CHAPTER 8: THE ELEMENTS OF ARF INVARIANT ONE

1. Introduction

One of the most important open problems in homotopy theory is whether or not there exist elements $\theta_N \in \pi^S_{2^{N+1}-2}$ of Arf invariant one. These elements arose in the work of Kervaire [26] and Kervaire and Milnor [27] as obstructions in surgery theory. Browder [14] showed that the nonvanishing of these obstructions is equivalent to the elements $h_N^2 \in E_2^{2^{N+1}-2,2} = \mathrm{Ext}^2_{\mathfrak{A}}(Z_2,Z_2)_{2^{N+1}-2}$ being infinite cylcles in the classical Adams spectral sequence for π^S_*. Thus, an element $\theta_N \in \pi^S_{2^{N+1}-2}$ has Arf invariant one if and only if the secondary cohomology operation Φ_N defined by the following Adem relation is nonzero in the mapping cone of θ_N:

$$(8.1.1) \qquad 0 = \sum_{i=0}^{N} Sq^{2^{N+1}-2^i} Sq^{2^i}.$$

The first three elements of Arf invariant one are merely η^2, ν^2 and σ^2. The next two elements of Arf invariant one, $\theta_4 \in \pi^S_{30}$ and $\theta_5 \in \pi^S_{62}$, have been shown to exist using the classical Adams spectral sequence [37], [11]. It is not known whether θ_N exists for $N \geq 6$. The reader can find a more detailed exposition of this problem in [12] and [13].

In Section 2 we show that the element $A[30] \in \pi^S_{30}$ has Arf invariant one by calculating that the secondary operation Φ_4 is nonzero in the mapping cone of A[30]. In Section 3 we identify θ_5 as A[62,1] by showing that $\theta_4^2 = 0$ using an argument of Mahowald based upon a generalization of [34A, Theorem 16]. The construction of Barratt, Jones and Mahowald [11] shows that θ_5 exists but does not determine the order of θ_5. The argument of Section 3 shows that there are choices of θ_5 of order two.

In [35] Mahowald showed that the elements $h_1 h_N \in E_2^{2^N,2}$ of the Adams spectral sequence are infinite cycles which are represented by the elements $\eta_N \in \pi^S_{2^N}$. In Section 2 we identify η_5 as A[32,1]. In Section 3, we identify η_6 as B[64,1].

2. The Existence of θ_4

Recall from Theorem 5.3.10 that $\pi^S_{30} = Z_2 A[30]$ and $A[30] = d^{12}(2\sigma M_1^{12})$. We will show that the secondary operation Φ_4 is nonzero in the mapping cone of A[30]. It follows that $A[30] = \theta_4$ has Arf invariant one. We will assume the definitions and basic properties of secondary cohomology operations and functional cohomology operations [47].

Let $f: S^{23} \longrightarrow BP^{(16)}$ be the attaching map of the cell represented in homology by $\langle M_1^4\rangle^3$. This map has image in the 16-skeleton becasause $\langle M_1^4\rangle^3$ survives to E^8. Let $i: BP^{(14)} \longrightarrow BP^{(16)}$ be the natural inclusion. Since $d^8(2\sigma\langle M_1^4\rangle^3) = 0$, there is a lifing F of $2\sigma f$ to $BP^{(14)}$:

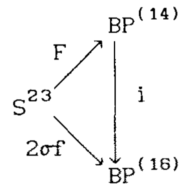

FIGURE 8.2.1: Definition of F

The next step is to define a map $G: \Sigma^7 C_f \longrightarrow C_F$. We begin by defining $G_1 = G|\Sigma^7 BP^{(8)}: \Sigma^7 BP^{(8)} \longrightarrow BP^{(8)}$ as the composite of the projection map $P: \Sigma^7 BP^{(8)} \longrightarrow \Sigma^7 BP^{(8)}/\ \Sigma^7 BP^{(6)} = S^{15} \vee S^{15}$ followed by $g \vee *$ where the first sphere is represented by $\langle M_1^4\rangle$ in homology, the second sphere is represented by $M_1 \bar{M}_2$ in homology and g is the attaching map of the cell represented by $\langle M_1^4\rangle^2$ in homology. Since $\langle M_1^4\rangle^2$ survives to E^8 and $d^8(\langle M_1^4\rangle^2) = 2\sigma\langle M_1^4\rangle$, the following diagram commutes:

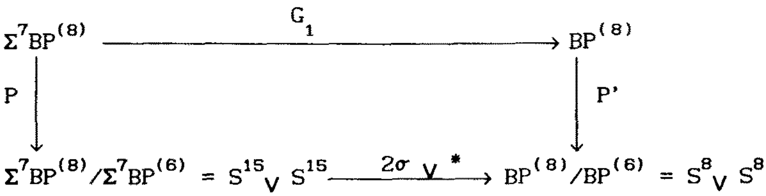

FIGURE 8.2.2: Definition of G_1

P' is the natural projection map above. Now define $G_2:\Sigma^7BP^{(16)} \longrightarrow BP^{(16)}$ as the composite of the projection map

$$P'':\Sigma^7BP^{(16)} \longrightarrow \Sigma^7BP^{(16)}/\ \Sigma^7BP^{(14)} = S^{23} \vee 3S^{23}$$

followed by $(g \wedge \alpha)' \vee 3*$. Here the first copy of S^{23} is represented by $<M_1^4>^3$ in homology and $\alpha:(D^8,S^7) \longrightarrow (BP^{(8)},S)$ represents $<M_1^4>$. Also, $(g \wedge \alpha)':S^{15} \wedge S^8 \longrightarrow BP^{(16)}$ is an extension of $S^{15} \wedge D^8 \xrightarrow{g\wedge\alpha} BP^{(8)} \wedge BP^{(8)} \xrightarrow{\varepsilon} BP^{(16)}$, thinking of D^8 as the upper hemisphere of S^8. This extension to S^{15} smash the bottom hemisphere exists as a map into $BP^{(8)}$ because $2\sigma^2 = 0$ in π_*^S. The top square in Figure 8.2.3 commutes because G_2 restricts to 2σ on Σ^7 of the cell C represented by $<M_1^4>^2$ in homology. Thus, the map G_3 must exist making the bottom square commute. Now G_3 maps all cells into $BP^{(14)}$ except for the cell Σ^7C, and G_2 on this cell is $2\sigma \wedge 1$. In Σ^7C_f, $C\Sigma^7S^{23}$ is attached to this cell by Σ^7f. Therefore G_3 on this cell is $2\sigma \wedge f$ which, as in Figure 8.2.1, lifts to $BP^{(14)}$. Thus G_3 lifts to a map G.

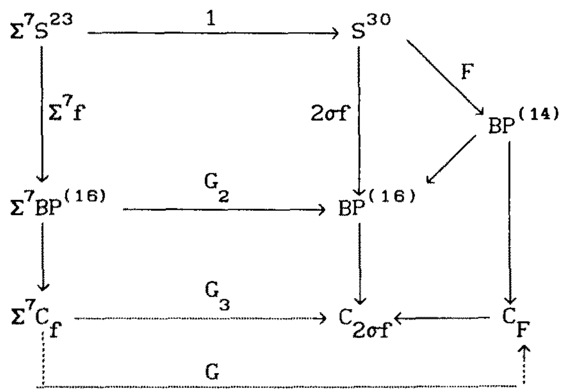

FIGURE 8.2.3: Definition of G

In $H^*(C_G;Z_2)$, let u(X) denote the element dual to $X \in H_*(C_G;Z_2)$. Let $Y \in H_{16}(C_G;Z_2)$ denote the element determined by the first sphere in Figure 8.2.2. By the definition of G_1, Y represents a cell with the same attaching map as $\langle M_1^4\rangle^2$. Therefore, $Sq^{16}u(1) = u(Y)$. Thus, the functional secondary cohomology operation Sq_G^{16} is defined on $u(1) \in H^0(C_F;Z_2)$ and equals $S^7u\langle M^4\rangle \in H^{15}(\Sigma^7C\ ;Z\ )$. By the Peterson-Stein formula: $G^*\circ\Phi\ (u(1))$
$= Sq^{31}Sq_G^1(u(1))+Sq^{30}Sq_G^2(u(1))+Sq^{28}Sq_G^4(u(1))+Sq^{24}Sq_G^8(u(1))+Sq^{16}Sq_G^{16}(u(1))$.
Since $H^k(\Sigma^7C_f;Z_2) = 0$ for $k = 0,1,3$, we must have $Sq_G^1(u(1)) = 0$, $Sq_G^2(u(1)) = 0$ and $Sq_G^4(u(1)) = 0$. Since $G_1|\Sigma^7BP^{(6)} = *$, $Sq_G^8(u(1))$ must be 0 not $S^7u(1)$. Thus, $G^*\circ\Phi_4(u(1)) = Sq^{16}Sq_G^{16}(u(1)) = Sq^{16}(S^7u\langle M_1^4\rangle) = S^7u(\langle M_1^4\rangle^3) \neq 0$. Thus, $\Phi_4(u(1)) \neq 0$ in $H^{31}(C_F;Z_2)$. Note that there is a unique top dimensional cell of degree 31 in C_F which determines a nononzero element $\tau \in H^{31}(C_F;Z_2)$. Hence $\tau = \Phi_4(u(1)) \neq 0$ Since $d^{12}(2\sigma M_1^{12}) = A[30]$ and F represents the boundary of $2\sigma M_1^{12}$, the triangle in the following diagram must commute up to homotopy. Therefore, there is an induced map J making the square commute.

