CHAPTER 2: TODA BRACKETS

1. Introduction

As we saw in the previous chapter, there is one very important step in our computation that is not algorithmic: the determination of the additive and multiplicative structure of π_*^S from the composition series which has been deduced from the Atiyah-Hirzebruch spectral sequence. One of the main tools we will use to determine these extensions is the relationship between Toda brackets in π_*^S and differentials in the spectral sequence. This idea was originated by J. P. May [40, Section 4]. May's three basic theorems regarding the behaviour of Massey products in spectral sequences defined from a filtered differential graded algebra were generalized to the Adams and Atiyah-Hirzebruch spectral sequences in [28]. In addition to these classical results, we will derive and use several new theorems of this type.

In Section 2 we give two definitions of Toda brackets in π_*^S: one using the composition product and one using the the smash product. By [29], these two Toda brackets are always equal. We will find that there are situations in which one point of view is advantageous over the other. In Section 3, we derive the basic properties of these Toda brackets. In Section 4, we prove several theorems which relate these Toda brackets to the differentials in the Atiyah-Hirzebruch spectral sequence. We will only be using three-fold and four-fold Toda brackets in our applications. Therefore, we do not hesitate to specialize to these cases.

2. Definitions

We will find it convenient to work with spectra in the coordinate-free setting of J. P. May [41]. After introducing coordinate-free notation, we give two defininitions of Toda brackets: one based on the smash product and one based on the composition product. These definitions were first given in [29]. Our composition Toda bracket generalizes Toda's orginal three-fold product [60] and Oguchi's four-fold product [51]. It agrees with Spanier's Toda bracket [58] but it is not clear whether it agree's with Gershenson's Toda bracket [21]. Our smash Toda bracket agrees with that of Porter [51] and corresponds under the Pontrjagin-Thom isomorphism to the Massey product of manifolds defined in [28]. In Theorem 2.2.3 we state the theorem from [29] that our two Toda brackets are equal. In addition, our Toda bracket is contained in Joel Cohen's Toda bracket [18]. We conclude this section with several practical criteria for concluding that a four-fold Toda bracket is defined.

The following notation will be used throughout. Let R^{∞} be the real inner product space with orthonormal basis $\mathcal{B} = \{b_1, b_2, \ldots\}$. We consider only finite dimensional subspaces W of R^{∞} which have a subset of $\mathcal{B}$ as a basis. Internal direct sum is denoted by +, and if W' is a subspace of W then $W'^{\perp}$ denotes the orthogonal complement of W' in W. All spaces are based CW complexes, all maps are based and all homotopies, cones and suspensions are reduced. Let S denote one point compactification. The n-sphere is defined as $S^n \equiv S(R^n)$. The isomorphism from a subspace V to $R^{\dim V}$ which preserves the ordered standard bases induces a canonical homeomorphism from SV to $S^{\dim V}$. Thus a map from SV to SW determines an element of $\pi_{\dim V}(S^{\dim W})$. If $i_1 < \cdots < i_t$ then define the disc $D(Rb_{i_1} + \cdots + Rb_{i_t})$ as $CS(Rb_{i_1}) \wedge S(Rb_{i_2} + \cdots + Rb_{i_t})$ where $C(-) = (I,\{1\}) \wedge (-)$ is the cone functor. If $1 \le j_1 < \cdots < j_k \le t$ and $f: SU_1 \wedge \cdots \wedge SU_t \wedge X \longrightarrow SU_1 \wedge \cdots \wedge SU_t \wedge Y$ then define $C_{j_1 \cdots j_k}(f)$ as the canonical map from $C_{j_1,\ldots,j_k}(SU_1 \wedge \cdots \wedge SU_t \wedge X)$ $\equiv SU_1 \wedge \cdots \wedge DU_{j_1} \wedge \cdots \wedge DU_{j_k} \wedge \cdots \wedge SU_t \wedge X$ to $C_{j_1,\ldots,j_k}(SU_1 \wedge \cdots \wedge SU_t \wedge Y)$ $\equiv SU_1 \wedge \cdots \wedge DU_{j_1} \wedge \cdots \wedge DU_{j_k} \wedge \cdots \wedge SU_t \wedge Y$ induced by f. Define an equivalence relation on ∂I^{t-1} by $(a_1,\ldots,a_{t-1}) \approx (b_1,\ldots,b_{t-1})$ if $\max(a_1,\ldots a_{t-1}) = 1$ and $\max(b_1,\ldots,b_{t-1}) = 1$. For $t \geq 3$ choose homeomorphisms $h_t : S^{t-2} \longrightarrow (\partial I^{t-1})/\approx$. Let T denotes the canonical interchange map. Then the maps $T\circ(h_t \wedge 1_{SV_1\wedge\cdots\wedge SV_t})$ define homeomorphisms

$$h: S(R^{t-2} + V_1 + \cdots + V_t) \longrightarrow \partial[DV_1\wedge\cdots\wedge DV_{t-1}\wedge SV_t]$$

Our spectra will be functors E defined on all finite dimensional subspaces W of R^{∞} with basis a subset of B. We will use the symbol ε to denote either the structure map $S \wedge E \longrightarrow E$ of a spectrum or the product $E \wedge E \longrightarrow E$ of a ring spectrum. Then $\pi_N E$ is defined as the direct limit over all W of the groups $[SW, EW']$ where W' is a subspace of W with $N = \dim(W/W')$. The structure maps of this direct limit are $\varepsilon\circ(SV \wedge -)$ where $V \perp W$. We now have the notation to give the two definitions of the Toda bracket $\langle X_1,\ldots,X_t\rangle$ where $X_1,\ldots,X_{t-1} \in \pi_*^S$, $X_t \in \pi_*(E)$ and E is any spectrum. We begin with the definition based on the composition of maps.

DEFINITION 2.2.1. Let E be a spectrum, let $X_1,\ldots,X_{t-1} \in \pi_*^S$ and let $X_t \in \pi_* E$. Let $G_{i-1,i} : SV_i\wedge\cdots\wedge SV_t\wedge SU \longrightarrow SV_{i+1}\wedge\cdots\wedge SV_t\wedge E_i U$ represent X_i, $1 \leq i \leq t$, where $R^{t-2} \perp V_1 \perp \ldots \perp V_t \perp U$, $E_i = S$ for $i \leq i \leq t-1$ and $E_t = E$. A defining system for $\langle G_{0,1},\ldots,G_{t-1,t}\rangle'_\circ$ consists of maps

$$G_{ij} : DV_{i+1}\wedge\cdots\wedge DV_{j-1}\wedge SV_j\wedge\cdots\wedge SV_t\wedge SU \longrightarrow SV_{j+1}\wedge\cdots\wedge SV_t\wedge E_j U$$

for $0 \leq i < j-1 < t$, $(i,j) \neq (0,t)$, such that

$$G_{ij}|\partial(DV_{i+1}\wedge\cdots\wedge DV_{j-1}\wedge SV_j\wedge\cdots\wedge SV_t\wedge SU) = \bigcup_{k=i+1}^{j-1} G_{ij}^k$$

where G_{ij}^k is the composite map

$$DV_{i+1}\wedge\cdots\wedge DV_{k-1}\wedge SV_k\wedge DV_{k+1}\wedge\cdots\wedge DV_{j-1}\wedge SV_j\wedge\cdots\wedge SV_t\wedge SU \xrightarrow{C_{k+1,\ldots,j-1}(G_{ik})}$$

$$DV_{k+1}\wedge\cdots\wedge DV_{j-1}\wedge SV_j\wedge\cdots\wedge SV_t\wedge SU \xrightarrow{G_{kj}} SV_{j+1}\wedge\cdots\wedge SV_t\wedge E_j U.$$

If $\langle G_{01},\ldots,G_{t-1,t}\rangle'_\circ$ has a defining system then define $\langle G_{01},\ldots,G_{t-1,t}\rangle'_\circ$ as the set of homotopy classes of the maps

$$\tilde{G}_{0t} \equiv \bigcup_{k=1}^{t-1} G^k_{0t}\circ(h \wedge 1_{SU}):S(R^{t-2}+V_1+\cdots+V_t) \wedge SU \longrightarrow EU$$

for all defining systems $\{G_{ij}\}$ of $\langle G_{01},\ldots,G_{t-1,t}\rangle'_{\circ}$. Define

$$\langle G_{01},\ldots,G_{t-1,t}\rangle_{\circ} = \varinjlim_{W} \langle G_{01} \wedge 1_{SW},\ldots,G_{t-2,t-1}\wedge 1_{SW},\varepsilon\circ(G_{t-1,t} \wedge 1_{SW})\rangle'_{\circ}.$$

This direct limit is taken over all W with $W \perp (R^{t-2}+V_1+\cdots+V_t+U)$. If W' is a subspace of W then the map $- \wedge 1_{S(W'^{\perp})}$ sends a defining system of $\langle G_{01}\wedge 1_{SW'},\ldots,\varepsilon\circ(G_{t-1,t}\wedge 1_{SW'})\rangle'_{\circ}$ to a defining system of $\langle G_{01}\wedge 1_{SW},\ldots,\varepsilon\circ(G_{t-1,t}\wedge 1_{SW})\rangle'_{\circ}$. Finally, define $\langle X_1,\ldots,X_t\rangle_{\circ}$ as the union of $\langle G_{01},\ldots,G_{t-1,t}\rangle_{\circ}$ for all choices of representatives $G_{i-1,i}$ of X_i, $1\le i\le t$.

The following definition of the Toda bracket based on the smash product is a direct analogue of the usual algebraic definition of the Massey product in the homology of a differential graded algebra.

