THIS IS THE (CO)END, MY ONLY (CO)FRIEND

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Abstract. The present note is a recollection of the most striking and useful applications of *co/end calculus*. We put a considerable effort in making arguments and constructions rather explicit: after having given a series of preliminary definitions, we characterize co/ends as particular co/limits; then we derive a number of results directly from this characterization. The last sections discuss the most interesting examples where co/end calculus serves as a powerful abstract way to do explicit computations in diverse fields like Algebra, Algebraic Topology and Category Theory. The appendices serve to sketch a number of results in theories heavily relying on co/end calculus; the reader who dares to arrive at this point, being completely introduced to the mysteries of *co/end fu*, can regard basically every statement as a guided exercise.

Contents

Introduction.

The purpose of this survey is to familiarize the reader with the so-called *co/end calculus*, gathering a series of examples of its application; the author would like to stress clearly, from the very beginning, that the material presented here makes no claim of originality: indeed, we put a special care in acknowledging carefully, where possible, each of the many authors whose work was an indispensable source in compiling this note. Among these, every erroneous or missing attribution must be ascribed to the mere ignorance of the author.

Key words and phrases. end, coend, dinatural transformation, operad, profunctor, Kan extension, weighted limit, nerve and realization, promonoidal category, Yoneda structure.

The introductory material is the most classical and comes almost *verbatim* from [ML98]; the nerve-realization calculus is a *patchwork* of various results, obtained in various flavours in the Algebraic Topology literature; for what concerns the theory of operads we mostly copied down all the computations in [Kel05]; *la théorie des distributeurs* comes from the work of J. Bénabou, whose theory of *profunctors* has been collected in some notes taken by T. Streicher, [Béner]; the chapter on weighted co/limits is taken almost verbatim from [Rie14, **II.7**]: only a couple of implicit conceptual dependencies on co/end calculus have been made explicit; the last chapter about promonoidal categories comes from [Day74, S^+12]; the appendices propose the reader to familiarize with the theory of promonoidal categories, and to show the initial results of [Day11, **§1-3**], a delightful and deep paper whose importance is far more than a source of unusual exercises.

Because of all these remarks, the reader has to keep in mind that the value of this work –if there is any– lies, rather than in the originality of the discussion, in having collected (albeit with a series of unforgivable, undavoidable sins of omission) as much material as possible about co/end calculus, exposed in an elementary way, to serve the inexperienced reader (either the beginner in the study of Category Theory, or the experienced student who exhausted the "primary" topics of her education) as a guide to familiarize with this extraordinarily valuable toolset producing a large number of *abstract-nonsense* proofs, most of which are "formally formal" strings of natural isomorphisms.

After the first examples, the keen reader will certainly prefer to re-write most of the proofs in the silence of her room, and we warmly invite her to do so; *imitation* of the basic techniques is for sure an unavoidable step in getting acquainted with the machinery of co/end calculus, and more generally with any machinery in mathematical language, in the same sense we all learned integration rules by imitation and training. Maybe it's not a coincidence (*nomen omen*?) that the one you are about to see is *another* integral calculus to be learned by means of examples and exercises.

It has been said that "Universal history is, perhaps, the history of a few metaphors" [BS64]: differential and integral calculus is undoubtedly one of such recurring themes. In some sense, the fortune of co/end calculus is based on the analogy which represents these universals by means of integrals [Yon60, DK69, zbM04], motivated by the Fubini theorem on the interchange of "iterated integrals"; unfortunately, the language of coends seems to be woefully underestimated and kept as a secret art everybody knows and manages, so that no elementary book in Category Theory (apart a single chapter in the aforementioned [ML98]) seems to contain something more than a bare introduction of the basic elements of the language.

What's missing, in the humble opinion of the author, is a source of examples, exercises and computations clear enough to show their readers how co/end calculus can literally *disintegrate* involved computations in a bunch of canonical isomorphisms. Trying to fill this gap in the literature has been the main motivation for the text you're about to read.

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suggesting improvements and corrections, having spotted an incredible number of errors, misprints and incoherent choices of notation. Their attentive proofreading has certainly increased the value –again, if there is any– of the document you're about to read.

> Los idealistas arguyen que las salas hexagonales son una forma necesaria del espacio absoluto o, por lo menos, de nuestra intuición del espacio.
>
> J. L. Borges, *La biblioteca de Babel.*

1. Dinaturality, extranaturality, co/wedges.

1.1. **Foundations, notation and conventions.** The main foundational convention we adopt throughout the paper is the assumption ([GV72]) that every set lies in a suitable Grothendieck universe. We implicitly fix such an universe $\mho$, whose elements are termed *sets*; categories are always considered to be small with respect to this universe (this assumption is not strictly necessary but we choose to follow it to ease up the discussion). Categories, 2-categories and bicategories are always denoted as boldface letters $\mathbf{C}, \mathbf{D}$ etc. Functors between categories are denoted as capital Latin letters like F, G, H, K etc.; the category of functors $\mathbf{C} \to \mathbf{D}$ is denoted as $\mathrm{Fun}(\mathbf{C}, \mathbf{D})$, $\mathbf{D}^{\mathbf{C}}$, $[\mathbf{C}, \mathbf{D}]$ and suchlike; following a tradition which goes back to Grothendieck, $\widehat{\mathbf{C}}$ is a shorthand for the category $[\mathbf{C}^{\mathrm{op}}, \mathbf{Sets}]$ of presheaves on $\mathbf{C}$; the canonical hom-bifunctor of a category $\mathbf{C}$ sending (c, c') to the set of all arrows $\hom(c, c') \subseteq \hom(\mathbf{C})$ is denoted as $\mathbf{C}(-, =) \colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Sets}$; morphisms in $\mathrm{Fun}(\mathbf{C}, \mathbf{D})$ (i.e. natural transformations) are often written in Greek, or Latin lowercase alphabet, and collected in the set $\mathsf{Nat}(F, G) = \mathbf{D}^{\mathbf{C}}(F, G)$. The simplex category $\boldsymbol{\Delta}$ is the *topologist's delta*, having objects *nonempty* finite ordinals $\Delta[n] := \{0 < 1 \cdots < n\}$; we denote $\Delta[n]$ the representable presheaf on $[n] \in \boldsymbol{\Delta}$, i.e. the image of $[n]$ under the Yoneda embedding of $\boldsymbol{\Delta}$ in the category $\mathbf{sSet} = \widehat{\boldsymbol{\Delta}}$ of simplicial sets.

1.2. **Dinaturality.** Let's start with a simple example. Let $\mathbf{Sets}$ be the category of sets and functions, considered with its natural cartesian closed structure: this means we have a bijection of sets

$$\mathbf{Sets}(A \times B, C) \cong \mathbf{Sets}(A, C^B) \tag{1}$$

where C^B is the set of all functions $B \to C$. The adjunction $- \times B \dashv (-)^B$ has a counit, i.e. a natural transformation

$$\epsilon_X \colon X^B \times B \to X \tag{2}$$

which can be considered "mute in the variable B".

Exercise 1.1. Show that for each $f \in \mathbf{Sets}(B, B')$ the following square is commutative:

$$\begin{array}{ccc} X^{B'} \times B & \xrightarrow{X^f \times B} & X^B \times B \\ {\scriptstyle X^{B'} \times f}\downarrow & & \downarrow{\scriptstyle \epsilon} \\ X^{B'} \times B' & \xrightarrow[\epsilon]{} & X \end{array} \tag{3}$$

The collection of functions $\{\epsilon_X \colon X^B \times B \to X\}$ is natural in the classical sense in the variable X; as for the variable B, the most we can say is the commutativity above; it doesn't remind naturality so much, and at a second glance *it's not even a functorial correspondence.* Fortunately a suitable generalization of naturality (a "supernaturality" condition), encoding the result of Exercise **1.1**, is available to describe this and other similar phenomena in the same common framework. A notion of supernaturality adapted to describe co/ends as suitable universal objects comes in two flavours: one of them is *di*naturality.

Definition 1.2 (Dinatural Transformation)**.** Given two functors $P, Q \colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$ a *dinatural transformation*, depicted as an arrow $\alpha \colon P \overset{\cdot\cdot}{\to} Q$, consists of a family of arrows $\{\alpha_c \colon P(c,c) \to Q(c,c)\}_{c \in \mathbf{C}}$ such that for any $f \colon c \to c'$ the following hexagonal diagram commutes

$$\begin{array}{ccccc} P(c',c) & \xrightarrow{P(f,c)} & P(c,c) & \xrightarrow{\alpha_c} & Q(c,c) \\ {\scriptstyle P(c',f)}\downarrow & & & & \downarrow{\scriptstyle Q(c,f)} \\ P(c',c') & \xrightarrow[\alpha_{c'}]{} & Q(c',c') & \xrightarrow[Q(f,c')]{} & Q(c,c') \end{array} \tag{4}$$

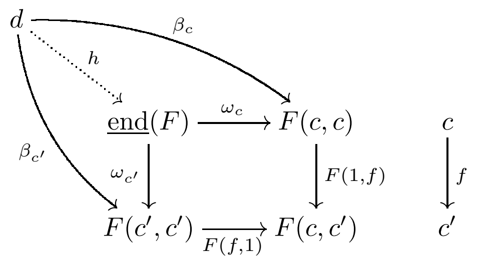
Exercise 1.3. A dinatural transformation serves to define natural transformations between functors having the same co/domain but different covariance; try to do this.

Exercise 1.4. Show with an example that dinatural transformations $\alpha \colon P \overset{\cdot\cdot}{\to} Q, \beta \colon Q \overset{\cdot\cdot}{\to} R$ cannot be composed in general. Nevertheless, there exists a "composition" of a dinatural $\alpha \colon P \overset{\cdot\cdot}{\to} Q$ with a natural $\eta \colon P' \to P$ which is again dinatural $P' \overset{\cdot\cdot}{\to} Q$, as well as a composition $P \overset{\cdot\cdot}{\to} Q \to Q'$ (hint: the appropriate diagram results as the pasting of a dinaturality hexagon and two naturality squares).

Definition 1.5 (Wedge for a functor)**.** Let $P \colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$; a *wedge* for P is a dinatural transformation $\Delta_d \overset{\cdot\cdot}{\to} P$ from the constant functor on the object $d \in \mathbf{D}$ (often denoted simply by $d \colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$), defined sending $(c, c') \mapsto d$, $(f, f') \mapsto 1_d$.

Definition 1.6 (End of a functor)**.** The *end* of a functor $F \colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$ consists of a universal wedge $\underline{\mathrm{end}}(F) \overset{\cdot\cdot}{\to} F$; the constant $\underline{\mathrm{end}}(F) \in \mathbf{D}$ is itself termed, by abuse, the end of the functor.

Spelled out explicitly, the universality requirement means that for any other wedge $\beta\colon d \stackrel{..}{\to} F$ the diagram



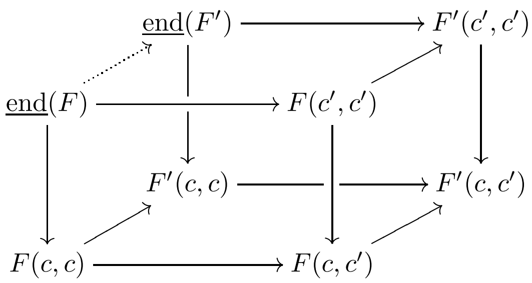
(5)

commutes for a unique arrow $h\colon d \to \underline{\mathrm{end}}(F)$, for any arrow $f\colon c \to c'$.

Exercise 1.7. Coends are obtained by a suitable dualization. State the definition of a *cowedge* for a functor $F\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$; a *coend* for F consists of a universal cowedge $\underline{\mathrm{coend}}(F)$ for P. Prove functoriality for coends. Show that coends are ends in the dual category: the coend of F is the end of $F^{\mathrm{op}}\colon \mathbf{C} \times \mathbf{C}^{\mathrm{op}} \cong \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}^{\mathrm{op}}$.

Exercise 1.8. Define a category $\mathrm{WD}(F)$ having objects the wedges for $F\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$ and show that the end of F is the terminal object of $\mathrm{WD}(F)$; dualize to coends (initial objects of a category $\mathrm{CWD}(F)$ of cowedges).

Remark 1.9. Uniqueness requirements imply functoriality: given a natural transformation $\eta\colon F \Rightarrow F'$ there is an induced arrow $\underline{\mathrm{end}}(\eta)\colon \underline{\mathrm{end}}(F) \to \underline{\mathrm{end}}(F')$ between their ends, as depicted in the diagram



(6)

This implies that taking the end of a functor is a (covariant) functor $\mathrm{Fun}(\mathbf{C}^{\mathrm{op}} \times \mathbf{C}, \mathbf{D}) \to \mathbf{D}$. The case of a coend is dually analogous: filling the details is an easy dualization exercise.

1.3. **Extranaturality.** A slightly less general, but better behaved (since it admits a graphical calculus) notion of supernaturality, allowing again to define co/wedges, is available: this is called *extranaturality* and it was introduced in [EK66].

Definition 1.10 (Extranatural transformation)**.** Let P, Q be functors

$$P\colon \mathbf{A} \times \mathbf{B}^{\mathrm{op}} \times \mathbf{B} \to \mathbf{D}$$
$$Q\colon \mathbf{A} \times \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}.$$

An *extranatural transformation* $\alpha\colon P \stackrel{..}{\to} Q$ consist of a collection of arrows

$$\{\alpha_{abc}\colon P(a,b,b) \longrightarrow Q(a,c,c)\} \tag{7}$$

such that the following hexagonal diagram commutes for every $f\colon a \to a'$, $g\colon b \to b'$, $h\colon c \to c'$, all taken in their suitable domains:

$$\begin{CD} P(a,b',b) @>{P(f,b',g)}>> P(a',b',b') @>{\alpha_{a'b'c}}>> Q(a',c,c) \\ @V{P(a,g,b)}VV @. @VV{Q(a',c,h)}V \\ P(a,b,b) @>>{\alpha_{abc'}}> Q(a,c',c') @>>{Q(f,h,c')}> Q(a',c,c'); \end{CD} \tag{8}$$

This commutative hexagon can be equivalently described as the juxtaposition of three distinguished commutative squares, depicted in [EK66]: they can be obtained simply putting either f and h, f and g, or g and h as identities in the former diagram, obtaining

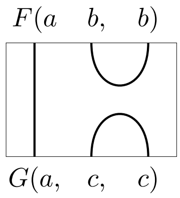
$$\begin{CD} P(a,b,b) @>{P(f,b,b)}>> P(a',b,b) \\ @V{\alpha_{abc}}VV @VV{\alpha_{a'bc}}V \\ Q(a,c,c) @>>{Q(f,c,c)}> Q(a',c,c) \end{CD} \qquad \begin{CD} P(a,b',b) @>{P(a,b',g)}>> P(a,b',b') \\ @V{P(a,g,b)}VV @VV{\alpha_{ab'c}}V \\ P(a,b,b) @>>{\alpha_{abc}}> Q(a,c,c) \end{CD} \qquad \begin{CD} P(a,b,b) @>{\alpha_{abc}}>> Q(a,c,c) \\ @V{\alpha_{abc'}}VV @VV{Q(a,c,h)}V \\ Q(a,c',c') @>>{Q(a,h,c')}> Q(a,c,c') \end{CD} \tag{9}$$

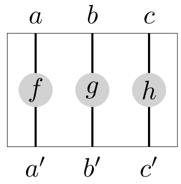
Remark 1.11. We can again define co/wedges in this setting: if $\mathbf{B} = \mathbf{C}$ and in $F(a,b,b) \to G(a,c,c)$ the functor F is constant in $d \in \mathbf{C}$, $G(a,c,c) = \bar{G}(c,c)$ is mute in a, we get a wedge condition for $d \stackrel{..}{\to} G$; dually we obtain a cowedge condition for $F(b,b) \to G(a,b,b) \equiv d'$ for all a,b,c.

Both notions give rise to the same notion of co/end as a universal co/wedge for a bifunctor $F\colon \mathbf{C}^{\text{op}} \times \mathbf{C} \to \mathbf{D}$. The main reason we should prefer extranaturality, as was pointed out to the author in a (semi)private conversation with T. Trimble, is that

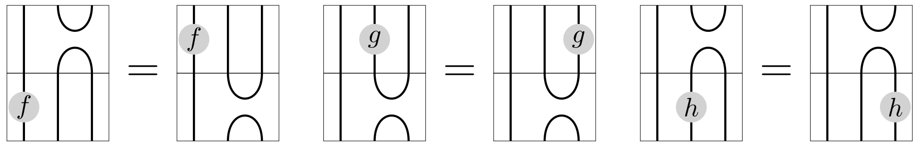
> For the purposes of describing end/coend calculus, I wouldn't emphasize dinatural transformations so much as I would extranatural transformations. Most dinatural transformations that arise in the wild can be analyzed in terms of extranatural (extraordinary natural, in the old lingo) transformations. [...]
>
> [Co/wedges can be regarded as particular examples of two slightly different, but related (see Prop **1.14**) constructions:] first, they are special examples of dinatural transformations. Second, they are special cases of an extranatural (*extraordinary natural*) transformation, which generally is a family of maps $F(a,a,b) \to G(b,c,c)$ which combines naturality in the argument b with a cowedge condition on a and a wedge condition on c.