(**)

$$\begin{array}{ccccc} & & BP^{(14)} & \longrightarrow & C_F \\ & \overset{F}{\nearrow} & \uparrow & & \vdots\ \uparrow J \\ S^{30} & & & & \\ & \underset{A[30]}{\searrow} & & & \\ & & S & \longrightarrow & C_{A[30]} \end{array}$$

Now $\Phi_4(u(\iota)) = \Phi_4\circ J^*(u(1)) = J^*\circ\Phi_4(u(1)) = J^*(\tau) \neq 0$. Thus, A[30] must have Arf invariant one. We have thus proved the following theorem.

THEOREM 8.2.1 A[30] has Arf invariant one.

We derive several Toda brackets involving elements related to θ_4. The first Toda bracket below was proved by Hoffman [24]. We give a proof using our spectral sequence.

THEOREM 8.2.2 (a) $\theta_4 = A[30] \in \langle\sigma, 2\sigma, 2\sigma, \sigma\rangle$

(b) $\nu\theta_4 = \nu A[30] \in \langle C[18], \sigma, 2\sigma\rangle$

(c) $\theta_4 = A[30] \in \langle\sigma, 2\sigma, \sigma^2, 2\rangle = \langle\sigma^2, 2, \sigma^2, 2\rangle$

(d) $\eta\theta_4 = \eta A[30] \in \langle A[16], 2, \sigma^2\rangle$

PROOF. (a) Represent $\langle M_1^4\rangle^2$ by μ_8 such that $\partial(\mu_8) = (\sigma \wedge 2\mu_4) \cup (B_{\sigma 2\sigma})$. Since $2\cdot\pi_{22}^S = 0$, $\sigma A[14] = 0$ and $\sigma\gamma_1 = 0$, it follows that $\langle\sigma, 2\sigma, 2\sigma\rangle = 2\langle\sigma, 2\sigma, \sigma\rangle = 0$. Thus, $2\sigma M_1^{12} \in E_{24,7}^{24}$ is represented by

$$\mathcal{M} = (\mu_4 \wedge \sigma \wedge 2\mu_8) \cup (B_{\sigma^2 2} \wedge \mu_8) \cup (\mu_4 \wedge B_{\sigma 2\sigma} \wedge 2\mu_4) \cup (B_{\langle\sigma,\sigma 2,\sigma 2\rangle} \wedge \mu_4) \cup (\mu_4 \wedge B_{\langle\sigma 2,\sigma 2,\sigma\rangle})$$

because $\partial\mathcal{M} = (B_{\sigma^2 2} \wedge B_{\sigma 2\sigma}) \cup (B_{\langle\sigma,\sigma 2,\sigma 2\rangle} \wedge \sigma) \cup (\sigma \wedge B_{\langle\sigma 2,\sigma 2,\sigma\rangle})$. Since $d^{24}(2\sigma M_1^{12}) = A[30]$, $\partial\mathcal{M}$ represents $A[30]$ and clearly $\partial\mathcal{M} \in \langle\sigma, \sigma 2, \sigma 2, \sigma\rangle$.

