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Cyclic homology and the Lie algebra homology of matrices

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In this paper we study a new homology theory for associative algebras called cyclic homology. We investigate its relations with Hochschild homology, de Rham cohomology and the homology of the Lie algebras of matrices.

In [3] A. Connes introduced the dual version: cyclic cohomology. One of his basic theorems, when formulated in homology, says that there is a long exact sequence

$$\cdots \longrightarrow H_n(A, A) \longrightarrow HC_n(A) \xrightarrow{S} HC_{n-2}(A) \longrightarrow H_{n-1}(A, A) \longrightarrow \cdots$$

where S is a kind of periodicity operator on the cyclic homology $HC_*(A)$, and where $H_n(A, A)$ is Hochschild homology. This result was found independently by Tsygan [12] whose proof shows that the periodicity comes from the degree two periodicity of the homology of cyclic groups.

In this paper we approach the subject of cyclic homology starting from a double complex suggested by Tsygan's work. On one hand this allows us to simplify, or at least to explain, the proofs of some of Connes' theorems using diagrams instead of cochain computations. On the other hand the double complex makes sense for an associative algebra over any commutative ground ring. One obtains a reasonable theory 'over the integers' by defining the cyclic homology to be the (total) homology of this double complex.

We note that the double complex appears in a more general context in Connes' recent theory [4] on cyclic objects in a category, and that this theory works 'over the integers'.

The contents of the paper are as follows. In the first section we construct the complex $\mathscr{C}(A)$ and use it to go between the cyclic quotient of the Hochschild complex and Connes' double complex with the b and B operators. We derive the long exact sequence and spectral sequence relating Hochschild and cyclic homology.

In the second section we construct maps from cyclic homology to, essentially, de Rham cohomology. In the case of a smooth commutative algebra over k, where k is of characteristic zero, we prove an algebraic version of a formula of

Connes: $HC_n = \Omega^n/d\Omega^{n-1} \oplus H_{DR}^{n-2} \oplus H_{DR}^{n-4} \oplus \ldots$. If the condition on the characteristic is dropped, then we show that Connes' formula gives at least the E^2-term of the spectral sequence from Hochschild to cyclic homology.

In the third section we exhibit a product $HC_n \otimes HC_p \to HC_{n+p+1}$ on cyclic homology and show that it is compatible with a similar product defined by Deligne on differential forms.

In the fourth section we develop a theory of reduced cyclic homology which is the cyclic homology of A relative to that of k. We show that the cyclic homology of a non unital algebra is the same as the reduced cyclic homology of the associated augmented algebra obtained by adjoining an identity. We also compute the cyclic homology for a ring of dual numbers.

The fifth section contains the computation of the cyclic homology for a tensor algebra.

The last section is devoted to the homology of the Lie algebras of matrices $\mathfrak{gl}(A)$, when the ground ring k is a field of characteristic zero. The main result, announced in [9] and independently by Tsygan in [12] claims that cyclic homology is the primitive part of the homology of the Lie algebra of matrices. A refinement of the technique gives stabilization results for the homology of $\mathfrak{gl}_r(A)$. This section ends up with another spectral sequence converging to cyclic homology and deduced from the rank filtration on $\mathfrak{gl}(A)$.

1. Hochschild and cyclic homology

Let A be an associative algebra (with identity) over a commutative ring k. We will use the abbreviation A^n for $A^{\otimes n}$, the n-fold tensor product of A over k, and write $(a_1, \ldots, a_n)$ for $a_1 \otimes \cdots \otimes a_n$. Let b and $b': A^{n+1} \to A^n$ denote the operators given by the formulas

$$b(a_0, \ldots, a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0, \ldots, a_i a_{i+1}, \ldots, a_n) + (-1)^n (a_n a_0, \ldots, a_{n-1})$$

$$b'(a_0, \ldots, a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0, \ldots, a_i a_{i+1}, \ldots, a_n)$$

The chain complex

$$\xrightarrow{b'} A^3 \xrightarrow{b'} A^2 \xrightarrow{b'} A$$

is the standard Hochschild resolution of A over $A \otimes A^{\text{op}}$ up to a dimension shift

[2]. It is acyclic because of the homotopy operator $s: A^n \to A^{n+1}$, $s(a_1, \ldots, a_n) = (1, a_1, \ldots, a_n)$ which satisfies

$$b's + sb' = id.$$

We shall refer to the complex (A^{*+1}, b') as the *acyclic Hochschild complex.*

Upon tensoring the Hochschild resolution (A^{*+2}, b') with A considered as a right module over $A \otimes A^{\text{op}}$, we obtain the chain complex

$$\xrightarrow{b} A^3 \xrightarrow{b} A^2 \xrightarrow{b} A$$

which we call the *Hochschild complex.* Its homology is the Hochschild homology $H_*(A, A)$, which we write simply $H_*(A)$. When A is flat over k one has

$$H_n(A) = \text{Tor}_n^{A \otimes A^{\text{op}}}(A, A).$$

We define an action of the cyclic group $\mathbb{Z}/n$ on A^n by letting the generator act as the operator

$$t(a_1, \ldots, a_n) = (-1)^{n-1}(a_n, a_1, \ldots, a_{n-1}).$$

Let $N = 1 + t + \cdots + t^{n-1}$ denote the corresponding norm operator on A^n.

We shall denote by $\mathscr{C}(A)$ the following double chain complex

$$\begin{array}{ccccccc}
\downarrow b & & \downarrow -b' & & \downarrow b & & \\
A^2 & \xleftarrow{1-t} & A^2 & \xleftarrow{N} & A^2 & \xleftarrow{1-t} & \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \\
A & \xleftarrow{1-t} & A & \xleftarrow{N} & A & \xleftarrow{1-t} &
\end{array}$$

in which the even degree columns are Hochschild complexes and the odd degree columns are acyclic Hochschild complexes with the sign of the differential changed. In the horizontal direction we have the standard complexes for the homology of $\mathbb{Z}/n$ with coefficients in A^n. The differential in the associated total complex Tot $\mathscr{C}(A)$ is the sum of the horizontal and vertical differentials and the following lemma shows that $d^2 = 0$.

LEMMA 1.1 [3, 12]. *One has $b(1-t) = (1-t)b'$ and $b'N = Nb$.*

Proof. If $j: A^{n+1} \to A^n$ is defined by $j(a_0, \ldots, a_n) = (-1)^n(a_na_0, a_1, \ldots, a_{n-1})$

then one has

$$b = \sum_{i=0}^{n} t^i j t^{-i-1} \quad \text{and} \quad b' = \sum_{i=0}^{n-1} t^i j t^{-i-1}$$

on A^{n+1}. Using these formulas it is easy to check the lemma, for example, $b'N$ and Nb are both equal to NjN.

DEFINITION. The *cyclic homology* $HC_*(A)$ of the associative k-algebra A is the homology of Tot $\mathscr{C}(A)$.

In order to show this definition agrees when k contains $\mathbb{Q}$ with the one used in [3, 7, 9], we note that there is an augmentation map

$$\operatorname{Tot} \mathscr{C}(A) \to C_*(A) = (A^{*+1}/(1-t), b)$$

to the quotient of the Hochschild complex obtained by taking the coinvariants for the actions of the various cyclic groups. The augmentation induces an edge homomorphism for the spectral sequence

$$E^1_{pq} = H_p(\mathbb{Z}/(q+1), A^{q+1}) \Rightarrow HC_*(A)$$

associated to the double complex. In characteristic zero the group homology vanishes in positive degrees, and the spectral sequence collapses, proving the following.

PROPOSITION 1.2. *If k contains $\mathbb{Q}$ the above augmentation map is a quasi-isomorphism* (i.e., *it induces an isomorphism on homology*):

$$HC_n(A) = H_n(A^{*+1}/(1-t), b).$$

Remark. From the spectral sequence it is easily seen that in order to have an isomorphism in degree n it is sufficient to assume that $n!$ is invertible.

The double complex $\mathscr{C}(A)$ can be simplified in two ways up to quasi-isomorphism. First of all the odd degree columns can be eliminated as they are acyclic. This leads to the double complex of Connes [3] with the differentials b and B, which we denote $\mathscr{B}(A)$. Secondly the Hochschild complexes can be normalized.

The double complex $\mathscr{B}(A)$ will be drawn in a somewhat unorthodox way

$$\begin{array}{ccccccc}
\downarrow b & \nwarrow^{B} & \downarrow b & \nwarrow^{B} & \downarrow b & & \\
A^2 & & A^2 & & A^2 & & \\
\downarrow b & \nwarrow^{B} & \downarrow b & \nwarrow^{B} & \downarrow & \nwarrow^{B} & \\
A & & A & & A & &
\end{array}$$

in order to show its relation to $\mathscr{C}(A)$. It consists of the even degree columns of $\mathscr{C}(A)$, where B is given by the composition

$$\begin{array}{ccc}
A^{n+1} & \xleftarrow{1-t} & A^{n+1} & & \\
 & & \uparrow s & & \\
 & & A^n & \xleftarrow{N} & A^n
\end{array}$$

More precisely $\mathscr{B}(A)_{pq} = A^{q+1}$ if p is even $\geqslant 0$ and $q \geqslant 0$; it is undefined if p is odd. One has

$$B^2 = (1-t)sN(1-t)sN = 0 \tag{1.3}$$

$$\begin{aligned} bB + Bb &= b(1-t)sN + (1-t)sNb \\ &= (1-t)(b's + sb')N = (1-t)N = 0, \end{aligned} \tag{1.4}$$

so $\operatorname{Tot} \mathscr{B}(A)$ is a chain complex for the differential $d = b + B$.