1.3.1. *A graphical calculus for extranaturality.* The graphical calculus for extranatural transformations depicts the components α_{abc}, and arrows $f\colon a \to a'$, $g\colon b \to b'$, $h\colon c \to c'$, as plain diagrams like

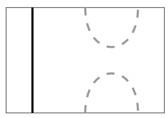




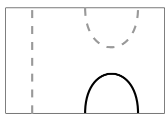
where wires are labeled by objects and must be thought oriented from top to bottom. The commutative squares of (**9**) become, in this representation, the following three string diagrams, whose equivalence is “graphically obvious”:

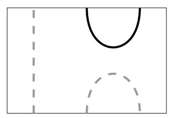


Remark 1.12. The notion of extranatural transformation can be specialized to encompass various other constructions: simple old naturality arises when F, G are both constant in their co/wedge components, like in



whereas wedge and cowedge conditions arise when either F, G are constant:





All the others mixed situations (a wedge-cowedge condition, naturality and a wedge, etc.) lacking a specified name, admit a graphical representation of the same sort.

Exercise 1.13 (Composition of extranaturals)**.** Show that extranatural transformations compose accordingly to these rules:

- (stalactites) Let F, G be functors of the form $\mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$. If $\alpha_{x,y} \colon F(x,y) \to G(x,y)$ is natural in x, y and $\beta_x \colon G(x,x) \to H$ is extranatural in x (for some object H of D), then
$$\beta_x \circ \alpha_{x,x} : F(x,x) \to H \tag{10}$$
is extranatural in x.
- (stalagmites) Let G, H be functors of the form $\mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$. If $\alpha_x \colon F \to G(x,x)$ is extranatural in x (for some object F of D) and $\beta_{x,y} \colon G(x,y) \to H(x,y)$ is natural in x, y, then
$$\beta_{x,x} \circ \alpha_x : F \to H(x,x) \tag{11}$$
is extranatural in x.

- (yanking) Let F, H be functors of the form $\mathbf{C} \to \mathbf{D}$, and let $G\colon \mathbf{C} \times \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$ be a functor. If $\alpha_{x,y}\colon F(y) \to G(x,x,y)$ is natural in y and extranatural in x, and if $\beta_{x,y}\colon G(x,y,y) \to H(x)$ is natural in x and extranatural in y, then

$$\beta_{x,x} \circ \alpha_{x,x}\colon F(x) \to H(x) \tag{12}$$

is natural in x.

Express these laws as equalities between suitable string diagrams.

Proposition 1.14. Extranatural are particular kinds of dinatural transformations.

Proof. Suppose you have functors $F\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \times \mathbf{C} \to \mathbf{D}$ and $G\colon \mathbf{C} \times \mathbf{C} \times \mathbf{C}^{\mathrm{op}} \to \mathbf{D}$. Now put $\mathbf{A} = \mathbf{C} \times \mathbf{C}^{\mathrm{op}} \times \mathbf{C}^{\mathrm{op}}$, and form two new functors $F', G' : \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{D}$ by taking the composites

$$F' = (\mathbf{C}^{\mathrm{op}} \times \mathbf{C} \times \mathbf{C}) \times (\mathbf{C} \times \mathbf{C}^{\mathrm{op}} \times \mathbf{C}^{\mathrm{op}}) \xrightarrow{\text{proj}} \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \times \mathbf{C} \xrightarrow{F} \mathbf{D}$$
$$(x', y', z'; x, y, z) \longmapsto (x', x, y') \overset{F}{\mapsto} F(x', x, y')$$

$$G' = (\mathbf{C}^{\mathrm{op}} \times \mathbf{C} \times \mathbf{C}) \times (\mathbf{C} \times \mathbf{C}^{\mathrm{op}} \times \mathbf{C}^{\mathrm{op}}) \xrightarrow{\text{proj}'} \mathbf{C} \times \mathbf{C} \times \mathbf{C}^{\mathrm{op}} \xrightarrow{G} \mathbf{D}$$
$$(x', y', z'; x, y, z) \mapsto (y', z', z) \overset{G}{\mapsto} G(y', z', z)$$

Now let's put $a' = (x', y', z')$ and $a = (x, y, z)$, considered as objects in $\mathbf{A}$. An arrow $\varphi\colon a' \to a$ in $\mathbf{A}$ thus amounts to a triple of arrows $f\colon x' \to x$, $g\colon y \to y'$, $h\colon z \to z'$ all in $\mathbf{C}$.

Following the instructions above, we have $F'(a', a) = F(x', x, y')$ and $G(a', a) = G(y', z', z)$.

Now if we write down a dinaturality hexagon for $\alpha : F' \ddot{\to} G'$, we get a diagram of shape

$$\begin{array}{ccccc} F'(a,a') & \xrightarrow{F'(1,\varphi)} & F'(a,a) & \xrightarrow{\alpha_a} & G'(a,a) \\ {\scriptstyle F(\varphi,1)}\big\downarrow & & & & \big\downarrow{\scriptstyle G'(\varphi,1)} \\ F'(a',a') & \xrightarrow[\alpha_{a'}]{} & G'(a',a') & \xrightarrow[G'(1,\varphi)]{} & G(a',a) \end{array} \tag{13}$$

which translates to a hexagon of shape

$$\begin{array}{ccccc} F(x,x',y) & \xrightarrow{F(1,f,1)} & F(x,x,y) & \longrightarrow & G(y,z,z) \\ {\scriptstyle F(f,1,g)}\big\downarrow & & & & \big\downarrow{\scriptstyle G(g,h,1)} \\ F(x',x',y') & \longrightarrow & G(y',z',z') & \xrightarrow[G(1,h,1)]{} & G(y',z',z) \end{array} \tag{14}$$

where the unlabeled arrows refer to the extranatural transformation. ☐

1.4. **The integral notation for co/ends.** A suggestive and useful notation alternative to the anonymous one $\underline{\text{co}}/\underline{\text{end}}(F)$ is due to N. Yoneda, which in [Yon60] introduces most of the notions we are dealing with, specialized to **Ab**-enriched functors $\mathbf{C}^{\text{op}} \times \mathbf{C} \to \mathbf{Ab}$.

The end of a functor $F \in \text{Fun}(\mathbf{C}^{\text{op}} \times \mathbf{C}, \mathbf{D})$ can be denoted as a "subscripted-integral" $\int_c F(c,c)$, and $\underline{\text{coend}}(F)$ as the "superscripted-integral" $\int^c F(c,c)$.

From now on we will systematically adopt this notation to denote $\underline{\text{co}}/\underline{\text{end}}(F)$, or slightly more pedantic variants of this, as

$$\int^{c\in\mathbf{C}} F(c,c), \qquad \int^{c:\mathbf{C}} F(c,c) \tag{15}$$

Remark 1.15. One should be aware that Yoneda's notation in [Yon60] is "reversed" as he calls *integration* our coends, which he denotes as $\int_{c\in\mathbf{C}} F(c,c)$, and *cointegrations* our ends, which he denotes $\int^*_{c\in\mathbf{C}} F(c,c)$.

Remark 1.16. Properties of co/ends acquire a particularly suggestive flavour when written in this notation:

- **i)** Functoriality (the *B-rule* for integrals or the *freshman's dream*): the unique arrow $\underline{\text{end}}(\eta)$ induced by a natural transformation $\eta\colon F \Rightarrow G$ between $F, G \in \text{Fun}(\mathbf{C}^{\text{op}} \times \mathbf{C}, \mathbf{D})$ can be written as $\int \eta\colon \int F \to \int G$, and uniqueness of this induced arrow entails that $\int(\eta \circ \sigma) = \int \eta \circ \int \sigma$.
- **ii)** The "Fubini theorem" for ends, which first appeared as equation (**4.0.1**) of [Yon60]: given a functor $F\colon \mathbf{C}^{\text{op}} \times \mathbf{C} \times \mathbf{E}^{\text{op}} \times \mathbf{E} \to \mathbf{D}$, we can form the end $\int_c F(c,c,-,=)$ obtaining a functor $\mathbf{E}^{\text{op}} \times \mathbf{E} \to \mathbf{D}$ whose end is $\int_e \int_c F(c,c,e,e) \in \mathbf{D}$; we can also form the ends $\int_c \int_e F(c,c,e,e) \in \mathbf{D}$ and $\int_{(c,e)} F(c,c,e,e)$ identifying $\mathbf{C}^{\text{op}} \times \mathbf{C} \times \mathbf{E}^{\text{op}} \times \mathbf{E}$ with $(\mathbf{C} \times \mathbf{E})^{\text{op}} \times (\mathbf{C} \times \mathbf{E})$. *Fubini's theorem* for ends states that there is a canonical isomorphism between the three:

$$\int_{(c,e)} F(c,c,e,e) \cong \int_e \int_c F(c,c,e,e) \cong \int_c \int_e F(c,c,e,e) \tag{16}$$

Exercise 1.17. Prove the Fubini theorem for ends embarking in a long exercise in universality; prove that if $F\colon \mathbf{C}^{\text{op}} \times \mathbf{C} \to \mathbf{D}$ is *mute* in one of the two variables (i.e. $F(c',c) = \bar{F}(c)$ or $\hat{F}(c')$ for each $c, c' \in \mathbf{C}$ and suitable functors $\bar{F}\colon \mathbf{C} \to \mathbf{D}$ or $\hat{F}\colon \mathbf{C}^{\text{op}} \to \mathbf{D}$), then the co/end of F is canonically isomorphic to its co/limit.

This gives an alternative proof[1] of a similar *Fubini rule for co/limits*: given a functor $F\colon I \times J \to \mathbf{D}$ we have

$$\varinjlim_I \varinjlim_J F \cong \varinjlim_J \varinjlim_I F \cong \varinjlim_{I\times J} F \tag{17}$$

(and similarly for limits).

Remark 1.18. In some sense, the Fubini rule for coends seems a rather weak analogy between integrals and coends; there is no doubt the fact that ([ML98, **IX.5**])

[1]There are, obviously, more direct and natural ways to prove this statement, and yet we believe that it's worth to mention this point of view on commutation of limits, rather than deriving the Fubini rule from the description (see below) of co/ends as co/limits.

> [...] the "variable of integration" c [in $\int_c F$] appears twice under the integral sign (once contravariant, once covariant) and is "bound" by the integral sign, in that the result no longer depends on c and so is unchanged if c is replaced by any other letter standing for an object of the category $\mathbf{C}$

motivates the integral notation, and nevertheless, this analogy seems quite elusive to justify in a precise way. In the eye of the author, it seems worthwhile to remember that in view of the characterization for co/ends in terms of co/equalizers given in **1.23**, $\int_c \colon \mathrm{Fun}(\mathbf{C}^{\mathrm{op}} \times \mathbf{C}, \mathbf{D}) \to \mathbf{D}$ can be thought as an *averaging* operation on a functor, giving the "fixed points" of the "action" induced by $F(\varphi, c'), F(c, \varphi)$ as $\varphi \colon c \to c'$ runs over $\hom(\mathbf{C})$.

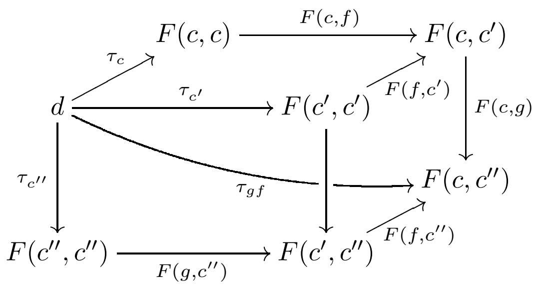
The author prefers to refrain from any further investigation, having no chance to give a valid (or rather, formal) explanation of the integral notation.

1.5. **Co/ends as co/limits.** A general tenet of elementary Category Theory is that you never escape from characterizing an universal construction as an element of the triad

limit - adjoint - representation of functors.

The formalism of co/ends is not an exception: the scope of the following subsection is to characterize, whenever it exists, the co/end of a functor $F \colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$ as a co/limit over a suitable diagram, and finally as the co/equalizer of a single pair of arrows.

First of all notice that given $F \colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$ and a wedge $\tau \colon d \stackrel{..}{\to} F$, we can build the following commutative diagram



(18)

where $c \xrightarrow{f} c' \xrightarrow{g} c''$ are two arbitrarily chosen arrows. From this commutativity we deduce the following relations:

$$\begin{aligned}\tau_{gf} &= F(gf, c'') \circ \tau_{c''} = F(c, gf) \circ \tau_c \\ &= F(f, c'') \circ F(g, c'') \circ \tau_{c''} = F(f, c'') \circ \tau_g \\ &= F(c, g) \circ F(c, f) \circ \tau_c = F(c, g) \circ \tau_f.\end{aligned}$$

where τ_f, τ_g are the common values $F(f, c')\tau_{c'} = F(c, f)\tau_c$ and $F(c', g)\tau_{c'} = F(g, c'')\tau_{c''}$ respectively, τ_{gf} is the common value $F(c, g)\tau_f = F(f, c'')\tau_g$. These relations imply that there is a link between co/wedges and co/cones, encoded in the following definition.

Definition 1.19 (Twisted arrow category of $\mathbf{C}$)**.** For every category $\mathbf{C}$ we define $\text{TW}(\mathbf{C})$, the category of *twisted arrows* in $\mathbf{C}$ as follows:

- $\text{Ob}(\text{TW}(\mathbf{C})) = \hom(\mathbf{C})$;
- Given $f: c \to c'$, $g: d \to d'$ a morphism $f \to g$ is given by a pair of arrows $(h: d \to c, k: c' \to d')$, such that the obvius square commutes (asking that the arrow between domains is in reversed order is *not* a mistake!).

Endowed with the obvious rules for composition and identity, $\text{TW}(\mathbf{C})$ is easily seen to be a category, and now we can find a functor

$$\text{Fun}(\mathbf{C}^{\text{op}} \times \mathbf{C}, \mathbf{D}) \longrightarrow \text{Fun}(\text{TW}(\mathbf{C}), \mathbf{D}) \tag{19}$$

defined sending F to the functor $\overline{F}: \text{TW}(\mathbf{C}) \to \mathbf{D}: f \mapsto F(\text{src}(f), \text{trg}(f))$; it is extremely easy to check that bifunctoriality for F exactly corresponds to functoriality for $\overline{F}$, but there is more.

Remark 1.20. The family $\{\tau_f\}_{f \in \hom(\mathbf{C})}$ constructed before is a cone for the functor $\overline{F}$, and conversely any such cone determines a wedge for F, given by $\{\tau_c := \tau_{1_c}\}_{c \in \mathbf{C}}$. Again, a morphism of cones goes to a morphism between the corresponding wedges, and conversely any morphism between wedges induces a morphism between the correspoding cones; these operations are mutually inverse and form an equivalence between the category $\text{CN}(\overline{F})$ of cones for $\overline{F}$ and the category $\text{WD}(F)$ of wedges for F (see Exercise **1.8**).

Equivalences of categories obviously respect initial/terminal objects, and since co/limits are initial/terminal objects in the category of co/cones, and (see Exercise **1.8**) co/ends are initial/terminal objects in the category of co/wedges, we obtain that

$$\int_c F \cong \varprojlim_{\text{TW}(\mathbf{C})} \overline{F}; \qquad \int^c F \cong \varinjlim_{\text{TW}(\mathbf{C})} \overline{F} \tag{20}$$

Remark 1.21. There is another (slightly *ad hoc* and cumbersome, in the humble opinion of the author) characterization of co/ends as co/limits, given by [ML98, Prop. **IX.5.1**], which relies upon the *subdivision (associative) plot* (see [Tri13, Def. **2**, **3**]) $\hat{\mathbf{C}}^{\S}$ of $\mathbf{C}$, whose

- objects are the set $\text{Ob}(\mathbf{C}) \sqcup \hom(\mathbf{C})$, in such a way that there exists a "marked" object $c^{\S}$ for each $c \in \mathbf{C}$, and another marked object $f^{\S}$ for each $f \in \hom(\mathbf{C})$. The reader must have clear in mind that $c^{\S}$ and $1_c^{\S}$ are *different* objects of $\mathbf{C}^{\S}$;
- arrows are the set of all symbols $\text{src}(f)^{\S} \to f^{\S}$, or $\text{trg}(f)^{\S} \to f^{\S}$, as f runs over $\hom(\mathbf{C})$;
- composition law is the empty function.

The *subdivision category* $\mathbf{C}^{\S}$ is obtained from $\hat{\mathbf{C}}^{\S}$ formally adding identities and giving to the resulting category the trivial composition law (composition is defined only if one of the arrows is the identity). The discussion before [ML98, Prop. **IX.5.1**] now sketches the proof that every functor $F: \mathbf{C}^{\text{op}} \times \mathbf{C} \to \mathbf{D}$ induces a functor $\overline{F}: \mathbf{C}^{\S} \to \mathbf{D}$, whose limit is isomorphic to the end of F.

Exercise 1.22. The categories $\text{TW}(\mathbf{C})$ and $\mathbf{C}^{\S}$ are linked by a *final* (see [Bor94a, **2.11.1**]) functor $K\colon \mathbf{C}^{\S} \to \text{TW}(\mathbf{C})$; this motivates the fact that the colimit is the same when indexed by one of the two. Define K and show that is is final, i.e. that for every object in $\text{TW}(\mathbf{C})$ the comma category $(\varphi \downarrow K)$ is nonempty and connected (see [Rie14, Remark **7.2.10**, Example **8.3.9**]).