(b), (c) The four-fold Toda bracket $\langle\sigma, 2\sigma, \sigma^2, 2\rangle$ is defined by Theorem 2.2.7(a) because $\langle\sigma, 2\sigma, \sigma^2\rangle \in \pi_{29}^S = 0$ and $\langle 2\sigma, \sigma^2, 2\rangle = \sigma\langle 2, \sigma^2, 2\rangle + 2\cdot\pi_{22}^S = \sigma(\eta\sigma^2) = 0$. Now $\nu A[30] \in \nu\langle\sigma, 2\sigma, 2\sigma, \sigma\rangle \subset \langle\langle\nu, \sigma, 2\sigma\rangle, 2\sigma, \sigma\rangle = \langle C[18], 2\sigma, \sigma\rangle$. Since $\mathrm{Cok}J_{26} = Z_2\nu^2 C[20]$, $\nu A[30] \in \langle C[18], \sigma, 2\sigma\rangle \subset \langle C[18], \sigma^2, 2\rangle = \langle\langle\nu, \sigma, 2\sigma\rangle, \sigma^2, 2\rangle = \nu\langle\sigma, 2\sigma, \sigma^2, 2\rangle$. Thus, $\langle\sigma, 2\sigma, \sigma^2, 2\rangle$ contains $A[30]$. Note that $\langle\sigma^2, 2, \sigma^2, 2\rangle$ is defined by Theorem 2.2.7(a) because $\langle\sigma^2, 2, \sigma^2\rangle \in \pi_{29}^S = 0$ and $\langle 2, \sigma^2, 2\rangle = \eta\sigma^2 = 0$. Now $\langle\sigma^2, 2, \sigma^2, 2\rangle \subset \langle\sigma, 2\sigma, \sigma^2, 2\rangle = \{A[30]\}$.

(d) $\eta A[30] \in \eta\langle 2, \sigma^2, 2\sigma, \sigma\rangle \subset \langle\langle\eta, 2, \sigma^2\rangle, 2\sigma, \sigma\rangle = \langle A[16], 2\sigma, \sigma\rangle + \langle\eta\gamma_1, 2\sigma, \sigma\rangle$. Now $\langle\eta\gamma_1, 2\sigma, \sigma\rangle \supset \eta\langle\gamma_1, 2\sigma, \sigma\rangle$. Since $\nu\langle\gamma_1, 2\sigma, \sigma\rangle = \langle\nu, \gamma_1, 2\sigma\rangle\sigma = 0$, $\langle\gamma_1, 2\sigma, \sigma\rangle$ can not equal $A[30]$ and must therefore equal zero. It follows that $\langle\eta\gamma_1, 2\sigma, \sigma\rangle = \eta\gamma_1\cdot\pi_{15}^S + \sigma\cdot\pi_{24}^S = \eta\xi$ where $\nu\xi = 0$. Thus, $\langle\eta\gamma_1, 2\sigma, \sigma\rangle = 0$ and $\eta A[30] \in \langle A[16], 2\sigma, \sigma\rangle$. ∎

We conclude this section by identifying the Mahowald element $\eta_5 \in \pi_{32}^S$ as $A[32,1]$.

THEOREM 8.2.3 Let η_5 be <u>any</u> element of π^S_{32} which projects to h_1h_5 in $E^{32,2}_\infty$ of the Adams spectral sequence. Then η_5 projects to A[32,1] in $E^{24}_{0,32}$ of the Atiyah-Hirzebruch spectral sequence.

PROOF. From the computation of E_2 of the Adams spectral sequence by Tangora [59], it follows from the fact that h_1h_5 is an infinite cycle that $h_1^3h_5$ is a nonbounding infinite cycle. Thus, if η_5 is any element that projects to h_1h_5 then $\eta^2\eta_5 \neq 0$. Since $\eta^2\cdot\pi^S_{32} = Z_2\eta^2 A[32,1]$ for any choice of A[32,1] modulo $Z_2A[32,2] \oplus Z_2A[32,3] \oplus Z_2\eta\gamma_3$, it follows that $\eta_5 \in A[32,1] + (Z_2A[32,2] \oplus Z_2A[32,3] \oplus Z_2\eta\gamma_3)$. Now the theorem follows from the observation that $Z_2A[32,2] \oplus Z_2A[32,3] \oplus Z_2\eta\gamma_3$ projects to zero in $E^{24}_{0,32}$ of the Atiyah-Hirzebruch spectral sequence. ∎

3. The Existence of θ_5

In this section we show that A[62,1] has Arf invariant one and is thus entitled to be denoted as θ_5. We also identify the Mahowald element η_6 as B[64,1]. In addition, we derive a few miscellaneous results which are relevant to the Arf invariant problem. We begin with the following well known lemma which can be proved from a computation of $\mathrm{Ext}_{\mathfrak{A}}(Z_2,Z_2)$ as the homology of the Λ-algebra.