PROPOSITION 1.5. *The complexes* $\operatorname{Tot} \mathscr{B}(A)$ *and* $\operatorname{Tot} \mathscr{C}(A)$ *are quasi-isomorphic.*

Proof. We define a map from $\mathscr{B} = \mathscr{B}(A)$ to $\mathscr{C} = \mathscr{C}(A)$ by sending x in $\mathscr{B}_{pq}$, p even, to (x, sNx) in $\mathscr{C}_{pq} \oplus \mathscr{C}_{p-1,q+1}$. This is a map of complexes because

$$\begin{aligned} d(x + sNx) &= (bx,\ Nx + (-b')sNx,\ (1-t)sNx) \\ &= (bx,\ sb'Nx,\ Bx) \\ &= (bx,\ sNbx + sNBx,\ Bx). \end{aligned}$$

Next we consider the increasing filtration of $\mathscr{C}$ and $\mathscr{B}$ by columns:

$$F_n\mathscr{C} = \bigoplus_{p \leqslant n} \mathscr{C}_{p*}, \qquad F_n\mathscr{B} = \bigoplus_{\substack{p \leqslant n \\ p \text{ even}}} \mathscr{B}_{p*}.$$

Since the odd degree columns of $\mathscr{C}$ are acyclic it is easily seen that the induced map on the associated graded complexes is a quasi-isomorphism. The proposition then follows by a standard induction.

Remark. It is clear from the above proof that one is not working in the category of double complexes, but rather with filtered complexes. Thus $\mathscr{B}(A)$ is a filtered subcomplex of $\mathscr{C}(A)$.

At this point we can easily prove the following basic results relating cyclic and Hochschild homology.

THEOREM 1.6. *For any associative k-algebra A there is a long exact sequence*

$$\cdots \longrightarrow H_n(A) \xrightarrow{I} HC_n(A) \xrightarrow{S} HC_{n-2}(A) \xrightarrow{B} H_{n-1}(A) \longrightarrow \cdots$$

It is clear from the picture of $\mathscr{B}(A)$ that one has an exact sequence of complexes

$$0 \to (A^{*+1}, b) \to \operatorname{Tot} \mathscr{B}(A) \to \operatorname{Tot} \mathscr{B}(A)[-2] \to 0$$

where $[-2]$ indicates that the degrees are shifted by -2: $(\operatorname{Tot} \mathscr{B}[-2])_n = \operatorname{Tot} \mathscr{B}_{n-2}$. Taking the associated long exact sequence in homology, we obtain the theorem from Proposition 1.5.

COROLLARY 1.7. *Cyclic homology is Morita invariant.*

Proof. This follows from the long exact sequence (1.6) and the Morita invariance of Hochschild homology (Cf. [14], theorem 3.7).

In order to simplify some further computations we change the indexing of the bicomplex $\mathscr{B}(A)$ and put

$$(\mathscr{B}(A)')_{pq} = \mathscr{B}(A)_{2p,q-p} = A^{q-p+1}. \tag{1.8}$$

In this setting the maps B go horizontally.

The increasing filtration of the bicomplex $\mathscr{B}(A)'$ by columns gives the following.

THEOREM 1.9. *There is a spectral sequence abutting to* $HC_n(A)$ *with* $E^1_{pq} = H_{q-p}(A)$ *and with* $d^1 : H_{q-p}(A) \to H_{q-p+1}(A)$ *induced by Connes' operator B.*

This theorem can also be obtained from Theorem 1.6. by interpreting the long exact sequence as an exact couple.

(1.10) Next we show that the complex $\mathscr{B}(A)'$ can be simplified further by replacing the Hochschild complexes by their normalizations. We recall that the Hochschild complex (A^{*+1}, b) is the chain complex associated to a simplicial abelian group. Hence it contains a degenerate subcomplex D_*, where $D_n \subset A^{n+1}$ is spanned by the elements $(a_0, \ldots, a_n)$ such that $a_i = 1$ for some i with $1 \leqslant i \leqslant n$. Upon dividing out by it we obtain the *normalized Hochschild complex* $A^{n+1}/D_n = A \otimes \bar{A}^n$, where $\bar{A} = A/k$, whose differential we denote again by b since it is given by the same formula. The degenerate subcomplex is known to be acyclic, so the projection $(A^{*+1}, b) \to (A \otimes \bar{A}^*, b)$ is a quasi-isomorphism.

We now normalize each column of $\mathscr{B}(A)'$ and obtain a double complex

$$\mathscr{B}(A)_{\text{norm}}: \quad \begin{array}{ccccc} \downarrow b & & \downarrow b & & \downarrow b \\ A\otimes\bar{A}^2 & \xleftarrow{B} & A\otimes\bar{A} & \xleftarrow{B} & A \\ \downarrow b & & \downarrow b & & \\ A\otimes\bar{A} & \xleftarrow{B} & A & & \\ \downarrow b & & & & \\ A & & & & \end{array}$$

PROPOSITION 1.11. *The projection of* $\operatorname{Tot}\mathscr{B}(A)$ *onto* $\operatorname{Tot}\mathscr{B}(A)_{\text{norm}}$ *is a quasi-isomorphism. The operator* $B: A\otimes\bar{A}^n \to A\otimes\bar{A}^{n+1}$ *is given by*

$$B(a_0, a_1, \ldots, a_n) = \sum_{i=0}^{n} (-1)^{in}(1, a_i, \ldots, a_n, a_0, \ldots, a_{i-1}).$$

Proof. We must check that the operator $B = (1-t)sN: A^{n+1} \to A^{n+2}$ passes to the quotient. Now the image of ts lies in D_{n+1}, so $B = sN$ from A^{n+1} to $A\otimes\bar{A}^{n+1}$. This gives the above formula, which can be used to show that B is well-defined on $A\otimes\bar{A}^n$. The rest is clear as the projection is columnwise a quasi-isomorphism.

Example 1. If $A = k$, then $\mathscr{B}(A)_{\text{norm}}$ reduces to copies of k in the diagonal hence $HC_n(k) = k$ for n even $\geqslant 0$ and $HC_n(k) = 0$ for n odd.

Example 2. Recall that if A is commutative the module of Kähler differentials $\Omega^1_A = \Omega^1_{A/k}$ is by definition the A-module generated by symbols dx for $x \in A$ with the relations $d(xy) = x\,dy + y\,dx$, $d(x+y) = dx + dy$ and $d(k) = 0$. It is easy to see that, when A is commutative, $HC_1(A) = \Omega^1_A/dA$ and $HC_0(A) = A$.

2. Relation to de Rham cohomology

Connes has calculated the continuous cyclic cohomology of the ring of smooth functions on a manifold in terms of currents. If we make the obvious translation to the algebraic setting of this paper, we obtain a formula for the cyclic homology of a smooth commutative algebra in characteristic zero in terms of algebraic differential forms. We are going to review the proof of this formula to see what can be said without assuming characteristic zero.

In this section the algebra A is assumed to be commutative. In this case the Hochschild complex is the chain complex associated to a simplicial commutative ring, and so it has a product, the so-called *shuffle product*, given by

$$(a, a_1, \ldots, a_p) \cdot (a', a_{p+1}, \ldots, a_{p+q}) = \sum \operatorname{sgn}(\sigma)(aa', a_{\sigma^{-1}1}, \ldots, a_{\sigma^{-1}(p+q)}) \tag{2.1}$$

where the sum is over all permutations σ of $\{1, 2, \ldots, p+q\}$ such that $\sigma 1 < \cdots < \sigma p$ and $\sigma(p+1) < \cdots < \sigma(p+q)$. In this way both the Hochschild and normalized Hochschild complexes become differential graded strictly anti-commutative A-algebras, where strict means that the square of any odd degree element is zero. Hence the Hochschild homology $H_*(A)$ is a graded strictly anti-commutative algebra over A.

Since

$$\begin{aligned} H_1(A) &= A \otimes \bar{A}/b(A \otimes \bar{A}^2) \\ &= A \otimes \bar{A}/\{(zx, y) - (z, xy) + (yz, x)\} \end{aligned}$$

it is easily seen that there is an isomorphism

$$\gamma : \Omega_A^1 \xrightarrow{\sim} H_1(A)$$

obtained by sending $a\,dx$ to the class of (a, x). Because of the multiplicative structure the map γ extends to an A-algebra map

$$\gamma : \Omega_A^n = \Lambda_A^n \Omega_A^1 \to H_n(A).$$

PROPOSITION 2.2. *One has a commutative square*

$$\begin{array}{ccc} \Omega_A^n & \xrightarrow{\gamma} & H_n(A) \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle B} \\ \Omega_A^{n+1} & \xrightarrow{\gamma} & H_{n+1}(A) \end{array}$$

where d is the exterior derivative on forms.