Remark 1.23. One of the most useful characterizations of co/ends, which turns out to be extremely useful in explicit computations, regards them as co/equalizers of pairs of maps. In fact it is rather easy to see that

$$\int_c F(c,c) \cong \mathrm{eq}\left(\prod_{c\in\mathbf{C}} F(c,c) \underset{F_*}{\overset{F^*}{\rightrightarrows}} \prod_{\varphi\colon c\to c'} F(c,c') \right) \tag{21}$$

where the product over $\varphi\colon c \to c'$ can be expressed as a double product (over the objects $c, c' \in \mathbf{C}$, and over the arrows φ between these two fixed objects), and the arrows F^*, F_* are easily obtained from the arrows whose (φ, c, c')-component is (respectively) $F(\varphi, c')$ and $F(c, \varphi)$.

Exercise 1.24. Dualize the above construction, to obtain a similar characterization for the coend $\int^c F(c,c)$, characterized as the coequalizer of a similar pair (F^*, F_*).

Definition 1.25. There is an obvious definition of *preservation* of co/ends mutuated from their description as co/limits, which can be regarded as the preservation of the particular kind of co/limit involved in the definition of $\underline{\mathrm{end}}(F)$ and $\underline{\mathrm{coend}}(F)$.

This remark entails easily that

Theorem 1.26. Every co/continuous functor $F\colon \mathbf{D} \to \mathbf{E}$ preserves every co/end that exists in $\mathbf{D}$, namely if $T\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$ has a co/end $\int_c^{/c} T(c,c)$, then

$$F\Big(\int_c^{/c} T(c,c) \Big) \cong \int_c^{/c} FT(c,c). \tag{22}$$

As a particular example of this, we have

Corollary 1.27 (The hom functor commutes with integrals)**.** Continuity of the hom bifunctor $\mathbf{C}(-,=)\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Sets}$ gives its co/end preservation properties: for every $c \in \mathbf{C}$ we have the canonical isomorphisms

$$\mathbf{C}\Big(\int^x F(x,x), c \Big) \cong \int_x \mathbf{C}(F(x,x), c)$$
$$\mathbf{C}\Big(c, \int_x F(x,x) \Big) \cong \int_x \mathbf{C}(c, F(x,x))$$

The power of this remark can't be overestimated: co/continuity of the hom functor is a fundamental *kata* of coend-fu. Basically *every* example in the rest of the paper involves a computation carried on using this co/end preservation property, plus the fully faithfulness of the Yoneda embedding.

Exercise 1.28. Show that the hom functors $\mathbf{C}(-, y)$ *jointly preserve* ends (see [ML98] for a definition).

1.6. **Natural transformations as ends.** A basic example exploiting the whole machinery introducted so far is the proof that the set of natural transformations between two functors $F, G\colon \mathbf{C} \to \mathbf{D}$ can be characterized as an end:

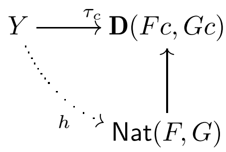
Theorem 1.29. Given functors $F, G\colon \mathbf{C} \to \mathbf{D}$ between small categories we have the canonical isomorphism of sets

$$\mathsf{Nat}(F, G) \cong \int_c \mathbf{D}(Fc, Gc). \tag{23}$$

Proof. Giving a wedge $\tau_c\colon Y \to \mathbf{D}(Fc, Gc)$ consists in giving a function $y \mapsto \tau_{c,y}\colon Fc \to Gc$, which is natural in $c \in \mathbf{C}$ (this is simply a rephrasing of the wedge condition):

$$G(f) \circ \tau_{c,y} = \tau_{c',y} \circ F(f) \tag{24}$$

for any $f\colon c \to c'$; this means that there exists a unique way to close the diagram



(25)

with a function sending $y \mapsto \tau_{-,y}$, and where $\mathsf{Nat}(F, G) \to \mathbf{D}(Fc, Gc)$ is the wedge sending a natural transformation to its c-component; the diagram commutes for a single $h\colon Y \to \mathsf{Nat}(F, G)$, and this is precisely the desired universal property for $\int_c \mathbf{D}(Fc, Gc)$. □

Remark 1.30. A suggestive way to express naturality as a "closure" condition is given in [Yon60, **4.1.1**], where for an **Ab**-enriched functor $F\colon \mathbf{C}^{\text{op}} \times \mathbf{C} \to \mathbf{Ab}$ we find that $\mathsf{Nat}(F, G) = \ker \delta$, for a suitable map δ defined among $\mathbf{Ab}(Fx, Gx) \oplus \mathbf{Ab}(Fy, Gy)$ and $\mathbf{Ab}(Fy, Gy)$.

Exercise 1.31. Prove again the above result, using the characterization of $\int_c \mathbf{D}(Fc, Gc)$ as an equalizer: the subset of $\prod_{c \in \mathbf{C}} \mathbf{D}(Fc, Gc)$ you will have to consider is precisely the subset of natural transformations $\{\tau_c\colon Fc \to Gc \mid Gf \circ \tau_c = \tau_{c'} \circ Ff,\ \forall f\colon c \to c'\}$.

After this, we can embark in more sophisticated and pervasive examples. In particular, the following section is the gist of the paper, demostrating the power of co/end calculus to prove highly technical and involved results by means of abstract-nonsense only.

2. Yoneda reduction, Kan extensions.

One of the most famous results about the category $[\mathbf{C}^{\text{op}}, \mathbf{Sets}]$ of presheaves on a category $\mathbf{C}$ is that every object in it can be canonically presented as a colimit of representable functors; see [ML98, Theorem **III.7.1**] for a description of this classical result (and its dual holding in $[\mathbf{C}, \mathbf{Sets}]$).

Now, co/end calculus allows us to rephrase this result in an extremely compact way, called *Yoneda reduction*; in a few words, it says that *every co/presheaf can be expressed as a co/end*[2].

Proposition 2.1 (Ninja Yoneda Lemma)**.** For every functor $K\colon \mathbf{C}^{\text{op}} \to \mathbf{Sets}$ and $H\colon \mathbf{C} \to \mathbf{Sets}$, we have the following isomorphisms (natural equivalences of functors):

$$(i)\ K \cong \int^c Kc \times \mathbf{C}(-,c) \qquad (ii)\ K \cong \int_c Kc^{\mathbf{C}(c,-)}$$
$$(iii)\ H \cong \int^c Hc \times \mathbf{C}(c,-) \qquad (iv)\ H \cong \int_c Hc^{\mathbf{C}(-,c)}$$

Remark 2.2. The name *ninja Yoneda lemma* is a pun coming from a `MathOverflow` comment by T. Leinster, whose content is basically the proof of the above statement:

> Th[e above one is] often called the *Density Formula*, [...] or (by Australian ninja category theorists) simply the Yoneda Lemma. (but Australian ninja category theorists call *everything* the Yoneda Lemma...).

Undoubtedly, there is a link between the above result and the Yoneda Lemma we all know: in fact, the proof heavily relies on the Yoneda isomorphism, and in enriched setting (see [Dub70, §**I.5**]) the ninja Yoneda lemma, read as a result in Kan extensions (see Exercise **2.8**), is *equivalent* to the Yoneda lemma.

Nevertheless, the author of the present note feels utterly unqualified to properly discuss the topic, as he lives in the wrong hemisphere of the planet to claim any authority on it, and to be acquainted with the unique taste of the Australian clan. Because of this, along the whole note, we keep the name "ninja Yoneda Lemma" as a mere (somewhat witty) nickname for the above isomorphisms.

Proof. We prove case (i) only, all the others being totally analogous. We put a certain emphasis on the style of this proof, as it is paradigmatic of all the subsequent ones. Consider the chain of isomorphisms

$$\begin{aligned}\mathbf{Sets}\Big(\textstyle\int^c Kc \times \mathbf{C}(x,c), y\Big) &\cong \textstyle\int_c \mathbf{Sets}\big(Kc \times \mathbf{C}(x,c), y\big)\\ &\cong \int_c \mathbf{Sets}(\mathbf{C}(x,c), \mathbf{Sets}(Kc,y))\\ &\cong \mathsf{Nat}(\mathbf{C}(x,-), \mathbf{Sets}(K-,y))\\ &\cong \mathbf{Sets}(Kx,y)\end{aligned}$$

where the first step is motivated by the coend-preservation property of the hom functor, the second follows from the fact that **Sets** is a cartesian closed category, where

$$\mathbf{Sets}(X \times Y, Z) \cong \mathbf{Sets}(X, \mathbf{Sets}(Y,Z)) \tag{26}$$

[2]The reader looking for a nifty explanation of this result should wait for a more torough discussion, which can be deduced from the material in Section **4**, thanks to the machinery of *weighted co/limits*.

for all three sets X, Y, Z (naturally in all arguments), and the final step exploits Theorem **1.29** plus the classical Yoneda Lemma.

Every step of this chain of isomorphisms is natural in y; now we have only to notice that the natural isomorphism of functors

$$\mathbf{Sets}\Big(\int^c Kc \times \mathbf{C}(x,c), y\Big) \cong \mathbf{Sets}(Kx, y) \tag{27}$$

ensures that there exists a (natural) isomorphism $\int^c Kc \times \mathbf{C}(x,c) \cong Kx$. This concludes the proof. □

Exercise 2.3. Prove in a similar way isomorphisms $(ii), (iii), (iv)$ (hint: for (ii) and (iv) start from $\mathbf{Sets}\Big(y, \int_c Hc^{\mathbf{C}(x,c)}\Big)$ and use again the end preservation property, cartesian closure of **Sets**, and Theorem **1.29**).

From now one we will make frequent use of the notion of (**Sets**-)*tensor* and (**Sets**-)*cotensor* in a category; these standard definitions are in the chapter of any book about enriched category theory (see for example [Bor94b, Ch. **6**], its references, and in particular its Definition **6.5.1**, which we report for the ease of the reader:

Definition 2.4 (Tensor and cotensor in a $\mathcal{V}$-category)**.** In any $\mathcal{V}$-enriched category $\mathbf{C}$ (see [Bor94b, Def. **6.2.1**]), the *tensor* $\cdot\colon \mathcal{V} \times \mathbf{C} \to \mathbf{C}$ is a functor $(V, c) \mapsto V \cdot c$ such that there is the isomorphism

$$\mathbf{C}(V \cdot c, c') \cong \mathcal{V}(V, \mathbf{C}(c, c')), \tag{28}$$

natural in all components; dually, the *cotensor* in an enriched category $\mathbf{C}$ is a functor $(V, c) \mapsto c^V$ (contravariant in V) such that there is the isomorphism

$$\mathbf{C}(c', c^V) \cong \mathcal{V}(V, \mathbf{C}(c', c)), \tag{29}$$

natural in all components.

Every co/complete, locally small category $\mathbf{C}$ is naturally **Sets**-co/tensored by choosing $c^V \cong \prod_{v \in V} c$ and $V \cdot c \cong \coprod_{v \in V} c$.

Remark 2.5. The tensor, hom and cotensor functors are the prototype of a THC *situation* (see Remark **3.7** and [Gra80, §**1.1**] for a definition); given the hom-objects of a $\mathcal{V}$-category $\mathbf{C}$, the tensor $\cdot\colon \mathcal{V} \times \mathbf{C} \to \mathbf{C}$ and the cotensor $(=)^{(-)}\colon \mathcal{V}^{\text{op}} \times \mathbf{C} \to \mathbf{C}$ can be characterized as adjoint functors: usual co/continuity properties of the co/tensor functors are implicitly derived from this characterization

2.1. **Kan extensions as co/ends.**

Definition 2.6. Given a functor $F\colon \mathbf{C} \to \mathbf{D}$, its *left* and *right Kan extensions* are defined to be, respectively, the left and right adjoint to the "precomposition" functor

$$F^*\colon \text{Fun}(\mathbf{D}, \mathbf{E}) \to \text{Fun}(\mathbf{C}, \mathbf{E}) \tag{30}$$

given by $H \mapsto F^*(H) = H \circ F$, in such a way that there are two isomorphisms

$$\mathsf{Nat}(\text{Lan}_F G, H) \cong \mathsf{Nat}(G, H \circ F)$$
$$\mathsf{Nat}(H \circ F, G) \cong \mathsf{Nat}(F, \text{Ran}_F G).$$

Now, we want to show that in "nice" situations it is possible to describe Kan extensions via co/ends: whenever the co/tensors (see Def. **2.4**) involved in the definition of the following co/ends exist in $\mathbf{D}$ for any choice of functors F, G (a blatant example is when $\mathbf{D}$ is co/complete, since in that case as said before it is always **Sets**-co/tensored by suitable iterated co/products), then the left/right Kan extensions of $G\colon \mathbf{C} \to \mathbf{E}$ along $F\colon \mathbf{C} \to \mathbf{D}$ exists and there are isomorphisms

$$\mathrm{Lan}_F G \cong \int^c \mathbf{D}(Fc, -) \cdot Gc \qquad\qquad \mathrm{Ran}_F G \cong \int_c Gc^{\mathbf{D}(-,Fc)}. \tag{31}$$

Proof. The proof consists of a string of canonical isomorphisms, exploiting simple remarks in elementary Category Theory and the results established so far: the same argument is offered in [ML98, Thm. **X.4.1, 2**]

$$\begin{aligned}
\mathsf{Nat}\Big(\textstyle\int^c \mathbf{D}(Fc, -) \cdot Gc, H\Big) &\overset{\mathbf{1.29}}{\cong} \textstyle\int_x \mathbf{D}\Big(\int^c \mathbf{D}(Fc, x) \cdot Gc, Hx\Big)\\
&\overset{\mathbf{1.27}}{\cong} \textstyle\int_{cx} \mathbf{D}(\mathbf{D}(Fc, x) \cdot Gc, Hx)\\
&\overset{(\mathbf{28})}{\cong} \textstyle\int_{cx} \mathbf{Sets}(\mathbf{D}(Fc, x), \mathbf{E}(Gc, Hx))\\
&\overset{\mathbf{1.29}}{\cong} \textstyle\int_c \mathsf{Nat}(\mathbf{D}(Fc, -), \mathbf{E}(Gc, H-))\\
&\overset{\text{Yon}}{\cong} \textstyle\int_c \mathbf{E}(Gc, HFc) \cong \mathsf{Nat}(G, HF)
\end{aligned}$$

the case of $\mathrm{Ran}_F G$ is dually analogous. □

Remark 2.7. This is the pattern of every "proof by coend-juggling" we will meet in the following; from now on we feel free to abandon a certain pedantry in declaring every single application of the results like the co/end preservation properties.

Exercise 2.8. Use equations (**31**) and the ninja Yoneda lemma that $\mathrm{Lan}_{\mathrm{id}}$ and $\mathrm{Ran}_{\mathrm{id}}$ are the identity functors, as expected. Use again (**31**) and the ninja Yoneda lemma to complete the proof that $F \mapsto \mathrm{Lan}_F$ is a pseudofunctor, by showing that for $\mathbf{A} \xrightarrow{F} \mathbf{B}$, $\mathbf{A} \xrightarrow{G} \mathbf{C} \xrightarrow{H} \mathbf{D}$ there is a uniquely determined laxity cell for composition

$$\mathrm{Lan}_H(\mathrm{Lan}_G(F)) \cong \mathrm{Lan}_{HG}(F) \tag{32}$$

(hint: coend-juggle with $\mathrm{Lan}_H(\mathrm{Lan}_G(F))d$ until you get $\int^{xy}(\mathbf{D}(Hx, d) \times \mathbf{C}(Gy, x)) \cdot Fy$; now use the ninja Yoneda lemma plus cocontinuity of the tensor, as suggested in Remark **2.5**).

Proposition 2.9. Left/right adjoint functors commute with left/right Kan extensions, whenever they can be expressed as the coends above.

Proof. An immediate corollary of Theorem **1.26**, once it has been proved that a left adjoint commutes with tensors, i.e. $F(X \cdot a) \cong X \cdot Fa$ for any $A(X, a) \in \mathbf{Sets} \times \mathbf{C}$. □

3. The nerve and realization paradigm.

3.1. **The classical nerve and realization.** The most fruitful application of the machinery of Kan extensions is the "Kan construction" for the *realization* of simplicial sets. We briefly sketch this construction.