LEMMA 8.3.1 The following elements are nonzero in $\mathrm{Ext}_{\mathfrak{A}}(Z_2,Z_2)$:

(a) h_N and h_N^2 for $N \geq 0$;

(b) $h_0h_N^2$ and h_1h_N for $N \geq 3$;

(c) $h_1h_N^2$ for $N \geq 4$;

(d) $h_1^2h_N^2$ for $N \geq 5$.

Adams's proof [2] of the nonexistence of elements of Hopf invariant one in degrees 2^N-1, $N \geq 4$, is equivalent to the following differentials in the Adams spectral sequence. The elements listed in Lemma 8.3.1 and the differentials of Theorem 8.3.2 for $N \geq 4$ are depicted in Figure 8.3.1. Note that there are other elements in the bidegrees of that figure which are not depicted.

THEOREM 8.3.2 $d^2(h_N) = h_0 h_{N-1}^2$ for $N \geq 4$.

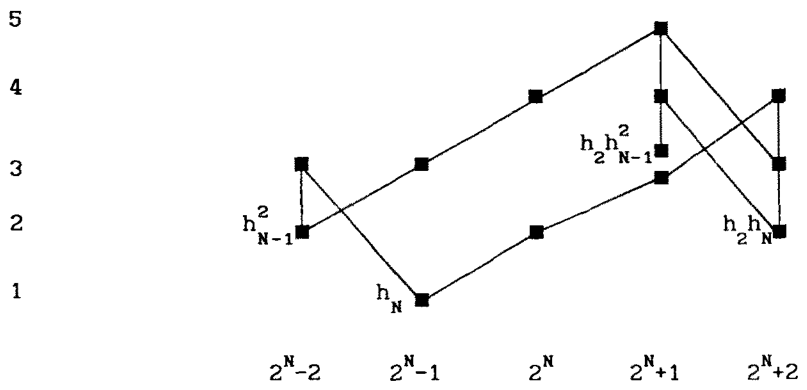

FIGURE 8.3.1: Part of E^2 of the Adams Spectral Sequence ($N \geq 6$)

The following lemmas will be used to identify θ_5 as A[62,1]. The entire argument is based upon ideas of Mahowald [34A] and is a rewording of a detailed proof which he sent to me.

LEMMA 8.3.3 $\langle\sigma^2, 2, A[30]\rangle \subset Z_2(\eta C[44]) \oplus Z_2(8D[45])$

PROOF. Note that $\eta^2\langle\sigma^2, 2, A[30]\rangle = \sigma^2\langle 2, A[30], \eta^2\rangle \subset \sigma^2\cdot\pi_{33}^S = 0$. Also, $\nu^2\langle\sigma^2, 2, A[30]\rangle = \sigma^2\langle 2, A[30], \nu^2\rangle \subset \sigma^2\cdot\pi_{37}^S = \sigma(4C[44]) = 0$. In addition, $2\langle\sigma^2, 2, A[30]\rangle = \sigma^2\langle 2, A[30], 2\rangle = 0$. The only elements of π_{45}^S which satisfy these three conditions are $Z_2(\eta C[44]) \oplus Z_2(8D[45])$. ∎

LEMMA 8.3.4 If $\xi \in \pi_*^S$ and $\xi A[36] = 0$ then

$$\xi C[44] \in \langle\eta\xi, \eta A[30], \nu, \sigma\rangle.$$

PROOF. By Theorem 2.4.6(a), if $\langle \eta, \eta A[30], \nu, \sigma \rangle$ were defined then it would contain C[44]. Now $\langle \eta A[30], \nu, \sigma \rangle \supset A[30]\langle \eta, \nu, \sigma \rangle = 0$, $\eta A[30] \cdot \pi_{11}^S = 0$ and $\sigma \cdot \pi_{35}^S = 0$. Thus, $\langle \eta A[30], \nu, \sigma \rangle = 0$. However, $A[36] \in \langle \eta, \eta A[30], \nu \rangle$. Since $\xi A[36] = 0$, $\langle \eta\xi, \eta A[30], \nu, \sigma \rangle$ is defined by Theorem 2.2.7(b). Thus, $\xi C[44] \in \langle \eta\xi, \eta A[30], \nu, \sigma \rangle$. ▮

LEMMA 8.3.5 (a) $A[16]A[36] = 0$.