Proof. Given a generator $w = a_0 da_1 \cdots da_n$ for Ω^n_A, $\gamma(w)$ is the class of

$$(a_0, a_1) \cdot (1, a_2) \cdots (1, a_n) = \sum_{\sigma \in \Sigma_n} \operatorname{sgn}(\sigma)(a_0, a_{\sigma^{-1}1}, \ldots, a_{\sigma^{-1}n}) \tag{2.3}$$

in $A \otimes \bar{A}^n$, where Σ_n is the group of permutations of $\{1, \ldots, n\}$. Similarly $\gamma(dw) = \gamma(da_0 \cdots da_n)$ is represented by

$$(1, a_0) \cdots (1, a_n) = \sum_{\rho \in \Sigma_{n+1}} \operatorname{sgn}(\rho)(1, a_{\rho^{-1}0}, \ldots, a_{\rho^{-1}n}) \tag{2.4}$$

where Σ_{n+1} denotes the permutations of $\{0, 1, \ldots, n\}$. By the formula of Proposition 1.11 B carries the shuffle product of (2.3) into

$$\sum_\sigma \operatorname{sgn}(\sigma) \sum_\tau \operatorname{sgn}(\tau)(1, a_{\sigma^{-1}\tau^{-1}0}, a_{\sigma^{-1}\tau^{-1}1}, \ldots, a_{\sigma^{-1}\tau^{-1}n}) \tag{2.5}$$

where τ ranges over the cyclic subgroup of Σ_{n+1} generated by $t : i \mapsto i+1$. Since Σ_{n+1} is the product of Σ_n and this cyclic subgroup, the expressions (2.4) and (2.5) are equal. Thus $B\gamma(w) = \gamma(dw)$ as required.

Now suppose that A is smooth over k in the sense of Grothendieck, for example, A is the ring of algebraic functions on a nonsingular variety over the perfect field k. Then it is known that the map from Ω^*_A to $H_*(A)$ is an isomorphism [5]. In effect, the ideal of the diagonal in $A \otimes A$ is locally generated by a regular sequence, so

$$H_*(A) = \operatorname{Tor}_*^{A \otimes A}(A, A)$$

can be computed using a Koszul sequence and shown to be an exterior algebra.

Consequently, in the spectral sequence of Theorem 1.9 the Hochschild homology can be identified with differential forms, and the differential d^1 can be identified with the exterior derivative by the proposition above. Hence we obtain the first part of the following.

THEOREM 2.6. *If A is smooth over k, then the spectral sequence of Theorem 1.9 becomes*

$$E^2_{pq} = \begin{Bmatrix} \Omega^q_A / d\Omega^{q-1}_A, & p = 0 \\ H^{q-p}_{DR}(A), & p > 0 \end{Bmatrix} \Rightarrow HC_{p+q}(A).$$

Remark. We do not know if the spectral sequence stops at E^2 when the characteristic is different from zero. The first possibility for a non trivial differential would be a natural map $d^2: H^n_{DR}(A) \to H^{n+3}_{DR}(A)$.

In the characteristic zero case there is defined a map $\mu: A \otimes \bar{A}^* \to \Omega^*_A$ by $\mu(a_0, \ldots, a_n) = (1/n!)a_0 da_1 \cdots da_n$. We leave to the reader to verify the identities

$$\mu b = 0 \quad \text{and} \quad d\mu = \mu B. \tag{2.7}$$

These identities show that μ induces a map from $\mathcal{B}(A)_{\text{norm}}$ to the following double complex $\mathcal{D}(A)$.

$$\begin{array}{ccccc}
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} \\
\Omega^2 & \xleftarrow{d} & \Omega^1 & \xleftarrow{d} & \Omega^0 \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \\
\Omega^1 & \xleftarrow{d} & \Omega^0 & & \\
\downarrow{\scriptstyle 0} & & & & \\
\Omega^0 & & & &
\end{array}$$

The total complex of $\mathcal{D}(A)$ is the direct sum of the truncated de Rham complexes

$$\Omega^{\leq i}: \Omega^i \leftarrow \Omega^{i-1} \leftarrow \cdots \leftarrow \Omega^0 \leftarrow 0 \leftarrow \cdots$$

suitably shifted: Tot $\mathcal{D}(A) = \bigoplus_i \Omega^{\leq i}[i]$.

THEOREM 2.8. *If k contains* $\mathbb{Q}$, *for any commutative k-algebra A, the map* μ *induces homomorphisms*

$$\mu_{n,i}: HC_n(A) \to H^{n-2i}_{DR}(A), \quad \text{for} \quad 1 \leq i \leq n/2,$$

and

$$\mu_{n,0}: HC_n(A) \to \Omega^n_A / d\Omega^{n-1}_A$$

Proof. These homomorphisms are obtained by composing μ with the projection on the truncated de Rham complex and then taking homology.

THEOREM 2.9. *If k contains* $\mathbb{Q}$ *and if A is smooth over k, then there is a* *canonical isomorphism*

$$\bigoplus_i \mu_{n,i} : HC_n(A) = \Omega^n_A/d\Omega^{n-1}_A \oplus H^{n-2}_{DR}(A) \oplus H^{n-4}_{DR}(A) \oplus \cdots .$$

Proof. It is clear that μ induces on homology a map inverse to γ, hence μ is a quasi-isomorphism. Therefore the spectral sequence of Theorem 2.6. is the spectral sequence of the double complex $\mathcal{D}(A)$, which proves the assertion.

3. Product structure

In this section we study a product $HC_n(A) \otimes HC_p(A) \xrightarrow{*} HC_{n+p+1}(A)$ for a commutative k-algebra A. We could as well define a product $HC_n(A) \otimes HC_p(A') \to HC_{n+p+1}(A \otimes A')$ for not necessarily commutative algebras, but we take $A = A'$ for simplicity.

First we investigate the properties of the map B defined on $A \otimes \bar{A}^*$ (cf. 1.11) with respect to the shuffle product.

LEMMA 3.1. $B(x \cdot B(y)) = B(x) \cdot B(y)$.

Proof. For $x = (a_0, a_1, \ldots, a_p)$ and $y = (a_{p+1}, \ldots, a_{p+q})$ we have $x \cdot B(y) = \sum \operatorname{sgn}(\sigma)(a_0, a_{\sigma^{-1}1}, \ldots, a_{\sigma^{-1}(p+q)})$ where the sum is over all permutations σ of $\{1, \ldots, p+q\}$ satisfying $\sigma 1 < \cdots < \sigma p$ and $\sigma k < \cdots < \sigma(p+q) < \sigma(p+1) < \cdots < \sigma(k+1)$ for some $k \in \{p+1, \ldots, p+q\}$. Therefore $B(x \cdot B(y)) = \sum \operatorname{sgn}(\tau)(1, a_{\tau^{-1}0}, \ldots, a_{\tau^{-1}(p+q)})$ where the sum is over all permutations τ such that there exist $i \in \{0, \ldots, p\}$ and $j \in \{p+1, \ldots, p+q\}$ for which $\tau i < \cdots < \tau p < \tau 0 < \cdots < \tau(i-1)$ and $\tau j < \cdots < \tau(p+q) < \tau(p+1) < \cdots < \tau(j-1)$. This last sum is easily seen to be equal to $B(x) \cdot B(y)$, whence the lemma.

We define a product on the total complex of $\mathcal{B}(A)_{\text{norm}}$ by the following formula. Let $x \in (\mathcal{B}(A)_{\text{norm}})_{lm} = A \otimes \bar{A}^{m-l}$ and $y \in (\mathcal{B}(A)_{\text{norm}})_{rs} = A \otimes \bar{A}^{s-r}$

$$x * y = \begin{cases} x \cdot B(y) & \text{when } r = 0, \\ 0 & \text{when } r \neq 0 \end{cases} \in (\mathcal{B}(A)_{\text{norm}})_{l+r, m+s+1}. \tag{3.2}$$

Then this formula is extended to $\operatorname{Tot} \mathcal{B}(A)_{\text{norm}} \otimes \operatorname{Tot} \mathcal{B}(A)_{\text{norm}}$ by linearity. For $x \in A \otimes \bar{A}^i$ the degree of x is i and is denoted $|x|$; it is also the degree mod 2 of x considered as an element in $\operatorname{Tot} \mathcal{B}(A)_{\text{norm}}$.