DEFINITION 2.2.2 Let E be a spectrum, let $X_1,\ldots,X_{t-1} \in \pi^S_*$ and let $X_t \in \pi_*E$. Let $G_{i-1,i}:SV_i \wedge SU_i \longrightarrow E_iU_i$ represent X_i for $1 \le i \le t$ where $R^{t-2} \perp V_1 \perp U_1 \perp \cdots \perp V_t \perp U_t$, $E_i = S$ for $1 \le i \le t-1$ and $E_t = E$. A defining system for $\langle G_{01},\ldots,G_{t-1,t}\rangle'_{\wedge}$ consists of maps

$$G_{ij}:DV_{i+1}\wedge SU_{i+1}\wedge\cdots\wedge DV_{j-1}\wedge SU_{j-1}\wedge SV_j\wedge SU_j \longrightarrow E_j(U_{i+1}+\cdots+U_j)$$

for $0 \le i < j-1 < t$, $(i,j)\ne(0,t)$, such that

$$G_{ij}|\partial(DV_{i+1} \wedge SU_{i+1}\wedge\cdots\wedge DV_{j-1} \wedge SU_{j-1} \wedge SV_j \wedge SU_j) = \bigcup_{k=i+1}^{j-1} G^k_{ij}$$

where G^k_{ij} is the composite map $\varepsilon\circ T\circ(G_{ik}\wedge G_{kj})$. If $\langle G_{01},\ldots,G_{t-1,t}\rangle'_{\wedge}$ has a defining system, then define $\langle G_{01},\ldots,G_{t-1,t}\rangle'_{\wedge}$ as the set of homotopy classes of the maps

$$\tilde{G}_{0t} \equiv (\bigcup_{k=1}^{t-1}G^k_{0t})\circ T\circ(h \wedge 1_{SU_1\wedge\cdots\wedge SU_t}):S(R^{t-2}+V_1+\cdots+V_t) \wedge SU_1\wedge\cdots\wedge SU_t \longrightarrow E(U_1+\cdots+U_t)$$

for all defining systems $\{G_{ij}\}$ of $\langle G_{01},\ldots,G_{t-1,t}\rangle'_{\wedge}$. Define

$$\langle G_{01},\dots,G_{t-1,t}\rangle_\wedge = \varinjlim_{W_1,\dots,W_t} \langle G_{01}\wedge 1_{SW_1},\dots,G_{t-2,t-1}\wedge 1_{SW_{t-1}},\varepsilon\circ(G_{t-1,t}\wedge 1_{SW_t})\rangle'_\wedge$$

where the direct limit is taken over all $W_1,\dots,W_t$ with $W_1 \perp \cdots \perp W_t \perp (R^{t-2}+V_1+U_1+\cdots+V_t+U_t)$. If W'_i is a subspace of W_i, $1 \le i \le t$, then the maps $\varepsilon\circ T\circ(-\wedge 1_{S(W'^{\perp}_{i+1})}\wedge\cdots\wedge_{S(W'^{\perp}_j)})\circ T$ send a defining system of $\langle G_{01}\wedge 1_{SW'_1},\dots,\varepsilon\circ(G_{t-1,t}\wedge 1_{SW'_t})\rangle'_\wedge$ to a defining system of $\langle G_{01}\wedge 1_{SW_1},\dots,\varepsilon\circ(G_{t-1,t}\wedge 1_{SW_t})\rangle'_\wedge$. Finally, define $\langle X_1,\dots,X_t\rangle_\wedge$ as the union of $\langle G_{01},\dots,G_{t-1,t}\rangle_\wedge$ for all choices of representatives $G_{i-1,i}$ of X_i, $1\le i\le t$.

The reader can find the proof of the following theorem in [29, Theorem 3.2].

THEOREM 2.2.3 Let E be a spectrum, let $X_1,\dots,X_{t-1} \in \pi_*^S$ and let $X_t \in \pi_*E$. Then $\langle X_1,\dots,X_t\rangle_\circ$ is defined if and only if $\langle X_1,\dots,X_t\rangle_\wedge$ is defined. Moreover, if these Toda brackets are defined then they are equal.

NOTATION: In view of this theorem, we will use the symbol $\langle X_1,\dots,X_t\rangle$ to denote $\langle X_1,\dots,X_t\rangle_\circ = \langle X_1,\dots,X_t\rangle_\wedge$.

We will try to imitate proofs of results for algebraic Massey products to construct proofs of the corresponding results for Toda brackets with defining systems constructed with the smash product. An obvious ingredient which we will require is the ability to add maps defined on cones.

DEFINITION 2.2.4 Let f and g be two maps from $C_{j_1,\dots,j_k}(X\wedge SU_1\wedge\cdots\wedge SU_t)$ to Y, where $U_1 \perp \cdots \perp U_t$ and $0\le k\le t$. Let $\{b_{i_1},\dots,b_{i_N}\}$ be a basis for $U_1+\cdots+U_t$ with $i_1 < \cdots < i_N$ and let $\mu(f) = \mu(X \wedge SU_1\wedge\cdots\wedge SU_t) = i_1$. Define

$$f \oplus g: C_{j_1,\dots,j_k}(X \wedge SU_1\wedge\cdots\wedge SU_t) \longrightarrow Y$$

in the usual way by pinching in the $\mu(f) = i_1$ coordinate. Also define -f in the usual way reversing the $\mu(f) = i_1$ coordinate. Let $f \ominus g = f \oplus (-g)$.

Now we have a sum $\oplus$ and a product $\wedge$ defined for the maps that arise in defining systems of Toda brackets. Unfortunately most of the usual algebraic identities only hold up to homotopy for these operations. However, there are five identities which these operations do satisfy.

THEOREM 2.2.5 The following identities hold whenever the expressions appearing in them are defined.

(a) $f \wedge (g \wedge h) = (f \wedge g) \wedge h$

(b) $-(f \oplus g) = (-f) \oplus (-g)$

(c) If $\mu(f) < \mu(W)$ then $1_{sw} \wedge (f \oplus g) = (1_{sw} \wedge f) \oplus (1_{sw} \wedge g)$.

(d) If $\mu(f) > \mu(g)$ then $f \wedge (g \oplus h) = (f \wedge g) \oplus (f \wedge h)$.

(e) If $\mu(f) > \mu(g)$ then $-(f \wedge g) = f \wedge (-g)$.

PROOF: The proofs of these properties are straightforward and are left to the reader. ∎

NOTATION: In view of property (e) above, $-f_1 \wedge \cdots \wedge f_t$ will mean $f_1 \wedge \cdots \wedge (-f_k) \wedge \cdots \wedge f_t$ where $\mu(f_k) = \min(\mu(f_1), \ldots, \mu(f_t))$.

We state next a useful technical result which says that $\langle X_1, \ldots, X_t \rangle_\wedge$ can be defined from any fixed set of representatives of $X_1, \ldots, X_t$.

THEOREM 2.2.6 Assume that $\langle X_1, \ldots, X_t \rangle$ is defined. Let $G_{i-1,i}$ represent X_i for $1 \le i \le t$. Then any element Z of $\langle X_1, \ldots, X_t \rangle$ has a representatives $\tilde{G}_{0t}$ where $\{G_{ij} | 0 \le i < j \le t, (i,j) \ne (0,t)\}$ is a defining system which contains the given $\{G_{i-1,i} | 1 \le i \le t\}$.

PROOF. Let $\{A_{ij} | 0 \le i < j \le t, (i,j) \ne (0,t)\}$ be a defining system such that $\tilde{A}_{ij}$ is a representative of Z. By induction on $k = j - i \ge 1$, we construct a defining system $\{G_{ij}\}$ and homotopies H_{ij} from A_{ij} to G_{ij} such that $H_{ij}|\text{Domain}(G_{ir} \wedge G_{rj}) = H_{ir} \wedge H_{rj}$ for $i < r < j$. When $k = 1$, the $G_{i-1,i}$ are given, and the $H_{i-1,i}$ can be found since $A_{i-1,i}$ and $G_{i-1,i}$ both represent X_i. Let $j-i = k$ and assume that the G_{st} and H_{st} have been constructed for $1 \le t-s < k$. Since (Domain G_{ij}, Domain $\tilde{G}_{ij}$) is homeomorphic to some (D^N, S^N), it has the homotopy extension property. By the induction hypothesis the homotopies $H_{ir} \wedge H_{rj}$, $i < r < j$, agree where their domains intersect and thus define a homotopy $H = \bigcup_{r=i+1}^{j-1}(H_{ir} \wedge H_{rj})$ from $\tilde{A}_{ij}$ to $\tilde{G}_{ij}$. By the homotopy extension property, there is a homotopy H_{ij} of A_{ij} which extends both H and A_{ij}. Define $G_{ij} = H_{ij}|\text{Domain}\ (G_{ij} \times \{1\})$. This completes the inductive step. Thus we have constructed a defining system $\{G_{ij}\}$ and a homotopy $\bigcup_{r=1}^{t-1} (H_{0r} \wedge H_{rt})$ from $\tilde{A}_{0t}$ to $\tilde{G}_{0t}$. ∎

Observe that the three-fold Toda bracket $\langle X_1, X_2, X_3\rangle$ is defined if and only if $X_1 \cdot X_2 = 0$ and $X_2 \cdot X_3 = 0$. The following theorem gives practical criteria for concluding that a four-fold Toda bracket is defined.

THEOREM 2.2.7 Assume that $0 \in \langle X_1, X_2, X_3\rangle$ and $0 \in \langle X_2, X_3, X_4\rangle$. Let N_i = Degree X_i, $1 \le i \le 4$. In addition assume that one of the following conditions is true.

(a) $\langle X_1, X_2, X_3\rangle = 0$.

(b) $\langle X_2, X_3, X_4\rangle = 0$.

(c) $X_1 \cdot \pi^S_{1+N_2+N_3} = 0$.

(d) $X_4 \cdot \pi^S_{1+N_2+N_3} = 0$.

(e) If $Y \in \pi^S_{1+N_2+N_3}$ then $Y = Y_1 + Y_2$ such that $X_1 \cdot Y_1 = 0$ and $X_4 \cdot Y_2 = 0$.

(f) $X_1 = X_3$.

(g) $X_2 = X_4$.

Then $\langle X_1, X_2, X_3, X_4\rangle$ is defined.

PROOF: We use the smash product and the smash product Toda bracket of Definition 2.2.2 throughout the proof.

(a) Let G_{12}, G_{23}, G_{34}, G_{13}, G_{24} be a defining system for $<X_2,X_3,X_4>$ which defines 0 in $<X_2,X_3,X_4>$. Extend this defining system by choosing any G_{01} and G_{02}. Then $\tilde{G}_{03} \in <X_1,X_2,X_3> = 0$, and thus we can find G_{03} to complete the defining system.