(b) $\eta A[16]A[30] = 0$.

(c) $A[16]C[44] = 0$.

PROOF. (a) $A[16]A[36] \in A[36]\langle \eta, 2, \sigma^2 \rangle = \langle A[36], \eta, 2 \rangle \sigma^2 \in \sigma^2 \cdot \pi_{38}^S = 0$.

(b) $\eta A[16]A[30] \in \eta A[30]\langle \eta, 2, \sigma^2 \rangle \subset \langle 0, 2, \sigma^2 \rangle = \sigma^2 \cdot \pi_{33}^S = 0$.

(c) Since $A[16]A[36] = 0$, $A[16]C[44] \in \langle \eta A[16], \eta A[30], \nu, \sigma \rangle$ $\supset \langle \eta A[16]A[30], \eta, \nu, \sigma \rangle = \langle 0, \eta, \nu, \sigma \rangle$. (Note that $\langle \eta A[16]A[30], \eta, \nu, \sigma \rangle$ is defined by Theorem 2.2.7(b) because $0 \in \langle \eta A[16]A[30], \eta, \nu \rangle$ and $0 = \langle \eta, \nu, \sigma \rangle$.) Since $\sigma \cdot \pi_{53}^S = 0$, $A[16]C[44] \in \langle \pi_{49}^S, \nu, \sigma \rangle + \eta A[16] \cdot \pi_{43}^S + \langle \eta A[16], \pi_{35}^S, \sigma \rangle$
$= \langle \alpha_6, \nu, \sigma \rangle + \langle \eta^2\gamma_5, \nu, \sigma \rangle + \langle \eta A[16], \eta A[14]C[20], \sigma \rangle + \langle \eta A[16], \nu A[32,3], \sigma \rangle + \langle \eta A[16], \beta_4, \sigma \rangle$
$= \langle \alpha_6, \nu, \sigma \rangle + \eta\gamma_5\langle \eta, \nu, \sigma \rangle + \langle \eta A[16], \eta C[20], 0 \rangle + A[16]A[32,3]\langle \eta, \nu, \sigma \rangle + A[16]\langle \eta, \beta_4, \sigma \rangle$
$= \langle \alpha_6, \nu, \sigma \rangle$. By Theorem 4.2.3 and Figure 4.2.2, it follows that $\langle \alpha_6, \nu, \sigma \rangle$ projects to an element of filtration degree at least 26 in the Adams spectral sequence. The only such element is $h_0^2P^5g = d^2(h_0P^4k)$. Thus, $0 = \langle \alpha_6, \nu, \sigma \rangle$ $= A[16]C[44]$. ▮

LEMMA 8.3.6 $A[30]^2 = 0$

PROOF. $A[30]^2 \in A[30]\langle 2, \sigma^2, 2, \sigma^2 \rangle \subset \langle\langle A[30], 2, \sigma^2 \rangle, 2, \sigma^2 \rangle$
$\subset \langle \eta C[44], 2, \sigma^2 \rangle + \langle 8D[45], 2, \sigma^2 \rangle \supset C[44]\langle \eta, 2, \sigma^2 \rangle + \langle 2\sigma B[38], 2, \sigma^2 \rangle$
$\supset C[44]A[16] + \sigma\langle 2B[38], 2, \sigma^2 \rangle \subset \sigma \cdot \pi_{53}^S = 0$. Since $\eta A[36] = 0$,
$A[30]^2 \in \eta C[44] \cdot \pi_{15}^S + 8D[45] \cdot \pi_{15}^S + \sigma^2 \cdot \pi_{46}^S = \eta\gamma_1 C[44] \in \gamma_1\langle \eta^2, \eta A[30], \nu, \sigma \rangle$
$\subset \langle \eta^2, \eta A[30], \langle \nu, \sigma, \gamma_1 \rangle\rangle \supset \langle \eta^2, A[30], \eta\langle \nu, \sigma, \gamma_1 \rangle\rangle = \langle \eta^2, A[30], \langle \eta, \nu, \sigma \rangle \gamma_1 \rangle$