Consider the category Δ of finite ordinals and monotone maps, as defined in [GJ09], and the Yoneda embedding $よ\colon \Delta \to \mathbf{sSet}$; we can define two functors $\rho\colon \Delta \to \mathbf{Top}$ and $i\colon \Delta \to \mathbf{Cat}$ which "represent" every object $[n] \in \Delta$ either as a topological space or as a small category:

- The category $i[n]$ is $\{0 \to 1 \to \cdots \to n\}$ (there is a similar functor regarding any poset $\mathbf{P} = (P, \leq)$ as a category where the composition function is induced by the partial order relation $\leq$, and i is the restriction of this functor to $\Delta \subset \mathbf{Pos}$);
- The topological space $\rho[n]$ is defined as the *standard n-simplex* embedded in $\mathbb{R}^{n+1}$,

$$\rho[n] = \Big\{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} : 0 \leq x_i \leq 1, \textstyle\sum_{i=0}^n x_i = 1\Big\}. \tag{33}$$

In a few words, we are in the situation depicted by the following diagrams:

$$\begin{array}{ccc} \Delta & \xrightarrow{\;i\;} & \mathbf{Cat} \\ {\scriptstyle よ}\downarrow & & \\ \mathbf{sSet} & & \end{array} \qquad \begin{array}{ccc} \Delta & \xrightarrow{\;\rho\;} & \mathbf{Top} \\ {\scriptstyle よ}\downarrow & & \\ \mathbf{sSet} & & \end{array} \tag{34}$$

The two functors i, ρ can be Kan-extended on the left, along the Yoneda embedding, and these extensions happen to be left adjoints (this can be proved directly, but we will present in a while a completely general statement), denoted respectively

$$\mathrm{Lan}_{よ} i \dashv N_i \text{ and } \mathrm{Lan}_{よ} \rho \dashv N_\rho; \tag{35}$$

these two right adjoint functors are called the *nerves* associated to i and ρ respectively, and are defined, respectively, sending a category $\mathbf{C}$ to the simplicial set $N_i(\mathbf{C})\colon [n] \mapsto \mathrm{Fun}(i[n], \mathbf{C})$ (the *classical nerve* of a category), and to the simplicial set $N_\rho(X)\colon [n] \mapsto \mathbf{Top}(\rho[n], X) = \mathbf{Top}(\Delta^n, X)$ (the *singular complex* of a space X[3]).

It should now be evident that there is a pattern (which we will call the "nerve-realization paradigm") acting behind the scenes, and yielding the classical/singular nerve as particular cases of a general construction interpreted from time to time in different settings; unraveling this machinery with the power of coend-calculus is the scope of the following section.

3.2. **Geometric, (higher-)categorical, toposophic realizations and their coherent nerves.** Algebraic topology, representation theory, and more generally every setting where a "well-behaved" categorical structure is involved constitute natural factories for examples of the nerve-realization paradigm. We now want to collect several examples (undoubtedly leaving outside many far more interesting others!) of nerve-realizations pairs, obtained varying the domain category of "geometric shapes" or the category where this fundamental shapes are "represented".

[3]The name is motivated by the fact that if we consider the free-abelian group on $N_\rho(X)_n$, the various $C_n = \mathbb{Z} \cdot N_\rho(X)_n = \coprod_{N_\rho(X)_n} \mathbb{Z}$ organize as a chain complex, whose homology is precisely the singular homology of X.

Definition 3.1. Any functor $\varphi\colon \mathbf{C} \to \mathbf{D}$ from a small category $\mathbf{C}$ to a (locally small) *cocomplete* category $\mathbf{D}$ is called a *nerve-realization context* (a NR-*context* for short).

Given a nerve-realization context φ, we can prove the following result:

Proposition 3.2 (Nerve-realization paradigm)**.** The left Kan extension of φ along the Yoneda embedding $\text{Y}\colon \mathbf{C} \to [\mathbf{C}^{\text{op}}, \mathbf{Sets}]$, i.e. the functor $R_\varphi = \text{Lan}_{\text{Y}}\varphi\colon [\mathbf{C}^{\text{op}}, \mathbf{Sets}] \to \mathbf{D}$ is a left adjoint, $R_\varphi \dashv N_\varphi$. R_φ is called the $\mathbf{D}$-*realization functor* or the *Yoneda extension* of φ, and its right adjoint the $\mathbf{D}$-*coherent nerve.*

Proof. The cocomplete category $\mathbf{D}$ is $\mathbf{Sets}$-tensored, and hence $\text{Lan}_{\text{Y}}\varphi$ can be written as the coend in equation (**31**); so the claim follows from the chain of isomorphisms

$$\begin{aligned}
\mathbf{D}\big(\text{Lan}_{\text{Y}}\varphi(P), d\big) &\cong \mathbf{D}\Big(\textstyle\int^c \widehat{\mathbf{C}}(\text{Y}c, P)\cdot \varphi c, d\Big)\\
&\cong \textstyle\int_c \mathbf{D}\big(\widehat{\mathbf{C}}(\text{Y}c, P)\cdot \varphi c, d\big)\\
&\cong \textstyle\int_c \mathbf{Sets}\big(\widehat{\mathbf{C}}(\text{Y}c, P), \mathbf{D}(\varphi c, d)\big)\\
&\cong \textstyle\int_c \mathbf{Sets}\big(Pc, \mathbf{D}(\varphi c, d)\big).
\end{aligned}$$

So, if we define $N_\varphi(d)$ to be $c \mapsto \mathbf{D}(\varphi c, d)$, this last set becomes canonically isomorphic to $\mathsf{Nat}(P, N_\varphi(d))$. ☐

Remark 3.3. The nerve-realization paradigm can be rewritten in the following equivalent form: there is an equivalence of categories, induced by the universal property of the Yoneda embedding,

$$\text{Fun}(\mathbf{C}, \mathbf{D}) \cong \text{RAdj}(\widehat{\mathbf{C}}, \mathbf{D}) \tag{36}$$

whenever $\mathbf{D}$ is a cocomplete locally small category (in such a way that "the category of nerve-realization contexts" is a high-sounding name for the category of functors $\text{Fun}(\mathbf{C}, \mathbf{D})$).

3.3. **Properties of the geometric realization.** A famous result in Algebraic Topology (see for example [GZ67, GJ09]) says that the geometric realization functor $R\colon \widehat{\Delta} \to \mathbf{Top}$ commutes with finite products: coend calculus gives a massive simplification of this result.

Proof. The main point of the proof is showing that the geometric realization commutes with products *of representables*: the rest of the proof relies on a suitable application of coend-fu. We could appeal conceptual ways to show this preliminary result ([ABLR02, §**2**]):

Proposition 3.4. The following properties for a functor $F\colon \mathbf{C} \to \mathbf{Sets}$ are equivalent:

- F commutes with finite limits;
- $\text{Lan}_{\text{Y}}F$ commutes with finite limits;
- F is a filtered colimit of representable functors;
- The *category of elements* $\mathbf{C}\!\int\! F$ of F (see Proposition **4.8**) is cofiltered.

Nevertheless, this result isn't powerful enough to show that the geometric realization ρ commutes with **Top**-products, since it gives only a *bijection* $|\Delta[n] \times \Delta[m]| \cong \Delta^n \times \Delta^m$; a certain amount of dirty work is necessary to show that this bijection is a homeomorphism. If we take the commutativity of R with finite products of representables for granted, the proof goes by nonsense, recalling that

- The geometric realization is a left adjoint, hence it commutes with colimits and tensors;
- Every simplicial set is a colimit of representables.

$$\begin{aligned} R(X \times Y) &\cong R\left[\left(\int^m X_m \cdot \Delta[m]\right) \times \left(\int^n Y_n \cdot \Delta[n]\right)\right] \\ &\cong R\left[\int^{mn} (X_m \cdot \Delta[m]) \times (Y_n \cdot \Delta[n])\right] \\ &\cong R\left[\int^{mn} (X_m \times Y_n) \cdot (\Delta[m] \times \Delta[n])\right] \\ &\cong \int^{mn} (X_m \times Y_n) \cdot R(\Delta[m] \times \Delta[n]) \\ &\cong \int^{mn} (X_m \times Y_n) \cdot \Delta^m \times \Delta^n \\ &\cong \int^{mn} (X_m \cdot \Delta^m) \times (Y_n \cdot \Delta^n) \\ &\cong \left(\int^m X_m \cdot \Delta^m\right) \times \left(\int^n Y_n \cdot \Delta^n\right) \\ &\cong R(X) \times R(Y) \end{aligned}$$

where we applied, respectively, the ninja Yoneda lemma, the colimit preservation property of R, its commutation with tensors, and its commutativity with finite products of representables. □

A natural factory of nerve-realization contexts seems to be Homotopical Algebra, as such functors are often used to build Quillen equivalences between model categories. But these are certainly not the only examples of NR-paradigms! We would like to gather here a (certainly incomplete) list of the more important and pervasive among the NR-contexts: for the sake of completeness, we repeat the description of the two above-mentioned examples of the topological and categorical realizations.

i) In the case of $\varphi = i\colon \Delta \to \mathbf{Cat}$, we obtain the *simplicial classical nerve* $N_{\mathbf{Cat}}$ of a (small) category $\mathbf{C}$, whose left adjoint is the *categorical realization* (the *fundamental category* $\tau_1 X$ of X described in [Joy02]). The nerve-realization adjunction

$$\tau_1 : \mathbf{sSet} \leftrightarrows \mathbf{Cat} : N_{\mathbf{Cat}} \tag{37}$$

gives a Quillen adjunction between the Joyal model structure on **sSet** (see [Joy02]) and the folk model structure on **Cat**.

ii) If $\varphi = \rho\colon \Delta \to \mathbf{Top}$ is the realization of a representable $[n]$ in the standard topological simplex, we obtain the adjunction between the *geometric realization* $|X|$ of a simplicial set X and the *singular complex* of a topological space Y, i.e. the simplicial set having as set of n-simplices the continuous functions $\Delta^n \to Y$.

iii) If $\varphi\colon \Delta \to \mathbf{Cat}_{\mathbf{\Delta}}$ is the functor which realizes every representable $[n]$ as a simplicial category having objects the same set $[n] = \{0, 1, \dots, n\}$ and as $\hom(i, j)$ the simplicial set obtained as the nerve of the poset $P(i, j)$

of subsets of the interval $[i,j]$ which contain both i and j[4], we obtain the *(Cordier) simplicially coherent nerve and realization*, which sends $\mathbf{C}$ into a simplicial set constructed "coherently remembering" that $\mathbf{C}$ is a simplicial category. This adjunction establishes a Quillen adjunction $\mathbf{sSet} \leftrightarrows \mathbf{Cat}_{\mathbf{\Delta}}$ which restricts to an equivalence between quasicategories (fibrant objects in the Joyal model structure on $\mathbf{sSet}$) and fibrant simplicial categories (with respect to the Bergner model structure on $\mathbf{Cat}_{\mathbf{\Delta}}$).

iv) The construction giving the topological realization of $\Delta[n]$ extends to the case of any "interval" in the sense of [Moe95, §**III.1**], i.e. any ordered topological space J having "endpoints" $0, 1$; indeed every such space J defines a "generalized" (in the sense of [Moe95, §**III.1**]) topological n-simplex $\Delta^n(J)$, i.e. a nerve-realization context $\varphi_J \colon \Delta \to \mathbf{Top}$.

Exercise 3.5 ([Moe95, Example **III.1.2**])**.** Compute the J-realization of $X \in \mathbf{sSet}$ in the case J is the Sierpiński space $\{0 < 1\}$ with topology $\{\varnothing, J, \{1\}\}$.

v) The correspondence $\delta \colon [n] \mapsto \mathrm{Sh}(\Delta^n)$ defines a *cosimplicial topos*, i.e. a cosimplicial object in the category of toposes, which serves as a NR-context. Several geometric properties of this nerve/realization are studied in [Moe95, §**III**].

vi) The well-known Dold-Kan correspondence, which asserts that there is an equivalence of categories between simplicial abelian groups $[\Delta^{\mathrm{op}}, \mathbf{Ab}]$ and chain complexes $\mathrm{CH}^+(\mathbf{Ab})$ with no negative homology, and it can be seen as an instance of the nerve-realization paradigm.

In this case, the functor $\Delta \to \mathrm{CH}^+(\mathbf{Ab})$ sending $[n]$ to $\mathbb{Z}^{\Delta[n]}$ (the free abelian group on $\Delta[n]$) and then to the *Moore complex* $M(\mathbb{Z}^{\Delta[n]})$ determined by any simplicial group $A \in [\Delta^{\mathrm{op}}, \mathbf{Ab}]$ as in [GJ09] is the nerve-realization context.

Example 3.6 (The tensor product as a coend)**.** Any ring R can be regarded as an $\mathbf{Ab}$-category with a single object, whose set of endomorphisms is the ring R itself; once noticed this, we obtain natural idenfitications for the categories of modules over R:

$$\mathrm{Mod}_R \cong \mathrm{Fun}(R^{\mathrm{op}}, \mathbf{Ab})$$
$$_R\mathrm{Mod} \cong \mathrm{Fun}(R, \mathbf{Ab}).$$

Given $A \in \mathrm{Mod}_R, B \in {}_R\mathrm{Mod}$, we can define a functor $T_{AB} \colon R^{\mathrm{op}} \times R \to \mathbf{Ab}$ which sends the unique object to the tensor product $A \otimes_{\mathbb{Z}} B$ of abelian groups. The coend of this functor can be computed as the coequalizer

$$\mathrm{coker}\Big(\textstyle\coprod_{r\in R} A \otimes_{\mathbb{Z}} B \overset{r\otimes 1}{\underset{1\otimes r}{\rightrightarrows}} \coprod_{r\in R} A \otimes_{\mathbb{Z}} B \Big), \tag{38}$$

or in other words, $\int^{*\in R} T_{AB} \cong A \otimes_R B$. This point of view on tensor products can be extremely generalized (see [ML98, §**IX.6**], but more on this has been written

[4] In particular if $i > j$ then $P(i,j)$ is empty and hence so is its nerve.

in [Yon60, §**4**]): given functors $F, G\colon \mathbf{C}^{\mathrm{op}}, \mathbf{C} \to \mathcal{V}$ having values in a cocomplete monoidal category, we can define the *tensor product* of F, G as the coend

$$F \boxtimes_{\mathbf{C}} G := \int^{c} Fc \otimes_{\mathcal{V}} Gc. \tag{39}$$

Remark 3.7. This can be regarded as part of a general theory which defines a THC-*situation* (see [Gra80, §**1.1**]; these are also called *adjunctions of two variables* in newer references) as a triple $\mathfrak{t} = \{\otimes, \wedge, [-,=]\}$ of (bi)functors between three categories $\mathbf{S}, \mathbf{A}, \mathbf{B}$, defined via the adjunctions

$$\hom_{\mathbf{B}}(S \otimes A, B) \cong \hom_{\mathbf{S}}(S, [A, B]) \cong \hom_{\mathbf{A}}(A, S \wedge B). \tag{40}$$

Such an isomorphism uniquely determines the variances of the three functors involved, in each variable; to be more clear, however, we notice that $\otimes\colon \mathbf{S} \times \mathbf{A} \to \mathbf{B}$, and then $\wedge\colon \mathbf{S}^{\mathrm{op}} \times \mathbf{B} \to \mathbf{A}$, and $[-,=]\colon \mathbf{A}^{\mathrm{op}} \times \mathbf{B} \to \mathbf{S}$.

Exercise 3.8. Use coend-fu to show that starting from a given THC-situation $\mathfrak{t} = \{\otimes, \wedge, [-,=]\}$, we can induce a new one $\mathfrak{t}' = \{\boxtimes, \curlywedge, \langle -,= \rangle\}$, on the categories $\mathbf{S}^{\mathbf{I}^{\mathrm{op}} \times \mathbf{J}}, \mathbf{A}^{\mathbf{I}}, \mathbf{B}^{\mathbf{J}}$, for any $\mathbf{I}, \mathbf{J} \in \mathbf{Cat}$; start defining $F \boxtimes G \in \mathbf{B}^{\mathbf{J}}$ out of $F \in \mathbf{S}^{\mathbf{I}^{\mathrm{op}} \times \mathbf{J}}, G \in \mathbf{A}^{\mathbf{I}}$, as the coend

$$\int^{i} F(i, -) \otimes Gi \tag{41}$$

and show that there is an adjunction

$$\mathbf{B}^{\mathbf{J}}(F \boxtimes G, H) \cong \mathbf{S}^{\mathbf{I}^{\mathrm{op}} \times \mathbf{J}}(F, \langle G, H \rangle) \cong \mathbf{A}^{\mathbf{I}}(G, F \curlywedge H) \tag{42}$$

developing $\mathbf{B}^{\mathbf{J}}(F \boxtimes G, H) = \dots$ in two ways.

4. Weighted limits

The theory of weighted co/limits, an extremely interesting argument even *per se*, can be easily reformulated and understood in terms of co/end calculus.

The whole discussion in this chapter comes from [Rie14, **II.7**]; even most of the notation is basically the same.

Remark 4.1. Let $F\colon \mathbf{C} \to \mathbf{A}$ be a functor between small categories. The *limit* $\varprojlim F$ of F can be characterized as the representing object of a suitable presheaf: indeed, we have the natural isomorphism

$$\mathbf{A}(a, \varprojlim F) \cong \mathbf{Sets}^{\mathbf{C}}(*, \mathbf{A}(a, F(-))) \tag{43}$$

where $*$ is a shorthand to denote the terminal functor $\mathbf{C} \to \mathbf{Sets}\colon X \mapsto *$ and $\mathbf{A}(a, F(-)) = \mathbf{A}(a, F)$ is the functor $\mathbf{C} \to \mathbf{Sets}$ sending c to $\mathbf{A}(a, Fc)$ (so we represent the functor $a \mapsto \mathbf{A}(a, F)$).

