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AN ALGEBRAIC FILTRATION OF $H_*(MO;\mathbb{Z}_2)$

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1. Introduction

Let $\mathscr{A}_{2*}$ denote the dual of the mod two Steenrod algebra. In [**5**] an algebraic filtration $B_*(n)$ of $H_*(BO;\mathbb{Z}_2)$ was constructed such that each $B_*(n)$ is a bipolynomial sub Hopf algebra and sub $\mathscr{A}_{2*}$-comodule of $H_*(BO;\mathbb{Z}_2)$. In Lemma 3.1 we prove that the Thom isomorphism determines a corresponding filtration of $H_*(MO;\mathbb{Z}_2)$ by polynomial subalgebras and sub $\mathscr{A}_{2*}$-comodules $M_*(n)$. Let $\mathscr{A}(n)$ denote the subalgebra of $\mathscr{A}_2$ generated by Sq^{2^k}, $0\leqq k<n$, and let $\mathscr{A}_*(n)$ be its dual, a quotient Hopf algebra of $\mathscr{A}_{2*}$. In Section 3 we construct a polynomial algebra and $\mathscr{A}_*(n)$-comodule $R(n)$ such that $M_*(n)\simeq\mathscr{A}_{2*}\square_{\mathscr{A}_*(n)}R(n)$ as algebras and $\mathscr{A}_{2*}$-comodules. Here $\square$ denotes the cotensor product defined in [**9**, §2]. Dually it will follow that $M^*(n)$ has a sub $\mathscr{A}(n)$-module and subcoalgebra $T(n)$ such that $M^*(n)\simeq\mathscr{A}_2\otimes_{\mathscr{A}(n)}T(n)$ as coalgebras and $\mathscr{A}_2$-modules. We also show that $M_*(n)$ can not be realised as the homology of a spectrum for $n\geqq 4$. Of course $M_*(0)=H_*(MO;\mathbb{Z}_2)$, $M_*(1)=H_*(MSO;\mathbb{Z}_2)$, $M_*(2)=H_*(M\mathrm{Spin};\mathbb{Z}_2)$ and $M_*(3)=H_*(MO\langle 8\rangle;\mathbb{Z}_2)$. Moreover, it follows from [**4**; Thm. 2.10, Cor. 2.11] that $M_*(n)=\mathrm{Image}[H_*(MO\langle\phi(n)\rangle;\mathbb{Z}_2)\to H_*(MO;\mathbb{Z}_2)]$ and $M^*(n)\simeq\mathrm{Image}[H^*(MO;\mathbb{Z}_2)\to H^*(MO\langle\phi(n)\rangle;\mathbb{Z}_2)]$. Here $MO\langle k\rangle$ id the Thom spectrum of $BO\langle k\rangle$, the $(k-1)$-connected covering of BO, and $\phi(n)=8s+2^t$ where $n=4s+t$, $0\leqq t\leqq 3$. In Section 4 we sketch the odd primary analogue—a filtration ${}_pM_*(n)$ of $H_*(MU_{p,0};\mathbb{Z}_p)$ for p an odd prime. $MU_{p,0}$ is the Thom spectrum of the $(2p\text{-}3)$-connected factor of the Adams splitting [**2**] of $\mathrm{BU}_{(p)}$.

Our structure theorems of Sections 3 and 4 follow from a general algebraic structure theorem which we prove in Section 2. That theorem generalizes the technique of Pengelley [**10**], [**11**] where he proved the special cases of our structure theorems for $M_*(n)$, $1\leqq n\leqq 3$.

2. A structure theorem for comodule algebras

The theorem below will be used in Sections 3 and 4 to determine the structure of $M_*(n)$ and ${}_pM_*(n)$. This theorem generalises the arguments of Pengelley [**11**] which in turn generalises the argument of Liulevicius [**7**].

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Theorem 2.1. *Let H be a connected Hopf algebra of finite type over a field F. Let M be a connected F-algebra of finite type and a left H-comodule with coaction ψ sich that ψ is an algebra homomorphism. Let H_0 be a commutative normal sub Hopf algebra of H. Assume that $H_0 \subset M$ is a sub-algebra of the centre of M and that M is a free H_0-module. Assume that $\psi|H_0 = \Delta|H_0$ where Δ is the coproduct of H. Then there is an F-algebra and left $H//H_0$-comodule N whose coaction ψ' is an algebra homomorphism such that $M \simeq H\Box_{H//H_0}N$ as algebras and H-comodules. Here $H\Box_{H//H_0}N$ has coaction $\Delta\Box 1$.*