(b) The proof of (b) is analogous to the proof of (a).

(c) As in the proof of (a) select G_{01}, G_{12}, G_{23}, G_{34}, G_{02}, G_{13}, G_{24} and G_{14}. By the previous theorem, there is a defining system G_{01}, G_{12}, G_{23}, G_{02}, G'_{13} of $<X_1,X_2,X_3>$ which defines $0 \in <X_1,X_2,X_3>$. Then $G_{01} \wedge (G_{13} \ominus G'_{13})$ represents an element of $X_1 \cdot \pi^S_{1+N_2+N_3} = 0$. Thus we can find G_{03} to complete the defining system.

(d) The proof of (d) is analogous to the proof of (c).

(e) As in the proof of (a) select G_{01}, G_{12}, G_{23}, G_{34}, G_{02}, G_{13}, G_{24} and G_{14}. By the previous theorem, there is a defining system G_{01}, G_{12}, G_{23}, G_{02}, G'_{13} of $<X_1,X_2,X_3>$ which defines $0 \in <X_1,X_2,X_3>$. Write $G_{13} \ominus G'_{13} = Y_2 \oplus Y_1$ where $X_1 \wedge Y_1$ and $X_4 \wedge Y_2$ are null homotopic. Then we can replace G_{13} by $(-Y_2 \oplus G_{13}) \oplus (-G'_{13} \oplus G'_{13})$ and find a new appropriate G_{14}. Since the new G_{13} equals $(-Y_2 \oplus Y_2) \oplus (Y_1 \oplus G'_{13})$ we can find a G_{03} to complete the defining system.

(f) Let G_{12}, G_{23}, G_{34}, G_{13}, G_{24} be a defining system for $<X_2,X_3,X_4>$ which defines 0 in $<X_2,X_3,X_4>$. Extend this defining system by choosing $G_{01} = G_{23}$ and any G_{02}. There are other choices $G'_{02} = G_{02} \oplus X$ and $G'_{13} = G_{13} \oplus Y$ such that the defining system G_{01}, G_{12}, G_{23}, G'_{02}, G'_{13} defines G which represents 0 in $<X_1,X_2,X_3>$. Replace G_{02} by $(G_{02} \oplus X \oplus Y) \cup (Y \cup_1 G_{23})$. Now $\tilde{G}_{03} = G$, and we can find a G_{03} to complete the defining system.

(g) The proof of (g) is analogous to the proof of (f). ∎

3. Properties of the Toda Bracket

In this section, we derive the indeterminacy as well as the additive and associative properties of the three-fold and four-fold Toda brackets defined in the previous section. Most of these results are direct analogues of the algebraic results for Massey products given by May in [39]. As with algebraic Massey products we say that $\langle X_1,\dots,X_t\rangle$ is strictly defined if $\langle X_m,\dots,X_n\rangle = 0$ whenever $1 \le m < n \le t$ and $n-m < t-1$. Note that every triple product which is defined is automatically strictly defined. We define the indeterminacy of a Toda bracket by

$$\text{Indet}\ \langle X_1,\dots,X_t\rangle = \langle X_1,\dots,X_t\rangle - \langle X_1,\dots,X_t\rangle.$$

In all of the proofs of this section we use defining systems as in Definition 2.2.2 which are based upon the smash product.

Before embarking on manipulating our Toda brackets, we should remark that there is a hidden sign convention built into our definitions. The easiest way to deal with this problem is to consider a defining system $\{G_{ij}\}$ of $\langle X_1,\dots,X_t\rangle_\wedge$ in which the $G_{01},\dots,G_{t-1,t}$ use subspace $V_1,\dots,V_t$ of R^∞ such that V_i has basis $\{b_{N(i,j)} \mid 1 \le j \le \dim(V_i)\}$ and $\{b_{N(i,j)} \mid 1 \le i \le t,\ 1 \le j \le \dim(V_i)\}$ in the lexicographical order of the $N(i,j)$ is the same ordering as the given ordering of B. Now think of $\tilde{G}_{0t}$ as using t-2 additional basis vectors $b_{k_1},\dots,b_{k_{t-2}}$ where $k_1 < N(1,j_1) < k_2 < N(2,j_2) < k_3 < \cdots < k_{t-2} < N(t-2,j_{t-2})$ for all $j_1,\dots,j_{t-2}$.

THEOREM 2.3.1 Let $X_i \in \pi^S_{N_i}$ for $1 \le i \le t$.

(a) Indet $\langle X_1,X_2,X_3\rangle$ is the ideal spanned by X_1 and X_3.

(b) If $X_3\cdot\pi^S_{N_1+N_2+1} \cap X_1\cdot\pi^S_{N_2+N_3+1} = 0$ and $X_2\cdot\pi^S_{N_3+N_4+1} \cap X_4\cdot\pi^S_{N_2+N_3+1} = 0$ then

$$\text{Indet}\ \langle X_1,X_2,X_3,X_4\rangle = \bigcup_A \langle A,X_3,X_4\rangle \cup \bigcup_B \langle X_1,B,X_4\rangle \cup \bigcup_C \langle X_1,X_2,C\rangle$$

where the first union is taken over all $A \in \pi^S_{N_1+N_2+1}$ such that $A\cdot X_3=0$, the second union is taken over all $B \in \pi^S_{N_2+N_3+1}$ such that $B\cdot X_1=B\cdot X_4=0$ and the third union is taken over all $C \in \pi^S_{N_3+N_4+1}$ such that $C\cdot X_2=0$.

PROOF: The proof of is this theorem is a direct analogue of the proof of the corresponding algebraic result for Massey products [40,Prop.2.4].∎

NOTE: The hypothesis in (b) above is satisfied if $\langle X ,X ,X ,X \rangle$ is strictly defined.

THEOREM 2.3.2 Assume that $\langle X_1,\ldots,X'_k+X''_k,\ldots,X_t\rangle$ is defined and $\langle X_1,\ldots,X'_k,\ldots X_t\rangle$ is strictly defined. Then $\langle X_1,\ldots,X''_k,\ldots,X_t\rangle$ is defined and $\langle X_1,\ldots,X'_k+X''_k,\ldots,X_t\rangle \subset \langle X_1,\ldots,X'_k,\ldots,X_t\rangle + \langle X_1,\ldots,X''_k,\ldots,X_t\rangle$.

PROOF. The proof is a direct analoge of the algebraic proof of [40,Prop.2.7].∎

The following associative properties of the three-fold Toda bracket are proved by Toda in [60].

THEOREM 2.3.3 Let degree $X_i = N(i)$ for $0 \le i \le 3$ and let degree $Y = M$.

(a) If $\langle X_1,X_2,X_3\rangle$ is defined then

$$Y\cdot\langle X_1,X_2,X_3\rangle \subset (-1)^M \langle Y\cdot X_1,X_2,X_3\rangle \text{ and } \langle X_1,X_2,X_3\rangle\cdot Y \subset \langle X_1,X_2,X_3\cdot Y\rangle.$$

(b) If $X_0\cdot X_1 = X_1\cdot X_2 = X_2\cdot X_3 = 0$ then

$$X_0\cdot\langle X_1,X_2,X_3\rangle = (-1)^{N(0)+N(1)} \langle X_0,X_1,X_2\rangle\cdot X_3.$$

(c) If the second of the three Toda brackets below is defined then they are all defined and

$$0 \in (-1)^{N(0)} \langle\langle X_0,X_1,X_2\rangle,X_3,X_4\rangle + \langle X_0,\langle X_1,X_2,X_3\rangle,X_4\rangle + (-1)^{N(1)} \langle X_0,X_1,\langle X_2,X_3,X_4\rangle\rangle.$$

(d) If $X_1\cdot Y\cdot X_2 = 0$ and $X_2\cdot X_3 = 0$ then $\langle X_1\cdot Y,X_2,X_3\rangle \subset (-1)^M \langle X_1,Y\cdot X_2,X_3\rangle$.

(e) If $X_1\cdot X_2 = 0$ and $X_2\cdot Y\cdot X_3 = 0$ then $\langle X_1,X_2,Y\cdot X_3\rangle \subset \langle X_1,X_2\cdot Y,X_3\rangle$.

In the next three theorems we give the analogous results for four-fold Toda brackets. Most of these results were proved by Oguchi [51] for his composition four-fold products. However, his Toda brackets are only defined under more restrictive conditions than ours. As a result some of his conclusions are sharper than ours.

THEOREM 2.3.4 Let degree $X_i = N(i)$ for $1 \le i \le 4$ and let degree $Y = M$.

(a) If $\langle X_1, X_2, X_3, X_4\rangle$ is defined then $\langle X_1, X_2, X_3, X_4\rangle = (-1)^P \langle X_4, X_3, X_2, X_1\rangle$ where $P = N(4)[N(1)+N(2)+N(3)+1] + N(3)[N(1)+N(2)] + N(1)[N(2)+1]$.

(b) If $\langle X_1, X_2, X_3, X_4\rangle$ is defined then

$$Y\cdot\langle X_1, X_2, X_3, X_4\rangle \subset (-1)^M \langle Y\cdot X_1, X_2, X_3, X_4\rangle \text{ and}$$

$$\langle X_1, X_2, X_3, X_4\rangle\cdot Y \subset \langle X_1, X_2, X_3, X_4\cdot Y\rangle.$$

(c) If $\langle X_1\cdot Y, X_2, X_3, X_4\rangle$ is defined then $\langle X_1, Y\cdot X_2, X_3, X_4\rangle$ is defined and

$$\langle X_1\cdot Y, X_2, X_3, X_4\rangle \subset (-1)^M \langle X_1, Y\cdot X_2, X_3, X_4\rangle.$$

(d) If $\langle X_1, X_2, X_3, Y\cdot X_4\rangle$ is defined then $\langle X_1, X_2, X_3\cdot Y, X_4\rangle$ is defined and

$$\langle X_1, X_2, X_3, Y\cdot X_4\rangle \subset \langle X_1, X_2, X_3\cdot Y, X_4\rangle.$$

(e) Assume that $\langle X_1, X_2\cdot Y, X_3, X_4\rangle$ and $\langle X_1, X_2, Y\cdot X_3, X_4\rangle$ are defined, and that $\langle X_1, X_2, YX_3\rangle = 0$. Then $I \equiv \langle X_1, X_2\cdot Y, X_3, X_4\rangle \cap \langle X_1, X_2, Y\cdot X_3, X_4\rangle \neq \phi$. Moreover the indeterminacy is given by $\mathrm{Indet}(I) \equiv I-I = \bigcup_A \langle A, X_3, X_4\rangle \cup \bigcup_B \langle X_1, X_2, B\rangle$ where the first union is taken over all $A \in \pi^S_{N(1)+N(2)+M+1}/\ Y\cdot\pi^S_{N(1)+N(2)+1}$ with $AX_3 = 0$ and the second union is taken over all $B \in \pi^S_{N(3)+N(4)+M+1}/\ Y\cdot\pi^S_{N(3)+N(4)+1}$ with $X_2B = 0$.