Dually, the colimit $\varinjlim F$ can be characterized, in the same notation, as the representing object in the natural isomorphism

$$\mathbf{A}(\varinjlim F, a) \cong \mathbf{Sets}^{\mathbf{C}}(*, \mathbf{A}(F(-), a)). \tag{44}$$

$\mathbf{Sets}^{\mathbf{C}}(*, \mathbf{A}(F, a))$ is a set of natural transformations: the leading idea behind the definition of weighted co/limit is to generalize this construction to admit shapes other than the terminal presheaf for the domain functor. We can couch the above discussion as the following definition:

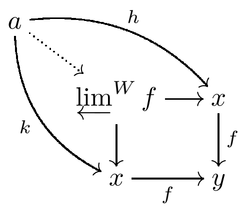
Definition 4.2 (Weighted limit and colimit)**.** Given functors $F\colon \mathbf{C} \to \mathbf{A}$ and $W\colon \mathbf{C} \to \mathbf{Sets}$, we define the *weighted limit* of F by W as a representative for the functor sending $a \in \mathbf{A}$ to $\mathbf{Sets}^{\mathbf{C}}(W, \mathbf{A}(a, F(-)))$, namely an object $\varprojlim^W F \in \mathbf{A}$ such that

$$\mathbf{A}\big(a, \varprojlim^W F\big) \cong \mathbf{Sets}^{\mathbf{C}}(W, \mathbf{A}(a, F(-))). \tag{45}$$

Dually we define the *colimit* of $F\colon \mathbf{C} \to \mathbf{A}$ weighted by $W\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$ to be an object $\varinjlim^W F \in \mathbf{A}$ such that

$$\mathbf{A}\big(\varinjlim^W F, a\big) \cong \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}(W, \mathbf{A}(F(-), a)). \tag{46}$$

Example 4.3. Let $f\colon \Delta^1 \to \mathbf{A}$ the functor choosing an arrow $f\colon x \to y$ in $\mathbf{A}$, and $W\colon \Delta^1 \to \mathbf{Sets}$ the functor sending $\{0 < 1\}$ to the single arrow $\{0, 1\} \to \{0\}$; then a natural transformation $W \Rightarrow \mathbf{A}(a, f)$ consists of arrows $W0 \to \mathbf{A}(a, x), W1 \to \mathbf{A}(a', y)$, namely on the choice of two arrows $h, k\colon a \to x$ such that $fh = fk$: the universal property for $\varprojlim^W f$ implies that this is the *kernel pair* of the arrow f, namely that h, k fill in the pullback



(47)

Proposition 4.4 (Weighted co/limits as co/ends)**.** When the indicated universal objects (the end below and the $\mathbf{Sets}$-cotensor $(X, a) \mapsto a^X$ used to define it) exist, we can express the weighted limit $\varprojlim^W F$ as an end:

$$\begin{aligned}\mathbf{Sets}^{\mathbf{C}}(W, \mathbf{A}(m, F)) &\cong \int_{c\in\mathbf{C}} \mathbf{Sets}(Wc, \mathbf{A}(m, Fc)) \\ &\cong \int_{c\in\mathbf{C}} \mathbf{A}\big(m, Fc^{Wc}\big) \\ &\cong \mathbf{A}\Big(m, \int_{c\in C} Fc^{Wc}\Big)\end{aligned}$$

This string of natural isomorphisms, y means of the uniqueness of the representative of a functor, implies that there is a canonical isomorphism

$$\varprojlim^W F \cong \int_{c\in\mathbf{C}} Fc^{Wc}. \tag{48}$$

Example 4.5. Consider the particular case of two parallel functors $W, F\colon \mathbf{C} \to \mathbf{Sets}$; then we can easily see that $\varprojlim^W F$ coincides with the set of natural transformations $W \Rightarrow F$, since the cotensor Fc^{Wc} amounts to the set $\mathbf{Sets}(Wc, Fc)$.

Example 4.6. The ninja Yoneda lemma, rewritten in this notation, says that $\varprojlim^{\mathbf{C}(c,-)} F \cong Fc$ (or, in case F is contravariant, $\varprojlim^{\mathbf{C}(-,c)} F \cong Fc$). This also suggests that *Kan extensions* can be expressed as suitable weighted co/limits, and more precisely that they can be characterized as those weighted co/limits where the weight is a representable functor:

$$\mathrm{Ran}_K F(\bullet) \cong \int_{c\in\mathbf{C}} Fc^{\mathbf{D}(\bullet, Kc)} \cong \varprojlim^{\mathbf{D}(\bullet, K-)} F. \tag{49}$$

The following Remark and Proposition constitute a central observation; the reader is advised to meditate on it for a long time.

Remark 4.7. Definition **4.2** can be extended in the case $F\colon \mathbf{C} \to \mathbf{A}$ is a $\mathcal{V}$-enriched functor between $\mathcal{V}$-categories, and $W\colon \mathbf{C} \to \mathcal{V}$ is a $\mathcal{V}$-co/presheaf; when $\mathcal{V} = \mathbf{Sets}$, the so-called *Grothendieck construction* sending a (co)presheaf into its category of elements turns out to trivialize almost completely the theory of **Sets**-weighted limits: as the following Propositon shows, in such a situation every weighted limit can be expressed as an old (we call them *conical*) limit. This is the main reason why the theory becomes a real new topic of study only in a genuinely $\mathcal{V}$-enriched setting.

Proposition 4.8 (**Sets**-weighted limits are limits)**.** Let's denote $\mathbf{C}\int W$ the *category of elements* of the weight $W\colon \mathbf{C} \to \mathbf{Sets}$, i.e. the category having objects the pairs $(c, x \in Wc)$, with the obvious choice of arrows. The category $\mathbf{C}\int W$ comes equipped with a canonical (Grothendieck) fibration $\Sigma\colon \mathbf{C}\int W \to \mathbf{C}$, such that for any functor $F\colon \mathbf{C} \to \mathbf{A}$ one has

$$\varprojlim^{W} F \cong \varprojlim_{(c,x)\in\mathbf{C}\int W} F \circ \Sigma. \tag{50}$$

Proof. The proof goes by inspection, using the characterization of the end $\int_{c\in\mathbf{C}} Fc^{Wc}$ as an equalizer (see Proposition **1.23**), and the characterization of **Sets**-cotensors as iterated products, showing that

$$\begin{aligned}
\int_{c\in\mathbf{C}} Fc^{Wc} &\cong \mathrm{eq}\Big(\prod_{c\in\mathbf{C}} Fc^{Wc} \rightrightarrows \prod_{c\to c'} Fc'^{Wc}\Big)\\
&\cong \mathrm{eq}\Big(\prod_{c\in\mathbf{C}}\prod_{x\in Wc} Fc \rightrightarrows \prod_{c\to c'}\prod_{x\in Wc} Fc'\Big)\\
(\star) &\cong \mathrm{eq}\Big(\prod_{(c,x)\in\mathbf{C}\int W} Fc \rightrightarrows \prod_{(c,x)\to(c',x')\in\mathbf{C}\int W} Fc'\Big)\\
&\cong \varprojlim_{(c,x)\in\mathbf{C}\int W} F\circ\Sigma
\end{aligned}$$

(equation $(\star)$ is motivated by the fact that every arrow $\Sigma(c,x) \to c'$ has a unique lift $(c,x) \to (c',x')$). □

Remark 4.9. When we consider Kan extensions as weighted co/limits, this result agrees with the classical theory: if the weight has the form $W = \mathbf{D}(d, K-)$ for an object $d \in \mathbf{D}$, and a functor $K\colon \mathbf{C} \to \mathbf{D}$, then the category of elements $\mathbf{C}\int W$ is precisely the *comma category* $(d \downarrow K)$: the right Kan extension of F along K can be computed as the conical limit of the functor FU, where $U\colon (d \downarrow K) \to \mathbf{C}$ is the obvious forgetful functor.

Obviously, when *every* weighted limit exists in $\mathbf{A}$, we can prove that the correspondence $(W, F) \mapsto \varprojlim^W F$ is a bifunctor:

$$\varprojlim^{-}(=)\colon \big(\mathbf{Sets}^{\mathbf{C}}\big)^{\mathrm{op}} \times \mathbf{A}^{\mathbf{C}} \longrightarrow \mathbf{A}. \tag{51}$$

A number of useful corollaries of this fact:

- The unique, terminal natural transformation $W \to *$ induces a *comparison arrow* between the weighted limit of any $F\colon \mathbf{C} \to \mathbf{A}$ and the classical (conical) limit: $\varprojlim F \to \varprojlim^W F$. For example, the classical limit of the functor $f\colon \Delta[1] \to \mathbf{A}$ described in Example **4.3** consists of the object $a = \mathrm{src}(f)$; hence the comparison arrow consists of the unique factorization of two copies of id_a along the kernel pair of f.
- The functor $\varprojlim^{(-)} F$ is continuous, namely we can prove the suggestive isomorphism
$$\varprojlim^{\left(\varinjlim_J W_j\right)} F \cong \varprojlim_J \big(\varprojlim^{W_j} F\big), \tag{52}$$
valid for any small diagram of weights $J \to [\mathbf{C}, \mathbf{Sets}]\colon j \mapsto W_j$.

Exercise 4.10. Prove Equation (**52**) using the characterization of $\varprojlim^W F \cong \int_c Fc^{Wc}$, plus its universal property.

Example 4.11 (Ends are weighted limits)**.** Formally speaking, ends are weighted limits: given $H\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$ we can take the hom functor $\mathbf{C}(-,=)\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Sets}$ as a weight, and if the weighted limit exists, we have the chain of isomorphisms

$$\varprojlim^{\mathbf{C}(-,=)} H \cong \int_{(c,c') \in \mathbf{C}^{\mathrm{op}} \times \mathbf{C}} H(c,c')^{\mathbf{C}(c,c')} \cong \int_c \Big(\int_{c'} H(c,c')^{\mathbf{C}(c,c')} \Big) \overset{\text{yon}}{\cong} \int_c H(c,c). \tag{53}$$

Exercise 4.12. Who is the category of elements of the hom functor? Compare with Definition **1.19**.

Remark 4.13. Aside from showing that "weighted limits are the true enriched-categorical limits", the above examples and the last characterization of ends as weighted limits are fundamental steps towards a sensible definition of *enriched* ends: given a cosmos $\mathcal{V}$ and a $\mathcal{V}$-functor $H\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathcal{V}$, we *define* $\int_c H(c,c)$ to be the limit of H weighted by $\mathbf{C}(-,=)\colon \mathbf{C}^{\mathrm{op}} \boxtimes \mathbf{C} \to \mathcal{V}$ (see Definition **4.15** for the notation $\mathbf{C} \boxtimes \mathbf{D}$).

Exercise 4.14. Every definition we gave until now can be dualized to obtain a theory of weighted *co*limits: fill in the details.

- (weighted colimits as coends) If $\mathbf{A}$ is cocomplete, we can express the weighted colimit $\varinjlim^W F$ as a coend: more precisely
$$\varinjlim^W F \cong \int^{c \in \mathbf{C}} Wc \cdot Fc \tag{54}$$
where we used, like everywhere else, the $\mathbf{Sets}$-tensoring of $\mathbf{A}$.

- (left Kan extensions as weighted colimits) Let $F\colon \mathbf{C}\to\mathbf{A}$ and $K\colon \mathbf{C}\to\mathbf{D}$ be functors; then

$$\mathrm{Lan}_K F(\bullet) \cong \int^{c\in\mathbf{C}} \mathbf{D}(Kc,\bullet)\cdot Fc \cong \varinjlim^{\mathbf{D}(K-,\bullet)} F \tag{55}$$

- (coends as hom-weighted colimits) The coend of $H\colon \mathbf{C}^{\mathrm{op}}\times\mathbf{C}\to\mathbf{D}$ can be written as $\varinjlim^{\mathbf{C}(-,-)}H$.
- If the weight W is **Sets**-valued, the colimit of F weighted by W can be written as a conical colimit over $\mathbf{C}^{\mathrm{op}}\int W$:

$$\varinjlim^{W} F \cong \varinjlim_{(c,x)\in\mathbf{C}^{\mathrm{op}}\int W} F\Sigma \tag{56}$$

- (functoriality) If the W-colimit of $F\colon \mathbf{C}\to\mathbf{A}$ always exists, then the correspondence $(W,F)\mapsto \varinjlim^{W}F$ is a functor, cocontinuous in its first variable:

$$\begin{aligned}&\varinjlim^{(-)}(=)\colon \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}\times\mathbf{A}^{\mathbf{C}}\longrightarrow\mathbf{A},\\ &\varinjlim^{\left(\varinjlim_J W_j\right)}F \cong \varinjlim_J\left(\varinjlim^{W_j}F\right)\end{aligned} \tag{57}$$

- (comparison) There is a canonical natural transformation $W\to *$, inducing a canonical *comparison arrow* from the W-colimit of any $F\colon \mathbf{C}\to\mathbf{A}$ to the conical colimit.

A fundamental step to write the theory of weighted limits relies upon the above-mentioned isomorphism

$$\mathbf{A}\big(m,\varprojlim^{W}F\big)\cong \mathcal{V}^{\mathbf{C}}(W,\mathbf{A}(m,F(-))) \tag{58}$$

valid in a $\mathcal{V}$-category $\mathbf{A}$ naturally in any object $m\in\mathbf{A}$; this isomorphism has to be interpreted in the base-cosmos $\mathcal{V}$, and this means that we have to find a way to interpret the category $\mathcal{V}^{\mathbf{C}}$ as an object $[\mathbf{C},\mathcal{V}]$ of $\mathcal{V}$: to do this, we must endow $\mathcal{V}$-**Cat** with a closed symmetric monoidal structure, such that

$$\mathcal{V}\text{-Fun}(\mathbf{C}\boxtimes\mathbf{E},\mathbf{D})\cong\mathcal{V}\text{-Fun}(\mathbf{E},[\mathbf{C},\mathbf{D}]). \tag{59}$$

Definition 4.15. Given two $\mathcal{V}$-categories $\mathbf{C},\mathbf{D}$ we define the $\mathcal{V}$-category $\mathbf{C}\boxtimes\mathbf{D}$ having

- as objects the set $\mathbf{C}\times\mathbf{D}$, and
- as $\mathcal{V}$-object of arrows $(c,d)\to(c',d')$ the object

$$\mathbf{C}(c,c')\otimes\mathbf{D}(d,d')\in\mathcal{V}. \tag{60}$$

The terminal category $*$ (or more precisely, the free $\mathcal{V}$-category associated to it) is the unit object for this monoidal structure.

Proposition 4.16. ($\mathcal{V}$-**Cat**, $\boxtimes$) is a closed monoidal structure, with internal hom denoted $[-,=]$.

Proof. Given $\mathbf{C}, \mathbf{D} \in \mathcal{V}\text{-}\mathbf{Cat}$ we define a $\mathcal{V}$-category whose objects are $\mathcal{V}$-functors $F, G\colon \mathbf{C} \to \mathbf{D}$ and where ('abstracting' Theorem **1.29** to the enriched setting) the $\mathcal{V}$-object of natural transformations $F \Rightarrow G$ is defined via the end

$$[\mathbf{C}, \mathbf{D}](F, G) := \int_{c \in \mathbf{C}} \mathbf{D}(Fc, Gc). \tag{61}$$

Recall that in the unenriched case, the end was better understood as the equalizer of a pair of arrows (see Exercise **1.31**):

$$\int_{c \in \mathbf{C}} \mathbf{D}(Fc, Gc) \cong \mathrm{eq}\Big(\prod_{c \in \mathbf{C}} \mathbf{D}(Fc, Gc) \rightrightarrows \prod_{c,c'} \prod_{c \to c'} \mathbf{D}(Fc, Gc') \Big) \tag{62}$$

In the enriched case, we can consider the same symbol, and re-interpret the product $\prod_{\mathbf{C}(c,c')}$ as a suitable *power* in $\mathcal{V}$:

$$\int_{c \in \mathbf{C}} \mathbf{D}(Fc, Gc) \cong \mathrm{eq}\Big(\prod_{c \in \mathbf{C}} \mathbf{D}(Fc, Gc) \rightrightarrows \prod_{c,c'} \mathbf{D}(Fc, Gc')^{\mathbf{C}(c,c')} \Big) \tag{63}$$

(see also [Gra80, §**2.3**], [Dub70] for a more detailed discussion about co/ends in enriched setting.)

It remains to prove, now, that the isomorphism (**59**) holds: this is rather easy, since in the above notations, any functor $F\colon \mathbf{C} \boxtimes \mathbf{E} \to \mathbf{D}$ defines a unique functor $\hat{F}\colon \mathbf{E} \to [\mathbf{C}, \mathbf{D}]$. □

Exercise 4.17. Fill in the details of the above proof; for those who need a hint, show that $\mathbf{E}(e, e') \to \mathbf{D}(F(x, e), F(x, e'))$ is a wedge in $x \in \mathbf{C}$.

The given definition for the enriched end allows us to state an elegant form of the $\mathcal{V}$-enriched Yoneda lemma:

Remark 4.18 ($\mathcal{V}$-Yoneda lemma)**.** Let $\mathbf{D}$ be a small $\mathcal{V}$-category, $d \in \mathbf{D}$ an object, and $F\colon \mathbf{D} \to \mathcal{V}$ a $\mathcal{V}$-functor. Then the canonical map

$$Fd \longrightarrow [\mathbf{D}, \mathcal{V}](\mathbf{D}(d, -), F) \tag{64}$$

induced by the universal property of the involved end[5] is a $\mathcal{V}$-isomorphism.