THEOREM 3.3. *The *-product defined above induces a degree* 1 *map of* *complexes*

$$\mathrm{Tot}\,\mathscr{B}(A)_{\mathrm{norm}} \otimes \mathrm{Tot}\,\mathscr{B}(A)_{\mathrm{norm}} \to \mathrm{Tot}\,\mathscr{B}(A)_{\mathrm{norm}}$$

which is associative. As a consequence it defines an associative product

$$* : HC_n(A) \otimes HC_p(A) \to HC_{n+p+1}(A).$$

Proof. We recall that the boundary δ of $\mathrm{Tot}\,\mathscr{B}(A)_{\mathrm{norm}}$ is given by

$$\delta(x) = \begin{Bmatrix} (B+b)(x) & \text{if } l \neq 0 \\ b(x) & \text{if } l = 0 \end{Bmatrix} \quad \text{for} \quad x \in (B(A)_{\mathrm{norm}})_{lm}.$$

We will prove the formula

$$\delta(x * y) = \delta(x) * y + (-1)^{|x|+1} x * \delta(y)$$

using (1.3), (1.4) and (3.1).

If $r \neq 0$ and 1 then both sides are equal to 0. If $r = 1$, then $x * y = 0$ and $\delta x * y = 0$. One has $x * \delta y = x * (B(y) + b(y)) = x \cdot BB(y) = 0$. If $r = 0$, then there are two cases. If $l \neq 0$, then

$$\begin{aligned}\delta(x * y) = \delta(x \cdot B(y)) &= B(x \cdot B(y)) + b(x \cdot B(y)) \\ &= Bx \cdot By + bx \cdot By + (-1)^{|x|+1} x \cdot bBy.\end{aligned}$$

On the other hand we have

$$\begin{aligned}\delta x * y + (-1)^{|x|+1} x * \delta y &= (Bx + bx) * y + (-1)^{|x|} x * by \\ &= Bx \cdot By + bx \cdot By + (-1)^{|x|} x \cdot Bby.\end{aligned}$$

The equality follows from $Bb + bB = 0$.

If $r = l = 0$, then

$$\delta(x * y) = \delta(x \cdot By) = b(x \cdot By) = bx \cdot By + (-1)^{|x|} x \cdot bBy.$$

On the other hand we have

$$\delta x * y + (-1)^{|x|+1} x * \delta y = bx * y + (-1)^{|x|+1} x * by = bx \cdot By + (-1)^{|x|+1} x \cdot Bby$$

and the proof of the first assertion is completed.

Associativity is proved in a similar way.
Example. For $n = p = 0$ the product is given by $A\otimes A \to \Omega^1_A/dA$, $a*b = adb$

PROPOSITION 3.4. *$H_*(A)$ is an $HC_*(A)$-module and the map $I: H_*(A)\to HC_*(A)$ is an $HC_*(A)$-map.*

Proof. The normalized Hochschild complex is a subcomplex of Tot $\mathcal{B}(A)_{\text{norm}}$ and it is easily seen that for $y\in\mathcal{B}(A)_{\text{norm}}$ the operation $x\mapsto x*y$ sends this subcomplex into itself. The formulas proved in (3.3) finish the proof.

(3.5) There is a similar product on the sum of the truncated de Rham complexes. In terms of the double complex $\mathcal{D}(A)$ (see section 2) this product is given by

$$x*y=\begin{cases} x\wedge dy & \text{when } r=0\\ 0 & \text{otherwise}\end{cases}\quad \begin{matrix}\text{for} & x\in(\mathcal{D}(A))_{lm}=\Omega_A^{m-l}\\ \text{and} & y\in(\mathcal{D}(A))_{rs}=\Omega_A^{s-r}.\end{matrix}$$

Deligne has remarked that this product is associative and homotopy graded commutative provided that one puts deg ((Tot $\mathcal{D}(A))_n) = n+1$ (unpublished notes by S. Bloch). The homotopy is given by $h(x\otimes y) = (-1)^{|x|+1}x\wedge y$. Therefore there is a graded commutative product on the homology.

When k contains $\mathbb{Q}$ it is immediately seen that the map μ is compatible with the products. This proves the following:

PROPOSITION 3.6. *The homomorphism*

$$\bigoplus_i \mu_{n,i}: HC_n \to \Omega^n/d\Omega^{n-1}\oplus H_{DR}^{n-2}\oplus H_{DR}^{n-4}\oplus\cdots$$

commutes with the products.

We now investigate the product on the complex $C_*(A) = (A^{*+1}/(1-t), b)$. It is defined by the same kind of formula:

$$x*y = x\cdot B(y),$$

where

$$B(a_0, a_1, \ldots, a_n) = \sum_{i=0}^{n} (-1)^{in}(1, a_i, \ldots, a_n, a_0, \ldots, a_{i-1})$$

and where the dot means shuffle product.

PROPOSITION 3.7. *Provided that we put* $\deg(C_n(A)) = n+1$, *the product* $*$ *induces on* $C_*(A)$ *a structure of commutative differential graded algebra.*

Proof. The derivation property of b and associativity are proved like in (3.3). Graded commutativity follows from the fact that $x \cdot B(y) - (-1)^{\deg x \deg y} y \cdot B(x)$ is in the image of $(1-t)$. To prove this we remark that $x \cdot B(y)$ (resp. $y \cdot B(x)$) is the sum of $(p+q)!/p!\,q!$ terms (where $p = |x|$ and $q = |y|$) and that for any such term there is a unique power of t which converts it into a term in $y \cdot B(x)$.

COROLLARY 3.8. *If* k *contains* $\mathbb{Q}$ *the* $*$*-product on* $HC_*(A)$ *verifies*

$$x * y = (-1)^{(n+1)(p+1)} y * x \quad \textit{for} \quad x \in HC_n(A) \quad \textit{and} \quad y \in HC_p(A).$$

Remark 3.9. The iterated $*$-product $(a_0, \ldots, a_n) \mapsto a_0 * \cdots * a_n$ from A^{n+1} to $HC_n(A)$ factors through $\Omega^n_A/d\Omega^{n-1}_A$ and defines a map $\Omega^n_A/d\Omega^{n-1}_A \to HC_n(A)$ whose composition with $\mu_{n,0}$ is, in view of (3.6), the identity.

4. Reduced cyclic homology

In this section we suppose that the homomorphism $k \to A$ given by the identity of A is injective. At the end of the first section we pointed out that the Hochschild homology of A can be computed using the normalized Hochschild complex. We now define the reduced Hochschild complex $(A \otimes \bar{A}^*, b)_{\text{red}}$ to be the quotient of the normalized Hochschild complex by the subcomplex given by the normalized Hochschild complex for the algebra k. As the latter complex consists of k in degree zero, we have an exact sequence

$$0 \to k[0] \to (A \otimes \bar{A}^*, b) \to (A \otimes \bar{A}^*, b)_{\text{red}} \to 0$$

and the reduced Hoschild complex is the same as the Hochschild complex except that the A in degree zero is replaced by $\bar{A}$. The homology of this reduced complex will be called the *reduced Hochschild homology* and denoted $\bar{H}_n(A)$. From the above exact sequence one obtains an exact sequence

$$0 \to H_1(A) \to \bar{H}_1(A) \to k \to H_0(A) \to \bar{H}_0(A) \to 0$$

and $\bar{H}_n(A) = H_n(A)$ for $n \geq 2$.

In a similar fashion we define the reduced cyclic homology $\bar{HC}_*(A)$ to be the homology of the double complex $\mathscr{B}(A)_{\text{red}}$ defined by an exact sequence

$$0 \to \mathscr{B}(k)_{\text{norm}} \to \mathscr{B}(A)_{\text{norm}} \to \mathscr{B}(A)_{\text{red}} \to 0$$

where $\mathscr{B}(A)_{\text{norm}}$ is the normalized version of Connes' double complex described in 1.10. This reduced Connes' complex is the same as $\mathscr{B}(A)_{\text{norm}}$ except that the diagonal of A's is replaced by $\bar{A}$'s.

PROPOSITION 4.1. *One has long exact sequences*

$$\to HC_n(k) \to HC_n(A) \to \bar{H}C_n(A) \to HC_{n-1}(k) \to$$
$$\to \bar{H}_n(A) \to \bar{H}C_n(A) \to \bar{H}C_{n-2}(A) \to \bar{H}_{n-1}(A) \to .$$

The first follows from the exact sequence defining $\mathscr{B}(A)_{\text{red}}$ and the fact that the homology of $\mathscr{B}(A)_{\text{norm}}$ is $HC_*(A)$. The second exact sequence can be derived as Theorem 1.6. but using the double complex $\mathscr{B}(A)_{\text{red}}$.

The reduced theory is a natural thing to consider when dealing with augmented algebras. We recall that an augmented algebra A is of the form $A = k \oplus I$ where I is the augmentation ideal, and that A is isomorphic to the algebra with identity obtained by adjoining an identity to the non-unital ring I. In fact the categories of augmented algebras and non-unital algebras are equivalent in this way.