PROOF. (a) If $\{G_{ij} | 0 \le i < j \le 4,\ (i,j) \neq (0,4)\}$ is a defining system for $\langle X_1, X_2, X_3, X_4\rangle$, let $A_{ij} = G_{4-j,4-i}$. Then $\{A_{ij} | 0 \le i < j \le 4,\ (i,j) \neq (0,4)\}$ is a defining system for $\langle X_4, X_3, X_2, X_1\rangle$. Since $\tilde{G}_{ij} = \tilde{A}_{ij}$, $\langle X_1, X_2, X_3, X_4\rangle \subset (-1)^P\langle X_4, X_3, X_2, X_1\rangle$, and by symmetry the two Toda brackets are equal.

(b) Let $\{G_{ij} | 0 \le i < j \le 4,\ (i,j) \neq (0,4)\}$ be a defining system for $\langle X_1, X_2, X_3, X_4\rangle$ and let J represent Y. Then the following display is a defining system for $\langle Y\cdot X_1, X_2, X_3, X_4\rangle$:

$$\begin{matrix} J\wedge G_{01} & & G_{12} & & G_{23} & & G_{34} \\ & J\wedge G_{02} & & G_{13} & & G_{24} & \\ & & J\wedge G_{03} & & G_{14} & & \end{matrix}$$

Thus, $\langle Y\cdot X_1, X_2, X_3, X_4\rangle$ is defined and contains $J \wedge \tilde{G}_{04}$. Therefore $Y\cdot\langle X_1, X_2, X_3, X_4\rangle \subset (-1)^M\langle Y\cdot X_1, X_2, X_3, X_4\rangle$. The second identity in (b) follows from the first one by (a).

(c) Let $\{G_{ij} | 0 \le i < j \le 4,\ (i,j) \ne (0,4)\}$ be a defining system for $\langle X_1\cdot Y, X_2, X_3, X_4\rangle$. Assume that $G_{01} = G'_{01} \wedge J$ where G'_{01}, J represents X_1, Y, resp. Then the following display is a defining system for $\langle X_1, Y\cdot X_2, X_3, X_4\rangle$:

$$\begin{matrix} G'_{01} & & J\wedge G_{12} & & G_{23} & & G_{34} \\ & G_{02} & & J\wedge G_{13} & & G_{24} & \\ & & G_{03} & & J\wedge G_{14} & & \end{matrix}$$

Thus $\langle X_1, Y\cdot X_2, X_3, X_4\rangle$ is defined and contains $\tilde{G}_{04}$ because $G'_{01} \wedge (J \wedge G_{14}) = G_{01} \wedge G_{14}$. Therefore $\langle X_1\cdot Y, X_2, X_3, X_4\rangle \subset (-1)^M \langle X_1, Y\cdot X_2, X_3, X_4\rangle$.

(d) This identity follows from the identity in (c) by applying the identity in (a).

(e) Let $G_{i-1,i}$ represent X_i for $1 \le i \le 4$, and let J represent Y. Extend G_{01}, $G_{12} \wedge J$, G_{23}, G_{34} to a defining system $\{G_{ij} | 0 \le i < j \le 4,\ (i,j) \ne (0,4)\}$ of $\langle X_1, X_2\cdot Y, X_3, X_4\rangle$. Extend G_{01}, G_{12}, $J \wedge G_{23}$, G_{13} by finding a G'_{02} to get a defining system of $\langle X_1, X_2, YX_3\rangle$. Since $\langle X_1, X_2, YX_3\rangle = 0$, we can find a G'_{03} such that $\partial G'_{03} = (G_{01} \wedge G_{13}) \cup (G'_{02} \wedge (J \wedge G_{23}))$. Then the following diagram exhibits two defining systems, one for $\langle X_1, X_2\cdot Y, X_3, X_4\rangle$ and the other for $\langle X_1, X_2, Y\cdot X_3, X_4\rangle$:

$$\begin{matrix} G_{01} & & G_{12}\wedge J & & G_{23} & & G_{34} \\ & G'_{02}\wedge J & & G_{13} & & G_{24} & \\ & & G'_{03} & & G_{14} & & \end{matrix} \qquad \begin{matrix} G_{01} & & G_{12} & & J\wedge G_{23} & & G_{34} \\ & G'_{02} & & G_{13} & & J\wedge G_{24} & \\ & & G'_{03} & & G_{14} & & \end{matrix}$$

Both of these defining systems define the same element, and thus the two Toda brackets have an element in common. The indeterminacy arises because not all defining systems of $\langle X_1\cdot Y, X_2, X_3, X_4\rangle$ have a (0,2) entry of the form $?\wedge J$ and not all defining systems of $\langle X_1, Y\cdot X_2, X_3, X_4\rangle$ have a (2,4) entry of the form $J\wedge ?$. ∎

THEOREM 2.3.5 Let degree $X_i = N(i)$ for $0 \le i \le 4$. Assume that $\langle X_1, X_2, X_3, X_4\rangle$ and $\langle X_0, X_1, X_2, X_3\rangle$ are strictly defined. Then

$$X_0\cdot\langle X_1, X_2, X_3, X_4\rangle = (-1)^{N(0)+N(1)} \langle X_0, X_1, X_2, X_3\rangle\cdot X_4.$$

PROOF. Let $\{G_{ij} | 0 \le i < j \le 4,\ (i,j) \ne (0,4)\}$ be a defining system for $\langle X_1, X_2, X_3, X_4\rangle$. Extend $\{G_{01}, G_{12}, G_{23}, G_{02}, G_{13}, G_{03}\}$ to a defining system $\{G_{ij} | -1 \le i < j \le 3,\ (i,j) \ne (-1,3)\}$ of $\langle X_0, X_1, X_2, X_3\rangle$. Then $(G_{-1,1}\wedge G_{14}) \cup (G_{-1,2}\wedge G_{24})$ restricted to the boundary of its domain is $(G_{-1,0}\wedge \tilde{G}_{04}) \cup (\tilde{G}_{-1,3}\wedge G_{34})$. Thus $X_0\cdot\langle X_1, X_2, X_3, X_4\rangle$ $\subset (-1)^{N(0)+N(1)}\langle X_0, X_1, X_2, X_3\rangle\cdot X_4$ and by symmetry the theorem follows. ∎

THEOREM 2.3.6 Let degree $X_i = N(i)$ for $0 \le i \le 4$.

(a) Assume that $\langle X_1, X_2, X_3, X_4\rangle$ is defined and that $X_0\cdot X_1 = 0$. Then

$$X_0\cdot\langle X_1, X_2, X_3, X_4\rangle \subset (-1)^{N(1)+1} \langle\langle X_0, X_1, X_2\rangle, X_3, X_4\rangle.$$

(b) Assume that $\langle X_0, X_1, X_2, X_3\rangle$ is defined and that $X_3\cdot X_4 = 0$. Then

$$\langle X_0, X_1, X_2, X_3\rangle\cdot X_4 \subset (-1)^{N(1)+1} \langle X_0, X_1, \langle X_2, X_3, X_4\rangle\rangle.$$

(c) Assume that $X_0\cdot X_1 = 0$, $X_1\cdot X_2 = 0$, $X_3\cdot X_4 = 0$ and $0 \in \langle X_0, X_1, X_2\rangle\cdot X_3$. Then $\langle X_0, X_1, X_2\cdot X_3, X_4\rangle$ is defined and contains $(-1)^{N(0)+1} \langle\langle X_0, X_1, X_2\rangle, X_3, X_4\rangle$.

(d) Assume that $X_0\cdot X_1 = 0$, $X_2\cdot X_3 = 0$, $X_3\cdot X_4 = 0$ and $0 \in X_1\cdot\langle X_2, X_3, X_4\rangle$. Then $\langle X_0, X_1\cdot X_2, X_3, X_4\rangle$ is defined and contains $(-1)^{N(1)+1} \langle X_0, X_1, \langle X_2, X_3, X_4\rangle\rangle$.

PROOF. (a) Let $\{G_{ij} | 0 \le i < j \le 4,\ (i,j) \ne (0,4)\}$ be a defining system for $\langle X_1, X_2, X_3, X_4\rangle$ and let $G_{-1,0}$ represent X_0. Then the following display is a defining system for $\langle\langle X_0, X_1, X_2\rangle, X_3, X_4\rangle$:

$$\begin{array}{ccc} \tilde{G}_{-1,2} & G_{23} & G_{34} \\ & (G_{-1,0}\wedge G_{03})\cup(G_{-1,1}\wedge G_{13}) \quad G_{24} & \end{array}$$

Now $G_{-1,1} \wedge G_{14}$ restricted to the boundary of its domain is the element of $\langle\langle X_0, X_1, X_2\rangle, X_3, X_4\rangle$ determined by the above defining system union $G_{-1,0} \wedge \tilde{G}_{04}$. Thus $X_0 \cdot \langle X_1, X_2, X_3, X_4\rangle \subset (-1)^{N(1)+1} \langle\langle X_0, X_1, X_2\rangle, X_3, X_4\rangle$.

(b) This identity follows from the one in (a) by Theorem 2.3.4(a).