Definition 4.19 (Enriched weighted co/limit)**.** Given a $\mathcal{V}$-functor $F\colon \mathbf{D} \to \mathbf{A}$ between $\mathcal{V}$-categories and a $\mathcal{V}$-presheaf $W\colon \mathbf{D} \to \mathcal{V}$ called a *weight*, the *limit of F weighted by W* is (provided it exists) the object $\varprojlim^W F$ endowed with a natural $\mathcal{V}$-isomorphism

$$\mathbf{A}\big(a, \varprojlim{}^W F\big) \cong [\mathbf{D}, \mathcal{V}](W, \mathbf{A}(a, F)). \tag{65}$$

Dually, the weighted *co*limit of F by $W\colon \mathbf{D}^{\mathrm{op}} \to \mathcal{V}$ realizes the representation

$$\mathbf{A}\big(\varinjlim{}^W F, a\big) \cong [\mathbf{D}^{\mathrm{op}}, \mathcal{V}](W, \mathbf{A}(F, a)). \tag{66}$$

[5]To be more precise, this arrow is induced by a wedge $\{Fd \to \mathcal{V}(\mathbf{D}(d, a), Fa)\}_{a \in \mathbf{C}}$, to find which it is enough to consider arrows $\mathbf{D}(d, a) \to \mathcal{V}(Fd, Fa)$.

5. Profunctors.

The lucid presentation in the notes [Béner] and in the book [CP08, **§4**] are standard references to follow this section.

Definition 5.1 (The bicategory of profunctors)**.** Define a bicategory **Prof** having

- objects those of **Cat** (small categories $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \dots$);
- 1-cells φ, depicted as arrows $\mathbf{A} \rightsquigarrow \mathbf{B}$, the functors $\mathbf{A}^{\mathrm{op}} \times \mathbf{B} \to \mathbf{Sets}$;
- 2-cells $\alpha\colon \varphi \Rightarrow \psi$ the natural transformations between functors.

Given two contiguous 1-cells $\mathbf{A} \overset{\varphi}{\rightsquigarrow} \mathbf{B} \overset{\psi}{\rightsquigarrow} \mathbf{C}$ we define their *composition* $\psi \diamond \varphi$ as the "matrix product" given by the coend

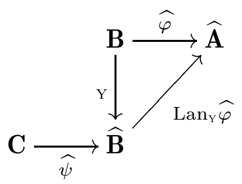
$$\psi \diamond \varphi(a, c) := \int^{x} \varphi(a, x) \times \psi(x, c) \tag{67}$$

The definition works well also with **Sets** replaced by an arbitrary Bénabou cosmos $\mathcal{V}$; the 1-cells of **Prof** are called *profunctors*, or more rarely *distributors* (following the equation funct*ions* : funct*ors* = distribut*ions* : distribut*ors*), *correspondences* (consider the case of $\mathcal{V} = \{0, 1\}$ where $\mathbf{A}, \mathbf{B}$ are discrete categories), *bimodules* (consider the case where $\mathcal{V} = \mathbf{Ab}$ and $\mathbf{A}, \mathbf{B}$ are rings).

Remark 5.2. There is an alternative, but equivalent definition for $\psi \diamond \varphi$ which exploits the universal property of $\widehat{\mathbf{C}}$: any profunctor $\varphi\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ can be identified with its mate under the adjunction giving the cartesian closed structure of **Cat**,

$$\mathrm{Fun}(\mathbf{A}^{\mathrm{op}} \times \mathbf{B}, \mathbf{Sets}) \cong \mathrm{Fun}(\mathbf{B}, [\mathbf{A}^{\mathrm{op}}, \mathbf{Sets}]) \tag{68}$$

i.e. with a functor $\widehat{\varphi}\colon \mathbf{B} \to \widehat{\mathbf{A}}$ obtained as $b \mapsto \varphi(-, b)$. Hence we can define the composition $\mathbf{A} \overset{\varphi}{\rightsquigarrow} \mathbf{B} \overset{\psi}{\rightsquigarrow} \mathbf{C}$ to be $\mathrm{Lan}_{\text{Y}}\widehat{\varphi} \circ \widehat{\psi}$:



(69)

This is equivalent to the previous definition, in view of the characterization of a left Kan extension as a coend in $\widehat{\mathbf{A}}$, given in Equation **31**:

$$\mathrm{Lan}_{\text{Y}}\widehat{\varphi} \cong \int^{b} \widehat{\mathbf{B}}(\text{Y}b, -) \cdot \widehat{\varphi}(b). \tag{70}$$

Since in **Sets** copower coincides with product (i.e. $X \cdot Y \cong X \times Y$, since $\mathbf{Sets}(X \cdot Y, B) \cong \mathbf{Sets}(X, \mathbf{Sets}(Y, B)) \cong \mathbf{Sets}(X \times Y, B)$, naturally in B), we have

$$\begin{aligned}\mathrm{Lan}_{\text{Y}}\hat{\varphi}(\hat{\psi}(c)) &\cong \int^{b} \widehat{\mathbf{B}}(\text{Y}b, \hat{\psi}(c)) \cdot \hat{\varphi}(b) \\ &\cong \int^{b} \hat{\psi}(c)(b) \cdot \hat{\varphi}(b) \cong \int^{B} \varphi(-, b) \times \psi(b, c).\end{aligned}$$

The properties of (strong) associativity and unitality for the composition of profunctors follow directly from the associativity of cartesian product, its cocontinuity as a functor of a fixed variable, and from the ninja Yoneda lemma **2.1**, as shown by the following computation:

- Composition of profunctors is associative (up to isomorphism):

$$\begin{aligned}
\varphi \diamond (\psi \diamond \eta) &= \int^{x} \varphi(b,x) \times (\psi \diamond \eta)(x,a) \\
&= \int^{x} \varphi(b,x) \times \Big(\int^{y} \psi(x,y) \times \eta(y,a) \Big) \\
&\cong \int^{xy} \varphi(b,x) \times \Big(\psi(x,y) \times \eta(y,a) \Big) \\
(\varphi \diamond \psi) \diamond \eta &= \int^{x} (\varphi \diamond \psi)(b,x) \times \eta(x,a) \\
&\cong \int^{xy} \Big(\varphi(b,y) \times \psi(y,x) \Big) \times \eta(x,a)
\end{aligned}$$

  and these results are clearly isomorphic, once we changed name to "integration" variables.
- Any object $\mathbf{A}$ has an identity arrow, given by the "diagonal" profunctor $\mathbf{A}(-,=) = \hom_{\mathbf{A}} \colon \mathbf{A}^{\text{op}} \times \mathbf{A} \to \mathbf{Sets}$: the fact that $\varphi \diamond \hom \cong \varphi$, $\hom \diamond \psi \cong \psi$ simply rewrites the ninja Yoneda lemma.

Definition 5.3 (Einstein notation)**.** There is a useful notation which can be implied to shorten involved computations with coends, and which is particularly evocative when dealing with profunctors; we choose to call it *Einstein convention* for evident graphical reasons[6].

Let $\varphi \colon \mathbf{A} \rightsquigarrow \mathbf{B}$, $\psi \colon \mathbf{B} \rightsquigarrow \mathbf{C}$ be two composable profunctors. If we adopt the notation φ^a_b, ψ^b_c to denote the images $\varphi(a,b), \psi(b,c) \in \mathbf{Sets}$ (keeping track that superscripts are contravariant and subscripts are covariant components), then composition of profunctors acquires more the familiar form of a "matrix product":

$$\varphi \diamond \psi(a,c) = \textstyle\int^{b} \varphi^a_b \times \psi^b_c = \textstyle\int^{b} \varphi^a_b \psi^b_c. \tag{71}$$

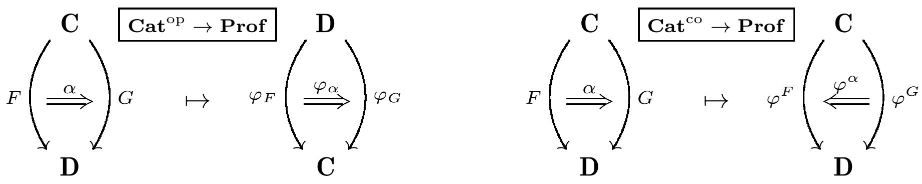
From now on, we feel free to adopt the Einstein summation convention during long calculations.

Remark 5.4 ((Co)presheaves are profunctors)**.** Presheaves on $\mathbf{C}$ obviously correspond to profunctors $\mathbf{C} \rightsquigarrow \mathbf{1}$; copresheaves, i.e. functors $\mathbf{C} \to \mathbf{Sets}$, correspond to profunctors $\mathbf{1} \rightsquigarrow \mathbf{C}$.

5.1. **Embeddings and adjoints.** There are two identity-on-objects embeddings $\mathbf{Cat} \to \mathbf{Prof}$ (respectively the *covariant* and the *contravariant* one, looking at the

[6] This notation has been adopted also in the beautiful [RV13], a valuable reading for several reasons, last but not least the fact that its authors heavily adopt coend-fu to simplify the discussion about "Reedy calculus".

behaviour on 2-cells), and send a diagram in **Cat** respectively to



(72)

(the notation is self-explanatory). This clearly defines a (pseudo)functor, since it's easy to see that

- $\varphi_{FG} \cong \varphi_F \diamond \varphi_G$, and $\varphi^{FG} \cong \varphi^G \diamond \varphi^F$;
- $\varphi_{1_\mathbf{A}} = \mathbf{A}(-,=)$.

Natural transformations $\alpha\colon F \Rightarrow G$ are obviously sent to 2-cells in **Prof**, and the covariancy of this assignment is uniqely determined as in the diagram above.

Remark 5.5. The 1-cells φ_F, φ^F are not independent: they are *adjoint* 1-cells in the bicategory **Prof**. Indeed, for every $F \in \mathbf{Cat}(\mathbf{A},\mathbf{B})$ we can define 2-cells

$$\epsilon = \epsilon_F \colon \varphi_F \diamond \varphi^F \Longrightarrow \mathbf{B}(-,=) \tag{73}$$

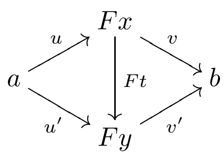
$$\eta = \eta_F \colon \mathbf{A}(-,=) \Longrightarrow \varphi^F \diamond \varphi_F \tag{74}$$

(*counit* and *unit* of the adjunction), whose nature is better understood unraveling the coends involved:

- For what concerns the counit, we write the coend as the quotient set

$$\int^x \mathbf{B}(a,Fx) \times \mathbf{B}(Fx,b) = \Big(\coprod_{x\in\mathbf{A}} \mathbf{B}(a,Fx) \times \mathbf{B}(Fx,b)\Big)/\simeq \tag{75}$$

  where $\simeq$ is the equivalence relation generated by $\big(a \xrightarrow{u} Fx, Fx \xrightarrow{v} b\big) \simeq \big(a \xrightarrow{u'} Fy, Fy \xrightarrow{v'} b\big)$ if there is $t\colon x \to y$ such that $v' = Ft \circ v$ e $Ft \circ u = u'$. This can be visualized as the commutativity of the square



(76)

  Now it's easily seen that sending $\big(a \xrightarrow{u} Fx, Fx \xrightarrow{v} b\big)$ in the composition $v \circ u$ descend to the quotient with respect to $\simeq$, hence $\epsilon\colon \varphi_F \diamond \varphi^F \to \mathbf{B}(-,-)$ is well defined. All boils down to notice that the composition

$$c\colon \mathbf{B}(a,Fx) \times \mathbf{B}(Fx,b) \to \mathbf{B}(a,b) \tag{77}$$

  defines a cowedge in the variable x.
- The unit of the adjunction is the 2-cell

$$\eta\colon \mathbf{A}(-,=) \Longrightarrow \varphi^F \diamond \varphi_F \tag{78}$$

obtained when we noticed that $\varphi^F \diamond \varphi_F(a,b) = \int^X \mathbf{B}(Fa,x)\times\mathbf{B}(x,Fb) \cong \mathbf{B}(Fa,Fb)$ (as a consequence of the ninja Yoneda lemma), is simply determined by the action of F on arrows, $\mathbf{A}(a,b)\to\mathbf{B}(Fa,Fb)$.

We now have to verify that the zig-zag identities (see [Bor94a, Thm **3.1.5.(2)**]) hold:

$$(\varphi^F \diamond \epsilon)\circ(\eta\diamond\varphi^F) = 1_{\varphi^F}$$
$$(\epsilon\diamond\varphi_F)\circ(\varphi_F\diamond\eta) = 1_{\varphi_F}$$

As for the first, we must verify that the diagram

$$\begin{array}{ccccc} \varphi^F & \xrightarrow{\sim} & \mathbf{B}(-,-)\diamond\varphi^F & \xrightarrow{\eta\diamond\varphi^F} & (\varphi^F\diamond\varphi_F)\diamond\varphi^F \\ \| & & & & \downarrow\cong \\ \varphi^F & \xleftarrow[\sim]{} & \varphi^F\diamond\mathbf{B}(-,-) & \xleftarrow[\varphi^F\diamond\epsilon]{} & \varphi^F\diamond(\varphi_F\diamond\varphi^F) \end{array} \tag{79}$$

commutes. One has to send $h\in\varphi^F(u,v)=\mathbf{B}(Fu,v)$ in the class $[(1_u,h)]\in \int^X \mathbf{B}(u,x)\times\mathbf{B}(Fx,v)$, which must go under $\eta\diamond\varphi^F$ in the class $[(F(1_u),h)]\in \int^{xy}\mathbf{B}(Fa,x)\times\mathbf{B}(x,Fy)\times\mathbf{B}(Fy,b)$, canonically identified with $\int^y \mathbf{B}(Fa,Fy)\times\mathbf{B}(Fy,b)$. Now $\varphi^F\diamond\epsilon$ acts composing the two arrows, and one obtains $F(1_A)\circ h = h$ back.

Similarly, to prove the second identity, the diagram

$$\begin{array}{ccccc} \varphi_F & \xrightarrow{\sim} & \varphi_F\diamond\mathbf{B}(-,-) & \xrightarrow{\varphi_F\diamond\eta} & \varphi_F\diamond(\varphi^F\diamond\varphi_F) \\ \| & & & & \downarrow\cong \\ \varphi_F & \xleftarrow[\sim]{} & \mathbf{B}(-,-)\diamond\varphi_F & \xleftarrow[\epsilon\diamond\varphi_F]{} & (\varphi_F\diamond\varphi^F)\diamond\varphi_F \end{array} \tag{80}$$

must commute (all the unlabeled isomorphisms are the canonical ones). This translates into

$$\Big(a\xrightarrow{u}Fb\Big)\longmapsto (u,1_b)_\sim \longmapsto (u,F(1_b))\longmapsto u\circ F(1_b)=u, \tag{81}$$

which is what we want; hence $\varphi_F \dashv \varphi^F$. □

Remark 5.6. Two functors $F\colon \mathbf{A}\leftrightarrows\mathbf{B}\colon G$ happen to be mutually adjoint iff $\varphi_F\cong\varphi^G$ (and hence $G\dashv F$) or $\varphi_G\cong\varphi^F$ (and hence $F\dashv G$).

Remark 5.7. It is a well-known fact (see [Bor94a, dual of Prop. **3.4.1**]) that if $F\dashv G$, then F is fully faithful if and only if the unit of the adjunction $\eta\colon 1\to GF$ is an isomorphism.

This criterion can be extended also to functors which do not admit a "real" right adjoint, once noticed that F is fully faithful if and only if $\mathbf{A}(a,b)\cong\mathbf{B}(Fa,Fb)$ for any two $a,b\in\mathbf{A}$, i.e. if and only if the unit $\eta\colon \hom_{\mathbf{A}}\Rightarrow\varphi^F\diamond\varphi_F$ is an isomorphism.

Example 5.8. Given a profunctor $\varphi\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ and a functor $F\colon \mathbf{B} \to \mathbf{D}$ we can define $F \otimes \varphi$ to be the functor $\mathbf{A} \to \mathbf{D}$ given by $\mathrm{Lan}_y F \circ \widehat{\varphi}$, where $\widehat{\varphi}\colon \mathbf{B} \to \widehat{\mathbf{A}}$ is the adjunct of φ. More explicitly,

$$F \otimes \varphi(a) = \int^b \mathsf{Nat}(y_b, \varphi(-,a)) \cdot Fb \cong \int^b \varphi_a^b \cdot F_b$$

Exploiting this definition, several things can be proved via coend-fu:

- $F \otimes \hom \cong F$ as a consequence of the ninja Yoneda lemma;
- If $\mathbf{C} \overset{\psi}{\rightsquigarrow} \mathbf{A} \overset{\varphi}{\rightsquigarrow} \mathbf{B} \xrightarrow{F} \mathbf{X}$, then $F \otimes (\varphi \diamond \psi) \cong (F \otimes \varphi) \otimes \psi$: indeed

$$\begin{aligned} F \otimes (\varphi \diamond \psi)a &= \textstyle\int^b (\varphi \diamond \psi)_a^b \times F_b \\ &\cong \textstyle\int^{bx} \varphi_x^b \times \psi_a^x \times F_b \\ &\cong \textstyle\int^x \psi_a^x \times \Big(\int^b \varphi_x^b \times F_b \Big) \\ &\cong \textstyle\int^x \psi_a^x \times (F \otimes \varphi)_x = ((F \otimes \varphi) \otimes \psi)a \end{aligned}$$

Example 5.9 (Kan extensions in **Prof**)**.** (see [CP08, p. **94**–]) Any profunctor has a "right Kan extension" in the sense the notion has in any bicategory, replacing the composition of functors with the composition of 1-cells, and natural transformations with 2-cells.