For an augmented algebra the first exact sequence in the above proposition splits yielding the isomorphism

$$HC_*(A) = HC_*(k) \oplus \bar{H}C_*(A)$$

At this point one might define the cyclic homology of non-unital algebra to be the reduced cyclic homology of the corresponding augmented algebra. On the other hand inspection of the arrows in the double complex $\mathscr{C}(A)$ of the first section shows that it makes sense for non-unital rings, hence we can make the definition $HC_*(I) = H_*(\text{Tot}\, \mathscr{C}(I))$. The following shows that these two definitions agree.

PROPOSITION 4.2. *If $A = k \oplus I$ is an augmented ring, then the complexes $\mathscr{C}(I)$ and $\mathscr{B}(A)_{\text{red}}$ are isomorphic, hence $HC_*(I) = \bar{H}C_*(A)$.*

Proof. We define an isomorphism from $\mathscr{C}(I)$ to $\mathscr{B}(A)_{\text{red}}$ by

$$\mathscr{C}(I)_{pn} \oplus \mathscr{C}(I)_{p+1,n-1} = I^{n+1} \oplus I^n \xrightarrow{\sim} A \otimes I^n = (\mathscr{B}(A)_{\text{red}})_{p,n+p}$$

where the isomorphism in the middle sends $(x_0, \ldots, x_n)$ in I^{n+1} and $(x_1, \ldots, x_n)$

in I^n to $(x_0, \ldots, x_n)$ and $1 \otimes (x_1, \ldots, x_n) = (1, x_1, \ldots, x_n)$ in $A \otimes \bar{A}^n$, respectively. By the formulas

$$\begin{aligned} b(1, x_1, \ldots, x_n) &= (x_1, \ldots, x_n) + \sum_{i=1}^{n-1} (-1)^i (1, \ldots, x_i x_{i+1}, \ldots) \\ &\quad + (-1)^n (x_n, x_1, \ldots, x_{n-1}) \\ &= (1-t)(x_1, \ldots, x_n) - 1 \otimes b'(x_1, \ldots, x_n) \end{aligned}$$

$$B(1, x_1, \ldots, x_n) = 0$$

$$B(x_0, \ldots, x_n) = \sum_{i=0}^{n} (-1)^{in} (1, x_i, \ldots, x_n, x_0, \ldots, x_{i-1}) = 1 \otimes N(x_0, \ldots, x_n)$$

the isomorphism respects the differentials.

(4.3) Example. Suppose $A = k \oplus I$ is a ring of dual numbers, that is, $xy = 0$ for x, y in I. Then the b and b' operators in $\mathscr{C}(I)$ are all zero, hence we have

$$\bar{H}C_n(A) = HC_n(I) = \bigoplus_{m=0}^{n} H_{n-m}(\mathbb{Z}/m+1, I^{m+1}).$$

In characteristic zero this becomes simply $\bar{H}C_n(A) = I^{n+1}/(1-t)$.

The remainder of this section will be devoted to proving the analogue for reduced cyclic homology of Proposition 1.2. Put $\bar{C}_n = \bar{A}^{n+1}/(1-t)$. As one has an exact sequence

$$1 \otimes A^n \to A^{n+1}/(1-t) \to \bar{C}_n \to 0$$

and $b(1, a_1, \ldots, a_n) \equiv (1-t)(a_1, \ldots, a_n) \bmod 1 \otimes A^{n-1}$, it follows that b induces a differential on $\bar{C}_*$.

PROPOSITION 4.4. *Assume that k contains $\mathbb{Q}$ and that k is a direct summand of A as a k-submodule. Then the complexes $\bar{C}_*$ and* $\operatorname{Tot} \mathscr{B}(A)_{\mathrm{red}}$ *are quasi-isomorphic, hence one has an isomorphism*

$$\bar{H}C_*(A) = H_*(\bar{A}^{*+1}/(1-t), b).$$

Proof. Put $\bar{\mathscr{B}} = \mathscr{B}(A)_{\mathrm{red}}$ and recall that

$$\bar{\mathscr{B}}_{pq} = \begin{cases} A \otimes \bar{A}^{q-p} & q-p>0 \\ \bar{A} & q-p=0 \\ 0 & q-p<0 \end{cases}$$

with horizontal differential B and vertical differential b. We define a map of complexes $\varepsilon : \mathrm{Tot}\, \bar{\mathcal{B}} \to \bar{C}$ by letting $\varepsilon : \bar{\mathcal{B}}_{0n} = A \otimes \bar{A}^n \to \bar{A}^{n+1}/(1-t)$ be the obvious surjection, and $\varepsilon(\bar{\mathcal{B}}_{pq}) = 0$ for $p > 0$. We define a filtration of $\bar{B}$ by

$$F_n\bar{\mathcal{B}} = \begin{cases} \bar{\mathcal{B}}_{pq} & q-p \leq n \\ 1 \otimes \bar{A}^{n+1} & q-p = n+1 \\ 0 & q-p > n+1 \end{cases}$$

where $1 \otimes \bar{A}^{n+1}$ denotes the k-submodule of $A \otimes \bar{A}^{n+1}$ spanned by the elements $(1, a_0, \ldots, a_n)$. As $B(A \otimes \bar{A}^n) \subset 1 \otimes \bar{A}^{n+1}$ by 1.11, $F_n\bar{\mathcal{B}}$ is a subcomplex of $\bar{\mathcal{B}}$. Moreover $\varepsilon(F_n\bar{\mathcal{B}}) \subset F_n\bar{C}$, where $F_n\bar{C} \subset \bar{C}$ coincides with $\bar{C}$ in degree less than n and is zero elsewhere.

As k is assumed to be a direct summand of A, we have $1 \otimes \bar{A}^{n+1} \cong \bar{A}^{n+1}$. One can now verify easily that $F_n\bar{\mathcal{B}}/F_{n-1}\bar{B}$ is isomorphic to the double complex

$$\begin{array}{ccc} & & \bar{A}^{n+1} \xleftarrow{\;N\;} \\ & & \downarrow{\scriptstyle 1-t} \\ \bar{A}^{n+1} & \xleftarrow{\;N\;} & \bar{A}^{n+1} \\ \downarrow{\scriptstyle 1-t} & & \\ \bar{A}^{n+1} & & \end{array}$$

(compare the formulas in the proof of 4.2). In characteristic zero, this is a resolution of $\bar{C}_n = F_n\bar{C}/F_{n-1}\bar{C}$. So the map ε induces quasi-isomorphisms on the quotients of the filtration, hence it is a quasi-isomorphism, proving the proposition.

Remark. When k is a field of characteristic zero, Proposition 4.4 can be derived using the interpretation of the cyclic homology in terms of the homology of the Lie algebra $\mathfrak{gl}(A)$ (cf. Remark 6.8).

5. Cyclic homology of a tensor algebra

Let A be a tensor algebra $T(V) = \bigoplus_{m \geq 0} V^m$, where V is a module over k. We first compute the Hochschild homology of A starting from the well-known

LEMMA 5.1. *One has an exact sequence*

$$0 \longrightarrow A \otimes V \otimes A \xrightarrow{\;b'\;} A \otimes A \xrightarrow{\;b'\;} A \longrightarrow 0.$$

Proof. In degree m for the tensor grading the three terms of this sequence are respectively m, $m+1$, and one copy of V^m. The lemma can then be proved by checking the arrows. Alternatively one can use the general fact that the kernel I of the multiplication $b': A\otimes A \to A$ is the module of noncommutative differentials of A, i.e., it represents derivations of A with values in A-bimodules. Since a derivation of the tensor algebra is specified by its restriction to V, it follows that $I = A\otimes V\otimes A$, whence the lemma.

As the above sequence of A-bimodules splits as a sequence of right A-modules, one gets a long exact sequence in Hochschild homology:

$$0 \longrightarrow H_1(A) \longrightarrow A\otimes V \xrightarrow{\ b\ } A \longrightarrow H_0(A) \longrightarrow 0$$

$$H_n(A)=0 \quad \text{for} \quad n\geqslant 2.$$

Here

$$\begin{aligned} b(v_1,\ldots,v_{m-1})\otimes v_m &= (v_1,\ldots,v_m)-(v_m,v_1,\ldots,v_{m-1}) \\ &= (1-\sigma)(v_1,\ldots,v_m), \end{aligned}$$

where σ is the cyclic permutation of V^m (without the sign). Thus we obtain

LEMMA 5.2. *The Hochschild homology of $A = T(V)$ is*

$$H_0(A) = \bigoplus_{m\geqslant 0} V^m/(1-\sigma), \qquad H_1(A) = \bigoplus_{m\geqslant 1} (V^m)^\sigma$$

$$H_n(A)=0 \quad \textit{for} \quad n\geqslant 2.$$

Next we look at the spectral sequence going from Hochschild to cyclic homology and note that it stops at E^2 because there are only two nonzero rows. This gives $\bar{H}C_0(A)=\bar{H}_0(A)$, $\bar{H}C_n(A)=\operatorname{Ker} B$ for n even >0, and $\bar{H}C_n(A)=\operatorname{Coker} B$ for n odd>0, where $B:\bar{H}_0(A)\to\bar{H}_1(A)$ is induced by $B:\bar{A}\to A\otimes\bar{A}$, $B(a)=1\otimes a$.