(c) Let $\{G_{ij} \mid -1 \le i < j \le 2,\ (i,j) \ne (-1,2)\}$ be a defining system for $\langle X_0, X_1, X_2\rangle$. Let G_{23}, G_{34} represent X_3, X_4, respectively. Find G_{24} such that $\tilde{G}_{24} = G_{23} \wedge G_{34}$ and find $G_{-1,3}$ such that $\tilde{G}_{-1,3} = \tilde{G}_{-1,2} \wedge G_{23}$. Then the following display is a defining system for $\langle X_0, X_1, X_2 \cdot X_3, X_4\rangle$:

$$\begin{array}{ccccccc} G_{-1,0} & & G_{01} & & G_{12} \wedge G_{23} & & G_{34} \\ & G_{-1,1} & & G_{02} \wedge G_{23} & & G_{12} \wedge G_{24} & \\ & & G_{-1,3} & & G_{02} \wedge G_{24} & & \end{array}$$

This defining system defines

$(G_{-1,0} \wedge G_{02} \wedge G_{24}) \cup (G_{-1,1} \wedge G_{12} \wedge G_{24}) \cup (G_{-1,3} \wedge G_{34})$

$= (\tilde{G}_{-1,2} \wedge G_{24}) \cup (G_{-1,3} \wedge G_{34})$, an arbitrary element of $\langle\langle X_0, X_1, X_2\rangle, X_3, X_4\rangle$.

Thus $\langle\langle X_0, X_1, X_2\rangle, X_3, X_4\rangle \subset (-1)^{N(0)+1} \langle X_0, X_1, X_2 \cdot X_3, X_4\rangle$.

(d) This identity follows from the identity in (c) by Theorem 2.3.4(a).∎

We conclude this section by recording a useful theorem of Toda [60,3.10].

THEOREM 2.3.7 Let α and β be elements of π_*^S.

(a) If degree α is odd then $\langle\alpha, \beta, \alpha\rangle \cap (-1)^{\deg \beta} \langle\beta, \alpha, 2\alpha\rangle \ne \emptyset$.

(b) If degree α is even then $\langle\alpha, \beta, \alpha\rangle \cap \beta \cdot \pi_*^S \ne \emptyset$.

4. The Atiyah-Hirzebruch Spectral Sequence

Toda brackets in the limit of a spectral sequence are related to the differentials in the spectral sequence. In this section we prove several theorems which depict this relationship in the Atiyah-Hirzebruch spectral sequence for the homotopy of a spectrum B:

$$E^2_{pq} = H_p(B;\pi^S_q) \Longrightarrow \pi^S_{p+q}(B)$$

Of course, the case in which we are inerested is when B = BP, and we specialize to that case in the last three theorems of this section. The idea of the following theorem is to analyze a Toda bracket by passing to an appropriate mapping cone. This idea is due to Joel Cohen [18] where he used it to decompose elements of π^S_* as Toda brackets of Hopf classes.

THEOREM 2.4.1 Let $X_0 \in \pi^S_{N(0)}$, $X_2 \in \pi^S_{N(2)}$, $X_3 \in \pi^S_{N(3)}$, $Y \in H_*B$ and let $r \geq 2$. Let C be the mapping cone of X_2. Assume that:

(i) $X_2 \cdot X_3 = 0$ in π^S_*.

(ii) $d^r(X_3 \cdot Y) = X_0$.

(iii) Y transgresses to the projection of $X_{02} \in C_*$ into the Atiyah-Hirzebruch spectral sequence for C_*B.

Let $X_1 = \sigma_*(X_{02}) \in \pi^S_{N(1)}$ where $\sigma: C \longrightarrow S^{N(2)+1}$ is the canonical collapsing map. Then $\langle X_1, X_2, X_3 \rangle$ is defined and contains X_0.

PROOF. We use the composition product Toda bracket of Definition 2.2.1 to prove this theorem. Let $G_{i-1,i}$ represent X_i for $0 \leq i \leq 3$, and let G_{02} represent X_{02}. Consider Figure 2.4.1. In that diagram, j is the canonical inclusion map and G_{13} exists by (i). Let G_{13*} be the map of spectral sequences induced by G_{13}. Then $X_0 = d^r(X_3 \cdot Y) = d^r \circ G_{13*}(Y) = G_{13*} \circ d^r(Y) = G_{13*}(X_{02})$. Thus X_0 is represented by $G_{13} \circ SG_{02}$ which is an element of $\langle X_1, X_2, X_3 \rangle$. ∎

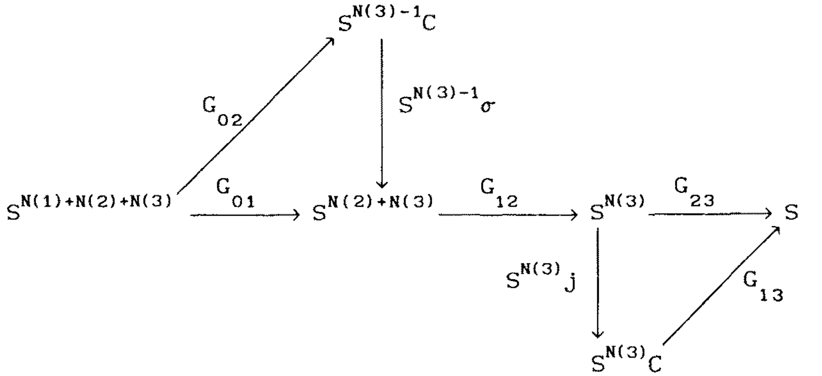

FIGURE 2.4.1

The next theorem is the most direct way of detecting a triple product in π_*^S from differentials in the Atiyah-Hirzebruch spectral sequence.

THEOREM 2.4.2 Assume that $<X_1, X_2, X_3>$ is defined in π_*^S. Assume that $d^{r(1)}(Y_1) = X_1$ and $d^{r(3)}(Y_3) = X_3$. Then $X_2 \cdot Y_1 \cdot Y_3$ survives to $E^{r(1)+r(3)}$ and there is an element of $<X_1, X_2, X_3>$ which projects to $d^{r(1)+r(3)}(X_2 \cdot Y_1 \cdot Y_3)$.

PROOF. We use the smash product Toda bracket of Definition 2.2.2 to prove this theorem. Let N(i) = degree X_i. For i=1,3, represent $Y_i \in E^{r(i)}_{r(i),p(i)}$ by $G_i : (SV_i \wedge DU_i, SV_i \wedge SU_i) \longrightarrow (SU_i \wedge B^{[r(i)]}, SU_i)$ where $G_i | SV_i \wedge SU_i \equiv G_{i-1,i}$ represents X_i. Represent X_2 by $G_{12} : SV_2 \wedge SU_2 \longrightarrow SU_2$. Find maps G_{02} and G_{13} as in Definition 2.2.2 to complete the defining system $\{G_{ij}\}$ of $<X_1, X_2, X_3>$. Define $F : (SV_1 \wedge DU_1 \wedge SV_2 \wedge SU_2 \wedge SV_3 \wedge DU_3) \cup (DV_1 \wedge SU_1 \wedge SV_2 \wedge SU_2 \wedge SV_3 \wedge DU_3)$

$\cup (SV_1 \wedge DU_1 \wedge DV_2 \wedge SU_2 \wedge SV_3 \wedge SU_3) \longrightarrow SU_1 \wedge SU_2 \wedge SU_3 \wedge B^{[r(1)+r(3)]}$

as $[\varepsilon \circ (G_1 \wedge G_{12} \wedge G_3)] \cup [\varepsilon \circ (G_{02} \wedge G_3)] \cup [\varepsilon \circ (G_1 \wedge G_{13})]$. Then Domain F is homeomorphic to a disc and

$F : (\text{Domain } F, \partial\text{Domain } F) \longrightarrow (SU_1 \wedge SU_2 \wedge SU_3 \wedge B^{[r(1)+r(3)]}, SU_1 \wedge SU_2 \wedge SU_3)$

represents $X_2 \cdot Y_1 \cdot Y_3$. Thus $X_1 \cdot Y_2 \cdot Y_3$ survives to $E^{r(1)+r(3)}_{r(1)+r(3),p(1)+p(3)+N(2)}$ and $d^{r(1)+r(3)}(X_2 \cdot Y_1 \cdot Y_3)$ is represented by $F|\partial\text{Domain}(F) = (G_{02} \wedge G_{23}) \cup (G_{01} \wedge G_{13}) = \tilde{G}_{03} \in <X_1, X_2, X_3>$. ▮

The previous theorem generalizes to longer Toda brackets. Unfortunately, technical hypotheses need to be added and the conclusion has indeterminacy. We give such a generalization for four-fold brackets.

THEOREM 2.4.3 Assume that $<X_1, X_2, X_3, X_4>$ is defined in π_*^S, and let $N(i) =$ degree X_i for $1 \le i \le 4$. Assume that $d^{r(i)}(Y_i) = X_i$ for i=1,3,4 where $Y_i \in E^{r(i)}_{r(i),p(i)}$. Assume that one of the following hypotheses hold:

(i) $E^{r(4)+h}_{r(1)-h,p(1)+N(2)+h} = 0$ for $0 \le h \le r(1)$.

(ii) $E^{r(4)+k}_{r(3)-k,p(3)+N(4)+k} = 0$ for $0 \le k \le r(3)$.

Then $X_2 \cdot Y_1 \cdot Y_3 \cdot Y_4$ survives to $E^{r(1)+r(3)+r(4)}_{N(1)+N(3)+N(4)+3,N(2)}$ and there is an element of $\langle X_1, X_2, X_3, X_4 \rangle$ which projects to $d^{r(1)+r(3)+r(4)}(X_2 \cdot Y_1 \cdot Y_3 \cdot Y_4)$.