One has the following chain of isomorphisms in **Prof** (see Definition **5.3** for the Einstein convention):

$$\begin{aligned} \mathrm{Nat}(\gamma \diamond \varphi, \eta) &\cong \textstyle\int_{ab} \mathbf{Sets}\big((\gamma \diamond \varphi)_b^a, \eta_b^a\big) \\ &\cong \textstyle\int_{ab} \mathbf{Sets}\Big(\int^x \gamma_x^a \times \varphi_b^x, \eta_b^a \Big) \\ &\cong \textstyle\int_{abx} \mathbf{Sets}\big(\gamma_x^a, \mathbf{Sets}(\varphi_b^x, \eta_b^a)\big) \\ &\cong \textstyle\int_{ax} \mathbf{Sets}\Big(\gamma_x^a, \int_b \mathbf{Sets}(\varphi_b^x, \eta_b^a)\Big) \\ &\cong \textstyle\int_{ax} \mathbf{Sets}\Big(\gamma_x^a, \mathrm{Ran}_\varphi \eta_x^a\Big) \\ &\cong \mathrm{Nat}(\gamma, \mathrm{Ran}_\varphi \eta) \end{aligned}$$

when we define $\mathrm{Ran}_\varphi \eta(a,x)$ to be $\mathrm{Nat}\big(\varphi(x,-), \eta(a,-)\big)$.

5.2. **The Yoneda structure on Cat and Prof.** Before we begin, we must familiarize with *Kan liftings*: in the same way a (global) Kan extension consists in an adjoint to the inverse image functor p^* (see Def **2.6**), a (global) Kan *lifting* consists of an adjoint to the *direct* image functor p_*. The definition is best appreciated in a general bicategory.

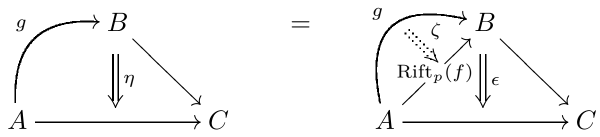
Definition 5.10 (Kan liftings)**.** Let $\mathcal{K}$ be a bicategory. Given 1-cells $p\colon B \to C$, $f\colon A \to C$ in $\mathcal{K}$, a *right Kan lift* of f through p, denoted $\mathrm{Rift}_p f$, is a 1-cell $\mathrm{Rift}_p(f)\colon A \to B$ equipped with a 2-cell

$$\varepsilon : p \circ \mathrm{Rift}_p(f) \Rightarrow f \tag{82}$$

satisfying the following universal property: given any pair $(g\colon A \to B, \eta\colon p \circ g \Rightarrow f)$, there exists a unique 2-cell

$$\zeta : g \Rightarrow \mathrm{Rift}_p(f) \tag{83}$$

such that the following diagram of 2-cells commutes for a unique $\zeta\colon g \Rightarrow \mathrm{Rift}_p(f)$

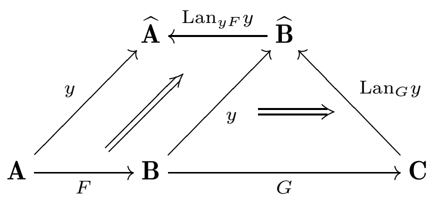


(84)

i.e. there is a unique factorization $\varepsilon \circ (p * \zeta) = \eta$. Dually, there exists a notion of *left* Kan lift of f through p, with all 2-cells going in the opposite direction, called $\mathrm{Lift}_p f$.

Now consider the diagram $\widehat{\mathbf{A}} \leftarrow \mathbf{A} \to \mathbf{B}$, where $F\colon \mathbf{A} \to \mathbf{B}$ is a functor to a cocomplete category, so that the nerve-realization paradigm (see Proposition **3.2**) applies, and $y\colon \mathbf{A} \to \widehat{\mathbf{A}}$ the Yoneda embedding. Then we can show that:

- **i**) $\mathrm{Lan}_y F \dashv \mathrm{Lan}_F y$ (another form of the NR-paradigm);
- **ii**) $F(-) \cong \mathrm{Lift}_{\mathbf{B}(F-,=)} y$ where "Lift" is the left Kan lifting of y with respect to $\mathbf{B}(F-,=)$;
- **iii**) $\mathrm{Lan}_y y \cong 1_{\widehat{\mathbf{A}}}$ (i.e., the Yoneda embedding is *dense*);
- **iv**) In the diagram



(85)

filled by canonical 2-cells, the arrow $\mathrm{Lan}_{yF} y \circ \mathrm{Lan}_G y$ coincides with $\mathrm{Lan}_{GF} y (\cong \mathrm{Lan}_G \mathrm{Lan}_F y)$.

Proof. Let's begin by showing property (i). We have to prove that $\mathrm{Lan}_F y \cong (B \mapsto \mathbf{B}(F-, B))$ (by uniqueness of adjoints). This is immediate if we write

$$\begin{aligned} \mathrm{Lan}_F y(b) &\cong \int^a \mathbf{B}(Fa, b) \cdot y(a) \\ &\cong \int^a \mathbf{B}(Fa, b) \times \mathbf{A}(-, a) \\ &\cong \mathbf{B}(F-, b) \end{aligned}$$

where the first isomorphism recognizes the **Sets**-tensoring of **Sets** as the cartesian product, and the second is motivated by the Ninja Yoneda lemma.

To show property (ii), we have to prove that using N_F as a shorthand for the functor $\mathrm{Lan}_F y = \mathbf{B}(F-,=)$, weh ave that $\mathrm{Lift}_{N_F} \dashv N_{F,*}$, where $N_{F,*}\colon P \mapsto N_F \circ$

$P$for any $P\colon \mathbf{A} \to B$. We have that

$$\begin{aligned}[\mathbf{A}, \widehat{\mathbf{A}}]\big(y, N_F \circ P\big) &\cong \int_{a'} \widehat{\mathbf{A}}\big(y_{a'}, N_F \circ P(a')\big)\\ &\cong \int_{a'} \widehat{\mathbf{A}}\big(y_{a'}, \mathbf{B}(F-, Pa')\big)\\ &\cong \int_{a'} \mathbf{B}(Fa', Pa')\\ &\cong [\mathbf{A}, \mathbf{B}](F, P)\end{aligned}$$

by the very definition of N_F and by Proposition **1.29**.

Property (iii) again follows from the coend-form for the left Kan extension, and from the ninja Yoneda lemma.

At this point property (iv) can be left as an exercise for the reader: it's enough to check that the functor $c \mapsto \Big(a \mapsto \mathrm{Nat}\big(y_{\mathbf{B},Fa}, \mathrm{Lan}_G y_{\mathbf{B}}(c)\big)\Big)$ corresponds, by Yoneda lemma, to $\mathrm{Lan}_G y_{\mathbf{B}}(c)(Fa)$, namely to $\mathbf{B}(GFa, c)$, namely to $N_{GF}(c)(a)$[7]. □

The above arguments sketches the proof that **Cat** is endowed with a canonical choice of a *Yoneda structure* in the sense of [SW78].

Exercise 5.11. Given profunctors $\kappa\colon \mathbf{C} \rightsquigarrow \mathbf{D}$ and $\lambda\colon \mathbf{C} \rightsquigarrow \mathbf{E}$ define

$$\kappa \rhd L = \int_c [\kappa(c, -), \lambda(c, =)]$$

Show that this operation behaves like a Kan lifting (of λ along κ); dually, given $\eta\colon \mathbf{D} \rightsquigarrow \mathbf{A}$ and $\lambda\colon \mathbf{E} \rightsquigarrow \mathbf{A}$ we can define

$$\lambda \lhd \eta = \int_a [\eta(=, a), \lambda(-, a)]. \tag{86}$$

Show that this second operation "behaves like a Kan extension" (vagueness is intended to be fixed as part of the exercise), and that these two operations "behave like an action" on $(\mathbf{Prof}, \diamond, \mathrm{hom})$ on the bicategory **Prof**:

- **i**) $(\kappa \diamond \eta) \rhd \lambda \cong \kappa \rhd (\eta \rhd \lambda)$;
- **ii**) $\lambda \lhd (\kappa \diamond \eta) \cong (\lambda \lhd \kappa) \lhd \eta$;
- **iii**) $\mathrm{hom} \rhd \lambda \cong \lambda \cong \lambda \lhd \mathrm{hom}$

6. Operads.

It has been known since the beginning that an *operad*, as defined by P. May in his [MM72], must be interpreted as a monoid in a suitable category of functors; making this analogy a precise statement, using the power of coend-fu, is precisely the content of Kelly's [Kel05], which we now follow almost *verbatim*.

A certain acquaintance with the machinery of operads is a fundamental prerequisite to follow the discussion; unfortunately, given the plethora of different interpretation of the theory, and different areas of Mathematics where the notion of operad arises, the beginners (one of which is the author of the present note) may

[7] The reader puzzled by the multiplication of variables on which these functors depend could be relieved by a moderate use of λ-notation.

eel rather disoriented when approaching any text on the subject, so it's extremely difficult to advise a single, comprehensive reference.

Among classical textbooks, we can't help but mention May's book [MM72], as well as more recent monographies like [LV12, MSS07] written respectively with an algebraic and a geometric taste. Among less classical and yet extremely valid points of view, the author profited a lot from a lucid, and unfortunately still unfinished, online draft [Tri] written by T. Trimble.

6.1. **Local conventions.** Along the whole section we will adopt the following notation and conventions:

- $\mathbf{P}$ is the *groupoid of natural numbers*, i.e. the category having objects the nonempty sets $[n]$ (denoted as n for short) $\{1, \dots, n\}$ and where $\mathbf{P}(m, n) = \varnothing$ if $n \neq m$ and $\mathrm{Sym}(n)$ (the group of bijections of n elements) if $n = m$.
- $\mathcal{V}$ is a fixed *Bénabou cosmos* (i.e. a bicomplete closed symmetric monoidal category, "a good setting to do enriched category theory").

Notice that $\mathbf{P}$ is symmetric monoidal, with respect to the sum of natural numbers; the action on arrows is given by $(\sigma, \tau) \mapsto \sigma + \tau$ defined acting as σ on the set $\{1, \dots, m\}$ and as τ on the set $\{m + 1, \dots, m + n\}$.

6.2. **Convolution product.** We begin presenting a general result about monoidal categories, outlined by B. Day: in the same way the set of "regular functions" on a topological group G acquires a *convolution* product given by $(f, g)(x) = \int_G f(xy^{-1})g(y)dy$ (the integral sign, here and in no other places, is not an end), we can endow the *category* of functors $F\colon \mathbf{C} \to \mathcal{V}$ with a monoidal structure different from the pointwise one, induced by the monoidality of $\mathcal{V}$.

This is called the *convolution product* of functors; appendix **A** will give a generalization of this point of view in form of exercises (see in particular Exercises **A.4**, **A.5**).

Definition 6.1 (Day convolution)**.** If $\mathbf{C}$ is a symmetric monoidal category, then the functor category $[\mathbf{C}, \mathcal{V}]$ is itself a Bénabou cosmos with respect to the monoidal structure given by *Day convolution product*: given $F, G \in [\mathbf{C}, \mathcal{V}]$ we define

$$F * G := \int^{cd} \mathbf{C}(c \otimes d, -) \cdot Fc \otimes Gd = \int^{cd} \mathbf{C}^{c\otimes d}_{(-)} F_c G_d \tag{87}$$

where we recall that $X \cdot V$ for $X \in \mathbf{Sets}, V \in \mathcal{V}$ is the *copower* (or *tensor*) $X \cdot V$ such that

$$\mathcal{V}(X \cdot V, W) \cong \mathbf{Sets}(X, \mathcal{V}(V, W)). \tag{88}$$

Proof. We have to show that this really defines a monoidal structure:

- Associativity follows from the associativity of the tensor product on $\mathbf{C}$ and the ninja Yoneda lemma (see Definition **5.3** for the Einstein convention; it is also harmless to suppress the distinction between monoidal products in

$\mathcal{V}$ and **Sets**-tensors):

$$\begin{aligned}[F*(G*H)]_x &= \textstyle\int^{ab} \mathbf{C}_x^{a\otimes b} F_a (G*H)_b \\ &\cong \textstyle\int^{ab} \mathbf{C}_x^{a\otimes b} \int^{cd} \mathbf{C}_b^{c\otimes d} F_a G_c H_d \\ &\cong \textstyle\int^{abcd} \mathbf{C}_x^{a\otimes b} \mathbf{C}_b^{c\otimes d} F_a G_c H_d \\ &\cong \textstyle\int^{acd} \mathbf{C}_x^{a\otimes (c\otimes d)} F_a G_c H_d \\ [(F*G)*H]_x &\cong \textstyle\int^{acd} \mathbf{C}_x^{(a\otimes c)\otimes d} F_a G_c H_d.\end{aligned}$$

- (Right) unitality : choose $J = よ_I = \mathbf{C}(I, -)$ and notice that the ninja Yoneda lemma implies that

$$\begin{aligned}[F*J]_x &\cong \textstyle\int^{cd} \mathbf{C}_x^{c\otimes d} F_c J_d \\ &\cong \textstyle\int^{cd} \mathbf{C}_x^{c\otimes d} \mathbf{C}_d^{I} F_c \\ &\cong \textstyle\int^{c} \mathbf{C}_x^{c\otimes I} F_c \cong F_x.\end{aligned}$$

- Left unitality is totally analogous.

The category $[\mathbf{C}, \mathcal{V}]$ becomes a Bénabou cosmos whose internal hom with respect to the convolution product is given by

$$[\![G, H]\!] := \int_c [Gc, H(c \otimes -)] \tag{89}$$

where $[-, =]$ is the internal hom in $\mathcal{V}$. Indeed, we can compute directly that

$$\begin{aligned}\mathrm{Nat}(F*G, H) &\cong \textstyle\int_c \mathcal{V}\big((F*G)c, Hc\big) \cong \int_c \mathcal{V}\left(\int^{ab} \mathbf{C}_c^{a\otimes b} F_a G_b,\ H_c\right) \\ &\cong \textstyle\int_{abc} \mathcal{V}\big(\mathbf{C}_c^{a\otimes b} F_a G_b,\ H_c\big) \cong \int_{abc} \mathcal{V}\big(F_a, [\mathbf{C}_c^{a\otimes b} G_b, H_c]\big) \\ &\cong \textstyle\int_{abc} \mathcal{V}\big(F_a, \big[G_b, [\mathbf{C}_c^{a\otimes b}, H_c]\big]\big) \cong \int_{ab} \mathcal{V}\big(F_a, \big[G_b, \int_c [\mathbf{C}_c^{a\otimes b}, H_c]\big]\big) \\ &\cong \textstyle\int_{ab} \mathcal{V}\big(F_a, [G_b, H_{a\otimes b}]\big) \cong \int_a \mathcal{V}\left(F_a, \int_b [G_b, H_{a\otimes b}]\right) \\ &\cong \textstyle\int_a \mathcal{V}(F_a, [\![G, H]\!]_a) \\ &\cong \mathrm{Nat}\big(F, [\![G, H]\!]\big).\end{aligned}$$

□

Remark 6.2. In the particular case $\mathbf{C} = \mathbf{P}$, this means that $[\mathbf{P}, \mathcal{V}]$ is a Bénabou cosmos if we define

$$\begin{cases}(F*G)_k := \int^{mn} \mathbf{P}(m+n, k) \otimes Fm \otimes Gn \\ [\![F, G]\!]_k := \int_n [Fn, G(n+k)]\end{cases} \tag{90}$$

In particular, we have the formula

$$F_1 * \cdots * F_n = \int^{k_1, \dots, k_n} \mathbf{P}\Big(\sum k_i, -\Big) \otimes F_1 k_1 \otimes \cdots \otimes F_n k_n. \tag{91}$$

for the convolution of $F_1, \dots, F_n \in [\mathbf{C}, \mathcal{V}]$, which will become useful along the discussion.

Now we define two functors

$$\begin{aligned} \Phi\colon [\mathbf{P},\mathcal{V}] \to \mathcal{V} &\qquad \text{(evaluation at 0)} \\ \Psi\colon \mathcal{V} \to [\mathbf{P},\mathcal{V}] &\qquad \text{(the left adjoint to } \Phi) \end{aligned}$$

Exercise 6.3. Prove that $\Psi(a \otimes b) \cong \Psi a \otimes \Psi b$ and $\Phi \circ \Psi \cong 1$; finally, if $a \in \mathcal{V}$ is identified with the constant functor in a, then $[\mathbf{P},\mathcal{V}](a * F, G) \cong \mathcal{V}(a, [F,G])$, where

$$[F,G] := \int_n \mathcal{V}(Fn, Gn). \tag{92}$$

The gist of the definition of a $\mathcal{V}$-operad lies in an additional monoidal structure on $[\mathbf{P},\mathcal{V}]$, defined by means of the Day convolution:

Definition 6.4 (Diamond product on $[\mathbf{P},\mathcal{V}]$)**.** Let $F, G \in [\mathbf{P},\mathcal{V}]$. Define

$$F \diamond G := \int^m Fm \otimes G^{*m}, \tag{93}$$

where $G^{*m} := G * \cdots * G$.

