\mathrm{LMod}_B(\mathcal{C}_A)$ is generated, under retracts and finite colimits, by the essential image of the free functor $F : \mathcal{C}_A \to \mathrm{LMod}_B(\mathcal{C}_A)$, it will suffice to show that for every pair of compact objects $C, D \in \mathcal{C}_A$, q induces an equivalence of left B-modules $\xi : \mathrm{Mor}_{\mathrm{LMod}_B(\mathcal{C}_A)}(F(C), F(D)) \to \mathrm{Mor}_{\mathcal{C}_B}(qF(C), qF(D))$. We can identify $\mathcal{C}^c_A$ with the fiber product $\mathcal{C}^c_A \simeq \mathcal{C}^c_B \times_{\mathcal{C}^c_{B'}} \mathcal{C}^c_{A'}$, so that C and D correspond to triples $(C_B, C_{A'}, \gamma)$ and $(D_B, D_{A'}, \delta)$. Let $C_{B'}$ denote the image of C in $\mathcal{C}_{B'}$ and let $D_{B'}$ be defined similarly. Using Lemma 5.3.25, we can identify ξ with the map $B \otimes \mathrm{Mor}_{\mathcal{C}_A}(C, D) \to \mathrm{Mor}_{\mathcal{C}_B}(C_B, D_B)$. Here we have an equivalence $\mathrm{Mor}_{\mathcal{C}_A}(C, D) \simeq \mathrm{Mor}_{\mathcal{C}_B}(C_B, D_B) \times_{\mathrm{Mor}_{\mathcal{C}_{B'}}(C_{B'}, D_{B'})} \mathrm{Mor}_{\mathcal{C}_{A'}}(C_{A'}, D_{A'})$. Using Lemma 5.3.25 and Proposition IX.7.4, we are reduced to proving that $\mathrm{Mor}_{\mathcal{C}_B}(C_B, D_B)$ and $\mathrm{Mor}_{\mathcal{C}_{A'}}(C_{A'}, D_{A'})$ are n-connective for some integer n. This follows from the fact that $\mathcal{C}_B$ and $\mathcal{C}_{A'}$ are tamely compactly generated (Lemma 5.3.31).

It remains to prove that q is essentially surjective. Note that the essential image of is closed under small colimits. Using Lemma 5.3.35, it will suffice to show that the essential image of q contains every object $C_B \in \mathcal{C}_B^c$ whose image in $\mathcal{C}$ is unobstructible. Let $C_{B'}$ be the image of $C_B \in \mathcal{C}_B^c$. To prove that C_B can be lifted to $\mathcal{C}_A^c$, it will suffice to show that $C_{B'}$ can be lifted to $\mathcal{C}_{A'}^c$. The existence of the desired lifting follows from Lemma 5.3.35. $\square$

Remark 5.3.38. The hypotheses of Theorem 5.3.33 are rather restrictive: many k-linear ∞-categories of interest (such as the ∞-categories of quasi-coherent sheaves on most algebraic varieties of dimension ≥ 2) cannot be generated by unobstructible objects. In these cases, the functor $\mathrm{CatDef}_{\mathcal{C},\omega} \to \mathrm{CatDef}_{\mathcal{C}}^{\wedge}$ need not be an equivalence. In these cases, it seems natural to ask if there is some explicit deformation-theoretic description of the formal moduli problem $\mathrm{CatDef}_{\mathcal{C}}^{\wedge}$, analogous to the explicit description of $\mathrm{ObjDef}_{M}^{\wedge}$ for an object $M \in \mathcal{C}$ given in Remark 5.2.16. To obtain a satisfactory answer, it is presumably necessary to allow *curved* deformations of the ∞-category $\mathcal{C}$.

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