PROOF. We use the smash product Toda bracket of Definition 2.2.2 to prove this theorem. Let $\{G_{ij} | 0 \le i < j \le 4, (i,j) \ne (0,4)\}$ be a defining system for $\langle X_1, X_2, X_3, X_4 \rangle$. For i=1,3,4 represent $Y_i \in E^{r(i)}_{r(i),p(i)}$ by $G_i : (SV_i \wedge DU_i, SV_i \wedge SU_i) \longrightarrow (SU_i \wedge B^{[r(i)]}, SU_i)$ where $G_i | SV_i \wedge SU_i \equiv G_{i-1,i}$ represents X_i. Let

$$F = (G_1 \wedge G_{12} \wedge G_3 \wedge G_4) \cup (G_{02} \wedge G_3 \wedge G_4) \cup (G_1 \wedge G_{13} \wedge G_4) \cup (G_{03} \wedge G_4) \cup (G_1 \wedge G_{14}):$$

$$\big[(SV_1 \wedge DU_1 \wedge SV_2 \wedge SU_2 \wedge SV_3 \wedge DU_3 \wedge SV_4 \wedge DU_4) \cup (DV_1 \wedge SU_1 \wedge SV_2 \wedge SU_2 \wedge SV_3 \wedge DU_3 \wedge SV_4 \wedge DU_4)$$
$$\cup (SV_1 \wedge DU_1 \wedge DV_2 \wedge SU_2 \wedge SV_3 \wedge SU_3 \wedge SV_4 \wedge DU_4) \cup (DV_1 \wedge SU_1 \wedge DV_2 \wedge SU_2 \wedge SV_3 \wedge SU_3 \wedge SV_4 \wedge DU_4)$$
$$\cup (SV_1 \wedge DU_1 \wedge DV_2 \wedge SU_2 \wedge DV_3 \wedge SU_3 \wedge SV_4 \wedge SU_4), (DV_1 \wedge SU_1 \wedge DV_2 \wedge SU_2 \wedge SV_3 \wedge SU_3 \wedge SV_4 \wedge SU_4)$$
$$\cup (SV_1 \wedge SU_1 \wedge DV_2 \wedge SU_2 \wedge DV_3 \wedge SU_3 \wedge SV_4 \wedge SU_4) \cup (SV_1 \wedge DU_1 \wedge SV_2 \wedge SU_2 \wedge SV_3 \wedge DU_3 \wedge SV_4 \wedge SU_4)$$
$$\cup (SV_1 \wedge DU_1 \wedge SV_2 \wedge SU_2 \wedge DV_3 \wedge SU_3 \wedge SV_4 \wedge SU_4) \cup (DV_1 \wedge SU_1 \wedge SV_2 \wedge SU_2 \wedge SV_3 \wedge DU_3 \wedge SV_4 \wedge SU_4)\big]$$
$$\longrightarrow (B^{[r(1)+r(3)+r(4)]}, B^{[r(1)+r(3)]}).$$

F has a disk as its domain and F restricted to the boundary of its domain is $[(G_{03} \wedge G_{34}) \cup (G_{01} \wedge G_{14})] \cup [(G_1 \wedge G_{12} \wedge G_3 \wedge G_{34}) \cup (G_1 \wedge G_{12} \wedge G_{24}) \cup (G_{02} \wedge G_3 \wedge G_{34})]$. Clearly F represents $X_2 \cdot Y_1 \cdot Y_3 \cdot Y_4$. Moreover, F restricted to the boundary of its domain is the sum of $(G_{01} \wedge G_{14}) \cup (G_{02} \wedge G_{24}) \cup (G_{03} \wedge G_{34})$ and the product $[(G_1 \wedge G_{12}) \cup G_{02}] \wedge [(G_3 \wedge G_{34}) \cup G_{24}]$. The first summand is an element of $\langle X_1, X_2, X_3, X_4 \rangle$. Under hypothesis (i), the first factor of the product is the boundary of a map of filtration degree less than r(1)+r(4) while the second factor is in filtration degree r(3) so that the product is the boundary of a map of filtration degree less than [r(1)+r(4)]+r(3). Under hypothesis (ii), the second factor of the product is the boundary of a map of filtration degree less than r(3)+r(4) while the first factor is in filtarion degree r(1) so that the product is the boundary of a map of filtration degree less than r(1)+[r(3)+r(4)]. Thus, in either case we can represent $X_2 \cdot Y_1 \cdot Y_3 \cdot Y_4$ by a map whose boundary is an element of $\langle X_1, X_2, X_3, X_4\rangle$. Thus, $X_2\cdot Y_1\cdot Y_3\cdot Y_4$ survives to $E^{r(1)+r(3)+r(4)}$ and $d^{r(1)+r(3)+r(4)}(X_2\cdot Y_1\cdot Y_3\cdot Y_4)$ is the projection into $E^{r(1)+r(3)+r(4)}_{0,N(1)+N(2)+N(3)+N(4)+2}$ of an element of $\langle X_1, X_2, X_3, X_4\rangle$.∎

We conclude this section with three theorems that refer only to our Atiyah-Hirzebruch spectral sequence, i.e., we take B = BP. As we shall see, the Toda brackets constructed there are common and useful for detecting nontrivial extensions in our spectral sequence. In Chapter 3, we shall see that we have elements of H_*BP with the following differentials: $d^2(M_1) = \eta$, $d^2(M_2) = \eta M_1^2$, $d^4(M_1^2) = \nu$, $d^4(\bar{M}_2) = \nu M_1$, $d^4(M_2^2) = \nu M_1^4$, $d^8\langle M_1^4\rangle = \sigma$ and $d^8\langle M_2^2\rangle = \sigma M_1^2$. We will represent M_1, M_2, M_1^2, $\bar{M}_2$, M_2^2, $\langle M_1^4\rangle$, $\langle M_2^2\rangle$ by μ_1, μ_{01}, μ_2, $\bar{\mu}_{01}$, μ_{02}, μ_4, $\langle\mu_{02}\rangle$, respectively. The reader may prefer to read the remainder of this section after reading Chapter 3.

THEOREM 2.4.4 Let $X \in \pi_*^S$.

(a) $X\cdot M_1^3$ survive to E^6 if and only if $\eta\cdot X = 0$ and $\nu\cdot X = 0$. In that case $\langle\eta, X, \nu\rangle$ is defined and projects to $d^6(X\cdot M_1^3)$.

(b) $X\cdot M_2$ survives to E^6 if and only if $\eta\cdot X = 0$. In that case $\langle\nu, \eta, X\rangle$ is defined and projects to $d^6(X\cdot M_2)$.

(c) $X\cdot\bar{M}_2$ survives to E^6 if and only if $\nu\cdot X = 0$. In that case $\langle\eta, \nu, X\rangle$ is defined and projects to $d^6(X\cdot\bar{M}_2)$.

PROOF. Represent $M_1 \in E^2_{2,0}$ by $\mu_1: (S^1\wedge DA, S^1\wedge SA) \longrightarrow (SA \wedge BP^{[2]}, SA)$ such that $\mu_1|S^1\wedge SA = \eta$. Represent $M_1^2 \in E^4_{4,0}$ by $\mu_2: (S^3\wedge DB, S^3\wedge SB) \longrightarrow (SB \wedge BP^{[4]}, SB)$ such that $\mu_2|S^3\wedge SB = \nu$. Let $G: SV \wedge SU \longrightarrow SU$ represent X. We use the smash product Toda bracket of Definition 2.2.2 throughout the proof. Observe that all three Toda brackets in this theorem have indeterminacy contained in (η, ν) which projects to zero in E^6.

(a) $d^2(X\cdot M_1^3) = \eta\cdot X\cdot M_1^2$ and if $\eta\cdot X = 0$ then $d^4(X\cdot M_1^3) = \nu\cdot X\cdot M_1$. Thus, $X\cdot M_1^3$ survives to E^6 if and only if $\eta\cdot X = 0$ and $\nu\cdot X = 0$. The latter condition is equivalent to $\langle\eta, X, \nu\rangle$ being defined. In this case we can apply Theorem 2.4.2 to conclude that $d^6(X\cdot M_1^3)$ is the projection of $\langle\eta, X, \nu\rangle$ into E^6.

(b) Represent $M_2 \in E^2_{6,0}$ by

$$\mu_{01}: \left(D^4\wedge DB \wedge S^1\wedge SA \wedge SC, (S^3\wedge DB \wedge S^1\wedge SA \wedge SC) \cup (D^4\wedge SB \wedge S^1\wedge SA \wedge SC)\right) \longrightarrow (SB \wedge SA \wedge SC \wedge BP^{[6]}, SB \wedge SA \wedge SC \wedge BP^{[4]})$$

such that μ_{01} restricted to the boundary of its domain is $(\mu_2 \wedge \eta) \cup B_{\nu\eta}$ where $B_{\nu\eta'} \mid S^3 \wedge SB \wedge S^1\wedge SA \wedge SC = \nu \wedge \eta$. Let

$B_{\eta X}: D^2\wedge SA \wedge SC \wedge SV \wedge SU \longrightarrow SA \wedge SC \wedge SU$ such that

$B_{\eta X}|S^1\wedge SA \wedge SC \wedge SV \wedge SU = \eta \wedge G \wedge 1_{SC}$. Then $X\cdot M_2 \in E^6$ is represented by

$F = (\mu_{01}\wedge G \wedge 1_{SC}) \cup (\mu_2\wedge B_{\eta X})$:

$$\left((D^4\wedge DB \wedge S^1\wedge SA \wedge SC \wedge SV \wedge SU) \cup (S^3\wedge DB \wedge D^2\wedge SA \wedge SC \wedge SV \wedge SU),\right.$$
$$\left.(D^4\wedge SB \wedge S^1\wedge SA \wedge SC \wedge SV \wedge SU) \cup (S^3\wedge SB \wedge D^2\wedge SA \wedge SC \wedge SV \wedge SU)\right)$$
$$\longrightarrow (SB \wedge SA \wedge SC \wedge SU \wedge BP^{[6]}, SB \wedge SA \wedge SC \wedge SU).$$

Thus, $d^6(X\cdot M_2)$ is represented by F restricted to the boundary of its domain which is $(B_{\nu\eta}\wedge G) \cup (\nu \wedge B_{\eta X}) \in \langle\nu, \eta, X\rangle$.