Associativity exploits the following

Lemma 6.5. There exists a natural equivalence $(F \diamond G)^{*m} \cong F^{*m} \diamond G$.

Proof. Calculemus:

$$\begin{aligned}
(F \diamond G)^{*m} &= \int^{\vec{n}_i} \mathbf{P}\Big(\sum n_i, -\Big) \otimes (F \diamond G)n_1 \otimes \cdots \otimes (F \diamond G)n_m \\
&\cong \int^{\vec{n}_i, k_i} \mathbf{P}\Big(\sum n_i, -\Big) \otimes Fk_1 \otimes G^{*k_1}n_1 \otimes \cdots \otimes Fk_m \otimes G^{*k_m}n_m \\
&\cong \int^{\vec{n}_i, \vec{k}_i} Fk_1 \otimes \cdots \otimes Fk_m \otimes \mathbf{P}\Big(\sum n_i, -\Big) \otimes G^{*k_1}n_1 \otimes \dots G^{*k_m}n_m \\
&\cong \int^{\vec{k}_i} Fk_1 \otimes \cdots \otimes Fk_m \otimes \big(G^{*k_1} * \cdots * G^{*k_m}\big) \\
&\cong \int^{\vec{k}_i} Fk_1 \otimes \cdots \otimes Fk_m \otimes G^{* \sum k_i} \\
\text{Ninja} &\cong \int^{\vec{k}_i, r} \mathbf{P}\Big(\sum k_i, r\Big) \otimes Fk_1 \otimes \cdots \otimes Fk_m \otimes G^{*r} \\
&\cong \int^r F^{*m}r * Gr = F^{*m} \diamond G
\end{aligned}$$

(we used a compact notation for $\int^{\vec{n}_i} = \int^{n_1,\dots,n_m}$; the Ninja Yoneda Lemma is used in the form $G^{*n} \cong \int^r \mathbf{P}(n,t) \otimes Gt = \mathbf{P}(n,-) \diamond G$, because $(n, G) \mapsto G^{*n}$ is a bifunctor). □

Associativity of the diamond product now follows at once: we have

$$\begin{aligned}(F\diamond(G\diamond H))(k) &= \int^m Fm\otimes(G\diamond H)^{*m}k\\ &\cong \int^m Fm\otimes(G^{*m}\diamond H)k\\ &\cong \int^{m,l} Fm\otimes G^{*m}l\otimes H^{*l}k\\ &\cong \int^l (F\diamond G)l\otimes H^{*l}k\\ &= ((F\diamond G)\diamond H)(k).\end{aligned}$$

A unit object for the $\diamond$-product is $J = \mathbf{P}(1,-)\otimes I$; indeed $J(1) = I$, $J(n) = \varnothing_{\mathcal{V}}$ for any $n \neq 1$ and the ninja Yoneda lemma applies on both sides to show unitality rules:

- On the left one has
$$J\diamond F = \int^m Jm\otimes F^{*m} = \int^m \mathbf{P}(1,m)\otimes F^{*m}\cong F^{*1} = F. \tag{94}$$
- On the right, $G\diamond J\cong G$ once noticed that $J^{*m}\cong\mathbf{P}(m,-)\otimes I$ since
$$\begin{aligned}J^{*m} &= \int^{\vec{n}_i}\mathbf{P}\Big(\sum n_i,-\Big)\otimes\mathbf{P}(1,n_1)\otimes\cdots\otimes\mathbf{P}(1,n_m)\otimes I\\ \text{Ninja}&\cong \mathbf{P}(1+\cdots+1,-)\otimes I = \mathbf{P}(m,-)\otimes I\end{aligned}$$
because
$$\int^{\vec{n}_i}\mathbf{P}(n_1+\cdots+n_m,-)\otimes\mathbf{P}(1,n_i)\cong\mathbf{P}(n_1+\cdots+n_{i-1}+1+n_{i+1}+\cdots+n_m,-), \tag{95}$$
for any $1\le i\le m$ (it is again an instance of the ninja Yoneda Lemma). One has
$$G\diamond J = \int^m Gm\otimes J^{*m}\cong\int^m Gm\otimes\mathbf{P}(m,-)\otimes I\cong G. \tag{96}$$

Theorem 6.6. The $\diamond$-monoidal structure is left closed, but not right closed.

Proof. Calculemus:

$$\begin{aligned}\mathsf{Nat}(F\diamond G,H) &\cong \mathsf{Nat}\Big(\int^m Fm\otimes G^{*m},H\Big)\\ &\cong \int_k \mathcal{V}\Big(\int^m Fm\otimes G^{*m},H\Big)\\ &\cong \int_{km}\mathcal{V}(Fm,[G^{*m}k,Hk])\\ &\cong \int_m \mathcal{V}(Fm,\int_k[G^{*m}k,Hk])\\ &\cong \mathsf{Nat}(F,\{G,H\})\end{aligned}$$

if we define $\{G, H\}m = \int_k [G^{*m}k, Hk]$. Hence the functor $- \diamond G$ has a right adjoint for any G.

The functor $F \diamond -$ can't have such an adjoint: why? (Incidentally, this shows also that the diamond product can't come from a convolution product with respect to a *promonoidal structure* in the sense of Proposition **A.3**. Re-read this result after having gone through all the exercises in Appendix **A**!). □

Definition 6.7. An *operad* in $\mathcal{V}$ consists of a monoid in the monoidal category $[\mathbf{P}, \mathcal{V}]$, with respect to the diamond structure $(\diamond, \{-,=\})$, i.e. of a functor $T \in [\mathbf{P}, \mathcal{V}]$ endowed with a natural transformation called *multiplication*, $\mu\colon T \diamond T \to T$ and a *unit* $\eta\colon J \to T$ such that

$$\begin{array}{ccc} T \diamond T \diamond T & \xrightarrow{T\diamond\mu} & T \diamond T \\ {\scriptstyle \mu\diamond T}\downarrow & & \downarrow{\scriptstyle \mu} \\ T \diamond T & \xrightarrow[\mu]{} & T \end{array} \qquad \begin{array}{ccccc} J \diamond T & \xrightarrow{\eta\diamond T} & T \diamond T & \xleftarrow{T\diamond\eta} & T \diamond J \\ & {\scriptstyle\sim}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle\sim} & \\ & & T & & \end{array} \tag{97}$$

are commutative diagrams.

Definition 6.8 (Endomorphism operad)**.** For any $F \in [\mathbf{P}, \mathcal{V}]$ the object $\{F, F\}$ is an operad whose multiplication is the adjunct of the arrow

$$\{F, F\} \diamond \{F, F\} \diamond F \xrightarrow{1\diamond \mathrm{ev}} \{F, F\} \diamond F \xrightarrow{\mathrm{ev}} F \tag{98}$$

and whose unit is the adjunct of the isomorphism $J \diamond F \cong F$.

Unraveling the previous definition, we can notice that an operad in $\mathcal{V}$ consists of

- Giving a natural transformation $\eta\colon J \to T$ amounts to a map $\eta_1 : I \to T(1)$, since $J(1) = I, J(n) = \varnothing$ for $n \neq 1$;
- Giving a natural transformation $\mu\colon T \diamond T \to T$, in view of the universal property of the two coends involved, amounts to give maps

$$\tau\colon Tm \otimes \mathbf{P}(n_1 + \cdots + n_m, k) \otimes Tn_1 \otimes \cdots \otimes Tn_m \longrightarrow Tk \tag{99}$$

  for any $m, n_1, \ldots, n_m, k \in \mathbb{N}$, natural in k and the n_i and such that $\tau(T\sigma \otimes 1 \otimes 1) = \tau(1 \otimes \mathbf{P}(\sigma, 1) \otimes \sigma)$ for every morphism $\sigma \in \mathbf{P}$. This is equivalent to giving a transformation

$$\tau\colon Tm \otimes Tn_1 \otimes \cdots \otimes Tn_m \longrightarrow [\mathbf{P}(n_1 + \cdots + n_m, -), T(-)] \tag{100}$$

  (considering the n_i fixed and the first functor constant in k) i.e., by the Yoneda Lemma a natural transformation

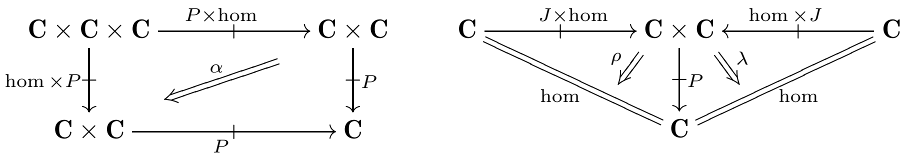
$$\tau\colon Tm \otimes Tn_1 \otimes \cdots \otimes Tn_m \longrightarrow T(n_1 + \cdots + n_m). \tag{101}$$

This concludes the discussion.

Appendix A. Promonoidal categories

A *promonoidal category* is what you obtain when you take the definition of a monoidal category and you replace every occurrence of the word *functor* with the word *profunctor*. More precisely, we define the following

Definition A.1 (Promonoidal structure on a category)**.** A *promonoidal category* consists of a category $\mathbf{C}$ which is a monoid object in $\mathbf{Prof}$, the category of profunctors defined at **??**. More explicitly, we are dealing with a category $\mathbf{C}$, endowed with a bi-profunctor $P\colon \mathbf{C}\times\mathbf{C} \rightsquigarrow \mathbf{C}$ and a profunctor $J\colon 1 \rightsquigarrow \mathbf{C}$, such that the following two diagrams are filled by the indicated 2-cells (respectively, the *associator* and *left/right unitor*) in $\mathbf{Prof}$:



(102)

These data are said to be a *promonoidal structure* on the category $\mathbf{C}$, and they are denoted as $\mathfrak{P} = (P, J, \alpha, \rho, \lambda)$.

Remark A.2. Coend calculus allows us to turn the conditions

$$P \diamond (P \times \hom) \cong P \diamond (\hom \times P) \tag{103}$$

$$P \diamond (J \times \hom) \cong \hom \cong P \diamond (\hom \times J) \tag{104}$$

giving the associativity and unit of the promonoidal structure into explicit relations involving the functors $P\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Sets}$ and $J\colon \mathbf{C} \to \mathbf{Sets}$; in particular we have the following rules, written in Einstein notation to save some space.

- The associativity condition for $P\colon \mathbf{C}\times\mathbf{C} \rightsquigarrow \mathbf{C}$ amounts to saying that the following boxed sets, obtained as coends, are naturally isomorphic (via a natural transformation $\alpha_{abc;d}$ having four components, three contravariant and one covariant.

$$\begin{aligned} P \diamond (\hom \times P) &= \int^y \int^z P^{yz}_a H^b_y P^{cd}_z \\ &\cong \int^z \Big(\int^y P^{yz}_a H^b_y \Big) P^{cd}_z \\ &\cong \boxed{\int^z P^{bz}_a P^{cd}_z} \\ P \diamond (P \times \hom) &\cong \int^y \int^z P^{yz}_d H^c_y P^{ab}_z \\ &\cong \int^z \Big(\int^y P^{yz}_d H^c_y \Big) P^{ab}_z \\ &\cong \boxed{\int^z P^{cz}_d P^{ab}_z}. \end{aligned}$$

- The left unit axiom is equivalent to the isomoprhism

$$(a,b) \mapsto \int^{yz} J_z H^a_y P^{yz}_b \int^z J_z \Big(\int^y H^a_y P^{yz}_b \Big) \cong \int^z J_z P^{az}_b \cong \hom(a,b) \tag{105}$$

The most interesting feature of (strong) promonoidal structure in categories is that they correspond bijectively with convolution structures on the category of functors $[\mathbf{C}, \mathcal{V}]$, heavily generalizing the Day convolution product of Definition **6.1**.

Proposition A.3. Let $\mathfrak{P} = (P, J, \alpha, \rho, \lambda)$ be a promonoidal structure on the category $\mathbf{C}$; then we can define a $\mathfrak{P}$-*convolution* on the category $[\mathbf{C}, \mathbf{Sets}]$ (or more generally, on the category $[\mathbf{C}, \mathcal{V}]$), via

$$[F *_{\mathfrak{P}} G]_c = \int^{ab} P(a, b; c) \times Fa \times Gb \tag{106}$$

$$J_{\mathfrak{P}} = J \tag{107}$$

Exercise A.4. Prove the above statement using associativity and unitality for $\mathfrak{P}$.

Exercise A.5 (Day and Cauchy convolutions)**.** Outline the promonoidal structure $\mathfrak{P}$ giving the Day convolution described in Definition **6.1**. If $\mathbf{C}$ is any small category, we define $P(a, b; c) = \mathbf{C}(a, c) \times \mathbf{C}(b, c)$ and J to be the terminal functor $\mathbf{C} \to \mathbf{Sets}$. Outline the convolution product on $[\mathbf{C}, \mathbf{Sets}]$, called the *Cauchy convolution*, obtained from this promonoidal structure.

Definition A.6. A functor $\Phi\colon [\mathbf{A}, \mathbf{Sets}]_{\mathfrak{P}} \to [\mathbf{B}, \mathbf{Sets}]_{\mathfrak{Q}}$ is said to *preserve the convolution product* if the obvious isomorphisms hold in $[\mathbf{B}, \mathbf{Sets}]_{\mathfrak{Q}}$:

- $\Phi(F *_{\mathfrak{P}} G) \cong \Phi(F) *_{\mathfrak{Q}} \Phi(G)$;
- $\Phi(J_{\mathfrak{P}}) = J_{\mathfrak{Q}}$.

Appendix B. Fourier transforms via coends.

Definition B.1. Let $\mathbf{A}, \mathbf{C}$ be two promonoidal categories; a *multiplicative kernel* from $\mathbf{A}$ to $\mathbf{C}$ consists of a profunctor $K\colon \mathbf{A} \rightsquigarrow \mathbf{C}$ such that there are the two natural isomorphisms

k_1) $\int^{yz} K^a_y K^b_z P^{yz}_x \cong \int^c K^c_x P^{ab}_c$;
k_2) $\int^c K^c_x J_c \cong J_x$.

A *multiplicative* natural transformation $\alpha\colon K \to H$ is a 2-cell between profunctors commuting with the structural isomorphisms k_1, k_2 in the obvious sense.

Exercise B.2. Define the category of multiplicative kernels $\ker(\mathbf{A}, \mathbf{C}) \subset \mathbf{Prof}(\mathbf{A}, C)$ showing that the composition of two kernels is again a kernel.

Exercise B.3. Show that a profunctor $K\colon \mathbf{A} \rightsquigarrow \mathbf{C}$ is a multiplicative kernel if and only if the cocontinuous functor $\mathrm{Lan}_{\mathrm{Y}} K = \hat{K}\colon [\mathbf{A}, \mathbf{Sets}] \to [\mathbf{C}, \mathbf{Sets}]$ preserves the convolution monoidal structure on both categories.

Describe the isomorphisms k_1, k_2 in the case of $\mathfrak{P} =$ Day convolution.

Exercise B.4. Show that a functor $F\colon (\mathbf{A}, \otimes_{\mathbf{A}}, i) \to (\mathbf{C}, \otimes_{\mathbf{C}}, j)$ between monoidal categories is strong monoidal, i.e.

- $F(a \otimes b) \cong Fa \otimes Fb$;
- $Fi \cong j$

if and only if $F^* = \hom(F, 1)$ is a multiplicative kernel.

Dually, show that for $\mathbf{A}, \mathbf{C}$ promonoidal, $F\colon \mathbf{C} \to \mathbf{A}$ preserves convolution on $[\mathbf{A}, \mathbf{Sets}]_{\mathfrak{P}}, [\mathbf{C}, \mathbf{Sets}]_{\mathfrak{Q}}$ precisely if $F_* = \hom(1, F)$ is a multiplicative kernel.

Definition B.5. Let $K\colon \mathbf{A} \rightsquigarrow \mathbf{C}$ be a multiplicative kernel between promonoidal categories; define the K-*Fourier transform* $f \mapsto \hat{K}(f)\colon \mathbf{C} \to \mathbf{Sets}$, obtained as the image of $f\colon \mathbf{A} \to \mathbf{Sets}$ under the left Kan extension $\mathrm{Lan}_{\mathrm{Y}}K\colon [\mathbf{A}, \mathbf{Sets}] \to [\mathbf{C}, \mathbf{Sets}]$.

Exercise B.6. Show the following properties of the K-Fourier transform:

- There is the canonical isomorphism

$$\hat{K}(f) \cong \int^a K(a, -) \times f(a) \tag{108}$$

- $\hat{K}$ preserves the convolution monoidal structure;
- $\hat{K}$ has a right adjoint defined by

$$\check{K}(g) \cong \int_x [K(-, x), g(x)]. \tag{109}$$

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