Proof. Let J be the ideal in M generated by the augmentation ideal of H_0, and let $N = M/J$ as an algebra. Then the H-coaction ψ on M induces a $H//H_0$-coaction ψ' on N. Clearly ψ' is an algebra homomorphism. Let $\pi: M \to N$ be the canonical map. Consider the following diagram.

$$\begin{array}{ccccc} M & \xrightarrow{\psi} & H\otimes M & \xrightarrow{1\otimes\pi} & H\otimes N \\ & \searrow^{\phi} & & & \uparrow \\ & & & & H\Box_{H//H_0}N \end{array}$$

Note that ϕ exists because $(\Delta\otimes 1 - 1\otimes\psi')(1\otimes\pi)\psi = (1\otimes 1\otimes\pi)(\Delta\otimes 1 - 1\otimes\psi)\psi = 0$. ϕ is a map of algebras and H-comodules because $(1\otimes\pi)\psi$ is and $H\Box_{H//H_0}N$ is a subalgebra and sub H-comodule of $H\otimes N$. Let $x\in M$. Write $x=\sum_{i=1}^{t} x_i h_i$ with $h_i\in H_0, x_i\notin J$ and $\deg x_i \leqq \deg x_{i+1}$ for all i. This is possible because H_0 is contained in the centre of M. Assume that x and all the h_i are nonzero and that $\{x_1,\ldots,x_t\}$ is linearly independent. Then $(1\otimes\pi)\psi(x)$ contains $h_t\otimes x_t$ as a nonzero summand. Thus $(1\otimes\pi)\psi(x)\neq 0$ and ϕ is one-to-one. By (9), $H\simeq H_0\otimes H//H_0$ as right $H//H_0$-comodules.

Thus as F-vector spaces we have

$$H\Box_{H//H_0}N \simeq (H_0\otimes H//H_0)\Box_{H//H_0}N \simeq H_0\otimes(H//H_0\Box_{H//H_0}N)\simeq H_0\otimes N\simeq M.$$

The last isomorphism holds because M is a free H_0-module. Thus the range and domain of ϕ have the same dimension in each degree and ϕ is an isomorphism.

3. The structure of $M_*(n)$ and $M^*(n)$

We begin by establishing that the $M_*(n)$ and $M^*(n)$ have the algebraic structure we wish to study.

Lemma 3.1. *The $M_*(n)$ are polynomial subalgebras and sub $\mathscr{A}_{2*}$-comodules of $H_*(MO;\mathbb{Z}_2)$. The $M^*(n)$ are quotient coalgebras and quotient $\mathscr{A}_2$-modules of $H^*(MO;\mathbb{Z}_2)$.*

Proof. We prove that the $M^*(n)$ are quotient $\mathscr{A}_2$-modules of $H^*(MO;\mathbb{Z}_2)$. The remaining assertions will then follow from the properties of the $B_*(n)$, $B^*(n)$, the Thom isomorphism and duality. Write $B^*(n)=H^*(BO;\mathbb{Z}_2)/I_n$ where I_n is an ideal and $\mathscr{A}_2$-submodule of $H^*(BO;\mathbb{Z}_2)$. (See [5, Theorem 2.1].) Let $x\in I_n$, let $\theta\in\mathscr{A}_2$ and let Φ denote the Thom isomorphism. Then $\theta\Phi(x)=\sum_i \Phi[\theta_i'(x)\Phi^{-1}(\theta_i''\Phi(1))]$ where $\Delta(\theta)=\sum_i \theta_i'\otimes\theta_i''$. Hence $\theta\Phi(x)\in\Phi(I_n)$ and thus $\Phi(I_n)$ is an $\mathscr{A}_2$-submodule of $H^*(MO;\mathbb{Z}_2)$. Therefore $M^*(n)=H^*(MO;\mathbb{Z}_2)/\Phi(I_n)$ is a quotient $\mathscr{A}_2$-module of $H^*(MO;\mathbb{Z}_2)$.

By [12], $H_*(MO;\mathbb{Z}_2)$ contains the dual of the Steenrod algebra $\mathscr{A}_{2*}=\mathbb{Z}_2[\xi_1,\ldots,\xi_n,\ldots]$. It follows from [8] that $[\mathscr{A}_2//\mathscr{A}(n)]^*$ is the sub Hopf algebra $S(n)=\mathbb{Z}_2[\bar\xi_1^{2^n},\bar\xi_2^{2^{n-1}},\ldots,\bar\xi_n^2,\bar\xi_{n+1},\bar\xi_{n+2},\ldots]$ of $\mathscr{A}_{2*}$ where $\bar\xi_k$ denotes the conjugate of ξ_k. Thus $\mathscr{A}_*(n)$ is the truncated polynomial algebra given as a quotient Hopf algebra of $\mathscr{A}_{2*}$ as having generators ξ_k, $1\leqq k\leqq n$, with ξ_k truncated at height 2^{n-k+1}.