$= \langle\eta^2, A[30], 0\rangle = \eta^2\cdot\pi^S_{58} = 0$. Thus, $A[30]^2 \in \langle\nu, \sigma, \gamma_1\rangle\cdot\pi^S_{34} \subset \{\nu^2C[20], \eta\alpha_3\}\cdot\pi^S_{34}$ $= \eta\alpha_3A[14]C[20] = \eta A[14]C[20]\langle 8\sigma, 2, \alpha_2\rangle = \eta A[14]\alpha_2\langle C[20], 8\sigma, 2\rangle \in (\eta\cdot\pi^S_{31})\cdot\pi^S_{28}$ $= (\eta\gamma_3)(A[8]C[20]) = 0.$ ∎

THEOREM 8.3.7 $A[62,1]$ + Span $\{A[62,2], A[62,3], A[62,4], B[62], \eta^2B[60]\}$ are all the elements of π^S_{62} of Arf invariant one. In particular, there are choices of θ_5 of order two.

PROOF. Since $\theta_4 = A[30]$ exists, $2\theta_4 = 0$ and $\theta_4^2 = 0$, it follows from [12, Theorem 2.1] that θ_5 exists and has order two. From Figure 8.3.1, we see that any element θ_5 of Arf invariant one satisfies $\eta^2\theta_5 \neq 0$. Since $\eta^2\cdot\pi^S_{62} = Z_2\eta^2A[62,1] \oplus Z_2\eta^2A[62,4]$, Span $\{A[62,2], A[62,3], B[62], \eta^2B[60]\}$ has Adams filtration at least three. Since $A[62,4] = d^{12}(\eta A[50,2]M_1^3\bar{M}_2)$ $= d^{12}(\nu^2A[45,1]M_1^3\bar{M}_2)$ and $C[20] = d^{12}(\nu^3M_1^3\bar{M}_2)$, $\nu A[62,4] = C[20]A[45,1]$. From Figure 8.3.1, $\nu\theta_5$ is nonzero and is represented in the Adams spectral sequence by $h_2h_5^2$ in filtration degree three while $C[20]A[45,1]$ has Adams filtration at least nine. Thus, $A[62,4]$ has Adams filtration at least three. Now all the elements of Span $\{A[62,2], A[62,3], A[62,4], B[62], \eta^2B[60]\}$ have Adams filtration at least three. Therefore, all the elements of $A[62,1]$+Span $\{A[62,2], A[62,3], A[62,4], B[62], \eta^2B[60]\}$ have Arf invariant one. ∎

Next we identify the Mahowald element η_6 in terms of the Atiyah-Hirzebruch spectral sequence. Recall that η_6 denotes any element of π^S_{64} which projects to h_1h_6 in $E_\infty^{64,2}$ of the Adams spectral sequence.

THEOREM 8.3.8 (a) Any choice of η_6 projects to B[64,1] in $E^{54}_{0,64}$ of the Atiyah-Hirzebruch spectral sequence.

(b) All the choices of η_6 are

$B[64,1] + (Z_2 A[64,1] \oplus Z_2 A[64,2] \oplus Z_2 A[64,3] \oplus Z_4 B[64,2] \oplus Z_2 \eta^2 A[62,1] \oplus Z_2 \eta\gamma_6)$.

(c) All the values of $2\eta_6$ are $\eta^2\theta_5 + Z_2\eta^2 A[62,4]$, and $4\eta_6 = 0$.

(d) There are choices of η_5 and η_6 such that $\eta_5^2 = 2\eta_6$.

PROOF. Since A[64,1], A[64,2], A[64,3], B[64,2], $\eta^2 A[62,1]$ and $\eta\gamma_7$ project to zero in $E_2^{64,2}$ of the Adams spectral sequence, all the choices for η_6 are

$\eta_6 = B[64,1] + pA[64,1] + qA[64,2] + rA[64,3] + sB[64,2] + t\eta^2 A[62,1] + u\eta\gamma_7$.

All of these elements project to B[64,1] in $E^{54}_{0,64}$. Moreover, $2\eta_6 =$ $2B[64,1] + 2sB[64,2] = \eta^2 A[62,1] + s\eta^2 A[62,4] = \eta^2\theta_5 + s\eta^2 A[62,4]$ and $4\eta_6 = 0$. Note that η_5^2 projects to $h_1^2 h_5^2$ in the Adams spectral sequence. Thus, η_5^2 is not zero, and by Mahowald [36] there are choices of η_5 and η_6 such that $2\eta_6 = \eta_5^2$. ∎