LEMMA 5.3. *With respect to the formulas of Lemma 5.2 the map $B:\bar{H}_0(A)\to\bar{H}_1(A)$ in degree m is given by the norm map*

$$\sum_{i=0}^{m-1}\sigma^i : V^m/(1-\sigma)\to (V^m)^\sigma.$$

Proof. Modulo $b(A\otimes\bar{A}^2)$ we have $(a_1, a_2a_3)\equiv(a_1a_2, a_3)+(a_3a_1, a_2)$ in $A\otimes\bar{A}$.

So if $a = v_1 \cdots v_m$ with v_i in V, then

$$\begin{aligned} B(v_1 \cdots v_m) &= (1, v_1 \cdots v_m) \\ &= (v_1, v_2 \cdots v_m) + (v_2 \cdots v_m, v_1) \\ &= (v_1 v_2, v_3 \cdots v_m) + (v_3 \cdots v_m v_1, v_2) + (v_2 \cdots v_m, v_1) \\ &= \sum_{i=1}^{m} (v_{i+1} \cdots v_m v_1 \cdots v_{i-1}, v_i) \quad \text{in} \quad A \otimes V. \end{aligned}$$

Upon identifying the degree m part of $A \otimes V$ with V^m the lemma follows.

PROPOSITION 5.4. *One has* $\bar{H}C_n(T(V)) = \bigoplus_{m>0} H_n(\mathbb{Z}/m, V^m)$ *where the cyclic group acts on* V^m *via* σ.

This follows by assembling the above lemmas and using the fact that the kernel and cokernel of the norm map gives the homology of a cyclic group.

In characteristic zero the proposition says that

$$\bar{H}C_0(T(V)) = \overline{T(V)}/[T(V), T(V)] = \bigoplus_{m>0} V^m/(1-\sigma)$$

and that $\bar{H}C_n(T(V)) = 0$ for $n > 0$. If one uses the interpretation of cyclic homology in terms of the Lie algebra homology of $\mathfrak{gl}(A)$ proved in section 6, then this formula for the cyclic homology of $T(V)$ was proved by W.-c. Hsiang and R. E. Staffeldt in [6].

6. Homology of Lie algebras of matrices

In this section k is a field of characteristic zero and A is an associative k-algebra (with identity) over k.

For any Lie algebra $\mathfrak{g}$ over k the homology of $\mathfrak{g}$ with coefficients in k is defined by $H_n(\mathfrak{g}) = \mathrm{Tor}_n^{U(\mathfrak{g})}(k, k)$ where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$ (cf. [2, 8]). There is a standard complex $(\Lambda^n\mathfrak{g}, d)$ which computes this homology, where $\Lambda^n\mathfrak{g}$ is the nth exterior product of $\mathfrak{g}$ over k and where

$$d(x_1 \wedge \cdots \wedge x_n) = \sum_{1 \le i < j \le n} (-1)^{i+j} [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_n.$$

Equipped with the Lie bracket $[x, y] = xy - yx$, the k-algebra $\mathcal{M}_r(A)$ of $r \times r$ matrices becomes a Lie algebra over k denoted $\mathfrak{gl}_r(A)$. The inclusions $\mathfrak{gl}_r(A) \hookrightarrow$

$\mathfrak{gl}_{r+1}(A)$ define $\mathfrak{gl}(A)=\varinjlim \mathfrak{gl}_r(A)$. We recall from section 1 that $C_*(A)=(A^{*+1}/(1-t), b)$.

LEMMA 6.1. *The map* $\lambda : \Lambda^{*+1}\mathfrak{gl}(A) \to C_*(\mathcal{M}(A))$ *defined by*

$$\lambda(x_0 \wedge \cdots \wedge x_n) = (-1)^n \sum \operatorname{sgn}(\sigma)(x_0, x_{\sigma 1}, \ldots, x_{\sigma n}),$$

where the sum is over all permutations of $\{1, 2, \ldots, n\}$, *is a map of complexes.*

Proof. We first remark that λ is well defined thanks to the cyclic permutation relation. To prove that $b\lambda = \lambda d$ one verifies easily that both composites applied to $x_0 \wedge \cdots \wedge x_n$ give $\sum \operatorname{sgn}(\sigma)(x_{\sigma 0}x_{\sigma 1}, x_{\sigma 2}, \ldots, x_{\sigma n})$ in $C_n(\mathcal{M}(A))$.

The trace map $\operatorname{Tr}: \mathcal{M}_r(A)^n \to A^n$, given by $\operatorname{Tr}(x, y, \ldots, z) = \sum (x_{i_1 i_2}, y_{i_2 i_3}, \ldots, z_{i_n i_1})$, where the sum is over all possible sets of indices $(i_1, \ldots, i_n)$, is compatible with b and with t. It induces the isomorphism $\operatorname{Tr}_*: HC_*(\mathcal{M}_r(A)) \to HC_*(A)$ (Morita invariance).

The homology of the Lie algebra $\mathfrak{gl}(A)$ is a Hopf algebra. The multiplication is induced by the direct sum $\oplus$ and the comultiplication by the diagonal Δ. An element x in a Hopf algebra is called primitive if $\Delta(x) = x \otimes 1 + 1 \otimes x$. Primitive elements form a graded Lie algebra. In the case of $H_*(\mathfrak{gl}(A))$ the primitive part is a commutative graded Lie algebra.

THEOREM 6.2. *Let k be a field of characteristic zero and A an associative k-algebra. The restriction of* $\operatorname{Tr}_*\lambda_*$ *to the primitive part of the homology of* $\mathfrak{gl}(A)$ *is an isomorphism*

$$\operatorname{Tr}_*\lambda_* : \operatorname{Prim} H_*(\mathfrak{gl}(A)) \xrightarrow{\approx} HC_{*-1}(A).$$

The proof involves invariant theory and a kind of "plus" construction (6.4) for algebraic complexes. We will use the abbreviation $\mathfrak{g}^n$ for $\mathfrak{g}^{\otimes n}$, the n-fold tensor product of $\mathfrak{g}$ over k.

(6.3) *Invariant theory.* Let Σ_n be the symmetric group of order n and let $k[\Sigma_n]$ be its group algebra over k. Suppose V is a vector space over k of dimension r and $\mathfrak{g} = \operatorname{Hom}(V, V)$ is the Lie algebra of endomorphisms of V. The homomorphism $k[\Sigma_n] \to \operatorname{Hom}(V^{\otimes n}, V^{\otimes n}) = \mathfrak{g}^n$ sends a permutation σ to the endomorphism of $V^{\otimes n}$ which permutes the variables according to σ. This endomorphism is invariant under the adjoint action of $\mathfrak{g}$ and the classical invariant theory of H. Weyl [13] asserts that $k[\Sigma_n] \to (\mathfrak{g}^n)^{\mathfrak{g}}$ is surjective. When $r \geq n$ this homomorphism is clearly injective and therefore bijective. By duality and using the natural isomorphism $\mathfrak{g}^* = \mathfrak{g}$ (the star is for dual) one deduces an isomorphism from $k[\Sigma_n] = k[\Sigma_n]^*$ to the module of coinvariants $(\mathfrak{g}^n)_{\mathfrak{g}}$, where Σ_n acts by conjugation on $k[\Sigma_n]$ and by permutation of the variables on $\mathfrak{g}^n$.

PROPOSITION 6.4. *Let $\mathfrak{g}'$ be a Lie algebra over k and $\mathfrak{g}$ a sub-Lie algebra of $\mathfrak{g}'$. Suppose that $\Lambda^n\mathfrak{g}'$ is semi-simple as a $\mathfrak{g}$-module for all n. Then taking the coinvariants with respect to $\mathfrak{g}$ gives a morphism of complexes*

$$\Lambda^*\mathfrak{g}' \twoheadrightarrow (\Lambda^*\mathfrak{g}')_{\mathfrak{g}}$$

which is a quasi-isomorphism.

Proof. There is a direct sum decomposition of complexes $\Lambda^n\mathfrak{g}' = (\Lambda^n\mathfrak{g}')_{\mathfrak{g}} \oplus L_n$ where L_n is made of simple modules on which $\mathfrak{g}$ does not act trivially. As $\mathfrak{g}$ acts trivially on the homology of $\mathfrak{g}'$ the complex L_* has to be acyclic and the proposition is proved.

(6.5) The important consequence of taking the coinvariants in the case of $\mathfrak{g} = \mathfrak{gl}(k)$ and $\mathfrak{g}' = \mathfrak{gl}(A)$ (with inclusion induced by $x \mapsto x \cdot 1$) is that the direct sum $\oplus$ becomes an associative operation. As a consequence $((\Lambda^*\mathfrak{gl}(A))_{\mathfrak{gl}(k)}, d)$ is a differential graded Hopf algebra.