(c) Represent $\bar{M}_2 \in E^4_{6,0}$ by

$$\bar{\mu}_{01}: \left(D^2\wedge DA \wedge S^3\wedge SB \wedge SH, (S^1\wedge DA \wedge S^3\wedge SB \wedge SH) \cup (D^2\wedge SA \wedge S^3\wedge SB \wedge SH)\right) \longrightarrow (SA \wedge SB \wedge SH \wedge BP^{[6]}, SA \wedge SB \wedge SH \wedge BP^{[2]})$$

such that $\bar{\mu}_{01}$ restricted to the boundary of its domain is $[(\mu_1 \wedge \nu) \cup B_{\eta\nu}] \wedge 1_{SH}$. Let $B_{\nu X}: D^4 \wedge SB \wedge SH \wedge SV \wedge SU \longrightarrow SB \wedge SH \wedge SU$ such that $B_{\nu X} \mid S^3\wedge SB \wedge SH \wedge SV \wedge SU = \nu \wedge G \wedge 1_{SH}$. Then $X\cdot\bar{M}_2 \in E^6$ is represented by $F = (\bar{\mu}_{01}\wedge G) \cup (\mu_1\wedge B_{\nu X})$:

$$\left((D^2\wedge DA \wedge S^3\wedge SB \wedge SH \wedge SV \wedge SU) \cup (S^1\wedge DA \wedge D^4\wedge SB \wedge SH \wedge SV \wedge SU),\right.$$
$$\left.(D^2\wedge SA \wedge S^3\wedge SB \wedge SH \wedge SV \wedge SU) \cup (S^1\wedge SA \wedge D^4\wedge SB \wedge SV \wedge SU)\right) \longrightarrow$$
$$(SA \wedge SB \wedge SH \wedge SU \wedge BP^{[6]}, SA \wedge SB \wedge SH \wedge SU).$$

Thus, $d^6(X\cdot\bar{M}_2)$ is represented by F restricted to the boundary of its domain which is $(B_{\eta\nu} \wedge G \wedge 1_{SH}) \cup (\eta \wedge B_{\nu X}) \in \langle\eta, \nu, X\rangle$. ∎

THEOREM 2.4.5 Let $X \in \pi_*^S$.

(a) $\langle \nu, \eta, X, \eta \rangle$ is defined if and only if $X \cdot M_1 M_2$ survives to E^8. In this case $\langle \nu, \eta, X, \eta \rangle$ projects to $d^8(X \cdot M_1 M_2)$. Moreover, νX is divisible by η.

(b) Assume that $\langle \eta, \nu, X, \nu \rangle$ is defined. Then $X \cdot M_1^2 \overline{M}_2$ survives to E^{10}, and $\langle \eta, \nu, X, \nu \rangle$ projects to $d^{10}(X \cdot M_1^2 \overline{M}_2)$. Moreover, σX is divisible by ν.

PROOF. Let $G: SV \wedge SU \longrightarrow SU$ represent $X \in \pi_*^S$. We use the smash product Toda bracket of Definition 2.2.2 throughout this proof and the notation of the proof of the preceding theorem.

(a) XM_1M_2 survives to E^4 if and only if $\eta X = 0$. In this case, XM_1M_2 survives to E^6 if and only if νX is divisible by η, i.e. $0 \in \langle \eta, X, \eta \rangle$. Then $d^6(XM_1M_2) = d^6(XM_2)M_1$ and $d^6(XM_2) \in \langle \nu, \eta, X \rangle$. Thus XM_1M_2 survies to E^8 if and only if $d^6(XM_2) \in (v)$, i.e. $0 \in \langle \nu, \eta, X \rangle$. Therefore, XM_1M_2 survives to E^8 if and only if $0 \in \langle \eta, X, \eta \rangle$ and $0 \in \langle \nu, \eta, X \rangle$. Then by Theorem 2.2.7(f), XM_1M_2 survives to E^8 if and only if $\langle \nu, \eta, X, \eta \rangle$ is defined. In that case let the following diagram depict a defining system for $\langle \nu, \eta, X, \eta \rangle_\wedge$:

$\nu \qquad \eta \qquad G \qquad \eta$

$B_{\nu\eta} \qquad B_{\eta X} \qquad B_{X\eta}$

$B_{\langle \nu, \eta, X \rangle} \qquad B_{\langle \eta, X, \eta \rangle}$

Here $B_{X\eta}: DV \wedge SU \wedge S^1 \wedge SA' \longrightarrow SU \wedge SA'$ such that $B_{X\eta} | SV \wedge SU \wedge S^1 \wedge SA' = G \wedge \eta$, $B_{\langle \nu, \eta, X \rangle}: D^4 \wedge SB \wedge D^2 \wedge SA \wedge SC \wedge SV \wedge SU \longrightarrow SB \wedge SA \wedge SC \wedge SU$ such that $B_{\langle \nu, \eta, X \rangle} | \partial$ [Domain $B_{\langle \nu, \eta, X \rangle}$] $= (B_{\nu\eta} \wedge G) \cup (\nu \wedge B_{\eta X})$ and $B_{\langle \eta, X, \eta \rangle}: D^2 \wedge SA \wedge SC \wedge DV \wedge SU \wedge S^1 \wedge SA' \longrightarrow SA \wedge SC \wedge SU \wedge SA'$ such that $B_{\langle \eta, X, \eta \rangle} | \partial$ [Domain $B_{\langle \eta, X, \eta \rangle}$] $= (\eta \wedge B_{X\eta}) \cup (B_{\eta X} \wedge \eta)$. Then the following map F represents $X \cdot M_1 M_2$ in E^8. F =

$(\mu_{02} \wedge G \wedge \mu_1) \cup (\mu_2 \wedge B_{\eta X} \wedge \mu_1) \cup (\mu_{02} \wedge B_{X\eta}) \cup (B_{\langle \nu, \eta, X \rangle} \wedge \mu_1) \cup (\mu_2 \wedge B_{\langle \eta, X, \eta \rangle})$:

$\big[(D^4 \wedge DB \wedge S^1 \wedge SA \wedge SC \wedge SV \wedge SU \wedge S^1 \wedge DA') \cup (S^3 \wedge DB \wedge D^2 \wedge SA \wedge SC \wedge SV \wedge SU \wedge S^1 \wedge DA')$

$\cup (D^4 \wedge DB \wedge S^1 \wedge SA \wedge SC \wedge DV \wedge SU \wedge S^1 \wedge SA') \cup (D^4 \wedge SB \wedge D^2 \wedge SA \wedge SC \wedge SV \wedge SU \wedge S^1 \wedge DA')$

$\cup (S^3 \wedge DB \wedge D^2 \wedge SA \wedge SC \wedge DV \wedge SU \wedge S^1 \wedge SA'), (S^3 \wedge SB \wedge D^2 \wedge SA \wedge SC \wedge DV \wedge SU \wedge S^1 \wedge SA')$

$\cup (D^4 \wedge SB \wedge S^1 \wedge SA \wedge SC \wedge DV \wedge SU \wedge S^1 \wedge SA') \cup (D^4 \wedge SB \wedge D^2 \wedge SA \wedge SC \wedge SV \wedge SU \wedge S^1 \wedge SA')\big]$

$$\longrightarrow (SB \wedge SA \wedge SC \wedge SU \wedge SA' \wedge BP^{[8]}, SB \wedge SA \wedge SC \wedge SU \wedge SA').$$

Thus $d^8(X\cdot M_1M_2)$ is represented by $F|\partial$ [Domain F]

$= (v \wedge B_{<\eta,X,\eta>}) \cup (B_{\nu\eta} \wedge B_{X\eta}) \cup (B_{<\nu,\eta,X>} \wedge \eta) \in <\nu,\eta,X,\eta>$.

The indeterminacy of $<\nu,\eta,X,\eta>$ is a sum of elements of the form ηA, νB, $<\nu,\eta,C>$ and $<\nu,D,\eta>$. All such elements project to zero in E^8. Thus, $<\nu,\eta,X,\eta>$ projects to a singleton in E^8. By Theorem 2.4.2, $v\cdot X \in <\eta,X,\eta>$. However, $0 \in <\eta,X,\eta>$ since $<\nu,\eta,X,\eta>$ is defined. Thus, $\nu\cdot X$ is in the indeterminacy of $<\eta,X,\eta>$ which is the ideal geneerated by η.

(b) Let the following diagram depict a defining system for $<\eta,\nu,X,\nu>$:

η		ν		G		ν
	$B_{\eta\nu}$		$B_{\nu X}$		$B_{X\nu}$	
		$B_{<\eta,\nu,X>}$		$B_{<\nu,X,>\nu}$		

Here $B_{X\nu}: DV \wedge SU \wedge S^3 \wedge SB' \longrightarrow SU \wedge SB'$ such that $B_{X\nu}|SV \wedge SU \wedge S^3 \wedge SB' = G \wedge \nu$, $B_{<\eta,\nu,X>}: D^2 \wedge SA \wedge D^4 \wedge SB \wedge SH \wedge SV \wedge SU \longrightarrow SA \wedge SB \wedge SH \wedge SU$ such that $B_{<\eta,\nu,X>}|[\partial$ Domain $B_{<\eta,\nu,X>}] = (\eta \wedge B_{\nu X}) \cup (B_{\eta\nu} \wedge G)$ and $B_{<\nu,X,\nu>}: D^4 \wedge SB \wedge SH \wedge DV \wedge SU \wedge S^3 \wedge SB' \longrightarrow SB \wedge SH \wedge SU \wedge SB'$ such that $B_{<\nu,X,\nu>}|[\partial$ Domain $B_{<\nu,X,\nu>}] = (\nu \wedge B_{X\nu}) \cup (B_{\nu X} \wedge \nu)$. Then the following map F represents $X\cdot M_1^2\overline{M}_2$ in E^{10}: F =

$$(\bar{\mu}_{02} \wedge G \wedge \mu_2) \cup (\mu_1 \wedge B_{\nu X} \wedge \mu_2) \cup (B_{<\eta,\nu,X>} \wedge \mu_2) \cup (\bar{\mu}_{02} \wedge B_{X\nu}) \cup (\mu_1 \wedge G_{<\nu,X,\nu>}):$$
$$\big[(D^2 \wedge DA \wedge S^3 \wedge SB \wedge SH \wedge SV \wedge SU \wedge S^3 \wedge DB') \cup (S^1 \wedge DA \wedge D^4 \wedge SB \wedge SH \wedge SV \wedge SU \wedge S^3 \wedge DB')$$
$$\cup (D^2 \wedge SA \wedge D^4 \wedge SB \wedge SH \wedge SV \wedge SU \wedge S^3 \wedge DB') \cup (D^2 \wedge DA \wedge S^3 \wedge SB \wedge SH \wedge DV \wedge SU \wedge S^3 \wedge SB')$$
$$\cup (S^1 \wedge DA \wedge D^4 \wedge SB \wedge SH \wedge DV \wedge SU \wedge S^3 \wedge SB'), (S^1 \wedge SA \wedge D^4 \wedge SB \wedge SH \wedge DV \wedge SU \wedge S^3 \wedge SB')$$
$$\cup (D^2 \wedge SA \wedge S^3 \wedge SB \wedge SH \wedge DV \wedge SU \wedge S^3 \wedge SB') \cup (D^2 \wedge SA \wedge D^4 \wedge SB \wedge SH \wedge DV \wedge SU \wedge S^3 \wedge SB')\big]$$
$$\longrightarrow (SA \wedge SB \wedge SH \wedge SU \wedge SB' \wedge BP^{[10]}, SA \wedge SB \wedge SH \wedge SU \wedge SB').$$