Lemma 3.2 $M_*(n)\supset S(n)$.

Proof. By [3] we can take $\bar\xi_k\in H_*(MO;\mathbb{Z}_2)$ to be $\Phi(\mathscr{P}_{2^k-1})$ where $\mathscr{P}_{2^k-1}\in PH_{2^k-1}(BO;\mathbb{Z}_2)$. By [5, Corollary 2.4] $B_*(k-1)$ has a unique nonzero primitive element in degree 2^k-1 which must be $\mathscr{P}_{2^k-1}$. If $k\leqq n$ then $\mathscr{P}_{2^k-1}^{2^{n-k+1}}\in B_*(n)$ by [5, Theorem 4.2]. Hence $\bar\xi_k\in M_*(n)$ for $k\geqq n+1$ and $\bar\xi_k^{2^{n-k+1}}\in M_*(n)$ for $n\geqq k\geqq 1$. Thus $S(n)\subset M_*(n)$.

We now apply the structure theorem of Section 2 to $M_*(n)$. If $k=2^{k_1}+\ldots+2^{k_t}$ with $0\leqq k_1<\ldots<k_t$ then write $L(k)=t$ and $M(k)=k_1$.

Theorem 3.3 *There is a left $\mathscr{A}_*(n)$-comodule and $\mathbb{Z}_2$-algebra*

$$R(n)=\mathbb{Z}_2[X_{k,n}\,|\,L(k)+M(k)>n, k\neq 2^{L(k)}-1, \text{ and } k2^{L(k)-n-1}\neq 2^{L(k)}-1]$$

such that degree $X_{k,n}=k$ and $M_*(n)\simeq\mathscr{A}_{2*}\square_{\mathscr{A}_*(n)}R(n)$ as $\mathbb{Z}_2$-algebras and $\mathscr{A}_{2*}$-comodules.

Proof. We apply Theorem 2.1 with $H=\mathscr{A}_2^*$, $H_0=S(n)$ and $M=M_*(n)$. Now the polynomial generators of $S(n)$ are a partial set of polynomial generators for $M_*(n)$. Thus $M_*(n)$ is a free $S(n)$-module. The remaining hypotheses of Theorem 2.1 are easily seen to hold. Thus our theorem holds with $R(n)=M_*(n)/J(n)$ and $J(n)$ the ideal in $M_*(n)$ generated by the augmentation ideal of $S(n)$. By [5, Corollary 2.4] $R(n)$ must be polynomial algebra with generators in the degrees asserted above.

Corollary 3.4 *There is a subcoalgebra and sub $\mathscr{A}(n)$-module $T(n)$ of $M^*(n)$ such that $M^*(n)\simeq\mathscr{A}_2\otimes_{\mathscr{A}(n)}T(n)$ as coalgebras and $\mathscr{A}$-modules.*

Proof. Set $T(n)=[M_*(n)/J(n)]^*$ in the notation of the proof of Theorem 3.3.

Corollary 3.5. *$\mathscr{A}_2//\mathscr{A}(n)$ is a direct summand of $M^*(n)$ simultaneously as a coalgebra and $\mathscr{A}_2$-module.*

Proof. $T(n)=Z_2\oplus T(n)^+$ so $M^*(n)\simeq\mathscr{A}_2\otimes_{\mathscr{A}(n)}T(n)=(\mathscr{A}_2\otimes_{\mathscr{A}(n)}Z_2)\oplus(\mathscr{A}_2\otimes_{\mathscr{A}(n)}T(n)^+)$. Now $\mathscr{A}_2\otimes_{\mathscr{A}(n)}Z_2=\mathscr{A}_2//\mathscr{A}(n)$.

We conclude by showing that the $M_*(n)$ can not be realised geometrically for $n\geqq 4$.

Theorem 3.6. *For $n \geqq 4$ there is no spectrum X whose $\mathbb{Z}_2$-homology is isomorphic to $M_*(n)$ as $\mathscr{A}_{2*}$-comodules.*

Proof. Assume that such a spectrum X exists Then $\mathrm{Sq}^{2^n}(1) \neq 0$ in $H^{2^n}(X; Z_2)$ and $H^k(X; Z_2) = 0$ for $0 < k < 2^n$. By [1], Sq^{2^n} factors using secondary operations for $n \geqq 4$, a contradiction.