PROPOSITION 6.6. *The primitive part of $((\Lambda^*\mathfrak{gl}(A))_{\mathfrak{gl}(k)}, d)$ is the complex $C_{*-1}(A)$.*

Proof. The k-vector space of rank 1 on which Σ_n acts by the signature will be denoted (sgn). Let $\mathfrak{g} = \mathfrak{gl}(k)$ and $\mathfrak{g} \otimes A = \mathfrak{gl}(A)$. There is a sequence of isomorphisms (see 6.3 for the last one):

$$\begin{aligned}(\Lambda^n(\mathfrak{g}\otimes A))_{\mathfrak{g}} &= ((\mathfrak{g}\otimes A)^n \otimes_{\Sigma_n} (\mathrm{sgn}))_{\mathfrak{g}} = ((\mathfrak{g}^n \otimes A^n) \otimes_{\Sigma_n} (\mathrm{sgn}))_{\mathfrak{g}} \\ &= ((\mathfrak{g}^n)_{\mathfrak{g}} \otimes A^n) \otimes_{\Sigma_n} (\mathrm{sgn}) = (k[\Sigma_n] \otimes A^n) \otimes_{\Sigma_n} (\mathrm{sgn}).\end{aligned}$$

It is important to remark that in the last term Σ_n acts on $k[\Sigma_n]$ by conjugation. This Σ_n-module splits into a direct sum of modules: one for each conjugacy class of Σ_n. Let U_n denote the conjugacy class of the cyclic permutations (i.e. with only one cycle). Now we will prove that the primitive part of $(k[\Sigma_n] \otimes A^n) \otimes_{\Sigma_n} (\mathrm{sgn})$ is $(k[U_n] \otimes A^n) \otimes_{\Sigma_n} (\mathrm{sgn})$. Let $x = [\sigma] \otimes (a_1, \ldots, a_n)$, $\sigma \in \Sigma_n$, $a_i \in A$. Then

$$\Delta(x) = \sum_{I,J} ([\sigma_I] \otimes (\ldots, a_i, \ldots)) \otimes ([\sigma_J] \otimes (\ldots, a_j, \ldots)),$$

where the sum is over all partitions (I, J) of $\{1, \ldots, n\}$ such that $\sigma(I) = I$ and $\sigma(J) = J$. In the formula σ_I (resp. σ_J) denotes the restriction of σ to I (resp. J) and $i \in I$ (resp. $j \in J$). We deduce from this formula that x is primitive if and only if $\sigma \in U_n$.

Any element of U_n is of the form $\sigma\tau\sigma^{-1}$ where $\tau = (12 \cdots n)$ and $\sigma \in \Sigma_n$. As a Σ_n-set U_n is isomorphic to $\Sigma_n/(\mathbb{Z}/n\mathbb{Z})$ where Σ_n acts by left multiplication. Explicitly one has $\sigma\tau\sigma^{-1} \mapsto$ (class of σ). From this we deduce the following sequence of isomorphisms

$$\begin{aligned}(\operatorname{Prim}(\Lambda^*(\mathfrak{g}\otimes A)_{\mathfrak{g}})_n &= (k[U_n]\otimes A^n)\otimes_{\Sigma_n}(\mathrm{sgn})\\ &= (k[\Sigma_n/(\mathbb{Z}/n\mathbb{Z})]\otimes A^n)\otimes_{\Sigma_n}(\mathrm{sgn})\\ &= A^n \otimes_{\mathbb{Z}/n\mathbb{Z}}(\mathrm{sgn}) = C_{n-1}(A)\end{aligned}$$

because $k[\Sigma_n/(\mathbb{Z}/n\mathbb{Z})]$ is induced from the trivial $\mathbb{Z}/n\mathbb{Z}$-module k by the inclusion of $\mathbb{Z}/n\mathbb{Z}$ in Σ_n sending the canonical generator to τ.

To compute the transformation of the differential d by this composition of isomorphisms we remark that the image of $(E_{12}^{a_1} \wedge E_{23}^{a_2} \wedge \cdots \wedge E_{n1}^{a_n})$ is $(-1)^{n-1}(a_1, \ldots, a_n)$, where E_{ij}^a denotes the matrix with exactly one non zero entry a in the ij-position. One easily shows that the image of $d(E_{12}^{a_1} \wedge \cdots \wedge E_{n1}^{a_n})$ by the sequence of isomorphisms is exactly $b(a_1, \ldots, a_n)$. This ends the proof of Proposition 6.6.

We now come back to the proof of Theorem 6.2. The primitive part of $H_*((\Lambda^*\mathfrak{gl}(A))_{\mathfrak{g}})$ is the homology of $\operatorname{Prim}(\Lambda^*\mathfrak{gl}(A))_{\mathfrak{g}}$ that is, in view of Proposition 6.6, the homology of $C_{*-1}(A)$, because we are in characteristic zero. The isomorphism of the theorem follows now from Proposition 6.4.

The computation $\operatorname{Tr}\lambda(E_{12}^{a_1} \wedge \cdots \wedge E_{n1}^{a_n}) = (-1)^{n-1}(a_1, \ldots, a_n)$ finishes the proof.

We now give some immediate consequences of Theorem 6.2. Let $\mathfrak{sl}(A)$ be the Lie algebra of matrices of trace zero (the trace being evaluated in $A/[A, A]$). This Lie algebra is perfect, i.e. $\mathfrak{sl}(A) = [\mathfrak{sl}(A), \mathfrak{sl}(A)]$, and so it has a universal central extension denoted $\mathfrak{st}(A)$ (cf. [7]).

COROLLARY 6.7. *In the characteristic zero case there are isomorphisms*

$$H_2(\mathfrak{sl}(A)) = HC_1(A) \quad \textit{and} \quad H_3(\mathfrak{st}(A)) = HC_2(A).$$

Proof. The exact sequence $0 \to \mathfrak{sl}(A) \to \mathfrak{gl}(A) \to HC_0(A) \to 0$ gives rise to a spectral sequence in homology from which one deduces the isomorphism $H_2(\mathfrak{sl}(A)) = \operatorname{Prim} H_2(\mathfrak{gl}(A))$. And so the first isomorphism follows from 6.2.

The exact sequence $0 \to H_2(\mathfrak{sl}(A)) \to \mathfrak{st}(A) \to \mathfrak{sl}(A) \to 0$ which characterizes the universal central extension gives rise to another spectral sequence in homology. These two spectral sequences together with the vanishing of the groups $H_1(\mathfrak{st}(A))$ and $H_2(\mathfrak{st}(A))$ (cf. [7]) gives an isomorphism $H_3(\mathfrak{st}(A)) = \operatorname{Prim} H_3(\mathfrak{gl}(A))$. And so the second isomorphism follows from 2.2.

Remark. The first isomorphism is true without any hypothesis on the characteristic of k. It was first proved in [1] for the commutative case and in [7] in general.

Remark 6.8. According to J.-L. Koszul [8] one has a spectral sequence

$$E^2_{pq} = H_p(\tilde{\mathfrak{g}}, \mathfrak{g}) \otimes H_q(\mathfrak{g}) \Rightarrow H_{p+q}(\tilde{\mathfrak{g}})$$

for any Lie algebra $\tilde{\mathfrak{g}}$ and sub-Lie algebra $\mathfrak{g}$ such that $\tilde{\mathfrak{g}}$ is semi-simple as a $\mathfrak{g}$-module. We apply this to $\mathfrak{gl}(k) \hookrightarrow \mathfrak{gl}(A)$. On the primitive parts the spectral sequence reduces to a long exact sequence involving (when we apply Theorem 6.2) $HC_*(k)$, $HC_*(A)$ and the homology of $(\bar{A}^{*+1}/(1-t), b)$. As a consequence we get another proof of Proposition 4.4 in characteristic zero: $H_*(\bar{A}^{*+1}/(1-t), b) = HC_*(A)/HC_*(k)$.

The following result gives informations on the stability of the homology of $\mathfrak{gl}_n(A)$ and was announced in [9].

THEOREM 6.9. *Let k be a field of characteristic zero and A an associative k-algebra with* 1. *The stabilization homomorphism $s_i : H_i(\mathfrak{gl}_{n-1}(A)) \to H_i(\mathfrak{gl}_n(A))$ is an isomorphism for $i < n-1$ and an epimorphism for $i = n-1$.*

Moreover, if A is commutative s_{n-1} is also an isomorphism and there is an exact sequence

$$H_n(\mathfrak{gl}_{n-1}(A)) \xrightarrow{s_n} H_n(\mathfrak{gl}_n(A)) \longrightarrow \Omega^{n-1}_A/d\Omega^{n-2}_A \longrightarrow 0.$$

Proof. We put $\mathfrak{g}_n = \mathfrak{gl}_n(k)$. By Proposition 6.4 the homology of $\mathfrak{g}_n \otimes A$ can be computed using the complex $L_* = (\Lambda^*(\mathfrak{g}_n \otimes A))_{\mathfrak{g}_n} = ((\mathfrak{g}_n^*)_{\mathfrak{g}_n} \otimes A^*) \otimes_{\Sigma_*} (\mathrm{sgn})$. We will compute the n first terms of the relative homology groups of the pair $(\mathfrak{g}_n \otimes A, \mathfrak{g}_{n-1} \otimes A)$ which are the homology groups of the quotient complex L_*/L'_*, where L'_* is the similar complex corresponding to $n-1$. By invariant theory (cf. 6.3) the map $(\mathfrak{g}^i_{n-1})_{\mathfrak{g}_{n-1}} \to (\mathfrak{g}^i_n)_{\mathfrak{g}_n}$ is an isomorphism when $i \leq n-1$. Therefore $L_i/L'_i = 0$ and $H_i(\mathfrak{gl}_n(A), \mathfrak{gl}_{n-1}(A)) = 0$ for $i \leq n-1$. It follows from the homology exact sequence that s_i is an isomorphism for $i < n-1$ and an epimorphism for $i = n-1$.