Thus $X\cdot M_1^2\overline{M}_2$ survives to E^{10} and $d^{10}(X\cdot M_1^2\overline{M}_2)$ is represented by $F|\partial$ [Domain F] $= (1_{SH} \wedge \eta \wedge B_{<\nu,X,\nu>}) \cup (1_{SH} \wedge B_{\eta\nu} \wedge B_{X\nu}) \cup (1_{SH} \wedge B_{<\eta,\nu,X>} \wedge \nu) \in <\eta,\nu,X,\nu>$. The indeterminacy of $<\eta,\nu,X,\nu>$ is the sum of elements of the form ηA, νB, $<\eta,\nu,C>$ and $<\eta,D,\nu>$. All such elements project to 0 in E^8. Therefore, $<\eta,\nu,X,\nu>$

projects to a singleton in E^{10}. By Theorem 2.4.2, $\sigma\cdot X \in \langle\nu,X,\nu\rangle$. However, $0 \in \langle\nu,X,\nu\rangle$ because $\langle\eta,\nu,X,\nu\rangle$ is defined. Therefore $\sigma\cdot X$ is in the indeterminacy of $\langle\nu,X,\nu\rangle$ which is the ideal generated by ν. ∎

The following theorem gives three special cases of Theorem 2.4.3 where no technical hypotheses are required.

THEOREM 2.4.6 (a) Let $X \in \pi_*^S$, and assume that $\langle\sigma,\nu,X,\eta\rangle$ is defined. Then $XM_1M_2^2$ survives to E^{14} and $d^{14}(XM_1M_2^2) \in \langle\sigma,\nu,X,\eta\rangle$.

(b) Let $X \in \pi_*^S$, and assume that $\langle\nu,\sigma,X,\eta\rangle$ is defined. Then $XM_1\langle M_2^2\rangle$ survives to E^{14} and $d^{14}(XM_1\langle M_2^2\rangle) \in \langle\nu,\sigma,X,\eta\rangle$.

(c) Let $d^{2r}(Y) = X \in \pi_*^S$ and let $\xi \in \pi_*^S$. Assume that $\langle X,\xi,\nu,\eta\rangle$ is defined. Then $\xi Y\bar{M}_2$ survives to E^{2r+6} and $d^{2r+6}(\xi Y\bar{M}_2) \in \langle X,\xi,\nu,\eta\rangle$.

(d) Let $d^{2r}(Y) = X \in \pi_*^S$ and let $\xi \in \pi_*^S$. Assume that $\langle X,\xi,\eta,\nu\rangle$ is defined. Then ξYM_2 survives to E^{2r+6} and $d^{2r+6}(\xi YM_2) \in \langle X,\xi,\eta,\nu\rangle$.

PROOF. (a) Let the following diagram depict a defining system for $\langle\sigma,\nu,X,\eta\rangle_\wedge$ using the same notation as in the previous theorems:

$$\begin{array}{ccccccc} \sigma & & \nu & & G & & \eta \\ & B_{\sigma\nu} & & B_{\nu X} & & B_{X\eta} & \\ & & B_{\langle\sigma,\nu,X\rangle} & & B_{\langle\nu,X,\eta\rangle} & & \end{array}$$

Let μ_4 represent $\langle M_1^4\rangle$ such that μ_4 restricted to the boundary of its domain is σ. Let μ_{02} represent M_2^2 such that μ_{02} restricted to the boundary of its domain is $(\mu_4\wedge\nu)\cup B_{\sigma\nu}$. Then $XM_1M_2^2$ is represented by $\phi =$ $(\mu_{12}\wedge G\wedge\mu_1)\cup(\mu_{02}\wedge B_{X\eta})\cup(\mu_4\wedge B_{\langle\nu,X,\eta\rangle})\cup(B_{\langle\sigma,\nu,X\rangle}\wedge\mu_1)\cup(\mu_4\wedge B_{\nu X}\wedge\mu_1)$.
Note that ϕ restricted to the boundary of its domain is $(\sigma\wedge B_{\langle\nu,X,\eta\rangle})\cup(B_{\sigma\nu}\wedge B_{X\eta})\cup(B_{\langle\sigma,\nu,X\rangle}\wedge\eta)$ which is an element of $\langle\sigma,\nu,X,\eta\rangle$.
Thus, $XM_1M_2^2$ survives to E^{14} and $d^{14}(XM_1M_2^2) \in \langle\sigma,\nu,X,\eta\rangle$.

(b) Let the following diagram depict a defining system for $\langle\nu,\sigma,X,\eta\rangle_\wedge$ using the above notation:

$$\begin{array}{ccccccc} \nu & & \sigma & & G & & \eta \\ & B_{\nu\sigma} & & B_{\sigma X} & & B_{X\eta} & \\ & & B_{\langle\nu,\sigma,X\rangle} & & B_{\langle\sigma,X,\eta\rangle} & & \end{array}$$

Let $\langle\mu_{02}\rangle$ represent $\langle M_2^2\rangle$ such that $\langle\mu_{02}\rangle$ restricted to the boundary of its domain is $(\mu_2 \wedge \sigma) \cup B_{\nu\sigma}$. Then $XM_1\langle M_2^2\rangle$ is represented by $\phi = (\langle\mu_{02}\rangle \wedge G \wedge \mu_1) \cup (\langle\mu_{02}\rangle \wedge B_{X\eta}) \cup (\mu_2 \wedge B_{\sigma X} \wedge \mu_1) \cup (\mu_2 \wedge B_{\langle\sigma,X,\eta\rangle})$ $\cup\ (B_{\langle\nu,\sigma,X\rangle} \wedge \mu_1)$. Note that ϕ restricted to the boundary of its domain is $(B_{\nu\sigma} \wedge B_{X\eta}) \cup (\nu \wedge B_{\langle\sigma,X,\eta\rangle}) \cup (B_{\langle\nu,\sigma,X\rangle} \wedge \eta)$ which is an element of $\langle\nu,\sigma,X,\eta\rangle$. Thus, $XM_1\langle M_2^2\rangle$ survives to E^{14} and $d^{14}(XM_1\langle M_2^2\rangle) \in \langle\nu,\sigma,X,\eta\rangle$.

(c) Let the following diagram depict a defining system for $\langle X,\xi,\nu,\eta\rangle_\wedge$ using the above notation:

$$\begin{array}{ccccccc} G & & \xi & & \nu & & \eta \\ & B_{X\xi} & & B_{\xi\nu} & & B_{\nu\eta} & \\ & & B_{\langle X,\xi,\nu\rangle} & & B_{\langle\xi,\nu,\eta\rangle} & & \end{array}$$

Then $\xi Y\bar{M}_2$ is represented by $\phi = (Y \wedge \xi \wedge \bar{\mu}_{01}) \cup (Y \wedge B_{\xi\nu} \wedge \mu_1) \cup (B_{X\xi} \wedge \bar{\mu}_{01})$ $\cup\ (B_{\langle X,\xi,\nu\rangle} \wedge \mu_1) \cup (Y \wedge B_{\langle\xi,\nu,\eta\rangle})$. Note that ϕ restricted to the boundary of its domain is $(B_{X\xi} \wedge B_{\nu\eta}) \cup (B_{\langle X,\xi,\nu\rangle} \wedge \eta) \cup (X \wedge B_{\langle\xi,\nu,\eta\rangle})$ which is an element of $\langle X,\xi,\nu,\eta\rangle$. Thus, $\xi Y\bar{M}_2$ survives to E^{2r+6} and $d^{2r+6}(\xi Y\bar{M}_2) \in \langle X,\xi,\nu,\eta\rangle$.

(d) Let the following diagram depict a defining system for $\langle X,\xi,\eta,\nu\rangle_\wedge$ using the above notation:

$$\begin{array}{ccccccc} G & & \xi & & \eta & & \nu \\ & B_{X\xi} & & B_{\xi\eta} & & B_{\eta\nu} & \\ & & B_{\langle X,\xi,\eta\rangle} & & B_{\langle\xi,\eta,\nu\rangle} & & \end{array}$$

Then ξYM_2 is represented by $\phi = (Y \wedge \xi \wedge \mu_{01}) \cup (Y \wedge B_{\xi\eta} \wedge \mu_2) \cup (B_{X\xi} \wedge \mu_{01})$ $\cup\ (B_{\langle X,\xi,\eta\rangle} \wedge \mu_2) \cup (Y \wedge B_{\langle\xi,\eta,\nu\rangle})$. Note that ϕ restricted to the boundary of its domain is $(B_{X\xi} \wedge B_{\eta\nu}) \cup (B_{\langle X,\xi,\eta\rangle} \wedge \nu) \cup (X \wedge B_{\langle\xi,\eta,\nu\rangle})$ which is an element of $\langle X,\xi,\eta,\nu\rangle$. Thus, ξYM_2 survives to E^{2r+6} and $d^{2r+6}(\xi YM_2) \in \langle X,\xi,\eta,\nu\rangle$. ∎