4. An algebraic filtration of $H_*(MU_{p,0}; \mathbb{Z}_p)$, p ODD

Let p be a fixed odd prime. By Adams [2] $BU_{(p)} = \Pi_{i=0}^{p-2} BU_{p,i}$ where $BU_{p,0}$ is (2p-3)-connected and hence $MU_{(p)} = \Pi_{i=0}^{p-2} MU_{p,i}$. Of course each $MU_{p,i}$ splits into suspensions of Brown-Peterson spectra. In [5, Section 6] we defined an algebraic filtration of $H_*(BU_{p,0}; \mathbb{Z}_p)$ by bipolynomial sub Hopf algebras and sub $\mathscr{A}_{p*}$-comodules ${}_pB_*(n)$. Arguing as in Lemma 3.1 we see that $H_*(MU_{p,0}; \mathbb{Z}_p)$ is filtered by polynomial subalgebras and sub $\mathscr{A}_p^*$-comodules ${}_pM_*(n)$. The duals ${}_pM^*(n)$ are quotient coalgebras and quotient $\mathscr{A}_p$-modules of $H^*(MU_{p,0}; \mathbb{Z}_p)$.

Let $\mathscr{A}_p(n)$ denote the subalgebra of $\mathscr{A}'_p$ generated by $\mathscr{P}^{p^k}$, $0 \leqq k < n$, where $\mathscr{A}'_p = \mathscr{A}_p/(\beta)$ is the Hopf algebra of reduced mod p Steenrod operations. Then $[\mathscr{A}'_p//\mathscr{A}_p(n)]^*$ is the sub Hopf algebra $S_p(n) = Z_p[\xi_1^{p^n}, \xi_2^{p^{n-1}}, \ldots, \xi_n^p, \xi_{n+1}, \xi_{n+2}, \ldots]$ of $\mathscr{A}'_{p*} = Z_p[\xi_1, \ldots, \xi_k, \ldots]$. As in Lemma 3.2, $S_p(n) \subset {}_pM_*(n)$. Write $k(p-1) = k_1 p^{e_1} + \ldots + k_t p^{e_t}$ with $0 \leqq e_1 < \ldots < e_t$ and $1 \leqq k_i \leqq p-1$. Define $L(k) = (k_1 + \ldots + k_t)/(p-1)$ and $M(k) = e_1$. Then Theorem 2.1 applies to ${}_pM_*(n)$ with $H = \mathscr{A}'_{p*}$, $H_0 = S_p(n)$ and $M = {}_pM_*(n)$ to produce the following theorem.

Theorem 4.1. *There is a left $\mathscr{A}_{p*}(n)$-comodule and $\mathbb{Z}_p$-algebra*

$$R_p(n) = \mathbb{Z}_2[Y_{k,n} \mid L(k) + M(k) > n,\ k(p-1) \neq p^{L(k)} - 1 \text{ and } k(p-1)p^{L(k)-n-1} \neq p^{L(k)} - 1]$$

such that $\deg Y_{k,n} = 2k(p-1)$ *and* ${}_pM_*(n) \simeq \mathscr{A}'_{p*} \square_{\mathscr{A}_{p*}(n)} R_p(n)$ *as $\mathbb{Z}_p$-algebras and $\mathscr{A}_{p*}$-comodules.*

Corollary 4.2. *There is a subcoalgebra and sub $\mathscr{A}_p(n)$-module $T_p(n)$ of ${}_pM^*(n)$ such that ${}_pM^*(n) \simeq \mathscr{A}'_p \otimes_{\mathscr{A}_p(n)} T_p(n)$ as coalgebras and $\mathscr{A}_p$-modules.*

Corollary 4.3. *$\mathscr{A}'_p//\mathscr{A}_p(n)$ is a direct summand of ${}_pM^*(n)$ simultaneously as a coalgebra and $\mathscr{A}_p$-module.*

Theorem 4.4 *For $n \geqq 1$ there is no spectrum X whose Z_p-homology is isomorphic to ${}_pM_*(n)$ as $\mathscr{A}_{p*}$-comodules.*

Proof. Assume that such a spectrum X exists. Then $\mathscr{P}^{p^n}(1) \neq 0$ in $H^{2p^n(p-1)}(X; Z_p)$ and $H^k(X; Z_p) = 0$ for $0 < k < 2p^n(p-1)$. By [6], $\mathscr{P}^{p^n}$ factors using secondary operations for $n \geqq 2$, a contradiction. Let $n = 1$. Observe that H^*X is p-torsion-free because $H^{\mathrm{odd}}(X; Z_p) = 0$. Thus Kane's argument with BP operations [4, p. 6] applies to produce a contradiction.

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