We will now compute the middle term of the homology exact sequence

$$H_n(\mathfrak{gl}_{n-1}(A)) \xrightarrow{s_n} H_n(\mathfrak{gl}_n(A)) \xrightarrow{p} H_n(L_*/L'_*)$$
$$\longrightarrow H_{n-1}(\mathfrak{gl}_{n-1}(A)) \xrightarrow{s_{n-1}} H_{n-1}(\mathfrak{gl}_n(A)) \longrightarrow 0$$

If V is a vector space of dimension $n-1$, then the kernel of the surjective homomorphism $k[\Sigma_n] \to \operatorname{Hom}(V^{\otimes n}, V^{\otimes n})^{\mathfrak{g}_n}$ is of dimension 1 and generated by $\sum_{\sigma\in\Sigma_n}(\operatorname{sgn}\sigma)\sigma$. By duality we deduce a short exact sequence

$$0 \longrightarrow (\mathfrak{g}^n_{n-1})_{\mathfrak{g}_{n-1}} \longrightarrow (\mathfrak{g}^n_n)_{\mathfrak{g}_n} = k[\Sigma_n] \xrightarrow{\operatorname{sgn}} k \longrightarrow 0.$$

Therefore we have $L_n/L'_n = (k\otimes A^n)\otimes_{\Sigma_n}(\operatorname{sgn}) = \Lambda^n A$. To determine the boundaries of $\Lambda^n A$ it is sufficient to compute the image of the composite

$$L_{n+1} \xrightarrow{d} L_n \longrightarrow\!\!\!\!\!\rightarrow \Lambda^n A.$$

We have seen in the proof of 6.6 that the restriction of the differential d to the primitive part is b. Thus the image in $\Lambda^n A$ is generated by the elements

$$\bar{b}(a_0,\ldots,a_n) = \sum_{i=0}^{n-1}(-1)^{i+1}(a_0\wedge\cdots\wedge a_i a_{i+1}\wedge\cdots\wedge a_n)$$
$$-(-1)^n(a_n a_0\wedge\cdots\wedge a_{n-1}).$$

Suppose now that A is commutative. Then the following formula proves that $\Lambda^n A/\operatorname{Im}\bar{b}$ is isomorphic to $\Omega^{n-1}_A/d\Omega^{n-2}_A$:

$$(a_0a_1\wedge a_2 - a_0\wedge a_1a_2 + a_0a_2\wedge a_1)\wedge a_3\wedge\cdots\wedge a_n$$
$$= 1/(n-1)!\sum \operatorname{sgn}(\sigma)\bar{b}(a_0, a_{\sigma 1},\ldots,a_{\sigma n}),$$

where the sum is over all permutations σ of $\{1,\ldots,n\}$ such that $\sigma^{-1}(1) < \sigma^{-1}(2)$.

Therefore $H_n(L_*/L'_*) = \Lambda^n A/\operatorname{Im}\bar{b} = \Omega^{n-1}_A/d\Omega^{n-2}_A$. To prove that the map p of the homology exact sequence is surjective it is sufficient to remark that the element $(1/n!)\sum_{\sigma\in\Sigma_n}\operatorname{sgn}(\sigma)\sigma\otimes(a_1,\ldots,a_n)\otimes 1$ of L_n is a cycle in L_* and maps to $a_1da_2\cdots da_n$ in $\Omega^{n-1}_A/d\Omega^{n-2}_A$. Thus the second assertion follows from the homology exact sequence.

Remark 6.10. The composition

$$HC_n(A) \longrightarrow H_{n+1}(\mathfrak{gl}(A)) = H_{n+1}(\mathfrak{gl}_{n+1}(A)) \xrightarrow{p} \Omega^n_A/d\Omega^{n-1}_A \quad \text{is} \quad (-1)^n\mu_{n,0}.$$

Remark 6.11. Let $GL(A)$ be the general linear group $\varinjlim GL_n(A)$. The homology $H_*(GL(A), \mathbb{Q})$ of this discrete group is a Hopf algebra and its primitive part is rational algebraic K-theory $K_*(A)\otimes\mathbb{Q}$ (cf. [10]). By analogy Prim $H_*(\mathfrak{gl}(A), k)$ should be called *additive algebraic K-theory*. Many results in algebraic K-theory have their counterpart in the additive framework. For instance the role of Milnor's K-theory is played by $\Omega^*/d\Omega^{*-1}$ and Theorem 6.9 is analogous to a theorem of A. Suslin.

(6.12) We now analyse a filtration on cyclic homology induced by the rank filtration on $\mathfrak{gl}(A)$. For $i \leq n$, $(\mathfrak{g}_i^n)^{\mathfrak{g}_i}$ is the 2-sided ideal F^{n-i} of $(\mathfrak{g}_n^n)^{\mathfrak{g}_n} = k[\Sigma_n]$. For instance $F^n = 0$, F^{n-1} is of dimension 1 generated by $\Sigma\, \mathrm{sgn}(\sigma)\sigma$, F^1 is the augmentation ideal and $F^0 = k[\Sigma_n]$. By duality we obtain a filtration on $(\mathfrak{g}_n^n)_{\mathfrak{g}_n} = k[\Sigma_n]^* = k[\Sigma_n]$:

$$0 = F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n = k[\Sigma_n],$$

where now F_1 is of dimension 1 generated by $\Sigma\,\sigma$ and F_{n-1} is the kernel of the signature homomorphism.

The modules F_i can be interpreted in terms of irreducible representations of Σ_n that is in terms of Young diagrams with less than i rows.

The filtration F_* determines a filtration of the submodule $k[U_n]$ and therefore a filtration of $C_{n-1}(A) = (k([U_n]\otimes A^n)\otimes_{\Sigma_n}(\mathrm{sgn}))$, that we denote by $F_*(C_{n-1}(A))$. In particular $F_iC_n(A) = C_n(A)$ as soon as $i > n$.

This filtration of $C_*(A)$ comes from the filtration of $\mathfrak{gl}$ by the $\mathfrak{gl}_i$. Thus, in view of Proposition 6.6 and using the fact that the boundary d preserves the rank filtration, it is immediately seen that the boundary operator b of $C_*(A)$ respect the filtration. As an immediate consequence we have:

PROPOSITION 6.13. *There exists a first quadrant spectral sequence*

$$E^1_{pq} = H_p(F_{p+q}C_*/F_{p+q-1}C_*) \Rightarrow HC_{p+q}.$$

As a corollary of Theorem 6.9 one can compute $E^1_{p0} = \Omega^p/d\Omega^{p-1}$ and the edge homomorphism is the map $(-1)^n\mu_{n,0}$ (cf. 2.8).

Considering the rank filtration in algebraic K-theory (cf. 6.11) C. Soulé has conjectured the vanishing of some K-groups [11, §2.10]. Similarly in the additive framework it is natural to conjecture the vanishing of $H_n(GL_i(A)) \cap \operatorname{Prim} H_n(GL(A)) = F_i HC_{n-1}(A)$ for $2i \leq n$. Translated in terms of symmetric groups this is equivalent to the following statement.

6.14. *The filtration F_* on $k[U_n]$ is such that*

$$F_i k[U_{2i+1}] = 0 \quad \textit{and} \quad F_i k[U_{2i}] = 0.$$

Moreover it is expected that $F_{i+1}k[U_{2i+1}]$ is of dimension 1 and generated by $\Sigma a(\sigma)\sigma$ where the sum is over U_{2i+1}, $a(\sigma) = \operatorname{sgn}(g)$ and g is such that $g\sigma g^{-1} = (12 \cdots 2i+1)$.

These assertions were stated as conjectures in the first draft of this paper. But C. Procesi informed us that the first one follows from a result of J. Levitzki on polynomial identities and that the second one follows from the Amitsur–Levitzki formula for matrices.

As a consequence the rank filtration on HC_{2n} and HC_{2n+1} is of length $n+1$, which is the same as the length of the filtration deduced from the filtration of $\mathcal{B}(A)'$ by columns. Proofs of these reults will appear elsewhere.

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